Subsidiary Mathematics

For Associate Nursing Program

Senior Five

Teacher's Guide

FOREWORD

Dear Teachers,

Rwanda Basic Education Board is honoured to present the teacher's guide for S5 Subsidiary Mathematics in Associate Nursing Program. This book serves as a guide to competence-based teaching and learning to ensure consistency and coherence in the learning of the mathematics content. The Rwandan educational philosophy is to ensure that learners achieve full potential at every level of education which will prepare them to be well integrated in society and exploit employment opportunities.

In line with efforts to improve the quality of education, the government of Rwanda emphasizes the importance of aligning teaching and learning materials with the syllabus to facilitate their learning process. Many factors influence what they learn, how well they learn and the competences they acquire. Those factors include the relevance of the specific content, the quality of teachers' pedagogical approaches, the assessment strategies and the instructional materials.

The ambition to develop a knowledge-based society and the growth of regional and global competition in the jobs market has necessitated the shift to a competence-based curriculum. This book provides active teaching and learning techniques that engage student teachers to develop competences.

In view of this, your role is to:

- Plan your lessons and prepare appropriate teaching materials.
- Organize group discussions for students considering the importance of social constructivism suggesting that learning occurs more effectively when the students works collaboratively with more knowledgeable and experienced people.
- Engage students through active learning methods such as inquiry methods, group discussions, research, investigative activities and group and individual work activities.

- Provide supervised opportunities for students to develop different competences by giving tasks which enhance critical thinking, problem solving, research, creativity and innovation, communication and cooperation.
- Support and facilitate the learning process by valuing students' contributions in the class activities.
- Guide students towards the harmonization of their findings.
- Encourage individual, peer and group evaluation of the work done in the classroom and use appropriate competence-based assessment approaches and methods.

To facilitate you in your teaching activities, the content of this book is self explanatory so that you can easily use it. It is divided in 3 parts:

The part I explain the structure of this book and give you the methodological guidance;

The part II gives a sample lesson plan;

The part III details the teaching guidance for concepts given in the student book.

Even though this teacher's guide contains the guidance on solutions for all activities given in the learner's book, you are requested to work through each question before judging student's findings.

I wish to sincerely express my appreciation to the people who contributed towards the development of this book, particularly, REB staff, Teachers from general education and experts from Local and international Organizations for their technical support.

Dr. MBARUSHIMANA Nelson

Director General, REB

ACKNOWLEDGEMENT

I wish to express my appreciation to the people who played a major role in the development of this teacher's guide for Senior Five Mathematics in Associate Nursing Program. It would not have been successful without active participation of different education stakeholders.

I owe gratitude to teachers whose efforts during the editing exercise of this book were very much valuable.

Finally, my word of gratitude goes to the Rwanda Education Board staffs who were involved in the whole process of in-house textbook production.

Joan MURUNGI

Head of CTLR Department

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PART I. GENERAL INTRODUCTION

1.1. The structure of the guide

The teacher's guide of Mathematics is composed of three parts:

The Part I concerns general introduction that discusses methodological guidance on how best to teach and learn Mathematics, developing competences in teaching and learning, addressing cross-cutting issues in teaching and learning and Guidance on assessment.

Part II presents a sample lesson plan. This lesson plan serves to guide the teacher on how to prepare a lesson in Mathematics.

The Part III is about the structure of a unit and the structure of a lesson. This includes information related to the different components of the unit. This part provides information and guidelines on how to facilitate student while working on learning activities. More other, many application activities from the textbook have answers in this part.

1.2. Methodological guidance

1.2.1 Developing competences

Since 2015 Rwanda shifted from a knowledge based to a competency-based curriculum for pre-primary, primary, secondary education. This called for changing the way of learning by shifting from teacher centred to a learner centred approach. Teachers are not only responsible for knowledge transfer but also for fostering learners' learning achievement and creating safe and supportive learning environment. It implies also that learners have to demonstrate what they are able to transfer the acquired knowledge, skills, values and attitude to new situations.

The competence-based curriculum employs an approach of teaching and learning based on discrete skills rather than dwelling on only knowledge or the cognitive domain of learning. It focuses on what learner can do rather than what learner knows. Learners develop competences through subject unit with specific learning objectives broken down into knowledge, skills and attitudes through learning activities.

In addition to the competences related to Mathematics, student teachers also develop generic competences which should promote the development of the higher order thinking skills and professional skills in Mathematics teaching. Generic competences are developed throughout all units of Mathematics as follows:

Generic competences	**Ways of developing generic competences**
Critical thinking	All activities that require learners to calculate, convert, interpret, analyse, compare and contrast, etc have a common factor of developing critical thinking into learners
Creativity and innovation	All activities that require learners to plot a graph of a given algebraic data, to organize and interpret statistical data collected and to apply skills in solving problems of economics have a common character of developing creativity into learners
Research and problem solving	All activities that require learners to make a research and apply their knowledge to solve problems from the real-life situation have a character of developing research and problem solving into learners.
Communication	During Mathematics class, all activities that require learners to discuss either in groups or in the whole class, present findings, debate …have a common character of developing communication skills into learners.
Co-operation, interpersonal relations and life skills	All activities that require learners to work in pairs or in groups have character of developing cooperation and life skills among learners.
Lifelong learning	All activities that are connected with research have a common character of developing into learners a curiosity of applying the knowledge learnt in a range of situations. The purpose of such kind of activities is for enabling learners to become life-long learners who can adapt to the fast-changing world and the uncertain future by taking initiative to update knowledge and skills with minimum external support.

The generic competences help learners deepen their understanding of Mathematics and apply their knowledge in a range of situations. As students develop generic competences, they also acquire the set of skills that employers look for in their employees, and so the generic competences prepare students for the world of work.

1.2.2 Addressing cross cutting issues

Among the changes brought by the competence-based curriculum is the integration of cross cutting issues as an integral part of the teaching learning process-as they relate to and must be considered within all subjects to be appropriately addressed. The eight cross cutting issues identified in the national curriculum framework are: Comprehensive Sexuality Education, Environment and Sustainability, Financial Education, Genocide studies, Gender, Inclusive Education, Peace and Values Education, and Standardization Culture.

Some cross-cutting issues may seem specific to particular learning areas/subjects but the teacher need to address all of them whenever an opportunity arises. In addition, learners should always be given an opportunity during the learning process to address these cross-cutting issues both within and out of the classroom.

Below are examples of how crosscutting issues can be addressed:

Cross-Cutting Issue	Ways of addressing cross-cutting issues
Comprehensive Sexuality Education: The primary goal of introducing Comprehensive Sexuality Education program in schools is to equip children, adolescents, and young people with knowledge, skills and values in an age appropriate and culturally gender sensitive manner so as to enable them to make responsible choices about their sexual and social relationships, explain and clarify feelings, values and attitudes, and promote and sustain risk reducing behaviour.	Using different charts and their interpretation, Mathematics teachers should lead students to discuss the following situation: "Alcohol abuse and unwanted pregnancies" and advise students on how they can fight those abuses. Some examples can be given when learning statistics, powers, logarithms and their properties.

Environment and Sustainability: Integration of Environment, Climate Change and Sustainability in the curriculum focuses on and advocates for the need to balance economic growth, society well-being and ecological systems. Learners need basic knowledge from the natural sciences, social sciences, and humanities to understand to interpret principles of sustainability.	Using Real life models or students' experience, Mathematics teacher should lead students to illustrate the situation of "population growth" and discuss its effects on the environment and sustainability.
Financial Education: The integration of Financial Education into the curriculum is aimed at a comprehensive Financial Education program as a precondition for achieving financial inclusion targets and improving the financial capability of Rwandans so that they can make appropriate financial decisions that best fit the circumstances of one's life.	Through different examples and calculations on interest rate problems, total revenue and total cost, Mathematics teacher can lead student to discuss how to make appropriate financial decisions.
Gender: At school, gender will be understood as family complementarities, gender roles and responsibilities, the need for gender equality and equity, gender stereotypes, gender sensitivity, etc.	Mathematics teacher should address gender as cross-cutting issue through assigning leading roles in the management of groups to both girls and boys and providing equal opportunity in the lesson participation and avoid any gender stereotype in the whole teaching and learning process.

Inclusive Education: Inclusion is based on the right of all learners to a quality and equitable education that meets their basic learning needs and understands the diversity of backgrounds and abilities as a learning opportunity.	Firstly, Mathematics teachers need to identify/recognize students with special needs. Then by using adapted teaching and learning resources while conducting a lesson and setting appropriate tasks to the level of students, they can cater for students with special education needs. They must create opportunity where student teachers can discuss how to support colleagues with special educational needs.
Peace and Values Education: Peace and Values Education (PVE) is defined as education that promotes social cohesion, positive values, including pluralism and personal responsibility, empathy, critical thinking and action in order to build a more peaceful society.	Through a given lesson, a teacher should: ▪ Set a learning objective which is addressing positive attitudes and values, ▪ Encourage students to develop the culture of tolerance during discussion and to be able to instil it in colleagues and cohabitants; ▪ Encourage students to respect ideas for others.
Standardization Culture: Standardization Culture in Rwanda will be promoted through formal education and plays a vital role in terms of health improvement, economic growth, industrialization, trade and general welfare of the people through the effective implementation of Standardization, Quality Assurance, Metrology and Testing.	With different word problems related to the effective implementation of Standardization, Quality Assurance, Metrology and Testing, students can be motivated to be aware of health improvement, economic growth, industrialization, trade and general welfare of the people.

1.2.3 Guidance on how to help students with special education needs in classroom

In the classroom, students learn in different way depending to their learning pace, needs or any other special problem they might have. However, the teacher has the responsibility to know how to adopt his/her methodologies and approaches in order to meet the learning need of each student in the classroom. Also teachers need to understand that student with special needs, have to be taught differently or need some accommodations to enhance the learning environment. This will be done depending on the subject and the nature of the lesson.

In order to create a well-rounded learning atmosphere, teachers need to:

- Remember that learners learn in different ways so they have to offer a variety of activities (e.g. role-play, music and singing, word games and quizzes, and outdoor activities);
- Maintain an organized classroom and limits distraction. This will help learners with special needs to stay on track during lesson and follow instruction easily;
- Vary the pace of teaching to meet the needs of each child. Some learners process information and learn more slowly than others;
- Break down instructions into smaller, manageable tasks. Learners with special needs often have difficulty understanding long-winded or several instructions at once. It is better to use simple, concrete sentences in order to facilitate them understand what you are asking.
- Use clear consistent language to explain the meaning (and demonstrate or show pictures) if you introduce new words or concepts;
- Make full use of facial expressions, gestures and body language;
- Pair a learner who has a disability with a friend. Let them do things together and learn from each other. Make sure the friend is not over

protective and does not do everything for the one with disability. Both learners will benefit from this strategy;

- Use multi-sensory strategies. As all learners learn in different ways, it is important to make every lesson as multi-sensory as possible. Learners with learning disabilities might have difficulty in one area, while they might excel in another. For example, use both visual and auditory cues.

Below are general strategies related to each main category of disabilities and how to deal with every situation that may arise in the classroom. However, the list is not exhaustive because each child is unique with different needs and that should be handled differently.

Strategy to help learners with developmental impairment:

- Use simple words and sentences when giving instructions;
- Use real objects that learners can feel and handle. Rather than just working abstractly with pen and paper;
- Break a task down into small steps or learning objectives. The learner should start with an activity that she/he can do already before moving on to something that is more difficult;
- Gradually give the learner less help;
- Let the learner with disability work in the same group with those without disability.

Strategy to help learners with visual impairment:

- Help learners to use other senses (hearing, touch, smell and taste) and carry out activities that will promote their learning and development;
- Use simple, clear and consistent language;
- Use tactile objects to help explain a concept;
- If the learner has some sight, ask him/her what he/she can see;
- Make sure the learner has a group of friends who are helpful and who allow him/her to be as independent as possible;

- Plan activities so that learners work in pairs or groups whenever possible;

Strategy to help learners with hearing disabilities or communication difficulties

- Always get the learner's attention before you begin to speak;
- Encourage the learner to look at your face;
- Use gestures, body language and facial expressions;
- Use pictures and objects as much as possible.
- Keep background noise to a minimum.

Strategies to help learners with physical disabilities or mobility difficulties:

- Adapt activities so that learners, who use wheelchairs or other mobility aids, can participate.
- Ask parents/caregivers to assist with adapting furniture e.g. the height of a table may need to be changed to make it easier for a learner to reach it or fit their legs or wheelchair under;
- Encourage peer support when needed;
- Get advice from parents or a health professional about assistive devices if the learner has one.

Adaptation of assessment strategies:

At the end of each unit, the tutor is advised to provide additional activities to help students achieve the key unit competence. These assessment activities are for remedial, consolidation and extension designed to cater for the needs of all categories of students; slow, average and gifted students respectively. Therefore, the tutor is expected to do assessment that fits individual student.

Remedial activities	After evaluation, slow students are provided with lower order thinking activities related to the concepts learnt to facilitate them in their learning. These activities can also be given to assist deepening knowledge acquired through the learning activities for slow students.
Consolidation activities	After introduction of any concept, a range number of activities can be provided to all students to enhance/ reinforce learning.
Extended activities	After evaluation, gifted and talented students can be provided with high order thinking activities related to the concepts learnt to make them think deeply and critically. These activities can be assigned to gifted and talented students to keep them working while other students are getting up to required level of knowledge through the learning activity.

1.2.4. Guidance on assessment

Assessment is an integral part of teaching and learning process. The main purpose of assessment is for improvement of learning outcomes. Assessment for learning/ Continuous/ formative assessment intend to improve students' learning and tutor's teaching whereas assessment of learning/summative assessment intends to improve the entire school's performance and education system in general.

Continuous/ formative assessment

It is an on-going process that arises during the teaching and learning process. It includes lesson evaluation and end of sub unit assessment. This formative assessment should play a big role in teaching and learning process. The teacher should encourage individual, peer and group evaluation of the work done in the classroom and uses appropriate competence-based assessment approaches and methods.

Formative assessment is used to:

- Determine the extent to which learning objectives are being achieved and competences are being acquired and to identify which students need remedial interventions, reinforcement as well as extended activities. The application activities are developed in the learner book and they are designed to be given as remedial, reinforcement, end lesson assessment, homework or assignment
- Motivate students to learn and succeed by encouraging students to read, or learn more, revise, etc.
- Check effectiveness of teaching methods in terms of variety, appropriateness, relevance, or need for new approaches and strategies. Mathematics tutors need to consider various aspects of the instructional process including appropriate language levels, meaningful examples, suitable methods and teaching aids/ materials, etc.
- Help students to take control of their own learning.

In teaching Mathematics, formative or continuous assessment should compare performance against instructional objectives. Formative assessment should measure the student's ability with respect to a criterion or standard. For this reason, it is used to determine what students can do, rather than how much they know.

Summative assessment

The assessment can serve as summative and informative depending to its purpose. The end unit assessment will be considered summative when it is done at end of unit and want to start a new one.

It will be formative assessment, when it is done in order to give information on the progress of learners and from there decide what adjustments need to be done.

The assessment done at the end of the term, end of year, is considered as summative assessment so that the teacher, school and parents are informed of the achievement of educational objective and think of improvement strategies. There is also end of level/ cycle assessment in form of national examinations.

When carrying out assessment?

Assessment should be clearly visible in lesson, unit, term and yearly plans.

- Before learning (diagnostic): At the beginning of a new unit or a section of work; assessment can be organized to find out what students already know / can do, and to check whether the students are at the same level.

- During learning (formative/continuous): When students appear to be having difficulty with some of the work, by using on-going assessment (continuous). The assessment aims at giving students support and feedback.
- After learning (summative): At the end of a section of work or a learning unit, the Mathematics Tutor has to assess after the learning. This is also known as Assessment of Learning to establish and record overall progress of students towards full achievement. Summative assessment in Rwandan schools mainly takes the form of written tests at the end of a learning unit or end of the month, and examinations at the end of a term, school year or cycle.

Instruments used in assessment.

- **Observation:** This is where the Mathematics teacher gathers information by watching students interacting, conversing, working, playing, etc. A teacher can use observations to collect data on behaviours that are difficult to assess by other methods such as attitudes, values, and generic competences and intellectual skills. It is very important because it is used before the lesson begins and throughout the lesson since the teacher has to continue observing each and every activity.
- **Questioning**

(a) Oral questioning: a process which requires a student to respond verbally to questions

(b) Class activities/ exercises: tasks that are given during the learning/ teaching process

(c) Short and informal questions usually asked during a lesson

(d) Homework and assignments: tasks assigned to students by their teacher to be completed outside of class.

Homework assignments, portfolio, project work, interview, debate, science fair, Mathematics projects and Mathematics competitions are also the different forms/instruments of assessment.

1.2.5. Teaching methods and techniques that promote active learning

The different learning styles for students can be catered for, if the teacher uses active learning whereby learners are really engaged in the learning process.

The main teaching methods used in mathematics are the following:

- **Inductive-deductive method:** Inductive method is to move from specific examples to generalization and deductive method is to move from generalization to specific examples.
- **Analytic-synthetic method:** Analytic method proceeds from unknown to known, 'Analysis' means 'breaking up' of the problem in hand so that it ultimately gets connected with something obvious or already known. Synthetic method is the opposite of the analytic method. Here one proceeds from known to unknown.
- **Skills Laboratory method:** Laboratory method is based on the maxim "learning by doing." It is a procedure for stimulating the activities of the students and to encourage them to make discoveries through practical activities.
- **Problem solving method, Project method and Seminar Method.**

The following are some active techniques to be used in Mathematics:

- Group work
- Research
- Probing questions
- Practical activities (drawing, plotting, interpreting graphs)
- Modelling
- Brainstorming
- Quiz Technique
- Discussion Technique
- Scenario building Technique

What is Active learning?

Active learning is a pedagogical approach that engages learners in doing things and thinking about the things they are doing. Learners play the key role in the active learning process. They are not empty vessels to fill but people with ideas, capacity and skills to build on for effective learning. Thus, in active learning, learners are encouraged to bring their own experience and knowledge into the learning process.

The role of the teacher in active learning	**The role of learners in active learning**
• The teacher engages learners through active learning methods such as inquiry methods, group discussions, research, investigative activities, group and individual work activities. • He/she encourages individual, peer and group evaluation of the work done in the classroom and uses appropriate competence-based assessment approaches and methods. • He provides supervised opportunities for learners to develop different competences by giving tasks which enhance critical thinking, problem solving, research, creativity and innovation, communication and cooperation. • Teacher supports and facilitates the learning process by valuing learners' contributions in the class activities.	A learner engaged in active learning: • Communicates and shares relevant information with peers through presentations, discussions, group work and other learner-centred activities (role play, case studies, project work, research and investigation); • Actively participates and takes responsibility for his/her own learning; • Develops knowledge and skills in active ways; • Carries out research/investigation by consulting print/online documents and resourceful people, and presents their findings; • Ensures the effective contribution of each group member in assigned tasks through clear explanation and arguments, critical thinking, responsibility and confidence in public speaking • Draws conclusions based on the findings from the learning activities.

Main steps for a lesson in active learning approach

All the principles and characteristics of the active learning process highlighted above are reflected in steps of a lesson as displayed below. Generally, the lesson is divided into three main parts whereby each one is divided into smaller steps to make sure that learners are involved in the learning process. Below are those main part and their small steps:

1) Introduction

Introduction is a part where the teacher makes connection between the current and previous lesson through appropriate technique. The teacher opens short discussions to encourage learners to think about the previous learning experience and connect it with the current instructional objective. The teacher reviews the prior knowledge, skills and attitudes which have a link with the new concepts to create good foundation and logical sequencings.

2) Development of the new lesson

The development of a lesson that introduces a new concept will go through the following small steps: discovery activities, presentation of learners' findings, exploitation, synthesis/summary and exercises/ application activities.

- **Discovery activity**

Step 1

- The teacher discusses convincingly with learners to take responsibility of their learning
- He/she distributes the task/activity and gives instructions related to the tasks (working in groups, pairs, or individual to instigate collaborative learning, to discover knowledge to be learned).

Step 2

- The teacher let learners work collaboratively on the task;
- During this period the teacher refrains to intervene directly on the knowledge;

  - He/she then monitors how the learners are progressing towards the knowledge to be learned and boosts those who are still behind (but without communicating to them the knowledge).

- **Presentation of learners' findings/productions**
  - In this episode, the teacher invites representatives of groups to present their productions/findings.
  - After three/four or an acceptable number of presentations, the teacher decides to engage the class into exploitation of learners' productions.

- **Exploitation of learner's findings/ productions**
  - The teacher asks learners to evaluate the productions: which ones are correct, incomplete or false
  - Then the teacher judges the logic of the learners' products, corrects those which are false, completes those which are incomplete, and confirms those which are correct.

- **Institutionalization or harmonization (summary/conclusion/ and examples)**
  - The teacher summarizes the learned knowledge and gives examples which illustrate the learned content.

- **Application activities**
  - Exercises of applying processes and products/objects related to learned unit/sub-unit
  - Exercises in real life contexts
  - Teacher guides learners to make the connection of what they learnt to real life situations. At this level, the role of teacher is to monitor the fixation of process and product/object being learned.

3) Assessment

In this step the teacher asks some questions to assess achievement of instructional objective. During assessment activity, learners work individually on the task/activity. The teacher avoids intervening directly. In fact, results from this assessment inform the teacher on next steps for the whole class and individuals. In some cases, the teacher can end with a homework/ assignment. Doing this will allow learners to relay their understanding on the concepts covered that day. Teacher leads them not to wait until the last minute for doing the homework as this often results in an incomplete homework set and/or an incomplete understanding of the concept.

Part II: Sample lesson plan

When teaching any lesson, you can follow the following steps.

Introduction

Start by reviewing previous lesson through asking some questions to learners. If there is no previous lesson, ask them prerequisites related questions for the lesson of the day.

Lesson development

In this step, activities can be more than one (exploration activity, explanation activity and elaboration activity). For each one, give an activity to learners that will be done in groups or individually. After a while, invite one or more groups for presentation of their work to other groups. If the activity is individual, ask one or more learners to present his/her work to others. After activities, capture the main points from the presentation of the learners and guide the whole class to summarize them. After this, provide application activity in their respective groups. Request learners to correct them on chalkboard where you guide every student by addressing eventual misconception.

Evaluation

Give students an activity to be done individually as an assessment. Correct every one and provide related feedback.

Conclusion

Conclude the lesson and remember to assign a home work to students.

This homework may include remedial activities, consolidation activities or extended activities depending on the feedback from the assessment. Sometimes when there is no problem in the assessment, a teacher can provide a homework which will arouse the curiosity of students for the next lesson.

See example of a planned lesson here bellow.

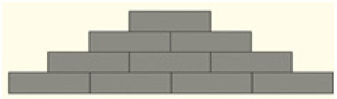
School: ……………… **Academic year:** ………………

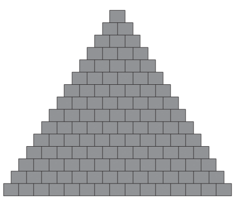
Teacher's name: …………

<table>
<tr><th>Term</th><th>Date</th><th>Subject</th><th>Class</th><th>Unit No</th><th>Lesson No</th><th>Duration</th><th>Class size</th></tr>
<tr><td>1</td><td>……</td><td>Mathematics</td><td>S5 PCB</td><td>2</td><td>1 of 12</td><td>80 minutes</td><td>35</td></tr>
<tr><td colspan="8">Type of Special Educational Needs and number of learners
4 low vision learners: to avail big printed documents and facilitate these learners. Avoid making a group of low vision only otherwise it can be considered as segregation.
Gifted learners: to encourage them to explain, to each other and help their classmates.</td></tr>
<tr><td colspan="2">Unit title</td><td colspan="6">SEQUENCES</td></tr>
<tr><td colspan="2">Key Unit Competence:</td><td colspan="6">Understand, manipulate and use arithmetic, geometric sequences.</td></tr>
<tr><td colspan="2">Title of the lesson</td><td colspan="6">Definition and properties of arithmic sequences</td></tr>
<tr><td colspan="2">Instructional objective</td><td colspan="6">Given a sequence of numbers, learners should be able to determine if it is an arithmetic sequence and find out a common difference of any arithmetic sequence accurately.</td></tr>
<tr><td colspan="2">Plan for this Class</td><td colspan="6">Location: Classroom
Learners are organised into groups and they have to do activity 2.1, from Learner's book page 24, in their groups</td></tr>
<tr><td colspan="2">Learning Materials</td><td colspan="6">Exercise books, pen and ruler</td></tr>
<tr><td colspan="2">References</td><td colspan="6">Learners Book</td></tr>
</table>

1.3. Description of teaching and learning activity

In groups, learners will do the activity 2.1 from learner's book page 24, make presentation of group findings. In conclusion, learners will do questions 1 and 2 of exercise 2.1 from learner's book page 26 in their respective groups and solve them on chalkboard. Learners will do question 3 of exercise 2.1 as individual quiz and questions 4 and 5 will be an assignment. At the end of the lesson learners are also given another assignment to be discussed as an activity of the next lesson "**General term of an arithmetic sequence**".

Timing for each step	Teacher's activities	Learners' activities	Competences and cross cutting issues to be addressed
Introduction 5 minutes	Ask questions on previous lesson.	**Question:** Suppose that you want to build a tower with blocks. a) On a piece of paper, draw that tower starting with 15 blocks for the bottom row until you are not able to add another row. b) How many rows are there? c) Write down the number of blocks that are in each row (from bottom row to the top row). d) In the numbers written down each number can be found by adding a constant number to the previous. Refer to the similar picture bellow and guess that constant number 	Students are developing communication skills when they are explaining and sharing ideas.

Body of the lesson **15 minutes**	**Step 1:** Form groups • Request the learners to do activity 2.1 from learner's book page 24 in their groups • Goes round to check the progress of the discussion, and intervenes where necessary. • Guides learners with special educational needs on how to do activity.	In their groups, learners will do activity 2.1. On a piece of paper, they will draw a tower with blocks starting with 15 blocks for the bottom row until they will not be able to add another row. • Reporter represents the work. • Learners interact through questions and comments.	• Cooperation and interpersonal management developed through working in groups. • Communication: learners communicate and convey information and ideas through speaking when they are presenting their work. • Self confidence: learners will gain self confidence competence when they are presenting their work.
10 minutes	**Step 2:** Request a reporter from each group to present the work on the chalkboard.	**Answers** a)  b) There are 15 rows c) 15, 14, 13, 12, 11, 10, 9, 8, 7, 6, 5, 4, 3, 2, 1 d) Observing this sequence the number that is added to the number to obtain the next is -1.	• In group activities, the fact of being convinced without fighting, peace and education values are developed too.

Conclusion 10 minutes	Request learners to give the main points of the learned lesson in summary.	Summarise the learned lesson: Sequences of numbers that follow a pattern of adding a fixed number, called common difference d, from one term to the next are called arithmetic sequences or arithmetic progressions. That is $d = u_{n+1} - u_n$ or $d = u_n - u_{n-1}$. For an arithmetic sequence u_{n-1}, u_n, u_{n+1}, we have $2u_n = u_{n-1} + u_{n+1}$.	• Learners develop critical thinking through generating a summary.
15 minutes	• Request learners to do questions 1 and 2 of exercise 2.1 in their respective groups • Goes round to check the progress of the discussion, and intervenes where necessary	Do questions 1 and 2 of exercise 2.1, from learner's book page 26, in their respective groups.	• Through group activities, cooperation is developed.

15 minutes	• Request some learners to answer to questions 1 and 2 of exercises 2.1 on chalkboard. • Ensures that the learners understood the learned lesson and decide to repeat the lesson or to continue with new lesson next time	Do questions 1 and 2 of exercise 2.1, from learner's book page 26, on chalkboard.	• Through presentation on chalkboard, communication skills are developped
10 minutes	Do question 3 of exercise 2.1, from learner's book page 26, as individual quiz.	Give to the learners an individual evaluation (quiz) and homework to the learned lesson. **Lead into next lesson** Request learners to do activity 2.2 at home.	
Teacher self evaluation	Even if the objective has been achieved, some learners had not the instruments of geometry. The time management has been disturbed by the fact of borrowing materials from their classmates. For this reason, next time each learner must have his/her own materials (instrument of geometry, calculator, ...)		

PART III: UNIT DEVELOPMENT

Unit 1 Trigonometric Formulae and Equations

1.1. Key unit competence

Solve trigonometric equations and related real-life problems.

