SUBSIDIARY MATHEMATICS

FOR

ASSOCIATE NURSING PROGRAM

Student's book

Senior 6

FOREWORD

Dear Student,

Rwanda Basic Education Board (REB) is honoured to present S6 subsidiary Mathematics book which serves as a guide to competence-based teaching and learning to ensure consistency and coherence in the learning of the Mathematics content. The Rwandan educational philosophy is to ensure that you achieve full potential at every level of education which will prepare you to be well integrated in society and exploit employment opportunities.

The government of Rwanda emphasizes the importance of aligning teaching and learning materials with the syllabus to facilitate your learning process. Many factors influence what you learn, how well you learn and the competences you acquire. Those factors include the relevance of the specific content, the quality of teachers' pedagogical approaches, the assessment strategies and the instructional materials available. In this book, we paid special attention to the activities that facilitate the learning process in which you can develop your ideas and make new discoveries during concrete activities carried out individually or with peers.

In competence-based curriculum, learning is considered as a process of active building and developing of knowledge and meanings by the learner where concepts are mainly introduced by an activity, situation or scenario that helps the learner to construct knowledge, develop skills and acquire positive attitudes and values.

For efficiency use of this textbook, your role is to:

- Work on given activities which lead to the development of skills;
- Share relevant information with other learners through presentations, discussions, group work and other active learning techniques such as role play, case studies, investigation and research in the library, on internet or outside;
- Participate and take responsibility for your own learning;
- Draw conclusions based on the findings from the learning activities.

To facilitate you in doing activities, the content of this book is self explanatory so that you can easily use it yourself, acquire and assess your competences. The book is made of units as presented in the syllabus. Each unit has the following structure: the key unit competence is given and it is followed by the introductory activity before the development of mathematical concepts that are connected to real world problems or to other sciences.

The development of each concept has the following points:

- It starts by a learning activity: it is a hand on well set activity to be done by students in order to generate the concept to be learnt;
- Main elements of the content to be emphasized;

- Worked examples; and
- Application activities: those are activities to be done by the user to consolidate competences or to assess the achievement of objectives.

Even though the book has some worked examples, you will succeed on the application activities depending on your ways of reading, questioning, thinking and grappling ideas of calculus not by searching for similar–looking worked out examples.

Furthermore, to succeed in Mathematics, you are asked to persevere; sometimes you will find concepts that need to be worked at before you completely understand. The only way to really grasp such a concept is to think about it and work related problems found in other reference books or media.

I wish to sincerely extend my appreciation to the people who contributed towards the development of this book, particularly REB staff and teachers. Any comment or contribution would be welcome to the improvement of this text book for the next edition.

Dr. MBARUSHIMANA Nelson

Director General, REB

ACKNOWLEDGEMENT

I wish to express my appreciation to the people who played a major role in the development and the editing of Subsidiary Mathematics book for Senior Six students. It would not have been successful without active participation of different education stakeholders.

I owe gratitude to Curriculum Officers and teachers whose efforts during the development and editing exercise of this book were very much valuable.

Finally, my word of gratitude goes to the Rwanda Basic Education Board staffs who were involved in the whole process of in-house textbook production.

Joan MURUNGI,

Head of Curriculum, Teaching and Learning Resources Department.

TABLE OF CONTENT

UNIT1: COMPLEX NUMBERS

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Key unit competence

Perform operations on complex numbers in different forms and use them to solve related problems in Physics, Engeneering, etc.

Introductory activity

Consider the extension of sets of numbers previously learnt from natural numbers to real numbers. It is actually very common for equations to be unsolved in one set of numbers but solved in another.

Let us find the solution of the following quadratic equations in the set of real numbers:

$a)\ x^2 - 4 = 0$

$b)\ x^2 + 4 = 0$

- Discuss the solution for each equation and provide the solution set where possible.
- Does every quadratic equation have solution in $\mathbb{R}$?
- What happens to the equation $x^2 + 4 = 0$ if we conventionally accept a number i such that $i^2 = -1$?

In this case, can any quadratic equation be solved?

Let us find the solution of the following quadratic equations in the set of real numbers $\mathbb{R}$:

a) $x^2 - 4 = 0$

b) $x^2 + 4 = 0$

1. 1 Algebraic form of Complex numbers and their geometric representation

1.1.1 Definition of complex number

Activity 1.1

Using the formula of solving quadratic equations in the set of real numbers and considering that $\sqrt{-1}=i$, find the solution set of the following equation $x^2+16=0$

What do you think about your answer? Is it an element of the set $\mathbb{R}$? Explain.

To overcome the obstacle of unsolved equation in $\mathbb{R}$, Bombelli, Italian mathematician of the sixteenth century, created new numbers which were given the name complex numbers.

The symbol "i" satisfying $i^2=-1$ was therefore created. The equation $x^2=-1$, which had not solution in $\mathbb{R}$ gets two in the new set, because $x^2=-1$ gives $x=i$ or $x=-i$ if we respect the properties of operations in $\mathbb{R}$.

Definition:

Given two real numbers a and b we define the complex number z as $z=a+ib$ with $i^2=-1$. The number a is called the real part of z denoted by $\text{Re}(z)$ and the number b is called the imaginary part of z denoted by $\text{Im}(z)$; the set of complex numbers is denoted by $\mathbb{C}$. Mathematically the set of complex numbers is defined as $\mathbb{C}=\left\{a+ib; \quad a,b\in\mathbb{R} \quad and\ i^2=-1\right\}$.

The expression $z=a+ib$ is known as the algebraic form of a complex number z.

If $b=0$, then $z=a$ and z is said to be a real number. Hence, any real number is a complex number.

This gives $\mathbb{R}\subset\mathbb{C}$ to mean that the set of real numbers is a subset of complex numbers.

If $a=0$ and $b\neq 0$, then $z=ib$, and the number z is said to be **pure imaginary**. As in the previous classes, we can write that $\mathbb{N}\subset\mathbb{Z}\subset\mathbb{Q}\subset\mathbb{R}\subset\mathbb{C}$.

Example 1.1

Find the real part and imaginary part of the following complex numbers and give your observations.

$a)\,4+7i \quad b)\,5-3i \quad c)\,\frac{1}{2}-\frac{\sqrt{3}}{2}i \quad d)\,\pi i \quad e)\,\sqrt{2} \quad f)\,-\frac{4}{7} \quad g)\,-11.6$.

Solution

Each of these numbers can be put in the form $a+ib$ where a and b are real numbers as detailed in the following table:

	Complex number	Real part	Imaginary part	Observations
a)	$4+7i$	4	7	A complex number
b)	$5-3i$	5	-3	A complex number
c)	$\frac{1}{2}-\frac{\sqrt{3}}{2}i$	$\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	A complex number
d)	πi	0	π	A complex number that is pure imaginary
e)	$\sqrt{2}$	$\sqrt{2}$	0	A complex number that is real number
f)	$-\frac{4}{7}$	$-\frac{4}{7}$	0	A complex number that is real number
g)	-11.6	-11.6	0	A complex number that is real number

Complex numbers are commonly used in electrical engineering, as well as in physics as it is developed in the last topic of this unit. To avoid the confusion between i representing the current and i for the imaginary unit, physicists prefer to use j to represent the imaginary unit.

As an example , the Figure 1.1 below shows a simple current divider made up of a capacitor and a resistor. Using the formula, the current in the resistor is given by

$$I_R = \frac{\frac{1}{j\omega C}}{R + \frac{1}{j\omega C}} I_T \Leftrightarrow I_R = \frac{1}{1 + j\omega CR} I_T$$ where $Z_C = \frac{1}{j\omega C}$ is the impedance of the capacitor and j is the imaginary unit.

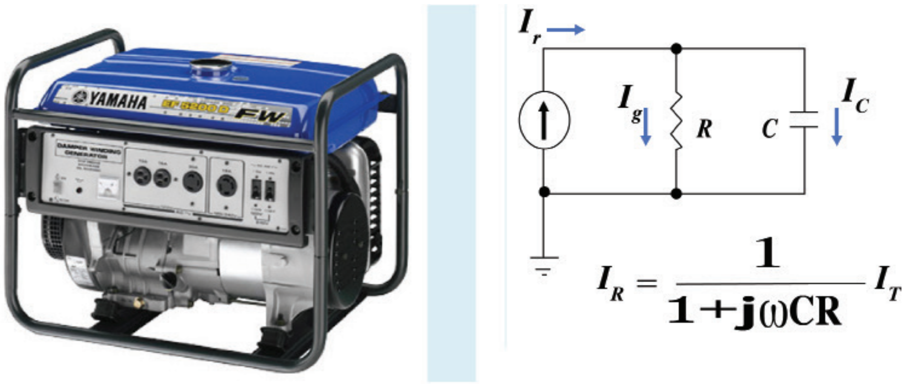


Figure 1. 1 A generator and the R-C current divider

The product τ = CR is known as the **time constant** of the circuit, and the frequency for which $\omega CR = 1$ is called the **corner frequency** of the circuit. Because the capacitor has zero impedance at high frequencies and infinite impedance at low frequencies, the current in the resistor remains at its DC value I_T for frequencies up to the corner frequency, whereupon it drops toward zero for higher frequencies as the capacitor effectively short-circuits the resistor. In other words, the current divider is a low pass filter for current in the resistor.

Properties of the imaginary number "i"

Activity 1.2

Use the definition of the complex number $\mathbb{C}$, and the fact that $i^2 = -1$ to find the following

a) i^3 b) i^4 c) i^5 d) i^7 e) i^8.

Generalize the value of i^n for $n \in \mathbb{N}$

From the activity 1.2, it easy to find that

$$i^1 = i; \quad i^2 = -1; \, i^3 = -i; \, i^4 = 1 \text{ and in general } i^{4n} = 1; \quad i^{4n+1} = i; \; i^{4n+2} = -1; \; i^{4n+3} = -i$$

In particular if $n = 0$ then $i^0 = 1$

Geometrically, we deduce that the imaginary unit, i , "cycles" through 4 different values each time we multiply as it is illustrated in Figure 1.2.

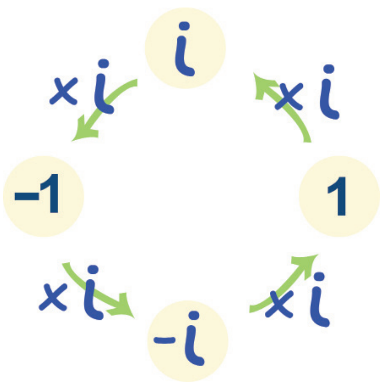


Figure 1. 2 Cycles of imaginary unit

From the figure 1.2, the following relations may be used:

$$\forall n \in \mathbb{N}, i^{4n} = 1, \quad i^{4n+1} = i, \, i^{4n+2} = -1, \, i^{4n+3} = -i.$$

Application activities 1.1

1. Observe the following complex numbers and identify the real part and imaginary part.

$a)\ z = 4+2i \quad b)\ z = i \quad c)\ z = \sqrt{2}-i \quad d)\ z = -3.5$

2. Use the properties of the number i to find the value of the following:

$a)\ i^{25} \quad b)\ i^{2310} \quad c)\ i^{71} \quad d)\ i^{51} \quad e)\ i^{28}$

3. In electricity when dealing with direct currents (DC), we encountered Ohm's law, which states that the resistance R is the ratio between voltage *V* and current *I* or $R = \frac{V}{I}$ With alternating currents (AC) both V and current I are expressed by complex numbers, so the resistance is now also complex. A complex resistance is called impedance and denoted by symbol Z. The building blocks of AC circuits are resistors (R, [Ω]), inductors (coils, L, [H=Henry]) and capacitors (C, [F=Farad]). Their respective impedances are $Z_R = R,\quad Z_L = j\omega L$ and $Z_C = \frac{1}{j\omega C}$; which of them has an imaginary part?

1.1.2 Geometric representation of a complex number

Activity 1.3

Draw the Cartesian plane and plot the following points: $A(2,3), B(-3,5)$ and $C\left(\frac{1}{2},7\right)$.

Consider the complex number $z = -3+5i$ and plot the point $Z(-3,5)$ in plane xoy.

Discuss if all complex numbers of the form $z = a+bi$ can be plotted in plane xoy.

The **complex plane** consists of two number lines that intersect in a right angle at the point (0,0). The horizontal number line (known as $x-axis$ in Cartesian plane) is the **real axis** while the vertical number line (the $y-axis$ in Cartesian plane) is the **imaginary axis**.

Every complex number $z = a+bi$ can be represented by a point $Z(a,b)$ in the complex plane.

The complex plane is also known as the **Argand diagram**. The new notation $Z(a,b)$ of the complex number $z=a+bi$ is the **geometric form** of z and the point $Z(a,b)$ is called the **affix** of $z=a+bi$. In the Cartesian plane, (a,b) is the coordinate of the extremity of the vector $\begin{pmatrix} a \\ b \end{pmatrix}$ from the origin (0,0).

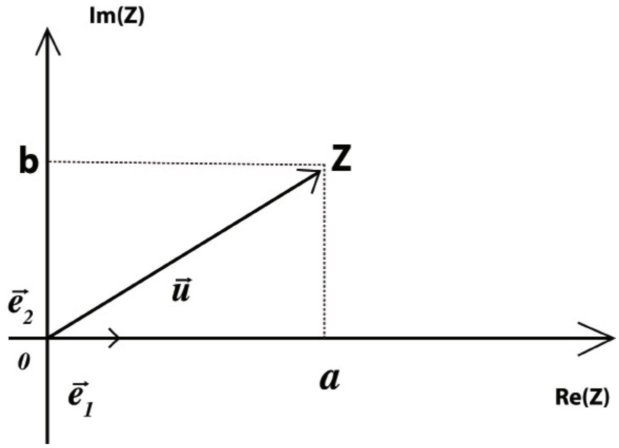


Figure 1. 3 The complex plane containing the complex number $z=a+bi$

Complex impedances in series

In electrical engineering, the treatment of resistors, capacitors, and inductors can be unified by introducing imaginary, frequency-dependent resistances for the latter two (capacitor and inductor) and combining all three in a single complex number called the **impedance**. If you work much with engineers, or if you plan to become one, you'll get familiar with the RC (Resistor-Capacitor) plane, just as you will with the RL (Resistor-Inductor) plane.

Each component (resistor, an inductor or a capacitor) has an impedance that can be represented as a vector in the RX plane. The vectors for resistors are constant regardless of the frequency.

Pure inductive reactances (X_L) and capacitive reactances (X_C) simply add together when coils and capacitors are in series. Thus , $X = X_L + X_C$.

In the RX plane, their vectors add, but because these vectors point in exactly opposite directions inductive reactance upwards and capacitive reactance downwards, the resultant sum vector will also inevitably point either straight up or down (Fig. 1.4).

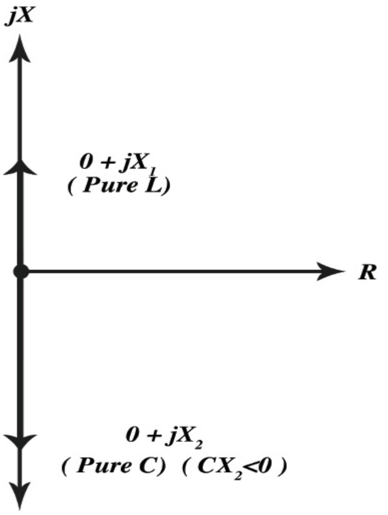


Figure 1. 4 Pure inductance and pure capacitance represented by reactance vectors that point straight up and down.

Example 1.2

a) Plot in the same Argand diagram the following complex numbers

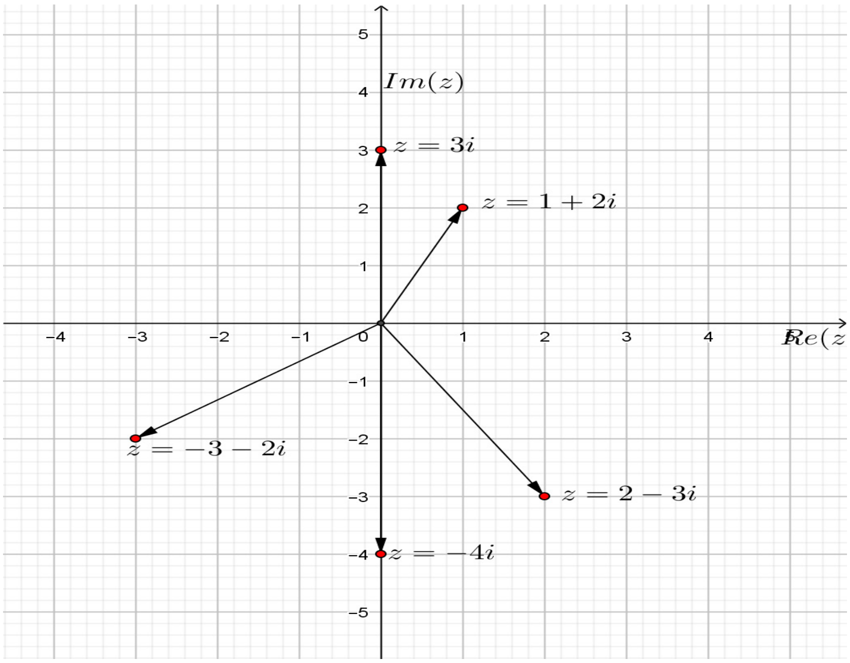
$z_1 = 1+2i,\ z_2 = 2-3i,\ z_3 = -3-2i,\ z_4 = 3i$ and $z_5 = -4i$

b) A coil and capacitor are connected in series, with $jX_L = 30j$ and $jX_C = -110j$.

What is the net reactance vector? Give comments on your answer.

Solution

a)



b) Since $X = X_L + X_C$, the net reactance vector is $jX_L + jX_C = 30j - 110j = -80j$.

This is a capacitive reactance, because it is negative imaginary.

Application activities 1.2

1. Represent in the complex plane the following numbers:

   $a)\ z = -1 + i \quad b)\ z = i \quad c)\ z = -4 - i \quad d)\ z = -3.5 + 1.2i$

2. A coil and capacitor are connected in series, with $jX_L = 200j$ and $jX_C = -150j$. What is the net reactance vector, if they are to be added? Interpret your answer.

1.1.3 Operation on complex numbers

1.1.3.1 Addition and subtraction in the set of complex numbers

Activity 1.4

a) Using the Cartesian plane, plot the point $A(1,2)$ and $B(-2,4)$; deduce the coordinate of the vector $\overrightarrow{OA} + \overrightarrow{OB}$.

b) Basing on the answer found in a), deduce the affix of the complex number $z_1 + z_2$ if $z_1 = 1 + 2i$ and $z_2 = -2 + 4i$.

c) Check your answer using algebraic method/technique.

d) Express your answer in words.

Complex numbers can be manipulated just like real numbers but using the property $i^2 = -1$ whenever appropriate. Many of the definitions and rules for doing this are simply common sense, and here we just summarise the main definitions.

Equality of complex numbers: $a + bi = c + di$ if and only if $a = c$ and $b = d$.

To perform **addition** and **subtraction** of complex numbers we combine real parts together and imaginary parts separately:

The **sum** of two complex numbers $z_1 = a + bi \text{ and } z_2 = c + di$ is a complex number whose real part is the sum of real parts of given complex numbers and the imaginary part is the sum of their imaginary parts. This means

$$z_1 + z_2 = \left[\operatorname{Re}(z_1) + \operatorname{Re}(z_2)\right] + i\left[\operatorname{Im}(z_1) + \operatorname{Im}(z_2)\right] \text{ or } z_1 + z_2 = (a + c) + (b + d)i.$$

The difference of $z_2 = c + di$ from $z_1 = a + bi$ is $z_1 - z_2 = (a - c) + (b - d)i$.

Example 1.3

Determine $z_1 + z_2$ and $z_1 - z_2$ given that

a) $z_1 = 5 + 6i$ and $z_2 = 3 + 7i$

b) $z_1 = 2 + 4i$ and $z_2 = 3 - 6i$

Solution

a) $z_1 + z_2 = (5 + 6i) + (3 + 7i) = (5 + 3) + (6 + 7)i = 8 + 13i$

$z_1 - z_2 = (5 + 6i) - (3 + 7i) = (5 - 3) + (6 - 7)i = 2 - i$

b) $z_1 + z_2 = (2 + 4i) + (3 - 6i) = (2 + 3) + (4 - 6)i = 5 - 2i$

$z_1 - z_2 = (2 + 4i) - (3 - 6i) = (2 - 3) + (4 + 6)i = -1 + 10i$

Adding impedance vectors

If you plan to become an engineer, you will need to practice adding and subtracting complex numbers. But it is not difficult once you get used to it by doing a few sample problems. In an alternating current series circuit containing a coil and capacitor, there is resistance, as well as reactance.

Whenever the resistance in a series circuit is significant, the impedance vectors no longer point straight up and straight down. Instead, they run off towards the "northeast" (for the inductive part of the circuit) and "southeast" (for the capacitive part). This is illustrated in Figure 1.5.

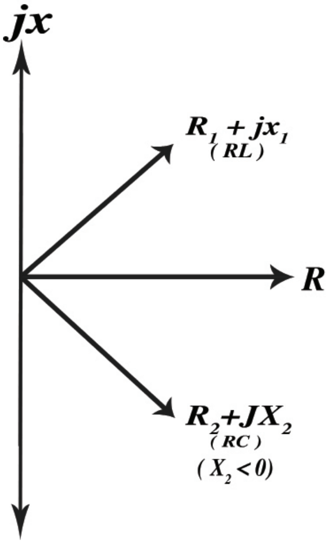


Figure 1. 5 Resistance with reactance and impedance vectors pointing "northeast "or "southeast."

When vectors don't lie along a single line, you need to use **vector addition** to be sure that you get the correct resultant. In Figure 1.6, the geometry of vector addition is shown by constructing a *parallelogram*, using the two vectors $Z_1 = R_1 + jX_1$ and $Z_2 = R_2 + jX_2$ as two of the sides. Then, the diagonal is the resultant.

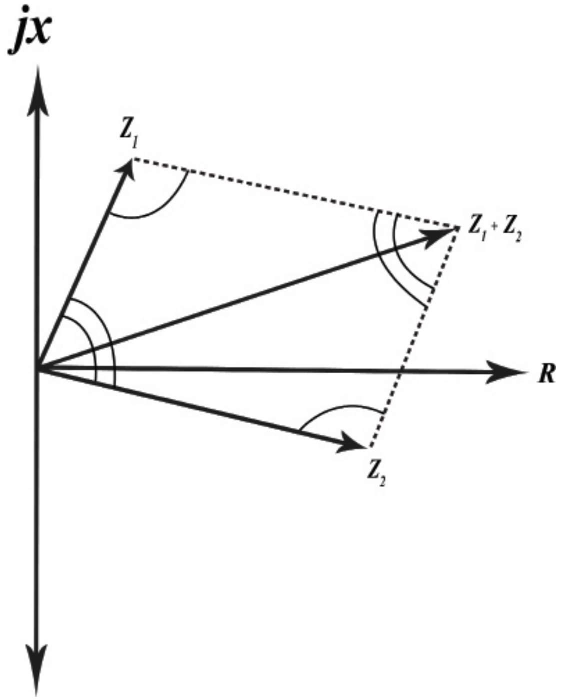


Figure 1. 6 Vector addition of impedances $Z_1 = R_1 + jX_1$ *and* $Z_2 = R_2 + jX_2$

Formula for complex impedances in series RLC circuits

Given two impedances, $Z_1 = R_1 + jX_1$ and $Z_2 = R_2 + jX_2$, the net impedance *Z* of these in series is their vector sum, given by $Z = (R_1 + R_2) + j(X_1 + X_2)$.

In calculating a vector sum using the arithmetic method the resistance and reactance components add separately. The reactances X_1 and X_2 might both be inductive (positive); they might both be capacitive (negative); or one might be inductive and the other capacitive.

When a coil, capacitor, and resistor are connected in series (Figure 1.7), the resistance *R* can be thought of as all belonging to the coil, when you use the above formulae. (Thinking of it all as belonging to the capacitor will also work.)

Then you have two vectors to sum up, when finding the impedance of a series RLC circuit: $Z = (R + jX_L) + (0 + jX_C) = R + j(X_L + X_C)$

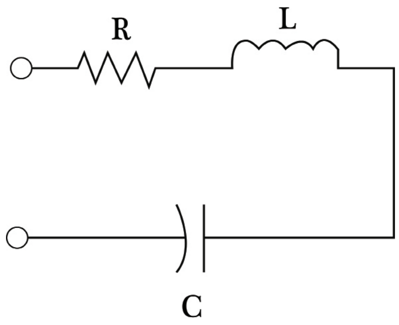


Figure 1. 7 A series RLC circuit

Example 1.4

A resistor, coil, and capacitor are connected in series with $R = 45\Omega$, $X_L = 22\Omega$ and $X_C = -30\Omega$. What is the net impedance, Z ?

Solution

Consider the resistor to be part of the coil (inductor), obtaining two complex vectors, $45 + 22j$ and $0 - 30j$. Adding these gives the resistance component of $(45+0)\Omega = 45\Omega$, and the reactive component of $(22j - 30j)\Omega = -8j\Omega$. Therefore, the net impedance is $Z = (45 - 8j)\Omega$.

Application activities 1.3

Represent graphically the following complex numbers, and then deduce the numerical answers from the diagrams.

$a)(5-i)+(2-7i)$

$b)(6+3i)-(10+8i)$

$c)(4+2i)-(4-2i)+(4-0.6i)$

$d)(8-i)-(2-i)$

1.1.3.2 Conjugate of a complex number

Activity 1.5

In the complex plane,

1. Plot the affix of complex number $z = 2 + 5i$
2. Find the image P' of the point P affix of z by the reflection across the real axis. What is the coordinate of P'?
3. Write the complex number z' associated to P' and discuss the relationship between z and z' of P'?
4. Write algebraically the complex number z' associated to P' and discuss the relationship between z and z'

Every complex number $z = a + bi$ has a corresponding complex number $\bar{z}$ called conjugate of z such that $\bar{z} = a - bi$ and affix of $\bar{z}$ is the "reflection" of affix of z about the real axis as illustrated in Figure 1.8.

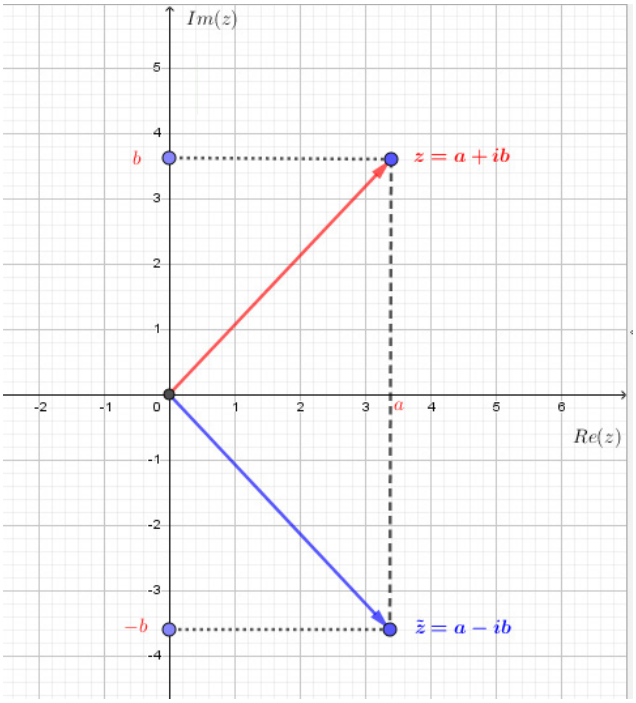


Figure 1. 8 Reflection of affix about the real axis

If $z = a + bi$, then $z + \bar{z} = 2a$ *and* $z - \bar{z} = 2bi$ which gives $\text{Re}(z) = \frac{z + \bar{z}}{2}$ and $\text{Im}(z) = \frac{z - \bar{z}}{2i}$.

Example 1.5

For each of the following complex numbers, find their conjugate

a) $2+4i$ b) $3-6i$ c) $2i-4$

Solution

a) $\overline{2+4i}=2-4i$ b) $\overline{3-6i}=3+6i$ c) $\overline{2i-4}=-2i-4$

Application activities 1.4

a) Represent in Argand diagram, the following complex numbers z and find $\overline{z}$ and $\overline{\overline{z}}$.

1) $z=2+4i$ 2) $z=2-4i$ 3) $z=-2+4i$ 4) $z=-2-4i$

b) Establish a relationship between z and $\overline{\overline{z}}$.

1.1.3.3 Multiplication and powers of complex number

Activity 1.6

Given two complex numbers $z_1=4-7i$ and $z_2=5+3i$

Apply the rules of product calculation in $\mathbb{R}$ and the convention $i^2=-1$

to find the product $z_1\cdot z_2$ and the powers z_1^2 and z_2^2.

Name and discuss the properties used while calculating the above product.

Express in your own words the property used and the answer you found.

Let $z_1=a+bi$ and $z_2=c+di$ be complex numbers. We define multiplication and powers as follows:

a) $z_1\cdot z_2=(a+bi)(c+di)=ac-bd+i(ad+bc)$

b) $z_1^{\ 2}=z_1\cdot z_1=(a+bi)(a+bi)=(a+bi)^2=a^2-b^2+2abi$

c) $z_1^n=\underbrace{z_1 \ldots z_1}_{n\ times}=(a+bi)\ldots(a+bi)=(a+bi)^n$

We use the distributive property of multiplication over addition to find the above products.

Example 1.6

Find the product/power of the following complex numbers

a) $(3-2i)(5+4i)$ $b)(4-3i)^2$ $c)(1+i)^4$

Solution

$a)\left(3-2i\right)\left(5+4i\right)=\left[\left(3\times5\right)-\left(-2\right)\times4\right]+i\left[\left(-2\right)\times5)+3\times4\right]=23+2i$

$b)\left(4-3i\right)^2=16-24i-9=7-24i$

$c)\left(1+i\right)^4=1+4i+6i^2+4i^3+i^4=1+4i-6-4i+1=-4$

Application activities 1.5

Perform the following operations to find z

a) $z=i(3-7i)(2-i)$ b) $z=(1+\mathrm{i})^2-3(2-\mathrm{i})^3$

1.1.3.4 Division in the set of complex numbers

Activity 1.7

Consider the complex number $z=\frac{2+3i}{5+i}$, apply the rules of rationalizing the denominator in $\mathbb{R}$ and the convention $i^2=-1$ to transform the denominator into real part without changing the value of z . Deduce the quotient of $2+3i$ by $5+i$.

Give the general rule of division in complex numbers. Express your answers in words.

Given two complex numbers $z_1=a+bi$ and $z_2=c+di$, the quotient $\frac{z_1}{z_2}$ is defined by: $\frac{z_1}{z_2}=\frac{\left(a+bi\right)\left(c-di\right)}{\left(c+di\right)\left(c-di\right)}=\frac{\left(ac+bd\right)+\left(bc-ad\right)i}{c^2+d^2}$

Example 1.7

Compute the following quotients

$a)\frac{1+2i}{3+4i}$ $b)\frac{1}{a+ib}$

Solution

$$a)\ z=\frac{1+2i}{3+4i}=\frac{\left(1+2i\right)\left(3-4i\right)}{\left(3+4i\right)\left(3-4i\right)}=\frac{11+2i}{25}$$

$$b)\ z=\frac{1}{a+ib}=\frac{a-ib}{\left(a+ib\right)\left(a-ib\right)}=\frac{a-ib}{a^2+b^2}=\frac{a}{a^2+b^2}-i\frac{b}{a^2+b^2}$$

Application activities 1.6

1. Find the value z in algebraic form

   $a)\ z=\dfrac{1}{(2+i)(1-2i)}$ $\qquad$ $b)\ z=\left(\dfrac{\sqrt{3}-i}{\sqrt{3}+i}\right)^4\left(\dfrac{1+i}{1-i}\right)^5$

2. Determine the real numbers x and y given that:

   $a)\ x+4y+xyi=12-16i$ $\qquad$ b) $x-7y+8xi=6y+(6y-100)i$

   $c)\ \dfrac{1}{x+yi}+\dfrac{1}{1+2i}=1 \quad ,(x\neq 0\ and\ y\neq 0)$

3. Given that $T=\dfrac{x-iy}{x+iy}$ where $x,y\in\mathbb{R}$ show that $\dfrac{1+T^2}{2T}=\dfrac{x^2-y^2}{x^2+y^2}$

1.1.4 Modulus of a complex number

Activity 1.8

Let $z=-2+3i$ be a complex number.

a) Plot z in the complex plane

b) Determine the distance between the origin $(0,0)$ to the point $(-2,3)$ affix of z. Express the formulae used in words.

The distance from origin to the point (a,b) which is the affix of the complex number $z=a+bi$ is called the **modulus** or **magnitude** of z and is denoted by $|z|$ or $|a+bi|$: $r=|z|=\sqrt{a^2+b^2}$ as illustrated on figure 1.9.

Notice that the modulus of a complex number is always a real number and in fact it will never be negative since square roots always return a positive number or zero depending on what is under the radical.

In addition, if *z* is a real number (*i.e.* $z=a+0i$) then, $|z|=\sqrt{a^2+0}=|a|$ (the absolute value of a).

We can compute the modulus of a complex number using its real and imaginary parts such that if $\text{Re}(z)=a$ and $Im(z)=b$ the modulus of z is

$$|z|=\sqrt{z.\bar{z}}=\sqrt{[\text{Re}(z)]^2+[\text{Im}(z)]^2}=\sqrt{a^2+b^2}$$

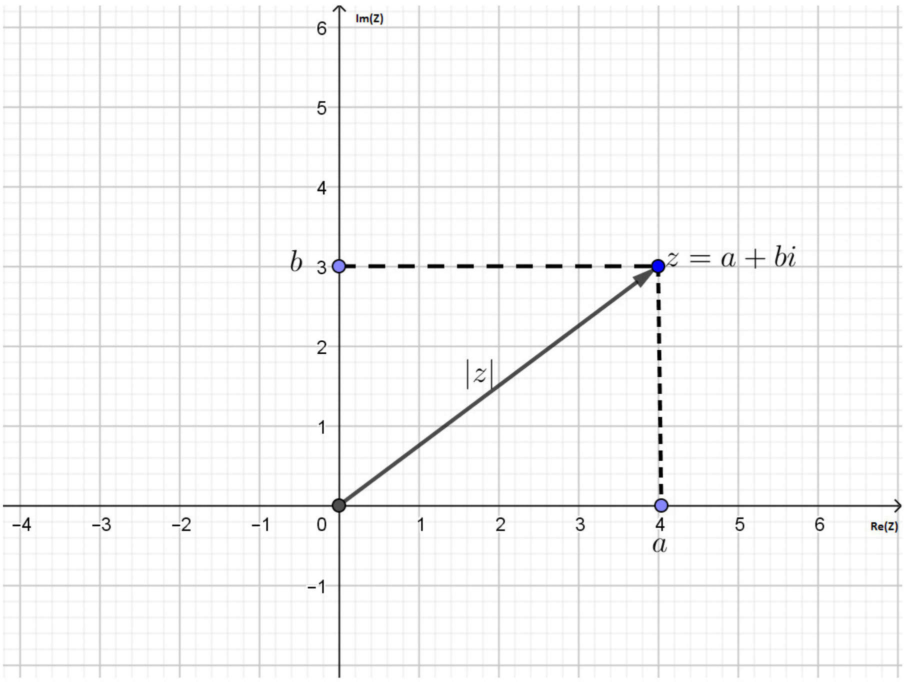


Figure 1. 9 Modulus of $z=a+bi$

Properties of modulus

For complex numbers z and z' and for any integer n, we have:

$i)\ |z|^2=|z^2|=z.\bar{z}$ $\quad ii)\ |z.z'|=|z||z'|$ $\quad iii)\ \left|\frac{1}{z}\right|=\frac{1}{|z|}, z\neq 0$

$iv)\ |z^n|=|z|^n$ $\quad v)\ \left|\frac{z}{z'}\right|=\frac{|z|}{|z'|}, z'\neq 0$ $\quad vi)\ |z+z'|\leq|z|+|z'|\ (triangular\ inequality)$

Example 1.8

Calculate the modulus of the following complex numbers

a) $4-3i$ $\quad b)\ 1+i\sqrt{3}$ $\quad c)\ -5i$ $\quad d)\ \frac{5}{1+i\sqrt{3}}$

Solution

a) $|4-3i|=\sqrt{16+9}=5$. b) $|1+i\sqrt{3}|=\sqrt{1+3}=2$. c) $|-5i|=\sqrt{25}=5$.

d) $\left|\dfrac{5}{1+i\sqrt{3}}\right| = \dfrac{5}{\left|1+i\sqrt{3}\right|} = \dfrac{5}{2}$

Application activities 1.7

1. Determine the modulus of the following complex numbers:

$z_1 = 2 - 3i$, $z_2 = 3 + 4i$, $z_3 = 6 + 4i$, and $z_4 = 15 - 8i$ then deduce the modulus of $z = \dfrac{(2-3i)(3+4i)}{(6+4i)(15-8i)}$

2. If $z_1 = 1 - i$, $z_2 = -2 + 4i$, $z_3 = 3 - 2i$, calculate:

a) $|2z_2 - 3z_1|$; b) $|z_1 . \overline{z}_2 + \overline{z}_1 . z_2|$; c) $\left|\dfrac{z_1 + z_2 + 1}{z_1 - z_2 + 1}\right|$, d) $\left|z_1^2 + z_2^2\right|^2 + \left|z_3^2 - z_2^2\right|^2$

3. If $\left|\dfrac{z+2}{z}\right| = 2$ and P represents z in the Argand plane, show that P lies on a circle and find the centre and radius of this circle.

4. Determine, in complex plane, the set of points M of affix z such that $|z-2i| = |z+2|$

1.1.5 Square root of a complex number

Activity 1.9

1.Consider the polynomials $P(x)$ and $Q(x)$ in one real variable defined as $P(x) = 2x^3 + 3x^2 - c$ and $Q(x) = ax^4 + bx^3 - dx^2 + fx + g$. Is it possible to have $P(x) = Q(x)$?

2. Given that $(a+bi)^2 = 6-4i$, where a and b are real numbers, discuss and determine the values of a and b. Using the values of a and b, calculate the square root of $z = 6-4i$. Express your answer in words.

A complex number $z = x + yi$ is a square root of a complex number $Z = a + bi$ if $z^2 = Z$. This means that $(x+yi)^2 = a + bi$ equivalently $x^2 + 2xyi - y^2 = a + bi$.

Then $\begin{cases} x^2 - y^2 = a \\ 2xy = b \end{cases}$. Therefore $\begin{cases} x = \pm\sqrt{\dfrac{1}{2}\left(a + \sqrt{a^2+b^2}\right)} \\ y = \pm\sqrt{\dfrac{1}{2}\left(\sqrt{a^2+b^2} - a\right)} \end{cases}$

The sign cannot be taken arbitrary because the product xy has the sign of b.

Thus,

- if $b > 0$ you take the same sign.
- if $b < 0$ you interchange the signs.
- In each case, a complex number has two square roots.

Example 1.9

Find the square root of

a) $8-6i$ b) $2i$ c) -2

Solution

a) Let $z^2 = (x+yi)^2 = 8-6i$ and develop the power: $(x^2-y^2)+2xyi = 8-6i$

Equate real parts and imaginary parts $\begin{matrix} x^2-y^2=8 & (1) \\ 2xy=-6 & (2) \end{matrix}$

Now, consider the modulus: $|z|^2 = |z^2|$, then $x^2+y^2=\sqrt{(8^2+6^2)} = 10$ (3)

Solving (1) and (3), we get $x^2=9$ and $y^2=1$ leading to $x=\pm 3$ and $y=\pm 1$.

From (2), x and y are of opposite signs, thus: $x=3$ and $y=-1$ or $x=3$ and $y=-1$

Finally: $z_1 = 3-i$ and $z_2 = -3+i$

b) $\begin{cases} a=0 \\ b=2>0 \end{cases}$, thus $\begin{cases} x=\pm\sqrt{\frac{2}{2}}=\pm 1 \\ y=\pm\sqrt{\frac{2}{2}}=\pm 1 \end{cases}$

Since b is positive, we take the same sign. This yields to $\sqrt{-2i}=1+i$ or $\sqrt{-2i}=-1-i$

c) $\begin{cases} a=-2<0 \\ b=0 \end{cases}$ $x=0$ *and* $y=\pm\sqrt{2}$ Thus, $\sqrt{-2}=\pm i\sqrt{2}$

Application activities 1.8

Find the square roots of the following complex numbers.

a) $z=-3+4i$ b) $z=-2i$ c) $z=2-2i\sqrt{3}$

1.1.6 Equations in the set of complex numbers

1.1.6.1 Simple linear equations of the form $Az+B=0$

Activity 1.10

Given the complex number z such that $4z+5i=12-i$,discuss how to find the value of z. Express your answer in words

To find the solution set of the equation $Az+B=0$ (where A and B are two complex numbers, A different from zero) follows the same process involved in solving equation of the form $ax+b=0$ in the set of real numbers.

Therefore $Az+B=0 \Rightarrow z=-\frac{B}{A}$. The remaing task is to express z in the form of $x+yi$.

Example 1.10

Solve each of the following equations in the set of complex numbers

a) $(1-i)z=2+i$ b) $iz+(2-10i)z=3z+2i$

Solution

a) $(1-i)z=2+i \Leftrightarrow z=\frac{2+i}{1-i}=\frac{(2+i)(1+i)}{(1-i)(1+i)}=\frac{1+3i}{2}=\frac{1}{2}+\frac{3}{2}i$

b) $iz+(2-10i)z=3z+2i \Leftrightarrow iz+(2-10i)z-3z=2i \Leftrightarrow (i+2-10i-3)z=2i \Leftrightarrow z=-\frac{9}{41}-\frac{i}{41}$

Application activities 1.9

Solve the following equation and the system of equations:

1) $(1+3i)z=2i+4i$

2) $\begin{cases} 7z+(8-2i)w=4-9i \\ (1+i)+(2-i)w=2+7i \end{cases}$

1.1.6.2 Quadratic equations

Activity 1.11

Given the quadratic equation $z^2-(1+i)=0$, you can write it as $z^2=1+i$. Calculate the square root of $1+i$ to get the value of z and discuss how to solve equations of the form $Az^2+C=0$ where A and C are complex numbers and A is different from zero. Express in words the formula used.

Solving simple quadratic equations in the set of complex numbers recalls the procedure of how to solve the quadratic equations in the set of real numbers considering that $\sqrt{-1}=i$.

Therefore, $Az^2+C=0 \Leftrightarrow z^2=\frac{-c}{A} \Leftrightarrow z=\sqrt{\frac{-c}{A}}$

Let's now discuss the general case $Az^2+Bz+C=0$ where the coefficient B is not zero. We are already familiar with finding square root of a complex number, the process of solving this equation is the same as the process for solving quadratic equation in the set of real numbers.

When solving equations of the form $Az^2+Bz+C=0,(A\neq 0)$; take $z=\frac{-B\pm\sqrt{B^2-4AC}}{2A}$ or $z=\frac{-B\pm\sqrt{\Delta}}{2}$, $\Delta=B^2-4AC$.

In particular, let a,b and c be real numbers $(a\neq 0)$, then the equation $az^2+bz+c=0$, has either two-real roots, one repeated real root or two conjugate complex roots.

- If $\Delta>0$, there are two distinct real roots: $z_1=\frac{-b+\sqrt{\Delta}}{2a}$ and $z_2=\frac{-b-\sqrt{\Delta}}{2a}$.
- If $\Delta=0$, there is a double real root: $z_1=z_2=-\frac{b}{2a}$
- If $\Delta<0$, there is no real roots. In this case there are two conjugate complex roots:

  $z_1=\frac{-b+i\sqrt{-\Delta}}{2a}$ and $z_2=\frac{-b-i\sqrt{-\Delta}}{2a}$ where $\Delta=b^2-4ac$

  If z_1 and z_2 are roots of quadratic equation $az^2+bz+c=0$, then

  $z_1+z_2=-\frac{b}{a}$ *and* $z_1.z_2=\frac{c}{a}$

Example 1.11

Solve the following equations a) $z^2+6z+10=0$ b) $z^2+\left(\sqrt{3}+i\right)z+1=0$

Solution

a) $z^2+6z+10=0$

$\Delta = 36 - 40 = -4$

Given that $i^2=-1$, $\Delta=4i^2$ and $\sqrt{\Delta}=2i$ or $\sqrt{\Delta}=-2i$. The roots of the given equation are: $z_1=\frac{-6-2i}{2}=-3-i$ and $z_2=\frac{-6+2i}{2}=-3+i$

b) $z^2+\left(\sqrt{3}+i\right)z+1=0$

$$z=\frac{-\left(\sqrt{3}+i\right)\pm\sqrt{\left(\sqrt{3}+i\right)^2-4}}{2}=\frac{-\left(\sqrt{3}+i\right)\pm\sqrt{-2+2\sqrt{3}i}}{2}$$

Now we have to solve $w^2=-2+2\sqrt{3}i$. Let $w=x+iy$, we find $x=\pm 1$ and $y=\pm 3$.

