Subsidiary Mathematics

For

Associate Nursing Program

Learner's Book

Senior Five

FOREWORD

Dear Student,

Rwanda Basic Education Board (REB) is honored to present Subsidiary Mathematics book for senior five students of of Associate Nursing Program. This book will serve as a guide to competence-based teaching and learning to ensure consistency and coherence in the learning of the Mathematics. The Rwandan educational philosophy is to ensure that you achieve full potential at every level of education which will prepare you to be well integrated in society and exploit employment opportunities.

The government of Rwanda emphasizes the importance of aligning teaching and learning materials with the syllabus to facilitate your learning process. Many factors influence what you learn, how well you learn and the competences you acquire. Those factors include the relevance of the specific content, the quality of teachers' pedagogical approaches, the assessment strategies and the instructional materials available. In this book, we paid special attention to the activities that facilitate the learning process in which you can develop your ideas and make new discoveries during concrete activities carried out individually or with peers.

In competence-based curriculum, learning is considered as a process of active building and developing of knowledge and meanings by the learner where concepts are mainly introduced by an activity, situation or scenario that helps the learner to construct knowledge, develop skills and acquire positive attitudes and values.

For efficiency use of this textbook, your role is to:

- Work on given activities which lead to the development of skills;
- Share relevant information with other learners through presentations, discussions, group work and other active learning techniques such as role play, case studies, investigation and research in the library, on internet or outside;
- Participate and take responsibility for your own learning;
- Draw conclusions based on the findings from the learning activities.

To facilitate you in doing activities, the content of this book is self explanatory so that you can easily use it yourself, acquire and assess your competences. The book is made of units as presented in the syllabus. Each unit has the following structure: the key unit competence is given and it is followed by the introductory activity before the development of mathematical concepts that are connected to real world problems or to other sciences.

The development of each concept has the following points:

- It starts by a learning activity: it is a hand on well set activity to be done by students in order to generate the concept to be learnt;
- Main elements of the content to be emphasized;
- Worked examples; and
- Application activities which are activities to be done by the user to consolidate competences or to assess the achievement of objectives.

Even though the book has some worked examples, you will succeed on the application activities depending on your ways of reading, questioning, thinking and grappling ideas of calculus not by searching for similar-looking worked out examples.

Furthermore, to succeed in Mathematics, you are asked to keep trying; sometimes you will find concepts that need to be worked at before you completely understand. The only way to really grasp such a concept is to think about it and work-related problems found in other reference books.

I wish to sincerely express my appreciation to the people who contributed towards the editing of this book, particularly, REB staffs and teachers for their technical support.

Any comment or contribution would be welcome to the improvement of this text book for the next edition.

Dr. MBARUSHIMANA Nelson

Director General, REB

ACKNOWLEDGEMENT

I wish to express my appreciation to the people who played a major role in the development and the editing of Subsidiary Mathematics book for Senior five students of Associate Nursing Program. It would not have been successful without active participation of different education stakeholders.

I owe gratitude to Curriculum Officers and teachers whose efforts during the editing exercise of this book were very much valuable.

Finally, my word of gratitude goes to the Rwanda Basic Education Board staffs who were involved in the whole process of in-house textbook production.

Joan MURUNGI

Head of Curriculum, Teaching and learning Resources Department

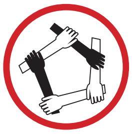
Contents

Icons used in this book

To guide you, each activity in the book is marked by a symbol or icon to show you what kind of activity it is. The icons are as follows:

Practical Activity icon

The hand indicates a practical activity such as curve sketching, draw figures, to have a selection of objects, individually or in a group, and then present your results or comments.



Group Work icon

Group work means that you are expected to discuss something in groups and report back on what your group discussed. In this way, you learn from each other and how to work together as a group to address or solve a problem.

Pairing Activity icon

This means that you are required to do the activity in pairs, exchange ideas and write down your results.