1.2. Objectives

By the end of this unit, learners should be able to:

- Solve trigonometric equations
- Use trigonometric formulae and equations in real life

1.3. List of lessons for unit 1

Week	Lessons	Content	Number of Periods
1	1	Recall on trigonometric formulae	1
	2	Addition and subtraction formulae;	1
	3	Double angle formulae	1
2	4	Half-angle formulae	1
	5	Transformation of product in sum and difference	1
	6	Transformation of sum in product	1
3	7	Trigonometric equation reducible to the form $\sin(x+\alpha)=k, \cos(x+\alpha)=k$ and $\tan(x+\alpha)=b$ for $\lvert k\rvert \le 1$ and $b \in IR$	2
	8	Trigonometric equation reducible to the form $\sin nx = k$	1
4-5	9	Trigonometric equation of the form $a\sin x + b\cos x = c$	1
	10	Applications of trigonometry: Simple harmonic motion in physics, Refraction of light, Medicine	3
	11	End unit assessment	1
	12	Remediation	1

1.4. Materials

Exercise books, pens, instrument of geometry, calculators

1.5. Guidance on the introductory activity

- Put learners in groups.
- Let them read and do the introductory activity in the Learner's book making sure that all learners are participating; help them where necessary
- Let learners present the solution of the activity
- Through class discussions, let learners give different ways of application of trigonometry in real life such as construction, satellite systems and astronomy, naval and aviation industries, land surveying, in cartography (creation of maps), medicine and so on.

Solution of the introductory activity

The height of the cathedral=485m

The base of the triangle =280m

$$\tan\theta = \frac{opposite\ side}{adjacent\ side} = 1.732142857142857$$

$$\theta = \tan^{-1} 1.73 = 60.001318460 \text{ degrees}$$

1.6. Content and activities

1.6.1 Trigonometric formulae

a) Content summary

- Addition and subtraction formulae (compound formulae)

$$\cos(x+y) = \cos x\cos y - \sin x\sin y$$

$$\cos(x-y) = \cos x\cos y + \sin x\sin y$$

$$\sin(x+y) = \sin x\cos y + \cos x\sin y$$

$$\sin(x-y) = \sin x\cos y - \cos x\sin y$$

$$\tan(x+y)=\frac{\tan x+\tan y}{1-\tan x\tan y}$$

$$\tan(x-y)=\frac{\tan x-\tan y}{1+\tan x\tan y}$$

$$\cot(x+y)=\frac{\cot x\cot y-1}{\cot y+\cot x}$$

$$\cot(x-y)=\frac{\cot x\cot y+1}{\cot y-\cot x}$$

- Double angles

$\cos^2 x=1-\sin^2 x$ and $\sin^2 x=1-\cos^2 x$

$$\sin 2x=2\sin x\cos x$$

$$\cos 2x=\cos^2 x-\sin^2 x$$

$$\tan 2x=\frac{2\tan x}{1-\tan^2 x}$$

$$\cot 2x=\frac{\cot^2 x-1}{2\cot x}$$

- Half angle formulae

$$\sin\frac{x}{2}=\pm\sqrt{\frac{1-\cos x}{2}}$$

$$\cos\frac{x}{2}=\pm\sqrt{\frac{\cos x+1}{2}}$$

$\tan\frac{x}{2}=\frac{1-\cos x}{\sin x}$ or $\tan\frac{x}{2}=\frac{\sin x}{1+\cos x}$

- Transformation of product in sum

$$\cos x\cos y=\frac{1}{2}\left[\cos(x+y)+\cos(x-y)\right]$$

$$\sin x\sin y=-\frac{1}{2}\left[\cos(x+y)-\cos(x-y)\right]$$

$$\sin x\cos y=\frac{1}{2}\left[\sin(x+y)+\sin(x-y)\right]$$

$$\cos x\sin y=\frac{1}{2}\left[\sin(x+y)-\sin(x-y)\right]$$

- Transformation of sum in product formulae

$$\cos p + \cos q = 2\cos\frac{p+q}{2}\cos\frac{p-q}{2}$$

$$\cos p - \cos q = -2\sin\frac{p+q}{2}\sin\frac{p-q}{2}$$

$$\sin p + \sin q = 2\sin\frac{p+q}{2}\cos\frac{p-q}{2}$$

$$\sin p - \sin q = 2\cos\frac{p+q}{2}\sin\frac{p-q}{2}$$

- t- Formulae

If $t = \tan\frac{A}{2}$, then $\sin\text{A} = \frac{2t}{1+t^2}, \cos A = \frac{1-t^2}{1+t^2}, \tan A = \frac{2t}{1-t^2}$

b) Teaching guidelines

- Organise class into groups. Request each group to have a group leader who will present their findings to the class.
- Request each group to do activity related to the lesson they are going to learn. The learners may need your assistance to do any activity. Help them to understand the activity. After group discussion, invite some or all groups for presentation of their work.
- After activity presentation, capture the main points from the presentation of the learners and summarise them. Guide the learners through given examples in Learner's book or through your own examples.
- Ask learners what they learned in day lesson to ensure that they understood what they have learned.
- Request learners to do exercises in their respective groups. Request learners to correct exercises on chalkboard and give them individual evaluation. Remember to give them homework.

c) Answers to activities

Activity 1.1

Materials

Exercise books, pens

Answers

a) $\sin(A+B)=\dfrac{PR}{OP}$

$$=\frac{PQ+QR}{OP}$$

$$=\frac{PQ+TS}{OP}$$

$$=\frac{PQ}{OP}+\frac{TS}{OP}$$

$$=\left(\frac{PQ}{PT}\times\frac{PT}{OP}\right)+\left(\frac{TS}{OT}\times\frac{OT}{OP}\right)$$

$$=\cos A\sin B+\sin A\cos B$$

So, $\sin(A+B)=\sin A\cos B+\cos A\sin B$.

b) Similarly, $\cos(A+B)=\dfrac{OR}{OP}$

$$=\frac{OS-RS}{OP}$$

$$=\frac{OS-QT}{OP}$$

$$=\frac{OS}{OP}-\frac{QT}{OP}$$

$$=\left(\frac{OS}{OT}\times\frac{OT}{OP}\right)-\left(\frac{QT}{PT}\times\frac{PT}{OP}\right)$$

$$=\cos A\ \cos B\text{-}\sin A\ \sin B$$

Thus, $\cos(A+B)=\cos A\cos B-\sin A\sin B$.

- $\tan(A+B)=\dfrac{\sin(A+B)}{\cos(A+B)}$

  From the identities for $\sin(A+B)$ and $\cos(A+B)$, you have $\tan(A+B)=\dfrac{\sin A\cos B+\cos A\sin B}{\cos A\cos B-\sin A\sin B}$

$$=\frac{\dfrac{\sin A}{\cos A}+\dfrac{\sin B}{\cos B}}{1-\dfrac{\sin A}{\cos A}\dfrac{\sin B}{\cos B}}$$

$$=\frac{\dfrac{\sin A\cancel{\cos B}}{\cos A\cancel{\cos B}}+\dfrac{\cancel{\cos A}\sin B}{\cancel{\cos A}\cos B}}{\dfrac{\cancel{\cos A}\,\cancel{\cos B}}{\cancel{\cos A}\,\cancel{\cos B}}-\dfrac{\sin A\sin B}{\cos A\cos B}}$$

$$=\frac{\tan A+\tan B}{1-\tan A\tan B}$$

  So, $\tan(A+B)=\dfrac{\tan A+\tan B}{1-\tan A\tan B}$.

- Replacing B by $-B$ in the identity for $\sin(A+B)$ gives

  $\sin(A-B)=\sin A\cos(-B)+\cos A\sin(-B)$

  Or $\sin(A-B)=\sin A\cos B-\cos A\sin B$

- Replacing B by $-B$ in the identity for $\cos(A+B)$ gives

  $\cos(A+B)=\cos A\cos(-B)-\sin A\sin(-B)$.

  Thus,

  $\cos(A-B)=\cos A\cos B+\sin A\sin B$

- Replacing B by $-B$ in the identity for $\tan(A+B)$ yields

  $\tan(A-B)=\dfrac{\tan A+\tan(-B)}{1-\tan A\tan(-B)}$

  Hence, $\tan(A-B)=\dfrac{\tan A-\tan B}{1+\tan A\tan B}$.

Application activity 1.1

1. a) $\frac{\sqrt{6}+\sqrt{2}}{4}$ b) $2+\sqrt{3}$

   c) $-\left(2+\sqrt{3}\right)$ d) $-\left(\sqrt{6}+\sqrt{2}\right)$

2. $\frac{1}{2}$

3. a) $\frac{\sqrt{6}-\sqrt{2}}{4}$ b) $\frac{\sqrt{6}+\sqrt{2}}{4}$

   c) $2-\sqrt{3}$ d) $2+\sqrt{3}$

4. a) $\frac{56}{33}$ b) $\frac{63}{16}$

Activity 1.2

Materials

Exercise books, pens

Answers

1. $\cos\left(x+x\right)=\cos x\cos x-\sin x\sin x$

   $\Rightarrow \cos 2x=\cos^2 x-\sin^2 x$

2. $\cos\left(x-x\right)=\cos x\cos x+\sin x\sin x$

   $\Leftrightarrow \cos 0=\cos^2 x+\sin^2 x$

   $\Leftrightarrow 1=\cos^2 x+\sin^2 x$

   $\Rightarrow \cos^2 x+\sin^2 x=1$

3. $\sin\left(x+x\right)=\sin x\cos x+\cos x\sin x$

   $\Rightarrow \sin 2x=2\sin x\cos x$

4. $\tan\left(x+x\right)=\frac{\tan x+\tan x}{1-\tan x\tan x}$

   $\Rightarrow \tan 2x=\frac{2\tan x}{1-\tan^2 x}$

5. $\cot\left(x+x\right)=\frac{\cot x\cot x-1}{\cot x+\cot x}$

   $\Rightarrow \cot 2x=\frac{\cot^2 x-1}{2\cot x}$

Application activity 1.2

1. $4\sin x\cos^3 x-4\cos x\sin^3 x$
2. $\cos^8 x+\sin^8 x-28\cos^2 x\sin^6 x+70\cos^4 x\sin^4 x-28\cos^6 x\sin^2 x$
3. a) $\frac{1}{2}$ b) $\frac{\sqrt{2}}{2}$ c) $\frac{\sqrt{2}}{4}$ d) $\frac{\sqrt{3}}{2}$

   a) $\sin 2x=\frac{2\tan x}{\tan^2 x+1}$

   b) $\cos 2x=\frac{1-\tan^2 x}{1+\tan^2 x}$

4. c) $\tan 2x=\frac{2\tan x}{1-\tan^2 x}$

Activity 1.3

Materials

Exercise books, pens

Answers

From the double angle formulae, you have

1. $\cos 2x=\cos^2 x-\sin^2 x$

   $=\left(1-\sin^2 x\right)-\sin^2 x$ from $\cos^2 x+\sin^2 x=1$

   $=1-2\sin^2 x$

   So, $\cos 2x=1-2\sin^2 x$

   Letting $\theta=2x$, $\cos 2x=1-2\sin^2 x$ gives

   $\cos\theta=1-2\sin^2\frac{\theta}{2}$

   Or $2\sin^2\frac{\theta}{2}=1-\cos\theta\Rightarrow \sin\frac{\theta}{2}=\pm\sqrt{\frac{1-\cos\theta}{2}}$

   So, $\sin\frac{\theta}{2}=\pm\sqrt{\frac{1-\cos\theta}{2}}$

2. $\cos 2x=\cos^2 x-\sin^2 x$

   $=\cos^2 x-\left(1-\cos^2 x\right)$ from $\cos^2 x+\sin^2 x=1$

   $=2\cos^2 x-1$

   So, $\cos 2x=2\cos^2 x-1$

   Letting $\theta=2x$, $\cos 2x=2\cos^2 x-1$ gives

   $\cos\theta=2\cos^2\frac{\theta}{2}-1\Leftrightarrow 2\cos^2\frac{\theta}{2}=1+\cos\theta$

$$\cos^2\frac{\theta}{2}=\frac{1}{2}(1+\cos\theta)\Rightarrow\cos\frac{\theta}{2}=\pm\sqrt{\frac{1+\cos\theta}{2}}$$

Thus, $\cos\frac{\theta}{2}=\pm\sqrt{\frac{1+\cos\theta}{2}}$

3. $\tan\frac{\theta}{2}=\dfrac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}}$

$$\Leftrightarrow\tan\frac{\theta}{2}=\frac{\pm\sqrt{\frac{1-\cos\theta}{2}}}{\pm\sqrt{\frac{1+\cos\theta}{2}}}\quad\Leftrightarrow\tan\frac{\theta}{2}=\frac{\sqrt{1-\cos\theta}}{\sqrt{1+\cos\theta}}$$

By rationalizing denominator, you get

$$\tan\frac{\theta}{2}=\frac{\sqrt{1-\cos\theta}}{\sqrt{1+\cos\theta}}\frac{\sqrt{1-\cos\theta}}{\sqrt{1-\cos\theta}}$$

$$\Leftrightarrow\tan\frac{\theta}{2}=\frac{\sqrt{(1-\cos\theta)^2}}{\sqrt{1-\cos^2\theta}}\qquad\Leftrightarrow\tan\frac{\theta}{2}=\frac{|1-\cos\theta|}{\sqrt{1-\cos^2\theta}}$$

$$\Leftrightarrow\tan\frac{\theta}{2}=\frac{\pm(1-\cos\theta)}{\sqrt{1-\cos^2\theta}}\qquad\Leftrightarrow\tan\frac{\theta}{2}=\frac{\pm(1-\cos\theta)}{|\sin\theta|}$$

$$\Leftrightarrow\tan\frac{\theta}{2}=\frac{1-\cos\theta}{\sin\theta}$$

So, $\tan\frac{\theta}{2}=\frac{1-\cos\theta}{\sin\theta}$

From $\tan\frac{\theta}{2}=\frac{\sqrt{1-\cos\theta}}{\sqrt{1+\cos\theta}}$, conjugating numerator, you get

$$\tan\frac{\theta}{2}=\frac{\sqrt{1-\cos\theta}}{\sqrt{1+\cos\theta}}\frac{\sqrt{1+\cos\theta}}{\sqrt{1+\cos\theta}}\qquad\Leftrightarrow\tan\frac{\theta}{2}=\frac{\sqrt{1-\cos^2\theta}}{\sqrt{(1+\cos\theta)^2}}$$

$$\Leftrightarrow\tan\frac{\theta}{2}=\frac{\sqrt{\sin^2\theta}}{\sqrt{(1+\cos\theta)^2}}\qquad\Leftrightarrow\tan\frac{\theta}{2}=\frac{|\sin\theta|}{|1+\cos\theta|}$$

$\Leftrightarrow \tan\frac{\theta}{2} = \frac{\sin\theta}{1+\cos\theta}$

So $\tan\frac{\theta}{2} = \frac{\sin\theta}{1+\cos\theta}$

In fact, $\tan\frac{\theta}{2} = \frac{1-\cos\theta}{\sin\theta}$ or $\tan\frac{\theta}{2} = \frac{\sin\theta}{1+\cos\theta}$

Application activity 1.3

1. $\pm\frac{\sqrt{6}}{3}, \pm\frac{\sqrt{3}}{3}$
2. If $\tan 2A = \frac{7}{24}$, $0 < A < \frac{\pi}{4}$, to find $\tan A$

$\tan 2A = \frac{2\tan A}{1-tan^2 A}$

$\Rightarrow \frac{7}{24} = \frac{2\tan A}{1-tan^2 A}$

$\Rightarrow 7 - 7tan^2 A = 48\tan A$

$\Rightarrow 7tan^2 A - 48\tan A - 7 = 0$

$\Rightarrow (7\tan A - 1)(\tan A + 7) = 0$

$\Rightarrow \tan A = \frac{1}{7}$ since $\tan A = 7$ is impossible for $0 < A < \frac{\pi}{4}$

3. $\pm\frac{\sqrt{7}}{4}, \pm\frac{3}{4}, \pm\frac{\sqrt{7}}{3}$

Activity 1.4

Materials

Exercise books, pens

Answers

1. $\sin(x+y) + \sin(x-y) = \sin x\cos y + \cancel{\cos x\sin y} + \sin x\cos y - \cancel{\cos x\sin y}$

$= 2\sin x\cos y$

2. $\sin(x+y)-\sin(x-y)=\cancel{\sin x\cos y}+\cos x\sin y-(\sin x\cos y-\cos x\sin y)$

$=\cancel{\sin x\cos y}+\cos x\sin y\cancel{-\sin x\cos y}+\cos x\sin y$

$=2\cos x\sin y$

3. $\cos(x+y)+\cos(x-y)=\cos x\cos y\cancel{-\sin x\sin y}+\cos x\cos y\cancel{+\sin x\sin y}$

$=2\cos x\cos y$

4. $\cos(x+y)-\cos(x-y)=\cos x\cos y-\sin x\sin y-(\cos x\cos y+\sin x\sin y)$

$=\cancel{\cos x\cos y}-\sin x\sin y\cancel{-\cos x\cos y}-\sin x\sin y$

$=-2\sin x\sin y$

Application activity 1.4

1. $\sin x\cos 3x=\frac{1}{2}(\sin 4x-\sin 2x)$
2. $\cos 12x\sin 9x=\frac{1}{2}(\sin 21x-\sin 3x)$
3. $2\cos\frac{5x}{2}\cos\frac{3x}{2}=\cos 4x+\cos x$

Activity 1.5

Materials

Exercise books, pens

Answers

The formulae for transforming product in sum are

$\cos x\cos y=\frac{1}{2}[\cos(x+y)+\cos(x-y)]$ (Equation 1)

$\sin x\sin y=-\frac{1}{2}[\cos(x+y)-\cos(x-y)]$ (Equation 2)

$\sin x\cos y=\frac{1}{2}[\sin(x+y)+\sin(x-y)]$ (Equation 3)

$\cos x\sin y=\frac{1}{2}[\sin(x+y)-\sin(x-y)]$ (Equation 4)

$$\begin{cases} x+y=p \\ x-y=q \end{cases} \Rightarrow \begin{cases} x=\dfrac{p+q}{2} \\ y=\dfrac{p-q}{2} \end{cases} \quad \text{(i)}$$

From equation (equation 1) and (i), you get

$$\cos\frac{p+q}{2}\cos\frac{p-q}{2}=\frac{1}{2}(\cos p+\cos q)$$

From equation (equation 2) and (i), you get

$$\sin\frac{p+q}{2}\sin\frac{p-q}{2}=-\frac{1}{2}(\cos p-\cos q)$$

From equation (equation 3) and (i), you get

$$\sin\frac{p+q}{2}\cos\frac{p-q}{2}=\frac{1}{2}(\sin p+\sin q)$$

From equation (equation 4) and (i), you get

$$\cos\frac{p+q}{2}\sin\frac{p-q}{2}=\frac{1}{2}(\sin p-\sin q)$$

Application activity 1.5

1. $\cos x+\cos 7x=2\cos 4x\cos 3x$
2. $\sin 4x-\sin 9x=-2\cos\frac{13x}{2}\sin\frac{5x}{2}$
3. $\sin 3x+\sin 4x=2\sin\frac{7x}{2}\cos\frac{x}{2}$

1.6.2 Trigonometric equations

a) Content summary

The solutions of a trigonometric equation for which $0\le x\le 2\pi$ are called **principle solutions** while the expression (involving integer k) of solution containing all values of the unknown angle is called the **general solution** of the trigonometric equation. When the interval of solution is not given, you are required to find general solution.

When solving trigonometric equation, note that

- general solution for
  1. $\sin x = 0$ is $x = k\pi, k \in \mathbb{Z}$
  2. $\cos x = 0$ is $x = (2k+1)\frac{\pi}{2}, k \in \mathbb{Z}$
  3. $\tan x = 0$ is $x = k\pi, k \in \mathbb{Z}$
  4. all angles having the same sine i.e. $\sin x = \sin\theta$ is $x = (-1)^k \theta + k\pi, k \in \mathbb{Z}$
  5. all angles having the same cosine i.e. $\cos x = \cos\theta$ is $x = \pm\theta + 2k\pi, k \in \mathbb{Z}$
  6. all angles having the same tangent i.e. $\tan x = \tan\theta$ is $x = \theta + k\pi, k \in \mathbb{Z}$
- The sum or difference of trigonometric functions containing unknown are transformed into the sum.

  Remember that

$$\cos p + \cos q = 2\cos\frac{p+q}{2}\cos\frac{p-q}{2}$$

$$\cos p - \cos q = -2\sin\frac{p+q}{2}\sin\frac{p-q}{2}$$

$$\sin p + \sin q = 2\sin\frac{p+q}{2}\cos\frac{p-q}{2}$$

$$\sin p - \sin q = 2\cos\frac{p+q}{2}\sin\frac{p-q}{2}$$

- To find the general solution of the equation of the form $a\sin x + b\cos x = c$ where $a, b, c \in \mathbb{Z}$ such that $|c| \le \sqrt{a^2+b^2}$

  Using t-formula $\sin x = \frac{2t}{1+t^2}, \cos x = \frac{1-t^2}{1+t^2}$, where $t = \tan\frac{x}{2}$, gives

$$a\frac{2t}{1+t^2} + b\frac{1-t^2}{1+t^2} = c \Rightarrow 2at + b - bt^2 = c(1+t^2)$$

$$\Leftrightarrow (b+c)t^2 - 2at + c - b = 0$$

  which is quadratic equation in t.

  Remember that $t = \tan\frac{x}{2}$.

Alternative method: in $a\sin x + b\cos x = c$

a) Divide each term by $\sqrt{a^2+b^2}$ and convert it in the form

$$\frac{a}{\sqrt{a^2+b^2}}\sin x+\frac{b}{\sqrt{a^2+b^2}}\cos x=\frac{c}{\sqrt{a^2+b^2}}$$

b) Let $\tan\theta=\frac{b}{a}$, then $\sin\theta=\frac{b}{\sqrt{a^2+b^2}},\cos\theta=\frac{a}{\sqrt{a^2+b^2}}$.

c) The given equation reduces to the form

$r\cos\theta\cos x+r\sin\theta\sin x=c$

or $\cos\theta\cos x+\sin\theta\sin x=\frac{c}{r}$

d) Then, $\cos(x-\theta)=\cos\alpha$, where $\cos\alpha=\frac{c}{r}=\frac{c}{\sqrt{a^2+b^2}}$

Therefore $x=\pm\alpha+\theta+2k\pi,k\in\mathbb{Z}$

Notice:

In solving the trigonometric equation, it is helpful to remember the following identities:

$\sin\alpha=\sin(\alpha+2k\pi),\ k\in\mathbb{Z}$	$\sin\alpha=\sin(\pi-\alpha)$
$\cos\alpha=\cos(\alpha+2k\pi),\ k\in\mathbb{Z}$	$\cos\alpha=\cos(-\alpha)$
$\tan\alpha=\tan(\alpha+k\pi),\ k\in\mathbb{Z}$	$\tan\alpha=\tan(\alpha+\pi)$

b) Teaching guidelines

Let learners know what inverse function is. Help them to recall that $f^{-1}\left[f(x)\right]=x$. Make sure that learners have scientific calculators.

- Organise class into groups. Request each group to have a group leader who will present their findings to the class.
- Request each group to do activity related to the lesson they are going to learn. The learners may need your assistance to do any activity. Help them to understand the activity. After group discussion, invite some or all groups for presentation of their work.

- After activity presentation, capture the main points from the presentation of the learners and summarise them. Guide the learners through given examples in Learner's book or through your own examples.
- Ask learners what they learned in day lesson to ensure that they understood what they have learned.
- Request learners to do exercises in their respective groups. Request learners to correct exercises on chalkboard and give them individual evaluation. Remember to give them homework.

c) Answers to activities

Activity 1.6

Materials

Exercise books, pens and calculator

Answers

1. $\begin{cases} \frac{\pi}{6}+2k\pi \\ \frac{5\pi}{6}+2k\pi \end{cases}, \ k\in\mathbb{Z}$ 2. $\pm\frac{\pi}{4}+2k\pi, \ k\in\mathbb{Z}$

Application activity 1.6

1. $\pm\frac{\pi}{12}+\frac{k\pi}{4}, k\in\mathbb{Z}$
2. $\left\{0,\frac{\pi}{14},\frac{\pi}{3}\right\}$
3. $\left\{\frac{\pi}{18},\frac{5\pi}{18},\frac{13\pi}{18},\frac{17\pi}{18},\frac{25\pi}{18},\frac{29\pi}{18}\right\}$
4. $\left(30^0,150^0,199.5^0,340.5^0\right)$
5. $\left\{\frac{\pi}{6},\frac{5\pi}{6}\right\}$

6. $\frac{\pi}{12}+\frac{k\pi}{3}$ $(k\in\mathbb{Z})$ or $\frac{5\pi}{12}+\frac{k\pi}{3}$

7. a) $(2k+1)\frac{\pi}{4}\ or\ k\pi,\ k\in\mathbb{Z}$

   b) $\frac{4k\pi}{5},\frac{4k\pi}{3},$

   $(1+2k)\frac{2\pi}{3},(1+2k)\frac{2\pi}{5}\ with\ k\in\mathbb{Z}$

   c) $\frac{k\pi}{3}\ or(2k+1)\frac{\pi}{2},\ k\in\mathbb{Z}$

   d) $(2k+1)\frac{\pi}{4}or(2k+1)\frac{\pi}{2},\ k\in\mathbb{Z}$

Activity 1.7

Materials

Exercise books, pens and calculator

Answers

1. $\sqrt{3}\cos x-\sin x=\sqrt{3}$

   We know that $\cos x=\frac{1-t^2}{1+t^2}$ and $\sin x=\frac{2t}{1+t^2}$ where $t=\tan\frac{x}{2}$

   $\sqrt{3}\cos x-\sin x=\sqrt{3}\Leftrightarrow\sqrt{3}\frac{1-t^2}{1+t^2}-\frac{2t}{1+t^2}=\sqrt{3}$

   $\Leftrightarrow\sqrt{3}(1-t^2)-2t=\sqrt{3}(1+t^2)$

   $\Leftrightarrow\sqrt{3}-\sqrt{3}t^2-2t=\sqrt{3}+\sqrt{3}t^2$

   $\Leftrightarrow\sqrt{3}t^2-2t=\sqrt{3}t^2$

   $\Leftrightarrow-2\sqrt{3}t^2-2t=0$

   $\Leftrightarrow\sqrt{3}t^2+t=0$

2. Solution of $\sqrt{3}t^2+t=0$

$\sqrt{3}t^2+t=0 \Leftrightarrow t\left(\sqrt{3}t+1\right)=0 \Rightarrow t=0 \text{ or } \sqrt{3}t+1=0$

$\Leftrightarrow t=0 \text{ or } t=-\frac{1}{\sqrt{3}}$

3. From t, let us find the value of x

$t=0 \Leftrightarrow \tan\frac{x}{2}=0 \quad \Rightarrow \frac{x}{2}=k\pi, k\in\mathbb{Z}$

So, $x=2k\pi, k\in\mathbb{Z}$

$t=-\frac{1}{\sqrt{3}} \Leftrightarrow \tan\frac{x}{2}=-\frac{1}{\sqrt{3}} \quad \Rightarrow \frac{x}{2}=-\frac{\pi}{6}+k\pi$

Thus, $x=-\frac{\pi}{3}+2k\pi,\ k\in\mathbb{Z}$

Application activity 1.7

1. $\left\{x=\frac{\pi}{6}+k\pi, x=\frac{\pi}{2}, k\in\mathbb{Z}\right\}$ 2. $\left\{x=\frac{\pi}{4}+k\pi, k\in\mathbb{Z}\right\}$

3. $\frac{\pi}{6}+2k\pi, \frac{\pi}{2}+2k\pi, k\in\mathbb{Z}$ 4. $119.7^0, 346.8^0$

1.6.3 Applications of trigonometric formulae and equations

Activity 1.8

Materials

Exercises book, pens

Answers

By reading text books or accessing internet, learners will discuss on harmonic motion and how trigonometry is used in harmonic motion.

An object that moves on a coordinate axis is in simple harmonic motion if its distance from the origin, d, at time t is given by either $d=a\cos\omega t$ or $d=a\sin\omega t$.

The motion has amplitude $|a|$, the maximum displacement of the object from its rest position. The period of the motion is $\frac{2\pi}{\omega}$, where $\omega>0$.

Activity 1.9

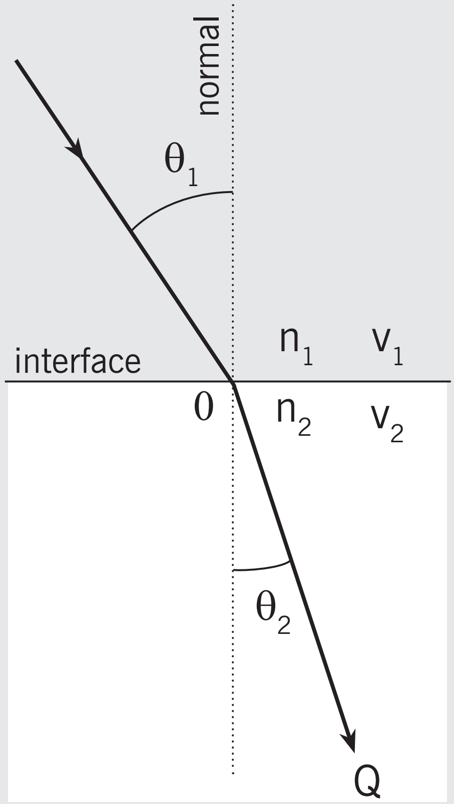
Materials

Exercises book, pens

Answers

By reading text books or accessing internet, learners will discuss on Snell's law.