Hence, $z=\frac{-\sqrt{3}-i\pm(1+\sqrt{3}i)}{2}$, that gives: $z_1=\frac{1-\sqrt{3}+(-1+\sqrt{3})i}{2}$ and

$z_2=\frac{-1-\sqrt{3}-(1+\sqrt{3})i}{2}$

Application activities 1.10

1) Solve the following equations

a) $z^2-(3+i)z+4+3i=0$ b) $z^2+9=0$

2) Prove algebraically that $(z-4)$ is a factor of $z^3-15z-4$ Based on your proof, discuss how to find the solution set of $z^3-15z-4=0$. List all solutions of the equations.

1. 2 Polar form of a complex number

1.2.1 Definition and properties of a complex number in polar form

Activity 1.12

Consider the vector $\overrightarrow{M}=\overrightarrow{e_1}+\overrightarrow{e_2}$, plot it in a Cartesian plane and show the position of the point M. Calculate $\left|\overrightarrow{M}\right|$, the modulus of $\overrightarrow{M}$ and write $\overrightarrow{M}$ in terms of $\left|\overrightarrow{M}\right|$ and the angle θ formed by $\overrightarrow{M}$ and $x-axis$. Express your answer in words.

A complex number $z=a+bi$ plotted in the complex plane has the modulus $r=\sqrt{a^2+b^2}$.

Let θ be the angle defined by the vector $\vec{r}=a\vec{i}+b\vec{j}$ and the real axis as shown in figure 1.10.

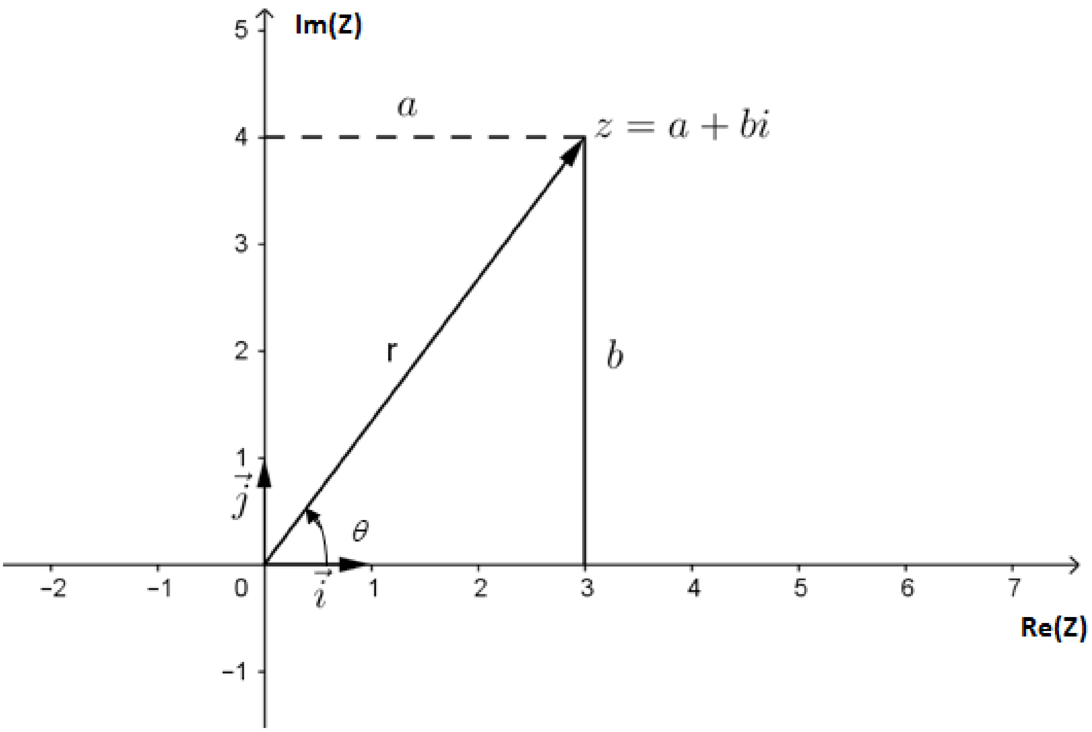


Figure 1. 10 Argument of complex number $z=a+bi$

Using trigonometric ratios, we have

$$\begin{cases}\sin\theta=\dfrac{b}{r}\\[2ex] \cos\theta=\dfrac{a}{r}\end{cases}\Leftrightarrow\begin{cases}b=r\sin\theta\\ a=r\cos\theta\end{cases}\Rightarrow z=r(\cos\theta+i\sin\theta)$$

From affix of a complex number $z = a + bi$, there is a connection between its modulus and angle between the corresponding vector and positive $x - axis$ as illustrated in figure 1.10. This angle is called the **argument** of z and denoted by $\arg(z)$.

Hence, $z = a + ib = r(\cos\theta + i\sin\theta)$ with $r = |z| = \sqrt{a^2 + b^2}$ and $\theta = \arg(z) = \arctan\frac{b}{a}$

The expression $z = r(\cos\theta + i\sin\theta)$ is called **polar form** or **trigonometric form** of the complex number z.

The complex number $z = r(\cos\theta + i\sin\theta)$ can be written in brief as $z = rcis\theta$ or $r\angle\theta$.

Depending on the position of the affix of the complex number z, x and y can be of the same or different signs. Therefore it is very necessary to choose $\theta = \arctan\frac{b}{a}$ carefully.

In summary, $\theta = arg(z) = \begin{cases} \arctan\frac{b}{a}, if\ a > 0 \\ \pi + \arctan\frac{b}{a}, if\ a < 0\ and\ b > 0 \\ -\pi + \arctan\frac{b}{a}, if\ a < 0\ and\ b < 0 \\ \frac{\pi}{2}, if\ a = 0\ and\ b > 0 \\ -\frac{\pi}{2}, if\ a = 0\ and\ b < 0 \\ Undefined, if\ a = 0\ and\ b = 0 \end{cases}$

The value of $\theta = \arg(z)$ must always be expressed in radians. It can change by any multiple of 2π and still give the same trigonometric ratios. Hence, the arg function is sometimes considered as multivalued. Normally, it is advised to consider the argument in the interval $]-\pi, \pi]$ also called principal argument and denoted by

$\theta = Arg\ z$.

The argument of the complex number 0 is undefined.

Properties

1. Let z and z' be two non-zero complex numbers. We have $z = z'$ if and only if $|z| = |z'|$ and $\arg(z) = \arg(z')[2\pi]$. This property is deduced from the definitions of modulus and an argument of a complex number.

2. The argument of z can be given by $\tan\theta = i\dfrac{\bar{z} - z}{\bar{z} + z}$.

Note that having a polar form of a complex number, you can get its corresponding algebraic form.

Example 1.12

a) Write the complex numbers in the polar form: i) $z = 1 + i$ ii) $z = \dfrac{1}{2} - i\dfrac{\sqrt{3}}{2}$

b) Convert in algebraic form: i) $\text{cis}\left(-\dfrac{\pi}{3}\right)$ ii) $2\ \text{cis}\left(-\dfrac{\pi}{2}\right)$

Solution

a) i) $z = 1 + i$, $|z| = \sqrt{2}$, Let θ be $Arg(z)$, thus $\begin{cases} \cos\theta = \dfrac{1}{\sqrt{2}} = \dfrac{\sqrt{2}}{2} \\ \sin\theta = \dfrac{1}{\sqrt{2}} = \dfrac{\sqrt{2}}{2} \end{cases}$, which gives that

θ lies in 1st quadrant and $Arg(z) = \theta = \dfrac{\pi}{4}$. Hence, z = $\sqrt{2}\left(\cos\dfrac{\pi}{4} + i\sin\dfrac{\pi}{4}\right)$

ii) $z = \dfrac{1}{2} - i\dfrac{\sqrt{3}}{2} \Rightarrow |z| = 1, Arg(z) = -\dfrac{\pi}{3}$. Thus $z = \dfrac{1}{2} - i\dfrac{\sqrt{3}}{2} = \cos\left(-\dfrac{\pi}{3}\right) + i\sin\left(-\dfrac{\pi}{3}\right)$

b) i) $cis\left(-\dfrac{\pi}{3}\right) = \cos\left(-\dfrac{\pi}{3}\right) + i\sin\left(-\dfrac{\pi}{3}\right) = \dfrac{1}{2} - i\dfrac{\sqrt{3}}{2}$

ii) $2cis\left(-\dfrac{\pi}{2}\right) = 2\left(\cos\left(-\dfrac{\pi}{2}\right) + i\sin\left(-\dfrac{\pi}{2}\right)\right) = 2(0 - i) = -2i$.

Application activities 1.11

1) Find the principal argument of the following complex numbers

a) $-2i$ b) $1 - i$ c) $1 - i\sqrt{3}$ d) $-1 + i\sqrt{3}$ e) $-\sqrt{3} - i$

2) Write the following complex numbers in the polar form

a) $z = -1 + i$ b) $z = -\dfrac{1}{2} + i\dfrac{\sqrt{3}}{2}$ c) $z = -2\left(\dfrac{1}{2} + i\dfrac{\sqrt{3}}{2}\right)$ d) $z = 2$ e) $z = -i$

3) Convert the following complex numbers in algebraic form

a) $5\,\text{cis}\,270^0$ b) $4\,\text{cis}\,300^0$ c) $\sqrt{2}\,\text{cis}\left(-\frac{\pi}{4}\right)$ d) $3\,\text{cis}\left(\frac{\pi}{2}\right)$

1.2.2 Multiplication and division of complex numbers in polar form

Activity 1.13

Given two complex numbers $z_1 = \sqrt{3} - i$ and $z_2 = 1 + i$,

a) Write z_1 and z_2 in polar form.

b) Determine the product $z_1.z_2$ in algebraic form and convert it in polar form

c) Deduce from (b) the argument and the modulus of $z_1.z_2$

d) Compare the argument of the product $z_1.z_2$ and those for z_1 and z_2, then establish any relationship among them. Express your answer in words

For two complex numbers $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$, their product is given by $z_1 z_2 = r_1 r_2\left[\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)\right]$ provided that 2π may be added to or substracted from $\theta_1+\theta_2$ if $\theta_1+\theta_2$ is outside the permitted range of the principal argument. That means the argument of $z_1.z_2$ is $(\theta_1+\theta_2)+2k\pi, k\in\mathbb{Z}$ and it can be denoted by $(\theta_1+\theta_2)[2\pi]$.

Similarly, the division is given by: $\frac{z_1}{z_2} = \frac{r_1}{r_2}\left[\cos(\theta_1-\theta_2) + i\sin(\theta_1-\theta_2)\right]$ provided that 2π may be added to or substracted from, $\theta_1-\theta_2$ if $\theta_1-\theta_2$ is outside the permitted range of the principal argument.

Therefore, for non-zero complex numbers z, z' and any integer n, we have:

i) $\arg(z.z') = \arg(z) + \arg(z')[2\pi]$

ii) $\arg\left(\frac{1}{z}\right) = -\arg(z)[2\pi]$

iii) $\arg(z^n) = n\arg(z)[2\pi]$

iv) $\arg\left(\frac{z}{z'}\right) = \arg(z) - \arg(z')[2\pi]$

Example 1.13

Compute using polar form

a) $z=\left(\sqrt{3}-i\right)\left(1+i\right)$ b) $z'=\dfrac{\sqrt{3}+i}{1+i}$

Solution

a) Let $z_1=\sqrt{3}-i$, thus $|z_1|=\sqrt{3+1}=2$ and $z_1=2\left(\frac{\sqrt{3}}{2}-\frac{1}{2}i\right)$.

Let $\alpha_1=\arg(z_1)$, thus $\left.\begin{array}{l}\cos\alpha_1=\frac{\sqrt{3}}{2}\\ \sin\alpha_1=-\frac{1}{2}\end{array}\right\}\Leftrightarrow \alpha_1=-\frac{\pi}{6}[2\pi]=arg(z_1)$.

$z_2=1+i\Rightarrow |z_2|=\sqrt{1+1}=\sqrt{2}$, and then $z_2=\sqrt{2}\left(\frac{\sqrt{2}}{2}+i\frac{\sqrt{3}}{2}\right)$

Let $\alpha_2=\arg(z_2)$, thus $\left.\begin{array}{l}\cos\alpha_2=\frac{\sqrt{2}}{2}\\ \sin\alpha_2=\frac{\sqrt{2}}{2}\end{array}\right\}\Leftrightarrow \alpha_2=\frac{\pi}{4}[2\pi]=\arg(z_2)$.

Then $\arg(z)=-\frac{\pi}{6}+\frac{\pi}{4}[2\pi]=\frac{\pi}{12}[2\pi]$. Therefore $z_1.z_2=2\sqrt{2}\left(\cos\frac{\pi}{12}+i\sin\frac{\pi}{12}\right)$

b) Let $z_3=\sqrt{3}+i$, thus $|z_3|=\sqrt{3+1}=2$ and $z_3=2\left(\frac{\sqrt{3}}{2}+\frac{1}{2}i\right)$.

Let $\alpha_3=\arg(z_3)$ $\left.\begin{array}{l}\cos\alpha_3=\frac{\sqrt{3}}{2}\\ \sin\alpha_3=\frac{1}{2}\end{array}\right\}\Rightarrow \alpha_3=\arg(z_3)=\frac{\pi}{6}\ [2\pi]$.

$$\frac{z_3}{z_2}=\frac{2\left(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right)}{\sqrt{2}\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)}=\frac{\sqrt{2}\left(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right)}{\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)}=\sqrt{2}\left(\cos\frac{\pi}{12}-i\sin\frac{\pi}{12}\right)$$

Application activities 1.12

Given 3 complex numbers z, y and w such that

$z=1+i,\ w=-\sqrt{3}+i\quad and\quad y=-3+i\sqrt{3}$

Convert them in polar form then compute $z.w,\ z.y,\ \frac{w}{y},\frac{y}{z}$

1.2.3 Powers in polar form

Activity 1.14

Given complex number $z = r(\cos\theta + i\sin\theta)$

1) Find the expression for $z^2 = z \cdot z$
2) Find the expression for $z^3 = z^2 \cdot z$
3) Using results from 1 to 2, guess the expression for z^n
4) Express your answers in words.

Power of a complex number z is given by

$$z^n = \left(r(\cos\theta + i\sin\theta)\right)^n = r^n(\cos n\theta + i\sin n\theta);\ n \in \mathbb{Z}_0$$

In particular if $r = 1$, we have the equality known as De Moivre's theorem

$$(\cos\theta + i\sin\theta)^n = (\cos n\theta + i\sin n\theta)$$

that is valid for any rational number n .

Example 1.14

Consider the complex number $z = -1 + i$; determine z^{20}

Solution

$z = -1 + i \Rightarrow |z| = \sqrt{2}$ and $\arg(z) = \frac{3\pi}{4}[2\pi]$ leading to $z = \sqrt{2}\left[\cos\left(\frac{3\pi}{4}\right) + i\sin\left(\frac{3\pi}{4}\right)\right]$

Hence, $z^{20} = \left(\sqrt{2}\right)^{20}\left[\cos\left(\frac{3\pi}{4}\right) + i\sin\left(\frac{3\pi}{4}\right)\right]^{20} = 2^{10}\left[\cos(15\pi) + i\sin(15\pi)\right] = -2^{10}$

Application activities 1.13

1) Simplify the following complex numbers using De Moivre's theorem

a) $(\cos 3\pi + i\sin 3\pi)^9$ b) $\left(\cos\frac{5\pi}{2} + i\sin\frac{5\pi}{2}\right)^{\frac{2}{5}}$ c) $\left(\cos\frac{\pi}{5} + i\sin\frac{\pi}{5}\right)^5$

d) $\left(\cos 45^0 + i\sin 45^0\right)^2$ e) $\left(\cos 60^0 + i\sin 60^0\right)^3$

2) Using De Moivre's theorem, perform the following powers

$a)\left(1 + i\sqrt{3}\right)^6$ $b)\left(-\sqrt{3} + i\right)^{10}$ $c)(1 - i)^7$

1.3 Exponential form of complex numbers

1.3.1 Definition of exponential form of a complex number

Activity 1.15

Conduct a research in the library or on internet to find out that the complex number $\cos\theta + i\sin\theta$ can be expressed by exponential function as follows:

$\cos\theta + i\sin\theta = e^{i\theta}$.

The voltage in AC circuit is expressed by $U(t) = U_0 e^{i\omega t}$ where ω is the angular frequency which is related to the frequency f by $\omega = 2\pi f$ and t the time the voltage appears somewhere in the circuit. Write this voltage as a complex number in polar form and deduce its modulus and argument.

Every complex number z of modulus $|z|$ and argument θ, can be written as $z = |z|(\cos\theta + i\sin\theta) = |z|e^{i\theta}$.

The expression $z = r.e^{i\theta}$ where r and θ are the modulus and the argument of z respectively is called **exponential form** of the complex number z.

Properties

1. Properties of powers learnt are used for complex numbers expressed in exponential form $e^{in\theta} = (\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta)$

This form is useful in altenating current to simplify calculation such that

$U(t) = e^{i\omega t} = U_0(\cos\omega t + i\sin\omega t)$

2. Given two complex numbers $z_1 = r_1 e^{i\theta_1}$ and $z_2 = r_2 e^{i\theta_2}$, the following identities are correct

a) $z_1.z_2 = r_1.r_2 e^{i(\theta_1+\theta_2)}$ b) $\frac{z_1}{z_2} = \frac{r_1}{r_2}.e^{i(\theta_1-\theta_2)}$ c) $z_1^n = r_1^n e^{in\theta_1}$

In electrical engineering, the treatment of resistors, capacitors and inductors can be unified by introducing imaginary, frequency-dependent resistances for capacitor, inductors and combining all three (resistors, capacitors, and inductors) in a single complex number called the **impedance**. This approach is called phasor calculus. As we have seen, the imaginary unit is denoted by j to avoid confusion with i which is generally in use to denote electric current. Since the voltage in an AC circuit is oscillating, it can be represented as

$$V = V_0 e^{jwt}$$
$$= V_0\left(\cos wt + j\sin wt\right)$$

Which denotes **Impedance**, V_o is peak value of impedance and $\omega = 2\pi f$ where f is the frequency of supply.

To obtain the measurable quantity, the real part is taken:
$\text{Re}\left(V\right) = V_0 \cos wt$ and is called **Resistance** while $\text{Im}\left(V\right) = V_o \sin \omega t$ denotes **Reactance** (inductive or capacitive).

One of the methods of analyzing the series LRC, is to start with "Ohm's Law" for reactive circuit: $I = \dfrac{V_0}{Z}$ with $Z = R + \dfrac{1}{j\omega C} + j\omega L$.

$$Z = R + \frac{1}{j\omega C} + j\omega L \Leftrightarrow Z = R - \frac{j}{\omega C} + j\omega L \Leftrightarrow Z = R + j\left(\omega L - \frac{1}{\omega C}\right).$$

To do division, convert the impedance to exponential form:

$$Z = R + j\left(\omega L - \frac{1}{\omega C}\right) = \sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}\, e^{j\phi}$$

with $\phi = \arctan\dfrac{\left(\omega L - \dfrac{1}{\omega C}\right)}{R} = \arctan\left(\dfrac{\omega^2 - \omega_0^2}{\gamma\omega}\right)$, $\omega_0 = \dfrac{1}{\sqrt{LC}}$, $and\ \gamma = \dfrac{R}{L}$.

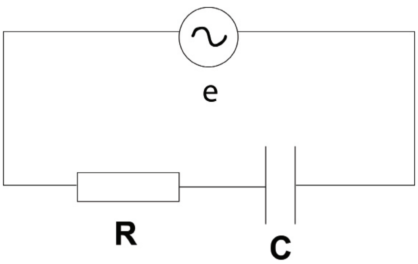
Thus, the current is given by $I = \dfrac{V_0}{\sqrt{R^2 + \left(\omega L - \dfrac{1}{\omega C}\right)^2}} e^{j\phi} = \dfrac{V_0 \cdot \dfrac{\omega}{L}}{\sqrt{\gamma^2\omega^2 + \left(\omega^2 - \omega_0^2\right)^2}} e^{j\phi}$

Example 1.15

a) Express the complex numbers in exponential form

i) $-\sqrt{3} + i$ ii) -3 iii) $-2i$

b) Consider the RC series of the alternating current circuits. The e.m.f (electromotive force) E that is supplied to the circuit is distributed between the resistor R and the capacitor C.



Given that the *same current* must flow in each element, the resistor and capacitor are in *series* such that the common current can often be taken to have the reference phase.

If the current is $I = I_m e^{j\omega t}$, find the expression of the applied electromotive force E.

Hint: $V_C = \frac{1}{j\omega C} I$

Solution

a) i) Exponential form of a complex number whose modulus r and argument θ is

$re^{i\theta}$.

Here $r = \left|-\sqrt{3}+i\right| = 2$

Let θ be argument of complex number $-\sqrt{3}+i$;we have

$\begin{cases} \cos\theta = -\frac{\sqrt{3}}{2} \\ \sin\theta = \frac{1}{2} \end{cases} \Rightarrow \theta$ lies in 2nd quadrant and $\theta = \pi - \frac{\pi}{6} = \frac{5\pi}{6}$.

Thus, $-\sqrt{3}+i = 2e^{\frac{5\pi i}{6}}$.

ii) Given $z=-3$, $r = \left|-3\right| = 3$.Let θ be argument of complex number -3;

Then, $\begin{cases} \cos\theta = -3 \\ \sin\theta = 0 \end{cases} \Rightarrow \theta$ lies on negative *real axis*, thus $\theta = \pi$. Thus, $-3 = 3e^{\pi i}$.

iii) $z=-2i$ $r = \left|-2i\right| = 2, \theta = -\frac{\pi}{2}$; so $-2i = 2e^{-i\frac{\pi}{2}}$.

b) In a series circuit, the potential differences are added up around the circuit:

$E = V_R + V_C$

$= RI - \frac{j}{\omega C} I = \left(R - \frac{j}{wc}\right) I_m e^{j\omega t}$

$= \sqrt{R^2 + \frac{1}{(\omega C)^2}} \left[e^{j\arctan\left(\frac{-1}{\omega CR}\right)} \right] I_m e^{j\omega t}$

Taking $\theta = \arctan\left(\frac{-1}{\omega CR}\right)$ and $|Z| = \sqrt{R^2 + \frac{1}{(\omega C)^2}}$, we find $E = Z.I_m.e^{j(\omega t - \theta)}$

When we apply De Moivre's theorem, we find

$$E = Z.I_m.e^{j(\omega t-\theta)} = Z.\mathrm{I}_m\left[\cos(\omega t-\theta)+j\sin(\omega t-\theta)\right]$$

This shows that $|Z| = \sqrt{R^2+\frac{1}{(\omega C)^2}}$ is the modulus of E and $\theta = \arctan\left(\frac{-1}{\omega CR}\right)$ is the phase between the electromotive force E and the current.

Application activities 1.14

1) Plot the following complex on the Argand diagram and put them on exponential form

a) $1-i$ b) $2i$ c) $\frac{\sqrt{3}}{2}+\frac{1}{2}i$ d) $-\sqrt{3}-i$

2) Express the following numbers in the algebraic form

a) $e^{i\frac{\pi}{3}}$ b) $e^{-i\frac{\pi}{4}}$ c) $3e^{\frac{\pi i}{6}}$ d) $2e^{\frac{2\pi i}{3}}$ e) $2e^{-\pi}$

3) Express the following complex numbers in exponential form:

$a) z=-1+i\sqrt{3}$ $b) z=3+4i$ $c) z=2-2i$ $d) z=-3+i\sqrt{3}$

1.3.2 Euler's formulae

Activity 1.16

From De Moivre's theorem, consider expressions of $e^{i\theta}$ and $e^{-i\theta}$ in algebraic form.

Discuss how to get the value of $\cos\theta$ and $\sin\theta$ in terms of $e^{i\theta}$ and $e^{-i\theta}$.

The following formulae are correct for the argument θ given in **radians** and called

Euler's formulae:

$$\cos\theta = \frac{1}{2}\left(e^{i\theta}+e^{-i\theta}\right)$$

$$Sin\theta = \frac{1}{2i}\left(e^{i\theta}-e^{-i\theta}\right)$$

The Euler's formulae are used to write the product of trigonometric expressions in form of the sum of trigonometric expressions. This method is called **linearization and is most used when integrating trigonometric functions**.

Example 1.16

Use Euler's formula to show that:

a) $2\sin x\cos y = \sin(x+y)+\sin(x-y)$ b) $\sin x\cos^2 x = \frac{1}{4}(\sin 3x+\sin x)$

Solution

Using Euler's formula for $\cos x$ and $\sin x$ we get

a) $2\sin x\cos y=2\left(\frac{e^{ix}-e^{-ix}}{2i}\right)\left(\frac{e^{iy}+e^{-iy}}{2}\right)=\frac{1}{2i}\left(e^{i(x+y)}+e^{i(x-y)}-e^{i(y-x)}-e^{i(-x-y)}\right)$

$=\frac{1}{2i}\left(e^{i(x+y)}-e^{-i(x+y)}+e^{i(x-y)}-e^{-i(x-y)}\right)\quad=\frac{1}{2i}\left(e^{i(x+y)}-e^{-i(x+y)}\right)+\frac{1}{2i}\left(e^{i(x-y)}-e^{-i(x-y)}\right)$

$=\sin(x+y)+\sin(x-y)$ (As requested)

b) $\sin x\cos^2 x=\left(\frac{e^{ix}-e^{-ix}}{2i}\right)\left(\frac{e^{ix}+e^{-ix}}{2}\right)^2=\left(\frac{e^{ix}-e^{-ix}}{2i}\right)\left(\frac{e^{i2x}+2+e^{-i2x}}{4}\right)$

$$=\frac{e^{i3x}+e^{-ix}+2e^{ix}-e^{ix}-e^{-i3x}-2e^{-ix}}{8i}=\frac{1}{4}\frac{e^{i3x}-e^{-i3x}}{2i}+\frac{e^{ix}-e^{-ix}}{2i}$$

Therefore, $\sin x\cos^2 x=\frac{1}{4}(\sin 3x+\sin x)$

Application activities 1.15

Apply Euler's formula to linearize the following:

$a)\sin^2 x\cos x$ $\quad$ $b)\cos^2 x\cos y$ $\quad$ $c)\cos^3 x$

1.3.3 Application of complex numbers in Physics

Activity 1.17

Conduct a research from different books of the library or browse internet to discover the application of complex numbers in other subjects such as physics, applied mathematics, engineering, etc

Complex numbers are applied in other subjects to express certain variables or facilitate the calculation in complicated expressions. They are mostly used in electrical engineering, electronics engineering, signal analysis, quantum mechanics, relativity, applied mathematics, fluid dynamics, electromagnetism, civil and mechanical engineering. Let look at an example from civil and mechanical engineering.

An alternating current is a current created by rotating a coil of wire through a magnetic field.

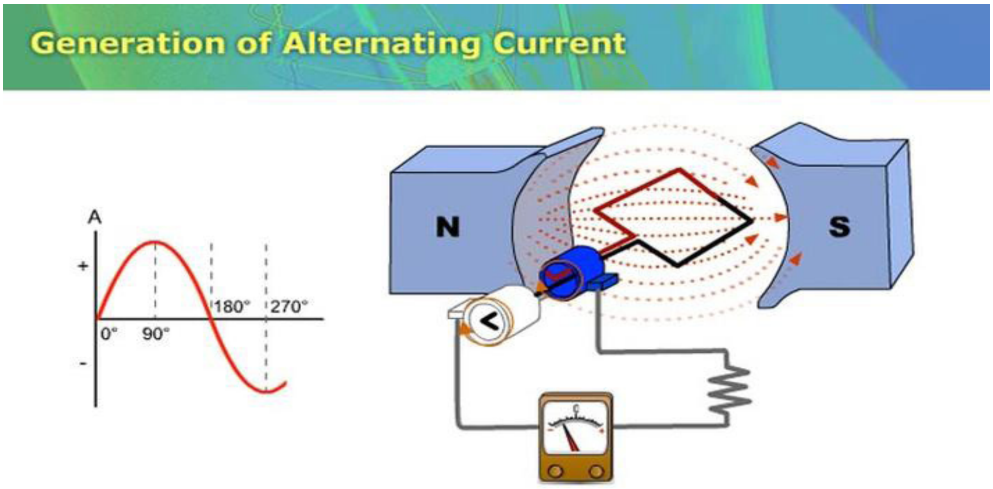


Figure1.11: generation of alternating current
(Source: https://www.google.com/imgres?imgurl=https://image.pbs.org/poster_images)

If the angular velocity of the wire is ω, respective impedances are $Z_R = R,\ Z_L = j\omega L$ and $Z_C = \dfrac{1}{j\omega C}$; their moduli are the resistance R, the capacitive reactance is $|Z_C| = \dfrac{1}{\omega C}$ and the inductive reactance is given by $|Z_L| = \omega L$.

End unit assessment

QUESTION ONE

i) Given two complex numbers $z_1 = 6+3i$ and $z_2 = 10+8i$, evaluate the following

$a) z_1 + z_2 \qquad b) \dfrac{z_1}{z_2} \qquad c) z_1 . z_2 \qquad d) \left(z_1 - \overline{z_2}\right)\left(z_1 + \overline{z_2}\right)$

ii) If $Z = R + j\omega L + \dfrac{1}{j\omega C}$, express Z in $(a+jb)$ form when $R = 10, L = 5, C = 0.04$ and $\omega = 4$

iii) Given the complex number $z = 3+3i$,

a) Convert z in polar form and in exponential form

b) Evaluate $(3+3i)^5$ and write the answer in algebraic form.

c) Transform the two square roots of $z = 3+3i$ into algebraic form

QUESTION TWO

Show that multiplication by i rotates a complex number through $\frac{\pi}{2}$ in the anticlockwise direction and division by i rotates it through $\frac{\pi}{2}$ in the clockwise direction.

QUESTION THREE

Using Euler's formula, linearize the following:

a) $\sin^2 x \cos x$ b) $\sin x \cos^2 x$ c) $\sin^2 x \cos^2 x$ d) $\sin^3 x$

QUESTION FOUR

A man travels 12 kms North-East, 20 kms 30° West of North, and then 18 kms 60° South of West. Determine analytically and graphically how far and in what direction he is from his starting point.

QUESTION FIVE

a) Make a research and explain the reason why engineers must learn complex numbers?

c) From what you read and learnt so far in this unit, write down any scientific/mathematical added value the set of complex numbers brings to the set of real numbers.

UNIT2: LOGARITHMIC AND EXPONENTIAL FUNCTIONS

UNIT2: LOGARITHMIC AND EXPONENTIAL FUNCTIONS

Key unit competence

Extend the concepts of functions to investigate fully logarithmic and exponential functions and use them to model and solve problems about interest rates, population growth or decay, magnitude of earthquake, etc.

Introductory activity

The Accountant for a Health Center receives money from patients in an interesting way so that the money he/she earns each day doubles what he/she earned the previous day. If he/she had 200USD on the first day and by taking t as the number of days, discuss the money he/she can have at the t^{th} day through answering the following questions:

a) Draw the table showing the money this Health Center Owner will have on each day starting from the first to the 10^{th} day.

b) Plot these data in rectangular coordinates

c) Based on the results in a), establish the formula for the Health Center Owner to find out the money he/she can earn on the n^{th} day. Therefore, if t is the time in days, express the money $F(t)$ for the economist.

d) Now the Health Center Owner wants to possess the money F under the same conditions, discuss how he/she can know the number of days necessary to get such money from the beginning of the business.

From the discussion, the function $F(t)$ found in c) and the function $Y(F)$ found in d) are respectively exponential function and logarithmic functions that are needed to be developed to be used without problems. In this unit, we are going to study the behaviour and properties of such essential functions and their application in real life situation.

2. 1 Logarithmic functions

2.1.1 Domain of definition for logarithmic function

Activity 2.1

Consider the following real numbers: 50, 100, ½, 0.7, 0.8, -30, -20, -5, 0.9, 10, 20, 40.

a) Draw and complete the table of values for $\log_{10}(x)$

b) Discuss the value of $\log_{10}(x)$ for $x<0$

c) Discuss the values of $\log_{10}(x)$ for $0<x<1$, $x=1$ and $x>1$.

d) Using the findings in a) plot the graph of $\log_{10}(x)$ for $x>0$

e) Explain in your own words what are the values of x for which $\log_{10}(x)$ is defined (the domain) and what are output values (the range).]

Given the function $y=\log_a x$, it is proven that if $x>0$ and a is a constant $(a>0, a\neq 1)$ then $\log_a x$ is a real number called the "logarithm to the base a of x"

Definition of logarithmic function

For a positive constant a (with a defferent from 1), we call logarithmic function, the function $f:\mathbb{R}_0^+ \to \mathbb{R}: x \mapsto \log_a x$. The domain of definition of the logarithm function is the set of positive real numbers and the range is the set of all real numbers. This means that $dom\, f=\{x\in\mathbb{R}: x>0\}=\left]0,+\infty\right[=\mathbb{R}_0^+$ and range f $=\mathbb{R}=\left]-\infty,+\infty\right[$.

The logarithmic function is neither even nor odd. If $u:\mathbb{R}\to\mathbb{R}: x\mapsto u(x)$ is any other function we can compose u and the logarithmic function as $y=\log_a(u(x))$ defined for x such that $u(x)\geq 0$.

In the expression $y=\log_a x$, y is referred to as the logarithm, is the base, and is the argument.

If the base is 10, it is not necessary to write the base, and we say decimal logarithm or common logarithm or Brigg's logarithm. So, the notation will become $y=\log x$. If the base is *e* (where *e* =2.718281828…), we have Neperean logarithm or natural logarithm denoted by $y=\ln x$ instead of $y=\log_e x$ as we might expect.

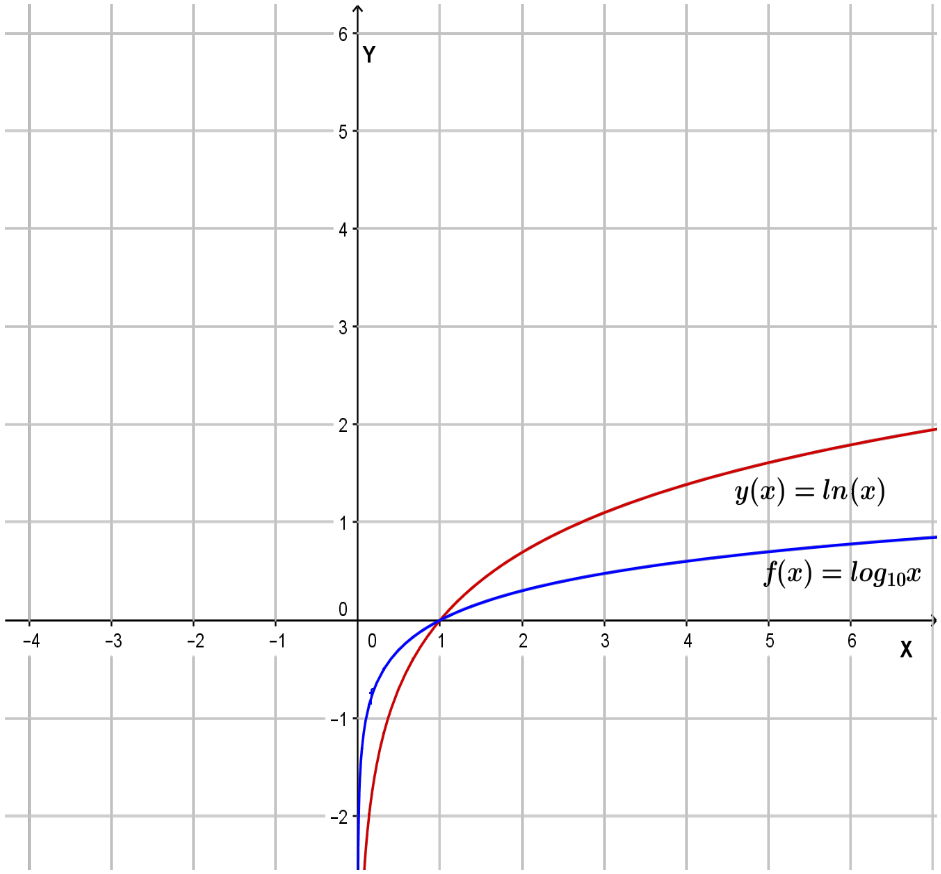


Figure 2.1: Graphs of logarithmic functions $f(x)=\log_{10}(x)$ *and* $y(x)=\ln(x)$

Example 2.1

Find the domain and range for the function

a) $f(x)=\log(x-4)$ b) $g(x)=\ln(x+6)$

Solution

a) To find the domain and the range of the function $y=\log(x-4)$, recalling that:

- **Domain:** Includes all values of x for which the function is defined
- **Range:** Includes all values y for which there is some such that $y=\log(x-4)$

Because $\log x$ is only defined for positive values of x.

So, in this problem $y=\log(x-4)$, is defined if and only if $x-4>0 \Leftrightarrow x>4$ and gives that $x \in]4,+\infty[$.

The range of y is still all real number $\mathbb{R}$

$Dom\ f = \{x \in \mathbb{R} : x-4>0\} = \{x \in \mathbb{R} : x>4\} =]4,+\infty[.\ Range\ f = \mathbb{R}.$.

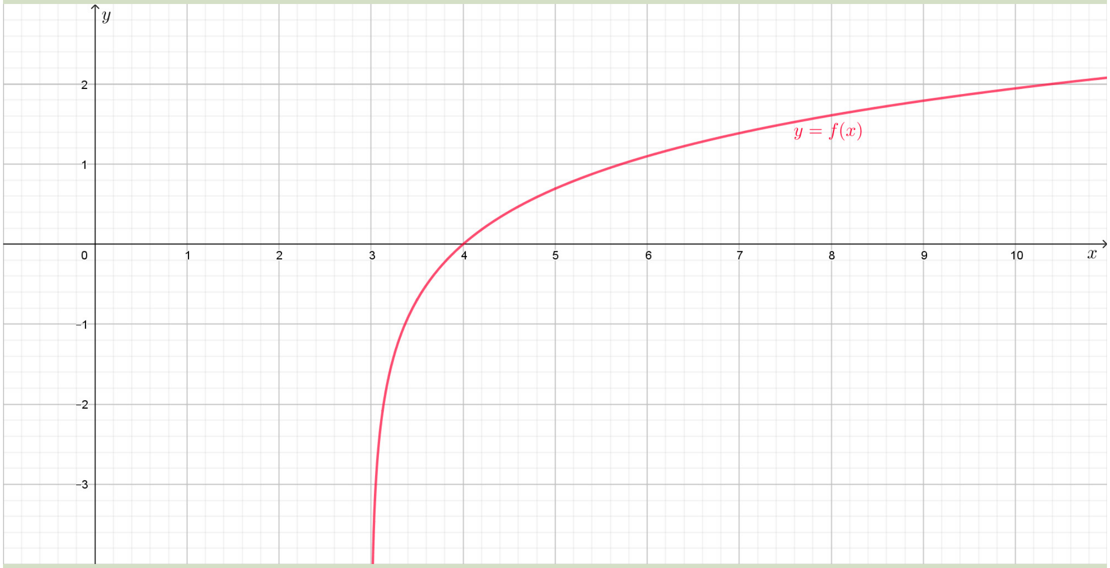
b) The function $y = \ln(x+6)$, is defined if and only if $x+6>0 \Leftrightarrow x>-6$ and gives that $x \in]-6,+\infty[$ which is the domain. The range is $\mathbb{R}$

$Dom\ g = \{x \in \mathbb{R} : x+6>0\} = \{x \in \mathbb{R} : x>-6\} =]-6,+\infty[.\ Range\ g = \mathbb{R}.$

Application activities 2.1.

1) State the domain and range of the following functions:
   a) $y = \log_3(x-2)+4$ b) $y = \log_5(8-2x)$

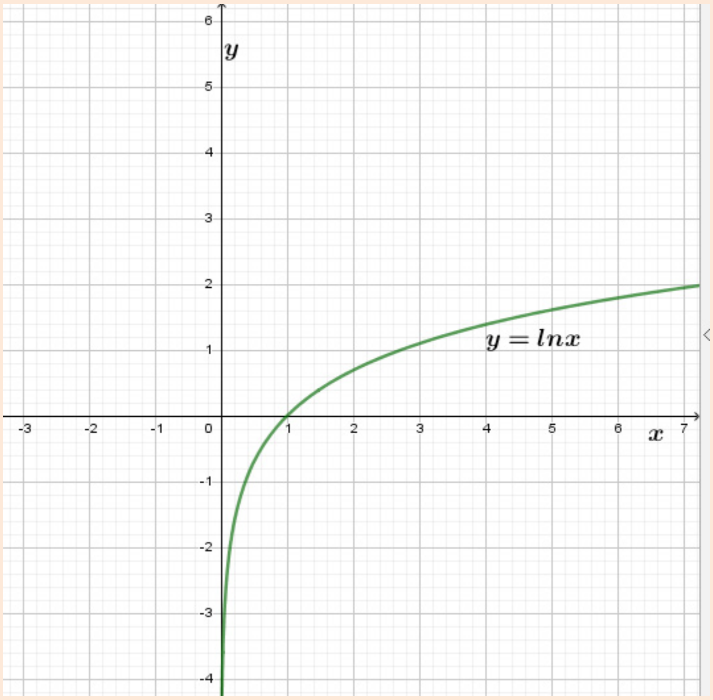
2) Observe the following graph of a given logarithmic function, then state its domain and range Justify your answers.



2.1.2 Limits and asymptotes of logarithmic functions

Activity 2.2

The graph below represents natural logarithmic function $f(x) = \ln x$



Consider the form of this graph then by using calculator, complete the table below to answer the questions that follow.

x	0.5	0.001	0.001	0.0001	2	100	1001	10000
$\ln x$								

1) Discuss the values of $\ln x$ when x takes values closer to 0 from the right and deduce $\lim_{x \to 0^+} \ln x$. Find equation of asymptotes of $f(x) = \ln x$ if any.

2) Discuss the values of $\ln x$ when x takes greater values and conclude about the $\lim_{x \to +\infty} \ln x$.

3) Explain why it is senseless to discuss $\lim_{x \to 0^-} \ln x$.

4) Express in your own words the meaning of the following sentence: "the line of equation $x = 0$ is a vertical asymptote".

The limit $\lim\limits_{x\to 0^+} \ln x = -\infty$ shows that the line OY with equation $x=0$ is the vertical asymptote. This means that as the independent variable x takes values approaching 0 from the right, the graph of the function approaches the line of equation $x=0$ without intercepting. In other words, the dependent variable y takes "bigger and bigger" negative values.

Then, $\lim\limits_{x\to +\infty} \ln x = +\infty$, which implies that there is no horizontal asymptote.

The $\lim\limits_{x\to 0^-} \ln x$ does not exist because values closer to 0 from the left are not included in the domain of the given function.

In general the limit of any logarithmic function can be determined in the same way as the limit of the natural function. If you feel more comfortable with the natural logarithmic function, use the relationship between logarthim in base a and natural logarthim: $f(x) = \log_a u(x) = \dfrac{\ln(u(x))}{\ln a}$ provided $a>0, a\neq 1$.