Research Activity icon

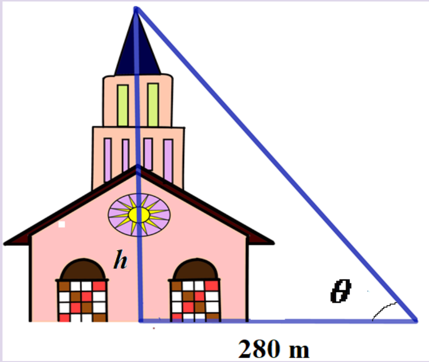
Some activities require you to do research either by reading textbooks or using the internet.

Good luck in using the book!

Unit 1 Trigonometric Formulae and Equations

Introductory activity

The height h of the cathedral is 485 m. The angle of elevation of the top of the Cathedral from a point 280 m away from the base of its steeple on level ground is θ. By using trigonometric concepts, find the value of this angle θ in degree.



Trigonometry studies relationship involving lengths and angles of a triangle. The techniques in trigonometry are used for finding relevance in navigation particularly satellite systems and astronomy, naval and aviation industries, land surveying, in cartography and (creation of maps) and medicine. Even if those are the scientific applications of the concepts in trigonometry, most of the mathematics we study would seem to have little real-life application. Trigonometry is really relevant in our day to day activities.

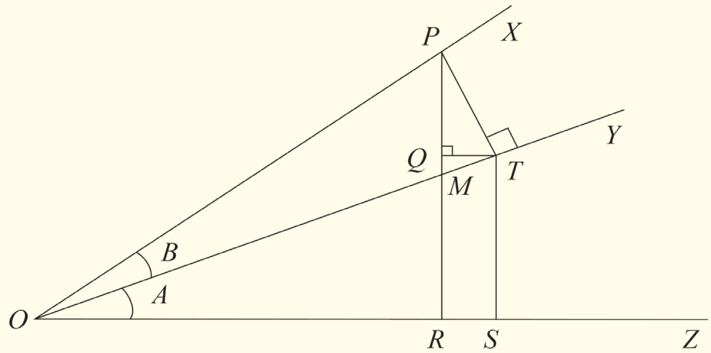
Objectives

By the end of this unit, a student will be able to:

- Solve trigonometric equations.
- Use trigonometric formulae and equations in real life.

1.1. Trigonometric formulae

Activity 1.1



In the diagram, the angle *MOR=A* and *POT=B* are each acute and the angle *POR=(A+B)* is also acute. PT is perpendicular to *OY*, *PR* is perpendicular to *OZ* and *QT* is perpendicular to *PR*.

Since *QT* is parallel to *OS*,

$\angle QTO = \angle TOS = A$

Since $\angle PTO = 90^\circ, \angle PTQ = 90^\circ - A.$

As PQT is a triangle, thus, $\angle PQT + \angle PTQ + \angle QPT = 180^\circ$

That is, $90^\circ + (90^\circ - A) + \angle QPT = 180^\circ$

$180^\circ - A + \angle QPT = 180^\circ$

Thus, $\angle QPT = A$.

Hence, use right triangles ORP, OST and OTP to find the formula of

a) $\sin(A+B)$

b) $\cos(A+B)$

Deduce the formula of

$\tan(A+B), \sin(A-B), \cos(A-B), \tan(A-B)$

Deduce the formula of

$\tan(A+B), \sin(A-B), \cos(A-B), \tan(A-B)$

1.1.1. Addition and subtraction formulae

From activity 1.1:

The addition and subtraction formulae are

$$\cos(x+y)=\cos x\cos y-\sin x\sin y$$
$$\cos(x-y)=\cos x\cos y+\sin x\sin y$$
$$\sin(x+y)=\sin x\cos y+\cos x\sin y$$
$$\sin(x-y)=\sin x\cos y-\cos x\sin y$$

Also,

$$\tan(x+y)=\frac{\tan x+\tan y}{1-\tan x\tan y}$$
$$\tan(x-y)=\frac{\tan x-\tan y}{1+\tan x\tan y}$$
$$\cot(x+y)=\frac{\cot x\cot y-1}{\cot y+\cot x}$$
$$\cot(x-y)=\frac{\cot x\cot y+1}{\cot y-\cot x}$$

Addition and subtraction formulae are useful for finding trigonometric number of some angles.