The degree of bending of the light's path depends on the angle that the incident beam of light makes with the surface, and on the ratio between the refractive indices of the two media.



Snell's law states that: $n_1 \sin\theta_1 = n_2 \sin\theta_2$

1.7. End of Unit Assessment

1. $\dfrac{4\tan x-4\tan^3 x}{1-6\tan^2 x+\tan^4 x}$

2. $\dfrac{\cot^4 x-6\cot^2 x+1}{4\cot^3 x-4\tan x}$

3. -1

4. $\tan(A+B)=\dfrac{\tan A+\tan B}{1-\tan A\tan B}=\dfrac{\dfrac{a}{a+1}+\dfrac{1}{2a+1}}{1-\dfrac{a}{a+1}\times\dfrac{1}{2a+1}}$

$=\dfrac{2a^2+a+a+1}{(a+1)(2a+1)}\times\dfrac{(a+1)(2a+1)}{(a+1)(2a+1)-a}$

$=\dfrac{2a^2+2a+1}{2a^2+a+2a+1-a}$

$=1$

$\tan(A+B)=1\Rightarrow A+B=\dfrac{\pi}{4}+k\pi,\ k\in\mathbb{Z}$

For $k=0,\ A+B=\dfrac{\pi}{4}$

5. a) $\dfrac{1}{2}$ b) $\dfrac{\sqrt{2}}{2}$ c) -1 d) $\dfrac{\sqrt{3}}{2}$

6. $-\dfrac{7}{25}$

7. $-\dfrac{120}{119}$

8. a) $2\cos\dfrac{17}{2}\cos\dfrac{x}{2}$ b) $2\sin 7x\cos 4x$

9. a) $\dfrac{1}{2}(\sin 15x-\sin 7x)$ b) $\dfrac{1}{2}(\sin 16x+\sin 2x)$

10. a) $\pm\dfrac{\pi}{3}+2k\pi,k\in\mathbb{Z}$ b) $\pm\dfrac{\pi}{3}+2k\pi,k\in\mathbb{Z}$

c) $\dfrac{5\pi}{6}+2k\pi,k\in\mathbb{Z}$ d) $\dfrac{\pi}{6}+k\pi\ or -\dfrac{\pi}{3}+k\pi,k\in\mathbb{Z}$

11. a) $(2k+1)\frac{\pi}{2},(2k+1)\frac{\pi}{8},k\in\mathbb{Z}$

b) $\frac{2k\pi}{5},(2k+1)\frac{\pi}{11},k\in\mathbb{Z}$

c) $\frac{k\pi}{4},\pm\frac{\pi}{3}+\mathrm{k}\pi,\ k\in\mathbb{Z}$

d) $\sin mx+\sin nx=0\ is$

$x=\frac{2k\pi}{m+n}\ or\ x=\frac{(2k+1)\pi}{m-n}$

12. a) $\frac{\pi}{2}+2k\pi,k\in\mathbb{Z}$ b) $-\frac{\pi}{6}\pm\frac{\pi}{3}+2k\pi,k\in\mathbb{Z}$

c) $\frac{\pi}{6}\pm\frac{\pi}{3}+2k\pi,k\in\mathbb{Z}$ d) $\frac{\pi}{4}+2k\pi,k\in\mathbb{Z}$

Unit 2 Sequences

2.1 Key unit competence:

Use arithmetic, geometric, harmonic sequences and their convergence to understand and solve problems arising in various contexts.

2.2 Objectives

By the end of this unit the learner will be able to:

- Define a sequence
- Identify an arithmetic, a harmonic or a geometric sequence
- Determine nth term and the sum of the first nth terms of an arithmetic or geometric sequence
- Apply the concepts of sequences to solve problems involving arithmetic, harmonic or geometric sequence
- Determine the convergence and divergence of a sequence

2.3 List of lessons for unit 2

Week	Lessons	Content	Number of Periods
6	1	Definition of sequences	1
	2	Convergent and divergent sequences	2
7	3	Arithmetic sequences	3
8	4	Geometric sequences	3
9	5	Application of sequences in solving real life problems: Problems including population growth, Problems including compound and simple interests, Half-life and Decay problems in Radioactivity, Bacteria growth problems in Biology …	2
	6	End unit assessment and remediation	1

2.4 Materials to be used

Exercise books, pens, instruments of geometry, calculator

2.5 Guidance to the introductory activity

- Invite learners to work in groups and give them instructions on how they can do the introductory activity found in unit 2 of the learner's book; help them where necessary
- Guide learners to read and analyse the questions insisting on the analysis of the given data and to determine the number of insects that will be there in second, third, fourth,…nth generation.
- Invite some group members to present groups' findings, then try to harmonize their answers; try to insist on the list formed by the number of insects at any generation and the generalisation (number of insects at nth generation).
- Basing on learner's experience, prior knowledge and abilities shown in answering the questions for this activity, use different questions to facilitate them to give their predictions and ensure that you arouse their curiosity on what is going to be leant in this unit.

Answer for the introductory activity

Number of insects is given by:

1^{st} generation $\to 126$, 2^{nd} generation $\to 126\times 2 = 252$,

3^{rd} generation $\to 252\times 2 = 504$, 4^{th} generation $\to 1008$,

$At\ n^{th}\ generation \to 126\times 2^{(n-1)} = 126.n^{(n-1)}$

5^{th} generation $\to 2016$, 6^{th} generation $\to 4032$,

7^{th} generation $\to 8064$, 8^{th} generation $\to 16128$,

9^{th} generation $\to 32256$, 10^{th} generation $\to 64512$

2.6 Content and activities

2.6.1 Arithmetic sequences and harmonic sequences

a) Content summary

Arithmetic progression

A finite or infinite sequence $a_1, a_2, a_3, \ldots, a_n$ or $a_1, a_2, a_3, \ldots, a_n, \ldots$ is said to be an Arithmetic Progression (A.P.) or Arithmetic Sequence if $a_k - a_{k-1} = d$, where d is a constant independent of k, for $k = 2, 3, \ldots, n$ or $k = 2, 3, \ldots, n, \ldots$ as the case may be.

Characteristics

If three consecutive terms, u_{n-1}, u_n, u_{n+1} are terms in arithmetic sequence, then, $2u_n = u_{n-1} + u_{n+1}$

Common difference

In A.P., the difference between any two consecutive terms is a constant d, called **common difference**

General term or n^{th} term

The n^{th} term, u_n, of an arithmetic sequence $\{u_n\}$ with common difference d and initial term u_1 is given by $u_n = u_1 + (n-1)d$

Generally, if u_p is any p^{th} term of a sequence, then the n^{th} term is given by $u_n = u_p + (n-p)d$

Arithmetic means

If three or more than three numbers are in arithmetic sequence, then all terms lying between the first and the last numbers are called arithmetic means. If B is arithmetic mean between A and C, then $B = \frac{A+C}{2}$.

To insert k **arithmetic means** between two terms u_1 and u_n is to form an arithmetic sequence of $n = k + 2$ terms whose first term is u_1 and the last term is u_n.

Sum of first nth terms or arithmetic series

The sum of first *n* terms of a finite arithmetic sequence with initial term u_1 is given by $S_n = \frac{n}{2}(u_1 + u_n)$ which is called **finite arithmetic series**

Harmonic sequence

A sequence is said to be in harmonic progression if the reciprocals of its terms form an arithmetic progression.

Characteristics

If three consecutive terms, h_{n-1}, h_n, h_{n+1} are terms in arithmetic sequence, then, $\frac{2}{h_n} = \frac{h_{n-1} + h_{n+1}}{h_{n-1}\, h_{n+1}}$ or

$$\frac{h_n}{2} = \frac{h_{n-1}h_{n+1}}{h_{n-1} + h_{n+1}} \Leftrightarrow h_n = \frac{2h_{n-1}h_{n+1}}{h_{n-1} + h_{n+1}}$$

General term or nth term of H.P.

Take the reciprocals of the terms of the given series; these reciprocals will be in A.P.

Find nth term of this A.P. using $u_n = u_1 + (n-1)d$ or $u_n = u_p + (n-p)d$

Take the reciprocal of the nth term of A.P., to get the required nth term of H.P.

Thus, the nth term of H.P. is $\frac{1}{u_1 + (n-1)d}$ or $\frac{1}{u_p + (n-p)d}$

Harmonic means

If three or more than three numbers are in harmonic sequence, then all terms lying between the first and the last numbers are called harmonic means. If B is harmonic mean between A and C, then $B = \frac{A+C}{2}$.

To insert *k* harmonic **means** between two terms h_1 and h_n is to form a harmonic sequence of $n = k + 2$ terms whose first term is h_1 and the last term is h_n.

b) Teaching guidelines

- Organise the class into groups. Request each group to have a group leader who will present their findings to the class.
- Ask each group to do activity related to the lesson they are going to learn. The learners may need your assistance to do any activity. Help them to understand the activity. After group discussion, invite some or all groups for presentation of their work.
- After activity presentation, capture the main points from the presentation of the learners and summarise them. Guide the learners through given examples in Learner's book or through your own examples.
- Ask learners what they learned in day lesson to ensure that they understood what they have learned.
- Request learners to do exercises in their respective groups. Request learners to correct exercises on chalkboard and give them individual evaluation. Remember to give them homework.

c) Answers to activities

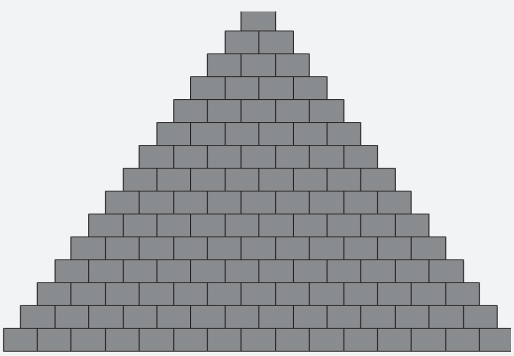
Activity 2.1

Materials

Exercise books, pens, instruments of geometry

Answers

a)



b) There are 15 rows

c) 15, 14, 13, 12, 11, 10, 9, 8, 7, 6, 5, 4, 3, 2, 1

d) That number is -1

Application activity 2. 1

1. a) $\{5,11,17,23\}$, $u_{10}=59$ b) $\{43,38,33,28\}$, $u_{10}=-2$
   c) $\{-7,-3,1,5\}$, $u_{10}=29$ d) $\{-1,-8,-15,-22\}$, $u_{10}=-64$
2. No. The number added to the 4th term to obtain the 5th term is not equal to the one used for previous first terms.
3. Common difference is 2
4. $x=3$ *or* 7, fourth term: 0 or 60
5. $a_{n+1}-a_n=4n-3$ As it is not a constant; the given sequence is not arithmetic.

Activity 2.2

Materials

Exercise books, pens

Answers

$$u_2=u_1+d$$
$$u_3=u_2+d=\left(u_1+d\right)+d=u_1+d$$
$$u_4=u_3+d=\left(u_1+2d\right)+d=u_1+3d$$
$$u_5=u_4+d=\left(u_1+3d\right)+d=u_1+4d$$
$$u_6=u_5+d=\left(u_1+4d\right)+d=u_1+5d$$
$$u_7=u_6+d=\left(u_1+5d\right)+d=u_1+6d$$
$$u_8=u_7+d=\left(u_1+6d\right)+d=u_1+7d$$
$$u_9=u_8+d=\left(u_1+7d\right)+d=u_1+8d$$
$$u_{10}=u_9+d=\left(u_1+8d\right)+d=u_1+9d$$

Generally,

$$u_n=u_{n-1}+d=\left(u_1+(n-2)d\right)+d=u_1+\left(n-1\right)d$$

Application activity 2. 2

1. 3 2. 9 3. 15

4. Answer is iii) 508th

5. 19, 36

Activity 2.3

Materials

Exercise books, pens, calculators

Answers

$u_1 = 2, u_7 = 20$

$u_n = u_1 + (n-1)d \Rightarrow u_7 = u_1 + 6d$

$\Rightarrow 20 = 2 + 6d$

$\Rightarrow d = 3$

The sequence is $2, 5, 8, 11, 14, 17, 20$

Application activity 2. 3

1. $-3, -1, 1, 3, 5, 7$

2. $2, 5, 8, 11, 14, 17, 20, 23, 26, 29, 32$

3. $-6\frac{1}{2}, -9\frac{1}{2}, -12\frac{1}{2}, -15\frac{1}{2}, \ldots, -39\frac{1}{2}$

4. 8 and 4

Activity 2.4

Materials

Exercise books, piece of paper (or manila paper), pens

Answers

$u_1 = u_1$

$u_2 = u_1 + d$

$\vdots$

$u_{n-1} = u_1 + (n-2)d$

$u_n = u_1 + (n-1)d$

Let s_n denote the sum of these terms.

We have

$$s_n = u_1 + [u_1 + d] + \ldots + [u_1 + (n-2)d] + [u_1 + (n-1)d]$$

Reversing the order of the sum, we obtain

$$s_n = [u_1 + (n-1)d] + [u_1 + (n-2)d] + \ldots + [u_1 + d] + u_1$$

Adding the left sides of these two equations and corresponding elements of the right sides,

we see that:

$$\begin{aligned} 2s_n &= [2u_1 + (n-1)d] + [2u_1 + (n-1)d] + \ldots + [2u_1 + (n-1)d] \\ &= n[2u_1 + (n-1)d] \end{aligned}$$

$$\Leftrightarrow s_n = \frac{n}{2}[2u_1 + (n-1)d] = \frac{n}{2}[u_1 + u_1 + (n-1)d]$$

By replacing $u_1 + (n-1)d$ with u_n, we obtain a useful formula for the sum:

$$s_n = \frac{n}{2}[u_1 + u_n]$$

or $s_n = \frac{n}{2}(u_1 + u_1(n-1)d) \Rightarrow s_n = \frac{n}{2}(2u_1 + (n-1)d)$

Application activity 2. 4

1. $2n(n+3)$
2. 860
3. 11
4. 144
5. 45045

Activity 2.5

Materials

Exercise books, piece of paper (or manila paper), pens

Answers

$\frac{1}{2},\frac{1}{4},\frac{1}{6},\frac{1}{8},\frac{1}{10},\frac{1}{12},\frac{1}{14},\frac{1}{16}$

The denominators are in arithmetic progression.

Application activity 2. 5

1. The sequence is $6,4,3,\frac{12}{5},2,\frac{12}{7},\frac{3}{2},\frac{4}{3}$.

   4^{th} *term is* $\frac{12}{5}$, 8^{th} *term is* $\frac{4}{3}$
2. $3,\frac{90}{23},\frac{90}{16},10$
3. The n^{th} term is $\frac{6}{n}$
4. The n^{th} term is $\frac{20}{5n+3}$
5. 22^{nd} term

2.6.2 Geometric sequences

a) Content summary

Definition

A Geometric Progression (G.P.) or Geometric Sequence is a sequence in which each term is a fixed multiple of the previous term i.e. $\frac{a_k}{a_{k-1}}=r$, where r is a constant independent of k, for

$k=2,3,\ldots,n$ or $k=2,3,\ldots,n,\ldots$.

Characteristics

If three consecutive terms, u_{n-1}, u_n, u_{n+1} are terms in geometric progression, then, $u_n^2 = u_{n-1}\, u_{n+1}$

Common ratio

In G.P., the ratio between any two consecutive terms is a constant r, called **common ratio**

General term or n^th term

The n^{th} term, u_n, of a geometric sequence $\{u_n\}$ with common ratio r and initial term u_1 is given by $u_n = u_1 r^{n-1}$

Generally, if u_p is any p^{th} term of a sequence then the n^{th} term is given by $u_n = u_p r^{n-p}$

Geometric means

If three or more than three numbers are in geometric sequence, then all terms lying between the first and the last numbers are called geometric means.

To insert *k* geometric **means** between two terms u_1 and u_n is to form a geometric sequence of $n = k+2$ terms whose the first term is u_1 and the last term is u_n.

Sum of first n^th terms or geometric series

The sum of first *n* terms of a finite geometric sequence with initial term u_1 is given by $S_n = \frac{u_1\left(1-r^n\right)}{1-r}$, $r<1$ or $S_n = \frac{u_1\left(r^n-1\right)}{r-1}$, $r>1$

which is called **finite geometric series**

If the initial term is u_0, then the formula is $s_n = \frac{u_0\left(1-r^{n+1}\right)}{1-r}$ with $r \neq 1$

If $r=1$, $s_n = nu_1$

Also, the product of first *n* terms of a geometric sequence with initial term u_1 and common ratio *r* is given by $P_n = \left(u_1\right)^n r^{\frac{n(n-1)}{2}}$

If the initial term is u_0 then $P_n = \left(u_0\right)^{n+1} r^{\frac{n}{2}(n+1)}$

b) Teaching guidelines

Let learners know what arithmetic sequence is. Recall that for an arithmetic sequence we add a constant number to the term to obtain the next term.

- Organise class into groups. Request each group to have a group leader who will present their findings to the class.
- Ask each group to do activity related to the lesson they are going to learn. The learners may need your assistance to do any activity. Help them to understand the activity. After group discussion, invite some or all groups for presentation of their work.
- After activity presentation, capture the main points from the presentation of the learners and summarise them. Guide the learners through given examples in Learner's book or through your own examples.
- Ask learners what they learned in day lesson to ensure that they understood what they have learned.
- Request learners to do exercises in their respective groups. Request learners to correct exercises on chalkboard and give them individual evaluation. Remember to give them homework.

c) Answers to activities

Activity 2.6

Materials

Exercise books, piece of paper (or manila paper), pens, calculator, scissors or blades

Answers

Learners will take a piece of paper and cut it into two equal part. Take one part and cut it again into two equal parts. When they continue in this way the fraction corresponding to the obtained parts according to the original piece of paper are as follows:

$\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}, \ldots$

Exercise 2.6

1. $x=-6 \ or \ 6$
2. No, the number multiplied to the fourth term to obtain fifth term is not the same as the one used for previous terms.
3. Common ratio is -2.

Activity 2.7

Materials

Exercise books, pens, calculators

Answers

Generation (n)	Number of ancestors	Observation
1st	2	$2\times1=2\times2^0$
2nd	4	2×2^1
3rd	8	2×2^2
4th	16	2×2^3
5th	32	2×2^4
6th	64	2×2^5
7th	128	2×2^6
8th	256	2×2^7
9th	512	2×2^8
10th	1024	2×2^9
11th	2048	2×2^{10}
12th	4096	2×2^{11}
⋮	⋮	⋮
n		$2\times2^{n-1}$

1. Up to 6th generation , this person has $64=2\times2^5$ ancestors
2. Up to 8th generation, this person has $256=2\times2^7$ ancestors
3. Up to 10th generation, this person has $1024=2\times2^9$ ancestors
4. Up to 12th generation, this person has $4096=2\times2^{11}$ ancestors

We have $2,4,8,\ldots$ 12th term which are in geometric progression.

Here, $a_1 = 2, r = 2$

The general formula which can be used is $2\times 2^{n-1} = 2^n$

Application activity 2. 7

1. $u_4 = 16$
2. 98304
3. $\frac{\sqrt[5]{16}}{4}$
4. $\frac{2^{10}}{3^6}$
5. i) $u_{12} = 1380$ ii) $u_{28} = 1190$

Activity 2.8

Materials

Exercise books, pens, calculators

Answers

$u_1 = 1, u_6 = 243$

$u_n = u_1 \cdot r^{n-1} \Rightarrow u_6 = u_1 \cdot r^5$

$\Rightarrow 243 = r^5 \quad \Rightarrow 3^5 = r^5 \quad \Rightarrow r = 3$

The sequence is $1,3,9,27,81,243$

Application activity 2. 8

1. $\frac{1}{4},\frac{1}{8},\frac{1}{16},\frac{1}{32},\frac{1}{64},\frac{1}{128},\frac{1}{256}$ or $\frac{1}{4},-\frac{1}{8},\frac{1}{16},-\frac{1}{32},\frac{1}{64},-\frac{1}{128},\frac{1}{256}$
2. $2,\frac{2}{3},\frac{2}{9},\frac{2}{27},\frac{2}{81},\frac{2}{243},\frac{2}{729}$ or $2,-\frac{2}{3},\frac{2}{9},-\frac{2}{27},\frac{2}{81},-\frac{2}{243},\frac{2}{729}$
3. $\frac{9}{2}$
4. a) $6\frac{1}{2}$ b) 6
5. $\frac{1}{3},1,3$

Activity 2.9

Materials
Exercise books, pens

Answers

1. Let $s_n = u_1 + u_2 + u_3 + \ldots + u_n$

$$s_n = u_1 + u_1 r + u_1 r^2 + \ldots + u_1 r^{n-1} \quad (1)$$

Multiply both sides of (1) by r, we obtain

$$s_n r = u_1 r + u_1 r^2 + u_1 r^3 + \ldots + u_1 r^n \quad (2)$$

Subtract (2) from (1), we get

$$\begin{array}{l} s_n = u_1 + u_1 r + u_1 r^2 + \ldots + u_1 r^{n-1} \\ \underline{-s_n r = \quad -u_1 r - u_1 r^2 - u_1 r^3 - \ldots - u_1 r^n} \\ s_n - s_n r = u_1 - u_1 r^n \end{array}$$

$\Leftrightarrow s_n\left(1-r\right) = u_1\left(1-r^n\right)$ or

$= \frac{u_1\left(1-r^n\right)}{\;}$ with $r \neq 1$

2. $P = u_1 \times u_1 r \times u_1 r^2 \times \ldots \times u_1 r^{n-1}$

$$\Leftrightarrow P = \left(u_1\right)^n \left(r \times r^2 \times \ldots \times r^{n-1}\right)$$

$$\Leftrightarrow P = \left(u_1\right)^n r^{(1+2+\ldots+n-1)}$$

We need the sum $S_{n-1} = 1 + 2 + \ldots + n - 1$

$$S_{n-1} = \frac{n-1}{2}\left(1 + n - 1\right) = \frac{n\left(n-1\right)}{2}$$

Then $P_n = \left(u_1\right)^n r^{\frac{n(n-1)}{2}}$

Application activity 2. 9

1) 21.25 2) 39.1

3) First term is 1, common ratio is $\frac{5}{4}$

4) -32 5) 8th

Activity 2.10

Materials

Exercise books, pen

Answers

If $-1<r<1$, $\lim_{n\to\infty} r^n = 0$

thus $\lim_{n\to\infty} \frac{u_1(1-r^n)}{1-r} = \frac{u_1}{1-r}$

Application activity 2. 10

1. a) $0<x<\frac{4}{3}$ b) $-\frac{190}{39}$
2. 115m

2.4.3 Applications

Activity 2.11

Materials

Exercises book, pens

Answers

1) $\lim_{n\to\infty}\left(\frac{3n^2-1}{n^2}\right)=3$

2) $\lim_{n\to\infty} n^2 = +\infty$

Application activity 2. 11

1. $\left\{2+(0.1)^n\right\}$ converges to 2

2. $\left\{\frac{1-2n}{1+2n}\right\}$ converge to -1

3. $\left\{\frac{1-5n^4}{n^4+8n^3}\right\}$ converges to -5

4. $\left\{-1^n\right\}$ diverge

5. $\left\{\frac{2n}{\sqrt{3n+1}}\right\}$ converges to $\frac{2}{\sqrt{3}}$

6. $\frac{\sqrt{7n^2+2}}{n^3+8}$ converge to 0.

Application activity 2. 12

1.
$$s_n = u_1\left(\frac{1-r^n}{1-r}\right)$$
$$\text{u}_1 = 25$$
$$r = 0.4$$
$$\text{s}_3 = 25\left(\frac{1-(0.4)^3}{1-0.4}\right) = 39$$
$$s_6 = 25\left(\frac{1-(0.4)^6}{1-0.4}\right) = 41.496$$
$$s_\infty = \frac{u_1}{1-r} = \frac{25}{1-0.4} = 41.666$$

a) The quantity of an anti-inflammatory drug in the body right after the 3rd. injection is 39 mg

b) The quantity of an anti-inflammatory drug in the body right after the 6th injection is 41.496 mg

c) The quantity of an anti-inflammatory drug in the body in the long run, right after an injection 41.666mg

2. From 7am to 11pm there are 17 hours

$u_1 = 0.4mg$

$n = 17$

$$s_n = u_1\left(\frac{1-r^n}{1-r}\right)$$

$$s_{17} = 0.4\left(\frac{1-(0.4)^{17}}{1-0.4}\right) = 1.375$$

There are 1.375 mg of nicotine is in the body right after the 11 pm cigarette.

2.7 End of Unit Assessment

1. a) $u_{20} = 78, S_{20} = 800$ b) $u_{20} = 23.5, S_{20} = 185$

2. a) $u_n = 2(n+1), S_n = n(n+3)$

   b) $u_n = 20-3n, S_n = \frac{n}{2}(37-3n)$

   c) $u_n = \frac{1}{n}, S_n = \frac{n+1}{2}$

3. a) $u_8 = 18, S_8 = 88$ b) $u_1 = 3, S_{10} = 210$

   c) $n = 10, d = 2$ d) $u_1 = 1, d = 2$

4. $\frac{157}{4}, \frac{79}{2}, \frac{159}{4}, 40, \frac{161}{4}, \frac{81}{2}, \frac{163}{4}, 41, \frac{165}{4}, \frac{83}{2}, \frac{167}{4}, 42, ...$

5. $5, 7, 9, 11, 13, 15, 17, 19, 21, 23, 25, 27, 29, 31, ...$

6. $-2, -\frac{7}{4}, -\frac{6}{4}, -\frac{5}{4}, -1, -\frac{3}{4}, -\frac{2}{4}, -\frac{1}{4}, 0, \frac{1}{4}$

7. 8,9,10

8. 9,11,13 or 13,11,9

9. a) $u_5 = 768, S_5 = 341$ b) $r = \frac{1}{2}, S_n = \frac{1533}{64}$

10. $u_5 = \frac{81}{2}$	11. $u_8 = \frac{2187}{4}$	
12. $2, 2\sqrt{2}, 4, 4\sqrt{2}, 8$ or $2, -2\sqrt{2}, 4, -4\sqrt{2}, 8$		
13. 3,6,12	14. 128 or -972	
15. 6, 6, 6 or 6, -3, -12.	16. 11,17,23	
17. 5,8,11,14	18. -4,-1,2,5,8	
19. 2,3	20. 6	
21. £11 million	22. $\frac{2}{3}$	23. $\frac{4}{5}, 5$
23. 2048000	25. $99.8^0 F$	26. 1800

Unit 3 Logarithmic and Exponential Equations

3.1 Key unit competence:

Solve equations involving logarithms or exponentials and apply them to model and solve related problems.

3.2 Objectives

By the end of this unit, student will be able to:

- Solve logarithmic equations
- Solve exponential equations
- Apply logarithmic and exponential equations in real life problems

3.3 List of Lessons for unit 3

Week	Lessons	Content	Number of Periods
10-11	1	Introduction to Exponential and logarithmic functions	1
	2	Logarithmic equations including natural logarithms	5
12	3	Exponential equations	3
13-14	4	Applications of logarithmic and exponential equations in solving real life problems: Interest rates problems, Mortgage problems, Population growth problems, Radioactive decay problems, Earthquake problems, Carbon dating problems, Problems about alcohol and risk of car accident.	5
	5	Assessment and Remediation	

3.4 Materials required

Exercise books, pens, instruments of geometry, calculator

3.5 Guidance to the introductory activity

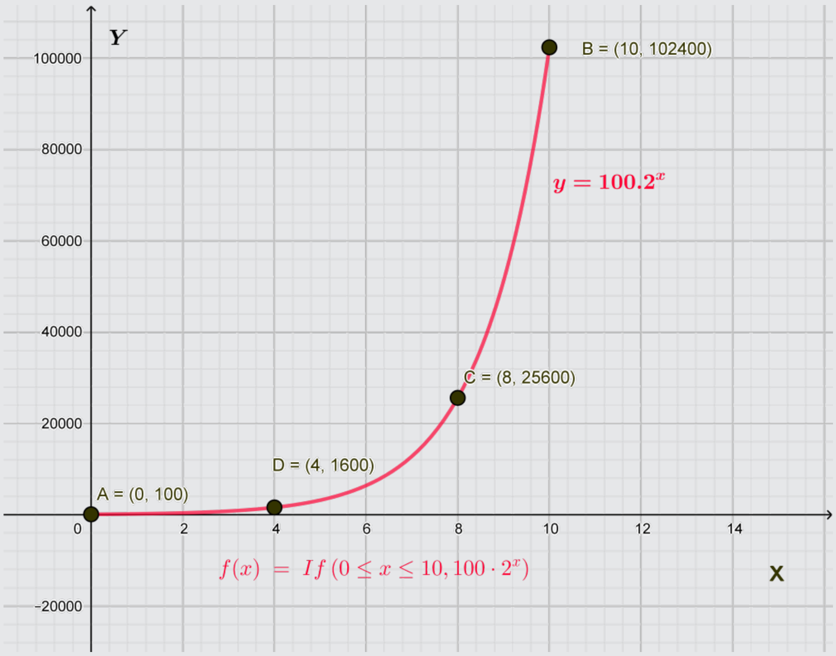
- Invite learners to work in groups and work on the introductory activity found in their Mathematics books;
- Move around in the class for facilitating students where necessary and give more clarification on eventual challenges they may face during their work; Verify and help groups where necessary
- Invite one member from each group with different working steps to present their work where they must explain the working steps;
- As a teacher, harmonize the findings from presentation and guide students to solve different types of logarithmic and exponential equations;

Solution:

a) Learners complete the table showing the money of the pharmacist from the 1st day up to 10th day.