Example 2.2

Determine each of the following limit

a) $\lim\limits_{x\to e} \ln x$ b) $\lim\limits_{x\to 2}(1-\ln x)$ c) $\lim\limits_{x\to +\infty} \log_3\left(\dfrac{x-4}{x+6}\right)$

Solution

a) $\lim\limits_{x\to e} \ln x = 1$ b) $\lim\limits_{x\to 2}(1-\ln x) = 1-\ln 2$

c) $\lim\limits_{x\to +\infty} \log_3\left(\dfrac{x-4}{2x+6}\right) = \log_3 \dfrac{1}{2}$=-$\log_3 2$ since $\lim\limits_{x\to \infty}\dfrac{x-4}{2x+6} = \dfrac{1}{2}$.

Alternatively, using natural logarithmic function, we have

$$\lim_{x\to +\infty} \log_3\left(\frac{x-4}{2x+6}\right) = \lim_{x\to +\infty} \frac{\ln\left(\dfrac{x-4}{2x+6}\right)}{\ln 3} = \frac{1}{\ln 3}\lim_{x\to +\infty} \ln\left(\frac{x-4}{2x+6}\right) = \frac{1}{\ln 3}\times \ln\frac{1}{2} = -\frac{\ln 2}{\ln 3} = -\log_3 2 = \log_3 \frac{1}{2}$$

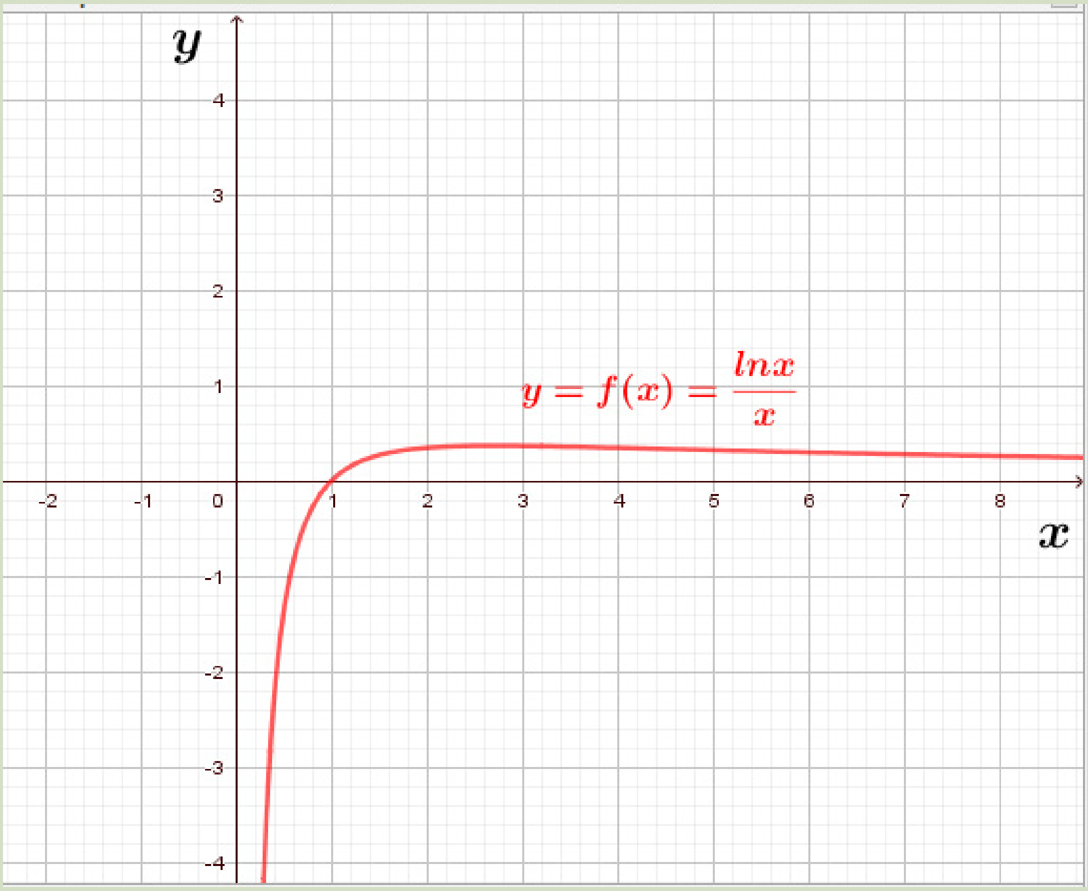
Application activities 2.2

i) Evaluate the following limits

1) $\lim\limits_{x\to+\infty} \ln(7x^3 - x^2 + 1)$ 2) $\lim\limits_{x\to1^+}\left(\ln\dfrac{1}{x-1}\right)$ 3) $\lim\limits_{x\to2^-} \log_5\left(x^2 - 5x + 6\right)$

4) $\lim\limits_{a\to4^+} \ln\dfrac{a}{\sqrt{a-4}}$ 5) $\lim\limits_{x\to+\infty} \ln\left(x^2 - 4x + 1\right)$ 6) $\lim\limits_{x\to+\infty} \dfrac{2+4\log x}{x}$

ii) observe the graph below of the function $p(x) = \dfrac{\ln x}{x}$ and deduce $\lim\limits_{x\to+\infty}\dfrac{\ln x}{x}$, $\lim\limits_{x\to0^+}\dfrac{\ln x}{x}$, $\lim\limits_{x\to1}\dfrac{\ln x}{x}$ and $\lim\limits_{x\to\frac{1}{5}}\left(\dfrac{\ln x}{x}\right)$



2.1.3 Continuity and asymptote of logarithmic functions

Activity 2.3

Let us consider the logarithmic function $f:\mathbb{R}_0^+ \to \mathbb{R},\quad y=f(x)=\log_2(x)$

1) Complete the following table:

$x=x_0$	$y=\log_2 x$	$\lim\limits_{x\to x_0}\log_2 x$
$\frac{1}{4}$		
$\frac{1}{2}$		
1		
2		
4		

3) Can you conclude that $\lim\limits_{x\to x_0}\log_2 x=\log_2(x_0)$ **?** What about the continuity of

$y=f(x)=\log_2(x)$?

4) By using the information drawn in the above table and the scientific calculator, plot the graph of $y=\log_2(x)$.

5) Give any justification that allows you to decide on the continuity of the function.

The graph of the logarithmic function $f:\mathbb{R}_0^+ \to \mathbb{R}, f(x)=\log_a(x),\quad a>1$ has the following characteristics:

- The domain is $]0,+\infty[$ and $f(x)$ is continuous on this interval.
- The range is $\mathbb{R}$
- The graph intersects the $x-axis$ at $(1,0)$
- As $x\to 0^+, y\to -\infty$, so the line of equation $x=0$ (the y- axis) is an asymptote to the curve
- As x increases, the graph rises more steeply for $x\in[0,1]$ and is flatter for $x\in[1,+\infty[$

- The logarithmic function is increasing and takes its values (range) from negative infinity to positive infinity.

Example 2.3

Let us consider the logarithmic function $y = \log_2(x-3)$

a) What is the equation of the asymptote line?

b) Determine the domain and range

c) Find the $x-$ intercept.

d) Determine other points through which the graph passes

e) Sketch the graph

Solution

a) The basic graph of $y = \log_2 x$ has been translated 3 units to the right, so the line $L \equiv x = 3$ is the vertical asymptote.

b) The function $y = \log_2(x-3)$ is defined for $x-3>0$

So, the domain is $]3,+\infty[$ .The range is $\mathbb{R}$

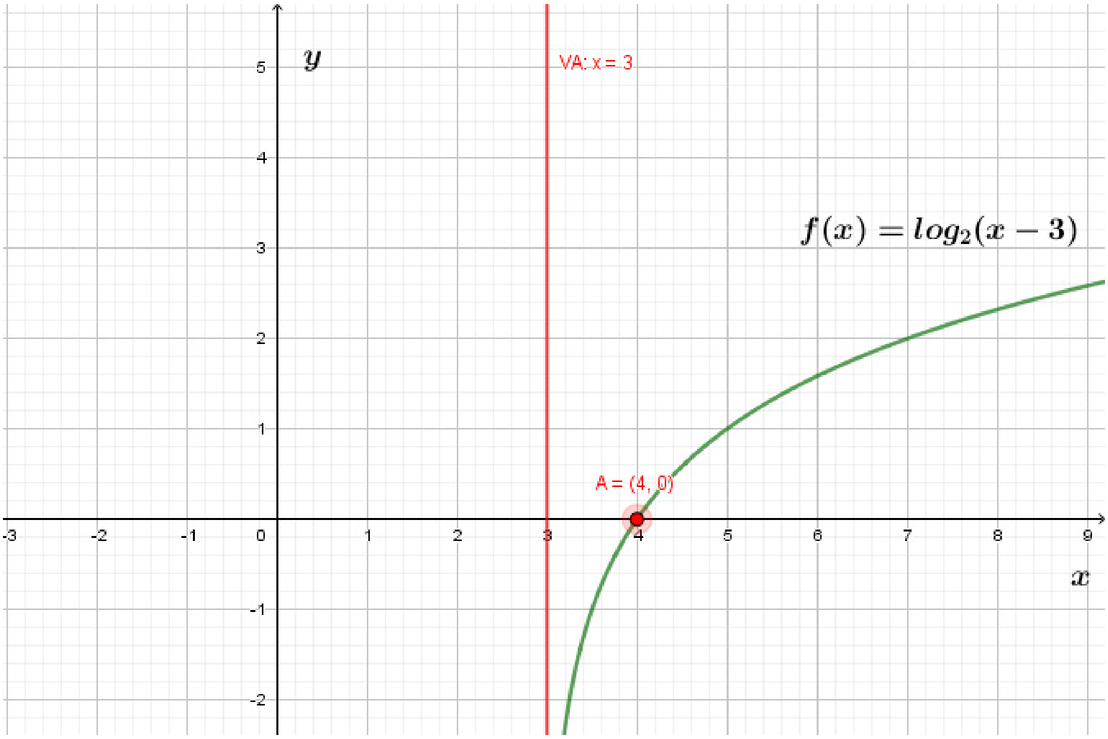
c) The intercept is (4, 0) since $\log_2(x-3) = 0 \Leftrightarrow x = 4$

d) Another point through which the graph passes can be found by allocating an arbitrary value to x in the domain then compute y.

For example, when $x = 5, y = \log_2(5-3) = \log_2 2 = 1$ which gives the point $(5,1)$.

Note that the graph does not intercept *y*-axis because the value 0 for *x* does not belong to the domain of the function.

The graph of $y = f(x) = \log_2(x-3)$



Application activities 2.3

1) Given the logarithmic function $y = -1 + \ln(x+1)$,

i) find equation of asymptote lines (if any)?

ii) State the domain and range

iii) Find the $x-$ intercept

iv) Find the $y-$ intercept

v) Determine another point belonging to the graph

vi) Sketch the graph

2) Sketch the graph of the logarithmic function $f(x) = \log_a x$ with $0 < a < 1$.

Precise the characteristics of the graph.

2.1.4. Differentiation of logarithmic functions

Activity2.4

Let $f(x)=\ln x$

a) Find $f(x+h)$ and $f(2+h)$

b) Complete the following table

h	$\frac{\ln(2+h)-\ln 2}{h}$
-0.1	
-0.001	
-0.00001	
0.1	
0.001	
0.00001	

From the results found in the above table approximate the value of $f'(2)=\lim_{h\to 0}\frac{\ln(2+h)-\ln 2}{h}$

And deduce the expression of $f'(\mathrm{x})$.

Based on your existing knowledge on dervatives, provide any interpretation of the number $f'(2)$.

The definition of derivative shows that if $y=\ln x$,

$$y'=\lim_{h\to 0}\frac{\ln(x+h)-\ln x}{h}=\lim_{h\to 0}\ln\left(\frac{x+h}{x}\right)^{\frac{1}{h}}$$

$$=\lim_{h\to 0}\ln\left(1+\frac{h}{x}\right)^{\frac{1}{h}}=\ln\lim_{h\to 0}\left(1+\frac{h}{x}\right)^{\frac{1}{h}}=\ln e^{\frac{1}{x}}=\frac{1}{x}$$

Then, the natural logarithmic function $y=\ln x$ is differentiable on $]0,+\infty[$ and $\frac{d}{dx}(\ln x)=\frac{1}{x}$, $(x>0)$.

Using the formula of base change, $\frac{d}{dx}(\log_a x)=\frac{1}{x.\text{lna}}$ for any base a provided the conditions are fulfilled. In the more general form, if $u(x)$ is any differentiable function such that $u(x)>0$, $\frac{d}{dx}[\ln u(x)]=\frac{1}{u(x)}\times u'(x)=\frac{u'(x)}{u(x)}$

and $\frac{d}{dx}[\log_a u(x)]=\frac{1}{u(x)}\times\frac{u'(x)}{\ln a}=\frac{u'(x)}{u(x).\ln a}$

Example 2.4

1. Differentiate each of the following functions with respect to x

a) $f(x)=\ln(x^3+3x-4)$

b) $f(x)=x^2\ln x$

c) $f(x)=\sin x\ln x$

d) $y=\log_2(5x^3)$

2. Find the slope of the line tangent to the graph of $y=\log_2(3x+1)$ at $x=1$

Solution

1. Differentiation

a) $\frac{d}{dx}\ln(x^3+3x-4)=\frac{1}{x^3+3x-4}(x^3+3x-4)'=\frac{3x^2+3}{x^3+3x-4}$

b) $\frac{d}{dx}(x^2\ln x)=\ln x\frac{d}{dx}x^2+x^2\frac{d}{dx}\ln x=\ln x\times 2x+x^2\times\frac{1}{x}=2x\ln x+x$

c) $\frac{d}{dx}(\sin x\ln x)=\ln x\frac{d}{dx}(\sin x)+\sin x\frac{d}{dx}(\ln x)=\ln x\times\cos x+\sin x\times\frac{1}{x}=\cos x\ln x+\frac{\sin x}{x}$

d) $\frac{d}{dx}\log_2(5x^3)=\frac{d}{dx}\left(\frac{\ln 5x^3}{\ln 2}\right)=\frac{1}{\ln 2}\frac{d}{dx}(\ln 5+3\ln x)$

$$=\frac{1}{\ln 2}\left(\frac{d}{dx}\ln 5+\frac{d}{dx}(3\ln x)\right)=\frac{1}{\ln 2}\left[0+3\frac{d}{dx}\ln x\right]=\frac{1}{\ln 2}\times 3\times\frac{1}{x}=\frac{3}{x\ln 2}$$

2. To find the slope, we must evaluate $\frac{dy}{dx}$ at $x=1$

$$\frac{d}{dx}\log_2(3x+1)=\frac{d}{dx}\left(\frac{\ln(3x+1)}{\ln 2}\right)=\frac{1}{\ln 2}\frac{d}{dx}(\ln 3x+1)=\frac{3}{(3x+1)\ln 2}$$

By evaluating the derivative at $x=1$, we see that the tangent line to the curve at the point $(1,\log_2 4)=(1,2)$ has the slope $\left.\frac{dy}{dx}\right|_{x=1}=\frac{3}{4\ln 2}=\frac{3}{\ln 16}$.

Application activities 2.4

1) Differentiate $y=\ln\sqrt{\frac{1+x}{1-x}}$ with respect to x.

2) An airplane takes off from an airport at sea level. If its altitude (in kilometres) at time t (in minutes) is given by $h=2000\ln(t+1)$, find the velocity of the airplane at time $t=3\,\text{min}$.

2.1.5 Variation of logarithmic function

Activity 2.5

Given two functions $f(x)=\ln x$ and $g(x)=\log_{10} x$,

1) Compare $f(2)$ and $f(10)$, $g(2)$ and $g(10)$ and deduce whether those functions are increasing or decreasing on $[2,10]$.

2) Use the tables of signs for $f'(x)$ and $g'(x)$ to establish the intervals and the variation of those functions.

3) Which function f or g is increasing or decreasing faster than another on [2,10]

The logarithmic function $f(x)=\log_a x$, $a>0, a\neq 1$ varies in the following way:

a) For $x>0$ then, $f'(x)=\frac{1}{x\ln a}$. The sign of $f'(x)$ depends on the value of the base a.

If $a>1$, $\ln a>0$ therefore, $f'(x)$ is always positive. Thus $f(x)=\log_a x$ is strictly increasing on $\mathbb{R}_0^+$

Variation table for $y=f(x)=\log_a x$ **for** $a>1$

x	0		1		a		$+\infty$
y'	‖	$+$	$\frac{1}{\ln a}$	$+$	$\frac{1}{a\ln a}$	$+$	
y	‖ $-\infty$	$\nearrow$	0	$\nearrow$	1	$\nearrow$	$+\infty$

If $0<a<1$, $\ln a<0$. Therefore $f'(x)$ is always negative.

Thus $f(x)=\log_a x$ is strictly decreasing on $\mathbb{R}_0^+$. This implies the absence of extrema (maxima or minima) values.

Table of variation for $y=f(x)=\log_a x$ for $0<a<1$

x	0		a		1		$+\infty$
y'	‖	-	$\frac{1}{a\ln a}$	-	$\frac{1}{\ln a}$	-	
y	‖ $+\infty$	$\searrow$	1	$\searrow$	0	$\searrow$	$-\infty$

Example 2.5

Discuss variations of the logarithmic function $f(x)=x-\ln x$

Solution

The function $f(x)=x-\ln x$ is defined for all $x>0$ and $f'(x)=\frac{d}{dx}(x-\ln x)=1-\frac{1}{x}$

Thus, $1-\frac{1}{x}=0 \Leftrightarrow \frac{1}{x}=1 \Rightarrow x=1$.

If $x=1$, $y=f(1)=1-\ln 1=1$, thus (1,1) is a point of the graph.

Variation table of $y = f\left(x\right) = x - \ln x$

x	0		1		$+\infty$
y'		-	0	+	
y	$+\infty$	$\searrow$	1	$\nearrow$	$+\infty$

From the table, one can observe that the function is decreasing for values when x lies in]0,1] and increasing for x greater than 1. The point (1,1) is minimum or equivalently the function takes the minimum value equal for x=1. The minimum value that is equal to 1 is absolute.

Application of differentiation: limits involving indeterminate forms

1) Evaluate $\lim\limits_{x\to+\infty} \dfrac{\ln x}{x} = 0$

Solution

$\lim\limits_{x\to+\infty} \dfrac{\ln x}{x}$ takes indeterminate form $\dfrac{\infty}{\infty}$. Apply Hospital rule:

$$\lim_{x\to+\infty} \frac{\ln x}{x} = \lim_{x\to+\infty} \frac{\frac{1}{x}}{1} = \lim_{x\to+\infty} \frac{1}{x} = 0$$

1) Evaluate $\lim\limits_{x\to0^+} \dfrac{\ln\left(1+x\right)}{x}$

Solution

$$\lim_{x\to0^+} \frac{\ln\left(1+x\right)}{x} \text{ (indeterminate form } \frac{0}{0}\text{).}$$

$$\lim_{x\to0^+} \frac{\ln\left(1+x\right)}{x} = \lim_{x\to0^+} \frac{1}{1+x} = 1$$

Application activity 2.5

1) Discuss variations of the function $f(x)=\frac{\ln(x-2)}{x-2}$

2) Suppose a satellite has been shot upward with an initial velocity of $20\,m/s$. Then the equation $h(t)=100\ln(t\ \ +1)$ gives the height of the satellite in m after t $\sec onds$

a) The derivative of the function for the height of the satellite gives the rate of change of the height or the velocity of the satellite. Find the velocity function.

b) Find the velocity function after 2 $\sec onds$

c) Is the velocity increasing or decreasing?

2. 2 Exponential functions

2.2.1 Domain of definition of exponential function

Activity 2.6

1) Let f(x) = and $g(x)$ denotes the inverse function of f(x).

i) complete the following table:

x	0	1	e	e^2		
$g(x)=f^{-1}(x)$					3	4

ii) Discuss and find out the set of all values of $g(x)$.

2) Consider the function $h(x)=3^x$ and complete the following table

X	-10	-1	0	1	10
$h(x)=3^x$					

a) Discuss whether $\forall x\in\mathbb{R}, h(x)\in\mathbb{R}$ and deduce the domain of $h(x)$

b) Discuss whether $h(x)$ can be negative or not and deduce the range of $h(x)$.

Remember that for $a>0, a\neq 1$ the logarithmic function is defined as

$\log:\mathbb{R}_0^+\to\mathbb{R}\ or\ x\to y=\log_a x$. The inverse of logarithmic function is called exponential function and defined as:

$\exp_a : \mathbb{R} \to \mathbb{R}_0^+ : x \mapsto y = \exp_a x$. For simplicity we write $\exp_a x = a^x$.

Therefore $a^x = y$ if and only if $\log_a y = x$. Obviously, the domain of the exponential function $y = f(x) = a^x$ is $]-\infty, +\infty[$ and its range is $]0, +\infty[$. In the expression $a^x = y$, a is the base, *x* the exponent and *y* the exponential of *x* in base *a*.

Generally, if $u(x)$ is a defined function of x, the function $f(x) = a^{u(x)}$ has the range $]0, +\infty[$ and its domain is the domain of $u(x)$.

Similarly, to logarithmic function, if the base "*a*" is the number "*e*", we have exponential function $y = e^x$ as the inverse of natural logarithm $y = \ln x$.

Example 2.6

Determine the domain of each of the following functions:

1) $f(x) = 3^{\sqrt{2x}}$ 2) $g(x) = e^{\frac{x+2}{x-3}}$ 3) $h(x) = e^{\sqrt{x^2-4}}$

Solution

1) Condition for the existence of $\sqrt{2x}$ in $\mathbb{R}$: $x \geq 0$. Thus, $Dom f = [0, +\infty[$

2) Condition for the existence of $\frac{x+2}{x-3}$ in $\mathbb{R}$: $x \neq 3$. Therefore $Dom\ g = \mathbb{R} \setminus \{3\} =]-\infty, 3[\cup]3, +\infty[$

3) Condition: $x^2 - 4 \geq 0 \Rightarrow x \in]-\infty, -2] \cup [2, +\infty[$.

Thus, $Dom\ h =]-\infty, -2] \cup [2, +\infty[$

Application activities 2.6

Discuss and determine the domain and range of the following functions

1) $f(x) = 5e^{2x}$ 2) $h(x) = 2^{\ln x}$ 3) $f(x) = 3^{\frac{x+1}{x-2}}$

2.2.2 Limits of exponential functions

Activity 2.7

1) You are familiar with the graph of $f(x)=\ln x$. Explain in your words how you can obtain the graph of its inverse $y=f^{-1}(x)=e^x$.

2) From the graph deduce $\lim\limits_{x\to-\infty} e^x$ and $\lim\limits_{x\to+\infty} e^x$. Are there any asymptotes?

3) Discuss $\lim\limits_{x\to-\infty}\left(\frac{1}{2}\right)^x$ *and* $\lim\limits_{x\to+\infty}\left(\frac{1}{2}\right)^x$.

4) Generalize above results to $\lim\limits_{x\to-\infty} a^x$ *and* $\lim\limits_{x\to+\infty} a^x$

Based on results on logarithmic functions, it is clear that: $\lim\limits_{x\to-\infty} e^x=0$ and $\lim\limits_{x\to+\infty} e^x=+\infty$

In general: If $a>1$, $\lim\limits_{x\to-\infty} a^x=0$ and $\lim\limits_{x\to+\infty} a^x=+\infty$

If $0<a<1$, $\lim\limits_{x\to-\infty} a^x=+\infty$ and $\lim\limits_{x\to+\infty} a^x=0$

Example 2.7

1) Evaluate

a) $\lim\limits_{x\to\infty} e^{1-4x-5x^2}$ b) $\lim\limits_{x\to1}\left(\frac{3}{5}\right)^{\frac{1}{x-1}}$ c) $\lim\limits_{x\to-\infty} 3^{\frac{1}{x}}$ d) $\lim\limits_{x\to1} 3^{\frac{1}{x-1}}$

Solution

a) $\lim\limits_{x\to\infty} e^{1-4x-5x^2}$

We know that $\lim\limits_{x\to\infty}(1-4x-5x^2)=-\infty$

Therefore, as the exponent goes to minus infinity in the limit and so the exponential must go to zero in the limit using the ideas from the previous formula.

Hence, $\lim\limits_{x\to\infty} e^{1-4x-5x^2}=0$

b) The exponent goes to infinity in the limit and so the exponential will also need to go to zero in the limit since the base is less than 1. Hence,

$\lim\limits_{x\to1}\left(\frac{3}{5}\right)^{\frac{1}{x-1}}=0$

c) $\lim\limits_{x \to -\infty} 3^{\frac{1}{x}} = 3^{\lim\limits_{x \to -\infty} \frac{1}{x}} = 3^0 = 1$

d) $\lim\limits_{x \to 1} 3^{\frac{1}{x-1}} = 3^{\lim\limits_{x \to 1} \frac{1}{x-1}} = 3^{\frac{1}{0}}$

Study one side limit:

x	$3^{\frac{1}{x-1}}$
0	0.33
0.2	0.25
0.4	0.16
0.6	0.06
0.8	0.004
0.9	0.00001

x	$3^{\frac{1}{x-1}}$
2	3
1.8	3.948
1.6	6.24
1.4	15.59
1.2	243
1.1	59,049

$\lim\limits_{x \to 1^-} 3^{\frac{1}{x-1}} = 0$ and $\lim\limits_{x \to 1^+} 3^{\frac{1}{x-1}} = +\infty$. Hence, $\lim\limits_{x \to 1} 3^{\frac{1}{x-1}}$ does not exist.

Alternatively:

Since $\lim\limits_{x \to 1^+} \frac{1}{x-1} = +\infty$ and $\lim\limits_{x \to 1^-} \frac{1}{x-1} = -\infty$, apply results on $\lim\limits_{x \to \pm\infty} a^x$ for $a > 1$ to have:

$\lim\limits_{x \to 1^-} 3^{\frac{1}{x-1}} = 0$ and $\lim\limits_{x \to 1^+} 3^{\frac{1}{x-1}} = +\infty$ .Hence, $\lim\limits_{x \to 1} 3^{\frac{1}{x-1}}$ does not exist.

2. Consider $f(x) = \frac{e^{-3x} - 2e^{8x}}{9e^{8x} - 7e^{-3x}}$, evaluate each of the following: $\lim\limits_{x \to -\infty} f(x)$ and $\lim\limits_{x \to +\infty} f(x)$.

Solution

$\lim\limits_{x \to -\infty} f(x) = \lim\limits_{x \to -\infty} \frac{e^{-3x} - 2e^{8x}}{9e^{8x} - 7e^{-3x}} = \lim\limits_{x \to -\infty} \frac{e^{-3x}\left(1 - 2e^{11x}\right)}{e^{-3x}\left(9e^{11x} - 7\right)} = \lim\limits_{x \to -\infty} \frac{1 - 2e^{11x}}{9e^{11x} - 7} = -\frac{1}{7}$ and

$\lim\limits_{x \to +\infty} f(x) = \lim\limits_{x \to +\infty} \frac{e^{-3x} - 2e^{8x}}{9e^{8x} - 7e^{-3x}} = \lim\limits_{x \to +\infty} \frac{e^{8x}\left(e^{-11x} - 2\right)}{e^{8x}\left(9 - 7e^{-11x}\right)} = -\frac{2}{9}$

3) Biologists consider a species of a plant or animal to be endangered if it is expected to become extinct in less than 20 years. The population y of a certain species is modelled by:

$y = 1096\, e^{-0.39t}$ (see the figure bellow)

Is this species endangered? Explain your answer.

Solution:

This species is endangered. This is because the value of the function y when t is approaching 20 is about 0 which means that the species is becoming extinct.

Application activities 2.7

For each given function , evaluate limit at $+\infty$ and $-\infty$

1) $f(x) = e^{8+2x-x^3}$

2) $f(x) = e^{\frac{6x^2+x}{5+3x}}$

3) $f(x) = 2e^{6x} - e^{-7x} - 10e^{4x}$

4) $f(x) = 3e^{-x} - 8e^{-5x} - e^{10x}$

5) $f(x) = \dfrac{e^{-3x} - 2e^{8x}}{9e^{8x} - 7e^{-3x}}$

2.2.3. Continuity and asymptotes of exponential function

Activity 2.8

Given the function $f\left(x\right) = 2^{(x-2)}$,

a. Find the domain and range of f.

b. Determine $\lim\limits_{x\to-\infty} f\left(x\right)$ and deduce the equation of horizontal asymptote for the graph.

c. Evaluate the value of $f\left(x\right)$ for $x = 0$ and deduce y- intercept.

d. Determine $\lim\limits_{x\to\infty} f(x)$ and $\lim\limits_{x\to-\infty} \dfrac{f(x)}{x}$

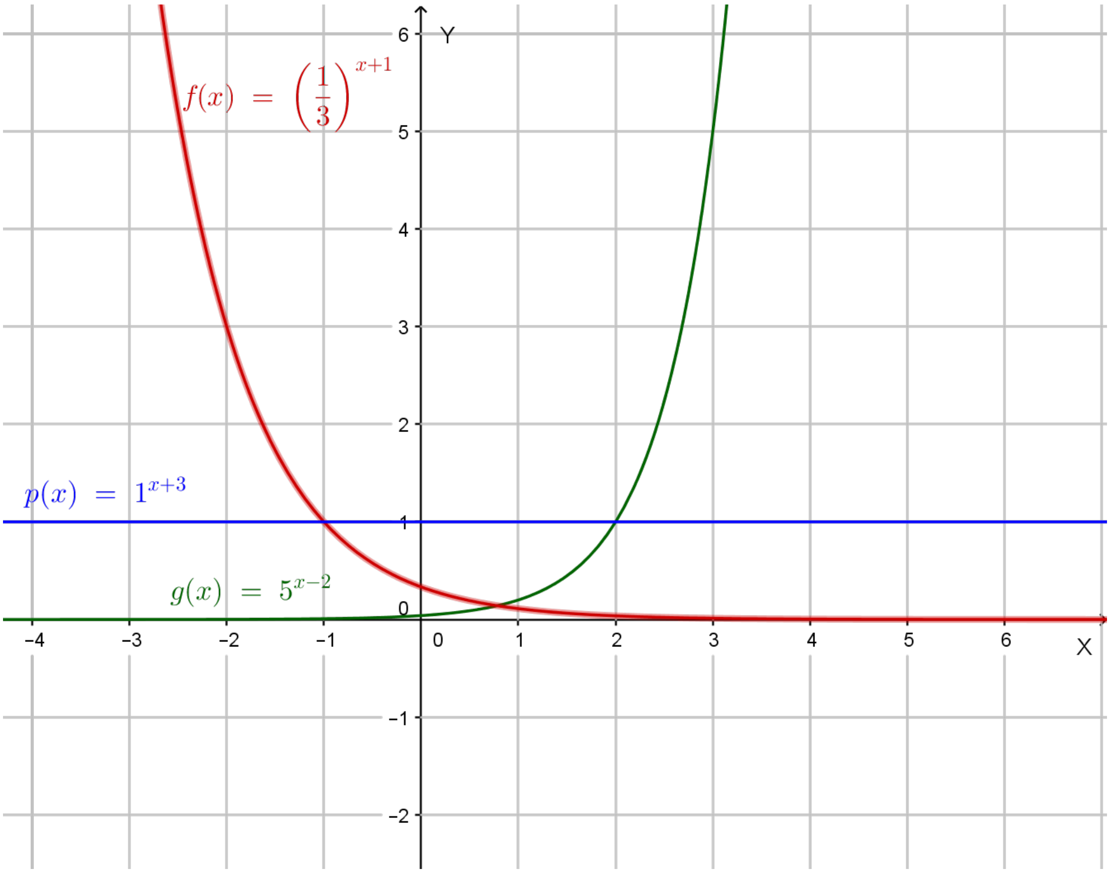
e. Evaluate $\lim\limits_{x\to 0^+} f(x)$ and $\lim\limits_{x\to 0^-} f(x)$. Discuss the continuity of this function at $x=0$.

f. Sketch the graph of $f(x)$.

For $a>0, a\neq 1$, the exponential function $f(x) = a^x$ is continuous on $\mathbb{R}$ and takes always nonnegative values. Its graphs admits the line of equation $y=0$ as horizontal symptote and intercepts y-axis at (0,1).. The function f is increasing from 0 to $+\infty$ if

a is greater than 1 and decreasing from $+\infty$ to 0 if a is smaller than 1. The function is the constant 1 if $a=1$ and its graph is the horizontal line of equation $y=1$.

Graphs of $g(x)=5^{x-2}$, $f(x)=\left(\dfrac{1}{3}\right)^{x+1}$ **and** $p(x)=1^{x+3}$



Example 2.8

Let $f(x)=3^{x+1}-1$. Find the domain, range and equation of the horizontal asymptote of the graph of f. Precise intercepts (if any) of the graph with axes. .

Solution

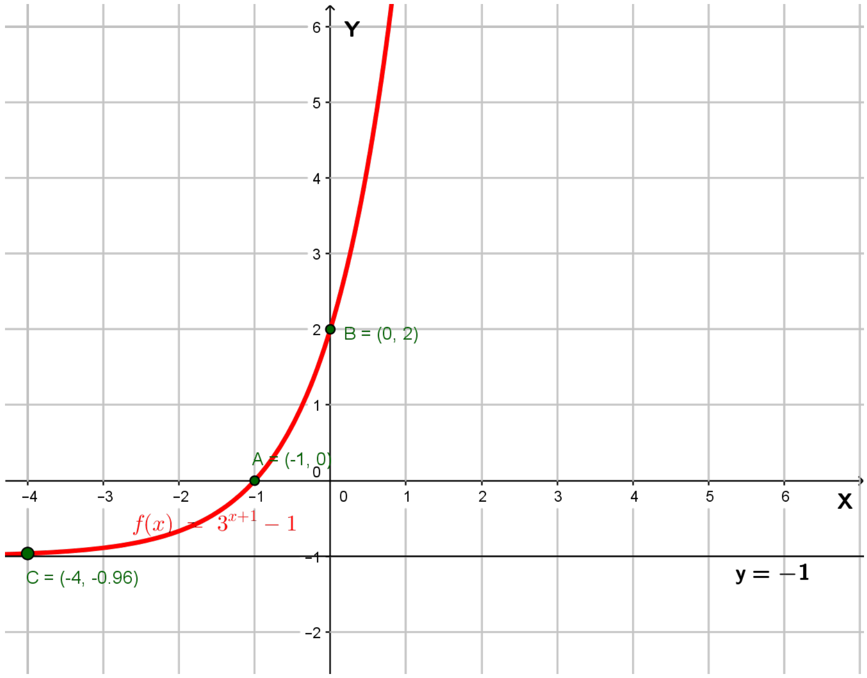
The domain of f is the set of all real numbers since the expression $x+1$ is defined for all real values. To find the range of f, we start with the fact that $3^{(x+1)}>0$ as exponential function.

Then, subtract 1 to both sides to get $3^{x+1}-1>-1$. Therefore, for any value of *x*, $f(x)>-1$. in other words, the range of *f* is $]-1,\infty[$. As x decreases without bound,

$f(x)=3^{x+1}-1$ approaches -1, in other words $\lim\limits_{x\to-\infty} f(x)=-1$. Thus, the graph of f has horizontal asymptote, the line of equation $y=-1$. To find the x intercept we need to solve the equation f(x) = 0 , which means $3^{(x+1)}-1=0$. Solving yields to $x=-1$.

The x- intercept is the point (-1,0) and the y- intercept is given by $(0, f(0)) = (0, 3^{(0+1)}-1) = (0, 2)$. Extra points: $(-2, f(-2)) = (-2, 3^{(-2+1)}-1) = (2, -\frac{3}{4})$ and $(-4, f(-4)) = (-4, 3^{(-4+1)}-1) = (-4, -\frac{26}{27})$

We can now use all the above information to plot $f(x)=3^{(x+1)}-1$:



Application activities 2.8

Given the function $f(x) = 2^x + 1$

a) Determine domain and range of $f(x)$.

b) Write the equation of horizontal asymptote of the graph of $f(x)$.

c) Find the x and y intercepts of the graph of $f(x)$ if there are any.

d) Sketch the graph of $f(x)$.

2.2.4. Differentiation of exponential functions

Activity 2.9

Given functions $f(x) = e^x$ and $g(x) = 2^x$

1) Determine the inverse of $f(x)$ and $g(x)$.

2) Use the derivative of logarithmic functions $p(x) = \ln x$ and $k(x) = \log_2 x$, then apply the rule of differentiating inverse functions to find the derivative of $f(x) = e^x$ and $g(x) = 2^x$

The derivative of $f(x) = e^x$ is noted by $\frac{d(e^x)}{dx} = e^x$ or $f'(x) = e^x$.

If u is a function of x, the derivative of $y = e^{u(x)}$ is $y' = d\left(\frac{e^{u(x)}}{dx}\right) = e^{u(x)}\frac{du(x)}{dx} = u'e^{u(x)}$

.

Thus, $\left(e^u\right)' = u'e^u$. The derivative of $g(x) = a^x$ is $g'(x) = a^x \ln a$.

Therefore, if u is a function of x, the derivative of $g(x) = a^{u(x)}$ is $g'(x) = u'(x)a^{u(x)} \ln a$

Example 2.9

1) Suppose that $f(x) = e^x$ and $f'(x) = e^x$

i) Complete the table below

x	$f(x)$	$f'(x)$
2		
4		
5		

iii) Deduce the slope of the tangent line at each value of x from the table above

iv) Graph the function $y = e^x$ indicating the slope of the tangent line at $x = 2$

Solution

i) Complete the table below

x	$f(x)$	$f'(x)$
2	7.38	7.38
4	54.59	54.59
5	148.41	148.41

iii) The equation of tangent line $T \equiv Y - y_o = f'(x_0)(x - x_0)$. Remember that $f'(x_0)$ is the slope of the line tangent to the graph of the function $f(x)$ at point (x_0, y_0).

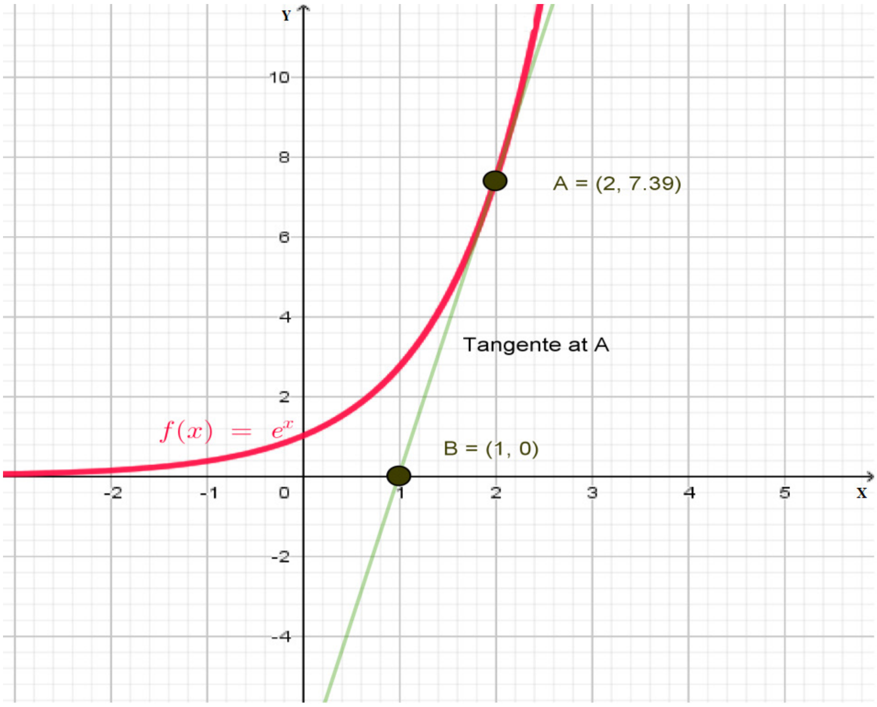
So for $x = 2$, y-value is $e^2 = 7.3890461..... \approx 7.39$,

Since the derivative of e^x is e^x then the slope of the tangent line at $x = 2$ is also $e^2 \approx 7.39$

For $x = 4$, $y - \text{value} = 54.61$, the slope of the tangent line at $x = 4$ is also 54.61

For $x = 5$, $y - \text{value} = 148.43$, the slope of the tangent line at $x = 5$ is also 148.43

Graph of $f(x) = e^x$ indicating the slope of the tangent line at $x = 2$



2) Find the derivative of $f(x)=e^{x^2}$

Solution

From the formula of derivative $(e^u)'=u'e^u$, we get $(e^{x^2})'=(x^2)'.e^{x^2}=2xe^{x^2}$

3) Given the function $f(x)=3^x$

i) Complete the table below

x	$f(x)$	$f'(x)$
2		
4		
5		

ii) Deduce the slope of the tangent line at each value of x from the table above

iii) Graph the function $f(x)=3^x$ indicating the slope of the tangent line at $x=4$

Solution

i) The derivative of $f(x)=3^x$ is $f'(x)=3^x\ln 3$

x	$f(x)$	$f'(x)$
2	9	9.88
4	81	88.98
5	243	266.96

The equation of tangent line at (x_0,y_0) is $T\equiv y-y_0=f'(x_0)(x-x_0)$.

ii) For $x=2$, *y*-value is $3^2=9$ and $f'(2)=3^2\ln 3=9.88$, then the slope of the tangent line at $x=2$ is also 9.88

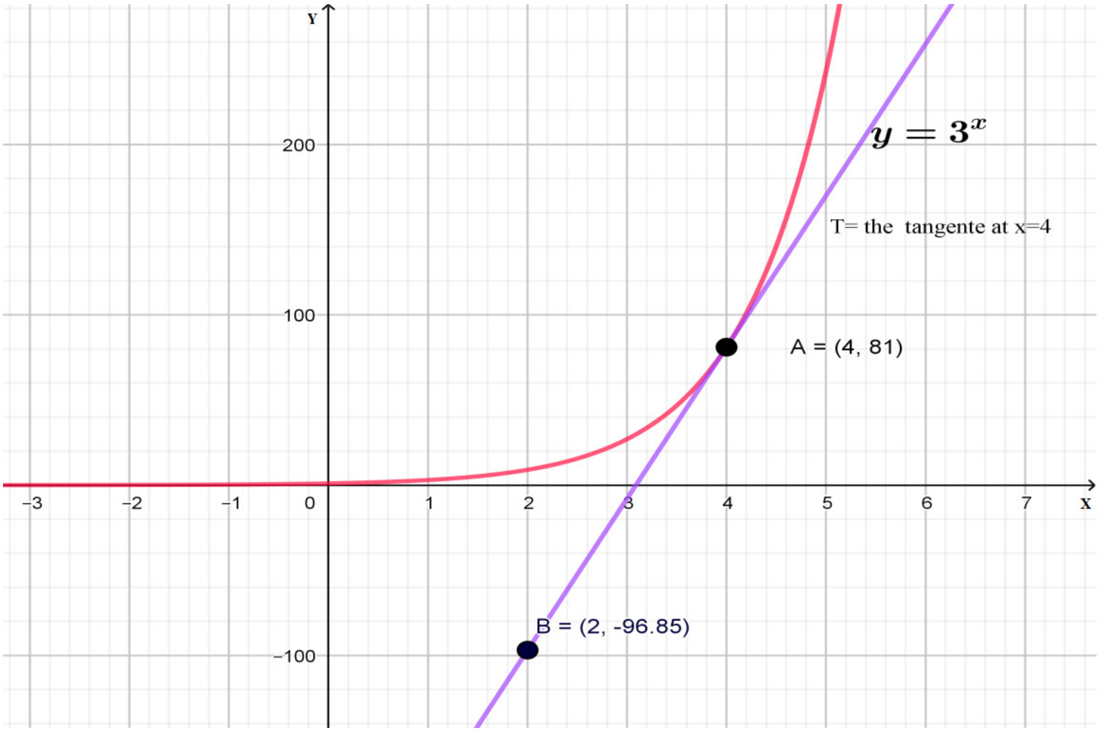
For $x=4$, *y*-value is $3^4=81$ and the slope of the tangent line at $x=4$ is also $f'(4)=3^4\ln 3=88.98$.

For $x=5$, *y*-value is $3^5=243$ and the slope of the tangent line at $x=5$ is $f'(5)=266.96$

Graph the function $f(x)=3^x$ indicating the slope of the tangent line at $x=4$.

From above calculations, the tangent line at the point $(4,81)$ has the equation $y = 88.98x - 274.92$.

iii) Graph of $f(x) = 3^x$ and its tangent at $x = 4$



Observe that the slope of the tangent line at a given point of the graph is the same as the derivative of the function at the $Ox-\text{coordinate}$ of the same point.