Example 1.1

Use addition and subtraction formulae to find $\cos 75^\circ$.

Solution

$$\cos 75^\circ=\cos\left(45^\circ+30^\circ\right)$$
$$=\cos 45^\circ\cos 30^\circ-\sin 45^\circ\sin 30^\circ$$
$$=\frac{\sqrt{2}}{2}\frac{\sqrt{3}}{2}-\frac{\sqrt{2}}{2}\frac{1}{2}$$
$$=\frac{\sqrt{6}-\sqrt{2}}{4}$$

Example 1.2

Find the value of $\sin 63^0 \cos 27^0 + \cos 63^0 \sin 27^0$

Solution

We know that $\sin A \cos B + \cos A \sin B = \sin(A+B)$

Therefore, $\sin 63^0 \cos 27^0 + \cos 63^0 \sin 27^0 = \sin(63^0 + 27^0) = \sin 90^0 = 1$

Example 1.3

Use addition and subtraction formulae to find $\tan\frac{5\pi}{3}$

Solution

Using the concept of opposite angle we have: $\frac{5\pi}{3} = 2\pi - \frac{\pi}{3}$

Then,

$$\tan\frac{5\pi}{3} = \tan\left(2\pi - \frac{\pi}{3}\right)$$

$$= \frac{\tan 2\pi - \tan\frac{\pi}{3}}{1 + \tan 2\pi \tan\frac{\pi}{3}}$$

$$= \frac{0-\sqrt{3}}{1+0} = -\sqrt{3}$$

Application activity 1.1

1. Without using calculator, find:

   a) $\sin 75^0$ b) $\tan 75^0$ c) $\tan 105^0$ d) $\sec 105^0$

2. Find the value of $\sin 40^0 \cos 10^0 - \cos 40^0 \sin 10^0$

3. Evaluate without using a calculator

   a) $\sin 15^0$ b) $\cos 15^0$ c) $\tan 15^0$ d) $\cot 15^0$

4. If $\tan\theta = \frac{3}{4}$ and $\tan\phi = \frac{5}{12}$, find;

   a) $\tan(\theta+\phi)$ b) $\cot(\theta-\phi)$

5. If $\tan A = \frac{5}{6}$ and $\tan B = \frac{1}{11}$, prove that $A+B = \frac{\pi}{4}$.

1.1.2. Double angle formulae

Activity 1.2

For each of the following relations, replace y by x and give your results.

1. $\cos(x+y) = \cos x \cos y - \sin x \sin y$
2. $\cos(x-y) = \cos x \cos y + \sin x \sin y$
3. $\sin(x+y) = \sin x \cos y + \cos x \sin y$
4. $\tan(x+y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}$
5. $\cot(x+y) = \frac{\cot x \cot y - 1}{\cot y + \cot x}$

From activity 1.2, we have

$$\cos^2 x + \sin^2 x = 1$$

This relation is called the **fundamental relation of trigonometry.**

From this relation, we can write

$\cos^2 x = 1 - \sin^2 x$ and $\sin^2 x = 1 - \cos^2 x$

$$\sin 2x = 2\sin x \cos x$$

$$\cos 2x = \cos^2 x - \sin^2 x$$

$$\tan 2x = \frac{2\tan x}{1-\tan^2 x}$$

$$\cot 2x = \frac{\cot^2 x - 1}{2\cot x}$$

The above four formulae are known as **double angle formulae**

Example 1.4

From double angle formulae and fundamental relation of trigonometry, prove that

$$\sin 2x = \frac{2\tan x}{1+\tan^2 x}$$

Solution

$$\sin 2x = 2\sin x\cos x$$

Dividing the right hand side by $\sin^2 x + \cos^2 x$, gives

$$\sin 2x = \frac{2\sin x\cos x}{\sin^2 x + \cos^2 x}$$

Dividing each term of the right hand side by $\cos^2 x$, we get

$$\sin 2x = \frac{\frac{2\sin x\cos x}{\cos^2 x}}{\frac{\sin^2 x + \cos^2 x}{\cos^2 x}} \Leftrightarrow \sin 2x = \frac{\frac{2\sin x}{\cos x}}{\frac{\sin^2 x}{\cos^2 x}+1}$$

$$\Leftrightarrow \sin 2x = \frac{2\tan x}{\tan^2 x + 1}$$

Hence, $\sin 2x = \dfrac{2\tan x}{1+\tan^2 x}$ as required.