Days	Amount s	USD
1st days	200USD	200
2nd day	$200x2 = 100x2x2 = 100x2^2$	400
3rd days	$100x2x2x2 = 100x2^3$	1600
4th days	$100x2x2x2x2 = 100x2^4$	3200
....		
10th day	$100x2x2x2x2... = 100x2^{10}$	102,400
Nth day	$100x2x2x2x2... = 100x2^n$	$100x2^n$

b) The graph plotted in a rectangular coordinate.



c) $f(n) = 100x2^n$ USD

- During the presentation let learners discover the concept of exponential function $F(t)$ starting with the property of a function with powers. $F(t) = 100x2^t$
- Learners establish the function $Y(F)$ inverse of $F(t)$

$$Y(F) = F^{-1}(t) = \ln\left(\frac{t}{100}\right) = -\ln(100) + \ln t$$

$$Y(t) = -4.6 + \ln(t)$$

d) The pharmacist wants to possess the money F under the same conditions, discuss how he/she can know the number of days necessary to get such money from the beginning of the business.

The economist wants to possess the money F, using the inverse function $Y(F) = -4.6 + \ln(F)$, she/he will use the equation $t = -4.6 + \ln(F)$ to calculate the number t of days required.

Conclude that $F(t)$ and $Y(t)$ are respectively exponential function and logarithmic functions that are needed to be well studied so that they may be used without problems. This unit deals with the behaviour and properties of such essential functions and their application in real life situation.

3.6 Content and activities

3.6.1 Exponential and logarithmic functions

a) Content summary

The graph of a logarithmic function is found by reflecting the graph of the corresponding exponential function to the **first bisector,** i.e. the line $y = x$.

Then the coordinates of the points for $y = a^x$ are reversed to obtain the coordinates of the points for $g(x) = \log_a x$.

b) Teaching guidelines

Let learners know how to draw linear function in 2-dimensions. Recall that to sketch a function you need a table of points.

- Organise class into groups. Request each group to have a group leader who will present their findings to the class.
- Ask each group to do activity related to the lesson they are going to learn. The learners may need your assistance to do any activity. Help them to understand the activity. After group discussion, invite some or all groups for presentation of their work.
- After activity presentation, capture the main points from the presentation of the learners and summarise them. Guide the learners through given examples in Learner's book or through your own examples.
- Ask learners what they learned in day lesson to ensure that they understood what they have learned.
- Request learners to do exercises in their respective groups. Request learners to correct exercises on chalkboard and give them individual evaluation. Remember to give them homework.

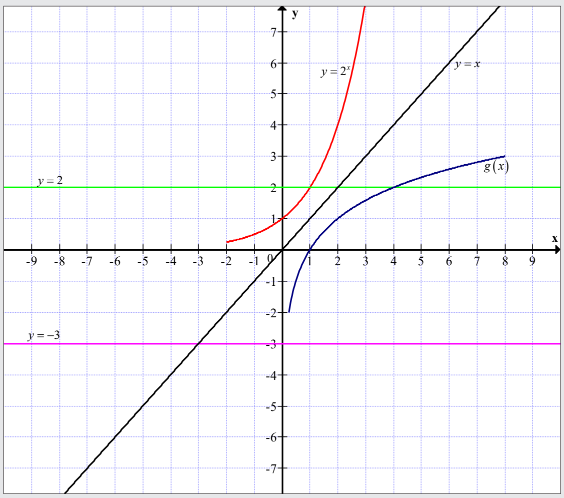
c) Answers to activities

Activity 3.1

Materials

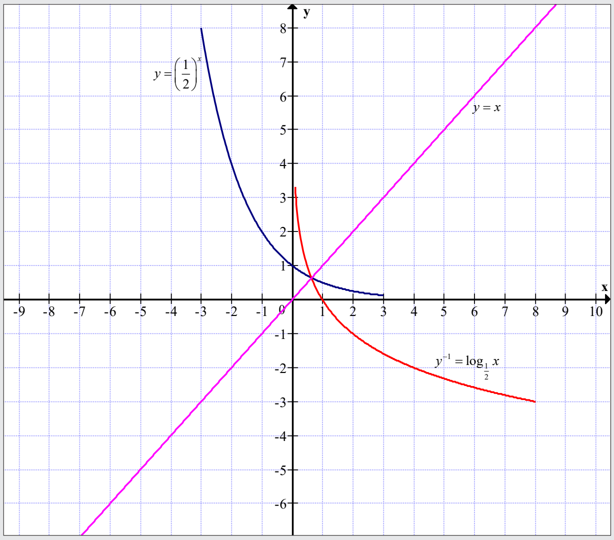
Exercise books, pens, instruments of geometry, calculator

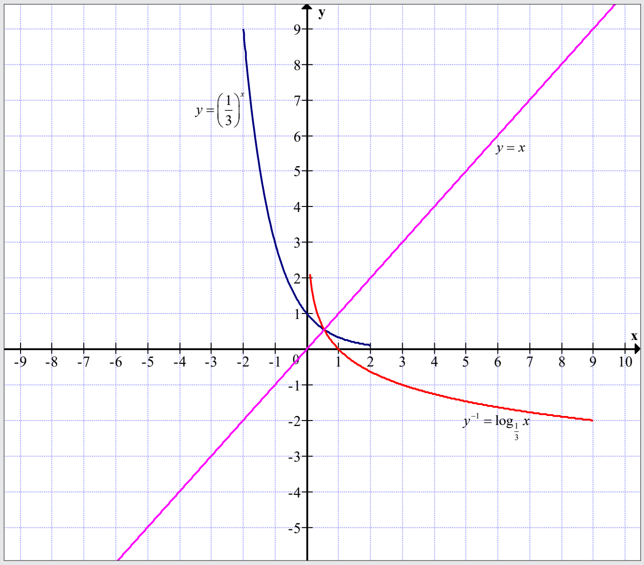
Answers



Any horizontal line crosses the curve at most once $y = 2^x$ is one to one function (is invertible function)

Application activity 3.1





3.6.2 Exponential and logarithmic equations

a) Content summary

In solving exponential or logarithmic equations, remember basic rules for exponents and/or logarithms.

Basic rules for exponents

For $a>0$ *and* $a\neq 1, m, n\in IR$

a) $a^m\times a^n=a^{m+n}$

b) $a^m : a^n=a^{m-n}$

c) $\left(a^m\right)^n=a^{mn}$

d) $a^{-n}=\dfrac{1}{a^n}$

e) $a^{\frac{1}{n}}=\sqrt[n]{a}$

f) $a^{\frac{m}{n}}=\sqrt[n]{a^m}$

g) $a^{\log_a b}=b$

Basic rules for logarithms

$\forall x, y\in]0,+\infty[, a\in]0,+\infty[\setminus\{1\}:$

a) $\log_a xy=\log_a x+\log_a y$

b) $\log_a \dfrac{1}{y}=-\log_a y$

c) $\log_a \dfrac{x}{y}=\log_a x-\log_a y$

d) $\log_a x^r=r\log_a x$

e) $\log_a b = \dfrac{\log_c b}{\log_c a}$

b) Teaching guidelines

Let learners know how to solve linear and quadratic equations. They should also know basic properties of powers. Help them to recall basic properties of powers.

- Organise class into groups. Request each group to have a group leader who will present their findings to the class.
- Ask each group to do activity related to the lesson they are going to learn. The learners may need your assistance to do any activity. Help them to understand the activity. After group discussion, invite some or all groups for presentation of their work.
- After activity presentation, capture the main points from the presentation of the learners and summarise them. Guide the learners through given examples in Learner's book or through your own examples.
- Ask learners what they learned in day lesson to ensure that they understood what they have learned.
- Request learners to do exercises in their respective groups. Request learners to correct exercises on chalkboard and give them individual evaluation. Remember to give them homework.

c) Answers to activities and exercises

Activity 3.2

Materials

Exercise books, pens, calculators

Answers

If $m=\log_a x, n=\log_a y$ and $z=\log_a xy$ then $x=a^m$, $y=a^n$ and $xy=a^z$.

Now, $xy=a^m a^n=a^{m+n}=a^z \Rightarrow z=m+n$.

Thus, $\log_a(xy)=\log_a x+\log_a y$.

If $m=\log_a x, n=\log_a y$ and $z=\log_a \frac{x}{y}$ then $x=a^m$, $y=a^n$ and $\frac{x}{y}=a^z$.

Now, $\frac{x}{y}=\frac{a^m}{a^n}=a^{m-n}=a^z \Rightarrow z=m-n$.

Thus, $\log_a(xy)=\log_a x-\log_a y$.

Application activity 3.2

1. a) $a^m\cdot a^n=\underbrace{a\cdot a\cdot a\cdots a}_{m\ factors}\times\underbrace{a\cdot a\cdot a\cdots a}_{n\ factors}=\underbrace{a\cdot a\cdot a\cdots a}_{m+n\ factors}=a^{m+n}$

   b) $\frac{a^m}{a^n}=a^m\frac{1}{a^n}=a^{m-n}$

   c) $\left(a^m\right)^n=\underbrace{\underbrace{a\cdot a\cdot a\cdots a}_{m\ factors}\times\underbrace{a\cdot a\cdot a\cdots a}_{m\ factors}\cdots\underbrace{a\cdot a\cdot a\cdots a}_{m\ factors}}_{n\ factors}=a^{mn}$

   d) $\frac{1}{a^m}=\frac{1}{a^m}{}^m=\left(\frac{1}{a}\right)^m=\left(a^{-1}\right)^m=a^{-m}$

   e) $a^{\frac{1}{n}}=\sqrt[n]{a}$ from definition

   f) $a^{\frac{m}{n}}=\left(a^m\right)^{\frac{1}{n}}=\sqrt[n]{a^m}$

2. a) $\log_4 64=3$ b) $\log_2\frac{1}{8}=-3$ c) $\log_{\frac{1}{2}} y=x$

   d) $\log_p q=3$ e) $\log_8 0.5=x$ f) $\log_5 q=p$

3. a) $\log_2 8 = x \Leftrightarrow 8 = 2^x \Leftrightarrow 2^x = 2^3 \Rightarrow x = 3$

   b) $\log_x 125 = 3 \Leftrightarrow 125 = x^3 \Rightarrow x = \sqrt[3]{125} \Leftrightarrow x = 5$

   c) $\log_x 64 = 0.5 \Leftrightarrow 64 = x^{0.5} \Rightarrow x = 64^2 \Leftrightarrow x = 4096$

   d) $\log_4 64 = x \Leftrightarrow 64 = 4^x \Leftrightarrow 4^x = 4^3 \Rightarrow x = 3$

   e) $\log_9 x = 3\frac{1}{2} \Leftrightarrow x = 9^{3\frac{1}{2}} \Leftrightarrow x = \left(9^{\frac{1}{2}}\right)^7 \Rightarrow x = 2187$

   f) $\log_2 \frac{1}{2} = x \Leftrightarrow \frac{1}{2} = 2^x \Leftrightarrow 2^x = 2^{-1} \Rightarrow x = -1$

4. a) 5 b) 1.5 c) -3

   d) –3 e) $\frac{1}{3}$ f) 1

   g) 1 h) 0

Activity 3.3

Materials

Exercise books, pens

Answers

1. $x = \log_a m$ and $z = \log_a \left(m^p\right)$ then $m = a^x$, $m^p = a^x$.

   Now, $m^p = \left(a^x\right)^p = a^{px} = a^z \Rightarrow z = px$.

   Thus, $\log_a \left(m^p\right) = p \log_a m$, as required.

2. $\log_a b = \dfrac{\log_c b}{\log_c a}$

   Let $\log_a b = x$, then $a^x = b$.

Take logarithms in base c of both sides $\log_c a^x = \log_c b$.

This gives $x \log_c a = \log_c b \Rightarrow x = \dfrac{\log_c b}{\log_c a}$.

Therefore, $\log_a b = \dfrac{\log_c b}{\log_c a}$

Application activity 3.3

1. a) $4p$ b) $-2p$ c) $1+p$
2. a) $\frac{e^2}{x}$ b) $\frac{x^2}{y^2}$ c) $-x^2+3\ln x$
3. a) $\left\{\frac{\ln 5}{\ln 2}\right\}$ b) $\left\{\frac{\ln 23}{\ln 3}\right\}$
   c) $\frac{1}{6}\left(\frac{\ln 17}{\ln 2}-1\right)$ d) $\{\ln 2, \ln 3\}$

3.6.3 Applications of logarithmic and exponential equations

Activity 3.4

Materials

Exercise books, pens, calculators

Answers

$P(t)=P_0 2^{kt}$

Here $P_0=2, k=2, t\,in\,hours \Rightarrow P(t)=2^{2t+1}$

a) $P(4)=2^9=512$

b) $P(t)=2^{13} \Leftrightarrow 2^{2t+1}=2^{13} \Rightarrow 2t+1=13 \Rightarrow t=6$

c) Number of cells left is $\frac{2^{22}}{2}\,or\,2^{21}$

Activity 3.5

Materials

Exercise books, pens, calculators

Answers

a) The original amount of material present is

$A(0)=80\left(2^0\right)=80\,gram$

b) For the half life, $A(t)=40$

$40=80\left(2^{-\frac{t}{100}}\right) \Rightarrow \frac{1}{2}=2^{-\frac{t}{100}} \Rightarrow 2^{-1}=2^{-\frac{t}{100}}$

$1=\frac{t}{100} \Rightarrow t=100$

Therefore the half life is 100 years

c) $A(t)=1$

$\Rightarrow 80\left(2^{-\frac{t}{100}}\right)=1 \qquad \Rightarrow 2^{-\frac{t}{100}}=\frac{1}{80} \qquad \Rightarrow 2^{-\frac{t}{100}}=80^{-1}$

$\Rightarrow \log 2^{-\frac{t}{100}}=\log 80^{-1} \qquad \Rightarrow -\frac{t}{100}\log 2=-\log 80$

$\Rightarrow t=\frac{\log 80}{\log 2}\times 100=632$

Therefore, it will take 632 years for material to decay to 1 gram.

Activity 3.6

Materials

Exercise books, pens

Answers

Suppose $P(t)$ has an exponential decay model so that

$P(t)=P_0e^{-kt} \quad (k<0)$.

At any fixed time t_1 let

$P_1=P_0e^{-kt_1}$

be the value of $P(t)$ and let T denote the amount of time required to reduce in value by half. Thus, at time t_1+T the value of $P(t)$ will be $2P_1$ so that $2P_1=P_0e^{-k(t_1+T)}=P_0e^{-kt_1}e^{-kT}$.

Since $P_1=P_0e^{-kt_1}$,

$2P_1=P_0e^{-k(t_1+T)}=P_0e^{-kt_1}e^{-kT} \xrightarrow{P_1=e^{-kt_1}} 2P_0e^{-kt_1}=P_0e^{-kt_1}e^{-kT}$

Or $2=e^{-kT}$, taking $\ln$ on both sides gives

$\ln 2=-kT$ or $T=-\frac{1}{k}\ln 2$ which does not depend on P_0 or t_1.

Application activity 3.4

1. About 1,012
2. 29.15 years
3. 1,250 bacteria
4. About 99.424; a little over 182 years.
5. 866 years
6. Frw 7,557.84
7. a) 10 years b) (i) 8 years (ii) 32.02 years.

3.7 End of Unit Assessment

1. 5	2. 1.5	3. 1.09
4. 1.5	5. -3	6. 1.05
7. $\{81\}$	8. $\{-1,6\}$	9. $\{6\}$
10. $\left\{\frac{1}{5},5\right\}$	11. $\left\{\frac{1}{4}\right\}$	12. $\{2\}$
13. a) $38.7\,million$	b) $46.4\,million$	c) $48.4\,million$
14. a) 20.8 years	b) 138 years	
15. 12.9^0		
16. a) 3.33sec	b) 4.72sec	c) 13.3sec

Unit 4 Trigonometric Functions and their Inverses

4.1 Key unit competence:

Apply theorems of limits and formulas of derivatives to solve problems including trigonometric functions.

4.2 Objectives

By the end of this unit, the learners should be able to:

- Find the domain and range of trigonometric function and their inverses.
- Study the parity of trigonometric functions.
- Study the periodicity of trigonometric functions.
- Evaluate limits of trigonometric functions and their inverse.
- Differentiate trigonometric functions and their inverse.

4.3 List of lessons for unit 4

Week	Content	Number of Periods
15	Introduction on trigonometric functions and their inverses	1
	Domain and range of trigonometric functions	2
16-17	Parity and periodicity of trigonometric functions	2
	Limits of trigonometric functions and their inverses	2
	Differentiation of trigonometric functions	2
18	Successive derivatives	1
	Application of trigonometric functions in the periodic motion and medicine.	1
	End unit assessment and Remediation	1

4.4 Materials required

Exercise books, pens, instruments of geometry, calculator

4.5 Guidance to the introductory activity

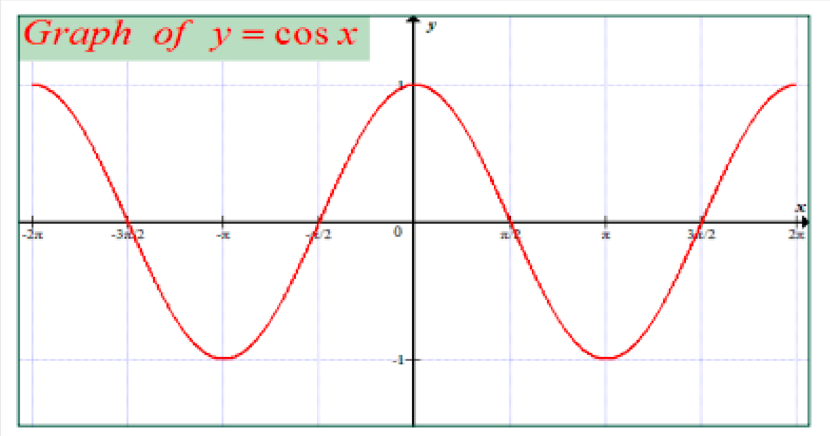
- Guide learners to form small groups and to work on the introductory activity;
- Walk around to monitor the work of each group and to assist any group in need;
- After a given time, invite learners to present their findings and harmonize them.
- Help learners to solve trigonometric equations

Answer to the introductory activity

a)

X	-2π	$\frac{-3\pi}{2}$	$-\pi$	$\frac{-\pi}{2}$	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
Y	1	0	-1	0	1	0	-1	0	1

b)



c) Values for which $f(x)=0$ are: $\frac{-3\pi}{2}, \frac{-\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}$

4.6 Contents and activities

4.6.1. Generalities on trigonometric functions and their inverses

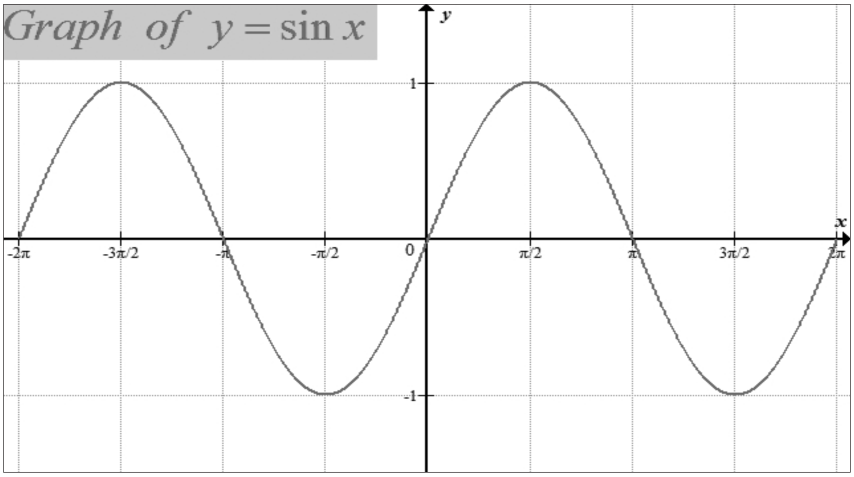
a) Content summary

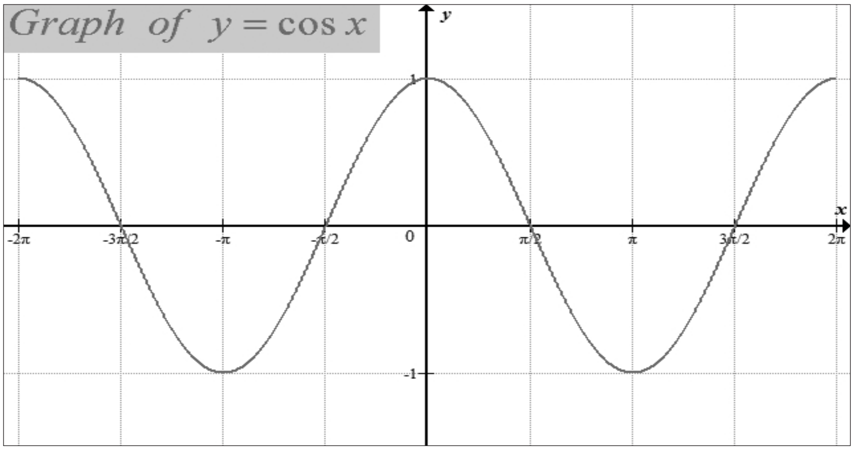
Domain and range of trigonometric functions

Cosine and sine

The domain of $\sin x$ and $\cos x$ is the set of real numbers.

The range of $\sin x$ and $\cos x$ is $[-1,1]$.

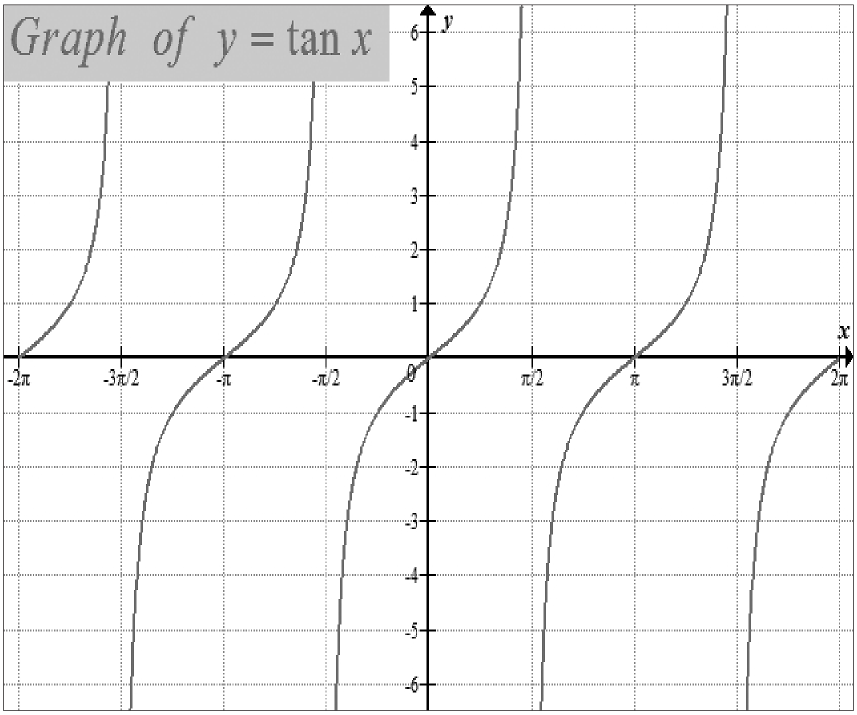


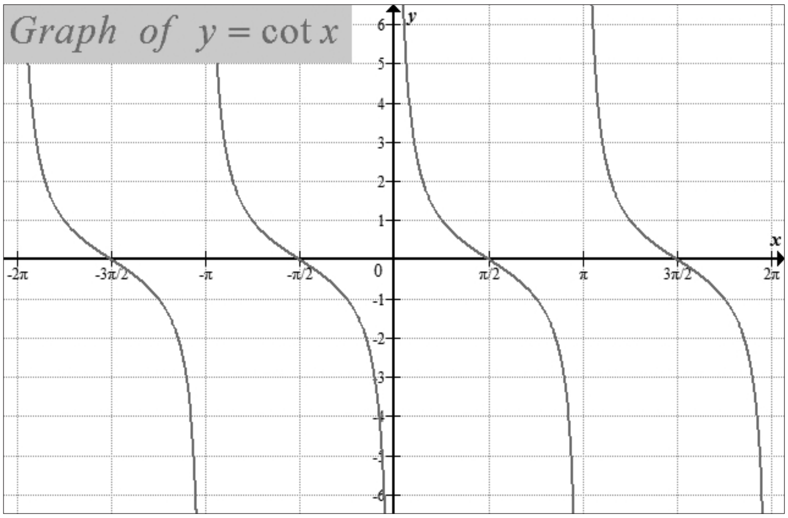


Tangent and cotangent

The domain of $\tan x$ is $\mathbb{R}\setminus\left\{\frac{\pi}{2}+k\pi\right\}, k\in\mathbb{Z}$. The range of $\tan x$ is the set of real numbers.

The domain of $\cot x$ is $\mathbb{R}\setminus\{k\pi\}, k\in\mathbb{Z}$. The range of $\cot x$ is the set of real numbers.

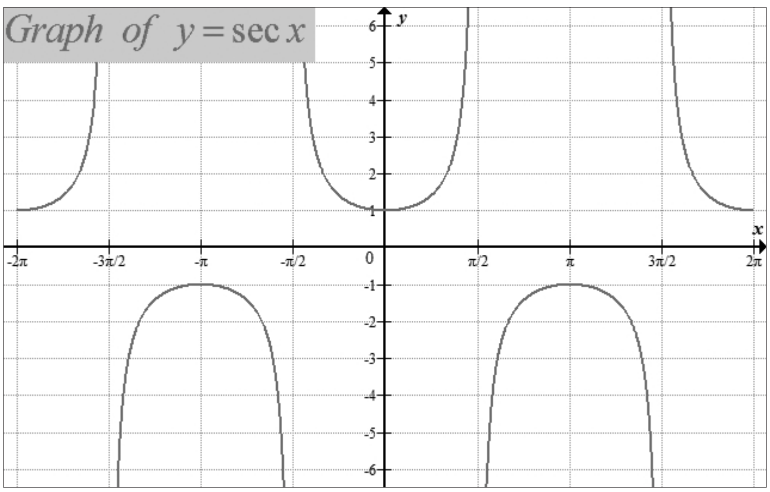


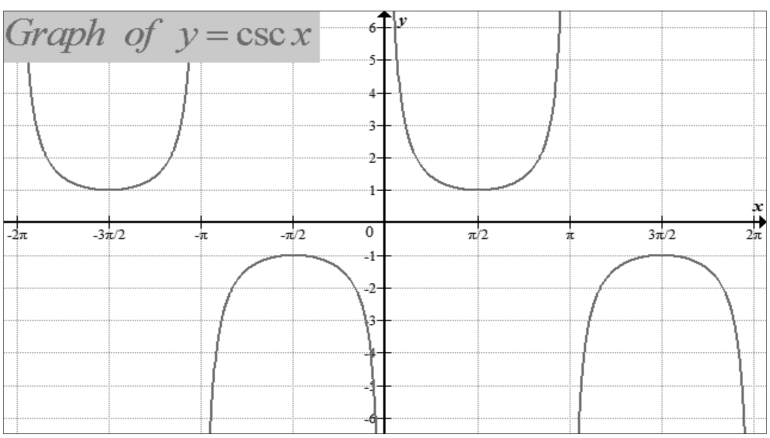


Secant and cosecant

The domain of $\sec x$ is $\mathbb{R}\setminus\left\{\frac{\pi}{2}+k\pi\right\},k\in\mathbb{Z}$. Since $\sec x=\frac{1}{\cos x}$ and range of cosine is $[-1,1]$, $\frac{1}{\cos x}$ will vary from negative infinity to -1 or 1 to plus infinity. Thus the range of $\sec x$ is $]-\infty,-1]\cup[1,+\infty[$

The domain of $\csc x$ is $\mathbb{R}\setminus\{k\pi\},k\in\mathbb{Z}$. Since $\csc x=\frac{1}{\sin x}$ and range of sine is $[-1,1]$, $\frac{1}{\sin x}$ will vary from negative infinity to -1 or 1 to plus infinity. Thus, the range of $\csc x$ is $]-\infty,-1]\cup[1,+\infty[$





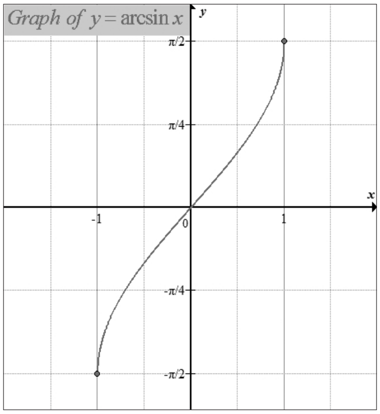
Inverse trigonometric functions

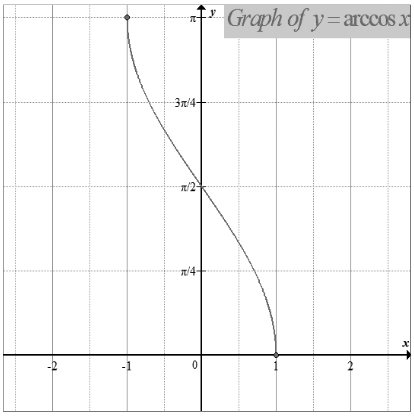
$\sin x$ and $\cos x$ have the inverses called **inverse sine** and **inverse cosine** denoted by $\sin^{-1} x$ and $\cos^{-1} x$ respectively.