Application of derivatives to remove indeterminate form $0^0, 1^\infty$, and ∞^0

These indeterminate forms are found in functions of the form $y = \left[f(x)\right]^{g(x)}$

To remove these indeterminate forms we change the function in the form $y = \left[f(x)\right]^{g(x)} = e^{g(x)\ln f(x)}$.Also $\lim\limits_{x\to k} e^{f(x)\ln g(x)} = e^{\lim\limits_{x\to k} f(x)\ln g(x)}$

Examples:

a) Show that $\lim\limits_{x\to 0^+} x^x = 1$

Solution

$\lim\limits_{x\to 0^+} x^x$ is an indeterminate form (IF) of the form 0^0

$$\lim_{x\to 0^+} x^x = \lim_{x\to 0^+} e^{x\ln x} = e^{\lim\limits_{x\to 0^+} x\ln x}$$

$$\lim_{x\to 0^+} x\ln x \ (0\times\infty\ IF)$$

$$\lim_{x\to 0^+} x\ln x = \lim_{x\to 0^+} \frac{\ln x}{\frac{1}{x}} (\frac{\infty}{\infty} IF)$$

$$\lim_{x\to 0^+} \frac{\ln x}{\frac{1}{x}} = \lim_{x\to 0^+} \frac{\frac{1}{x}}{-\frac{1}{x^2}} = 0 \text{ (Hospital rule).}$$

Finally, $\lim\limits_{x\to 0^+} x^x = 1$

b) Show that $\lim\limits_{x\to +\infty} \left(1+\frac{1}{x}\right)^x = e$

Solution

$$\lim_{x\to +\infty} \left(1+\frac{1}{x}\right)^x \quad \left(1^\infty \quad IF\right).$$

$$\lim_{x\to +\infty} \left(1+\frac{1}{x}\right)^x = \lim_{x\to +\infty} e^{x\ln\left(1+\frac{1}{x}\right)} \quad = e^{\lim\limits_{x\to +\infty} x\ln\left(1+\frac{1}{x}\right)}$$

But, $\lim\limits_{x\to +\infty} x\ln\left(1+\frac{1}{x}\right) = \lim\limits_{x\to +\infty} \dfrac{\ln\left(1+\frac{1}{x}\right)}{\frac{1}{x}} \quad \left(\frac{0}{0}\ IF\right)$

$$\lim_{x\to +\infty} \frac{\ln\left(1+\frac{1}{x}\right)}{\frac{1}{x}} = \lim_{x\to +\infty} \frac{-\frac{1}{x^2}}{\left(1+\frac{1}{x}\right)\left(-\frac{1}{x^2}\right)} \quad \text{(Hospital rule)}$$

$= \lim\limits_{x\to +\infty} \dfrac{1}{1+\frac{1}{x}} = 1$ **Thus,** $\lim\limits_{x\to +\infty} \left(1+\frac{1}{x}\right)^x = e^1 = e$

c)Show that $\lim\limits_{x\to -1} \left(\dfrac{1}{x+1}\right)^{x+1} = 1.$

Solution

$$\lim_{x\to -1}\left(\frac{1}{x+1}\right)^{x+1} (\infty^0 IF) \Rightarrow \lim_{x\to -1}\left(\frac{1}{x+1}\right)^{x+1} = \lim_{x\to -1} e^{(x+1)\ln\left(\frac{1}{x+1}\right)} = e^{\lim_{x\to -1}(x+1)\ln\left(\frac{1}{x+1}\right)}$$

$$\text{Or } \lim_{x\to -1}(x+1)\ln\left(\frac{1}{x+1}\right) = \lim_{x\to -1}\frac{\ln\left(\frac{1}{x+1}\right)}{\frac{1}{x+1}} = \lim_{x\to -1}\frac{-\frac{1}{x+1}}{-\frac{1}{(x+1)^2}} = 0 \text{ (Hospital rule).}$$

$$\text{Finally, } \lim_{x\to -1}\left(\frac{1}{x+1}\right)^{x+1} = e^0 = 1$$

Application activity 2.9

1) Given the function $f(x) = 4^x$

i) Find $f'(x)$ the derivative function of $f(x)$

ii) Find $f(5)$

iii) Deduce the slope of the tangent line at $x = 5$

iv) Plot the function $f(x)$ and tangent line to $f(x)$ at $x = 4$.

2) Find the derivative of each of the following function

a) $f(x) = 10^{3x}$ b) $f(x) = xe^{x^2+1}$ c) $f(x) = \dfrac{3^{4x+2}}{x}$

2.2.5 Variations of exponential functions

Activity 2.10

Given two functions $f(x)=2^x$ and $g(x)=0.5^x$,

1) Compare $f(1)$ and $f(10)$, deduce whether the function $f(x)$ is increasing or decreasing on the interval $[1,10]$.

2) Compare $g(1)$ and $g(10)$ and deduce whether the function $g(x)$ is increasing or decreasing on the interval $[1,10]$.

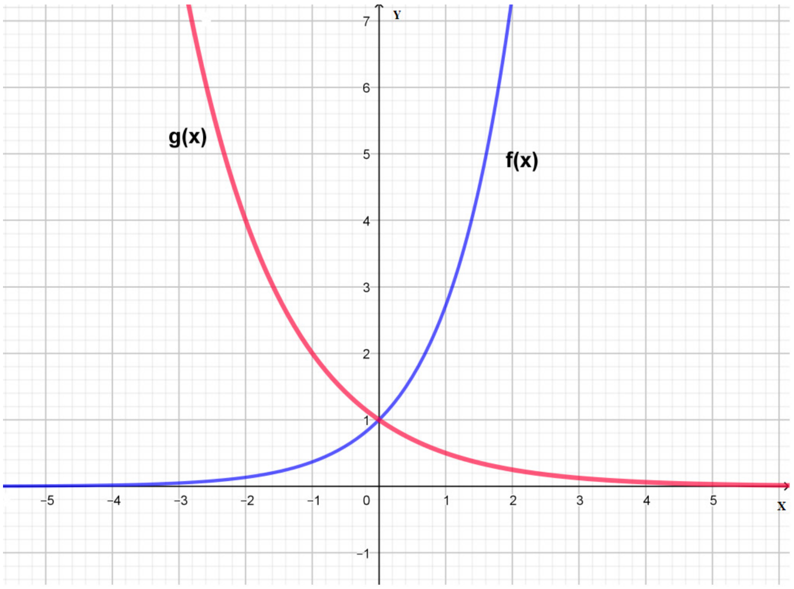
3) Use derivatives $f'(x)$ and $g'(x)$ to discuss variations of each of the functions.

4) Plot the graphs of $f(x)$ and $g(x)$.

5) Express in your own words the variations of exponential function

The function $g(x)=a^x, a>1$ defined on $\mathbb{R}$ is always increasing. When $0<a<1$, the function $g(x)=a^x$ is always decreasing. This means the exponential functions $g(x)=a^x$ does not have extremum (maximum or minimum); this means that the function increases or decreases "indefinitely".

Graph of $g(x)=(0.5)^x$ **and** $f(x)=2^x$



Example 2.10

Given the function $f(x)=xe^x$

i) Find the derivative of $f(x)=xe^x$

ii) Solve $f'(x)=0$

iii) Discuss extrema of the function.

iv) Establish the sign diagram of $f'(x)$ and variations of $f(x)$

v) Plot the graph of the function $f(x)$.

Solution

i) The domain of the function is $\mathbb{R}$.

The derivative of $f(x)=xe^x$ is defined by $f^{'}(x)=e^x+xe^x=(1+x)e^x$

ii) $f'(x)=0$ if $x=-1$

iii) Sign diagram for $f(x)$

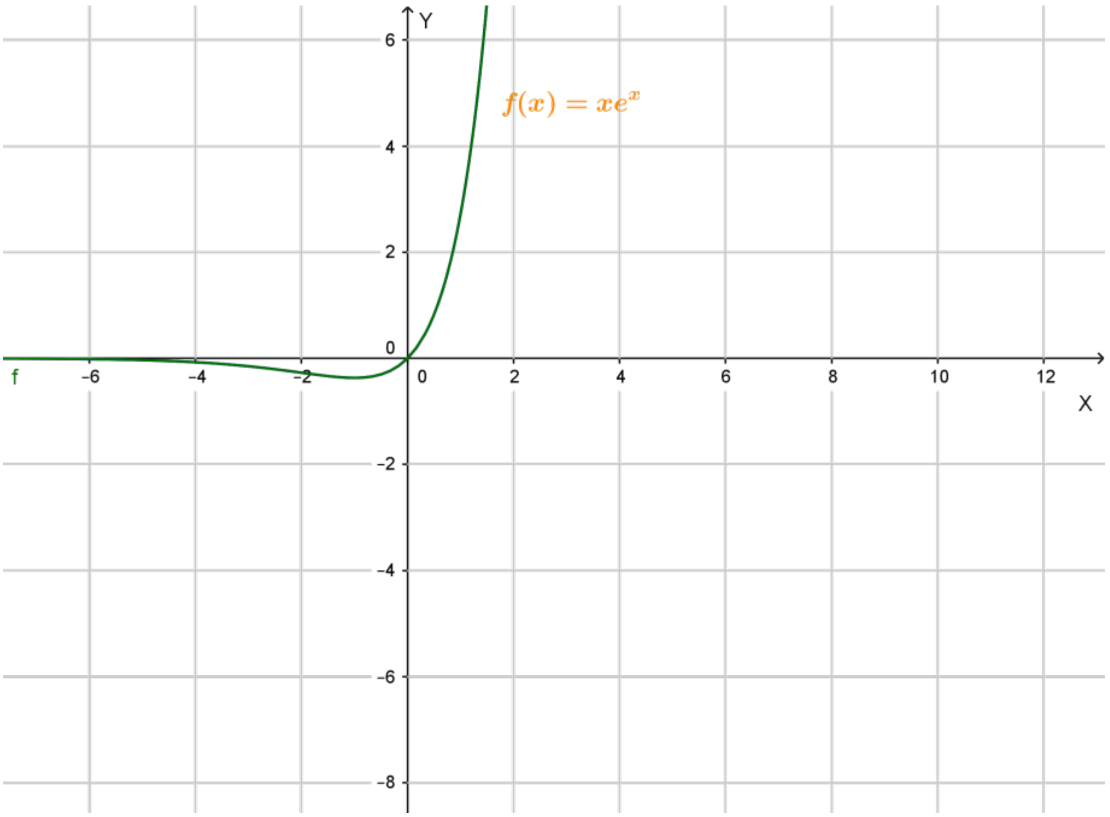
There is need to find limit of the function at the boundaries of the domain: $\lim\limits_{x\to-\infty} xe^x=0$ and $\lim\limits_{x\to+\infty} xe^x=+\infty$. The limit at $-\infty$ tells us that line of equation

$y=0$ is horizontal asymptote when x is taking "indefinitely" negative values.

X	$-\infty$		-1		0			$+\infty$
$f'(x)$	$-$		0	$+$		$+$	$+$	
$f(x)$	0 $\searrow$		-1/e	$\nearrow$	0	$\nearrow$		$+\infty$

Therefore, $f(x)$ is decreasing from 0 to $-\frac{1}{e}$ on the interval $]-\infty,-1]$ and increases from $-\frac{1}{e}$ to $+\infty$ on $[-1,+\infty[$. The function has minimum (absolute) equal to $-\frac{1}{e}$ when $x=-1$.

iv) Graph of $f(x) = xe^x$



Application activity 2.10

1) Given the function $f\left(x\right) = xe^{x^2}$

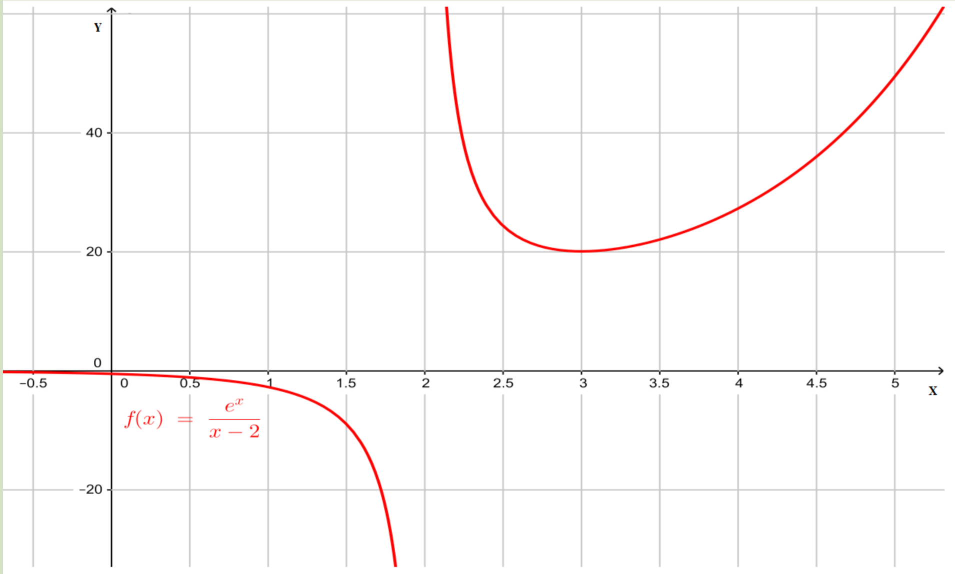
a) Find the derivative of $f(x)$

b) Discuss the table of variation of $f\left(x\right)$, deduce whether $f(x)$ is increasing or decreasing and write down the interval where the function increasing or decreasing.

c) Indicate the extrema points and asymptotes to the graph (if any)

d) Using calculator, complete a table of values and plot the graph of $f(x)$ and compare this graph with the one you find by the use of a mathematical software if possible.

2) Observe the following graph representing a certain function $f(x)=\frac{e^x}{x-2}$ and answer the proposed questions.



a) Precise the domain of definition and the range of the function
b) Discuss continuity of the function and existence of asymptotes to the graph
c) Discuss variations of the function

3) The consumption of natural mineral resource M has risen from 4 million tonnes at the rate of 20% per year. Assume that growth of the consumption has been continuous and governed by the function $M = M_0 e^{rt}$ where M is the final value, M_0 the initial consumption value, r the annual rate of growth and t the time in years.

a) Find the consumption after 6 years
b) Draw the graph illustrating the consumption in function of time.

2. 3 Applications of logarithmic and exponential functions

Logarithmic and exponential functions are very essential in pure sciences, social sciences and real life situations. They are used by bank officers to deal with interests on loans they provide to clients. Economists and demographists use such functions to estimate the number of population after a certain period and many researchers use them to model certain natural phenomena. We are going to develop some of these applications.

2.3.1 Interest rate problems

Activity 2.11

An amount of 2,000 US dollars ($2,000) is invested at a bank that pays an interest rate of 10% compounded once annually. Find the total amount at the end of t years by proceeding as follows:

Complete the table below:

At the end of	The total amount
The first year	2,000+0.1(2,000)=2,000(1+0.1)
The second year	2,000(1+0.1)+0.1[2,000(1+0.1)]=2,000(1+0.1)^2
The third year	2,000(1+0.1) ⋯ + …= 2,000(1+0.1)^3
The fourth year	…
The fifth year	…
...	…
The t^{th} year	…

If a principal P is invested at an interest rate r for a period of t years, then the amount A (how much you make) of the investment can be calculated by the following generalised formula of the interest rate problems:

a) $A = P(1+r)$ Simple interest for one year

b) $A = P\left(1+\frac{r}{n}\right)^{nt}$ Interest compounded n times per year

c) $A = Pe^{rt}$ Interest compounded continuously.

Example 2.11

An amount of 500,000 FRW is invested at a bank that pays an interest rate of 12% compounded annually.

a) How much will the owner have at the end of 10 years, in each of the following cases? The interest rate is compounded:
 i) once a year.
 ii) twice a year
b) What type of interest rate among the two would the client prefer? Explain why.

Solution

a)

i) For once a year, at the end of 10 years the owner will have

$$A = P(1+r)^t = 500,000(1+0.12)^{10}$$
$$= 500,000(1.12)^{10} = 1,552,924.10 Frw$$

ii) For twice a year, at the end of 10 years the owner will have

$$A = P(1+\frac{r}{2})^{2t} = 500,000(1+\frac{0.12}{2})^{2(10)}$$
$$= 500,000(1.06)^{20} = 1,603,567 Frw$$

b) Since $1,603,567 > 1,552,924.10$, the client will prefer compounding many times per year as it results in more money.

Application activity 2.11

Your aunt would like to invests 300,000 FRW at a bank. The Bank I pays an interest rate of 10% compounded once annually. The Bank II pays an interest rate of 9.8% compounded continuously. Your aunt will withdraw the money plus interest after 10 years.

At which bank do you advice your aunt to invest her money so as to get much money at the end of 10 years?

2.3.2 Mortgage problems

Activity 2.12

1) Go to conduct a research in the library, on internet or conduct a conversation with a bank officer to write down the meaning of the following when you get a loan from the bank:

i) the periodic payment P

ii) the annual interest rate r

iii) the mortgage amount M

iv) the number t of years to cover the mortgage

v) the number n of payments per year.

vi) Among all these elements/components, what is the most useful for the client to be informed about by the bank once he/she is given the mortgage loan ?

2) Your elder brother is newly employed at a company and earns 500,000 FRW per month. He would like to know if he can afford monthly payments on a mortgage of 20,000,000 FRW with an interest rate of 6% that runs for 20 years. Given that the quantities above are governed by the relation

$$P = \frac{\frac{rM}{n}}{1-\left(1+\frac{r}{n}\right)^{-nt}}$$

show your brother that he can afford the monthly payments by determining the following :

i) The monthly payment, that will be retained at the bank

ii) The balance that your brother can withdraw each month from the bank

iii) How much interest your brother will pay to the bank by the end of 20 years.

When a person gets a loan (mortgage) from the bank, the mortgage amount ***M***, the number of payments or the number ***t*** of years to cover the mortgage, the amount of the payment ***P***, how often the payment is made or the number ***n*** of payments per year, and the interest rate ***r***, it is proved that all the 5 components are related by the following formula: $P = \frac{\frac{rM}{n}}{1-\left(1+\frac{r}{n}\right)^{-nt}}$

If P is a monthly payment $P = \frac{M\left(\frac{r}{12}\right)}{1-\left[\frac{1}{1+\frac{r}{12}}\right]^{12t}}$

Example 2.12

A business woman wants to apply for a mortgage of 75,000 US dollars with an interest of 8% per month that runs for 20 years. How much interest will she pay over the 20 years?

Solution

Substituting for *M=75,000, r =0.08, t=20, n=12* in the equation

$$P=\frac{\frac{rM}{n}}{1-\left(1+\frac{r}{n}\right)^{-nt}},\ \text{we have}\quad P=\frac{\frac{(0.08)(75,000)}{12}}{1-\left(1+\frac{0.08}{12}\right)^{-(12).(20)}}=627.33$$

Each month she will be paying 627.33 US dollars.

The total amount she will pay is $627.33\times 12\times 20\text{US dollars}=150,559.2$ US dollars

The interest will be $(150,559.2-75,000)\text{US dollars}=75,559.2$ US dollars

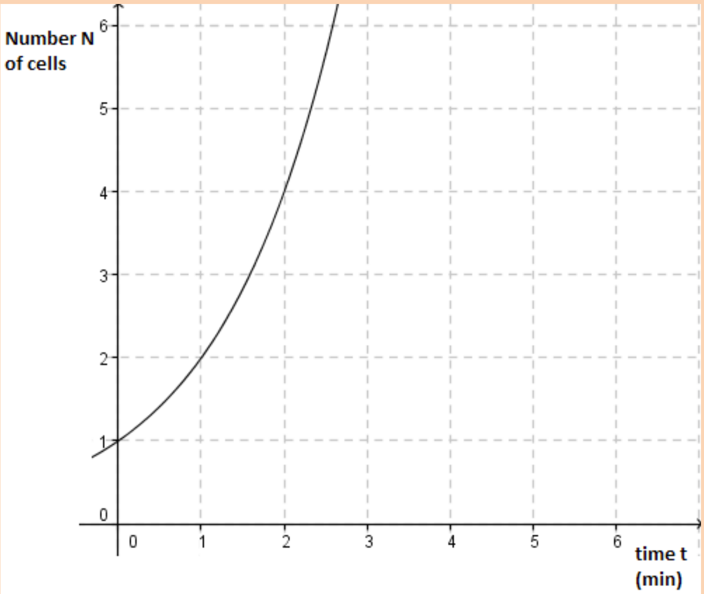
Application activity 2.12

A bank can offer a mortgage at 10 % interest rate to be paid back with monthly payments for 20 years. After analysis, a potential borrower finds that she can afford monthly payment of 200,000FRW. How much of mortgage can she ask for?

2.3.3 Population growth problems

Activity 2.13

Analyze the graph below showing the number of cells recorded by a student in a biology laboratory of his/her school during an experiment as function of time t .



a. Complete the table below:

Time t(minutes)	0	1	2	3	4			
Number of cells	…	…	…					

b. Given that $N(t)=N_0e^{kt}$, where N (t) is the quantity at time t, N_0 is the initial quantity and k is a positive constant, what is the value of N_0 ? Predict the number of cells after 5 minutes if $k=2$ c. What happens to the number of cells as the time becomes larger and larger? Is the number of cells growing or not? Explain your answer.

If P_0 is the population at the beginning of a certain period and r is the constant rate of growth per period, the population for *n* periods will be $P_n=P_0(1+r)^n$. This is similar to the final value (*F*) of an initial investment (*A*) deposited for *t* discrete time periods at an interest rate of $i\%$ *which is* calculated using the formula $F=A(1+i)^t$

To derive a formula that will give the final sum accumulated after a period of continuous growth, we first assume that growth occurs at several discrete time intervals throughout a year. We also assume that A is the initial sum, r is the nominal

annual rate of growth, n is the number of times per year that increments are accumulated and y is the final value. This means that after t years of growth the final sum will be: $y=A\left(1+\frac{r}{n}\right)^t$

Growth becomes continuous as the number of times per year that increments in growth are accumulated increases towards infinity.

When $n\to\infty$, we get $\lim\limits_{x\to\infty} y=\lim\limits_{x\to\infty} A\left(1+\frac{r}{n}\right)^t=Ae^{rt}$.

This is similar to $N(t)=N_0e^{kt}$ where A and N_0, r and k take respectively the same meanings.

Therefore, the final value $A(t)$ of any variable growing continuously at a known annual rate r from a given original value A_0 is given by the following formula

$A(t)=A_0e^{rt}$

Examples 2.13

1) The number of bacteria in a culture increases according to an equation of the type $N(t)=N_0e^{kt}$ Given that the number of bacteria triples in 2 hours,

   a) find an equation free of N_0 and solve the equation for k

b) How long would it take for the number of bacteria to be 5 times the initial number?

Solution

a) $N(2)=3N_0 \Leftrightarrow 3N_0 = N_0e^{k(2)} \Leftrightarrow e^{2k} = 3 \Rightarrow 2k = \ln 3 \Leftrightarrow k = \frac{\ln 3}{2} = 0.5493$

b) $5N_0 = N_0e^{0.5493t} \Leftrightarrow e^{0.5493t} = 5$, thus $t = \frac{\ln 5}{0.5493} \approx 2.93$

It will take 2.93 hours for the number of bacteria to be 5 times the initial number.

2) Population in a developing country is growing continuously at an annual rate of 3%. If the population is now 4.5 million, what will it be in 15 years' time?

Solution

The final value of the population (in millions) is found by using the formula $y = Ae^{rt}$ and substituting the given numbers: initial value A = 4.5; rate of growth r = 3% = 0.03; number of time periods *t* = 15, giving $y = 4.5e^{0.03(15)} = 7.0574048$ million

3) A lake is stocked with 500 fish and the fish Population P begins to increase according to the logistic growth model

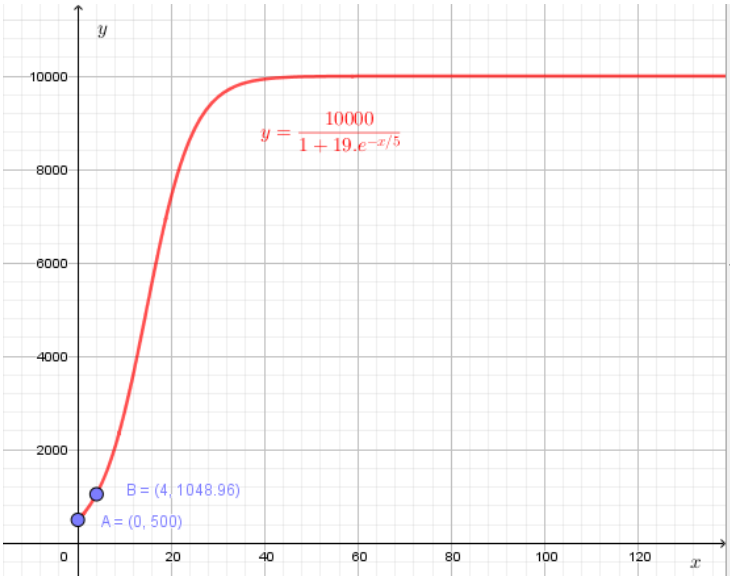
$$P = \frac{10000}{1+19\,e^{-\frac{x}{5}}}, \quad x \geq 0$$

Where x is measured in months.

a) Use Geogebra or other graphing utility to graph the function P.
b) Estimate the number of fish in the lake after 4 months.
c) Does the population have a limit as t increases without bound? Explain your reasoning.
d) After how many months is the population increasing most rapidly? Explain your reasoning.

Solution

a) the graph is the following:



b) At the beginning, (t = 0), the number of fish is 500. After 4 months (t = 4), the number of fish is 4,1048.

c) As the time x increases, the number of fish will be 10,000.

d) The population is increasing most rapidly after 4 months. This is because the increment of fish after 1 month is greater.

4) On a college campus of 5000 students, the spread of a flu virus (less dangerous than COVID-19) through the student body is modelled by

$$P = \frac{5\,000}{1+4999\,e^{-0.8t}}, \quad t \geq 0$$

Where P is the total number of infected people and t is the time, measured in days.

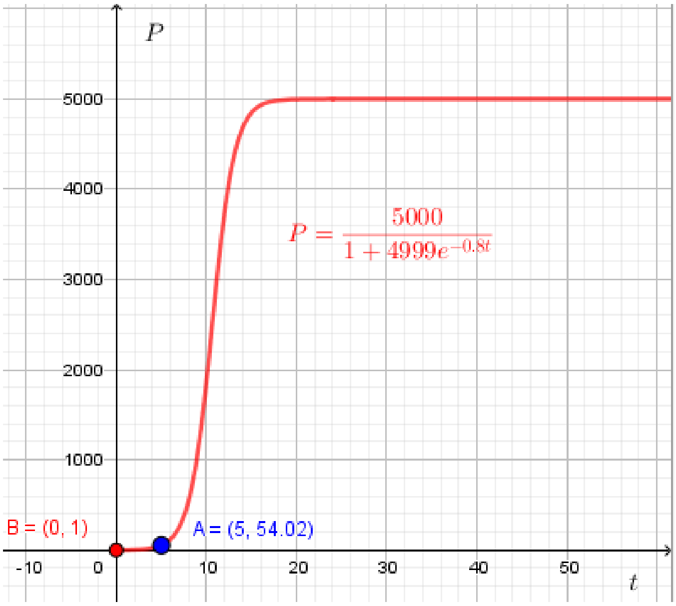
a) Use Geogebra or other graphing utility to graph the function P.

b) How many students will be infected after 5 days?

c) According to this model, can all the students on campus become infected with the flu? Explain your reasoning.

Solution

a) The graph of the function is the following:



b) After 5 days, the calculator and the graph show that 54 students will be infected.

c) According to this model, when the time increases without bound, the graph shows that all students can be infected. However, in real life, the infinite time is not possible. Therefore, all students cannot be infected.

Application activity 2.13

1) The population of a city increases according to the law of uninhibited growth. If the population doubles in 5 years and the current population is one million, what will be the size of the population in ten years from now?

2) A country economy is forecast to grow continuously at an annual rate of 2.5%. If its Gross National product (GNP) is currently 56 billion of US Dollars, what will the forecast for GNP be after 1.75 years (at the end of the third quarter the year after Next)?

3) One town of a given country had a population of 11,000 in 2,000 and 13,000 in 2017. Assuming an exponential growth model, determine the constant rate of growth per year.

2.3.4 Uninhibited decay and radioactive decay problems

Activity 2.14

The annual catch of fish from a specific dam is declining continually at a constant rate. Five years ago the total catch was 1000 kilograms, if the rate of decline is 20%, graph the process and deduce the total catch of this year considering that the number of fish reduces with $N(t) = N_0 e^{kt}$, *(Where* N_0 *is the total catch at initial time period and* $k = -20\%$ *the constant rate of decline).*

A phenomenon that can be modelled by an equation of the type $N(t) = N_0 e^{kt}$, where N (t) is the quantity at time t,

N_0 is the initial quantity and k is a negative constant, is said to follow the law of uninhibited decay. Radioactive materials follow the law of uninhibited decay.

Example 2.14

1) Suppose that you start an experiment in the biology laboratory of your school with 5,000,000 cells. The cells die according to the equation of the type $N(t) = N_0 e^{kt}$

After one minute you observe that there are 2,750,000 cells.

a) How many cells will be remaining after two minutes?

b) How long will it take for the number of cells to be less than 1,000?

Solution

a) $N(t) = N_0 e^{kt}$; $N(0) = N_0 e^{k(0)} = N_0 = 5,000,000$ thus $N(t) = 5,000,000 e^{kt}$.

$$N(1) = 5,000,000 e^{k} = 2,750,000$$

$$\Leftrightarrow e^{k} = \frac{2,750,000}{5,000,000} = 0.55 \Rightarrow k = \ln 0.55 = -0.597837$$

$$N(t) = 5,000,000 e^{-0.597837t}$$

After 2 minutes: $N(2) = 5,000,000 e^{-0.597837(2)} = 1,512,500$

There will be 1,512,500 cells remaining after two minutes

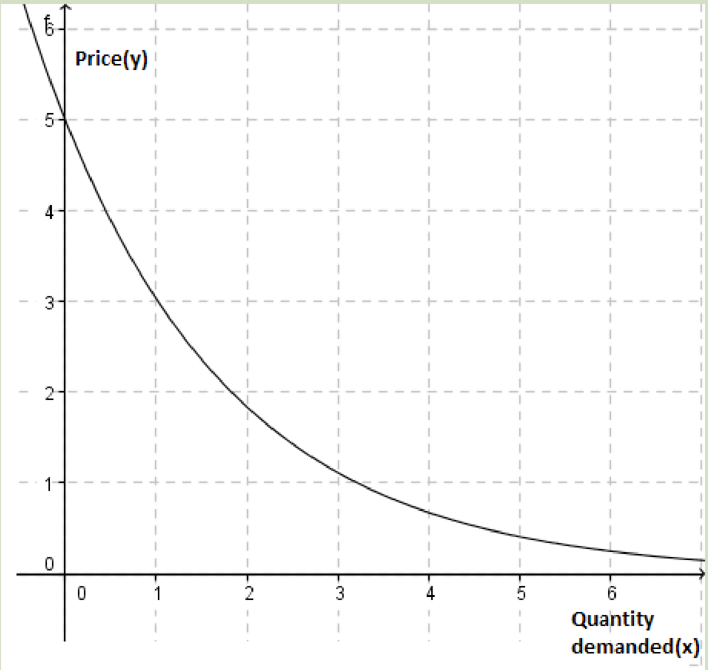
b) $1,000 = 5,000,000 e^{-0.597837t} \Leftrightarrow e^{-0.597837t} = \frac{1,000}{5,000,000} = \frac{1}{5,000}$

$$\Rightarrow -0.597837t = -\ln 5000 \Leftrightarrow t = \frac{\ln 5000}{0.597837} \approx 14.25$$

The time is 14.25 minutes.

Application activity 2.14

1) Analyze the graph below showing the price for a particular commodity:



a) What is the fixed price ?
b) What happens to the price as the time becomes larger and larger?
c) Model the problem by an equation of the type $N(t) = N_0 e^{kt}$. Write down the value of N_0 and the value of k. Precise the sign of k.

2) The normal healing of a wound is modelled by the equation $W(t) = 50e^{-0.2t}$, where $w(t)$ is the surface area ,in cm^2,of the wound t days following the injury when there is no infection to retard the healing.
a) What is the initial surface area of the wound?
b) Use the model to predict how large should the area of the wound be after 4 days if the healing is taking place

3) An object is heated to 800C and then allowed to cool in a room whose temperature is200C.Given that $u(t) = t + (u_0 - T)e^{kt}$,$k < 0$,u is the temperature of the heated object at a given time t and T is the constant temperature of the surrounding medium, if the temperature of the object is 60⁰C after 4 minutes, when will the temperature be 25⁰C?

2.3.5 Earthquake problems

Activity 2.15

Do the research in the library or explore internet to find out how Charles Richter tried to compare the magnitude of two earthquakes by the use of logarithmic function.

An earthquake is characterized by its **epicenter** and its **magnitude**.

Seismographic readings are made at a distance of 100 kilometers from the epicenter of an earthquake. If there is no earthquake, the seismographic reading is $x_0 = 0.001$ millimeter.

For an earthquake, the **Richter's scale** converts the seismographic reading x millimeters into magnitude through the formula $M(x) = \log \frac{x}{x_0}$, where $M(x)$ is the magnitude of the earthquake, x is the intensity of the earthquake (measured by the amplitude of a seismograph reading taken 100 *km* from the epicenter of the earthquake), and x_0 the intensity of a standard earthquake. The ratio of the seismographic readings is used to compare two earthquakes

Example 2.15

Two earthquakes took place at A and at B. Their magnitudes, on Richter's scale, were 8.9 and 8.3, respectively. Compare the two earthquakes by finding the ratio of their seismographic readings.

Solution

Let x and y be the seismographic readings of the earthquakes at A and at B, respectively.

Then, $\log \frac{x}{0.001} = 8.9$ **and** $\log \frac{y}{0.001} = 8.3$

This is equivalent to: $\frac{x}{0.001} = 10^{8.9}$ **and** $\frac{y}{0.001} = 10^{8.3}$

Dividing side by side, $\dfrac{\frac{x}{0.001}}{\frac{y}{0.001}} = \dfrac{10^{8.9}}{10^{8.3}} \Leftrightarrow \dfrac{x}{y} = 10^{8.9-8.3} = 10^{0.6} = 3.981$

This means that the earthquake at A is about 4 times heavy than the one happened at B.

Application activity 2.15

The earthquake that took place in Ecuador in April 2016 was of magnitude 7.8 on Richter's scale. How intense was that earthquake compared to the one that took place in:

a) The Mexico City in 1985, which was of magnitude 8.1 on Richter's
b) San Francisco in 1906, which was of magnitude 6.9 on Richter's scale.

2.3.6 Carbon dating problems

Activity 2.16

Carbon-14, a radioactive isotope of the element that, unlike other more stable forms of carbon, decays away at a steady rate. Organisms capture a certain amount of carbon-14 from the atmosphere when they are alive. By measuring the ratio of the radio isotope to non-radioactive carbon, the amount of carbon-14 decay can be worked out, thereby giving an age for the specimen in question. Scientists found that the time necessary for the amount of carbon-14 to reduce to its half is about 5700 years, and the radioactive material decays according to the equation $N(t)=N_0e^{kt},\ k<0$,where $N(t)$ is the level of Carbon-14 in the remains, and t is the time (in years) from the moment of the death of the human, $t_{1/2}$ is the half-life of carbon-14 (5,700 ±30 years), and the constant $k=\frac{-\ln 2}{5700}=-\frac{693}{5700}$

Discuss how to find the time t elapsed from the death of the human to the moment of the discovery of the remains.

The half-life of a substance is the amount of time it takes for half of that substance to decay. It is only a property of substances that decay at a rate proportional to their mass. Through research, scientists have agreed that the half-life of C^{14} is approximately 5700 years.

A formula to calculate how old a sample is by carbon-14 dating is:

$t=\frac{\ln\left(\frac{N(t)}{N_0}\right)}{-\ln 2}.t_{1/2}$ $k=\frac{-\ln 2}{5700}=-\frac{693}{5700}$ Where $t_{1/2}$ is the half-life of carbon-14 that is (5,700 ±30 years) and $\frac{N(t)}{N_0}$ the per cent of carbon-14 in the sample compared to the amount in living tissue?

Example 2.17

A scientist determines that a sample of petrified wood has a carbon-14 decay rate of 8.00 counts per minute per gram. What is the age of the piece of wood in years? The decay rate of carbon-14 in fresh wood today is 13.6 counts per minute per gram, and the half- life of carbon-14 is 5730 years.

Solution

$$t=\frac{\ln\left(\frac{8}{13.6}\right)}{-0.693}\times 5730=4,387.4 \textbf{ Years.}$$

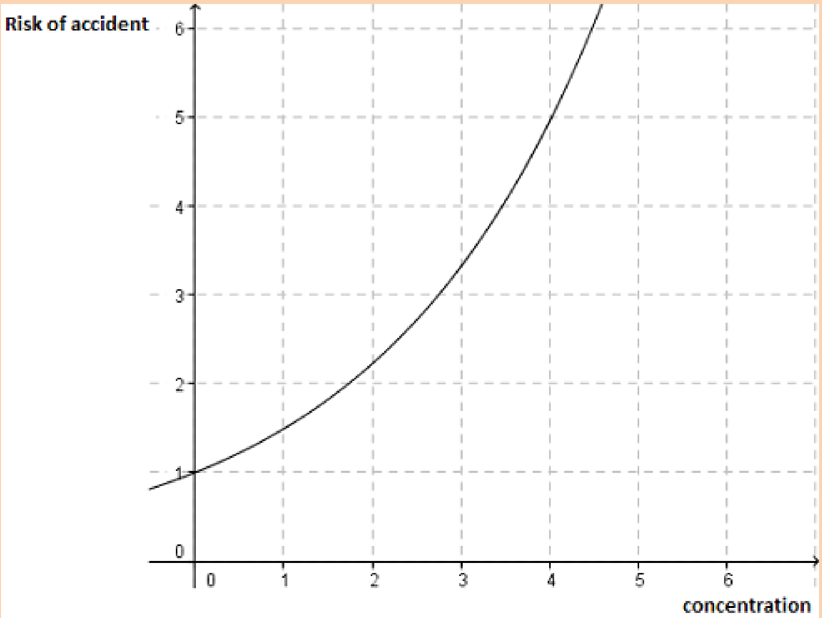
Application activity 2.16

A scrap of paper taken from the Dead Sea animal was found to have a C^{14}/C^{12} ratio of 0.79 times that found in plants living today. Estimate the age of the animal given that the half-life of carbon-14 is 5,700 years.

2.3.7 Problems about alcohol and risk of car accident

Activity 2.17

a) Discuss the dangers caused by the drivers who drink alcohol in excess.
b) The following graph shows the risk of a car accident with respect to the driver's blood concentration of alcohol:



i) What is the risk when there is no alcohol in the blood? Why is that risk not 0?
ii) Comment on the variation of the risk with respect to the concentration of alcohol in the driver's blood.

c) Write down approximately the type of the equation that can be used to model the risk.

Science shows that the concentration of alcohol in a person's blood is measurable. Recent medical research suggests that the risk R (given as a present) of having an accident while driving can be modelled by an equation of the type $R(x) = R_0 e^{kx}$ where x is the variable concentration of alcohol in the blood and k is a constant.

Example 2.17

The risk R of having an accident while driving is modelled by the equation $R(x) = 2e^{kx}$ where x is the concentration of alcohol in the driver's blood.

Suppose that a concentration of alcohol in the blood of 0.06 results in 4% risk (R=4) of an accident.

a) Find the value of the constant k in the equation $R(x) = 2e^{kx}$
b) Using this value of k, what is the risk if the concentration of alcohol is 0.08?
c) Using the same value of k, what concentration of alcohol corresponds to a risk of 100%?
d) If the law stipulates that anyone with a risk of having an accident of 10% or more should not drive, at what concentration of alcohol should the driver be arrested and charged?

Solution

a) From the equation $R(x) = R_0 e^{kx}$, substituting,

$$4 = 2e^{k(0.06)} \Leftrightarrow e^{0.06k} = 2 \Rightarrow 0.06k = \ln 2 \Leftrightarrow k = \frac{\ln 2}{0.06} = 11.552453.$$

b) The equation becomes $R(x) = 2e^{11.552453x}$. Now

$R(0.08) = 2e^{(11.552453).(0.08)} = 0.5$. Therefore, the risk is 5%

$$\text{c) } 100 = 2e^{11.552453x} \Leftrightarrow e^{11.552453x} = 50 \Rightarrow x = \frac{\ln 50}{11.552453} = 0.33.$$

$$\text{d) } 10 = 2e^{11.553453x} \Leftrightarrow 5 = e^{11.553453x} \Rightarrow x = \frac{\ln 5}{11.553453}$$

$$= 0.1393036274 \approx 0.14$$

Thus, the concentration of alcohol that the driver is arrested and charged is 0.14.

Application activity 2.17

Suppose that the risk R of having accident while driving a car is modelled by the equation $R(x) = 4e^{kx}$ .Suppose the concentration of alcohol of 0.05 results in 8% of risk of accident, what is the risk if the concentration is 0.18 and what concentration yields to 100% of risk of accident

END UNIT ASSESSMENT

QUESTION ONE

Determine the domain and range of the following functions

a) $f(x) = \log_2(3x-2)$ b) $f(x) = \ln(x^2-1)$

c) $f(x) = 2e^{3x+1}$ d) $f(t) = 4^{\sqrt{3t+1}}$

QUESTION TWO

Evaluate each of the following limits and give the equation of the asymptotes if any

a) $\lim\limits_{x\to 0^+} \frac{\ln x}{x}$ b) $\lim\limits_{x\to +\infty} (3 + x^2 \ln x)$

QUESTION THREE

1) Differentiate with respect to x the following functions

a) $f(x) = \log_2 \sqrt{\frac{x^2-4}{x+2}}$ b) $h(x) = \frac{1}{3}\left(4^{2x+5}\right)$

2) The tangent line touches the function $y = e^{2x+1}$ at the point A whose abscissa equal to $-\frac{1}{2}$,

i) Determine the coordinates of the intersection point of the graph and its tangent line

ii) Determine the equation of the tangent line and the equation of the normal line at A

iii) Sketch the graph of the function and the tangent in the same xy- coordinates system using a table of values and/or mathematical software if possible.

QUESTION FOUR

Investigate the nature of extrema (if any) of the curve $y = xe^{-x}$ and sketch the graph.

QUESTION FIVE

1) Carry out a research in the library or on internet and explain at least 5 applications of logarithmic or exponential functions in other human sciences.

2) The population of the world in 1995 was 5.7 billion, and the estimated relative growth rate is 2% per year. If the population continues to grow at this rate, when will it reach 114 billion?

3) Discuss how this unit inspired you in relation to learning other subjects or to your future. If no inspiration at all, explain why.

Unit 3: INTEGRATION

Unit 3: INTEGRATION

Key unit competence

Use integration as an inverse of differentiation and then apply definite integrals to find area of plane shapes.

Introductory activity

Two groups of students were asked to calculate the area of a quadrilateral field BCDA shown in the following figure:

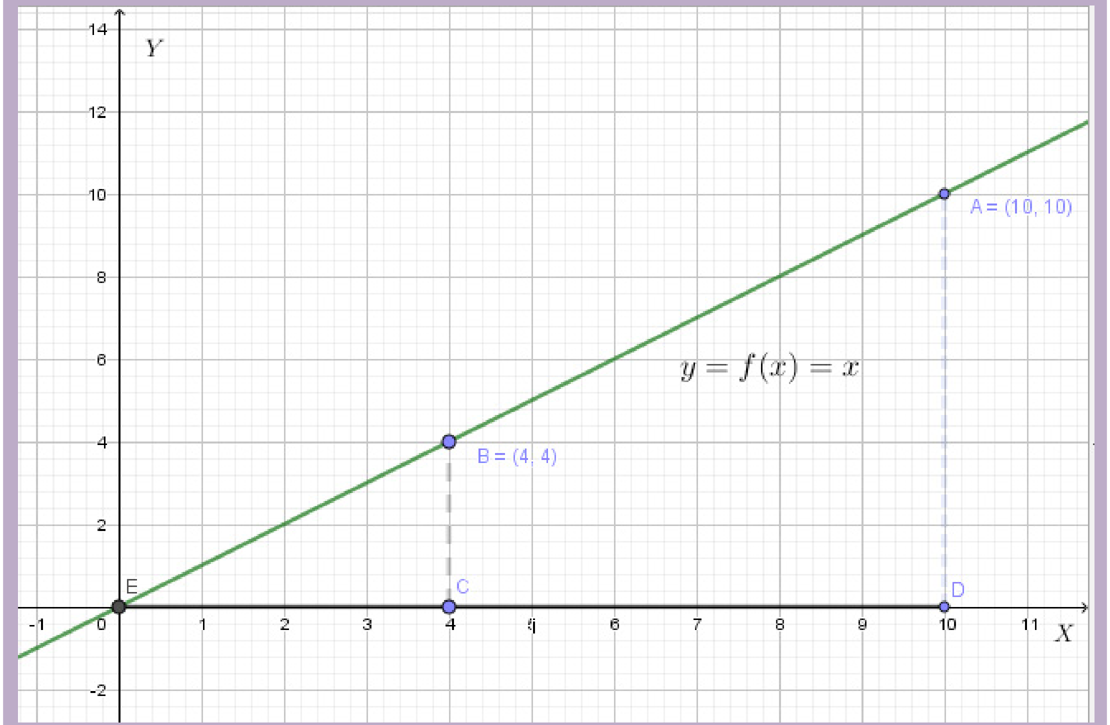


Figure 3.1: A quadrilateral field BCDA

The first group calculated the difference of the area for two triangles EDA and ECB

$A_1 = area(\Delta EDA) - area(\Delta ECB)$

The second group with high critical thinking skills used a function $F(x)$ that was differentiated to find $f(x) = x$ (which means $F'(x) = f(x)$) and the x-coordinate d of D and the x- coordinate c of C in the following way: $A_2 = F(d) - F(c)$.

1) Determine the area A_1 found by the first group.
2) Discuss and determine the function $F(x)$ used by the second group. What is the name of $F(x)$ if you relate it with $f(x)$?
3) Determine A_2 the area found by the second group using $F(x)$
4) Compare A_1 and A_2. Discuss if it is possible to find the area bounded by a function $f(x)$, the x-axis and lines with equation $x = x_1$ and $x = x_2$.

In this unit we are going to study the anti-derivatives of a given function $f(x)$ generally called integrals and their application in other sciences and real life situations such as the calculation of area of plane regions, etc.

3. 1 Differential of a function

Activity 3.1

The total consumption of a company is modeled by the function $y = f(x) = 4 + 0.5x + 0.1\sqrt{x}$, where x is the total disposable income (one unit representing 10^6 FRW).

a) What is the consumption of the company at $x = 2$ and at $x = 10$?
b) If Δx is the increment of x from 2 to 10, what is the corresponding increment Δy of the consumption of the company? Represent graphically this situation.
c) Discuss the increment of f if x changes from x_0 to x_1 where $(x_1 > x_0)$.
d) Given that the variation of f when x changes from x_0 to $x_0 + \Delta x$ is $\Delta y = f'(x_0)\Delta x$ determine the limit of Δy as Δx becomes very small.
e) Represent graphically the increment on x and the increment on f showing x_0 and x_1 and compare Δy and its limit when $\Delta x \to 0$.

Let $y = f(x)$ be a given continuous function on a certain real interval. When the variable x changes from x to $x + h$ within the interval, $f(x)$ changes from $f(x)$ to $f(x+h)$. The variation in x is $\Delta x = h$ while the corresponding variation in y becomes $\Delta y = f(x + \Delta x) - f(x)$.

When Δx becomes very small, the change in y can be approximated by the differential of y, that is, $\Delta y \approx dy$ and $\Delta x = dx$.

The rate of change $\frac{\Delta y}{\Delta x} = \frac{f(x+h) - f(x)}{h}$ means that $\Delta y = f'(x)\Delta x$

Therefore, $dy = f'(x)dx$. The differential of a function $f(x)$ is the approximated increment of that function when the variation in x becomes very small. It is given by $dy = f'(x)dx$ .f ' (x) {\displaystyle f′(x)}

Geometrically, the ratio $\frac{\Delta y}{\Delta x} = \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}$ represents the slope of the line AB passing through $A(x_0, f(x_0))$ and $B(x_0 + \Delta x, f(x_0 + \Delta x))$ as illustrated in Figure 3.2. When the change in x becomes smaller and smaller, that is Δx approaches 0, the line L becomes the tangent line (T) to the graph at the point $(x_0, f(x_0))$.

This means that the ratio $\frac{\Delta y}{\Delta x} = \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}$ becomes the slope of this tangent or equivalently $\lim\limits_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = f'(x_0)$.

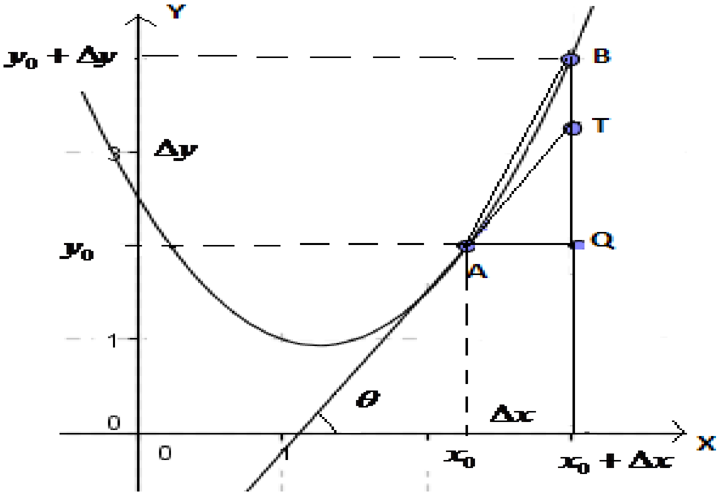


Figure 3. 2: differential and increment of a function $y = f(x)$ *when* x *varies to* $x + \Delta x$

The derivative of function $y = f(x)$ at x_0 is the slope of the geometric tangent line to the graph of the function at the point A; it is such that $y' = \tan\theta = \frac{QT}{AQ} = \frac{QT}{dx}$.

Examples 3.1.

1) Consider the function $y = f(x) = 2x^2$. Illustrate the increment of y when x increases from 1 to 2.

Solution

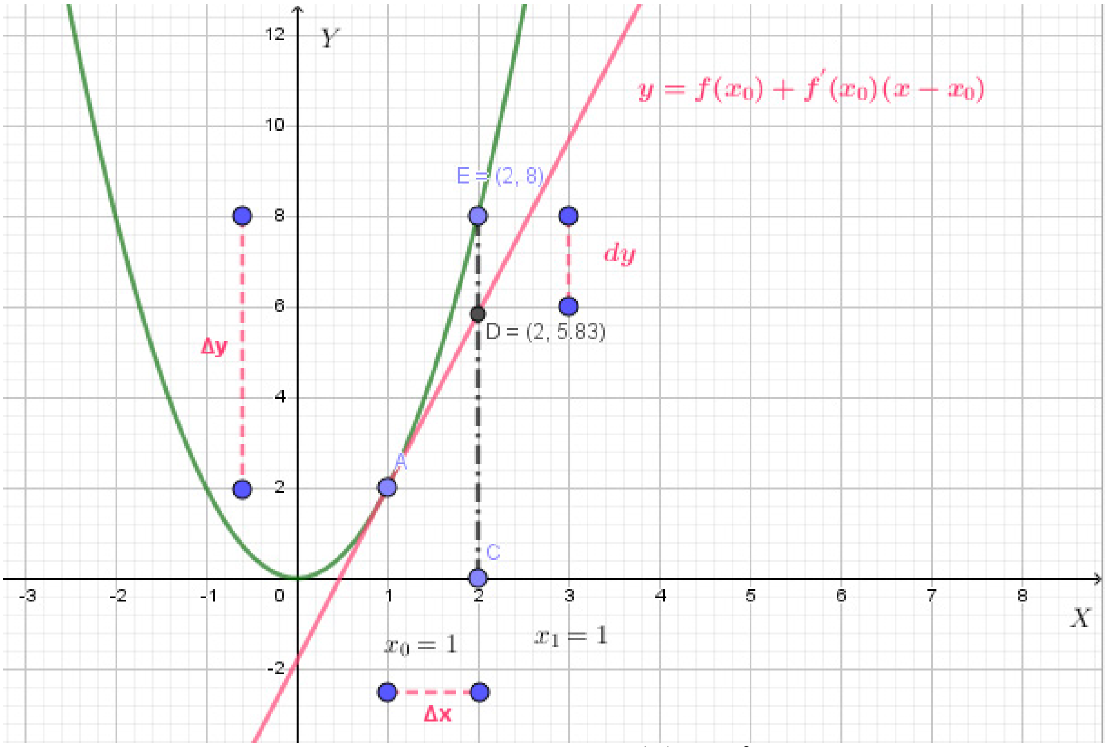


Figure 3.3 Illustration of increment of the function $y = f(x) = 2x^2$

This graph shows that from the point $x = x_0 = 1$ to $x = x_1 = 2$ where the increment is $\Delta x = 2 - 1 = 1$, the function $y = f(x) = 2x^2$ varies from $y_0 = f(1) = 2$ to $y_1 = f(2) = 8$

That means the increment of the function $\Delta y = 8 - 2 = 6$ which is different to the differential dy measured from the tangent to the graph as it illustrated on the figure 3.3

2) Find the differential of function $y = f(x)$ in each of the following cases:

a) $y = \cos x$ b) $y = e^{3x}$ c) $y = \ln^2 x$ d) $y = \frac{1}{x^2} + \sqrt{x} - 3$

Solution

a) $dy = \sin x dx$ b) $dy = 3e^{3x} dx$ c) $dy = \frac{2\ln x dx}{x}$ d) $dy = (\frac{-2}{x^3} + \frac{1}{2\sqrt{x}})dx$

3) The demand function of an item is modeled by the equation $y = \frac{2}{\sqrt[4]{x}}$, where x the number of units is demanded and y is the price in thousand of Frw. Given that $x = 16$, with a maximum error of 2, use differentials to approximate the maximum error in y and interpret your result.

Solution

$dy = \frac{-dx}{2\sqrt[4]{x^5}}$. For $x = 16$ and $dx = 2$, we have: $dy = \frac{-2}{2\sqrt[4]{16^5}} = -\frac{1}{32} = -0.03125$

For 16 units demanded $(\text{for } x = 16)$, with an error of 2, the corresponding price is $y = \frac{2}{\sqrt[4]{16}} = 1$ thousand (1, 000 Frw), with an approximate error of 31 Frw.

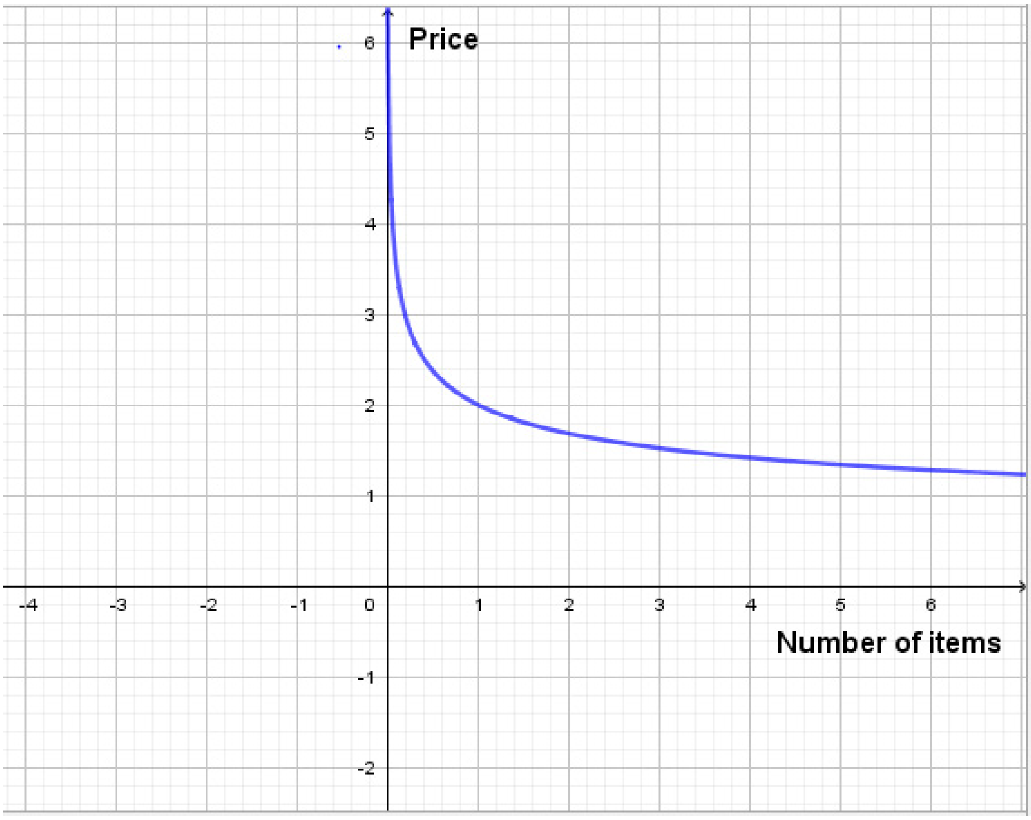


Figure3. 4: The price in thousand of Frw as function of the number of units demanded

X	0.5	1	2	3	4	….
Y	2.4	2	1.7	1.5	1.4	….

One can observe that as the number of units demanded increases, the price decreases.

Application activity 3.1:

1) Find the differential of each of the following function:
   a) $f(x) = \sin 3x$ b) $f(x) = x^2 e^x$ c) $f(x) = \frac{\ln x}{x}$

2) A company designed a tank in the shape of a cube. It claims that the side measures 4 meters, with an error of 0.02. Approximate, in litres, the capacity of the container and use differentials to approximate the error on the measurement of the volume.

3.2 Anti-derivatives

Activity 3.2:

Suppose that three caterpillars are moving on a straight line with constant velocity $v = 2$ (in meters per min)

1) Write down the position of each caterpillar at time *t* if their respective initial positions are:
   i) 1 meter ii) 2 meters iii)4 meters.

2) If $e(t)$ is the position in function of time, draw the graph of $e(t)$ for the third caterpillar and verify whether or not $e'(t) = \text{v}(t)$ where $\text{v}(t)$ is the velocity.

3) In the same way:
   i) Find a function $F(x)$ whose derivative is $\text{f}(x) = \cos x$, that is, $F'(x) = \text{f(x)} = \text{cosx}$
   ii) Discuss the number of possibilities for $F(x)$ which are there and the relationship among them.
   iii) How do functions $F(x)$ differ?

Let $y = f(x)$ be a continuous function of variable x. An anti-derivative of $f(x)$ is any function $F(x)$ such that $F'(x) = f(x)$. A function has infinitely many anti-derivatives, all of them differing by an additive constant.

It means that if $F(x)$ is an anti-derivative of $f(x)$, $F(x) + \text{c}$, (*c* is an arbitrary constant) is also an anti-derivative of $f(x)$.

Example 3. 2.

Given the function $f(x) = x\ln x - x$,

a) Find the derivative of $f(x)$

b) From the answer in (a), deduce the anti-derivative of $g(x) = \ln x$ whose graph passes through point $(e,1)$. Plot the graph of the function g and its anti-derivative on the same rectangular coordinate.

Solution

a) $f'(x) = (x\ln x - x)' = \ln x$

b) The anti-derivatives of $g(x) = \ln x$ are of the type $F(x) = x\ln x - x + C$

$F(e) = e\ln e - e + C = 1 \Leftrightarrow C = 1$

Therefore, the required anti-derivative is $F(x) = x\ln x - x + 1$

The figure below shows function $g(x) = \ln x$ and three of its anti derivatives

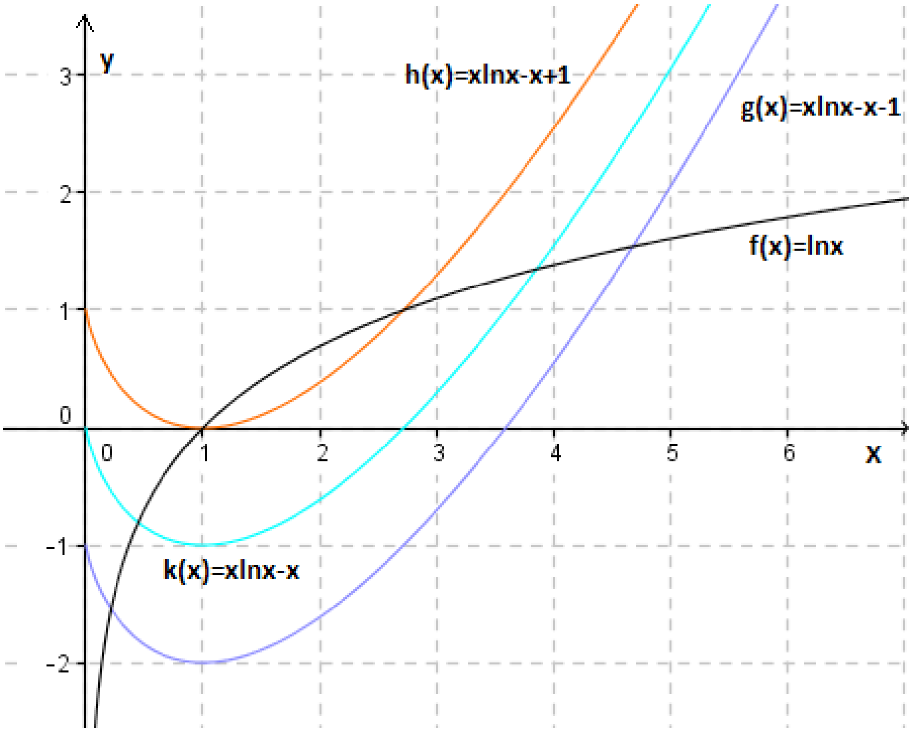
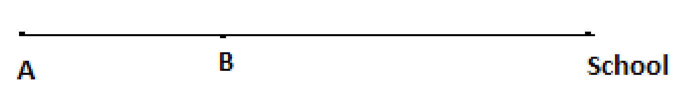


Figure 3.5: Graph of the function $f(x) = \ln x$ *and 3 of its anti-derivatives*

Application activity 3.2:

A student leaves his/her home B at 7:00 a.m. for school and moves at a variable speed modelled by the equation $v(t) = 2t + 1$ meters per minute. At the same time, his/her elder brother leaves his office A to bring him / her the school fees, at a constant speed of 50 meters per minute



a) Find the position of the student at time t minutes.
b) Given that A and B are 0.5 kilometre apart and the student takes 20 minutes to reach the school, will his/her brother meet him before reaching the school?

3.3 Indefinite integrals

Activity 3.3:

A student picked the work of an absent classmate and discovered that it was about differentiation. He/she got the answer $f'(x) = x + 2\cos x$ but could not know which function was differentiated to obtain the answer.

a. Use your knowledge and skills about anti-derivatives to find a possible expression for the original function differentiated to obtain $f'(x) = x + 2\cos x$

b. Is the expression of the original function unique? If not, write down three other expressions for possible original function.

d. Explain the similarity and the contrast between the different answers you have got.

e. Is there any systematic way of finding the set of all anti derivatives of a given function?

Let $y = f(x)$ be a continuous function of variable x. The **indefinite integral** of $f(x)$ is the set of all its anti-derivatives. If $F(x)$ is any anti-derivative of function $f(x)$, then the indefinite integral of $f(x)$ is denoted and defined as follows:
$\int f(x)dx = F(x) + C$ where C is an arbitrary constant called the **constant of integration**.

Thus, $\int f(x)dx = F(x) + C$ if and only if $F'(x) = f(x)$.

The process of finding the indefinite integral of a function is called **integration.**

The symbol $\int$ is the sign of integration while $f(x)dx$ is **the integrand.** Note that the integrand is a differential, dx shows that one is integrating with respect to variable x.

Example 3.3

Evaluate the following indefinite integrals:

a) $\int 5dx$ b) $\int e^t dt$ c) $\int \cos u \, du$ d) $\int \frac{1}{x} dx$ where $x > 0$

Solution

a) $\int 5dx = 5x + C$ b) $\int e^t dt = e^t + C$ c) $\int \cos u \, du = \sin u + C$

d) $\int \frac{1}{x} dx = \ln|x| + C$

Properties of indefinite integrals:

Let $y = f(x)$ and $y = g(x)$ be continuous functions and k a constant. Integration obeys the following properties:

1) $\int kf(x)dx = k\int f(x)dx$: the integral of the product of a constant by a function is equal to the product of the constant by the integral of the function.
2) $\int [f(x) + g(x)]dx = \int f(x)dx + \int g(x)dx$: the integral of a sum of two functions is equal to the sum of the integrals of the terms.

Example 3.4

1) Determine whether each of the following is correct or not. In any case explain your answer:
   a) $\int 2x \cos x dx = \int 2x dx \int \cos x dx = x^2 \sin x + C$
   b) $\int xe^x dx = x\int e^x dx = xe^x + C$
   c) $\int (2x + \cos x)dx = \int 2x dx + \int \cos x dx = x^2 + \sin x + C$

Solution

a) $\int 2x \cos x dx = \int 2x dx \int \cos x dx = x^2 \sin x + C$ not correct because the integral of product is not the product of integrals

b) $\int xe^x dx = x\int e^x dx = xe^x + C$ not correct because the variable x is considered as a constant

c) $\int (2x + \cos x)dx = \int 2x dx + \int \cos x dx = x^2 + \sin x + C$ correct because the integral of sum is the sum of integrals

2) Evaluate

a) $\int (3x^2+4x-5)dx$ b) $\int 8e^{-2x}dx$ c) $\int (3^{2x}-\frac{1}{x})dx$, where $x>0$

Solution

a) $\int (3x^2+4x-5)dx = x^3+2x^2-5x+C$ b) $\int 8e^{-2x}dx = -4e^{-2x}+C$

c) $\int (3^{2x}-\frac{1}{x})dx = \frac{1}{2\ln 3}e^{2x} - \ln x + C$

3) Suppose that the Volume $V(t)$ of a cell at time t changes according to

$V(t)=3+\int \sin t dt$; Find $V(t)$ if $V(0)=3$.

Solution:

$V(t)=3+\int \sin t dt = 3-\cos t + c$ as c is a constant.

Given that $V(0)=3$, $3-\cos t + c = 3$ which gives $c=1$.

Then, $V(t)=3-\cos t+1$

Or $V(t)=4-\cos t$

Application activity 3.3

1) Evaluate:

a) $\int (x^3+3\sqrt{x}-7)dx$ b) $\int (4x-12x^2+8x-9)dx$ c) $\int (\frac{1}{x^2}+e^{-x}-\frac{2}{x})dx$

2) A student calculated $\int \frac{x^3-2}{x^3}dx$ as follows: $\frac{\int (x^3-2)dx}{\int x^3 dx} = \frac{\frac{1}{4}x^4-2x}{\frac{1}{4}x^4}+C$, which is not correct. Show the mistake and suggest the correct working step and solution.

3) Function $y=f(x)$ is such that $\frac{dy}{dx}=\frac{x^3-5}{x^2}$. Find the expression of $y=f(x)$ if $f(1)=\frac{1}{2}$

4) In Economics, if $f(x)$ is the total cost of producing x units of a certain item, then the marginal cost is the derivative, with respect to x, of the total cost. Given that the marginal cost is $M(x)=1+50x-4x^2$, graph $f(x)$ and $M(x)$ on the same diagram.

3.4. Techniques of integration

3.4.1 Basic integration formulae

Activity 3.4

Given $(Arc\tan x)' = \frac{1}{1+x^2}$ and $\left(\frac{a^x}{\ln a}\right)' = a^x$. Discuss how to find $\int \frac{1}{1+x^2}dx$ and $\int a^x dx$. What is the formula that can be used to find these integrals?

Given any anti-derivative F of a function f, every possible anti-derivative of f can be written in the form of $F(\mathrm{x})+\mathrm{C}$,where C is any constant. This means that when you remember formulae used to differentiate some functions, it is easy to determine integrals. Roughly speaking, the integration is backward of the differentiation.

List of basic integration formulae

1) If k is constant, $\int kdx = kx + C$
2) $\int u^n du = \frac{1}{n+1}u^{n+1} + C$,where $n \neq -1$, n is a constant
3) If $b \neq -1$, and u a differentiable function, $\int u^b du = \frac{u^{b+1}}{b+1} + C$
4) By definition, $\int x^{-1}dx = \int \frac{1}{x}dx = \ln|x| + c$ for x nonzero
5) $\int e^x dx = e^x + c$, the integral of exponential function of base e
6) $If\ a > 0\ and\ a \neq 1, \int a^x dx, = \frac{a^x}{\ln a} + c$
7) $\int \frac{1}{x-1}dx = \ln|x-1| + C\ If\ a \neq 0, \int \frac{dx}{ax+b} = \frac{1}{a}\ln|ax+b| + C$
8) $if\ a \neq 0\ and\ n \neq -1, \int (ax+b)^n dx = \frac{(ax+b)^{n+1}}{a(n+1)} + c$

Integration involving trigonometric functions

9) $\int \cos xdx = \sin x + C$ 10) $\int \sin xdx = -\cos x + C$ 11) $\int \frac{dx}{1+x^2} = Arc\tan x + C$

12) $\int \frac{dx}{\sqrt{1-x^2}} = Arc\sin x + C$ 13) $\int \frac{dx}{\cos^2 x} = \tan x + C$ 14) $\int \frac{dx}{\sin^2 x} = -\cot x + C$

15) $If\ a \neq 0, \int \sin(ax+b)dx = -\frac{1}{a}\cos(\text{ax}+\text{b})+C$ 16) $\int \sec^2 xdx = \tan x + C$

17) $\int \text{cosec}^2\text{x}dx = -\cot x + C$ 18) $\int \sec^2(ax+b)dx = \frac{1}{a}\tan(\text{ax}+\text{b})+C$

19) $\int \text{cosec}^2(ax+b)dx = -\frac{1}{a}\cot(\text{ax}+\text{b})+C$ 20) $\int \cos(\text{ax}+\text{b})dx = \frac{1}{a}\sin(ax+b)+C$

Those formulae are easily used while integrating simple functions but there may be need of further techniques to integrate functions involving trigonometric functions.

Example

Find $\int \cos^4 xdx$

To integrate a polynomial involving trigonometric functions, apply linearization principle learnt in Unit 1 whereby

$$\cos^4 x = \left(\frac{e^{ix}+e^{-ix}}{2}\right)^4 = \frac{1}{8}\left[\left(\frac{e^{4ix}+e^{-4ix}}{2}\right)+4\left(\frac{e^{2ix}+e^{-2ix}}{2}\right)+3\right] = \frac{1}{8}\cos 4x + \frac{1}{2}\cos 2x + \frac{3}{8}.$$

Therefore $\int \cos^4 xdx = \int (\frac{1}{8}\cos 4x + \frac{1}{2}\cos 2x + \frac{3}{8})dx = \frac{1}{32}\sin 4x + \frac{1}{4}\sin 2x + \frac{3}{8}x + C$

Application activity 3.4

Compute the following indefinite integrals:

1) $\int e^{3x+1}dx$ 2) $\int \text{cosec}^2(2x+3)dx$ 3) $\int 3^x dx$

4) $\int (10+\sin^3 x)dx$ 5) $\int \frac{4dx}{\cos^2 x}$ 6) $\int (8-x^5)dx$

3.4.2 Integration by changing variables

Activity 3.5.

1) Using the basic integration formula, integrate the following:

i) $\int x^5 dx$ *ii)* $\int 2x(x^2+4)^5 dx$

2) Explain the problems faced when integrating (ii) above if any.

3) Let $u = x^2+4$ what is the derivative of u? Deduce dx in function of u and discuss how to determine $\int 2x(x^2+4)^5 dx$ using expression of u.

Some functions could be difficult to integrate by using anti-derivatives and basic integration formula immediately. To overcome this problem, other techniques such as change of variable or integration by substitution could be used. It is the method in which the original variables are expressed as functions of other variables.

Generally, if we cannot integrate $\int h(x)dx$ directly, it is possible to find a new variable u and function $f(u)$ for which $\int h(x)dx = \int f(u(x))dx = \int f(u)du$

The process of integration by changing variable or integration by substitution can be described as follows:

If we have to integrate the following,

$$\int f(u)u'dx = F(u) + C$$

1) Change variable and differentiate. For example, let $t = u$ then $dt = u'dx$.

2) Find out the value u and du by substituting these values into the integral and get $\int f(\mathrm{t})\mathrm{u}'\frac{dt}{u'} = \int f(\mathrm{t})\,\mathrm{dt}$

3) Integrate $\int f(t)dt$ by using anti-derivative method and basic immediate integration formula to get $\int f(t)dt = g(t) + C$

4) Return to the initial variable by replacing variable t by variable u, to get $f(t) + C = f(u) + C$

Examples 3.5

Evaluate

1) $\int (2x^2 - 5)xdx$ 2) $\int \frac{x}{\sqrt{x-2}}dx$ 3) $\int \frac{\ln x}{x}dx$

Solution

1) Given $\int (2x^2 - 5)xdx$ and suppose that $u = 2x^2 - 5$ *then* $du = 4xdx \Leftrightarrow \frac{1}{4}du = xdx.$

Hence $\int (2x^2 - 5)dx = \frac{1}{4}\int udu = \frac{1}{4}\frac{u^2}{2} + C = \frac{1}{8}u^2 + C$. Substituting u by $2x^2 - 5$ we get $\int (2x^2 - 5)xdx = \frac{1}{8}(2x^2 - 5)^2 + C$

2) For $\int \frac{x}{\sqrt{x-2}}dx$ we put $u = x - 2$ or $u + 2 = x \Rightarrow du = dx$. Thus

$$\int \frac{x}{\sqrt{x-2}}dx = \int \frac{(u+2)}{\sqrt{u}}dx \Rightarrow \int \frac{(u+2)}{u^{\frac{1}{2}}}du = \int (u^{\frac{1}{2}} + 2u^{\frac{-1}{2}})du = \frac{2}{3}u^{\frac{3}{2}} + 4u^{\frac{1}{2}} + C$$

Replacing u by $x-2$, we get $\int (2x^2-5)xdx = \frac{2}{3}(x-2)^{\frac{3}{2}} + 4(x-2)^{\frac{1}{2}} + C$

To integrate $\int \frac{\ln x}{x}dx$, we let $u = \ln x$. Then, $du = \frac{1}{x}dx$. Thus $\int \frac{\ln x}{x}dx = \int udu = \frac{1}{2}u^2 + C$

Substituting u by $\ln x$ yields $\int \frac{\ln x}{x}dx = \frac{1}{2}\ln^2 x + C$

Application activities 3.5

Determine the following integrals

1) $\int xe^{x2}dx$ 2) $\int \frac{dx}{(1-2x)^2}$ 3) $\int \frac{x+x^2}{(-3x^2+4-2x^3)^2}dx$ 4) $\int \frac{x}{(1-2x^2)^{\frac{1}{3}}}dx$

5) $\int x\sqrt{-1+x^2}dx$

3.4.3 Integration by parts

Activity 3.6

Use the integration by changing variable to evaluate the following :

1) $\int 3x^2(x^3+1)dx$ 2) $\int xe^x dx$

i) Was it easy for you to integrate (2) using changing variable methods?

ii) Let $u = x$ and $dv = e^x dx$. Find v

iii) Compare $\int xe^x dx$ and $\int udv$

iv) Determine $\int udv = uv - \int vdu$ and deduce $\int xe^x dx$

When we integrate, we can find some functions which can't be integrated immediately by using integration by changing variable method. To overcome that problem, you should use integration by parts or partial integration technique. In this, you have to find the integral of a product of two functions in terms of the integral of their derivative and anti-derivative.

If u and v are two functions of x, the product rule for differentiation can be used to integrate the product udv or vdu in the following way. Since

$d(uv) = udv + vdu$ it comes that $\int d(uv) = \int udv + \int vdu$. This leads to: $uv = \int udv + \int vdu$.

Thus, $\int udv = u.v - \int vdu$

When using integration by parts, keep in mind that you are splitting up the integrand into two parts. One of these parts, corresponding to u will be differentiated and the other, corresponding to dv, will be integrated. Since you can differentiate easily both parts, you should choose a dv for which you know an anti-derivative to make easier the integration.

Examples 3.6.

Calculate the following integral

1) $\int x\sin xdx$ 2) $\int \ln xdx$ 3) $\int x\cos 2xdx$

Solution

1) $\int x\sin xdx = ?$ Let $\begin{cases} u = x \\ dv = \sin xdx \end{cases}$ *then* $\begin{cases} du = dx \\ v = \int \sin xdx = -\cos x \end{cases}$

Let's use the integration by parts formula: $\int udv = uv - \int vdu$. We have

$\int udv = -x\cos x - \int(-\cos x)dx = -x\cos x + \sin x + C$

Finally, $\int x\sin xdx = -x\cos x + \sin x + C$

2) $\int \ln x dx = ?$

Choose $\begin{cases} u = \ln x \\ dv = dx \end{cases}$ then $\begin{cases} du = \dfrac{1}{x} \\ v = x \end{cases}$. Using the integration by parts rule, we get

$\int udv = \ln x(x) - \int x\dfrac{dx}{x} = x\ln x - \int dx = x\ln x - x + C$.

Thus, $\int \ln xdx = x\ln x - x + C$

3) $\int x\cos 2x dx = ?$

Let $\begin{cases} u = x \\ dv = \cos 2x dx \end{cases}$, then $\begin{cases} du = dx \\ v = \frac{1}{2}\sin 2x \end{cases}$, applying the integration by parts, we

find $= \frac{x}{2}\sin 2x - \frac{1}{2}\int \sin 2x dx = \frac{x}{2}\sin 2x - \frac{1}{2}(-\frac{1}{2}\cos 2x) + C$

Therefore, $\int x\cos 2x dx = \frac{x}{2}\sin 2x + \frac{1}{4}\cos 2x + C$

Application activities 3.6

Compute the following integrals using integration by parts

1) $\int 3x^2 e^{-x} dx$ 2) $\int x^2 \ln x dx$ 3) $\int \frac{x}{3}\sin 2x dx$ 4) $\int x\sqrt{x+5} dx$ 5) $\int 8x\cos x dx$

3.5 Applications of indefinite integrals

Activity 3.7:

A 50 Newton leaky gallon of water is lifted 10 meters into the air at constant speed. Given that the water leaks at constant rate and by the time the gallon reaches the height of 10 meters it contains no more drop of water (neglect the weight of the empty gallon),

a. Complete the table below

Height above the ground (m)	Weight (N)
0	
	0

If (x, y) is the point where x is the height and y the weight, write the coordinates of the gallon at the beginning (x_0, y_0) and at the end point (x_1, y_1).

Hence, model the weight by a function of the type $F(x) = ax + b$.

b. Graph the weight F as function of the height *x* above the ground level and interpret your result.

c. Discuss how to find an expression for the work W done in lifting the gallon given that $dW = F(x)dx$.

d. Given that the work done when the gallon is 2 meters above the ground is 40 Joules, find the work done when the gallon is 6 meters above the ground level.

e. Plot the graph of the work and interpret your graph.

Indefinite integrals are used to solve some problems encountered in daily life

1. Determination of the work done by a force moving through an axis

The **position** s, the speed v and the acceleration a are all functions of variable time t and related by $s = \int v(t)dt$, $v = \int a(t)dt$

The **work** W done by a force F moving through x-axis is given by $W = \int F(x)dx$

2. The cost related to the marginal cost M in Economics or other businesses

The **cost** C (respectively the revenue R, utility U and profit P) is related to the marginal cost M (respectively marginal revenue, marginal utility, and marginal profit) by the formula $C(x) = \int M(x)dx$, where x is the number of units produced. The marginal cost is the additional cost to produce one extra unit.

Example 3.6

The force required to extend an elastic spring to x units longer is proportional to x.

a) Given that a force of 50 Newton is required to extend the spring to 10 centimetres longer than its natural length, write the expression of the force $F(x)$ as a function of x and plot it.

b) Find the expression of the work done to extend the length of the spring by x units, given that it requires a work of 10 $N \bullet cm$ to extend the length of the spring by 2 cm

c)Plot the work $W(x)$ against the extension length x and interpret your result.

Solution

a) Since the force F is proportional to the extension length x, we have $F(x) = kx$, where k is a constant of proportionality? Substituting for x in the equation $F(x) = kx$,

we have $50N = (10cm)k \Rightarrow k = 5Ncm^{-1}$ $50 = k10$ giving $k = 5$. Therefore $F(x) = 5x$

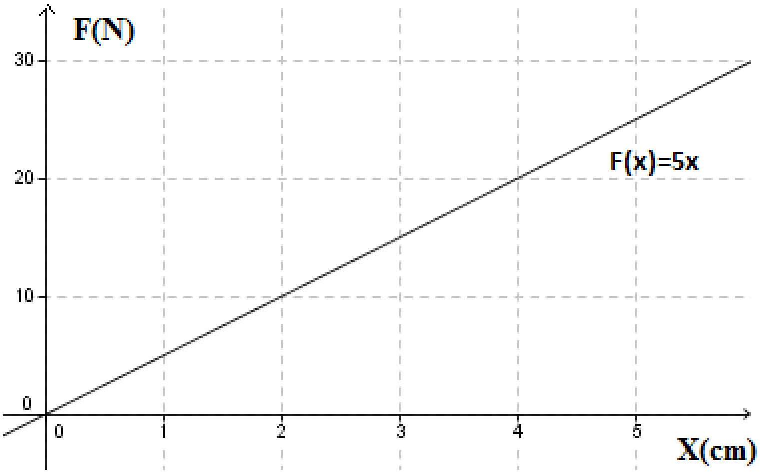


Figure 3.6. Graph of the force $F(x)$ *as a function of* x

b) The work done is given by $W(x)=\int 5x\,dx=\frac{5}{2}x^2+C$, that is $\mathrm{W}(x)=\frac{5}{2}x^2+C$

Then, $\frac{5}{2}(2)^2+C=10 \Leftrightarrow C=0$. It follows that $\mathrm{W}(x)=\frac{5}{2}x^2$

c) The graph:

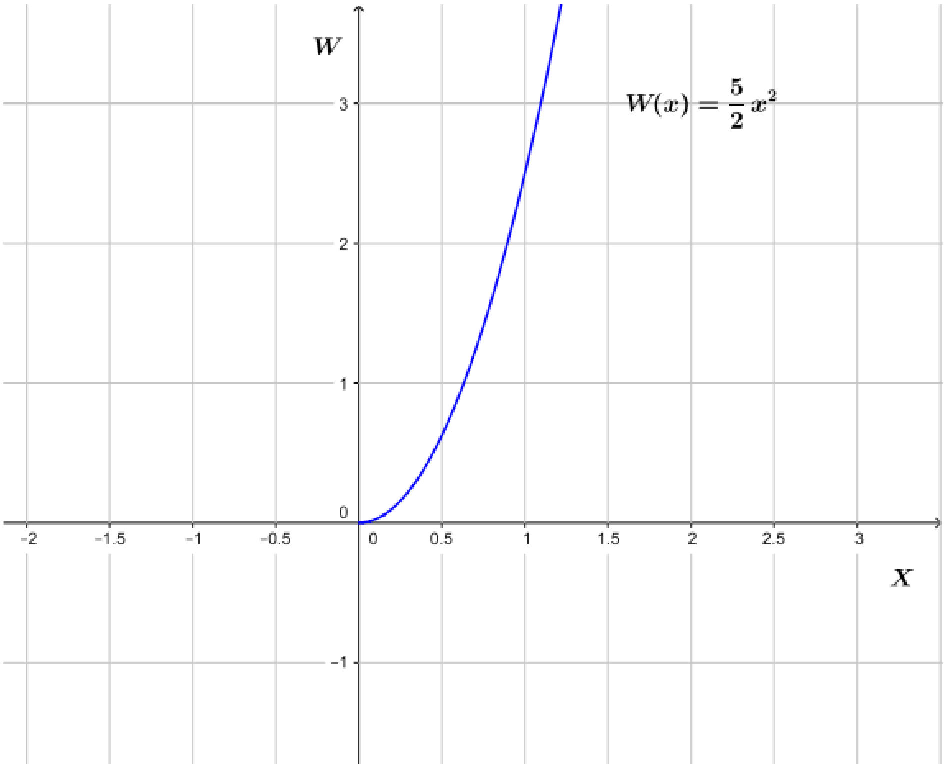


Figure 3.7 The work $\mathrm{W}(x)$ *against the extension length* x

As the extension length increases linearly, the work increases in a quadratic way from $x=0$

Application activity 3.7

1) A supplier of a certain item realized a marginal revenue modelled by the equation $M(x)=30-10e^{-\frac{1}{40}x}$, when she sold x units of the item. Find the expression for her total revenue as function of the number of units sold, given that her initial revenue is 100.
2) The marginal demand for food at a restaurant is modelled by the equation $M(x)=-0.0002x+\frac{1}{400}$, where x is the price in FRW, per plate of food. Given that for a plate of food with cost 800 FRW, the demand is 98,
   a) Represent graphically the demand function
   b) What would be the expected demand if the price per plate is increased to 1,200 FRW?
   c) Interpret this result
   d) Which advice do you give to the restaurant owner?

3.6 Definite integrals

3. 6.1 Definition and properties of definite integrals

Activity 3.8

A Senior Six learner is preparing to sit an end of year exam of Mathematics. He/she draws on the same axes the linear function defined by $f(x)=2x$, $y=0$, and two vertical lines, $x=0$, and $x=4$.

a) Draw the shape obtained and prove that it is in the form of a triangle.
b) By using the formula for the area of a triangle, calculate the area enclosed by the functions $y=2x$, $y=0$, $x=0$, and $x=4$
c) Let consider the function $F(x)$ as an antiderivative of $f(x)=2x$. Find $F(x)$ and carry out $F(4)-F(0)$ Compare the findings of b) to the area obtained in c).

Let f be a continuous function defined on a close interval $[a,b]$ and F be an anti-derivative of f. For any anti-derivative *F(x)* of *f(x)* on $[a,b]$ the difference $F(b)-F(a)$ has a unique value. This value is defined as a definite integral of *f(x)* for $a \le x \le b$. We write, $\int_a^b f(x)dx = F(b)-F(a)$.

Thus, if $F(x)$ is an anti-derivative of $f(x)$, then $\int_a^b f(x)dx = [F(x)+c]_a^b$

$= [(F(b)+c)-(F(a)+c)] = [F(b)+c-F(a)-c] = F(b)-F(a)$

$\int_a^b f(x)dx$ is read as the "integral from a to b of $f(x)$, a is called **lower limit** and b is called **upper limit**. The interval $[a,b]$ is called **the range of integration**.

Geometrically, the definite integral $\int_a^b f(x)dx$ is the area of the region enclosed by the curve $y = f(x)$, the vertical lines $x = a$, $x = b$ and the $x-axis$ as illustrated in the following figure.