Note that the symbols $\sin^{-1} x$ and $\cos^{-1} x$ are never used to denote $\frac{1}{\sin x}$ and $\frac{1}{\cos x}$ respectively. If desired, $\frac{1}{\sin x}$ and $\frac{1}{\cos x}$ can be written as $(\sin x)^{-1}$ and $(\cos x)^{-1}$ (or $\csc x$ and $\sec x$) respectively.

To define $\sin^{-1} x$ and $\cos^{-1} x$ we restrict the domain of $\sin x$ and $\cos x$ to the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ and $[0, \pi]$ respectively to obtain a one-to-one function.

In older literature, $\sin^{-1} x$ and $\cos^{-1} x$ are called **arcsine of** x and **arccosine of** x and they are denoted by $arc\sin x$ and $arc\cos x$ respectively.





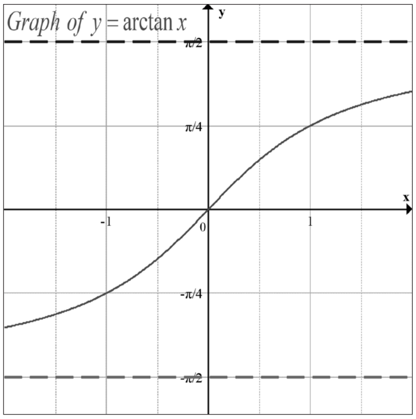
Inverse tangent and inverse cotangent

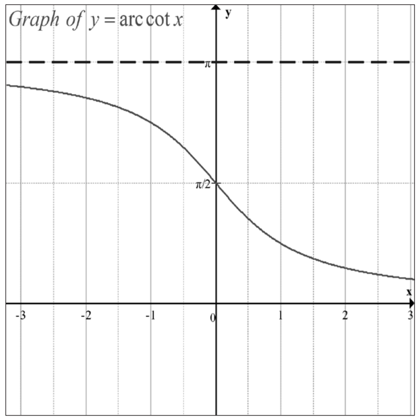
Tangent of x, denoted $\tan x$, is a function which is defined for all positive and negative values of x except $\pm 90^0, \pm 270^0, \ldots$. The range of $\tan x$ is $(-\infty, +\infty)$. It has the inverse called inverse tangent and is denoted by $\tan^{-1} x$.

To define $\tan^{-1} x$, we restrict the domain of $\tan x$ to $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

We will summarize the property of inverse cotangent briefly:

$y = \cot^{-1} x$ is equivalent to $x = \cot y$ if $0 < y < \pi$ and $-\infty < x < +\infty$





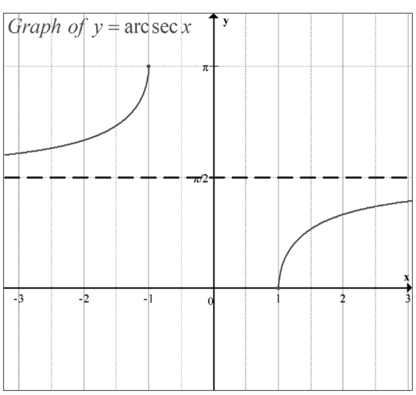
Inverse secant and inverse cosecant

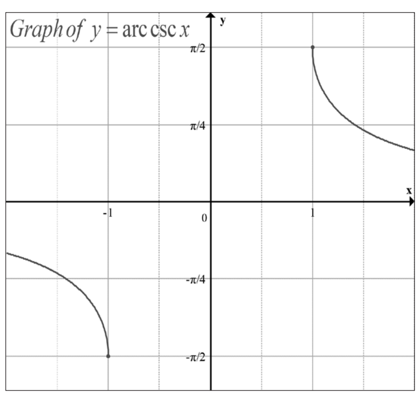
The inverse secant, denoted $\sec^{-1} x$, is defined to be the inverse of restricted secant function.

The domain of $\sec^{-1} x$ is $(-\infty,-1] \cup [1,+\infty)$ and the range is $\left[0,\frac{\pi}{2}\right[\cup \left]\frac{\pi}{2},\pi\right]$

We will summarize the property of inverse cosecant briefly:

$y = \csc^{-1} x$ is equivalent to $x = \csc y$ if $-\frac{\pi}{2} \le y < 0$ or $0 < y \le \frac{\pi}{2}$ and $|x| \ge 1$





Domain restrictions that make the trigonometric functions one to one

Function	Domain restriction	Range
Sine	$\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$	[-1,1]
Cosine	$[0,\pi]$	[-1,1]
Tangent	$\left]-\frac{\pi}{2},\frac{\pi}{2}\right[$	$\mathbb{R}$
Cotangent	$]0,\pi[$	$\mathbb{R}$
Secant	$\left[0,\frac{\pi}{2}\right[\cup\left]\frac{\pi}{2},\pi\right]$	$]-\infty,-1]\cup[1,+\infty[$
Cosecant	$\left[-\frac{\pi}{2},0\right[\cup\left]0,\frac{\pi}{2}\right]$	$]-\infty,-1]\cup[1,+\infty[$

Because $\sin x$ (restricted) and $\sin^{-1} x$; $\cos x$ (restricted) and $\cos^{-1} x$ are inverses to each other, it follows that:

- $\sin^{-1}(\sin y)=y$ if $-\frac{\pi}{2}\le y\le\frac{\pi}{2}$; $\sin\left(\sin^{-1} x\right)=x$ if $-1\le x\le 1$
- $\cos^{-1}(\cos y)=y$ if $0\le y\le\pi$; $\cos\left(\cos^{-1} x\right)=x$ if $-1\le x\le 1$

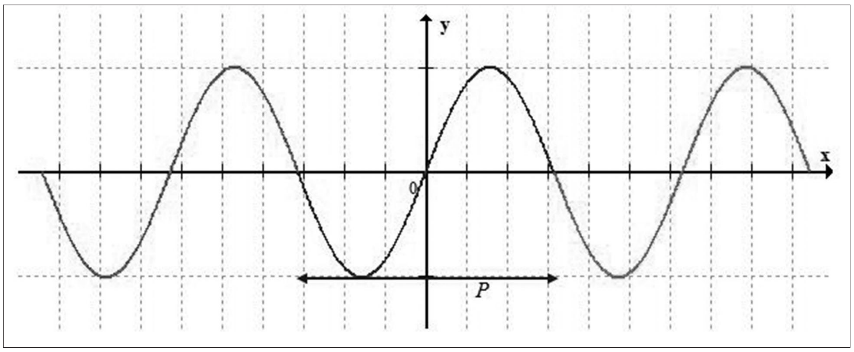
From these relations, we obtain the following important result:

Theorem 1

- If $-1\le x\le 1$ and $-\frac{\pi}{2}\le y\le\frac{\pi}{2}$, then $y=\sin^{-1} x$ and $\sin y=x$ are equivalent.
- If $-1\le x\le 1$ and $0\le y\le\pi$, then $y=\cos^{-1} x$ and $\cos y=x$ are equivalent.

Periodic functions

A function *f* is called **periodic** if there is a positive number *P* such that $f(x+P)=f(x)$ whenever *x* and $x+P$ lie in the domain of *f*.



Any function which is not periodic is called **aperiodic.**

The period of sum, difference or product of trigonometric function is given by the **Lowest Common Multiple (LCM)** of the periods of each term or factor.

b) Teaching guidelines

Let learners know how to find domain of definition of a polynomial, rational and irrational functions. Recall that the domain of definition of a function is the set of elements where the function is defined.

- Organise class into groups. Request each group to have a group leader who will present their findings to the class.
- Ask each group to do activity related to the lesson they are going to learn. The learners may need your assistance to do any activity. Help them to understand the activity. After group discussion, invite some or all groups for presentation of their work.
- After activity presentation, capture the main points from the presentation of the learners and summarise them. Guide the learners through given examples in Learner's book or through your own examples.
- Ask learners what they learned in day lesson to ensure that they understood what they have learned.
- Request learners to do exercises in their respective groups. Request learners to correct exercises on chalkboard and give them individual evaluation. Remember to give them homework.

c) Answers to activities

Activity 4.1

Materials

Exercise books, pens, calculators

Answers

1. No value of x
2. No value of x
3. $x=\frac{\pi}{2}+k\pi, k\in\mathbb{Z}$
4. $x=k\pi, k\in\mathbb{Z}$

Application activity 4.1

1. $\mathbb{R}$
2. $\mathbb{R}\setminus\{0\}$
3. $\mathbb{R}\setminus\{0\}$

Activity 4.2

Materials

Exercise books, pens, calculators

Answers

1. $]-\infty,-1[\cup]1,+\infty[$
2. $]-\infty,-1[\cup]1,+\infty[$
3. No value of x

Application activity 4.2

1. $\left[-\frac{1}{2},0\right[\cup\left]0,\frac{1}{2}\right]$
2. $[-1,1]$
3. $]0,1]$
4. $[-1,0[\cup]0,1]$

Activity 4.3

Materials

Exercise books, pens

Answers

1. $f(-x)=\frac{\sin x}{x}, -f(x)=-\frac{\sin x}{x}, f(-x)\neq -f(x), f(-x)=f(x)$
2. $g(-x)=-\frac{\cos x}{x}, -g(x)=-\frac{\cos x}{x}, g(-x)=-g(x), g(-x)\neq g(x)$

Application activity 4.3

1. Even 2. Odd 3. Odd
4. Neither even nor odd, 1 is in domain but -1 is not in domain.

Activity 4.4

Materials

Exercise books, pens, calculators

Answers

1. $2k\pi, k \in \mathbb{Z}$ 2. $2k\pi, k \in \mathbb{Z}$ 3. $k\pi, k \in \mathbb{Z}$

Application activity 4.4

1. π 2. 3π 3. $\frac{\pi}{3}$
4. 2π 5. $\frac{2\pi}{w}$ 6. $\frac{\pi}{2}$

Activity 4.5

Materials

Exercise books, pens, calculators

Answers

1. 2π 2. π

Application activity 4.5

1. π 2. 2π 3. 2π 4. $\frac{2\sqrt{3}\pi}{3}$

4.6.2 Limits of trigonometric functions

a) Content summary

$$\lim_{x\to 0}\frac{\sin x}{x}=1$$

Teaching guidelines

Let learners know how to find the limits of polynomial, rational and irrational functions. They also know trigonometric identities.

- Organise class into groups. Request each group to have a group leader who will present their findings to the class.
- Ask each group to do activity related to the lesson they are going to learn. The learners may need your assistance to do any activity. Help them to understand the activity. After group discussion, invite some or all groups for presentation of their work.
- After activity presentation, capture the main points from the presentation of the learners and summarise them. Guide the learners through given examples in Learner's book or through your own examples.
- Ask learners what they learned in day lesson to ensure that they understood what they have learned.
- Request learners to do exercises in their respective groups. Request learners to correct exercises on chalkboard and give them individual evaluation. Remember to give them homework.

c) Answers to activities

Activity 4.6

Materials

Exercise books, pens, calculators

Answers

A. 1) $\lim_{x\to 0}\sin x=\sin 0=0$ 2) $\lim_{x\to 0}x\sin x=0\sin 0=0$

3) $\lim_{x\to 0}\cos x=\cos 0=1$ 4) $\lim_{x\to 0}\frac{1}{x}=\infty$

5) $\lim_{x\to 0}\frac{\cos x}{x}=\lim_{x\to 0}\frac{1}{x}\lim_{x\to 0}\cos x=\infty\times 1=\infty$

B. 1)

x	$\frac{\sin x}{x}$
1	0.841470985
0.9	0.870363233
0.8	0.896695114
0.7	0.920310982
0.6	0.941070789
0.5	0.958851077
0.4	0.973545856
0.3	0.985067356
0.2	0.993346654
0.1	0.998334166
0.01	0.999983333
0.001	0.999999833
0.0001	0.999999998

x	$\frac{\sin x}{x}$
-1	0.841470985
-0.9	0.870363233
-0.8	0.896695114
-0.7	0.920310982
-0.6	0.941070789
-0.5	0.958851077
-0.4	0.973545856
-0.3	0.985067356
-0.2	0.993346654
-0.1	0.998334166
-0.01	0.999983333
-0.001	0.999999833
-0.0001	0.999999998

2) $\lim_{x\to 0^+} \frac{\sin x}{x} = 1$ 3) $\lim_{x\to 0^-} \frac{\sin x}{x} = 1$

4) Since both limits on each side are equal to 1 then $\lim_{x\to 0} \frac{\sin x}{x} = 1$

Application activity 4.6

1) $\frac{\pi}{4}$ 2) 2 3) 0 4) 3

Activity 4.7

Materials

Exercises book, pens, calculator

Answers

1. a) Let $y = \cos^{-1}(-1)$. This is equivalent to $\cos y = -1, 0 \le y \le \pi$. The only value of y satisfying these conditions is π. So $\cos^{-1}(-1) = \pi$

b) Let $y=\tan^{-1}(-1)$. This is equivalent to $\tan y=-1, -\frac{\pi}{2}<y<\frac{\pi}{2}$. The only value of y satisfying these conditions is $-\frac{\pi}{4}$. So $\tan^{-1}(-1)=-\frac{\pi}{4}$

c) Let $y=\csc^{-1}(1)$. This is equivalent to $\csc y=1, \frac{\pi}{2}\le y\le\frac{\pi}{2}, \mathrm{y}\ne 0$. The only value of y satisfying these conditions is $\frac{\pi}{2}$. So $\csc^{-1}(1)=\frac{\pi}{2}$

d) Let $y=\cos^{-1}\left(\frac{1}{2}\right)$. This is equivalent to $\cos y=\frac{1}{2}, 0\le y\le\pi$. The only value of y satisfying these conditions is $\frac{\pi}{3}$. So $\cos^{-1}\left(\frac{1}{2}\right)=\frac{\pi}{3}$

2. a) $\lim\limits_{x\to 1}\cos^{-1}\left(1-2x^2\right)=\cos^{-1}(-1)=\pi$

b) $\lim\limits_{x\to 0}\tan^{-1}(x-1)=\tan^{-1}(-1)=-\frac{\pi}{4}$

c) $\lim\limits_{x\to 1}\cos^{-1}\frac{x-1}{1-x^2}=\frac{0}{0}\ \mathrm{I.C}$. Remove this I.C

$\lim\limits_{x\to 1}\frac{x-1}{1-x^2}=\lim\limits_{x\to 1}\frac{1}{-2x}=-\frac{1}{2}$. But $\cos^{-1}\left(-\frac{1}{2}\right)=\frac{2\pi}{3}$.

Thus, $\lim\limits_{x\to 1}\cos^{-1}\frac{x-1}{1-x^2}=\frac{2\pi}{3}$

d) $\lim\limits_{x\to 1}\frac{x^2-1}{\sin^{-1}(1-x)}=\frac{0}{0}\ \mathrm{I.C}$. Remove this I.C

$$\lim_{x\to 1}\frac{x^2-1}{\sin^{-1}(1-x)}=\lim_{x\to 1}\frac{2x}{-\frac{1}{\sqrt{1-(1-x)^2}}}=\lim_{x\to 1}\frac{2x}{-\frac{1}{\sqrt{2x-x^2}}}=-2$$

Thus $\lim\limits_{x\to 1}\frac{x^2-1}{\sin^{-1}(1-x)}=-2$

Application activity 4.7

1) $\frac{3\pi}{4}$ 2) $\frac{\pi}{6}$ 3) $\frac{3\pi}{4}$ 4) $\frac{\pi}{4}$

4.6.3 Differentiation of trigonometric functions and their inverses

a) Content summary

This section looks at the derivative of trigonometric functions and their inverses.

Derivative of trigonometric functions

1. $\frac{d(\sin u)}{dx}=\cos u\frac{du}{dx}$
2. $\frac{d(\cos u)}{dx}=-\sin u\frac{du}{dx}$
3. $\frac{d(\tan u)}{dx}=\sec^2 u\frac{du}{dx}$
4. $\frac{d(\cot u)}{dx}=-\csc^2 u\frac{du}{dx}$
5. $\frac{d(\sec u)}{dx}=\sec u\tan u\frac{du}{dx}$
6. $\frac{d(\csc u)}{dx}=-\csc u\cot u\frac{du}{dx}$

Derivative of inverse trigonometric functions

1. $\frac{d(\arcsin u)}{dx}=\frac{1}{\sqrt{1-u^2}}\frac{du}{dx}$
2. $\frac{d(\arccos u)}{dx}=-\frac{1}{\sqrt{1-u^2}}\frac{du}{dx}$
3. $\frac{d(\arctan u)}{dx}=\frac{1}{1+u^2}\frac{du}{dx}$
4. $\frac{d(\text{arccot}\, u)}{dx}=-\frac{1}{1+u^2}\frac{du}{dx}$
5. $\frac{d(\text{arcsec}\, u)}{dx}=\frac{1}{|u|\sqrt{u^2-1}}\frac{du}{dx}$
6. $\frac{d(\text{arccsc}\, u)}{dx}=-\frac{1}{|u|\sqrt{u^2-1}}\frac{du}{dx}$

c) Teaching guidelines

Let learners know trigonometric identities. The trigonometric identities will be used in this section

- Organise class into groups. Request each group to have a group leader who will present their findings to the class.
- Ask each group to do activity related to the lesson they are going to learn. The learners may need your assistance to do any activity. Help them to understand the activity. After group discussion, invite some or all groups for presentation of their work.

- After activity presentation, capture the main points from the presentation of the learners and summarise them. Guide the learners through given examples in Learner's book or through your own examples.
- Ask learners what they learned in day lesson to ensure that they understood what they have learned.
- Request learners to do exercises in their respective groups. Request learners to correct exercises on chalkboard and give them individual evaluation. Remember to give them homework.

Answers to activities

Activity 4.8

Materials

Exercise books, pens

Answers

1. $\forall x_0 \in \mathbb{R}$

$$(\sin x_0)' = \lim_{x\to x_0} \frac{\sin x - \sin x_0}{x - x_0}$$

$$= \lim_{x\to x_0} \frac{2\cos\frac{x+x_0}{2}\sin\frac{x-x_0}{2}}{x-x_0}$$

$$= \lim_{x\to x_0} \cos\frac{x+x_0}{2} \lim_{x\to x_0} \frac{2\sin\frac{x-x_0}{2}}{x-x_0}$$

$$= \lim_{x\to x_0} \cos\frac{x+x_0}{2} \lim_{x\to x_0} \frac{2\sin\frac{x-x_0}{2}}{2\frac{x-x_0}{2}}$$

$$= \lim_{x\to x_0} \cos\frac{x+x_0}{2} \lim_{x\to x_0} \frac{\sin\frac{x-x_0}{2}}{\frac{x-x_0}{2}}$$

$$= \left(\cos\frac{x_0+x_0}{2}\right)\times 1$$

$$= \cos x_0$$

Thus, $\forall x \in \mathbb{R}, \ (\sin x)' = \cos x$

2. $\cos x = \sin\left(\frac{\pi}{2} - x\right)$

$$(\cos x)' = \left[\sin\left(\frac{\pi}{2} - x\right)\right]'$$

$$= \left(\frac{\pi}{2} - x\right)' \cos\left(\frac{\pi}{2} - x\right)$$

$$= -\cos\left(\frac{\pi}{2} - x\right)$$

$$= -\sin x$$

Thus, $\forall x \in \mathbb{R},\ (\cos x)' = -\sin x$

3. $\cos x = \sin\left(\frac{\pi}{2} - x\right)$

$$(\cos x)' = \left[\sin\left(\frac{\pi}{2} - x\right)\right]'$$

$$= \left(\frac{\pi}{2} - x\right)' \cos\left(\frac{\pi}{2} - x\right)$$

$$= -\cos\left(\frac{\pi}{2} - x\right)$$

$$= -\sin x$$

Thus, $\forall x \in \mathbb{R},\ (\cos x)' = -\sin x$

Application activity 4.8

1. $2x\cos(x^2+3)$
2. $6x\cos\left(x^2+4\right)\sin^2\left(x^2+4\right)$
3. $-6x\sin 3x^2$
4. $-6\cos^2 2x\sin 2x$

Activity 4.9

Materials

Exercise books, pens

Answers

1. $\tan x = \dfrac{\sin x}{\cos x}$

$$(\tan x)' = \left(\frac{\sin x}{\cos x}\right)'$$

$$= \frac{(\sin x)' \cos x - \sin x (\cos x)'}{(\cos x)^2}$$

$$= \frac{\cos x \cos x + \sin x \sin x}{(\cos x)^2}$$

$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x}$$

$$= \frac{1}{\cos^2 x}$$

$$(\tan x)' = \frac{1}{\cos^2 x}$$

$$= \sec^2 x$$

$$= 1 + \tan^2 x$$

2. $\cot x = \tan\left(\dfrac{\pi}{2} - x\right)$ $\quad (\cot x)' = \left[\tan\left(\dfrac{\pi}{2} - x\right)\right]'$

$$= \frac{\left(\frac{\pi}{2} - x\right)'}{\cos^2\left(\frac{\pi}{2} - x\right)}$$

$$= \frac{-1}{\sin^2 x}$$

$$(\cot x)' = \frac{-1}{\sin^2 x}$$

$$= -\csc^2 x$$

$$= -\left(1 + \cot^2 x\right)$$

Application activity 4.9

1. $\tan x + x\left(1+\tan^2 x\right)$
2. $3\left[1+\tan^2\left(3x+2\right)\right]$
3. $-2x\left[1+\cot^2\left(x^2-5\right)\right]$
4. $-4\sin x\left(1+\cot^2 4x\right)+\cos x\cot 4x$

Activity 4.10

Materials

Exercise books, pens

Answers

1. $\sec x = \dfrac{1}{\cos x}$

$$\left(\sec x\right)' = \left(\frac{1}{\cos x}\right)'$$
$$= \frac{\sin x}{\cos^2 x}$$
$$= \frac{1}{\cos x}\frac{\sin x}{\cos x}$$
$$= \sec x \tan x$$

2. $\csc x = \dfrac{1}{\sin x}$

$$\left(\csc x\right)' = \left(\frac{1}{\sin x}\right)'$$
$$= \frac{-\cos x}{\sin^2 x}$$
$$= \frac{-1}{\sin x}\frac{\cos x}{\sin x}$$
$$= -\csc x \cot x$$

Application activity 4.10

1. $3\sec\left(3x+2\right)\tan\left(3x+2\right)$
2. $\theta^2 \csc 2\theta\left(3-2\theta\cot 2\theta\right)$
3. $12\sec^4 3x\tan 3x$

Activity 4.11

Materials

Exercise books, pens

Answers

1. $f(x)=\sin^{-1}x$ for $x\in[-1,1]$ and $x=\sin y$ for $y\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$ where $y=f(x)$.

$$(\sin^{-1}x)'=\frac{1}{(\sin y)'}=\frac{1}{\cos y}=\frac{1}{\cos(\sin^{-1}x)}$$

$$=\frac{1}{\sqrt{1-\sin^2(\sin^{-1}x)}} \quad \text{since } \cos x=\sqrt{1-\sin^2 x}$$

$$=\frac{1}{\sqrt{1-x^2}}$$

2. $f(x)=\cos^{-1}x$ for $x\in[-1,1]$ and $x=\cos y$ for $y\in[0,\pi]$ where $y=f(x)$

$$(\cos^{-1}x)'=\frac{1}{(\cos y)'}=\frac{1}{-\sin y}=\frac{-1}{\sin(\cos^{-1}x)}$$

$$=\frac{-1}{\sqrt{1-\cos^2(\cos^{-1}x)}} \quad \text{since } \sin x=\sqrt{1-\cos^2 x}$$

$$=\frac{-1}{\sqrt{1-x^2}}$$

Application activity 4.11

1. $\frac{1}{|x|\sqrt{x^2-1}}$
2. $\frac{-2x}{\sqrt{1-x^4}}$
3. $\frac{-1}{\sqrt{2x-x^2}}$
4. $\frac{1}{\sqrt{2x}\sqrt{1-2x}}$

Activity 4.12

Materials

Exercise books, pens

Answers

1. $f(x)=\tan^{-1}x$ for $x\in\mathbb{R}$ and $x=\tan y$ for $y\in\left]-\frac{\pi}{2},\frac{\pi}{2}\right[$ where $y=f(x)$.

$$(\tan^{-1}x)'=\frac{1}{(\tan y)'}=\frac{1}{1+\tan^2 y}=\frac{1}{1+\tan^2(\tan^{-1}x)}=\frac{1}{1+x^2}$$

2. $f(x)=\cot^{-1}x$ for $x\in\mathbb{R}$ and $x=\cot y$ for $y\in\left]0,\pi\right[$ where $y=f(x)$

$$\left(\cot^{-1}x\right)'=\frac{1}{(\cot y)'}=\frac{-1}{1+\cot^2 y}=\frac{-1}{1+\cot^2\left(\cot^{-1}x\right)}=\frac{-1}{1+x^2}$$

Application activity 4.12

1. $f'(x)=\frac{1}{2\sqrt{x}(1+x)}$ 2. $f'(x)=\frac{1}{|x|\sqrt{x^2-1}}+\frac{1}{1+x^2}$

3. $f'(x)=\frac{-1}{2x\sqrt{x-1}}$

Activity 4.13

Materials

Exercise books, pens

Answers

1. $f(x)=\sec^{-1}x$ for $x\le -1$ or $x\ge 1$ and $x=\sec y$ for $y\in[0,\pi], y\ne\frac{\pi}{2}$ where $y=f(x)$

$$\left(\sec^{-1}x\right)'=\frac{1}{(\sec y)'}=\frac{1}{\sec y\tan y}=\frac{1}{x\sqrt{x^2-1}}$$

2. $f(x)=\csc^{-1}x$ for $x\le -1$ or $x\ge 1$ and $x=\csc y$ for $y\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right], y\ne 0$ where $y=f(x)$

$$\left(\csc^{-1}x\right)'=\frac{1}{(\csc y)'}=\frac{-1}{\csc y\cot y}=\frac{-1}{x\sqrt{x^2-1}}$$

Application activity 4.13

1. $\frac{1}{(2x+1)\sqrt{x^2+x}}$ 2. $\frac{1}{x\sqrt{25x^2-1}}$

3. $\frac{-2x}{\left(x^2+1\right)\sqrt{x^4+2x^2}}$

Activity 4.14

Materials

Exercises book, pens

Answers

1. $g'(x) = 4\cos(4x)$
2. $-16\sin(4x)$
3. $-64\cos(4x)$
4. $256\sin(4x)$
5. $1024\cos(4x)$

Application activity 4.14

1. a) $y' = \tan^2 x + 1,\ y'' = 2\tan^3 x + 2\tan x,$

   $y''' = 6\tan^4 x + 8\tan^2 x + 2$

   b) $y' = \sec x \tan x,\ y'' = \sec x\left(1 + 2\tan^2 x\right),$

   $y''' = \sec x\left(6\tan^3 x + 5\tan x\right)$

   c) $y' = -2x\sin\left(x^2\right),\ y'' = -2\sin\left(x^2\right) - 4x^2\cos\left(x^2\right),$

   $y''' = -12x\cos\left(x^2\right) + 8x^3\sin\left(x^2\right)$

   d) $y' = \frac{\cos x}{x} - \frac{\sin x}{x^2},\ y'' = -2\frac{\cos x}{x^2} - \frac{\sin x}{x} + 2\frac{\sin x}{x^3},$

   $y''' = -\frac{\cos x}{x} + 6\frac{\cos x}{x^3} + 3\frac{\sin x}{x^2} - 6\frac{\sin x}{x^4}$

2. a) $2^{n-1}\cos\left(2x + \frac{n\pi}{2}\right)$

   b) $\frac{1}{2}\left[5^n \sin\left(5x + \frac{n\pi}{2}\right) - \sin\left(x + \frac{n\pi}{2}\right)\right]$

4.6.4 Applications of trigonometric functions in real life

Activity 4.15

Materials

Exercises book, pens

Answers

By reading text books or accessing internet, learners will discuss on harmonic motion and how differentiation of trigonometry functions is used to find velocity, acceleration and jerk of an object if the function representing its position is known.