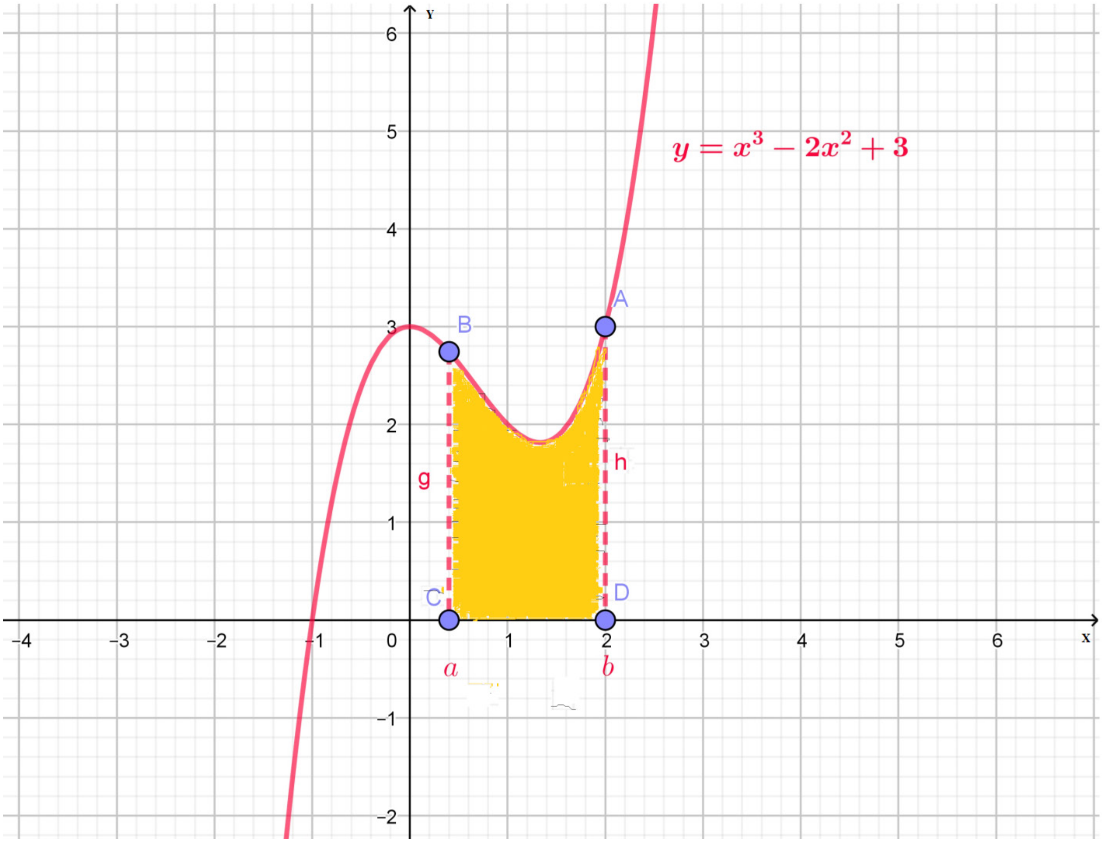


Figure3.8: Definite integral of a function $f(x)$ on a given interval $[a,b]$

The area of coloured region is given by $\int_a^b (x^3 - 2x^2 + 3)dx$. If measurement units are provided for axes, then the area of the region is the product of this definite integral and the area of square unit.

Fundamental theorem of integral calculus:

Let $F(x)$ and $f(x)$ be functions defined on an interval $[a,b]$. If $f(x)$ is continuous and $F'(x) = f(x)$, then $\int_a^b f(x)dx = [F(x)]_a^b = F(b) - F(a)$.

Properties of definite integrals

If $f(x)$ *and* $g(x)$ are continuous functions on a closed interval $[a,b]$ then:

1) $\int_a^b 0dx = 0$

2) $\int_a^b f(x)dx = -\int_b^a f(x)dx$ (Permutation of bounds)

3) $\int_a^b [\alpha f(x) \pm \beta g(x)]dx = \alpha\int_a^b f(x)dx \pm \beta\int_a^b g(x)dx,\ \alpha \text{ and } \beta \in \mathbb{R}$ (Linearity)

4) $\int_a^a f(x)dx = 0$ (Bounds are equal)

5) $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx \text{ with } a < c < b$ (Chasles relation)

6) $\forall x \in [a,b], f(x) \le g(x) \Rightarrow \int_a^b f(x)dx \le \int_a^b g(x)dx$ it follows that

$f(x) \ge 0 \Rightarrow \int_a^b f(x)dx \ge 0 \text{ and } \left|\int_a^b f(x)dx\right| \le \int_a^b |f(x)|dx$ (Positivity)

7) $\int_{-a}^{a} f(x)dx = \begin{cases} 2\int_{0}^{a} f(x)dx, \; if \; f(x), \; is \; even \; function \\ 0 \; if \; f(x) \; is \; an \; odd \; function \end{cases}$

Example3.7

Evaluate the following definite integrals:

1) $\int_{2}^{3} xdx$ 2) $\int_{1}^{4}\left(e^x - \sqrt{x}\right)dx$ 3) $\int_{0}^{\pi}\left[\frac{1}{2}\sin x + x\right]dx$

Solution

1) $\int_{2}^{3} xdx = \left[\frac{x^2}{2}\right]_2^3 = \left(\frac{3^2}{2} - \frac{2^2}{2}\right) = \frac{9}{2} - \frac{4}{2} = \frac{5}{2}$

2) $\int_{1}^{4}\left(e^x - 2\sqrt{x}\right)dx = \int_{1}^{4} e^x dx - 2\int_{1}^{4}\sqrt{x}dx = \left[e^x\right]_1^4 - 2\left[\frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1}\right]_1^4 = \left(e^4 - e^1\right) - 2\left(\frac{4^{\frac{3}{2}}}{\frac{3}{2}} - \frac{1^{\frac{3}{2}}}{\frac{3}{2}}\right)$

$= \left(e^4 - e\right) - 2\left(\frac{16}{3} - \frac{2}{3}\right) = e^4 - e - \frac{2}{3}\text{x}14 = e^4 - e - \frac{28}{3}$

3) $\int_{0}^{\pi}\left[\frac{1}{2}\sin x + x\right]dx = \int_{0}^{\pi}\frac{1}{2}\sin xdx + \int_{0}^{\pi} xdx = \frac{1}{2}\int_{0}^{\pi}\sin xdx + \int_{0}^{\pi} xdx = \frac{1}{2}\left[-\cos x\right]_0^{\pi} + \left[\frac{x^2}{2}\right]_0^{\pi}$

$= \frac{1}{2}\left(-\cos\pi - \left(-\cos 0\right) + \left(\frac{\pi^2}{2} - \frac{0^2}{2}\right)\right) = \frac{1}{2}\left(\left(1+1\right) + \frac{\pi^2}{2}\right) = \frac{2}{2} + \frac{\pi^2}{2} = 1 + \frac{\pi^2}{2}$

Application activity 3.8

1) Evaluate each of the following definite integrals

a) $\int_{1}^{2}(4x^2 - 3x)dx$ b) $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}(\cos 2x - 2e^x)dx$

2. Find a function $F(x)$ satisfying $F'(x) = 5x^2 + 1 \; and \; \text{F}(0) = 2$, then plot its graph by joining its main points.

3. In business and economics it is known that when $f(x)$ is the demand function (the quantities of a commodity that would be purchased at various prices), the consumer's surplus (total consumer gain) given by $\int_0^{x_0} f(x)dx - x_0 y_0$ is represented by the area below the demand curve and above the line $y = y_0$ where y_0 is the market price corresponding to the market demand x_0 as shown in the figure 3.8 below.

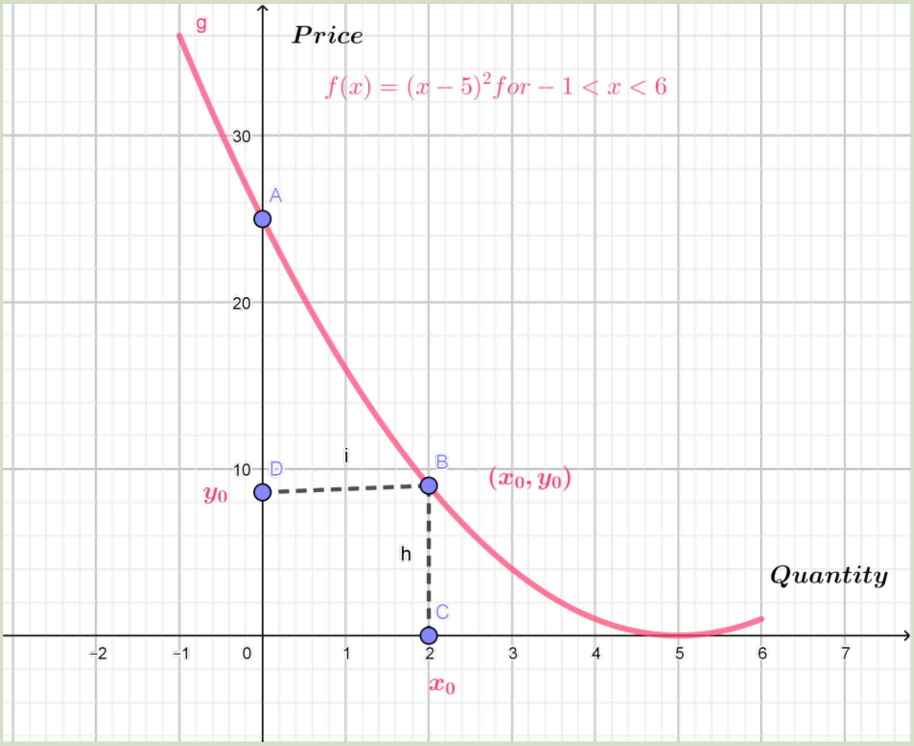


Figure 3.8. the graph of the demand function

a) Find the consumer's surplus for $x_0 = 3$, if the demand function is

$f(x) = y = 30 - 2x - x^2$

b) Plot the demand function $f(x) = y = 30 - 2x - x^2$

3.6.2 Techniques of Integration of definite integrals

Activity 3.9

1) Consider the continuous function $f(x)=e^{x^2}$ on a closed interval $[a,b]$

i) Let $t=x^2$, determine the value of t when $x=0$ and when $x=2$

ii) Determine the value of dx in function of dt

iii) Evaluate the integral $\int_a^b 2xe^{x^2}dx$ using expression of t, considering the results found in i) and ii).

iv) Explain what happens to the boundaries of the integral when you apply the substitution method

2) Evaluate $\int_1^e x^2 \ln x dx$

Many times, some functions can not be integrated directly. In that case we have to adopt other techniques in finding the integrals. The fundamental theorem in calculus tells us that computing definite integral of *f(x)* requires determining its antiderivtive, therefore the techniques used in determining indefinite integrals are also used in computing definite integrals.

a) Integration by substitution

The method in which we change the variable to some other variable is called "**Integration by substitution**".

When definite integral is to be found by substitution then change the lower and upper limits of integration. If substitution is $\varphi(x)$ and lower limit of integration is a and upper limit is b then new lower and upper limits will be $\varphi(a)$ and $\varphi(b)$ respectively.

Example 3.8

1) $\int_0^2 x\sqrt{5-x^2}\,dx$ 2) $\int_0^3 6xe^{x^2+1}\,dx$

Solution

1) $\int_0^2 x\sqrt{5-x^2}\,dx$ put $5-x^2=t,\ \text{then} -2xdx=dt,\ or\ xdx=-\frac{1}{2}dt$

$when\ x=0,\ t=5,\ when\ x=2,\ t=5-4=1$

$$\int_0^2 x\sqrt{5-x^2}\,dx = \int_5^1 -\sqrt{t}\,\frac{dt}{2} = \frac{1}{2}\int_1^5 \sqrt{t}\,dt$$

$$= \frac{1}{2}\int_1^5 t^{\frac{1}{2}}dt = \frac{1}{2}\left[\frac{t^{\frac{1}{2}+1}}{\frac{1}{2}+1}\right]_1^5 = \frac{1}{2}\left[\frac{t^{\frac{3}{2}}}{\frac{3}{2}}\right]_1^5 = \frac{1}{2}\times\frac{2}{3}\left[5^{\frac{3}{2}}-1^{\frac{3}{2}}\right] = \frac{1}{3}\left(\sqrt{125}-1\right)$$

2) $\int_0^3 6xe^{x^2+1}\,dx$ put $x^2+1=t,$ then $2xdx = dt$ or $xdx = \frac{1}{2}dt$

$when\ x=0\ ,\ t=1\ and\ when\ x=3,\ t=10$

$$\int_0^3 6xe^{x^2+1}dx = \int_1^{10} 6e^t\,\frac{dt}{2} = 3\int_1^{10} e^t dt = 3\left[e^t\right]_1^{10} = 3\left(e^{10}-e^1\right) = 3\left(e^{10}-e\right)$$

b) Integration by parts

To compute the definite integral of the form $\int_a^b f(x)g(x)dx$ using integration by parts, simply set $u = f\left(x\right)$ and $dv = g(x)dx$. Then $du = f'(x)dx$ and $v = G(x),$ antiderivative of $g(x)$ so that the integration by parts becomes

$$\int_a^b udv = \left[uv\right]_a^b - \int_a^b vdu$$

Example 3.9

Evaluate the following definite integrals:

1) $I = \int_0^3 xe^x dx$ 2) $J = \int_0^{\frac{\pi}{6}} \left(4+5x^2\right)\cos 3x dx$

Solution

1) $I = \int_0^3 xe^x dx$. Let $\begin{cases} u = x, & du = dx \\ dv = e^x dx, & v = \int e^x dx = e^x + c \end{cases}$

Applying the integration by parts formula $I = uv\big|_a^b - \int_a^b vdu$ yields to

$$I = xe^x\big|_0^3 - \int_0^3 e^x dx = xe^x\big|_0^3 - e^x\big|_0^3 = \left[3e^3 - 0e^0\right] - \left[e^3 - e^0\right] = 3e^3 - e^3 + 1 = 2e^3 + 1$$

2) $J=\int_0^{\frac{\pi}{6}}\left(4+5x^2\right)\cos 3x dx$ Let

$$\begin{cases} u=4+5x^2, & du=10xdx \\ dv=\cos 3xdx, & v=\int \cos 3xdx \Rightarrow v=\frac{1}{3}\sin 3x+c \end{cases}$$

Applying the the integration by parts formula $I=uv|_a^b-\int_a^b vdu$ to get

$$J=\left[\left(4+5x^2\right)\text{x}\frac{1}{3}\sin 3x\right]_0^{\frac{\pi}{6}}-\int_0^{\frac{\pi}{6}}\left(\frac{1}{3}\sin 3x\right).10x\,dx=\left[\frac{1}{3}\left(4+5\frac{\pi^2}{36}\right)\sin\frac{3\pi}{6}-0\right]-\frac{10}{3}\int_0^{\frac{\pi}{6}}x\sin 3\text{xdx}$$

$$=\left[\frac{1}{3}\left(4+\frac{5\pi^2}{36}\right)-\frac{10}{3}\int_0^{\frac{\pi}{6}}x\sin 3\text{xdx}\right]=\left[\frac{1}{3}\left(4+\frac{5\pi^2}{36}\right)-10\left[\frac{-x\cos 3x}{3}\Bigg|_0^{\frac{\pi}{6}}+\frac{1}{3}\int_0^{\frac{\pi}{6}}1\text{x}\cos 3xdx\right]\right]$$

$$=\frac{1}{3}\left[\left(4+\frac{5\pi^2}{36}\right)-10\left[\frac{-x\cos 3x}{3}+\frac{1}{9}\sin 3x\right]_0^{\frac{\pi}{6}}\right]=\frac{1}{3}\left[\left(4+\frac{5\pi^2}{36}\right)+0-\frac{10}{9}\right]=\frac{1}{27}\left(\frac{5\pi^2}{4}+26\right)$$

Application activity 3.9

Evaluate the following definite integrals by using the indicated technique

1) $\int_0^{\pi}\cos xe^{\sin x}dx$ (Use integration by substitution)

2) $\int_0^{1}\ln(1+x)dx$ (Use integration by parts)

3.6.3 Applications of definite integrals

Activity 3.10

1) The plane region bounded by the curve $y=16-x^2$, $the\ x-axis\ and\ x=1,\ x=3$ is shown in the diagram.

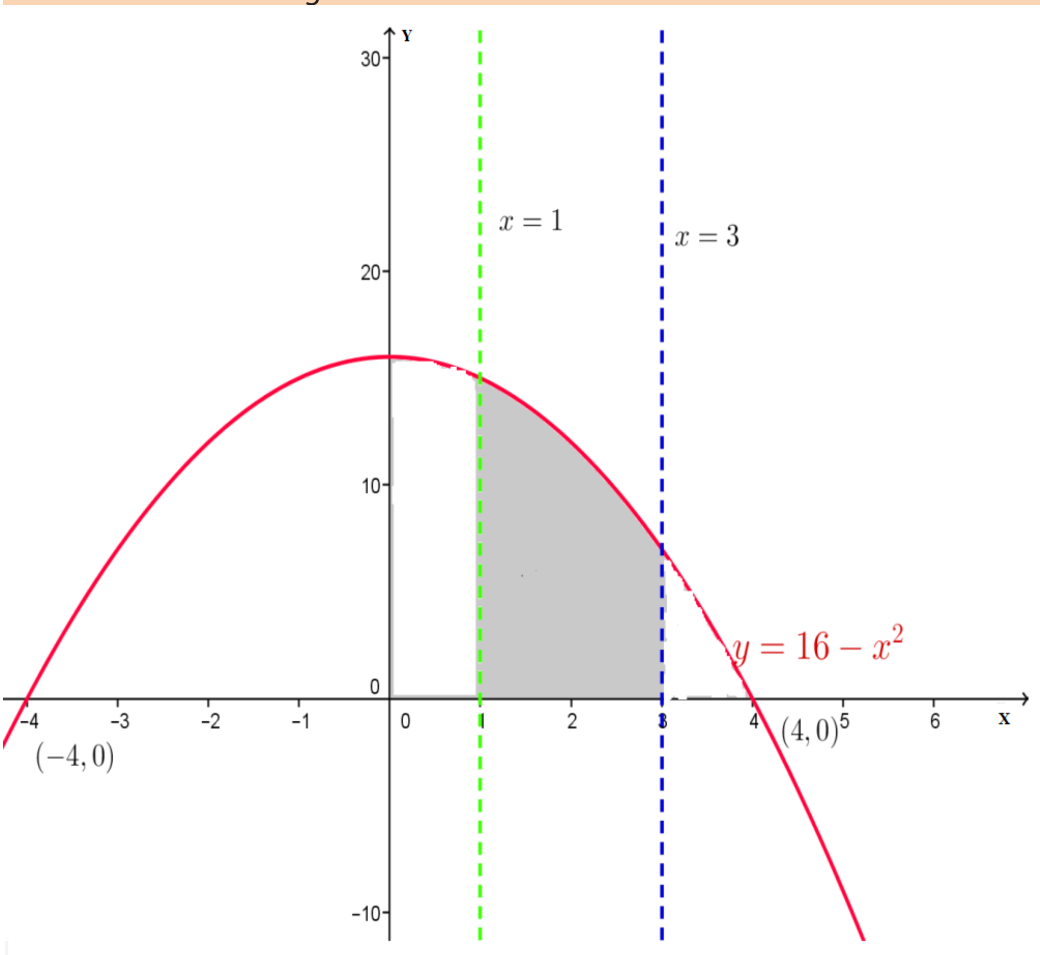


Figure 3.9: Region bounded by the curve $y=16-x^2$, $the\ x-axis\ and\ x=1,\ x=3$

a) Write down the definite integral which represents the measure of this surface area.

b) Hence, calculate the area

2) At a certain factory, the marginal cost is $3(q-4)^2$ dollars per unit when the level of production is q units. By how much will the total manufacturing cost increase if the level of production is raised from 6 units to 10 units.

3) The force when the spring is compressed by x units is given by $F(x)=16x$ *Newton* Determine the work done on a spring when it is compressed from its natural length of 1m to the length of 0.75.

a) Area of a region between two curves

We can apply the definite integrals to evaluate the area bounded by the graph of function and lines $x=a, x=b$ and $y=0$ on the interval where the function is defined.

Suppose that a plane region M is bounded by the graphs of two continuous functions $y=f(x)$ *and* $y=g(x)$ and the vertical straight lines $x=a$ *and* $x=b$ as shown in figure below

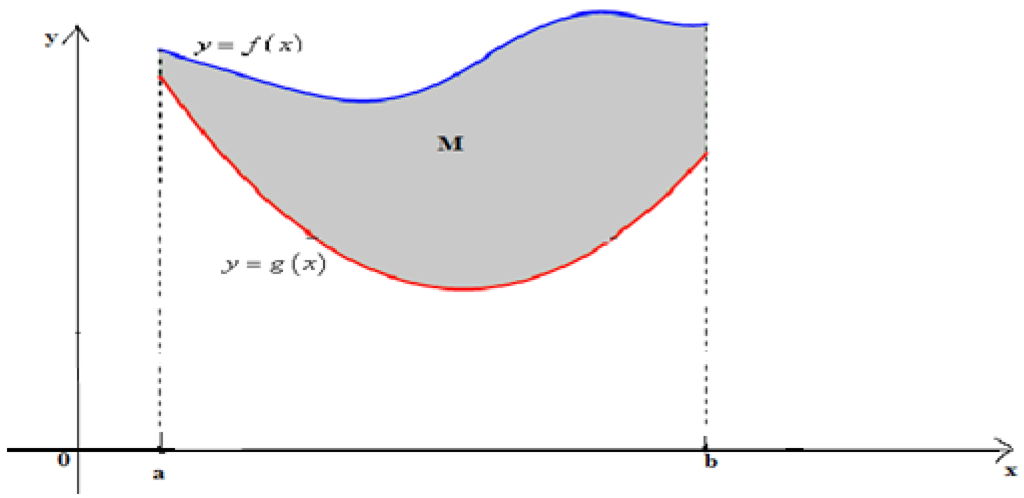


Figure 3.10 Area between two curves

Assume that $a<b$ and that $f(x)\geq g(x)$ on $[a,b]$, so the graph of f lies above the graph of g If $g(x)\geq 0$ on $[a,b]$, then the area A *of* M is the area above the $x-axis$

under the graph of f minus the area above the $x-axis$ under the graph of g :

$$A=\int_a^b f(x)dx-\int_a^b g(x)dx=\int_a^b [f(x)-g(x)]dx$$

From the above figure the area $A=\int_a^b [f(x)-g(x)]dx$ is calculated as follows

$$A=\int_a^b \left[\begin{pmatrix} upper \\ function \end{pmatrix}-\begin{pmatrix} lower \\ function \end{pmatrix}\right]dx$$ with $f(x)\geq g(x)$ for $a \leq x \leq b$

Even if f *and* g can take negative values on $[a,b]$, this interpretation and resulting area formula $A=\int_a^b [f(x)-g(x)]dx$ remain valid, provided that $f(x)\geq g(x)$ on $[a,b]$.

Hence the total area lying between the graphs $y=f(x)$ *and* $y=g(x)$ and between the vertical lines $x=a$ *and* $x=b$ is given by using the absolute value form:

$$A=\left|\int_a^b f(x)-g(x)dx\right|$$

Remember to plot graphs of the functions to locate the upper and lower function before starting calculations..

Example 3.10

1) Find the area of plane region M lying between the curves

$f(x)=8-3x^2$ *and* $g(x)=x^2-4x$

Solution

First, we have to solve the equation $f(x)=g(x)$ to find the intersections of the curves.

We get now: $8-3x^2=x^2-4x$.

Simple calculations lead to $x^2-x-2=0$ (Quadratic equation)

Solve the equation to get $x=2$ or $x=-1$.

So, the two curves intersect at two points of respective abscissa $x=2$ *and* $x=-1$.

The graphs of the two functions are parabola.

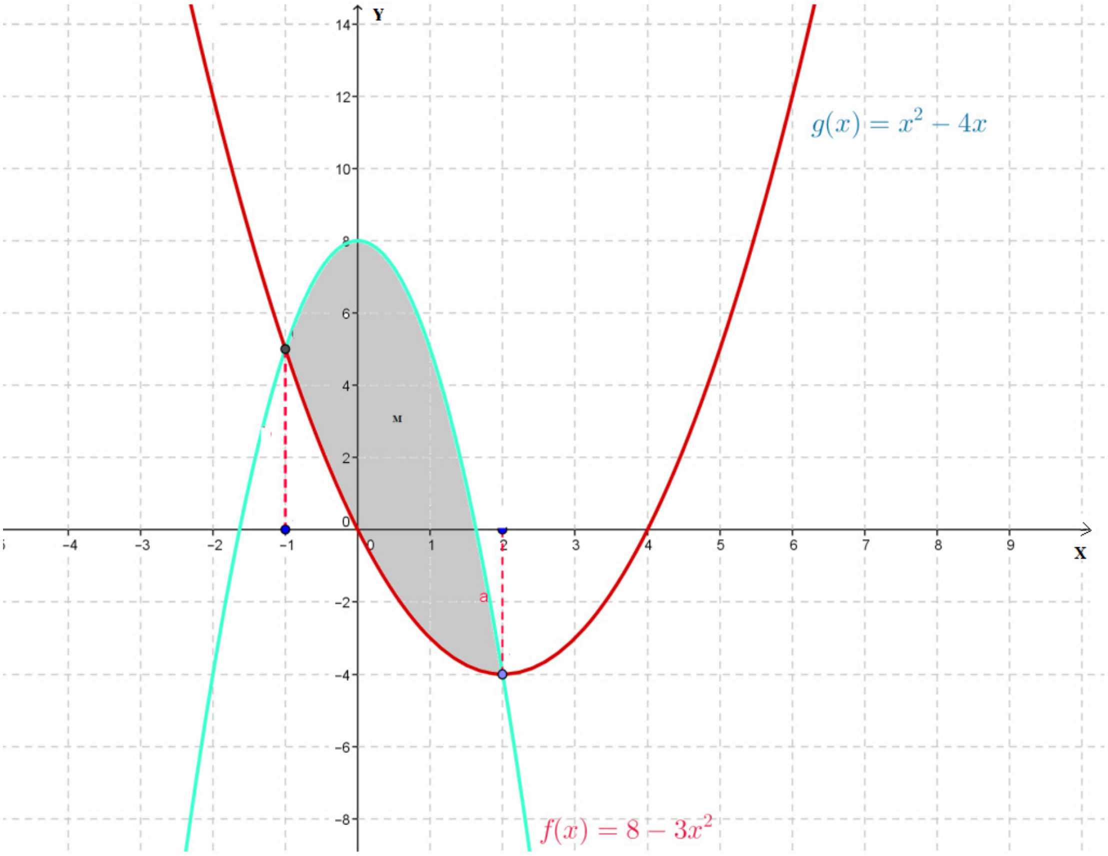


Figure 3.11. Graph of area between the curves of $f(x) = 8 - 3x^2$ *and* $g(x) = x^2 - 4x$

The bounded region M between $f(x) = 8 - 3x^2$ *and* $g(x) = x^2 - 4x$ is shaded.

Since $f(x) \geq g(x)$ *for* $-1 \leq x \leq 2$, the area A *of* M is given by:

$$A = \int_{-1}^{2} [f(x) - g(x)]dx = \int_{-1}^{2} \left[\left(8 - 3x^2\right) - \left(x^2 - 4x\right)\right]dx$$

$$= \int_{-1}^{2} \left(8 - 4x^2 + 4x\right)dx = \left[8x - \frac{4x^3}{3} + \frac{4x^2}{2}\right]_{-1}^{2}$$

$$= 8(2) - \frac{4}{3}(8) + 8 - \left[-8 + \frac{4}{3} + 2\right]$$

$$= 16 - \frac{32}{3} + 8 - \left(-\frac{14}{3}\right) = 24 - \frac{18}{3} = \frac{72 - 18}{3} = \frac{54}{3} = 18$$

The final answer is expressed in term of surface area as: A=18 $\times$square units

For sexample, if the unit on axis stands for 2 cms. The area of the square unit is then 4 cm^2. In this case the area of the region is $18 \times 4cm^2 = 72cm^2$.

2) Determine the area of the region K bounded by $y = 2x^2 + 10$ *and* $y = 4x + 16$

Solution

To find the intersection points, we have to equate the two equations: $2x^2 + 10 = 4x + 16 \Leftrightarrow 2x^2 - 4x - 6 = 0 \Leftrightarrow 2(x+1)(x-3) = 0$

So, the two curves will intersect when $x = -1$ *and* $x = 3$,

The graph of *g* is a parabola while the graph of *f* is a line.

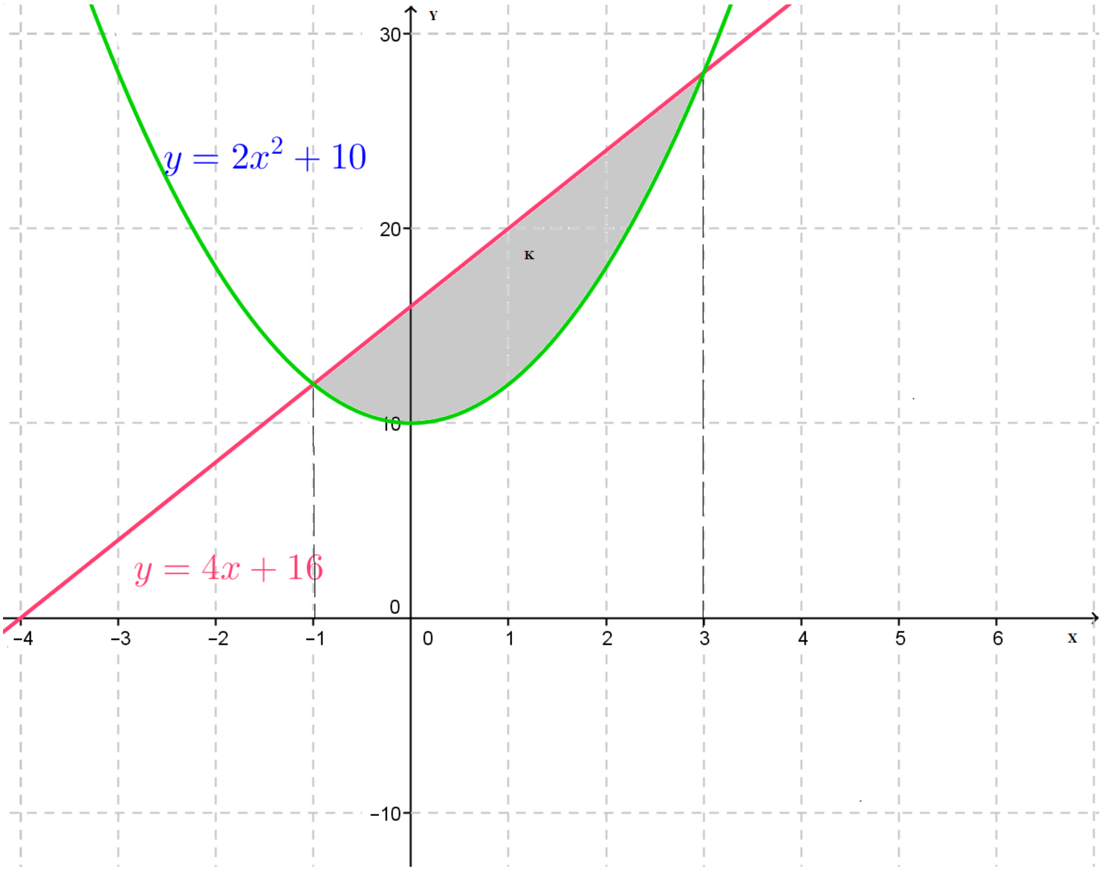


Figure 3.11 Graph of area between the curves of $g(x) = 2x^2 + 10$ *and* $f(x) = 4x + 16$

$$Area\ \ A = \int_a^b \left[\begin{pmatrix} upper \\ function \end{pmatrix} - \begin{pmatrix} lower \\ function \end{pmatrix} \right] dx$$

$$A = \int_{-1}^{3} \left[(4x+16) - (2x^2+10) \right] dx = \int_{-1}^{3} (-2x^2 + 4x + 6) dx$$

Therefore, $A=\left[-\frac{2x^2}{3}+2x^2+6x\right]_{-1}^{3}=\frac{64}{3}$ Unit of Area (U.A)

b) Determination of the work done in Physics

When a force acts on an object to move that object, it is said to have done work on the object. The amount of the work done by a constant force is measured by the product of the magnitude of the force and distance moved in the direction of the force. This assumes that the force is in the direction of the motion.

Work = (Force)(Distance)

Suppose that a force in the direction of the $x-axis$ moves an object from $x=a$ to $x=b$ on that axis and that force varies continuously with the position x of the object, that is $F=F(x)$ is a continuous function. The element of work done by the force in moving the object through a very short distance from x to $x+dx$ is $dW=F(x)dx$, so the total work done by the force is $W=\int_{x=a}^{x=b} dW=\int_a^b F(x)dx$

The unit of work is the joule (J), the force is in newtons (N) and the distance/displacement is in metres (m).

Example 3.11

A variable force (F in Newton) modeled by the equation $F=4x-3$ is applied over a certain distance (x). What is the work (W in Joules) done in moving the object for a displacement of $2m$ to $4m$

Solution

$$W=\int_2^4 (4x-3)dx=\left[4\frac{x^2}{2}-3x\right]_2^4=\left[2x^2-3x\right]_2^4=[2(16)-3(4)]-[2(4)-3(2)]$$

$=(32-12)-(8-6)=20-2=18$. The work done is 18 joules.

c) Determination of cost function in Economics

In economics, the marginal function is obtained by differentiating the total function. Now, when marginal function is given and initial values are given, the total function can be obtained using integration.

If C denotes the total cost and $M(x)=\frac{dC}{dx}$ is the marginal cost, we can write

$C=C(x)=\int M(x)dx+K$, where the constant of integration K represents the fixed cost.

Example 3.12

The marginal cost function of manufacturing $x\ units$ of a product is $5-16x+3x^2$ FRW. Find the total cost of producing 5 up to 20 items.

Solution

$$C=\int_5^{20}\left(5-16x+3x^2\right)dx=\left[5x-16\frac{x^2}{2}+3\frac{x^3}{3}\right]_5^{20}$$

Therefore, $C=\left[5(20)-8(20)^2+20^3\right]-\left[5(5)-8(25)+5^3\right]$

= (100-3200+8000) - (25-200+125) = 4900 - (-50) = 4950

The required cost is 4,950 Frw.

Application activity 3.10

1) Evaluate the area of the plane region bounded by the graphs of the functions $f(x)=-x^2-2x+2\ and\ g(x)=x^2+x-3$
2) Calculate the total area A lying between the curves $y=\sin x\ and\ y=\cos x\ from\ x=0\ to\ y=2\pi$.
3) Determine the area A of the region bounded by the parabola $y=x^2$ and the straight lines $y=0,\ x=0\ and\ x=c,\ where\ c>0$. Take 1.5 cm as unit on axes.

4) The marginal profit for a product is model by $\frac{dP}{dx}=40-3\sqrt{x}$ where P is the profit and x the sales. Find the change in profit when sales increase from 100 to 121 units.

End unit assessment

QUESTION ONE

Calculate the following integrals

a) $\int (9x^7 + \frac{1}{x-1} + \frac{2}{\cos^2 x} - \frac{1}{2}e^x)dx =$ b) $\int \frac{x}{\sqrt{x+3}}dx =$ c) $\int \frac{1}{4}\sin 3\,\mathrm{x}\mathrm{dx} =$

QUESTION TWO

Discuss and solve the following problems

a) The marginal cost function of producing x units of soft drink are given by the function $MC = \frac{x}{\sqrt{x^2+1600}}$ .Given that the fixed cost is 500 Frw, determine

i) The total cost function

ii) An average cost function

b) Consider the function f defined by $f(x) = 4 - \sqrt{x}$

i) Plot the graph of $y = f(x)$ showing the intercepts with the coordinate axes.

ii) On the diagram, shade the area which is bounded by the curve and the coordinate axes.

iii) Express the shaded area in terms of a definite integral

iv) Calculate the area of the shaded part.

QUESTION THREE

Discuss how this unit inspired you in relation to learning other subjects or to your future. If no inspiration at all, explain why.

Unit 4. ORDINARY DIFFERENTIAL EQUATIONS

Unit 4. ORDNINARY DIFFERENTIAL EQUATIONS

Key unit competence

Use ordinary differential equations of first and second order to model and solve related problems in Physics, Economics, Chemistry, Biology, Demography, etc.

Introductory activity

A quantity $y(t)$ is said to have an exponential growth model if it increases at a rate that is proportional to the amount of the quantity present, and it is said to have an exponential decay model if it decreases at a rate that is proportional to the amount of the quantity present.

Thus, for an exponential growth model, the quantity $y(t)$ satisfies an equation of the form $\frac{dy}{dt}=ky$ (k is a non-negative constant called annual growth rate). Given that $\frac{dy}{dt}=ky$ can be written as $\frac{dy}{y}=kdt$, solve this equation and apply the answer $y(t)$ obtained in the following problem:

The size of the resident Rwandan population in 2018 is estimated to 12,089,721 with a growth rate of about 2.37% comparatively to year 2017 (www.statistics.gov.rw/publication/demographic-dividend).

Assuming an exponential growth model and constant growth rate,

1) Estimate the national population at the beginning of the year 2020, 2030, 2040 and 2050.
2) Discuss your observations on the behavior of the national population along these 4 years.
3) What are pieces of advice would you provide to policy makers?
4) Draw a graph representing your observations mentioned in 1)

4.1 Definition and classification of differential equations

Activity 4.1

For each of the following equations

1) $y = 4kx$
2) $y = kx + bx^2$
3) $y = k\cos 2x - b\sin 2x$

Differentiate the given equations (once or twice), deduce the value of the constant k or b then substitute the obtained value of k or b in the initial equation.

Discuss the new equation obtained and write down the highest order of the derivative that occurs in that equation

A differential equation is any equation which contains derivatives of the unknown function; it shows the relationship between an **independent variable** *x*, a **dependent variable** y (unknown) and one or more differential coefficients of *y* with respect to *x*.

An **ordinary differential equation** (ODE) for a **dependent variable** *y* (unknown) in terms of an **independent variable** *x* is any equation which involves first or higher order derivatives of *y* with respect to *x*, and possibly *x* and *y*.

The general ordinary differential equation of the n^{th} order is

$$F\left(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, \ldots, \frac{d^ny}{dx^n}\right) = 0$$

or $F(x, y, y', y'', \ldots, y^{(n)}) = 0$ in implicite form.While in explicite form , we have $\frac{d^ny}{dx^n} = f\left(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, \ldots, \frac{d^{n-1}y}{dx^{n-1}}\right)$ or $y^{(n)} = f(x, y, y', y'', \ldots y^{(n-1)})$.

Particularly, the differential equation of the 1st order is of the form

$F\left(x, y, \frac{dy}{dx}\right) = 0$ or $F(x, y, y') = 0$ or $\frac{dy}{dx} = f(x, y)$ or $y' = f(x, y)$

Order of a differential equation: Differential equations are classified according to the highest derivative which occurs in them.

The **order** of a differential equation is the highest derivative present in the differential equation.

The **degree** of an ordinary differential equation is the algebraic degree of its highest ordered derivative after simplification.

Examples 4.1

1) $2\left(y'\right)^2+2x+y'''=0 \; or \; 2\left(\frac{dy}{dx}\right)^2+2x+\frac{d^3y}{dx^3}$; the order is 3 and the degree is 1.

2) $y''+2y'+x^2=0$; the order is 2 and the degree is 1.

3) $y''+2x+y^2=0$; the order is 2 and the degree is 1.

4) $L\frac{d^2q}{dt^2}+R\frac{dq}{dt}+\frac{q}{C}=E\sin\omega t \; or \; Lq''+Rq'+\frac{q}{c}=E\sin\omega t$; order is 2, the degree is 1.

5) $\frac{dy}{dx}=x^2$ is of order 1 and degree 1

6) $y\frac{d^2y}{dx^2}+\cos x=0$, Order 2 and degree 1

7) $\left(\frac{dy}{dx}\right)^2+y=x$, Order 1 and degree 2

Application activities 4.1

Discuss and state the order and the degree of each of the following differential equations. Explain your answer.

a) $\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^4-4x+y=1$

b) $\left(\frac{dy}{dx}\right)^3-2x=\cos y-2\sin x$

c) $\left(y''\right)^3+\left(y'\right)-2y=x$

d) $y\frac{d^2y}{dx^2}=-\cos x$

e) $x^2\left(\frac{d^2y}{dx^2}\right)^4+y\left(\frac{dy}{dx}\right)+y^4=0$

4.2 Differential equations of first order with separable variables

Activity 4.2

1) Consider $4y' - 2x = 0$,

   a) Solve the equation for y' and integrate both sides to deduce the value of the dependent variable y.

   b) What can you say if you were given $4\frac{dy}{dx} - 2x = 0$?

   c) Check whether y is solution of the given equation.

2) Apply the technique so-called separation of variables used in (1) to solve the following:

   a) $\sin x dx - \sin y dy = 0$

   b) $x\frac{dy}{dx} = 1$

3) Discuss how to solve $f(y)\frac{dy}{dx} = g(x)$

A **separable differential equation** is an equation of the form $\frac{dy}{dx} = f(x)h(y)$.

These are called separable variables because the expression for $\frac{dy}{dx}$ or y' can be separated into a product of separate functions of *x* and *y* alone. This means that they can be rewritten so that all terms involving y are on one side of the equation and all terms involving x are on the other side.

That is: $\frac{dy}{h(y)} = f(x)dx$

Hence, solving the equation requires simply integrating both sides with respect to their respective variables;

$$\int \frac{dy}{h(y)} = \int f(x)dx + c$$

Of course, the left-hand side is now an integral with respect to *y*, the right-hand side with respect to *x*. Note that we only need one arbitrary constant.

In particular if $h(y) = m$ (a constant), the differential equation of the form $\frac{dy}{dx} = mf(x)$ is solved by direct integration. That is:

$$dy = mf(x)dx \Leftrightarrow \quad y = m\int f(x)dx + c.$$

Similarly, equation of the form $\frac{dy}{dx}=mf(y)$ is solved by direct integration:

$$\frac{dy}{f(y)}=mdx \quad \Leftrightarrow \int \frac{dy}{f(y)}=mx+c$$

A **solution** to a differential equation on an interval $\alpha < x < \beta$ is any function which satisfies the differential equation in question on that interval. It is important to note that solutions are often accompanied by intervals and these intervals can impart some important information about the solution.

Example 4.2

Show that $y=\frac{x}{2}\left(\frac{x^2}{6}+1\right)$ is a solution of $\frac{dy}{dx}=\frac{x^2+2}{4}$ on $]-\infty,+\infty[$.

Solution

Given that $y=\frac{x}{2}\left(\frac{x^2}{6}+1\right)=\frac{x^3+6x}{12}$, we have $\frac{dy}{dx}=\frac{3x^2+6}{12}=\frac{x^2+2}{4}$. Therefore,

$$y=\frac{x}{2}\left(\frac{x^2}{6}+1\right) \text{ is solution of } \frac{dy}{dx}=\frac{x^2+2}{4} \text{ on }]-\infty,+\infty[.$$

It is easily checked that for any constant *c*, $y=\frac{x}{2}\left(\frac{x^2}{6}+1\right)+c$ is also a solution to the equation called general solution of the given equation.

A **general solution** to a given differential equation is the most general form that the solution can take and doesn't take any initial conditions into account. In this way, there are an infinite number of solutions to a differential equation depending on the value of the constant; it is better (especially in applied problems) to precise conditions which lead to a particular solution.

Initial Conditions are conditions or set of conditions imposed to the general solution that will allow us to determine one **particular solution** also called **actual solution** that we are looking for.

In other words, initial conditions are values of the solution and/or its derivative(s) at specific points which help to determine values of arbitrary constants that appear in the general solution. Since the number of arbitrary constants in general solution to a given differential equation is equal to the order of the ODE, it follows that it requires *n* conditions to determine values for all *n* arbitrary constants in the general solution of an n^{th}-order differential equation (one condition for each constant). For a first order equation, the single arbitrary constant can be determined by specifying the value of the unknown function *y(x)* at an arbitrary *x*-value x_0, say $y(x_0)=y_0$.

Geometrically, initial condition of a first order differential equation ($\frac{dy}{dx}=f(x,y)$)

enables us to identify a specific function $\left(y = y(x)\right)$ whose curve passes through the point $\left(x_0, y_0\right)$ and the slope is $f\left(x_0, y_0\right)$.

A differential equation along which an appropriate number of initial conditions are given is called **Initial Value Problem** (**IVP**). Therefore, the **actual solution or particular solution** to a differential equation is the specific solution that not only satisfies the differential equation, but also satisfies the given initial condition(s).

Examples 4.3

1) Solve

a) $\dfrac{dy}{dx} = xy$ b) $\dfrac{dy}{dx} = \dfrac{x^2+1}{4}$

Solution

a) $\dfrac{dy}{dx} = xy$ we separate to give $\dfrac{dy}{y} = xdx$ so, $\int \dfrac{dy}{y} = \int xdx + c$ and integrating both sides with respect to their respective variables gives $\ln y = \dfrac{x^2}{2} + c$ or $y = e^{\ln y} = e^{\frac{x^2}{2}+c}$ Then, $y = e^{\frac{x^2}{2}+c} \Rightarrow y = e^c \times e^{\frac{x^2}{2}}$

b) $\dfrac{dy}{dx} = \dfrac{x^2+1}{4} \Leftrightarrow 4dy = \left(x^2+1\right)dx$. It means that:

$$\int 4dy = \int (x^2+1)dx \Rightarrow 4y = \frac{x^3}{3} + x + c \Rightarrow y = \frac{x^3}{12} + \frac{x}{4} + k \text{ (where we set } k = \frac{c}{4}).$$

2) Solve the IVP: $\dfrac{dy}{dx} = \dfrac{y}{x-3}$ with $y\left(0\right) = -3$

Solution

(**IVP**): **If** $\dfrac{dy}{dx} = \dfrac{y}{x-3}$; $y\left(0\right) = -3$ then $\dfrac{dy}{dx} = \dfrac{y}{x-3} \Rightarrow \dfrac{dy}{y} = \dfrac{dx}{x-3} \Rightarrow \int \dfrac{dy}{y} = \int \dfrac{dx}{x-3}$

Simple integrations yield to $\ln y = \ln\left(x-3\right) + c$. For simplicity and aesthetic purpose, set $c = \ln k$. Thus $\ln y = \ln\left(x-3\right) + \ln k$ or equivalently $y = k(x-3)$ (general solution).

Let's apply initial condition to the general solution. If $y\left(0\right) = -3$ it follows that $-3 = k\left(0-3\right)$ giving $k = 1$. Therefore, $y = x-3$ is the required particular solution that represents equation of the unique line passing through (0, -3) and whose slope is $\dfrac{y_0}{x_0-3} = \dfrac{-3}{0-3} = 1$.

3) Consider a falling apple with mass m illustrated on the figure 4.1, assume that only gravity and air resistance with the coefficient $\alpha = 0.38$ are acting upon it when falling.

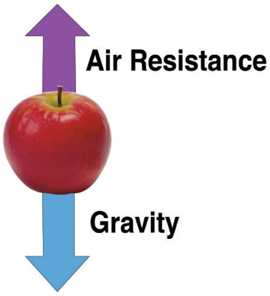


Figure 4.1: Forces acting upon the falling apple

a) Derive a differential equation expressing Newton's Second Law of motion for the apple and solve it to determine its velocity at any time t.

b) If the apple weights 0.2 kg and the coefficient for air resistance $\alpha = 0.38$ and supposing that at the initial time ($t = 0$) the velocity was null ($v = 0$), determine the velocity of the apple at any time t.

c) Plot the function velocity and discuss its limit as the time increases towards infinity.

Solution

a) Recall that Newton's Second Law of motion can be written as $m\frac{dv}{dt} = F(t,v)$ where $F(t,v)$ is the sum of forces that act on the apple and may be a function of the time t and its velocity v.

For this situation, we have two forces acting on the apple:

- Force of gravity $F_G = mg$ (where g is gravitational acceleration) acting in the downward positive direction, and
- Air resistance $F_A = -\alpha v$ (where α is a coefficient and v the velocity) acting in the upward direction and hence in negative direction.

Putting all of these together into Newton's second law, we find the following $m\frac{dv}{dt} = mg - \alpha v$ and the acceleration $\frac{dv}{dt} = g - \frac{\alpha}{m}v$. This is a differential equation with separable variables: the time t and the velocity v .Separating variables, we have

$$\frac{dv}{g-\frac{\alpha}{m}v}=dt$$

b) So, let's assume that the apple has a mass of 0.2 kg and that $\alpha=0.38$.

Plugging these values into (1) gives the following differential equation

$$\frac{dv}{9.8-\frac{0.38}{0.2}v}=dt \text{ or equivalently } \frac{dv}{9.8-1.9v}=dt$$

Integrating both sides, the velocity of the falling object at the time t becomes:

$v=5.15+ke^{-1.9t}$ where k is an arbitrary constant.

c) Supposing that at the initial time ($t=0$) the velocity was null ($v=0$), the constant becomes $k=-5.15$.

Therefore, the velocity v of the apple of 0.2kg released with the initial velocity $v_0=0$ in the air of resistance $\alpha=0.38$, is given by: $v=5.15\left(1-e^{-1.9t}\right)$ Unit of velocity.

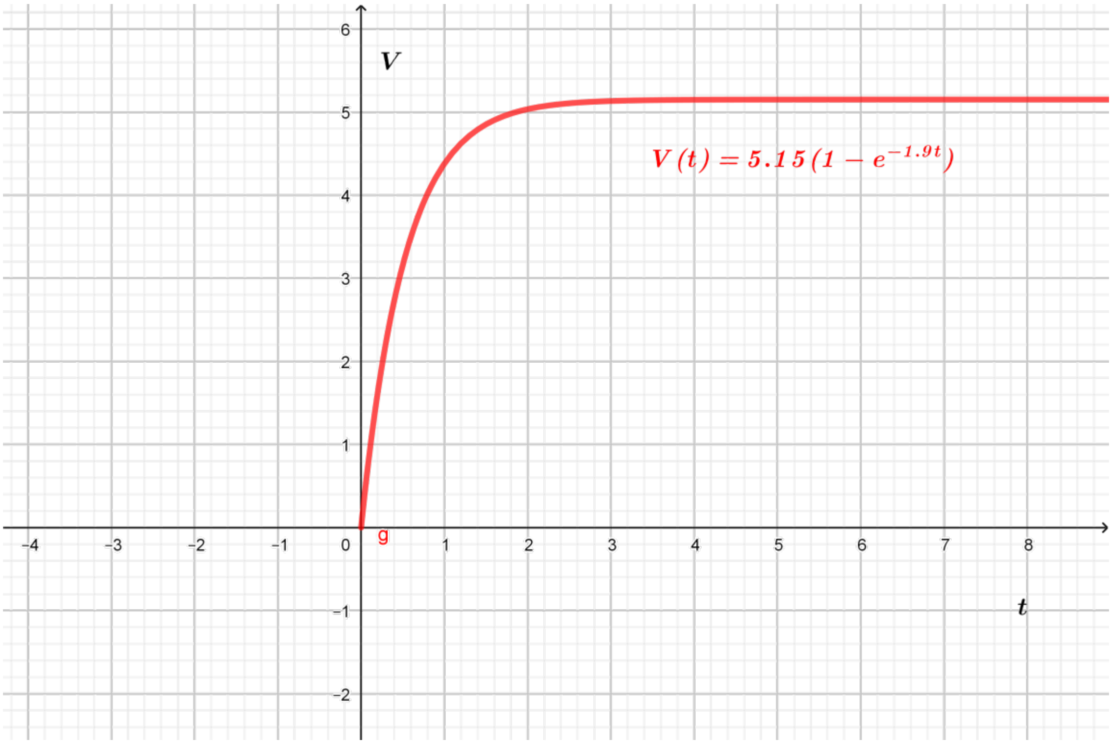


Figure 4.2: Velocity of a falling apple

The figure shows that the velocity increases with time towards $V = 5.15\ m/s$ as time increases indefinitely.

4) Assume that $w(t)$ denotes the amount of radioactive material in a substance at the time t. Radioactive decay is described by the differential equation

$\frac{dw}{dt} = -\lambda w(t)$ with $w(0) = w_0$ where λ is a positive constant called the decay constant.

a) Solve this equation to find $w(t)$

b) Assume that $w(0) = w_0 = 123gr$ and $w(5) = 20gr$ and that the time is measured in minutes. Find the decay constant λ and determine the half-life time of the radio-active substance.

Solution

a) $\frac{dw}{dt} = -\lambda w(t)$

$\frac{dw}{w(t)} = -\lambda dt$

$\ln w = -\lambda t + c$, where c is a constant.

$w(t) = Ce^{-\lambda t}$

b) Given that $w(0) = w_0 = 123gr$

we have C = 123.

$w(5) = 20gr$ gives $20 = 123e^{-\lambda.5}$

Then, $\ln\left(\frac{20}{123}\right) = -5\lambda$

$$\lambda = \frac{-\ln\left(\frac{20}{123}\right)}{5} = 0.363$$

Application activities 4.2

1) Determine the general solution for: a) $\frac{dy}{dx} = x\cos x$ b) $x\frac{dy}{dx} = 2 - 4x^3$.

2) Solve the following initial value problem: $(x+1)\frac{dy}{dx} = x(y^2+1)$, $y(0) = 0$

3) a) The graph of a differentiable function $y = y(x)$ passes through the point (0, 1) and at every point $P(x, y)$ on the graph the tangent line is perpendicular to the line through P and the origin. Find an initial-value problem whose solution is $y(x)$.

b) Explain why the differential equation in part (a) is separable. Solve the initial-value problem using either separation of variables and describe the curve

4) Determine the particular solution of $\left(y^2-1\right)\frac{dy}{dt}=3y$ given that $y=1$ when $t=2\frac{1}{6}$.

5) Determine the particular solution of $xy=\left(1+x^2\right)\frac{dy}{dx}$ given that $y=1$ when $x=0$

6) a) The variation of resistance R in ohms of an aluminum conductor with temperature $\theta^{\circ}C$ is given by $\frac{dR}{d\theta}=\alpha R$, where α is the temperature coefficient of resistance of aluminum. If $R=R_o$ when $\theta=0^oC$, solve the equation for R.

b) If $\alpha=38\times10^{-4}/^oC$, determine the resistance of an aluminum conductor at 50^oC, correct to 3 significant figures, when its resistance at 0^oC is 24.0Ω.

4.3. Linear differential equations of the first order

Activity 4.3

Consider the differential equation $\frac{dy}{dx}+2xy=x$ (1)

Assume that there exists a function $I(x)$ called an **integrating factor** that must help us to solve easily the given equation (1).

1) Compute $I(x)=e^{\int 2xdx}$. For the time being, set the integration constant to 0.
2) Multiply both sides in the differential equation (1) by $I(x)$ and verify that the left side becomes the product rule and write it as follows $\left(I(x).y(x)\right)'$
3) Integrate both sides, make sure you properly deal with the constant of integration
4) Solve for the function $y(x)$
5) Verify if the value of $y(x)$ obtained in 4) is solution of (1).

A first ordinary differential equation (ODE) in which the only power to which y' or any of its derivatives occurs is zero or one is called a **linear ordinary differential equation of first order** .Any other first order ODE that is not linear is said to be **nonlinear.**

Thus, if p and q are functions in x or constants the general linear equation of first order can take the form $\frac{dy}{dx}+py=q$ (2). There exists a "magical" function $I(x)$ called **integrating factor** that helps to solve the equation (2).

The solution process for a first order linear differential equation is as follows:

a) Determine an **integrating factor** $I(x)=e^{\int pdx}$ taking the integrating constant $c\ =0$.

b) Multiply both sides in the differential equation (2) by $I(x)$ and get the left side equivalent to $\left(I(x).y(x)\right)'$.

c) Integrate both sides, make sure you properly deal with the constant of integration

d) Obtain the final solution of the form $y(x)=\frac{\int I(x)q(x)dx+c}{I(x)}$

or simply $y(x)=\left(\int e^{\int p(x)dx}q(x)dx\right)e^{-\int p(x)dx}$

This process can be simplified by letting $y=uv$ where u and v are functions in x to be determined in the following ways: $v=e^{-\int pdx}$ by taking the constant $c=0$ and $u=\int q\, e^{\int pdx}\, dx$

Therefore, the solution of the equation $\frac{dy}{dx}+py=q$ becomes $y=uv$ where $u=\int q\, e^{\int pdx}\, dx$ and $v=e^{-\int pdx}$.

Examples 4.4

1) State the order and degree of each ODE, and state which are linear or non-linear:

i) $\frac{dy}{dx}+y=x$, ii) $x''+3t^2=0$ iii) $R\frac{dq}{dt}+\frac{q}{C}=3$

iv) $\frac{d^3y}{dx^3}+\frac{d^2y}{dx^2}-2x\sin 2x\frac{dy}{dx}=y+4x^4$

Solution

i) $\frac{dy}{dx}+y=x$ is a linear differential equation in y of first order and its degree is 1

ii) $x''+3t^2=0$ is linear differential equation of second order in x and the degree is 1.

iii) $R\frac{dq}{dt}+\frac{q}{C}=3$ is linear differential equation of first order in q and its degree is 1.

iv) $\frac{d^3y}{dx^3}+\frac{d^2y}{dx^2}-2x\sin 2x\frac{dy}{dx}=y+4x^4$ is linear differential equation of third order in y and its degree is 1.

2) Consider the equation for the falling apple $\frac{dv}{dt}+1.9v=9.8$, $v(0)=0$, previously seen (see Example 4.3, figure 4.1) and solve it using integrating factor. Compare the two processes (the one used in example 4.3, question 3 and the use of integrating factor).

Solution

$$\frac{dv}{dt}+1.9v=9.8,\quad v(0)=0$$

Relating this equation to $\frac{dy}{dx}+py=q$, the functions $p(x)$ and $q(x)$ are constant such that $p=1.9$ and $q=9.8$.

Therefore, the integrating factor $I(t)=e^{\int 1.9dt}=e^{1.9t}$

And $v(t)=\frac{\int I(t)q(t)dt+C}{I(t)}$. Given that

$$\int I(t)q(t)dt+C=\int 9.8e^{1.9t}dt+C=\frac{9.8}{1.9}e^{1.9t}+c,$$

$$v(t)=\frac{5.15e^{1.9t}+c}{e^{1.9t}}=5.15+ce^{-1.9t}$$

Applying the initial condition, we get $v(t)=5.15\left(1-e^{-1.9t}\right)$ which is the same as the answer seen in example 4.3.

The solving process to get the general solution seems to involve more steps than the method for separable variables

3) Use integrating factor to solve $y'-x^2y=x^2$.

Solution

$$y'-x^2y=x^2$$

The equation becomes $\frac{dy}{dx}-x^2y=x^2$ with $p=-x^2 and\ q=x^2$

Therefore, $I(t)=e^{\int -x^2dx}=e^{-\frac{x^3}{3}}$ and $y(x)=\frac{\int I(x)q(x)dx+C}{I(x)}=\frac{\int e^{-\frac{x^3}{3}}(x^2)dx+C}{e^{-\frac{x^3}{3}}}$,

thus $y(x)=\frac{\int e^{-\frac{x^3}{3}}.(x^2)dx+c}{e^{-\frac{x^3}{3}}}=\frac{-e^{-\frac{x^3}{3}}+c}{e^{-\frac{x^3}{3}}}=-1+ce^{\frac{x^3}{3}}$.

Finally, $y(x)=-1+ce^{\frac{x^3}{3}}$, where *c* is a constant.

Application activities 4.3

1) Determine the general solution of the following equations

a) $y'+\frac{y}{x}=1$ b) $y'+xy=x$ c) $y'+\frac{y}{x}=x$

d) $y'+2y=e^x$ e) $y'-2x\,y=e^{x^2}$ f) $y'+\frac{3}{x}y=\frac{\sin x}{x^3}$

2) The Voltage potential difference (p.d.V) between the plates of a capacitor *C* charged by a steady voltage *E* through a resistor *R* is given by the equation $CR\frac{dV}{dt}+V=E$.

a) Solve the equation for V given that at $t=0$, $V=0$.

b) Calculate V, rounding to 3 decimal places, when $E=25V$, $C=20\times10^{-6}F$, $R=200\times10^3\Omega$ and $t=3$ seconds

4.4 Applications of ordinary differential equations

4.4.1 Differential equations and the population growth

Activity 4.4

The number of individuals (population) P present at a given time *t* is a function of time. Given that the rate of change (in time) of this population $\frac{dP}{dt}$ is proportional to the population P present,

1) Write a differential equation expressing this model assuming that the constant of proportionality K is positive;
2) Solve the obtained equation considering that $K=5\%$ and the population was P_0 at initial time $t=0$;

3) Let the population of a given town be now $P_0 = 11,500,000$. Assume that the variation remains the same (constant), what will be its size after 5 years? Plot the related graph and give your interpretation in your own words.
What would you advice police makers of that town? (Give many suggestions as you can)

The growth of a population is usually modeled with an equation of the form $\frac{dP}{dt} = KP$ where P represents the number of individuals on a given time *t*. The constant K is a positive when the population increases and negative when the population decreases.

Separating variables and integrating both sides of $\frac{dP}{dt} = KP$, we get: $\int \frac{dP}{P} = k\int dt \Rightarrow \ln P = Kt + c \Rightarrow P = ce^{Kt}$. If the initial population at time $t = 0$ is P_0, then $P_0 = ce^{k(0)} = c$. Therefore, $P_0 = c$ and we have $P = P_0 e^{Kt}$

A solution to this equation is the exponential function: $P(t) = P_0 e^{kt}$ where P_0 is the initial population at time $t = 0$, K the annual growth rate or the annual decay rate.

Example 4.5

Consider the population P of a region where there is no immigration or emigration. The rate at which the population is growing is often proportional to the size of the population. This means larger populations grow faster, as we expect since there are more people to have babies.

If the population has a continuous growth rate of 2% per unit time, what is its population at any time t?

Solution

We know that $\frac{dP}{dt} = KP$. Separating variables and integrating both sides to get:

$\int \frac{dP}{P} = k\int dt \Rightarrow \ln P = Kt + c \Rightarrow P = ce^{Kt}$ **for** $K = 0.02$ **and has the general solution**

$P = ce^{0.02t}$

If the initial population at time $t = 0$ is P_0, then $P_0 = ce^{(0.02)(0)} = c$. So, $P_0 = c$ and we have $P = P_0 e^{0.02t}$

Application activity 4.4

1) The population of a given city is double in 20 years. We suppose that the rate of increasing is proportional to the number of population. In how many times the population will be three times, if on $t = 0$ we have the population P_0.

2) In laboratory, it is observed that the population of bacteria is increasing from $1,000$ to $3,000$ during 10 hours. If the rate of increasing of bacteria is proportional to the present number of bacteria on the time t, find the number of bacteria after 5 hours.

4.4.2 Differential equations and Crime investigation

Activity 4.5

Consider the object of temperature T that is cooling with time t in a given environment. Newton's law of cooling states that the rate of change (in time) of the temperature is proportional to the difference between the temperature T of the object and the temperature Te of the environment surrounding the object.

1) Write the differential equation expressing this model;
2) Solve the obtained equation considering that the constant of proportionality $K = \ln\left(\frac{1}{2}\right)$;
3) Assume that the object that had a normal temperature of $98.6^0 F$ at the initial time was put in a room with a constant temperature 70.0^0F; plot the graph of T and give its interpretation in your own words.

The time of death of a murdered person can be determined with the help of modeling through differential equation. Police personnel discovers the body of a dead person presumably murdered and the problem is to estimate the time of death.

The Newton's Law of Cooling stating that the rate of change (in time) of the temperature is proportional to the difference between the temperature T of the object and the temperature Te of the environment surrounding the object is very essential in solving such a problem.

Therefore, the time of death of a murdered person can be determined with the help of the differential equation $\frac{dT}{dt} = -K\left(T - T_e\right)$

The given equation is equivalent to $\frac{dT}{dt}+KT=KT_e$.This is a first order linear differential equation, its solution $T(t)=T_e+Be^{Kt}$ where B is a constant.

Example 4.6

Police discovers a murder victim in a hotel room at $9:00\,am$ one morning. The temperature of the body is $80.0^0 F$. One hour later, at $10:00\,am$, the body has cooled to $75.0^0 F$. The room is kept at a constant temperature of 70.0⁰F. Assume that the victim had a normal temperature of $98.6^0 F$ at the time of death.

a) Formulate the differential equation for the temperature of the body as function of time.
b) Solve the differential equation.
c) Use your solution in b) to estimate the time the murder took place.

Solution

Let T be the temperature of the body after t hours. By Newton's Law of Cooling we have the differential equation $\frac{dT}{dt}=K(T-70)$ where K is a constant. Separating variables we get

$\frac{dT}{T-70}=Kdt$. Integrating both sides $\int\frac{dT}{T-70}=\int Kdt \Rightarrow \ln(T-70)=Kt+c$ for $T-70>0$ because the body will never be cooler than the room. Thus $T-70=e^{kt+c}=e^c e^{kt}$. Taking $A=e^c\geq 0$ we get $T=Ae^{kt}+70$. Consideling $A=e^c$ (a positive constant), we get $T-70=e^{\ln(T-70)}=e^{kt+c}=e^c e^{kt}$

Take $t=0$ when the body was found at 9:00am. Plug in $t=0$ and $T=80.0^0 F$ and solve for c (It's easier to solve for $A=e^c$ and use this in the formula).

Therefore $80=70+Ae^{k\times 0}=70+A$. Then, $A=10$ and $T=70+10e^{kt}$.

Plug $t=1$ hour and $T=75.0^0 F$ in the equation and solve for K.

We get $75=70+10e^{1\times k}$ $so\ \frac{1}{2}=e^k\ and\ K=\ln\left(\frac{1}{2}\right)=-\ln 2$. Thus, $T=70+10e^{-t\ln 2}$

Therefore $98.6=70+10e^{-t\ln 2}$, so $2.86=e^{-t\ln 2}\ and\ t=-\frac{\ln 2.86}{\ln 2}\approx -1.516$

This negative sign implies that the murder occurred 1.516h before the time the police discorvered that body.

Finally, 1.516 hour is about 1 hour and 31 minutes. The murder took place about

$7:29\,a.m.$

Application activity 4.5

The body is located in a room that is kept at a constant 70⁰F. For some time after the death, the body will radiate heat into the cooler room, causing the body's temperature to decrease assuming that the victim's temperature was normal 98.6⁰F at the time of death.

a) Solve the related differential equation.
b) Use your solution in to estimate the time the murder took place if $K=28.10^{-4}$ and the officer arrived at 10.40 p.m. and the body temperature was 94.4⁰F.

4.4.3 Differential equations and the quantity of a drug in the body

Activity 4.6

To combat the infection in a human body, appropriate dose of medicine is essential. Since the amount of the drug in the human body decreases with time, the medicine must be given in multiple doses. The rate $\frac{dQ}{dt}$ at which the level of the drug in a patient's blood decays is proportional to the quantity Q of the drug left in the body. If a patient is given an initial dose Q_0 at time $t=0$,

1) Establish an equation for modeling the situation
2) Solve the obtained equation and find the quantity of drug $Q(t)$ left in the body at the time t
3) Draw $Q(t)$ and interpret the graph given that the drug provided was 100mg at $t=0$.
4) Discuss what happens when the patient does not respect the dose of medicine as prescribed by the Doctor.

The rate at which a drug leaves a patient's body is proportional to the quantity of the drug left in the body. If we let Q represent the quantity of drug left, then $\frac{dQ}{dt}=-kQ$
The negative sign indicates that the quantity of drug in the body is decreasing.
The solution to this differential equation is $Q=Q_0e^{-kt}$ and the quantity decreases exponentially.

The constant k depends on the drug and Q_0 is the amount of drug in the body at time zero. Sometimes physicians convey information about the relative decay rate with a *half -life*, which is the time it takes for Q to decrease by a factor of 1/2.

Example 4.7

A patient having major surgery is given the antibiotic vancomycin (an antibiotic used to treat a number of bacterial infections) intravenously at a rate of $85\, mg\ per\ hour$. The rate at which the drug is excreted from the body is proportional to the quantity present with proportionality constant 0.1, if time is in hours. Write a differential equation for the quantity, Q in mg, of vancomycin in the body after t hours.

Solution

The quantity of vancomycin, Q, is increasing at a constant rate of $85\, mg\ per\ hour$ and is decreasing at a rate of 0.1 times Q. The administration of $85\, mg\ per\ hour$ makes a positive contribution to the rate of change $\frac{dQ}{dt}$. The excretion at a rate of $0.1\,Q$ makes a negative contribution to $\frac{dQ}{dt}$. Putting these together, we have:

rate of change of a quantity = rate in – rate out,

So, $\frac{dQ}{dt} = 85 - 0.1Q$.

Application activity 4.6

Valproic acid is a drug used to control epilepsy; its half-life ($Q = \frac{1}{2}Q_0$) in the human body is about 15 hours.

a) Use the half-life as initial condition to find the constant K in the differential equation;
b) At what time will 10% of the original dose remain?

4.4.4 Differential equations in economics and finance

Activity 4.7

Assume that in a perfectly competitive market the speed with which price P adjusts towards its equilibrium value depends on how much excess demand there is. Given that the rate of change of the price $P(t)$ of a product at time t is proportional to the difference of the demand and the supply for the commodity $(Q_d - Q_s)$,

a) Write a differential equation modeling the rate of change of the price if the constant of proportionality $k = 0.08$ is in proportion to excess demand.
b) Assuming that $Q_d = 280 - 4P(t)$ and $Q_s = -35 + 8P(t)$ solve the equation obtained in (a).
c) Determine and plot $P(t)$ at the time t if the price is currently 19.
d) Compare the price at $t = 1$ and the price as t gets larger i.e. $\lim_{t\to\infty} P(t)$.

In a perfectly competitive market the speed with which price *P* adjusts towards its equilibrium value depends on how much excess demand there is. This is quite a reasonable proposition. If consumers wish to purchase a lot more produce than suppliers are willing to sell at the current price, then there will be great pressure for price to rise, but if there is only a slight shortfall then price adjustment may be sluggish. If excess demand is negative this means that quantity supplied exceeds quantity demanded, in which case price would tend to fall.

To derive the differential equation that describes this process, assume that the demand and supply functions are Q_d and Q_s

If r represents the rate of adjustment of *P* in proportion to excess demand, then we can write $\frac{dP}{dt} = r\left(Q_d - Q_s\right)$.

The solution of this equation leads to the function of the price $P(t)$ that changes over time.

Example 4.8

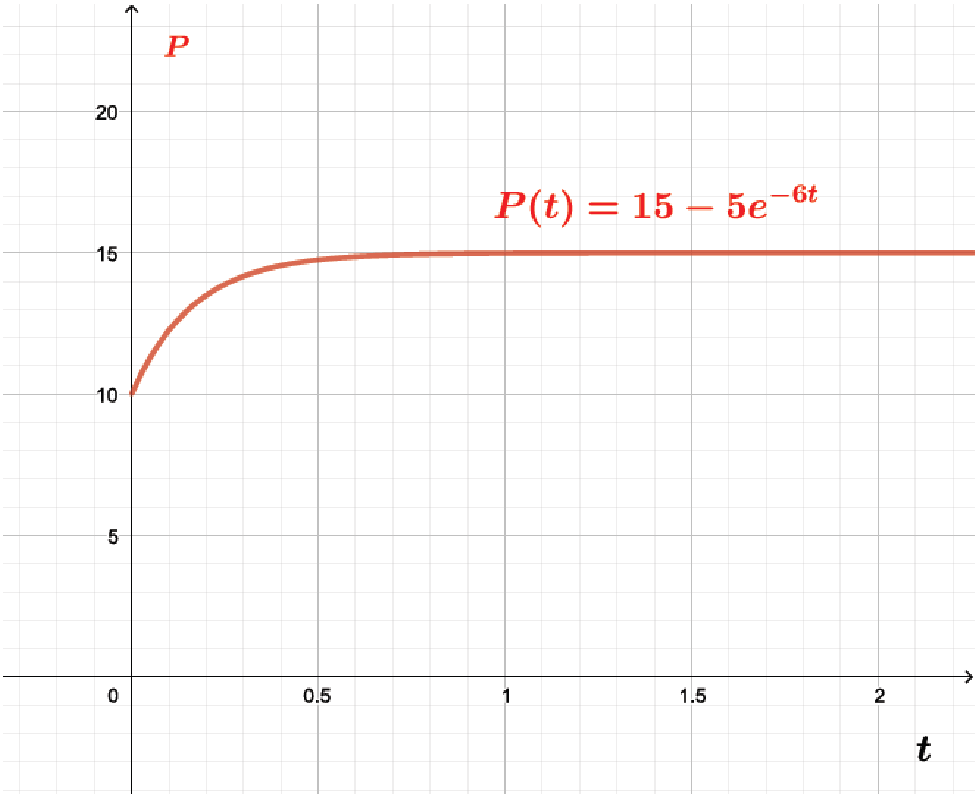
A perfectly competitive market has the demand and supply functions $Q_d = 170 - 8P$ and $Q_s = -10 + 4P$ respectively.

When the market is out of equilibrium the rate of adjustment of price is a function of excess demand such that $r = \frac{1}{2}$. Given that in the initial time period price P_0 is 10 which is not its equilibrium value, express P as function of *t* and comment on the stability of this market.

Solution

$\frac{dP}{dt} = r\left(Q_d - Q_s\right) \Rightarrow \frac{dP}{dt} = \frac{1}{2}\left(170 - 8P - \left(-10 + 4P\right)\right)$. Simplifying we obtain $\frac{dP}{dt} = -6P + 90$ which is a linear first-order differential equation. Solving the obtained ODE we have $P(t) = Ae^{-6t} + 15$ and Considering the initial condition, $P(0) = 10$ we find $A = -5$.

Therefore, $P(t) = -5e^{-6t} + 15$



The coefficient of t in this exponential function is the negative number -6. This means that the first term of the solution called the complementary function, will get closer to zero as t gets larger and so $P(t)$ will converge on its equilibrium value of 15. This market is therefore stable.

You can check this by using the above solution to calculate $P(t)$ for example taking $t = 3$, you can see that this is extremely close to the equilibrium price of 15 and so we can say that price returns to its equilibrium value within the first few time periods in this particular market.

Application activity 4.7

If the demand and supply functions in a competitive market are $Q_d = 50 - 0.2P$ and $Q_s = -10 + 0.3P$. Given that the rate $r = 0.04$, derive and solve the relevant difference equation to get a function for $P(t)$ given that price is 100 in time period 0. Comment on the stability of this market.

4.4.5 Differential equations in electricity (Series Circuits)

Activity 4.8

Let a series circuit contain only a resistor and an inductor as shown in Figure 4.3

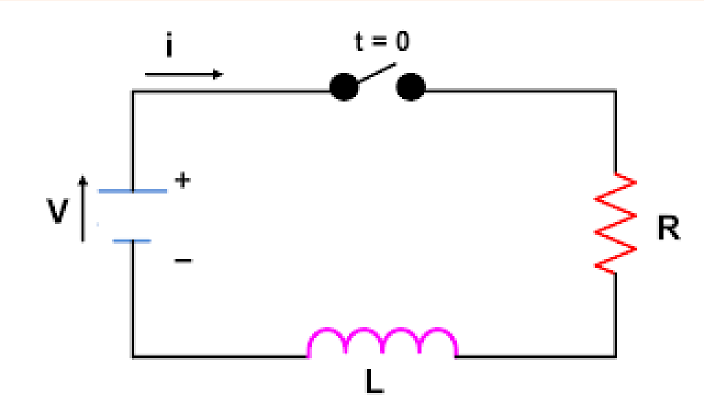


Figure 4.3: The RL series circuit
Source: electricalacademia.com/basic-lutrical/rl-series-circuit-analysis

By Kirchhoff's second law the sum of the voltage drop $L\frac{di}{dt}$ across the inductor L and the voltage drop iR across the resistor R is the same as the impressed voltage $E(t)$ on the circuit.

a) write down the equation modeling the situation where the current $i(t)$ varies with time t,

b) Solve the obtained equation considering that the voltage is constant and equals 110 volts and the current was zero before switching on.

c)What can you say about the value of the current as t gets larger i.e. $\lim_{t\to\infty} i(t)$.

The voltage in the circuit is modeled by Kirchhoff's second law saying that the voltage in the circuit is the sum of the voltage drop across the components of the circuit. It is known from physics that the voltage drops across the resistor, inductor, and capacitor are RI, $L\frac{dI}{dt}$ and $\frac{Q}{C}$ respectively.

Example 4.9

The voltage drop across a capacitor with capacitance C is given by $\frac{Q(t)}{C}$, where Q is the charge on the capacitor. Hence, for the series circuit composed of a resistor and a capacitor as shown in figure bellow

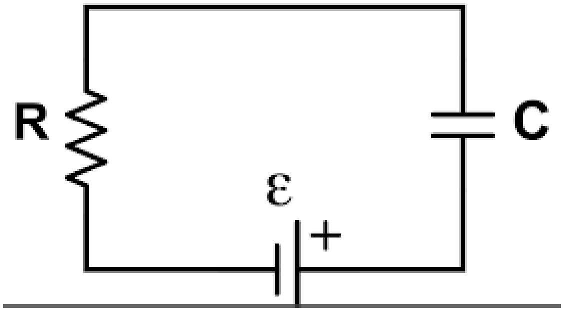


Figure 4.4: The RC series circuit
Source: electricalacademia.com/rc-series-circuit-analysis/

a) Determine the differential equation modeling the voltage of the circuit taking the charge $Q(t)$ as the dependent variable;

b) Solve the equation obtained if 100-volt electromotive force is applied to the circuit

c)If the resistance is 200 ohms and the capacitance is 10^{-4} farads. Find the charge $Q(t)$ on the capacitor if $Q(0)=0$. Deduce the current $i(t)$.

Solution

a) Applying Kirchhoff's second law in the circuit composed of the capacitor and the resistor, we get $Ri+\frac{Q}{C}=E(t)$

Since $i=\frac{dQ}{dt}$, our differential equation can be written as $R\frac{dQ}{dt}+\frac{Q}{C}=E(t)$.

b) If $E(t)=100$ then $R\frac{dQ}{dt}+\frac{Q}{C}=100$ or $R\frac{dQ}{dt}+\frac{1}{C}Q=100$

$$\Rightarrow\frac{dQ}{dt}+\frac{1}{RC}Q=\frac{100}{R}$$

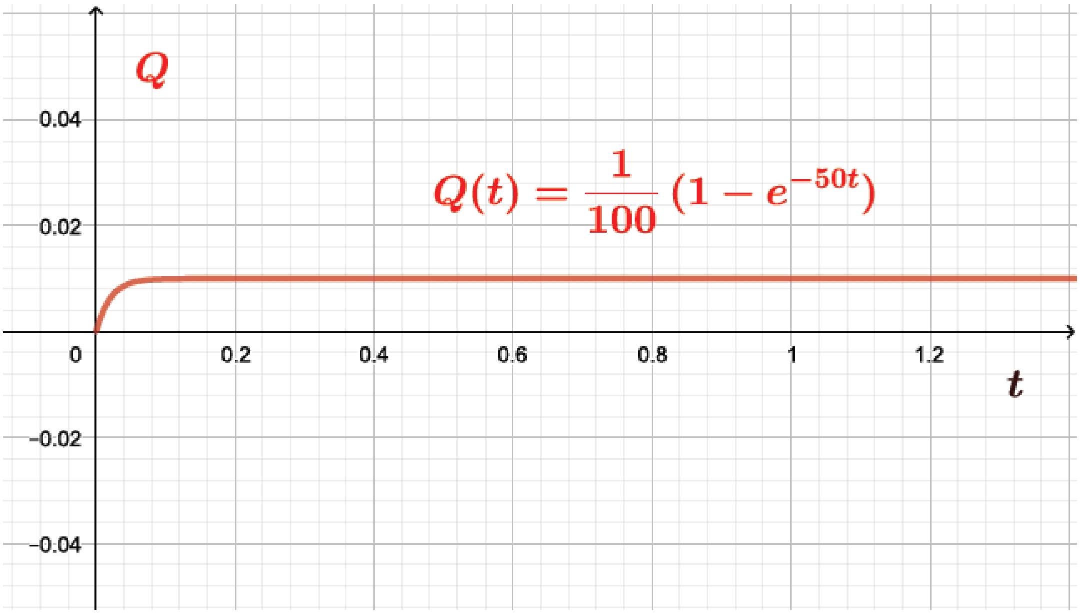
This is a first order linear differential equation of the form $\frac{dQ}{dt}+pQ=q$ where P and Q are constant $p=\frac{1}{RC}$, $q=\frac{100}{R}$.The integrating factor is $I(t)=e^{\int\frac{1}{RC}dt}=e^{\frac{1}{RC}t}$

and $Q(t)=\dfrac{\int I(t)q(t)dt+K}{I(t)}=\dfrac{\int e^{\frac{1}{RC}t}\dfrac{100}{R}dt+K}{e^{\frac{1}{RC}t}}=\dfrac{RC\dfrac{100}{R}e^{\frac{1}{RC}t}+K}{e^{\frac{1}{RC}t}}$

$Q(t)=100C+Ke^{-\frac{1}{RC}t}$ where K is a constant of integration. The charge is $Q(t)=100C+Ke^{-\frac{1}{RC}t}$ where K is a constant.

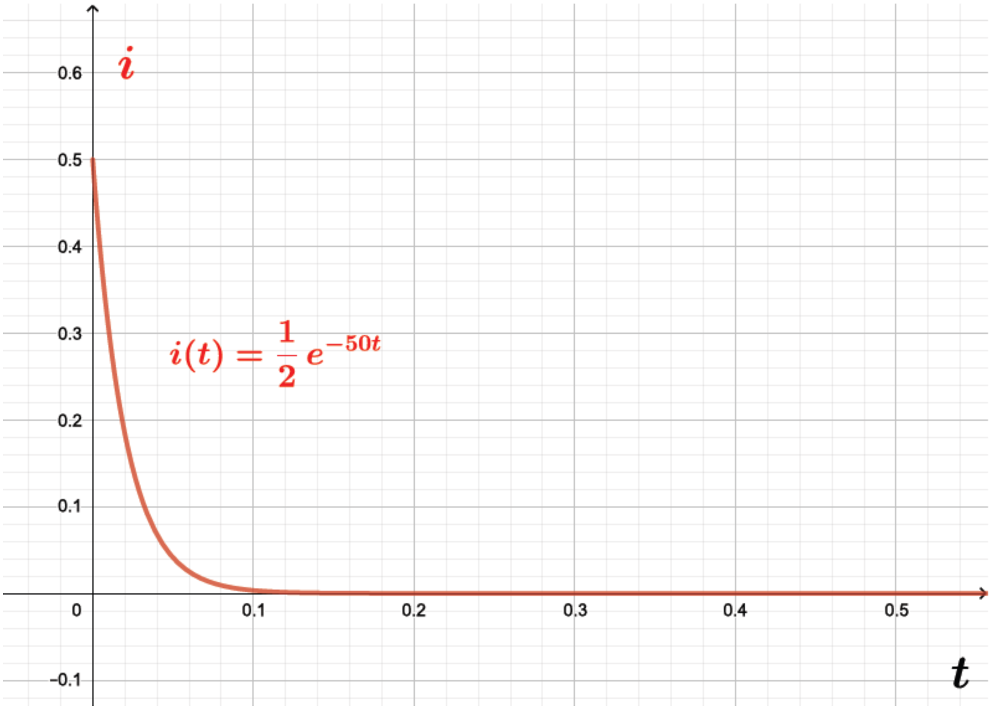
c)Given that $R=200$, $C=10^{-4}$ and $Q(0)=0$, we have

$Q(0)=0\Rightarrow 100C+K=0\Rightarrow K=-100C$. Therefore, $Q(t)=\dfrac{1}{100}\left(1-e^{-50t}\right)$



The charge is increasing towards $Q=\dfrac{1}{100}$ as the time increases indefinitely

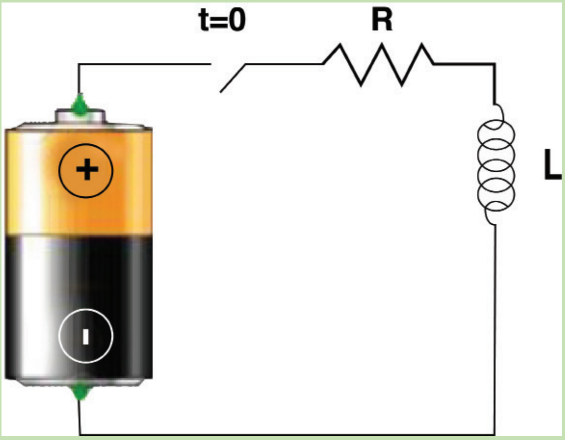
Given that $i=\dfrac{dQ}{dt}$, The current in the circuit is $i=\dfrac{dQ}{dt}=\dfrac{1}{2}e^{-50t}$



The current is decreasing.

Application activity 4.8

Let a series circuit contain only a resistor and an inductor as shown on the following figure:



Source: electricalacademia.com/rc-series-circuit-analysis/

a) Establish a differential equation modeling the current i in the closed circuit

b) If $R = 12\ ohms$, $L = 4H$ are connected to a battery that gives a constant voltage of 60 V and the switch is closed when $t = 0$ (i.e the current starts with $i(0) = 0$); Find the current after 1 s.

c)Find what happens to the current after a long time.

4.5. Introduction to second order linear homogeneous ordinary differential equations

Activity 4.9

1) Write down different examples of second order differential equations (some of degree 1, other of degree greater than 1) Identify their similarities and differences
2) Given the differential equation $2\frac{d^2y}{dt^2} + 0.1y = 0$

   a) Verify that $y = A\cos\sqrt{\frac{1}{20}}t + B\sin\sqrt{\frac{1}{20}}t$ is its general solution.

   b) Compare the two differential equations: $\frac{d^2y}{dt^2} + 0.05y = 0$ and $\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y = r(x)$ and identify the value of $p(x), q(x)\, and\ r(x)$ in $\frac{d^2y}{dt^2} + 0.05y = 0$.

The general second order linear differential equation is of the form

$$\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y = r(x)$$

or more simply, $y'' + p(x)y' + q(x)y = r(x)$ where $p(x), q(x)$ and $r(x)$ are functions of x alone (or perhaps constants).

If $r(x)$ is identically zero, the differential equation is said to be **homogeneous** equation $y'' + p(x)y' + q(x)y = 0$; otherwise it is said to be **non-homogeneous or inhomogeneous**

A second order differential equation which can not be written in the form $\frac{d^2y}{dx^2}+p(x)\frac{dy}{dx}+q(x)y=r(x)$ is said to be **non-linear.**

In this part, we limit our study to the particular type of linear equation of second order of the form $a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=0$

where *a*, *b*, *c* are constants (and $a\neq 0$, otherwise it wouldn't be second order).

This equation is called **second order linear homogeneous differential equation** with constant coefficients and occurs everywhere in science and engineering, most notably in the modeling of vibrating springs in a resisting medium, and in electrical circuits.

Example 4.10

Discuss the characteristics of the following equations and identify linear homogeneous differential equations among them.

a) $\frac{d^2y}{dx^2}-9y=x$ b) $x\frac{d^2y}{dx^2}-4\frac{dy}{dx}+x^2y=0$

c) $\frac{dy}{dx}\frac{d^3y}{dx^3}+2y^2=0$ d) $(y'')^3+(y')^5-2y=x^4$

Solution

We are going to compare each equation with $\frac{d^2y}{dx^2}+p(x)\frac{dy}{dx}+q(x)y=r(x)$ where $p(x)$ is the function coefficient of $\frac{dy}{dx}$, $q(x)$ the function coefficient of y and $r(x)$ the right hand side function.

a) $\frac{d^2y}{dx^2}-9y=x$, the function coefficient of $\frac{dy}{dx}$ is zero but the right-hand side is different from zero; thus, the equation is a linear but not homogeneous.

b) The equation $x\frac{d^2y}{dx^2}-4\frac{dy}{dx}+x^2y=0$ can be written as $\frac{d^2y}{dx^2}-\frac{4}{x}\frac{dy}{dx}+xy=0$, the coefficient of $\frac{dy}{dx}$ is function of *x* only and the second-hand side function is zero; it is a linear homogeneous differential equation.

c) $\frac{dy}{dx}\frac{d^3y}{dx^3}+2y^2=0$,can not be written in the form of $\frac{d^2y}{dx^2}+p(x)\frac{dy}{dx}+q(x)y=r(x)$.

This is not a second order linear differential equation.

d) $(y'')^3+(y')^5-2y=x^4$, It is a second order differential equation of third degree.

It is not **a second order** linear differential equation.

Application activity 4.9

Given the following equations, indicate which are linear homogeneous differential equations

a) $\frac{d^2y}{dx^2}-9y=0$ b) $x\frac{d^2y}{dx^2}+\cos x=0$ c) $x\frac{d^2y}{dx^2}+\frac{dy}{dx}+xy=0$

4.6 Solving linear homogeneous differential equations

4.6.1. Linear independence and superposition principle

Activity 4.10

Given the following differential equation $\frac{d^2y}{dx^2}+3\frac{dy}{dx}+2y=0$.