If we have the function representing the position, say $S(t)$, then:

- The velocity of the object is $v = \frac{ds}{dt}$
- The acceleration of the object is $a = \frac{d^2 s}{dt^2}$
- The jerk of the object is $j = \frac{d^3 s}{dt^3}$

Application activity 4.15

1. a) $3m$ b) $-9\pi\ {}^{m}/_{s}$ c) $-27\pi\ {}^{m}/_{s^2}$

   d) $\left(3\pi t + \frac{\pi}{3}\right)$ e) $\frac{3}{2}Hz$ f) $\frac{2}{3}$

2. a) amplitude x_m is $4m$

   frequency f is $0.5\,Hz$

   period T is 2

   angular frequency ω is π

   b) velocity is $\frac{dx}{dt} = -4\pi \sin\left(\pi t + \frac{\pi}{4}\right)$

   acceleration $\frac{d^2 x}{dt^2} = -4\pi^2 \cos\left(\pi t + \frac{\pi}{4}\right)$

   c) displacement at $t = 1$ is $-2\sqrt{2}\pi\, m$

   velocity at $t = 1$ is $2\pi\sqrt{2}\ {}^{m}/_{s}$

   acceleration at $t = 1$ is $2\pi^2\sqrt{2}\ {}^{m}/_{s^2}$

   d) the maximum speed $4\pi\ {}^{m}/_{s}$

   maximum acceleration $4\pi^2\ {}^{m}/_{s^2}$

4.7 End of Unit Assessment

1. a) $\frac{2\pi}{3}$ b) π c) 2π
   d) not periodic e) π f) not periodic
2. a) neither even nor odd b) even c) odd
   d) odd
3. a) 2 b) $\frac{3}{2}$ c) 1 d) 2
   e) 7 f) $\frac{2}{3}$ g) $\frac{1}{16}$ h) $\frac{-11}{9}$
   i) $\frac{15}{7}$ j) $\frac{1}{2}$ k) 1 l) $\cos a$
   m) $-\sin a$ n) 0 o) 4 p) 0
   q) 2 r) $\frac{1}{2}$ s) $\frac{3}{4}$ t) $\frac{1}{2}$
   u) $\frac{\pi^2}{2}$ v) 0 w) $\frac{\sqrt{2}}{32}$
4. a) $3\sec x\tan x+10\csc^2 x$ b) $-12x^{-5}-2x\tan x-x^2\sec^2 x$
   c) $5\cos 2x-4\csc x\cot x$ d) $\frac{3\cos t-2}{(3-2\cos t)^2}$
   e) $-48x\sin\left(6x^2+5\right)$ f) $72x^3\sin^2\left(2x^4+1\right)\cos\left(2x^4+1\right)$
   g) $4\left(x-\cos^2 x\right)^3\left(1+\sin 2x\right)$ h) $\frac{2\sin 4x-4(2x+3)\cos 4x}{\sin^2 4x}$
   i) $\frac{-2x^2}{\sqrt{1-x^2}}$ j) $\frac{-1}{\sqrt{1-x^2}}$
   k) $\frac{-2}{x\sqrt{x^2-4}}$ l) $\frac{x^2-1}{x\sqrt{x^2-1}}$
   m) $\sin^{-1} x$
5. The amount of money in the bank account will be increasing during the following intervals:
   $2.1588<t<5.3004$, $8.4420<t<10$
6. a) $-\frac{\pi}{3}$ b) $\sqrt{1-t^2}$ c) $\frac{\pi}{2}$
   d) -1 e) $\frac{\pi}{3}$ f) $\frac{\pi}{4}$

Unit 5 Vector Space of Real Numbers

5.1 Key unit competence:

Apply properties of vectors and their operations in $\mathbb{R}$ to solve problems related to angles between vectors.

5.2 Objectives

By the end of this unit, the learners should be able to:

- Define and apply different operations on vectors
- Find the norm of a vector
- Calculate the scalar and vector product of two vectors
- Calculate the angle between two vectors
- Apply and transfer the skills of vectors to other area of knowledge.

5.3 List of lessons for unit 5

Week	Lessons	Content	Number of Periods
19	1	Vector of $\mathbb{R}^3$ and examples (e.g.: gravitational force).	1
	2	Operations of vectors of $\mathbb{R}^3$ (addition, subtraction, scalar multiplication by a scalar)	2
20-21	3	Introduction to Euclidian vector space $\mathbb{R}^3$	1
	4	Scalar or Dot product of two vectors and properties	1
	5	Magnitude (or norm or length) of a vector	1
	6	Angle between two vectors	
	7	Vector product and its properties	1
	8	Applications of scalar and vector products: Work done by the force, area of a parallelogram.	1
	9	End unit assessment and Remediation	1

5.4 Materials Required

Exercise books, pens, instrument of geometry, calculator

5.5 Guidance to the introductory activity

- Guide learners to form small groups and to work on the introductory activity;
- Walk around to monitor the work of each group and to assist any group in need;
- After a given time, invite learners to present their findings and harmonize them.

Answers to the introductory activity

For this activity the teacher should let the students discuss in their groups, through their discussion the answers of sub-questions a), b) and c) will be extracted from it. The discussion is also based on the prior knowledge of vector space in two dimensions that they learned in senior four.

For example

a) Addition of vectors is: associative, commutative, closure in the set of vectors; ….

b) The vector can increase, decrease or remain constant depending on the scalar multiplied by it.

c) Example of vectors in real life; velocity, force, acceleration.

5.6 Content and activities

5.6.1 Scalar product of two vectors

a) Content summary

Scalar product

The scalar product of two vectors of space is the application $E_0 \times E_0 \to \mathbb{R}$.

Algebraically, the scalar product of vectors $\vec{u} = (a_1, a_2, a_3)$ and $\vec{v} = (b_1,\ b_2,\ b_3)$ of space is defined by $\vec{u} \cdot \vec{v} = a_1b_1 + a_2b_2 + a_3b_3$.

b) Teaching guidelines

Let learners know how to find scalar product of two vectors and magnitude of a vector in two dimensions. In three dimensions, there is a third component, z

- Organise class into groups. Request each group to have a group leader who will present their findings to the class.
- Ask each group to do activity related to the lesson they are going to learn. The learners may need your assistance to do any activity. Help them to understand the activity. After group discussion, invite some or all groups for presentation of their work.
- After activity presentation, capture the main points from the presentation of the learners and summarise them. Guide the learners through given examples in Learner's book or through your own examples.
- Ask learners what they learned in day lesson to ensure that they understood what they have learned.
- Request learners to do exercises in their respective groups. Request learners to correct exercises on chalkboard and give them individual evaluation. Remember to give them homework.

c) Answers to activities

Activity 5.1

Materials

Exercise books, pens, calculators

Answers

a) -5 b) 2

Application activity 5.1

1. 26 2. -9 3. -195 4. -125

5.6.2 Magnitude (or norm or length) of a vector

Magnitude of a vector

The magnitude of the vector $\vec{u}$ noted by $\left\|\vec{u}\right\|$ is defined as its length.

If $\vec{u}=(a,b,c)$ then $\left\|\vec{u}\right\|=\sqrt{a^2+b^2+c^2}$.

Distance between two points

The distance between two points $A(a_1,a_2,a_3)$ and $B(b_1,b_2,b_3)$ denoted, $d(A,B)$ is

$$d(A,B)=\left\|\overrightarrow{AB}\right\|=\sqrt{(b_1-a_1)^2+(b_2-a_2)^2+(b_3-a_3)^2}$$

Activity 5.2

Materials

Exercise books, pens, calculators

Answers

a) $5\sqrt{2}$ b) $\sqrt{46}$

Application activity 5.2

1. 7 2. $\sqrt{26}$ 3. $\sqrt{610}$ 4. $\sqrt{746}$

5.6.3 Angle between two vectors

a) Content summary

Angle between two vectors $\vec{u}$ and $\vec{v}$ is such that

$$\cos(\vec{u},\vec{v})=\frac{\vec{u}\cdot\vec{v}}{\left\|\vec{u}\right\|\cdot\left\|\vec{v}\right\|}$$

b) Teaching guidelines

Let learners know how to find scalar product of two vectors and magnitude of a vector in two dimensions. In three dimensions, there is a third component, z Organise class into groups. Request each group to have a group leader who will present their findings to the class.

- Ask each group to do activity related to the lesson they are going to learn. The learners may need your assistance to do any activity. Help them to understand the activity. After group discussion, invite some or all groups for presentation of their work.
- After activity presentation, capture the main points from the presentation of the learners and summarise them. Guide the learners through given examples in Learner's book or through your own examples.
- Ask learners what they learned in day lesson to ensure that they understood what they have learned.
- Request learners to do exercises in their respective groups. Request learners to correct exercises on chalkboard and give them individual evaluation. Remember to give them homework.

c) Answers to activities

Activity 5.3

Materials

Exercise books, pens, calculators

Answers

1. $\vec{u}\cdot\vec{v} = -21$
2. $\|\vec{u}\|\|\vec{v}\| = 21$
3. $\cos^{-1}\left(\frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\|\vec{v}\|}\right) = \pi + 2k\pi = (1+2k)\pi,\ k \in \mathbb{Z}$

Application activity 5.3

1. a) 47.5^0 b) 47^0 c) 70.6^0 d) 31.7^0
2. a) $\cos\alpha = \frac{2}{\sqrt{29}},\ \cos\beta = \frac{3}{\sqrt{29}},\ \cos\gamma = \frac{4}{\sqrt{29}}$

b)b) $\cos\alpha = \frac{4}{\sqrt{17}}$, $\cos\beta = \frac{-1}{\sqrt{17}}$, $\cos\gamma = \frac{4}{3\sqrt{17}}$

c) $\cos\alpha = \frac{1}{\sqrt{201}}$, $\cos\beta = \frac{-2}{\sqrt{201}}$, $\cos\gamma = \frac{-14}{\sqrt{201}}$

d) $\cos\alpha = 1$, $\cos\beta = 0$, $\cos\gamma = 0$

5.6.4 Vector product

a) Content summary

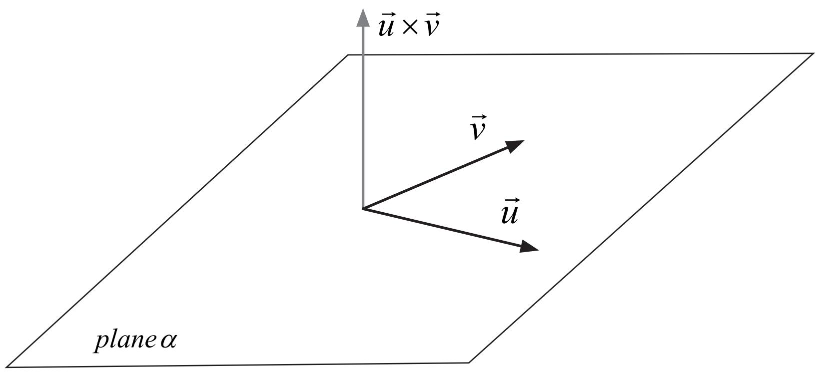
The vector product (or cross product or Gibbs vector product) of

$\vec{u} = (a_1, a_2, a_3)$ and $\vec{v} = (b_1, b_2, b_3)$ is denoted and defined by

$$\vec{u} \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \vec{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \vec{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \vec{k}$$

Or

$$\vec{u} \times \vec{v} = \left(\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}, -\begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix}, \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \right)$$



The vector product of any two vectors is perpendicular to each of these vectors.

Mixed product

The mixed product of the vectors $\vec{u} = (a_1, a_2, a_3)$, $\vec{v} = (b_1, b_2, b_3)$ and $\vec{w} = (c_1, c_2, c_3)$ is denoted and defined by $[\vec{u}, \vec{v}, \vec{w}] = \vec{u} \cdot (\vec{v} \times \vec{w})$

b) Teaching guidelines

Let learners know how to find determinant of order three and scalar product of two vectors.

- Organise class into groups. Request each group to have a group leader who will present their findings to the class.
- Ask each group to do activity related to the lesson they are going to learn. The learners may need your assistance to do any activity. Help them to understand the activity. After group discussion, invite some or all groups for presentation of their work.
- After activity presentation, capture the main points from the presentation of the learners and summarise them. Guide the learners through given examples in Learner's book or through your own examples.
- Ask learners what they learned in day lesson to ensure that they understood what they have learned.
- Request learners to do exercises in their respective groups. Request learners to correct exercises on chalkboard and give them individual evaluation. Remember to give them homework.

c) Answers to activities

Activity 5.4

Materials

Exercise books, pens, calculators

Answers

1. Let $\vec{w}=(a,b,c)$ be that vector. $\vec{w}$ is perpendicular to both $\vec{u}=(4,2,1)$ and $\vec{v}=(-2,4,2)$ if $4a+2b+c=0$ and $-2a+4b+2c=0$. Solving we have $\vec{w}=(0,1,-2)$ or any other multiple of this vector.

2. $\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 4 & 2 & 1 \\ -2 & 4 & 2 \end{vmatrix} = \vec{i}\begin{vmatrix} 2 & 1 \\ 4 & 2 \end{vmatrix} - \vec{j}\begin{vmatrix} 4 & 1 \\ -2 & 2 \end{vmatrix} + \vec{k}\begin{vmatrix} 4 & 2 \\ -2 & 4 \end{vmatrix}$

$= -10\vec{j} + 20\vec{k}$

3. Vector obtained in 2 is a multiple of vector obtained in 1.

Application activity 5.4

1. $(2,-10,2)$ 2. $(-2,-2,0)$ 3. $(2,-10,-8)$

4. $(-1,1,3)$ 5. $(2,-12,1)$

Activity 5.5

Materials

Exercise books, pens

Answers

1. $\left(\begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix}, -\begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix}, \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}\right)$ or

$(b_2c_3 - c_2b_3, -b_1c_3 + c_1b_3, b_1c_2 - c_1b_2)$

2. $\left(a_1\begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix}, -a_2\begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix}, a_3\begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}\right)$ or

$(a_1b_2c_3 - a_1c_2b_3, -a_2b_1c_3 + a_2c_1b_3, a_3b_1c_2 - a_3c_1b_2)$

Application activity 5.5

1. -16 2. -8 3. -82

4. 17 5. -7

5.6.5 Applications of vector space in real life

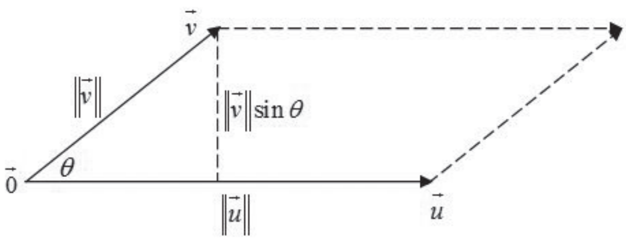
a) Content summary

Work done as a scalar product

If a constant force F acting on a particle displaces it from A to B, the work done is given by $work\ done = \overrightarrow{F} \cdot \overrightarrow{AB}$

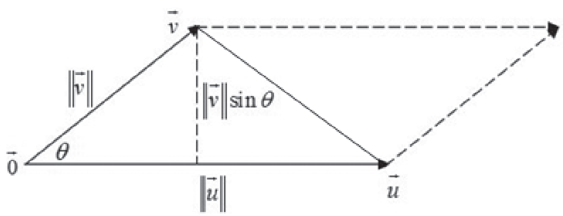
Area of a parallelogram

Area of a parallelogram with vectors $\vec{u}$ and $\vec{v}$ as two adjacent sides is $S_{\square} = \left\|\vec{u} \times \vec{v}\right\|$

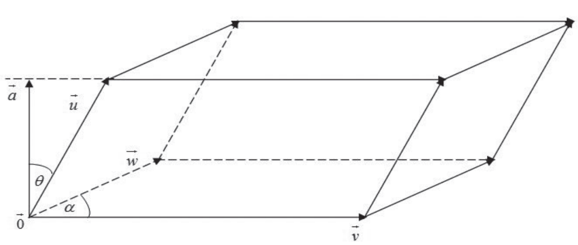


Area of a triangle

Thus, the area of triangle with vectors $\vec{u}$ and $\vec{v}$ as two sides is $S_{\triangle} = \frac{1}{2}\left\|\vec{u} \times \vec{v}\right\|$



Volume of a parallelepiped



The volume of a parallelepiped which has vectors $\vec{u}$, $\vec{v}$ and $\vec{w}$ as three concurrent edges, where $\vec{v}$ and $\vec{w}$ define its base, is given by

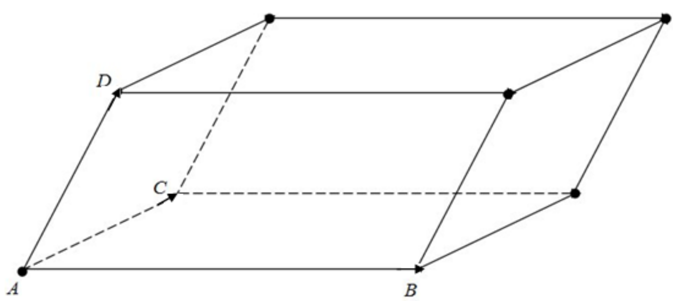
$$V = \left|\vec{u}\cdot\left(\vec{v}\times\vec{w}\right)\right|$$

Remember that if $\vec{u} = (a_1, a_2, a_3)$, $\vec{v} = (b_1, b_2, b_3)$ and $\vec{w} = (c_1, c_2, c_3)$,

then $\vec{u}\cdot\left(\vec{v}\times\vec{w}\right) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$.

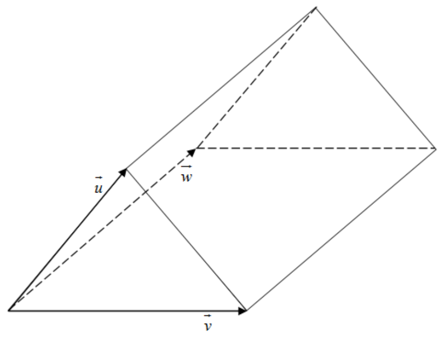
If the parallelepiped is defined by four points $A(a_1, a_2, a_3)$, $B(b_1, b_2, b_3)$, $C(c_1, c_2, c_3)$ and $D(d_1, d_2, d_3)$, its volume is

$$V = \left|\overrightarrow{AD}\cdot\left(\overrightarrow{AB}\times\overrightarrow{AC}\right)\right|$$



Volume of a triangular prism

The parallelepiped can be split into 2 triangular prism of equal volume. Since the volume of a parallelepiped is the magnitude of the mixed product, then the volume of a triangular prism is equal to ½ of the magnitude of the mixed product.

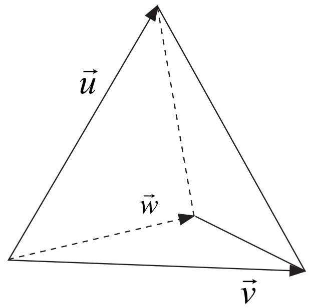


$$V = \frac{1}{2}\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Volume of a tetrahedron

The parallelepiped can be split into 6 tetrahedra of equal volume.

Since the volume of a parallelepiped is the magnitude of the mixed product, then the volume of a tetrahedron is equal to $\frac{1}{6}$ of the magnitude of the mixed product.



$$V = \frac{1}{6}\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Remark

A tetrahedron is also called **triangular pyramid**.

b) Teaching guidelines

Let learners know how to find scalar product, vector product, mixed product and magnitude of a vector.

- Organise class into groups. Request each group to have a group leader who will present their findings to the class.
- Ask each group to do activity related to the lesson they are going to learn. The learners may need your assistance to do any activity. Help them to understand the activity. After group discussion, invite some or all groups for presentation of their work.
- After activity presentation, capture the main points from the presentation of the learners and summarise them. Guide the learners through given examples in Learner's book or through your own examples.
- Ask learners what they learned in day lesson to ensure that they understood what they have learned.
- Request learners to do exercises in their respective groups. Request learners to correct exercises on chalkboard and give them individual evaluation. Remember to give them homework.

c) Answers to activities

Activity 5.6

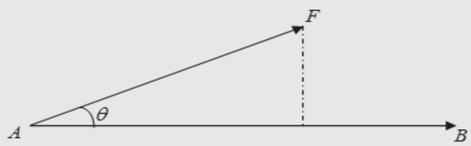
Materials

Exercise books, pens

Answers

If a constant force F acting on a particle displaces it from A to B, then,

$$\begin{aligned} work\ done &= (component\ of\ F)\cdot Displacement \\ &= (F\cos\theta)\cdot AB \\ &= \vec{F}\cdot\overrightarrow{AB} \end{aligned}$$



Application activity 5.6

1. 16 unit of work
2. 55 unit of work
3. $\frac{141}{2}$ unit of work
4. 20 unit of work

Activity 5.7

Materials

Exercise books, pens

Answers

Since the base and the height of this parallelogram are $\|\vec{u}\|$ and $\|\vec{v}\|\sin\theta$ respectively, the area is $S_{\square} = \|\vec{u}\|\|\vec{v}\|\sin\theta$. But $\|\vec{u}\|\|\vec{v}\|\sin\theta = \|\vec{u}\times\vec{v}\|$.

Then area is $S_{\square} = \|\vec{u}\times\vec{v}\|$.

Thus, the magnitude of the vector product of two vectors $\vec{u}$ and $\vec{v}$ represents the area of a parallelogram with vectors $\vec{u}$ and $\vec{v}$ as two adjacent sides.

Application activity 5.7

1. a) $220\sqrt{2}$ Sq. units b) $\sqrt{445}$ Sq. units
2. $\frac{3\sqrt{2}}{2}$ Sq. units
3. $\sqrt{29}$ Sq. units
4. $\frac{\sqrt{6}}{2}$ Sq. units

Activity 5.8

Materials

Exercise books, pens

Answers

The base of this parallelepiped is defined by the vectors $\vec{v}$ and $\vec{w}$.

Then, the area of the base is $S = \|\vec{v} \times \vec{w}\| = \|\vec{v}\|\|\vec{w}\|\sin\alpha$.

The height of this parallelepiped is $\|\vec{a}\|$.

Since the vector $\vec{a}$ is not known we can find the height in terms of $\|\vec{u}\|$. We see that $\cos\theta = \frac{\|\vec{a}\|}{\|\vec{u}\|} \Leftrightarrow \|\vec{a}\| = \|\vec{u}\|\cos\theta$.

The angle θ is the angle between the vector $\vec{a}$ and vector $\vec{u}$ but it is also the angle between the vector $\vec{u}$ and the vector given by the vector product $\vec{v} \times \vec{w}$ since this cross product is perpendicular to both $\vec{v}$ and $\vec{w}$.

Now, the volume of the parallelepiped is product of the area of the base and the height.

Then,

$$V = \|\vec{v}\|\|\vec{w}\|\sin\alpha\|\vec{u}\|\cos\theta \Leftrightarrow V = \|\vec{v}\|\|\vec{w}\|\|\vec{u}\|\sin\alpha\cos\theta$$

$$\Leftrightarrow V = \|\vec{u}\|\left(\|\vec{v}\|\|\vec{w}\|\sin\alpha\right)\cos\theta$$

$$\Leftrightarrow V = \|\vec{u}\|\|\vec{v} \times \vec{w}\|\cos\theta \quad \Leftrightarrow V = \left|\vec{u} \cdot \left(\vec{v} \times \vec{w}\right)\right|$$

Thus the volume of a parallelepiped which has vectors $\vec{u}$, $\vec{v}$ and $\vec{w}$ as three concurrent edges, where $\vec{v}$ and $\vec{w}$ define its base, is given by

$V = \left|\vec{u}\cdot\left(\vec{v}\times\vec{w}\right)\right|$

Application activity 5.8

1. a) $\frac{3}{2}$ cubic units   b) $\frac{35}{2}$ cubic units
2. a) 2 cubic units   b) $\frac{10}{3}$ cubic units
3. 20 cubic units
4. $7\frac{1}{3}$ cubic units

5.7 End of Unit Assessment

1. a) 0   b) $\vec{k}$ c) $-\vec{j}$
   d) $\vec{i}$ e) 0   f) 0
2. a) 7 sq. units   b) 30 sq. units
   c) 15 sq. units   d) $7\sqrt{3}$ sq. units
3. a) 20   b) 13   c) 39
4. a) $(-20,-67,-9)$ b) $(-78,52,-26)$
   c) $(24,0,-16)$ d) $(-12,-22,-8)$
   e) $(0,-56,-392)$ f) $(0,56,392)$
5. a) $\frac{\sqrt{374}}{2}$ sq. units   b) $9\sqrt{13}$ sq. units
6. ambiguous, needs parentheses
7. a) 16 cubic units   b) 45 cubic units
8. a) 9 cubic units   b) $\sqrt{122}$ sq. units
9. a) $\frac{9\sqrt{2}}{2}$ sq. units   b) $\frac{1}{2}\sqrt{b^2c^2+c^2a^2+a^2b^2}$ sq. units
10. $\frac{1}{2}\sqrt{\left(y_1z_2-y_2z_1\right)^2+\left(z_1x_2-z_2x_1\right)^2+\left(x_1y_2-x_2y_1\right)^2}$ square units
11. 6 cubic units
12. 77.88^0

Unit 6 Matrices and Determinant of Order 3

6.1 Key unit competence:

Apply matrix and determinant of order 3 to solve related problems.

6.2 Objectives

By the end of this unit, the learners should be able to:

- define and give example of matrix of order three.
- perform different operations on matrices of order three.
- find the determinant of order three.
- find the inverse of matrix of order three.
- solve system of three linear equations by matrix inverse method.

6.3 List of lessons unit 6

Week	Lessons	Content	Number of Periods
22	1	Introduction on square matrices of order 3: definition and examples	1
	2	Types of matrices and equality of matrices	2
23	3	Operations on matrices and properties: addition, subtraction and multiplication	3
24	4	Transpose of matrix	3
	5	Multiplication by a scalar	
25	6	Determinants of a matrix of order 3 and properties	3
26	7	Matrix inverse	3
27	8	Applications of determinants in solving problems from physics, medicine, or buying and selling	2
	9	End unit assessment and Remediation	1

6.4 Materials required

Exercise books, pens, calculators

6.5 Guidance to the introductory activity

- Guide learners to form small groups and to work on the introductory activity; learners will present given data in the matrix
- Walk around to monitor the work of each group and to assist any group in need;
- After a given time, invite learners to present their findings and harmonize them.
- Learners will use the given guidance and calculate the determinant of the matrix
- Learners will give other example of where matrices are used in real life

Answer to the introductory activity

$$A=\begin{pmatrix}7 & 2 & 9\\ 5 & 4 & 6\\ 8 & 7 & 9\end{pmatrix}$$

$$Det\,A=\begin{vmatrix}7 & 2 & 9\\ 5 & 4 & 6\\ 8 & 7 & 9\end{vmatrix}=-9$$

6.6 Content and activities

6.6.1 Square matrices of order three

a) content summary

- Addition and subtraction (only matrices of the same type can be subtracted)

If $A=\begin{pmatrix}a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{pmatrix}$ and $B=\begin{pmatrix}b_{11} & b_{12} & b_{13}\\ b_{21} & b_{22} & b_{23}\\ b_{31} & b_{32} & b_{33}\end{pmatrix}$, then

$$A+B=\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}+\begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix}=\begin{pmatrix} a_{11}+b_{11} & a_{12}+b_{12} & a_{13}+b_{13} \\ a_{21}+b_{21} & a_{22}+b_{22} & a_{23}+b_{23} \\ a_{31}+b_{31} & a_{32}+b_{32} & a_{33}+b_{33} \end{pmatrix}$$

$$A-B=\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}-\begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix}=\begin{pmatrix} a_{11}-b_{11} & a_{12}-b_{12} & a_{13}-b_{13} \\ a_{21}-b_{21} & a_{22}-b_{22} & a_{23}-b_{23} \\ a_{31}-b_{31} & a_{32}-b_{32} & a_{33}-b_{33} \end{pmatrix}$$

- Transpose

If $A=\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$, then $A^t=\begin{pmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{pmatrix}$

Properties of transpose of matrices

1. $\left(A^t\right)^t = A$
2. $(A + B)^t = A^t + B^t$
3. $(\alpha \cdot A)^t = \alpha \cdot A^t$

Multiplication

Two matrices A and B can be multiplied together if and only if the number of columns of A is equal to the number of rows of B.

$$M_{m\times n} \times M_{n\times p} = M_{m\times p}$$

$$A\times B=\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}\times\begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix}$$

$$=\begin{pmatrix} a_{11}b_{11}+a_{12}b_{21}+a_{13}b_{31} & a_{11}b_{12}+a_{12}b_{22}+a_{13}b_{32} & a_{11}b_{13}+a_{12}b_{23}+a_{13}b_{33} \\ a_{21}b_{11}+a_{22}b_{21}+a_{23}b_{31} & a_{21}b_{12}+a_{22}b_{22}+a_{23}b_{32} & a_{21}b_{13}+a_{22}b_{23}+a_{23}b_{33} \\ a_{31}b_{11}+a_{32}b_{21}+a_{33}b_{31} & a_{31}b_{12}+a_{32}b_{22}+a_{33}b_{32} & a_{31}b_{13}+a_{32}b_{23}+a_{33}b_{33} \end{pmatrix}$$

Properties of multiplication of matrices

Let A,B,C be matrices of order three

1. Associative

$$A\times(B\times C)=(A\times B)\times C$$

2. Multiplicative Identity

   $A \times I = A$, where I is the identity matrix with the same order as matrix *A*.

3. Not Commutative

   $A \times B = B \times A$

4. Distributive

   $A \times (B + C) = A \times B + A \times C$

5. $(A \times B)^t = B^t \times A^t$

Notice

- If $AB = 0$, it does not necessarily follow that $A = 0$ or $B = 0$.
- **Commuting matrices in multiplication**

  In general the multiplication of matrices is not commutative, i.e, $AB \neq BA$, but we can have the case where two matrices A and B satisfy $AB = BA$. In this case, A and B are said to be commuting.

b) Teaching guidelines

Let learners know what square matrix of order two is. Square matrix of order three will have three rows and three columns.

- Organise class into groups. Request each group to have a group leader who will present their findings to the class.
- Ask each group to do activity related to the lesson they are going to learn. The learners may need your assistance to do any activity. Help them to understand the activity. After group discussion, invite some or all groups for presentation of their work.
- After activity presentation, capture the main points from the presentation of the learners and summarise them. Guide the learners through given examples in Learner's book or through your own examples.

- Ask learners what they learned in day lesson to ensure that they understood what they have learned.
- Request learners to do exercises in their respective groups. Request learners to correct exercises on chalkboard and give them individual evaluation. Remember to give them homework.

c) Answers to activities

Activity 6.1

Materials

Exercise books, pens

Answers

$$\begin{pmatrix} -12 & 0 & -5 \\ 3 & -2 & 1 \\ 6 & 2 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

Application activity 6.1

1. a) $\begin{pmatrix} 2 & -3 & 11 \\ 2 & 13 & 1 \\ 0 & 11 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 23 & 24 \\ 1 & 43 & 44 \\ 1 & 1 & 2 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 12 \\ 1 & -20 & 4 \\ 0 & 18 & 6 \end{pmatrix},$

$\begin{pmatrix} -4 & 2 & 1 \\ -9 & 1 & 0 \\ 7 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 3 & 9 & 2 \\ 5 & 6 & 4 \\ 7 & 3 & -8 \end{pmatrix}$

b) $\begin{pmatrix} -1 & 2 & 4 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & -1 & 5 \\ 0 & 3 & 1 \\ 0 & 0 & 2 \end{pmatrix}, \begin{pmatrix} 11 & 2 & 7 \\ 0 & 10 & 9 \\ 0 & 0 & 3 \end{pmatrix},$

$\begin{pmatrix} 8 & -2 & 3 \\ 0 & 9 & -5 \\ 0 & 0 & 4 \end{pmatrix}, \begin{pmatrix} 2 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 7 \end{pmatrix}$

c) $\begin{pmatrix} -1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $\begin{pmatrix} -1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{pmatrix}$, $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 10 & 0 \\ 0 & 0 & 6 \end{pmatrix}$,

$\begin{pmatrix} -3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}$, $\begin{pmatrix} 4 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 6 \end{pmatrix}$

There are many possible answers

2. $x = 5,\ y = -2$

Activity 6.2

Materials

Exercise books, pens, calculators

Answers

1. $\begin{pmatrix} 7 & 2 & -2 \\ 0 & 10 & 7 \\ 6 & 0 & 2 \end{pmatrix}$ 2. $\begin{pmatrix} 0 & 4 & 3 \\ 0 & -1 & 0 \\ 5 & 0 & -3 \end{pmatrix}$ 3. $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

4. $\begin{pmatrix} 3 & 2 & 0 \\ 0 & 4 & 3 \\ 4 & 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 3 & 2 & 0 \\ 0 & 4 & 3 \\ 4 & 0 & 0 \end{pmatrix}$. Addition of matrices is commutative

5. $\begin{pmatrix} 4 & 4 & -1 \\ 5 & 5 & 5 \\ 5 & 4 & 0 \end{pmatrix}$ and $\begin{pmatrix} 4 & 4 & -1 \\ 5 & 5 & 5 \\ 5 & 4 & 0 \end{pmatrix}$. Addition of matrices is associative

6. $\begin{pmatrix} 1 & 0 & 3 \\ 2 & 1 & 0 \\ 1 & 1 & -1 \end{pmatrix}$, $\begin{pmatrix} 2 & 0 & 1 \\ 0 & 3 & 0 \\ -1 & 2 & 1 \end{pmatrix}$, and $\begin{pmatrix} 1 & 5 & 1 \\ 2 & 1 & 4 \\ -1 & 2 & 0 \end{pmatrix}$

Application activity 6.2

1. $\begin{pmatrix} 5 & 2 & -1 \\ 8 & 5 & 5 \\ 3 & -2 & 2 \end{pmatrix}$ 2. $\begin{pmatrix} 3 & 0 & 1 \\ 1 & 2 & 2 \\ 1 & -3 & 1 \end{pmatrix}$ 3. $\begin{pmatrix} 6 & 8 & 6 \\ -13 & -15 & -4 \\ 0 & 6 & -4 \end{pmatrix}$

Activity 6.3

Materials

Exercise books, pens, calculators

Answers

1. $\begin{pmatrix} 3 & 1 & 2 \\ 3 & -1 & -2 \\ 1 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & -2 \\ 1 & -1 & -1 \\ -1 & 1 & 0 \end{pmatrix}$
2. $\begin{pmatrix} 3 & 2 & 0 \\ 4 & -2 & -3 \\ 0 & 2 & 1 \end{pmatrix}$
3. $\begin{pmatrix} 3 & 4 & 0 \\ 2 & -2 & 2 \\ 0 & -3 & 1 \end{pmatrix}$
4. $\begin{pmatrix} 3 & 2 & 0 \\ 4 & -2 & -3 \\ 0 & 2 & 1 \end{pmatrix}$
5. Matrices obtained in 2 and 4 are the same
6. $\begin{pmatrix} 3 & 1 & 2 \\ 3 & -1 & -2 \\ 1 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 3 & 3 & 1 \\ 1 & -1 & 1 \\ 2 & -2 & 1 \end{pmatrix}$ The second matrix is equal to matrix A

Application activity 6.3

1. $\begin{pmatrix} 1 & -3 & 3 \\ 2 & 1 & 0 \\ 2 & 4 & 3 \end{pmatrix}$ 2. $\begin{pmatrix} 1 & 4 & 9 \\ -1 & 1 & -3 \\ 6 & 5 & 5 \end{pmatrix}$ 3. $\begin{pmatrix} -3 & 16 & 12 \\ 1 & -3 & -14 \\ 8 & -5 & -2 \end{pmatrix}$

4. $M^t = \begin{pmatrix} 1 & 2 & 4 \\ 4 & 1 & 0 \\ 1 & 1 & 8 \end{pmatrix}, \begin{pmatrix} 1 & 2 & x^2 \\ 4 & 1 & 0 \\ 1 & x+3 & 8 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 4 \\ 4 & 1 & 0 \\ 1 & 1 & 8 \end{pmatrix}$

$$\begin{cases} x^2 = 4 \\ x+3 = 1 \end{cases} \Rightarrow x = -2$$

Activity 6.4

Materials

Exercise books, pens, calculators

Answers

$$A \times B = \begin{pmatrix} -1 & 1 & 1 \\ 2 & 1 & 2 \\ 0 & 3 & -1 \end{pmatrix} \times \begin{pmatrix} 1 & 1 & -1 \\ 1 & 2 & 3 \\ -1 & 0 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} -1\times1+1\times1+1\times(-1) & -1\times1+1\times2+1\times0 & -1\times(-1)+1\times3+1\times2 \\ 2\times1+1\times1+2\times(-1) & 2\times1+1\times2+2\times0 & 2\times(-1)+1\times3+2\times2 \\ 0\times1+3\times1+(-1)\times(-1) & 0\times1+3\times2+(-1)\times0 & 0\times(-1)+3\times3+(-1)\times2 \end{pmatrix} = \begin{pmatrix} -1 & 1 & 6 \\ 1 & 4 & 3 \\ 4 & 6 & 7 \end{pmatrix}$$

Application activity 6.4

1. $A\times B=\begin{pmatrix}-28 & 36 & 39\\ 28 & -6 & -5\\ 56 & 64 & 80\end{pmatrix}$

2. $A\times C=\begin{pmatrix}47 & 4 & -36\\ 1 & -9 & 31\\ 112 & 8 & -28\end{pmatrix}$

3. $B\times C=\begin{pmatrix}161 & 9 & -21\\ 276 & -22 & -18\\ 123 & 7 & -17\end{pmatrix}$

Activity 6.5

Materials

Exercise books, pens, calculators

Answers

1. $A\times B=\begin{pmatrix}-1 & 3 & -2\\ -2 & -1 & 3\\ 1 & 1 & -2\end{pmatrix}, B\times A=\begin{pmatrix}2 & -4 & -1\\ 7 & -7 & -2\\ -5 & 3 & 1\end{pmatrix}$

   $A\times B\neq B\times A$.

   Multiplication of matrices is not commutative

2. $\left(A\times B\right)^t=\begin{pmatrix}-1 & -2 & 1\\ 3 & -1 & 1\\ -2 & 3 & -2\end{pmatrix}, B^t\times A^t=\begin{pmatrix}-1 & -2 & 1\\ 3 & -1 & 1\\ -2 & 3 & -2\end{pmatrix}$

   $\left(A\times B\right)^t=B^t\times A^t$

3. $A\times(B\times C)=\begin{pmatrix}3&0&1\\0&-1&1\\-2&3&2\end{pmatrix}\begin{pmatrix}2&-3&1\\4&-3&0\\-2&0&1\end{pmatrix}=\begin{pmatrix}4&-9&4\\-6&3&1\\4&-3&0\end{pmatrix}$,

$(A\times B)\times C=\begin{pmatrix}-1&3&-2\\-2&-1&3\\1&1&-2\end{pmatrix}\begin{pmatrix}1&1&-1\\1&-2&1\\-1&1&0\end{pmatrix}=\begin{pmatrix}4&-9&4\\-6&3&1\\4&-3&0\end{pmatrix}$

$A\times(B\times C)=(A\times B)\times C$.

Multiplication of matrices is associative

4. $A\times(B+C)=\begin{pmatrix}3&0&1\\0&-1&1\\-2&3&2\end{pmatrix}\begin{pmatrix}1&2&-2\\2&-1&-1\\-2&1&1\end{pmatrix}=\begin{pmatrix}1&7&-5\\-4&2&2\\0&-5&3\end{pmatrix}$,

$A\times B+A\times C=\begin{pmatrix}-1&3&-2\\-2&-1&3\\1&1&-2\end{pmatrix}+\begin{pmatrix}2&4&-3\\-2&3&-1\\-1&-6&5\end{pmatrix}=\begin{pmatrix}1&7&-5\\-4&2&2\\0&-5&3\end{pmatrix}$

$A\times(B+C)=A\times B+A\times C$.

Multiplication of matrices is distributive over addition

Application activity 6.5

1. $A\times B=\begin{pmatrix}-3&0&2\\-4&0&0\\0&0&0\end{pmatrix},\ B\times A=\begin{pmatrix}-2&0&-2\\1&0&1\\-1&0&-1\end{pmatrix}$

2. $(A\times B)\times C=\begin{pmatrix}-3&0&2\\-4&0&0\\0&0&0\end{pmatrix}\begin{pmatrix}1&0&-1\\-2&2&0\\-1&1&0\end{pmatrix}=\begin{pmatrix}-5&2&3\\-4&0&4\\0&0&0\end{pmatrix}$

$A\times(B\times C)=\begin{pmatrix}1&0&1\\1&-1&1\\0&0&0\end{pmatrix}\begin{pmatrix}-3&1&2\\-1&2&-1\\-2&1&1\end{pmatrix}=\begin{pmatrix}-5&2&3\\-4&0&4\\0&0&0\end{pmatrix}$

3. $A\times\left(B+C\right)=\begin{pmatrix}1&0&1\\1&-1&1\\0&0&0\end{pmatrix}\begin{pmatrix}-1&0&0\\-1&2&2\\-2&1&1\end{pmatrix}=\begin{pmatrix}-3&1&1\\-2&-1&-1\\0&0&0\end{pmatrix}$,

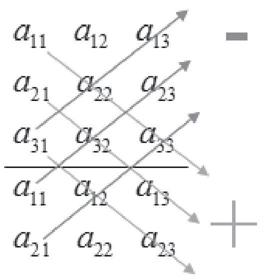
$A\times B+A\times C=\begin{pmatrix}-3&0&2\\-4&0&0\\0&0&0\end{pmatrix}+\begin{pmatrix}0&1&-1\\2&-1&-1\\0&0&0\end{pmatrix}=\begin{pmatrix}-3&1&1\\-2&-1&-1\\0&0&0\end{pmatrix}$

4. $tr\left(A\times B\right)=tr\begin{pmatrix}-3&0&2\\-4&0&0\\0&0&0\end{pmatrix}=-3$

6.6.2 Determinant of order three

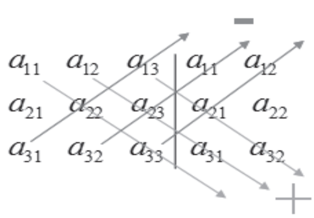
a) Content summary

To calculate the 3x3 determinant we rewrite the first two rows below the determinant (or first two columns to the right of the determinant).



$$\det=a_{11}a_{22}a_{33}+a_{21}a_{32}a_{13}+a_{31}a_{12}a_{23}-a_{31}a_{22}a_{13}-a_{11}a_{32}a_{23}-a_{21}a_{12}a_{33}$$

Or



$$\text{detA}=a_{11}a_{22}a_{33}+a_{12}a_{23}a_{31}+a_{13}a_{21}a_{32}-a_{31}a_{22}a_{13}-a_{32}a_{23}a_{11}-a_{33}a_{21}a_{12}$$

General rule for $n\times n$ matrices (minor and cofactor)

General method of finding the determinant of matrix with $n\times n$ dimension $\left(2\times2,\ 3\times3,\ 4\times4,\ 5\times5,\ldots\right)$ is the use of cofactors.

Minor

An element, a_{ij}, to the value of the determinant of order $n-1$, obtained by deleting the row i and the column j in the matrix is called a **minor.**

$$\begin{vmatrix} 1 & 2 & 1 \\ 2 & [5] & 4 \\ 3 & 6 & 2 \end{vmatrix} \rightarrow \begin{vmatrix} 1 & 1 \\ 3 & 2 \end{vmatrix}$$

Cofactor

The **cofactor** of the element a_{ij} is its minor prefixing:

The **+** sign if **i+j** is **even.**

The **–** sign if **i+j** is **odd.**

$$\begin{vmatrix} 1 & 2 & 1 \\ [2] & 5 & 4 \\ 3 & 6 & 2 \end{vmatrix} \rightarrow -\begin{vmatrix} 2 & 1 \\ 6 & 2 \end{vmatrix}$$

The value of a determinant is equal to the sum of the products of the elements of a line (row or column) by its corresponding cofactors:

$$|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

b) Teaching guidelines

Let learners know how to find determinant of order two. For determinant of order three, we have three rows and three columns.

- Organise class into groups. Request each group to have a group leader who will present their findings to the class.
- Ask each group to do activity related to the lesson they are going to learn. The learners may need your assistance to do any activity. Help them to understand the activity. After group discussion, invite some or all groups for presentation of their work.

- After activity presentation, capture the main points from the presentation of the learners and summarise them. Guide the learners through given examples in Learner's book or through your own examples.
- Ask learners what they learned in day lesson to ensure that they understood what they have learned.
- Request learners to do exercises in their respective groups. Request learners to correct exercises on chalkboard and give them individual evaluation. Remember to give them homework.

c) Answers to activities

Activity 6.6

Materials

Exercise books, pens, calculators

Answers

1. $(1\times6\times1)+(3\times0\times2)+(5\times1\times(-4))-(2\times6\times5)-(1\times0\times1)-(1\times3\times(-4))=-62$
2. $(10\times2\times4)+((-6)\times5\times2)+(0\times3\times1)-(4\times5\times0)-(2\times3\times10)-(1\times(-6)\times2)=-70$

Application activity 6.6

1) 9 2) 6 3) 8

Activity 6.7

Materials

Exercise books, pens, calculators

Answers

1. $|A| = 0, |B| = 0$
2. $|C \cdot D| = -36, |C| \cdot |D| = 6 \times (-6) = -36$
   $|C \cdot D| = |C| \cdot |D|$
   Determinant of product is equal to the product of determinants.
3. Product of leading diagonal elements $1 \times 2 \times 3 = 6$, $|C| = 6$
   Determinant of a triangular matrix is equal to the product of leading diagonal elements.

Application activity 6.7

1. $|A| = 0$ 2. $|B| = 0$ 3. $|C| = 14$

Activity 6.8

Materials

Exercise books, pens, calculators

Answers

1. $|A| = -1$
2. Cofactor of each element:

$cofactor(1) = 3,$ $cofactor(1) = -5,$ $cofactor(1) = 1$

$cofactor(2) = 1,$ $cofactor(1) = -2,$ $cofactor(-1) = 1$

$cofactor(3) = -2,$ $cofactor(2) = 3,$ $cofactor(1) = -1$

Cofactor matrix

$$\begin{pmatrix} 3 & -5 & 1 \\ 1 & -2 & 1 \\ -2 & 3 & -1 \end{pmatrix}$$

3. Transpose of cofactor matrix is

$$\begin{pmatrix} 3 & 1 & -2 \\ -5 & -2 & 3 \\ 1 & 1 & -1 \end{pmatrix}$$

4. $\frac{1}{-1}\begin{pmatrix} 3 & 1 & -2 \\ -5 & -2 & 3 \\ 1 & 1 & -1 \end{pmatrix} = \begin{pmatrix} -3 & -1 & 2 \\ 5 & 2 & -3 \\ -1 & -1 & 1 \end{pmatrix}$

5. $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, the obtained product is the identity matrix.

Application activity 6.8

1. No inverse

2. $\begin{pmatrix} \frac{23}{268} & -\frac{29}{268} & \frac{5}{268} \\ -\frac{3}{268} & -\frac{37}{268} & \frac{11}{268} \\ -\frac{9}{268} & \frac{23}{268} & \frac{33}{268} \end{pmatrix}$

3. $\begin{pmatrix} \frac{6}{7} & -\frac{45}{14} & \frac{16}{7} \\ -\frac{5}{7} & \frac{24}{7} & -\frac{18}{7} \\ \frac{1}{7} & -\frac{11}{14} & \frac{5}{7} \end{pmatrix}$

4. $\begin{pmatrix} -\frac{2}{5} & -\frac{3}{5} & 1 \\ \frac{1}{5} & \frac{4}{5} & -1 \\ -\frac{6}{5} & -\frac{29}{5} & 8 \end{pmatrix}$

6.6.3 Applications of matrices in real life

a) Content summary

Consider the following simultaneous linear equations.

$$\begin{cases} a_{11}x + a_{12}y + a_{13}z = c_1 \\ a_{21}x + a_{22}y + a_{23}z = c_2 \\ a_{31}x + a_{32}y + a_{33}z = c_3 \end{cases}$$

One of the methods of solving this, is the use Cramer's rule.

$$\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \neq 0 \qquad \Delta_x = \begin{vmatrix} c_1 & a_{12} & a_{13} \\ c_2 & a_{22} & a_{23} \\ c_3 & a_{32} & a_{33} \end{vmatrix}$$

$$\Delta_y = \begin{vmatrix} a_{11} & c_1 & a_{13} \\ a_{21} & c_2 & a_{23} \\ a_{31} & c_3 & a_{33} \end{vmatrix} \qquad \Delta_z = \begin{vmatrix} a_{11} & a_{12} & c_1 \\ a_{21} & a_{22} & c_2 \\ a_{31} & a_{32} & c_3 \end{vmatrix}$$

$x = \frac{\Delta_x}{\Delta}$, $y = \frac{\Delta_y}{\Delta}$ and $z = \frac{\Delta_z}{\Delta}$

Remember that if $\Delta = 0$, there is no solution or infinity of solution

Teaching guidelines

Let learners know how to rewrite a system of linear equation in matrix form, how to find inverse of matrix and how to multiply to matrices of oder two.

- Ask each group to do activity related to the lesson they are going to learn. The learners may need your assistance to do any activity. Help them to understand the activity. After group discussion, invite some or all groups for presentation of their work.
- After activity presentation, capture the main points from the presentation of the learners and summarise them. Guide the learners through given examples in Learner's book or through your own examples.
- Ask learners what they learned in day lesson to ensure that they understood what they have learned.
- Request learners to do exercises in their respective groups. Request learners to correct exercises on chalkboard and give them individual evaluation. Remember to give them homework.

Answers to activities

Activity 6.9

Materials

Exercise books, pens

Answers

1. $\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}\begin{pmatrix} x \\ y \\ x \end{pmatrix} = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$

2. $\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}^{-1}\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}\begin{pmatrix} x \\ y \\ x \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}^{-1}\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$

$\begin{pmatrix} x \\ y \\ x \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}^{-1}\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$

Application activity 6.9

1. $S = \{(0,0,0)\}$ 2. $S = \{\ \}$ 3. $S = \{(1,2,0)\}$

6.7. End of Unit Assessment

1. a) $\begin{pmatrix} -7 & -3 & 0 \\ 0 & 4 & -12 \\ 0 & -10 & -2 \end{pmatrix}$ b) $\begin{pmatrix} -9 & -23 & 6 \\ 0 & -4 & -16 \\ -4 & 2 & -8 \end{pmatrix}$

c) $\begin{pmatrix} 7 & 8 & 3 \\ 2 & 4 & -10 \\ 2 & -14 & 7 \end{pmatrix}$ d) $\begin{pmatrix} 29 & 28 & 15 \\ 10 & -34 & -21 \\ -4 & 28 & -16 \end{pmatrix}$

e) $\begin{pmatrix} 38 & 36 & 13 \\ -1 & 12 & -42 \\ 0 & 0 & -18 \end{pmatrix}$ f) $\begin{pmatrix} 118 & 120 & 37 \\ 19 & 12 & 10 \\ 14 & 0 & 76 \end{pmatrix}$

2. a) $\begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}$ b) no inverse

c) $\begin{pmatrix} \frac{44}{207} & -\frac{8}{207} & \frac{1}{69} \\ \frac{1}{207} & -\frac{19}{207} & \frac{11}{69} \\ -\frac{13}{207} & \frac{40}{207} & -\frac{5}{69} \end{pmatrix}$

3. $X = \begin{pmatrix} 3 & -2 & -2 \\ -5 & 5 & 2 \\ 5 & -3 & 1 \end{pmatrix}$

4. a) $S = \{(0,0,0)\}$ b) $S = \{(1,1,1)\}$ c) $S = \{(3,0,1)\}$

5. Let x represent chicken breast
y represent potato
z represent spinach

$$A = \begin{pmatrix} 24 & 4 & 5 \\ 0 & 26 & 7 \\ 1.5 & 0 & 0.5 \end{pmatrix}$$

$$Det\,A = \begin{vmatrix} 24 & 4 & 5 \\ 0 & 26 & 7 \\ 1.5 & 0 & 0.5 \end{vmatrix} = 159$$

$$\Delta_x = \begin{vmatrix} 38 & 4 & 5 \\ 40 & 26 & 7 \\ 2.5 & 0 & 0.5 \end{vmatrix} = 159$$

$$x = \frac{\Delta_x}{\Delta} = \frac{159}{159} = 1$$

$$\Delta_y = \begin{vmatrix} 24 & 38 & 5 \\ 0 & 40 & 7 \\ 1.5 & 2.5 & 0.5 \end{vmatrix} = 159$$

$$y = \frac{\Delta_y}{\Delta} = \frac{159}{159} = 1$$

$$\Delta_z = \begin{vmatrix} 24 & 4 & 38 \\ 0 & 26 & 40 \\ 1.5 & 0 & 2.5 \end{vmatrix} = 318$$

$$z = \frac{\Delta_z}{\Delta} = \frac{318}{159} = 2$$

To create a meal containing 38 grams of protein, 40 grams of carbohydrates and 2.5 grams of fat, there will be a mixture of 1 chicken breast, 1 potato and 2 spinach

Unit 7 Bivariate Statistics

7.1 Key unit competence:

Extend understanding, analysis and interpretation of bivariate data to correlation coefficients and regression lines .

7.2 Objectives

By the end of this unit, learners will be able to:

- Find measures of variability of two quantitative variables
- Draw the scatter diagram of a given statistical series in two quantitative variables
- Determine the linear regression line of a given series
- Calculate a linear correlation coefficient of a given double series and interpret it.

7.3 List of lessons for unit 7

Week	Lessons	Content	Number of Periods
28-29	1	Introduction to bivariate statistics	2
	2	Covariance	2
	3	Regression lines	2
30-31	4	Coefficient of correlation	3
	5	Applications: Data analysis, interpretation and prediction problems in various areas (biology, business and medicine)	2
	6	End unit assessment and remediation	1

7.4 Materials required

Exercise books, pens, calculators

7.5 Guidance to the introductory activity

- Invite learners to work in groups and do the introductory activity found in their Mathematics books;

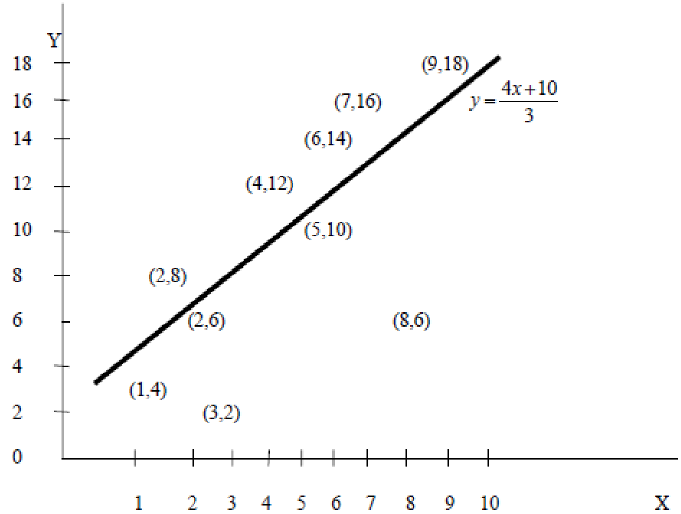
- Move around in the class for facilitating learners where necessary and give more clarification on eventual challenges they may face during their work;
- Invite one member from each group with different working steps to present their work where they must explain the working steps;
- As a teacher, harmonize the findings from presentation;
- Use different probing questions and guide them to explore the content and examples given in the student's book and lead them to discover how to determine and explain the covariance for bivariate statistics data.

Answer for introductory activity:

	X_i	Y_i	X_i^2	$X_i Y_i$
	1	4	1	4
	2	8	4	64
	3	2	9	6
	4	12	16	48
	5	10	25	50
	6	14	36	84
	7	16	49	112
	8	6	64	48
	9	18	81	162
$\sum$	45	88		

a) Scatter diagram: plotting the 9 sample points (1,4), (2,8), (3,2), (4,12), (5,10), (6,14), (7,16), (8,6), (9,18).

The first point on the line is (2,8). Another point on the line is (x,y)=(4,12) so the regression line of y on x passes through the two points (2,8) and (4,12) plot these points and join them the required line of regression of y on x is obtained.



For x=2, y=8

For x=4, y=12

The slope of the line is 2

The equation of the line is given by

$$y - y_1 = m(x - x_1)$$
$$y - 8 = 2(x - 2)$$
$$y = 2x - 4 + 8$$
$$y = 2x + 4$$

7.6 Content and activities

7.6.1 Covariance

a)Content summary

$$\operatorname{cov}(x, y) = \frac{1}{n}\sum_{i=1}^{k} f_i x_i y_i - \bar{x}\,\bar{y}$$

a) Teaching guidelines

Let learners know how to find variance and standard deviation of a series. In bivariate statistics we use two series.

- Organise class into groups. Request each group to have a group leader who will present their findings to the class.

- Ask each group to do activity related to the lesson they are going to learn. The learners may need your assistance to do any activity. Help them to understand the activity. After group discussion, invite some or all groups for presentation of their work.
- After activity presentation, capture the main points from the presentation of the learners and summarise them. Guide the learners through given examples in Learner's book or through your own examples.
- Ask learners what they learned in day lesson to ensure that they understood what they have learned.
- Request learners to do exercises in their respective groups. Request learners to correct exercises on chalkboard and give them individual evaluation. Remember to give them homework.