1) Verify that e^{-x} and e^{-2x} are two solutions to the given equation.
2) Multiply e^{-x} by any constant and e^{-2x} by another constant and take the function y as the sum of these results.
3) Substitute the function obtained in (2) in the given homogeneous differential equation. Write down your observations and discuss the results.

In order to discuss the solutions to a second-order linear homogeneous differential equation, it is very useful to introduce some terminology.

Two functions are said to be ***linearly dependent*** if one is a ***constant*** multiple of the other, which means that $\frac{y_1(x)}{y_2(x)}=\text{constant}$. Otherwise, they are called ***linearly independent*** i.e. $\frac{y_1(x)}{y_2(x)}\neq\text{constant}$

Thus, $f(x)=e^{2x}$ and $g(x)=3e^{2x}$ are linearly dependent, but $f(x)=e^{2x}$ and $h(x)=xe^{2x}$ are linearly independent.

There is a useful result (known as superposition or linearity principle) which states that "if y_1 and y_2 are two solutions of the homogeneous linear differential equations $a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=0$, then any other linear combination $(y=Ay_1+By_2)$ of these two solutions is also a solution of the equation".

Two solutions y_1 and y_2 are called **basis** of solution of $a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=0$, if y_1 and y_2 are linearly independent. In this case the corresponding general solution is

$y=c_1y_1+c_2y_2$ where c_1 and c_2 are arbitrary constant.

A mathematical theorem enables us to check whether or not a set of given functions is linearly independent. Two differentiable functions $y_1(x)$ and $y_2(x)$ are linearly independent if and only if the determinant called the Wronskian of y_1 and y_2 denoted by $W(y_1(x), y_2(x))$ and is defined as follows:

$$W(y_1,y_2)=\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}=y_1y_2'-y_1'y_2\neq 0 \text{ (is not zero).}$$

Example 4.11

Verify that $y=\cos 2x$ and $y=\sin 2x$ are solutions of $y''+4y=0$ and that their sum is also a solution. Deduce the general solution of $y''+4y=0$.

Solution

First of all, the two functions are linearly independent since

$\begin{vmatrix} \cos 2x & \sin 2x \\ -2\sin 2x & 2\cos 2x \end{vmatrix}=2\cos^2 2x+2\sin^2 2x=2\neq 0$. From $y=\cos 2x$, we have

$$y'=-2\sin 2x, \qquad y''=-4\cos 2x.$$

Hence, $y''+y=(\cos 2x)''+4\cos 2x=-4\cos 2x+4\cos 2x=0$.

Thus, $y=\cos 2x$ is a solution of $y''+4y=0$. Similarly, for $y=\sin 2x$, we have $y''+y=(\sin 2x)''+4\sin 2x=-4\sin 2x+4\sin 2x=0$. Hence, $y=\sin 2x$ is a solution of $y''+4y=0$. Their sum is $y=\cos 2x+\sin 2x$, then $y'=-2\sin 2x+2\cos 2x$ and

$y'' = -4\cos 2x - 4\sin 2x$.

Hence, $y'' + 4y = (\cos 2x + \sin 2x)'' + 4(\cos 2x + \sin 2x)$

$$= -4\cos 2x - 4\sin 2x + 4\cos 2x + 4\sin 2x = 0.$$

Finally, the corresponding general solution is $y = c_1 \cos 2x + c_2 \sin 2x$, where c_1 and c_2 are arbitrary constant.

Application activities 4.10

1) Verify that $y_1 = 1 + \cos x$ and $y_2 = 1 + \sin x$ are solutions of $y'' + y = 1$ but their sum is not a solution. Explain why.

2) Are the following functions linearly independent or depedent ?

a) $\cos^2 x$ and $\sin^2 x$ b) e^{-x} and e^{2x} c) e^{ax} and $5e^{ax}$

d) $5\sin x \cos x$ and $4\sin 2x$ e) $e^{ax}\cos 2x$ and $e^{ax}\sin 2x$

f) $\ln x$ and $\ln\sqrt{x}$ g) e^{ax} and xe^{ax} h) $2\sin^2 x$ and $1-\cos^2 x$

4.6.2. Characteristic equation of a second order differential equation

Activity 4.11

1) Find the solution of the equation $y' - ky = 0, k \text{ is a constant}$

2) Put the solution obtained in 1) in the equation $y'' - 3y' - 4y = 0$ and give the condition so that the solution obtained in 1) is a solution of $y'' - 3y' - 4y = 0$. What can you say about the solution of $y'' - 3y' - 4y = 0$?

From differential equation $a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$, if we put $a = 0$ then we are back to the simple first order linear equation and we know that this has an exponential solution.

This encourages us to try a similar exponential function for the second order equation.

We therefore take a trial solution $y = e^{\lambda x}$ where λ is certain constant parameter to be determined.

Substituting $y' = \lambda e^{\lambda x}$, $y'' = \lambda^2 e^{\lambda x}$ into the equation $a\dfrac{d^2y}{dx^2} + b\dfrac{dy}{dx} + cy = 0$,

we get $\left(a\lambda^2 + b\lambda + c\right)e^{\lambda x} = 0, \Leftrightarrow a\lambda^2 + b\lambda + c = 0$, since $e^{\lambda x} \neq 0$.

So λ satisfies a quadratic equation with the same coefficients as the DE itself.

This equation in λ is called the **auxiliary** or **characteristic** equation (AE) of $a\dfrac{d^2y}{dx^2} + b\dfrac{dy}{dx} + cy = 0$.

Example 4.12

For a given differential equation, write down the characteristic equation, verify by substitution whether the given functions are solutions and deduce the general solution.

a) $y'' - 9y = 0,\ y = e^{3x}\ and\ \ y = e^{-3x}$

b) $y'' + 2y' + 2y = 0,\ y = e^{-x}\cos x\ and\ \ y = e^{-x}\sin x$

Solution

a) We get characteristic equation by considering the order of derivative as the power of λ that is $\lambda^2 = y'', \lambda = y'$ and $\lambda^0 = y$. Thus, characteristic equation of $y'' - 9y = 0$ is $\lambda^2 - 9\lambda^0 = 0$ or $\lambda^2 - 9 = 0$.

For $y = e^{3x}$,

$$y'' - 9y = \left(e^{3x}\right)'' - 9\left(e^{3x}\right) = 3\left(e^{3x}\right)' - 9\left(e^{3x}\right) = 9\left(e^{3x}\right) - 9\left(e^{3x}\right) = 0.$$

Therefore, $y = e^{3x}$ is a solution of $y'' - 9y = 0$.

For $y = e^{-3x}$, we have

$$y'' - 9y = \left(e^{-3x}\right)'' - 9\left(e^{-3x}\right) = -3\left(e^{-3x}\right)' - 9\left(e^{-3x}\right) = 9\left(e^{-3x}\right) - 9\left(e^{-3x}\right) = 0.$$

Thus, $y = e^{3x}$ is a solution of $y'' - 9y = 0$.

As, $y = e^{3x}$ and $y = e^{-3x}$ are linearly independent from superposition principle, the general solution is $y = c_1e^{3x} + c_2e^{-3x}$ where c_1 and c_2 are arbitrary constant.

b) Characteristic equation of $y'' + 2y' + 2y = 0$ is $\lambda^2 + 2\lambda + 2\lambda^0 = 0$ or $\lambda^2 + 2\lambda + 2 = 0$.

For $y=e^{-x}\cos x$, we have

$$
\begin{aligned}
y''+2y'+2y &= \left(e^{-x}\cos x\right)''+2\left(e^{-x}\cos x\right)'+2e^{-x}\cos x \\
&= \left(-e^{-x}\cos x-e^{-x}\sin x\right)'+2\left(-e^{-x}\cos x-e^{-x}\sin x\right)+2e^{-x}\cos x \\
&= \left(-e^{-x}\cos x-e^{-x}\sin x\right)'-2e^{-x}\sin x \\
&= e^{-x}\cos x+e^{-x}\sin x+e^{-x}\sin x-e^{-x}\cos x-2e^{-x}\sin x \\
&= 0
\end{aligned}
$$

Therefore, $y=e^{-x}\cos x$ is a solution of $y''+2y'+2y=0$, similarly, if $y=e^{-x}\sin x$,

we get
$$
\begin{aligned}
y''+2y'+2y &= \left(e^{-x}\sin x\right)''+2\left(e^{-x}\sin x\right)'+2e^{-x}\sin x \\
&= \left(-e^{-x}\sin x+e^{-x}\cos x\right)'+2\left(-e^{-x}\sin x+e^{-x}\cos x\right)+2e^{-x}\sin x \\
&= \left(-e^{-x}\sin x+e^{-x}\cos x\right)'+2e^{-x}\cos x \\
&= e^{-x}\sin x-e^{-x}\cos x-e^{-x}\cos x-e^{-x}\sin x+2e^{-x}\sin x \\
&= 0
\end{aligned}
$$

Hence $y=e^{-x}\sin x$ is a solution of $y''+2y'+2y=0$. In conclusion $y=e^{-x}\cos x$ and $y=e^{-x}\sin x$ are linearly independent.

Since $\dfrac{e^{-x}\cos x}{e^{-x}\sin x}=\cot x\neq$ constant, thus the general solution of $y''+2y'+2y=0$ is $y=c_1e^{-x}\cos x+c_2e^{-x}\sin x$ or $y=e^{-x}(c_1\cos x+c_2\sin x)$.

Application activity 4.11

Verify by substitution whether the given functions are linearly independent; if so, determine the general solution of a corresponding differential equation. Write down and solve the characteristic equation for each ODE.

a) $y''+4y=0,\ y=\cos 2x\ \ and\ \ y=2\sin x\cos x$

b) $y''-2y'+y=0,\ y=e^{x}\ \ and\ \ y=3e^{x}$

c) $4y''+4y'+y=0,\ y=e^{-\frac{x}{2}}\ \ and\ \ y=xe^{-\frac{x}{2}}$

The basic tools that you need in solving **linear homogeneous differential equations** with constant coefficients are simply **solution of quadratic** equations (including complex roots) and **superposition principle.** Since the characteristic equation is quadratic, it may have either two distinct real solutions, or repeated real solutions or two complex solutions (Unit 1).

Therefore, solutions of ODE: $ay'' + by' + cy = 0$ depends on solutions of the characteristic equation: $a\lambda^2 + b\lambda + c = 0$. This is the purpose of the following paragraphs.

4.6.3. Solving DE whose Characteristic equation has two distinct real roots

Activity 4.12

Determine the roots λ_1 and λ_2 of characteristic equation of the following differential equation $y'' + 7y' + 6y = 0$.

From λ_1 and λ_2, review example 4.9 and use superposition principle to find the general solution of $y'' + 7y' + 6y = 0$.

From auxiliary equation of differential equation $ay'' + by' + cy = 0$ having two distinct real roots λ_1 and λ_2, we get $y_1 = e^{\lambda_1 x}$ and $y_2 = e^{\lambda_2 x}$ as the basis of solution of differential equation $ay'' + by' + cy = 0$.

By superposition principle, the corresponding general solution is $y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$.

Example 4.13

1) Solve the following differential equation: $y'' + y' - 2y = 0$

Solution

$y'' + y' - 2y = 0$, the characteristic equation is $r^2 + r - 2 = 0$

Resolution of characteristic equation: simple factorization yields to $(r-1)(r+2) = 0$.

Then, $r_1 = 1$ and $r_2 = -2$ are solutions of the characteristic equation.

Therefore, the general solution is $y = c_1 e^{x} + c_2 e^{-2x}$.

2) Determine the particular solution of $y'' + 3y' + 2y = 0$, for which

$y(0) = y'(0) = 1$ and plot the curve of this solution.

Solution

$y'' + 3y' + 2y = 0$, the characteristic equation is $m^2 + 3m + 2 = 0$. Solving the equation gives $m_1 = -1$ and $m_2 = -2$ as solutions.

The general solution is $y = c_1 e^{-x} + c_2 e^{-2x}$.

Let's apply initial conditions: as $y = c_1 e^{-x} + c_2 e^{-2x}$, $y' = -c_1 e^{-x} - 2c_2 e^{-2x}$.

From $y(0) = y'(0) = 1$, we have $c_1 + c_2 = 1$ and $-c_1 - 2c_2 = 1$.

Solving these simultaneous equations gives $c_1 = 3$ and $c_2 = -2$.

Hence, the particular solution is $y = 3e^{-x} - 2e^{-2x}$.

Graphical representation

x	-1	-0.8	-0.6	-0.4	-0.2	0	0.3	0.6	0.8	1	2	3
y	-6.6	-3.2	-1.2	0.0	0.7	1.0	1.1	1.0	0.9	0.8	0.4	0.1

Figure 4.3: Graphical presentation of $y = 3e^{-x} - 2e^{-2x}$

Application activity 4.12

1) Solve the following differential equations

a) $y'' - 3y' = 0$ b) $y'' - 8y = 0$ c) $y'' + 7y' + 6y = 0$

d) $y'' + y' - 2y = 0$ e) $y'' - 5y' + 6y = 0$ f) $2y'' + 3y' - 2y = 0$

2) Solve the following initial value problem and represent the solution graphically.

a) $y'' + 5y' + 6y = 0, y(0) = 1; y'(0) = -2$

b) $3y'' + 5y' - 2y = 0, \quad y(0) = 2, y'(0) = 3$

4.6.4. Solving DE whose Characteristic equation has a real double root repeated roots

Activity 4.13

Suppose that the auxiliary equation $y'' + py' + qy = 0$ has distinct real roots m and n,

a) Show that the function $f(x) = \dfrac{e^{mx} - e^{nx}}{m-n}$ is a solution of the equation $y'' + py' + qy = 0$.

b) Using Hospital rule, show that $\lim_{m\to n} f(x) = xe^{nx}$ and check if $\lim_{m\to n} f(x)$ is a solution of $y'' + py' + qy = 0$, where $f(x) = \dfrac{e^{mx} - e^{nx}}{m-n}$).

c) Check whether $y = \lim_{m\to n} f(x)$ and $y = e^{nx}$ are linearly independent or not and then deduce the general solution of $y'' - 2y' + y = 0$.

When the roots of auxiliary equation of the differential equation $ay'' + by' + cy = 0$ are equal, that is, $\lambda_1 = \lambda_2 = -\dfrac{b}{2a}$, we obtain only one solution $y_1 = e^{\lambda x}$.

The second linearly independent solution will be $y_2 = xe^{\lambda x}$ such that $\dfrac{xe^{\lambda x}}{e^{\lambda x}} = x \neq cte$; and then, the basis of $ay'' + by' + cy = 0$ is made by $y_1 = e^{\lambda x}$ and $y_2 = xe^{\lambda x}$ and therefore, the corresponding general solution is given by $y = c_1 e^{\lambda x} + c_2 xe^{\lambda x}$.

Example 4.14

1) Find the general solution of $y'' + 4y' + 4y = 0$

Solution

The characteristic equation of $y'' + 4y' + 4y = 0$ is $m^2 + 4m + 4 = 0$,

and $m_1 = m_2 = -2$.

The basis of solution $y'' + 4y' + 4y = 0$ is made by $y_1 = e^{-2x}$ and $y_2 = xe^{-2x}$;

Hence, the general solution is $y = c_1 e^{-2x} + c_2 xe^{-2x}$.

2) Solve the following initial value problem and give its graphical interpretation.

$y'' - 6y' + 9y = 0$, $y(0) = -2$ and $y'(0) = -12$.

Solution

The characteristic equation of $y''-6y'+9y=0$ is $m^2-6m+9=0$, and $m_1=m_2=3$.

The basis of solution of $y''-6y'+9y=0$ is $\left\{y=e^{2x}, y=xe^{2x}\right\}$

and thus its corresponding general solution is $y=c_1e^{3x}+c_2xe^{3x}$.

Considering the system of initial conditions $y(0)=-2$ and $y'(0)=-12$, we obtain $y(0)=-2$ gives $c_1+c_2=-2$; from $y'(0)=-12$ and $y'=3c_1e^{3x}+c_2\left(e^{3x}+3xe^{3x}\right)$ we get $3c_1+c_2=-12$.

Solving the simultaneous equations

$\begin{cases} c_1+c_2=-2 \\ 3c_1+c_2=-12 \end{cases}$, we get $c_1=-5, c_2=3$ and then, the particular solution is

$y=-5e^{3x}+3xe^{3x}$.

Table of values for $y=-5e^{3x}+3xe^{3x}$

x	**-1.5**	**-1**	**-0.5**	**0**	**0.5**	**1.33**	**1.5**
y	-0.1	-0.4	-1.5	-5.0	-15.7	-54.6	-45

Figure 4.4: the graph for $y=-5e^{3x}+3xe^{3x}$

From the graph for $y=-5e^{3x}+3xe^{3x}$, it is clear that the solution for the initial value problem $y''-6y'+9y=0$, $y(0)=-2$ and $y'(0)=-12$ is an exponential function that is strictly decreasing from $-\infty$ to $x=1.33$ and strictly increasing on $]1.33,+\infty[$.

Application activity 4.13

1) Solve the following differential equations

a) $y''+8y'+16=0$ b) $4y''-12y'+9y=0$ c) $4y''+4y'+y=0$

d) $\frac{d^2y}{dx^2}-\frac{1}{3}\frac{dy}{dx}+\frac{1}{36}y=0$ e) $4\frac{d^2y}{dx^2}-4\pi\frac{dy}{dx}+\pi^2y=0$

2) Solve the following initial value problem and represent the solution graphically.

a) $y''+2y'+y=0,\quad y(0)=3,\ y'(0)=-0.5$

b) $10y''+5y'+0.625y=0,\ y(0)=2\ \ and\ \ y'(0)=-4.5$

4.6.5. Solving DE whose characteristic equation has complex roots

Activity 4.14

Assume that the roots m_1 and m_2 of the characteristic equation of the differential equation $y''-y'+5y=0$ are conjugate complex numbers. In this case m_1 and m_2 can be written in the form $a\pm ib$

1) Write down the two independent solutions y_1 and y_2
2) Use polar form of a complex number to write the two solutions.
3) Since the obtained solutions are not real valued functions, find the two real valued functions that form a real basis by suitably combining the two solutions obtained in 2) by the use of **Euler formula**. Hence determine the general solution of the differential equation $y''-y'+5y=0$.

For auxiliary equation having complex roots, the bases of the given differential equation are $y_1=e^{(\alpha+i\beta)x}$ and $y_2=e^{(\alpha-i\beta)x}$ giving general solution $y=e^{\alpha x}\left(c_1e^{i\beta x}+c_2e^{-i\beta x}\right)$

From **Euler's formula**, the basis of real solution are $y_1=e^{\alpha x}\cos\beta x$ and $y_2=e^{\alpha x}\sin\beta x$, and then, the corresponding general solution is $y=e^{\alpha x}\left(A\cos\beta x+B\sin\beta x\right)$.

Example 4.15

1) Find the general solution of $y'' + 2y' + 2y = 0$

Solution

Characteristic equation of $y'' + 2y' + 2y = 0$ is $r^2 - 2r + 2 = 0$; $\Delta = 4 - 8 = -4$ and $\sqrt{\Delta} = 2i$

$r_1 = \frac{2+2i}{2} = 1+i$ and $r_2 = \frac{2-2i}{2} = 1-i$. Here $\alpha = 1$ and $\beta = 1$, then the general solution is $y = e^x(c_1 \cos x + c_2 \sin x)$.

2) Solve the initial value problem $y'' + 4y' + 13y = 0$, $y(0) = 0, y'\left(\frac{\pi}{2}\right) = 1$ and present solution graphically for $x \geq 0$.

Solution

Characteristic equation of $y'' + 4y' + 13y = 0$ is $r^2 + 4r + 13 = 0$; $\Delta = 16 - 52 = -36$

Then, $r_1 = \frac{-4+6i}{2} = -2+3i$, $r_2 = \frac{-4-6i}{2} = -2-3i$. Here $\alpha = -2$ and $\beta = 3$,

Therefore, the corresponding general solution is $y = e^{-2x}(c_1 \cos 3x + c_2 \sin 3x)$.

As $y(0) = 0$ it follows $c_1 = 0$

We have that $y' = -2e^{-2x}(c_1 \cos 3x + i \sin 3x) + e^{-2x}(-3c_1 \sin 3x + 3i \cos 3x)$

Thus, $y'\left(\frac{\pi}{2}\right) = 1 \Leftrightarrow 1 = 2c_2 e^{-\pi} + 3c_1 e^{-\pi} \Leftrightarrow 1 = 2c_2 e^{-\pi} \Leftrightarrow c_2 = \frac{1}{2}e^{\pi}$.Hence, particular solution is $y = e^{-2x}\left(\frac{e^{\pi}}{2}\sin 3x\right)$ or $y = \frac{1}{2}e^{\pi-2x}\sin 3x$.

Physically, the solution represents an oscillation with inconsistent amplitude as it is shown by the figure below.

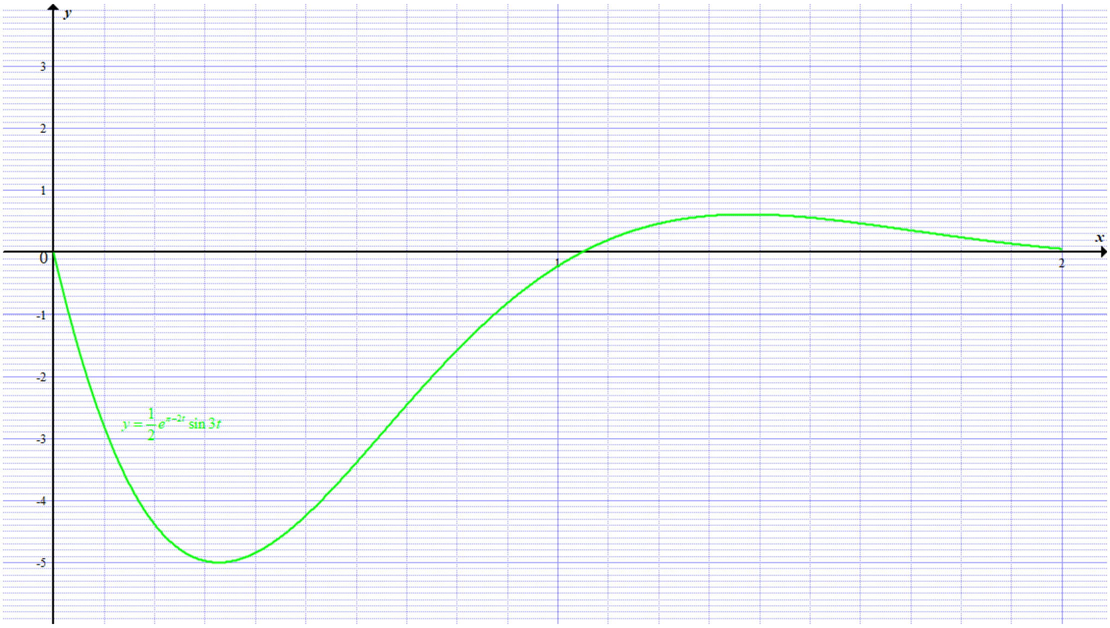


Figure 4.5: Graphical presentation of $y = e^{-2x}\left(\dfrac{e^{\pi}}{2}\sin 3x\right)$

Application activities 4.14

1) Determine the solution of the following equations

a) $y''+25y=0$ b) $y''-4y'+5y=0$ c) $y''+4y'+13y=0$

d) $y''+6y'+11y=0$ e) $y''-2y'+10y=0$ f) $10y''+2y'+1.7y=0$

2) Solve the initial value problem and represent the particular solution graphically.

a) $20y''+4y'+y=0,\ y(0)=2 \ \ and \ \ y'(0)=-4.5$

b) $y''+2y'+2y=0, y(0)=2\, and \ \ y'(0)=-3$

3) In solving second order linear homogeneous differential equation $a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=0$, you get different forms of solution depending on the number of roots of auxiliary equation. Make a summary by completing the following table, taking $y=e^{\alpha x}$ as a solution.

Case	Roots	Basis	General solution
1	Distinct real: λ_1 and λ_2		
2	Real double: λ		
3	Complex conjugate $\alpha \pm i\omega$		

4.7. Applications of second order linear homogeneous differential equation

Activity 4.15

1) Consider the motion of an object with mass m at the end of a spring as illustrated by the figure 4.5.

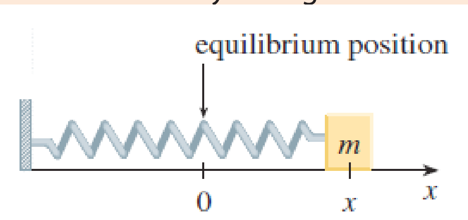


Figure 4.5: non-damped spring
Source: https:// phys.libretexts.org/

If we ignore any external resisting forces (due to air resistance or friction) then, by Newton's Second Law (force equals mass times acceleration), the mass m makes a simple harmonic motion with the equation $m\frac{d^2x}{dt^2}=-kx$ where k is a positive constant called the spring constant.

a) Write the equation of motion in the form of $a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=0$ and deduce the value of *a*, *b* and *c*.

b) Solve $m\frac{d^2x}{dt^2}=-kx$ to determine the general solution.

c)If the mass of 2 kg is moving on a spring with the spring constant k equals 128, determine and illustrate the position $x(t)$ of the mass at any time t given that at $t=0$, it is initially stretched at 0.2m and released with initial velocity 0.

In solving second order linear homogeneous differential equations, we essentially have just three distinct types of functions: $e^{\alpha x}$, $xe^{\alpha x}$ and $e^{\alpha x}\sin\omega x$. Each of these functions has a particular type of behavior for which you can identify the class it belongs to, and the type of physical system it represents:

1) $e^{\alpha x}$ gives an increasing ($\alpha > 0$) or decreasing ($\alpha < 0$) exponential function

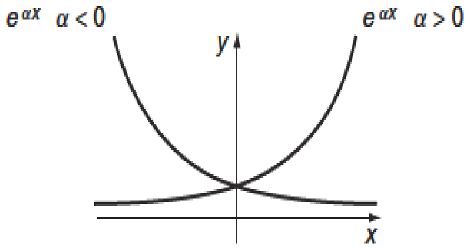


Figure 4.7: Graph of $y=e^{\alpha x}$

2) $xe^{\alpha x}$ gives a similar type of function, and it also passes through the origin.

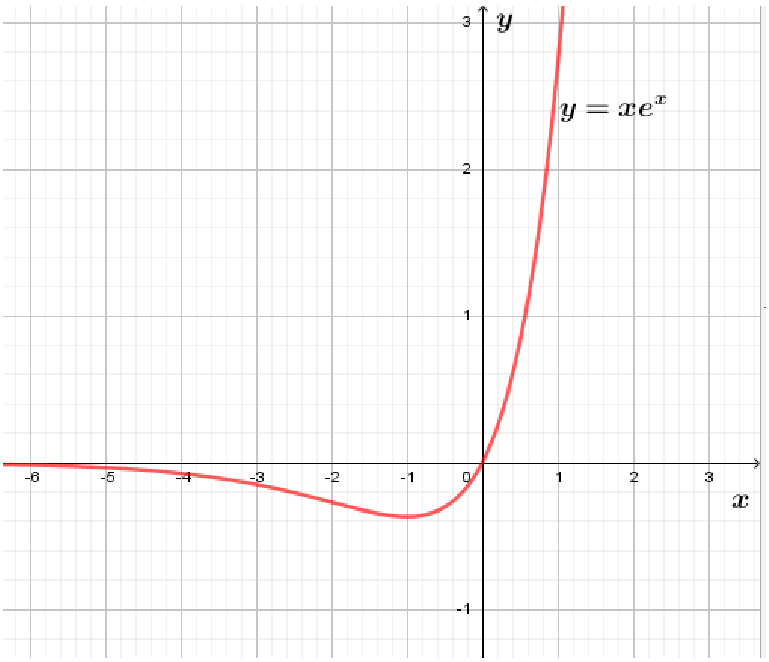


Figure 4.8: Graph of $y=xe^{\alpha x}$

3) $e^{\alpha x}\sin\omega x$ gives either a simple oscillating wave ($\alpha = 0$), or a sinusoidal wave with amplitude that decreases ($\alpha < 0$) or increases ($\alpha > 0$) as x increases.

This could represent an oscillating system in a resisting medium where the sign of the exponent of the exponential function determines the amplitude which is increasing or decreasing with the time as illustrated on the following figures.

Amplitude is increasing with time t

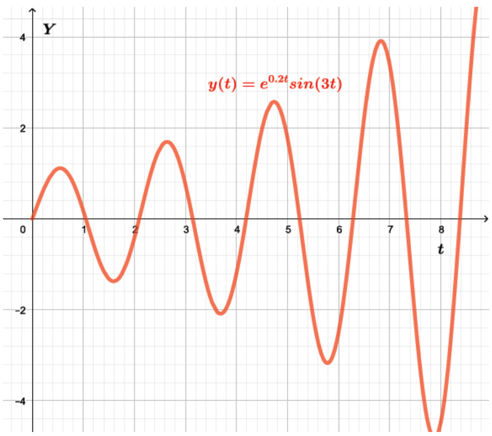


Figure 4.9: Graph of $y = e^{0.2t}\sin(3t)$

Amplitude is decreasing with time t

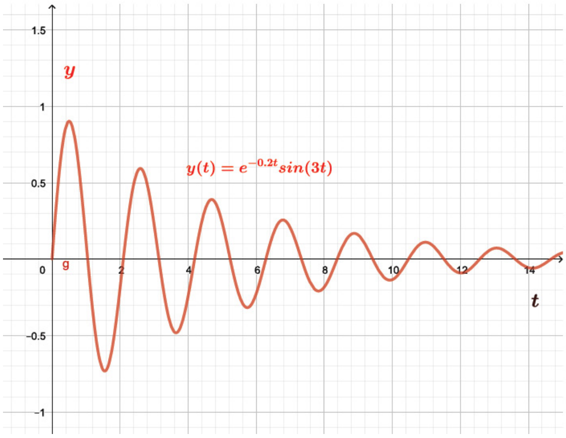


Figure 4.10: Graph of $y = e^{-0.2t}\sin(3t)$

There are many areas of science and engineering where second order linear differential equations provide useful models.

In practice, the amplitude of vibration in **simple harmonic motion** does not remain constant but becomes progressively smaller as the time increases. Such vibration is said to be **damped**.

The differential equations learnt above are used to describe some type of **oscillatory** behavior, with some degree of **damping** as they are illustrated bellow.

Consider the motion of a spring that is subject to a frictional force or a damping force (in the case where a vertical spring moves through a fluid as on the figure 4.11.

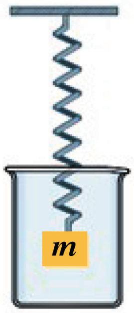


Figure 4.11: spring moves through a fluid
Source: https:// phys.libretexts.org/

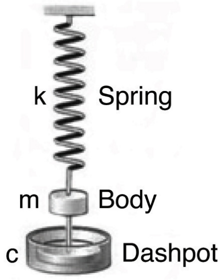


Figure 4.12: the body to a dashpot

Physically this can be done by connecting the body to a dashpot (which acts to resist displacement) as illustrated on the figure 4.12.

An example is the damping force supplied by a shock absorber in a car or a bicycle.

Apart from the restoring force $-kx$, the damping force proportional to the velocity of the mass and acts also in the direction opposite to the motion and equals $-c\frac{dx}{dt}$ where c is a positive constant, called the **damping constant**.

Therefore, Newton's Second Law of motion gives $m\frac{d^2x}{dt^2}=-kx-c\frac{dx}{dt}$ or $m\frac{d^2x}{dt^2}+c\frac{dx}{dt}+kx=0$

This is a second-order linear differential equation; its auxiliary equation is $m\lambda^2+c\lambda+k=0$. Therefore, $\lambda_1=\frac{-c+\sqrt{c^2-4mk}}{2m}$ and $\lambda_2=\frac{-c-\sqrt{c^2-4mk}}{2m}$ .Depending on the value of c^2-4mk, 3 cases can occur:

a) $c^2-4mk>0$, in this case, the spring is said to be over damping

In this case λ_1 and λ_2 are distinct real roots and $x(t)=c_1e^{\lambda_1 t}+c_2e^{\lambda_2 t}$

As $t>0$, both exponents are negative, hence both terms approach zero as $t\to+\infty$,in terms of physics, after a sufficiently long time the mass will be at rest at the static equilibrium position $(y=0)$. This is requiring a strong damping force (high-viscosity oil or grease) compared with a weak spring or small mass.

b) $c^2-4mk=0$, in this case, the spring is said to be in critical damping

It corresponds to equal roots $\lambda_1=\lambda_2=-\frac{c}{2m}$ and the solution is given by $x(t)=(c_1+c_2t)e^{-\frac{c}{2m}t}$

It is similar to the first case, but it is the border case between non-oscillatory motion (over damping) and oscillations; the damping is just sufficient to suppress vibrations.

The solution $x(t)=(c_1+c_2t)e^{-\frac{c}{2m}t}$ can pass through the equilibrium position $(x=0)$ at most once since $e^{-\frac{c}{2m}t}$ is never zero and c_1+c_2t can have at most one root.

c) $c^2 - 4mk < 0$ in this case, the spring is said to be under damping

It is the most interesting case and it occurs if the damping coefficient c is so small that $c^2 < 4mk$

The roots of the auxiliary equation are complex numbers $\lambda_1 = \frac{-c}{2m} + \omega i; \quad \lambda_2 = \frac{-c}{2m} - \omega i$ where $\omega = \frac{\sqrt{4mk - c^2}}{2m}$.

The corresponding general solution is given by $x = e^{-\frac{c}{2m}t}\left(A\cos(\omega t) + B\sin(\omega t)\right)$

This sum can be combined into a phase-shifted cosine, $y = Ce^{-\frac{c}{2m}t}\cos(\omega t - \delta)$, with amplitude $C = A^2 + B^2$ and phase angle $\delta = \arctan\frac{B}{A}$.

Equation $y = Ce^{-\frac{c}{2m}t}\cos(\omega t - \delta)$, is physically more informative since it exhibits the amplitude and phase of oscillation. These oscillations are damped by factor $y = e^{-\frac{c}{2m}t}$ which means graphically that the oscillations are limited by the curves representing $y = Ce^{-\frac{c}{2m}t}$ and $y = -Ce^{-\frac{c}{2m}t}$.

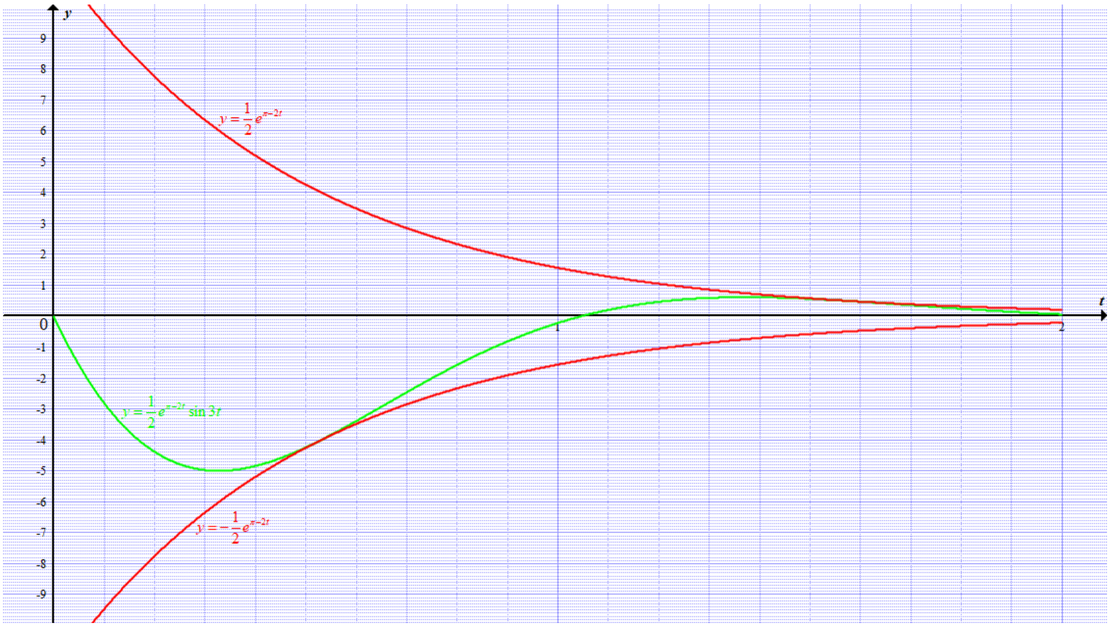


Figure 4.13: comparison of the graph of $y = \frac{1}{2}e^{-2t}$, $y = \frac{1}{2}e^{-2t}\sin 3x$ *and* $y = -\frac{1}{2}e^{-2t}$

Hyperphysics..phy-astr.gsu.edu/hbase/oscda2.html.

Examples 4.16

Solve graphically, the initial value problem

$$\frac{d^2y}{dt^2}+0.4\frac{dy}{dt}+9.04y=0, \qquad y(0)=0,\ y'(0)=3.$$

In the same diagram, plot the graph of the functions $f(t)=e^{-0.2t}$ and $g(t)=-e^{-0.2t}$.

What is your observation?

Solution

Given the ODE $\frac{d^2y}{dt^2}+0.4\frac{dy}{dt}+9.04y=0$ with characteristic equation:

$\lambda^2+0.4\lambda+9.04=0$ and

$\Delta=0.16-36.16=-36$. Thus $\lambda_1=\frac{-0.4+6i}{2}=-0.2+3i$, $\lambda_2=\frac{-0.4-6i}{2}=-0.2-3i$

The corresponding general solution is $y(t)=e^{-0.2t}\left(c_1\cos 3t+c_2\sin 3t\right)$.

From the system of the initial conditions $\begin{cases} y(0)=0 \\ y'(0)=3 \end{cases}$ we get $c_1=0$ and $c_2=1$.

Therefore, $y(t)=e^{-0.2t}\sin 3t$

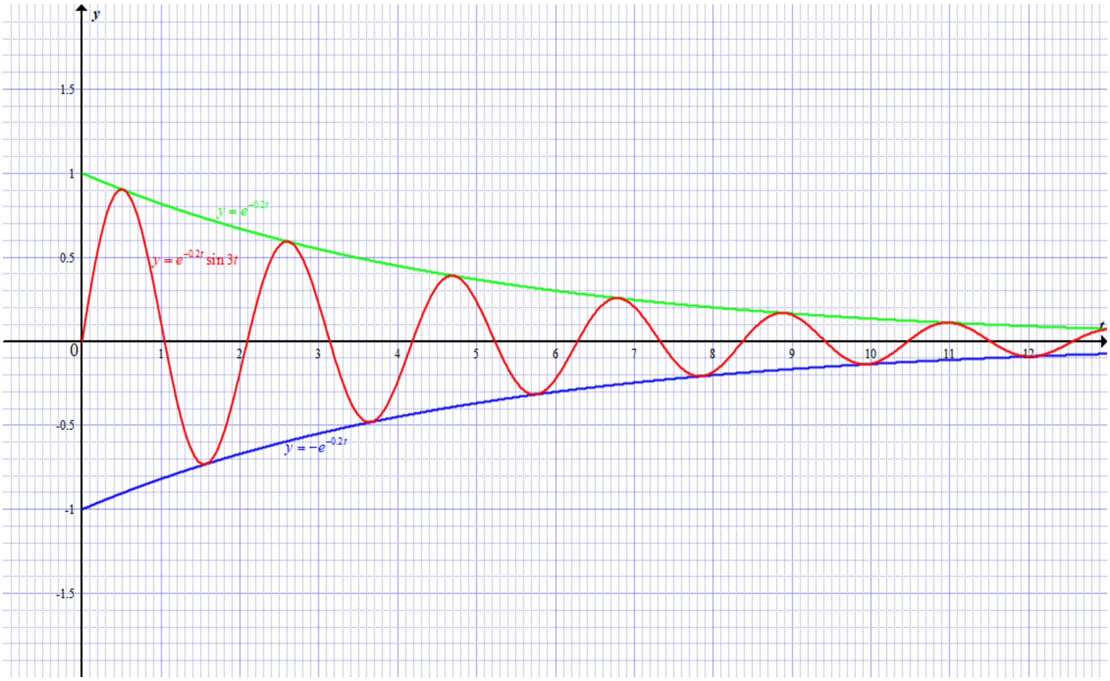


Figure 4.14: Graph of $y(t)=e^{-0.2t}\sin 3t$, $f(t)=e^{-0.2t}$ *and* $g(t)=-e^{-0.2t}$

Application activity 4.15

1) A mass of 2k is oscillating on a spring of constant $k = 128$ in a fluid with damping constant $c = 40$. Find and plot the graph for the position of the mass at any time t if it starts from the equilibrium position and is given a push to start it with an initial velocity of $0.6m/s$.

2) An electric Circuit contains an electromotive force E (supplied by a battery or generator), a resistor R, an inductor L, and a capacitor C in series. If the charge on the capacitor at time is $Q = Q(t)$, then the current is the rate of change of $Q(t)$ with respect to the time t, which means $I = \frac{dQ}{dt}$.

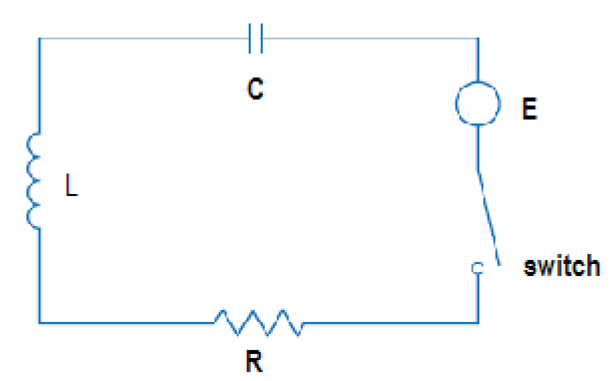


Figure 4.15: Electric circuit

It is known from physics that the voltage drops across the resistor, inductor, and capacitor are $RI, L\frac{dI}{dt}$ and $\frac{Q}{C}$. The Kirchhoff's voltage law says that the sum of these voltage drops is equal to the supplied voltage $E(t)$ and expressed in the following equation $L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{1}{C}Q = E(t)$

a) Write this equation considering that $R = 40$, $L = 1$, $C = 16.10^{-4}$ and

$E(t) = 100\cos 10t$

b) What type of equation obtained if you consider $E(t)$ for $t = \frac{\pi}{20}$?

c) Determine the general solution for the equation obtained in (b).

End unit assessment

QUESTION ONE

Given the differential equation $\frac{d^2y}{dx^2}+\frac{dy}{dx}-6y=0,\ y(0)=10, y'(0)=0$

a) What is the order and the degree of the equation?
b) Solve it to establish the general solution
c) If $y(0)=10, y'(0)=0$, deduce the solution $y(x)$
d) Plot the graph of the solution $y(x)$ and discuss its variation in your own words.

QUESTION TWO

If the demand and supply functions in a competitive market are $Q_d=35-5P$ and $Q_s=-23+6P$ and the rate of adjustment of price when the market is out of equilibrium is $\frac{dp}{dt}=0.2(Q_d-Q_s)$,

a) Solve the relevant differential equation to get a function for P in terms of t given that the price is 100 in time period 0.
b) Plot the graph of P and comment on the stability of this market.

QUESTION THREE

A spring has a mass of 1 kg and its spring constant $k=100$. Given that the spring is released at a point 0.1 m above its equilibrium position with the initial velocity $V=0$,

a) Write the differential equation modeling the movement of that mass.
b) Solve that equation and graph the position function for the mass if the damping constant of the spring $C=25$.
c) What type of damping occurs?

QUESTION FOUR

The police discover the body of a murdered victim at 12 p.m and find the temperature of the body to be 94.6^0F. The body temperature of the victim is then 93.4^0F one hour later and the temperature of the room that stays constant is 70^0F.

a) Write the differential equation that models the temperature of the murdered person.
b) Solve the equation taking into account that T (0) = 94.6^0F and T (1) = 93.4^0F
c) Calculate the time at which the victim was murdered assuming that his body was 98.6^0F when the murder occurred.

QUESTION FIVE

Discuss how this unit inspired you in relation to learning other subjects or to your future. If no inspiration at all, explain why.

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