Answers to activities

Activity 7.1

Materials

Exercise books, pens

Answers

i	x_i	y_i	$x_i - \overline{x}$	$y_i - \overline{y}$	$(x_i - \overline{x})(y_i - \overline{y})$
1	3	6	0.5	0.5	0.25
2	1	1	-1.5	-4.5	6.75
3	4	3	1.5	-2.5	-3.75
4	3	8	0.5	2.5	1.25
5	2	7	-0.5	1.5	-0.75
6	2	8	-0.5	2.5	-1.25
	$\sum_{i=1}^{6} x_i = 15$	$\sum_{i=1}^{6} y_i = 33$			$\sum_{i=1}^{6}(x_i - \overline{x})(y_i - \overline{y}) = 2.5$
	$\overline{x} = 2.5$	$\overline{y} = 5.5$			

1. If you divide by total frequency you get variance
2. If you divide by total frequency you get **covariance**

Application activity 7.1

1. $\text{cov}(x,y)=5$
2. $\text{cov}(x,y)=\frac{71}{12}$
3. $\text{cov}(x,y)=98.75$

7.6.2 Regression lines

a) Content summary

The regression line *y* on *x* is written as

$$L_{y/x} \equiv y-\bar{y}=\frac{\text{cov}(x,y)}{\sigma_x^2}(x-\bar{x})$$

The regression line *x* on *y* is written as

$$L_{x/y} \equiv x-\bar{x}=\frac{\text{cov}(x,y)}{\sigma_y^2}(y-\bar{y})$$

b) Teaching guidelines

Let learners know how to find mean, standard deviation and covariance.

- Organise class into groups. Request each group to have a group leader who will present their findings to the class.
- Ask each group to do activity related to the lesson they are going to learn. The learners may need your assistance to do any activity. Help them to understand the activity. After group discussion, invite some or all groups for presentation of their work.
- After activity presentation, capture the main points from the presentation of the learners and summarise them. Guide the learners through given examples in Learner's book or through your own examples.
- Ask learners what they learned in day lesson to ensure that they understood what they have learned.

- Request learners to do exercises in their respective groups. Request learners to correct exercises on chalkboard and give them individual evaluation. Remember to give them homework.

c) Answers to activities

Activity 7.2

Materials

Exercise books, pens

Answers

1. $D_b^{/}=2\sum_{i=1}^{k}(y_i-ax_i-b)(-1)$ or $D_b^{/}=-2\sum_{i=1}^{k}(y_i-ax_i-b)$

2. $\sum_{i=1}^{k}(y_i-ax_i-b)=0$ or $\sum_{i=1}^{k}y_i-\sum_{i=1}^{k}ax_i-\sum_{i=1}^{k}b=0$

   or $\sum_{i=1}^{k}b=\sum_{i=1}^{k}y_i-\sum_{i=1}^{k}ax_i$

   Dividing both sides by n gives

   $\frac{1}{n}\sum_{i=1}^{k}b=\frac{1}{n}\sum_{i=1}^{k}y_i-\frac{1}{n}\sum_{i=1}^{k}ax_i$ or

   $\frac{b}{n}\sum_{i=1}^{k}1=\frac{1}{n}\sum_{i=1}^{k}y_i-\frac{a}{n}\sum_{i=1}^{k}x_i$ or $b=\overline{y}-a\overline{x}$

3. $\sum_{i=1}^{k}(y_i-ax_i-b)^2=\sum_{i=1}^{k}\left(y_i-ax_i-\overline{y}+a\overline{x}\right)^2$

   Or

   $\sum_{i=1}^{k}(y_i-ax_i-b)^2=\sum_{i=1}^{k}\left[\left(y_i-\overline{y}\right)-a\left(x_i-\overline{x}\right)\right]^2$

Differentiation with respect to a and equating to zero:

$$\sum_{i=1}^{k} 2\left[\left(y_i-\bar{y}\right)-a\left(x_i-\bar{x}\right)\right]\left[-\left(x_i-\bar{x}\right)\right]=0$$

$$-2\sum_{i=1}^{k}\left[\left(y_i-\bar{y}\right)-a\left(x_i-\bar{x}\right)\right]\left(x_i-\bar{x}\right)=0$$

$$\Leftrightarrow \sum_{i=1}^{k}\left[\left(y_i-\bar{y}\right)-a\left(x_i-\bar{x}\right)\right]\left(x_i-\bar{x}\right)=0$$

$$\Leftrightarrow \sum_{i=1}^{k}\left[\left(x_i-\bar{x}\right)\left(y_i-\bar{y}\right)-a\left(x_i-\bar{x}\right)^2\right]=0$$

$$\Leftrightarrow \sum_{i=1}^{k}\left(x_i-\bar{x}\right)\left(y_i-\bar{y}\right)-\sum_{i=1}^{k}a\left(x_i-\bar{x}\right)^2=0$$

$$\Leftrightarrow \sum_{i=1}^{k}a\left(x_i-\bar{x}\right)^2=\sum_{i=1}^{k}\left(x_i-\bar{x}\right)\left(y_i-\bar{y}\right)$$

$$\Leftrightarrow a\sum_{i=1}^{k}\left(x_i-\bar{x}\right)^2=\sum_{i=1}^{k}\left(x_i-\bar{x}\right)\left(y_i-\bar{y}\right)$$

Dividing both sides by n gives

$$\Leftrightarrow \frac{a}{n}\sum_{i=1}^{k}\left(x_i-\bar{x}\right)^2=\frac{1}{n}\sum_{i=1}^{k}\left(x_i-\bar{x}\right)\left(y_i-\bar{y}\right)$$

$$\Rightarrow a=\frac{\frac{1}{n}\sum_{i=1}^{k}\left(x_i-\bar{x}\right)\left(y_i-\bar{y}\right)}{\frac{1}{n}\sum_{i=1}^{k}\left(x_i-\bar{x}\right)^2}$$

4. The variance for variable x is $\sigma_x^2=\frac{1}{n}\sum_{i=1}^{k}\left(x_i-\bar{x}\right)^2$ and the variance for variable y is $\sigma_y^2=\frac{1}{n}\sum_{i=1}^{k}\left(y_i-\bar{y}\right)^2$ and the covariance of these two variables is $\operatorname{cov}(x,y)=\frac{1}{n}\sum_{i=1}^{k}\left(x_i-\bar{x}\right)\left(y_i-\bar{y}\right)$

Then $a=\frac{\operatorname{cov}(x,y)}{\sigma_x^2}$

5. Now, we have seen that the regression line y on x is $y = ax + b$, where

$$\begin{cases} a = \dfrac{\operatorname{cov}(x,y)}{\sigma_x^2} \\ b = \overline{y} - a\overline{x} \end{cases}$$

Or

$$y = \frac{\operatorname{cov}(x,y)}{\sigma_x^2}x + \left(\overline{y} - \frac{\operatorname{cov}(x,y)}{\sigma_x^2}\overline{x}\right)$$

Application activity 7.2

1. a) $y = 0.19x - 8.098$ b) $y = 4.06$
2. $x = -5.6y + 163.3, \quad y = -0.06x + 21.8$

7.4.3 Coefficient of correlation

a) Content summary

The **Pearson's coefficient of correlation** also called **product moment coefficient of correlation** or **simply coefficient of correlatio**n, denoted by r, is a measure of the strength of linear relationship between two variables.

The **correlation coefficient** between two variables x and y is given by $r = \dfrac{\operatorname{cov}(x,y)}{\sigma_x \sigma_y}$.

The **Spearman's coefficient** of rank correlation is given by

$$\rho = 1 - \frac{6\sum_{i=1}^{k} d_i^2}{n(n^2-1)}$$

Where, *d* refers to the difference of ranks between paired items in two series.

b) Teaching guidelines

- Organise class into groups. Request each group to have a group leader who will present their findings to the class.
- Ask each group to do activity related to the lesson they are going to learn. The learners may need your assistance to do any activity. Help them to understand the activity. After group discussion, invite some or all groups for presentation of their work.
- After activity presentation, capture the main points from the presentation of the learners and summarise them. Guide the learners through given examples in Learner's book or through your own examples.
- Ask learners what they learned in day lesson to ensure that they understood what they have learned.
- Request learners to do exercises in their respective groups. Request learners to correct exercises on chalkboard and give them individual evaluation. Remember to give them homework.

c) Answers to activities

Activity 7.3

Materials

Exercise books, pens, calculators

Answers

1. $\sigma_x = 3.3,\ \sigma_y = 3.4$
2. $\operatorname{cov}(x, y) = 4.1$
3. $\dfrac{\operatorname{cov}(x, y)}{\sigma_x \sigma_y} = \dfrac{4.1}{(3.3)(3.4)} = 0.37$

Application activity 7.3

1. $r = 0.9$, as the correlation coefficient is very close to 1, the correlation is very strong.
2. $r = 0.94$, as the correlation coefficient is very close to 1, the correlation is very strong.
3. $r = -0.26$, as the correlation coefficient is very close to 0, the correlation is very weak.

7.6.4 Application of bivariote statistics in real life

Activity 7.4

Materials

Exercises book, pens

Answers

By reading textbooks or accessing internet, learners will discuss how bivariate statistics is used in daily life. Bivariate statistics can help in prediction of a value for one variable if we know the value of the other by using regression lines.

7.7 End of Unit Assessment

1. Data set 1

   a) $y = 4.50 + 0.64x$ b) $x = 4.42 + 0.75y$

   Data set 2

   a) $y = 90.31 - 1.78x$ b) $x = 37.80 - 0.39y$

2 . $y = -2.59 + 0.65x; 36.4(1\text{dp})$

3. $r = 0.918,$

4. $y = 0.611x + 10.5, x = 1.478y - 1.143, y = 28.83$

5. $y = 0.94x + 92.26, Blood\ pressure = 134.56$

6. $y = 3.8 + 1.6x, x = -2.06 + 0.59y$

7. $y = -8 + 1.2x$; For x =10, we have y = 4

8. $c = 15, d = -5$

9. a) 0.60, b) $w = 0.89h - 76$

10. $\bar{x} = -\frac{3}{29}, \bar{y} = \frac{15}{29}, r = \frac{3}{4}$

11. $r = 0.4$

12. $r = 0.82$

13. $r = 0.77$

14. $r = -0.415$

15. a) $r = 0.954$ b) $\bar{x} = 2$, $\bar{y} = 3$

16. a) $\bar{x} = 13, \bar{y} = 17$ b) $\sigma_y = 4$ c) $r = 0.6$

17. a) $\bar{x} = 13, \bar{y} = 17$ b) $\sigma_y = 4$

18. $\sigma = 0.26$

19. a) $\sigma = 0.43$

b) Somme agreement between average attendances ranking a position in league, high position in league correlating with high attendance.

20. a) (i) -0.976 (ii) $-0.292(or\ 0.292)$

b) The transport manager's order is more profitable for the seller, saleswomen is unlikely to try to dissuade.

c) (i) No, maximum value is 1

(ii) Yes, higher performing cars generally do less mileage to the gallon.

(iii) No, the higher the engine capacity, the dearer the car.

d) When only two rankings are known; when relationship is non-linear.

21. a) There is a strong positive correlation

b) 54.5

Unit 8 Conditional Probability and Bayes Theorem

8.1 Key unit competence:

Solve problems using Bayes theorem and use data to make decisions about likelihood and risk.

8.2 Objectives

By the end of this unit, the learners should be able to:

- find probability of independent events.
- find probability of one event given that the other event has occurred.
- use and apply Bayes theorem.

8.3. List of lessons for unit 8

Lesson number	Lesson title	Number of periods
1	Introductory activity	1
2	Independent events and multiplication rule	1
3	Conditional probability: Probability of event B occurring when event A has already taken place	2
4	Basic formulae and properties of conditional probability	2
5	Bayes theorem and its applications	2
6	End unit assessment	1

8.4. Materials required

Exercise books, pens, ruler, calculator

8.5. Guidance to the introductory activity

A box contains 3 red pens and 4 blue pens. One pen is taken from the box and is not replaced. Another pen is taken from the box. Let A be the event "the first pen is red" and B be the event "the second pen is blue."

Is the occurrence of event B affected by the occurrence of event A? Explain.

Give more other examples of real life problems involving probability.

Answer for introductory activity

a) Yes, the occurrence of event B will be affected by the occurrence of event A because when the pen is taken, the number of remaining pens to be considered in the second event(sample space) reduces. This is because the pen taken is not replaced.

b) Some academic fields based on the probability theory are statistics, communication theory, computer performance evaluation, signals and image processing, game theory. In medical decision-making, clinical estimate of probability strongly affects the physician's belief as to whether or not a patient has a disease, and this belief, in turn, determines actions: to rule out, to treat, or to do more tests, doctors may use conditional probability to calculate the probability that a particular patient has a disease, given the presence of a particular set of symptoms....

More applications of the probability theory are character recognition, speech recognition, opinion survey, missile control and seismic analysis, etc.

8.6 Content and activities

8.6.0 Recall on tree diagram

Given that the content was not provided in the student's book for S4, try to use the following content to help students understand the use of tree diagrams when calculating the probability.

a) Content summary

A **tree diagram** is a means which can be used to show the probabilities of certain **outcomes** occurring when two or more **trials** take place in succession.

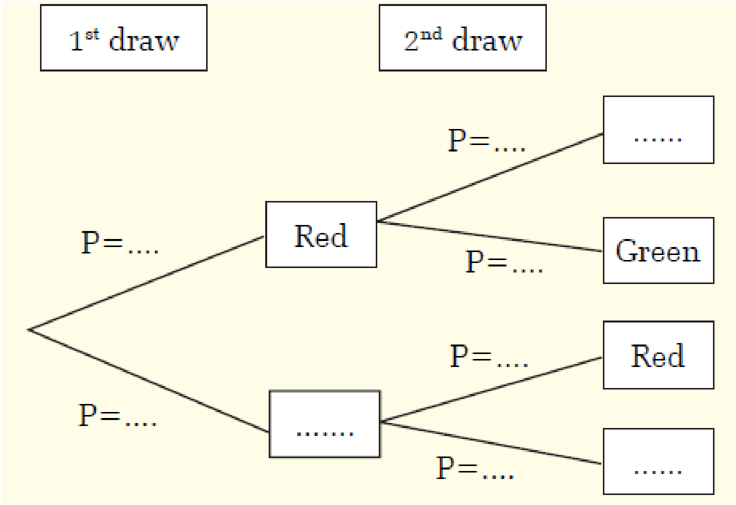
The **outcome** is written at the end of the branch and the fraction on the branch gives the probability of the outcome occurring.

For each **trial** the number of branches is equal to the number of possible outcomes of that trial. In the diagram there are two possible outcomes, A and B, of each trial.

Example 1

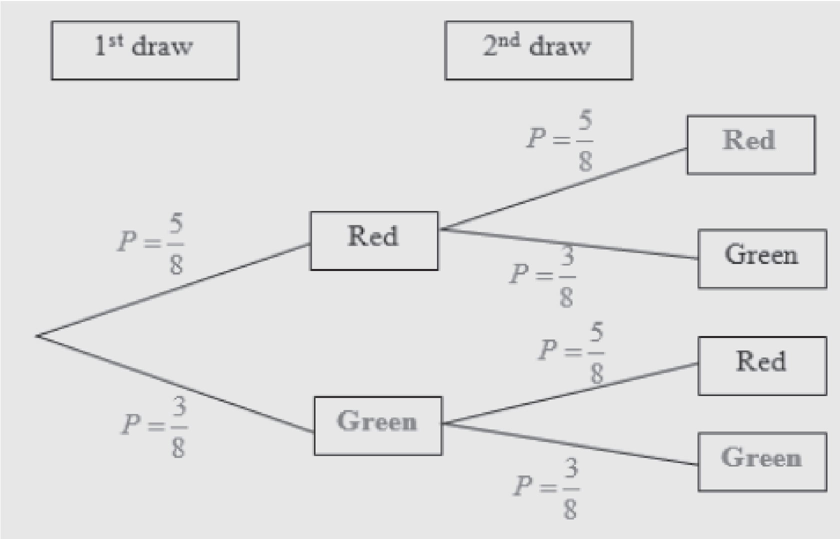
A bag contains 5 red balls and 3 green balls. A ball is drawn at random and then replaced. Another ball is drawn.

1. For the first draw, what is the probability of choosing a red ball and the probability of choosing a green ball?
2. For the second draw, what is the probability of choosing a red ball and probability of choosing a green ball? Remember that after the first draw the ball is replaced in the bag.
3. In the following figure, complete the missing colors and probabilities



Solution

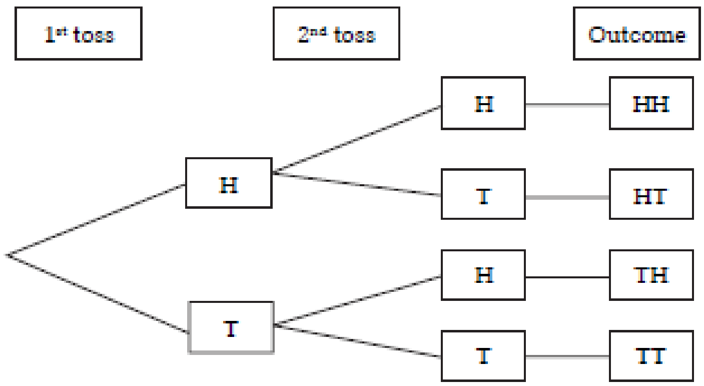
1. Total number of balls is 8. There are 5 red balls and 3 green balls, then probability of choosing a red ball is 5/8 and probability of choosing a green ball is 3/8.
2. Since after the first draw the ball is replaced in the bag, we have the same number of balls as in question 1. The total number of balls is 8. There are 5 red balls and 3 green balls, then probability of choosing a red ball is 5/8 and probability of choosing a green ball is 3/8 .
3. We have:



Example 2

One coin is tossed twice. Find the set of possible outcomes (one face of a coin is Head and the second is Tail).

Solution:

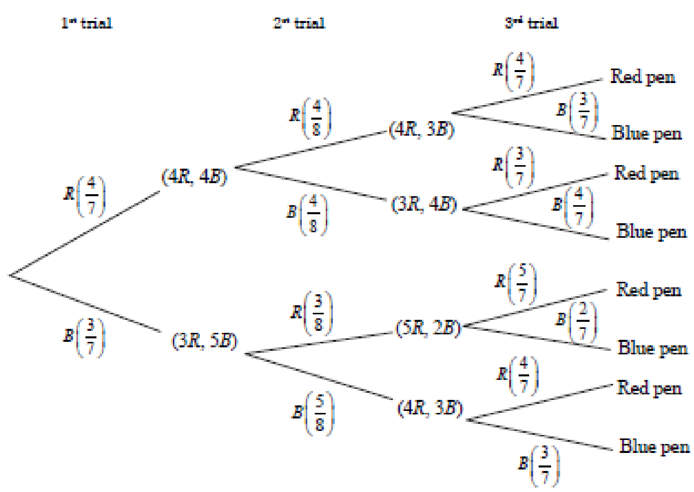


Example 3:

A bag (1) contains 4 red pens and 3 blue pens. Another bag (2) contains 3 red pens and 4 blue pens. A pen is taken from the first bag (1) and placed into the second bag (2). The second bag (2) is shaken and a pen is taken from it and placed in the first bag (1). If now a pen is taken from the first bag, use the tree diagram to find the probability that it is a red pen.

Solution:

Tree diagram is given below:



From tree diagram, the probability to have a red pen is:

$$P(R)=\frac{4}{7}\times\frac{4}{8}\times\frac{4}{7}+\frac{4}{7}\times\frac{4}{8}\times\frac{3}{7}+\frac{3}{7}\times\frac{3}{8}\times\frac{5}{7}+\frac{3}{7}\times\frac{5}{8}\times\frac{4}{7}$$

$$=\frac{64}{392}+\frac{48}{392}+\frac{45}{392}+\frac{60}{392}\ =\frac{31}{56}$$

Example 4

A bag contains 8 balls of which 3 are red and 5 are green. One ball is drawn at random, its color is noted and the ball replaced in the bag. A ball is again drawn from the bag and its color is noted. Find the probability the ball drawn will be;

a) red followed by green,

b) red and green in any order,

c) of the same color.

Solution:

a) $\frac{15}{64}$ b) $\frac{15}{32}$ c) $\frac{17}{32}$

Example 5:

A class consists of six girls and 10 boys. If a committee of three is chosen at random, find the probability of:

a) Three boys being chosen.

b) Exactly two boys and a girl being chosen.

c) Exactly two girls and a boy being chosen.

d) Three girls being chosen.

Solution

a) $P(3\text{boys})=\frac{10}{16}\times\frac{9}{15}\times\frac{8}{14}=0.214$

b) $P(2 \text{ boys and } 1 \text{ girl}) = \frac{10}{16}\times\frac{9}{15}\times\frac{6}{14}+\frac{10}{16}\times\frac{6}{15}\times\frac{9}{14}+\frac{6}{16}\times\frac{10}{15}\times\frac{9}{14}=0.482$

c) $P(2 \text{ girls and } 1 \text{ boy}) = \frac{10}{16}\times\frac{6}{15}\times\frac{5}{14}+\frac{6}{16}\times\frac{10}{15}\times\frac{5}{14}+\frac{6}{16}\times\frac{5}{15}\times\frac{10}{14}=0.268$

d) $P(3 \text{ girls}) = \frac{6}{16}\times\frac{5}{15}\times\frac{4}{14}=0.0357$

Example 6:

A bag contains 10 discs; 7 are black and 3 white. A disc is selected, and then replaced. A second disc is selected. Find the probability of the following;

a) Both discs are black

b) Both discs are white.

Solution:

$a)\frac{49}{100}$ b) $\frac{9}{100}$

8.6.1 Independent events

a) Content summary

If probability of event B is not affected by the occurrence of event A, events A and B are said to be independent and $P(A\cap B)=P(A)\times P(B)$

This rule is the simplest form of the multiplication law of probability.

b) Teaching guidelines

- Organise class into groups. Request each group to have a group leader who will present their findings to the class.
- Ask each group to do activity related to the lesson they are going to learn. The learners may need your assistance to do any activity. Help them to understand the activity. After group discussion, invite some or all groups for presentation of their work.

- After activity presentation, capture the main points from the presentation of the learners and summarise them. Guide the learners through given examples in Learner's book or through your own examples.
- Ask learners what they learned in day lesson to ensure that they understood what they have learned.
- Request learners to do exercises in their respective groups. Request learners to correct exercises on chalkboard and give them individual evaluation. Remember to give them homework.

Answers to activities

Activity 8.1

Materials

Exercise books, pens

Answers

The occurrence of the event for the second selection is not affected by the event for the first selection because for the first selection the book is replaced. It means that the sample space does not change.

Application activity 8.1

1. $P(\text{red and red}) = \frac{1}{5} \times \frac{1}{5} = \frac{1}{25}$

2. $P(\text{head and } 3) = \frac{1}{2} \times \frac{1}{6} = \frac{1}{12}$

8.6.2 Conditional probability

a) Content summary

Dependent events

When the outcome or occurrence of the first event affects the outcome or occurrence of the second event in such a way that the probability is changed, the events are said to be ***dependent***.

Example 1

Suppose that you have a deck of cards; then draw a card from that deck, not replacing it, and then draw a second card.

a) What is the sample space for each event?

b) Suppose you select successively two cards, what is the probability of selecting two red cards?

c) Explain if there is any relationship (Independence or dependence) between those two events considering the sample space. Does the selection of the first card affect the selection of the second card?

Solution:

a) Sample space for the first drawing is $\Omega = 52$, But for the second drawing the sample space is $\Omega = 51$.

b) The outcomes of the first draw has affected the outcome of the second i.e. the outcome of the second dependent on the first drawing.

Example 2:

1. Suppose a card is drawn from a deck and not replaced, and then the second card is drawn. What is the probability of selecting an ace on the first card and a king on the second card?

Solution:

The probability of selecting an ace on the first draw is $\frac{4}{52}$. But since that card is not replaced, the probability of selecting a king on the second card is $\frac{4}{51}$, since there are 51 cards remaining.

The outcomes of the first draw has affected the outcome of the second. By multiplication rule , the probability of both events occurring is :

$$\frac{4}{52}\times\frac{4}{51}=\frac{16}{2652}=\frac{4}{663}=0.006 .$$

Note that:

The event of getting a king on the second draw given that an ace was drawn the first time is called a **conditional probability**.

Example 3

The world wide Insurance Company found that 53% of the residents of a city had home owner's Insurance with its company of the clients, 27% also had automobile Insurance with the company. If a resident is selected at random, find the probability that the resident has both home owner's and automobile Insurance with the world wide Insurance Company.

Solution:

$$P(H\ and\ A)=P(H)\times P(A\setminus H)=0.53\times 0.27=0.1431$$

The probability of an event B given that event A has occurred is called the **conditional probability** of B given A and is written $P(B\mid A)$.

In this case, $P(B\mid A)$ is the probability that B occurs considering A as the sample space, and since the subset of A in which B occurs is $A\cap B$, then

$$P(B\mid A)=\frac{P(B\cap A)}{P(A)}$$

From this result, we have a general statement of the multiplication law: $P(A\cap B)=P(A)\times P(B\mid A)$

b) Teaching guidelines

- Organise class into groups. Request each group to have a group leader who will present their findings to the class.
- Ask each group to do activity related to the lesson they are going to learn. The learners may need your assistance to do any activity. Help them to understand the activity. After group discussion, invite some or all groups for presentation of their work.
- After activity presentation, capture the main points from the presentation of the learners and summarise them. Guide the learners through given examples in Learner's book or through your own examples.
- Ask learners what they learned in day lesson to ensure that they understood what they have learned.
- Request learners to do exercises in their respective groups. Request learners to correct exercises on chalkboard and give them individual evaluation. Remember to give them homework.

Answers to activities

Activity 8.2

Materials

Exercise books, pens

Answers

The occurrence of the event for the second selection is affected by the event for the first selection because for the first selection the book is not replaced. It means that the sample space has been changed.

Application activity 8.2

1. $\frac{1}{2}$

2. $P(6 \mid even) = \frac{\frac{1}{6}}{\frac{3}{6}} = \frac{1}{3}$

3. $P(White \mid Black) = \frac{P(Black \text{ and } White)}{P(Black)} = \frac{0.34}{0.47} = 0.72$

8.6.3 Bayes theorem and applications

a) Content summary

Let $B_1, B_2, B_3, ..., B_n$ be incompatible and exhaustive events and let *A* be an arbitrary event.

We have:

$$P(B_i \mid A) = \frac{P(B_i \cap A)}{P(A)} = \frac{P(A \mid B_i)P(B_i)}{\sum_{i=1}^{n} P(A \mid B_i)P(B_i)}$$

This formula is called **Bayes' formula**.

Remark

We also have (**Bayes' rule)**

$$P(B \mid A) = \frac{P(A \mid B)P(B)}{P(A)}$$

Teaching guidelines

- Organise class into groups. Request each group to have a group leader who will present their findings to the class.
- Ask each group to do activity related to the lesson they are going to learn. The learners may need your assistance to do any activity. Help them to understand the activity. After group discussion, invite some or all groups for presentation of their work.

- After activity presentation, capture the main points from the presentation of the learners and summarise them. Guide the learners through given examples in Learner's book or through your own examples.
- Ask learners what they learned in day lesson to ensure that they understood what they have learned.
- Request learners to do exercises in their respective groups. Request learners to correct exercises on chalkboard and give them individual evaluation. Remember to give them homework.

Answers to activities

Activity 8.3

Materials

Exercise books, pens

Answers

1. $P(A)=P(A|B_1)P(B_1)+P(A|B_2)P(B_2)+P(A|B_3)P(B_3)$

2. $$P(B_1|A)=\frac{P(B_1\cap A)}{P(A)}=\frac{P(A|B_1)P(B_1)}{P(A|B_1)P(B_1)+P(A|B_2)P(B_2)+P(A|B_3)P(B_3)}$$

$$P(B_2|A)=\frac{P(B_2\cap A)}{P(A)}=\frac{P(A|B_2)P(B_2)}{P(A|B_1)P(B_1)+P(A|B_2)P(B_2)+P(A|B_3)P(B_3)}$$

$$P(B_3|A)=\frac{P(B_3\cap A)}{P(A)}=\frac{P(A|B_3)P(B_3)}{P(A|B_1)P(B_1)+P(A|B_2)P(B_2)+P(A|B_3)P(B_3)}$$

Generally,

$$P(B_i|A)=\frac{P(B_i\cap A)}{P(A)}=\frac{P(A|B_i)P(B_i)}{\sum_{i=1}^{3}P(A|B_i)P(B_i)}$$

Application activity 8.3

1. a) $\frac{67}{120}$ b) $\frac{24}{53}$

2. $P(engeneer \mid managerial) = \frac{0.2\times 0.75}{0.2\times 0.75 + 0.2\times 0.5 + 0.6\times 0.2} = 0.405$

3. $P(No\ accident \mid Triggered\ alarm) = \frac{0.9\times 0.02}{0.1\times 0.97 + 0.9\times 0.02} = 0.157$

8.7 End of Unit Assessment

1. 0.15
2. 0.13
3. 0.56
4. $\frac{1}{169}$
5. $\frac{15}{128}$
6. $\frac{729}{1000}$
7. 0.37
8. $\frac{10}{21}$
9. a) 0.34 b) 0.714 c) 0.0833
10. a) 0.43 b) 0.1166 c) 0.8966
11. a) 0.0001 b) 0.0081
12. a) 0.384 b) 0.512
13. 0.1083
14. a) 0.5514 b) 0.2941
15. $\frac{3}{13}$

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