Example 1.5

Express $\cos 4x$ in function of $\sin x$ only

Solution

$$\cos 4x = \cos 2(2x) = 1 - 2\sin^2 2x$$

$=1-2(2\sin x\cos x)^2$

$=1-2(4\sin^2 x\cos^2 x)$

$=1-8\sin^2 x\cos^2 x$

$=1-8\sin^2 x(1-\sin^2 x)$

$=1-8\sin^2 x+8\sin^4 x$

Application activity 1.2

1. Express $\sin 4x$ in function of $\sin x$ and $\cos x$.
2. Express $\cos 8x$ in function of $\sin x$.
3. Evaluate each of the following without using a calculator:
   a) $2\sin 15^0 \cos 15^0$ b) $2\sin\left(22\frac{1}{2}\right)^0 \cos\left(22\frac{1}{2}\right)^0$
   c) $\sin\frac{\pi}{8}\times\cos\frac{\pi}{8}$ d) $\cos^2\frac{\pi}{12}-\sin^2\frac{\pi}{12}$
4. From double angle formulae, prove the following formulae known as t-formulae, where $t=\tan x$:
   a) $\sin 2x=\frac{2t}{1+t^2}$ b) $\cos 2x=\frac{1-t^2}{1+t^2}$ c) $\tan 2x=\frac{2t}{1-t^2}$

1.1.3. Half angle formulae

Activity 1.3

1. Show that $\cos 2x=1-2\sin^2 x$. By letting $\theta=2x$, deduce the value of $\sin\frac{\theta}{2}$.
2. Show that $\cos 2x=2\cos^2 x-1$. By letting $\theta=2x$, deduce the value of $\cos\frac{\theta}{2}$.
3. Using results in 1 and 2, deduce the value of $\tan\frac{\theta}{2}$. (Recall that $\tan x=\frac{\sin x}{\cos x}$).

From activity 1.3:

$\sin\frac{\theta}{2}=\pm\sqrt{\frac{1-\cos\theta}{2}}$. By permuting θ *by* x. We can write

$$\sin\frac{x}{2}=\pm\sqrt{\frac{1-\cos x}{2}}$$

Also, $\cos\frac{\theta}{2}=\pm\sqrt{\frac{\cos\theta+1}{2}}$. We can write $\cos\frac{x}{2}=\pm\sqrt{\frac{\cos x+1}{2}}$

The half angle formulae are:

$$\sin\frac{x}{2}=\pm\sqrt{\frac{1-\cos x}{2}}$$

$$\cos\frac{x}{2}=\pm\sqrt{\frac{\cos x+1}{2}}$$

,the sign + or − is chosen depending on the quadrant in which $\frac{x}{2}$ lies.

$\tan\frac{x}{2}=\frac{1-\cos x}{\sin x}$ or $\tan\frac{x}{2}=\frac{\sin x}{1+\cos x}$

Example 1.6

Using the half angle formula, find the exact value of $\cos 15^0$.

Solution

15^0 is in first quadrant, then $\cos 15^0$ must be positive

$$\cos 15^0=\cos\left(\frac{1}{2}\left(30^0\right)\right)$$

$$=\sqrt{\frac{1+\cos 30^0}{2}}\quad=\sqrt{\frac{1+\frac{\sqrt{3}}{2}}{2}}$$

$$=\frac{\sqrt{2+\sqrt{3}}}{2}$$

Example 1.7

If $\cos A=-\frac{7}{25}$, find the values of $\sin\frac{1}{2}A$, $\cos\frac{1}{2}A$ and $\tan\frac{1}{2}A$.

Solution

$$\sin\frac{1}{2}A=\pm\sqrt{\frac{1-\cos A}{2}}$$

$$\Rightarrow \sin\frac{1}{2}A=\pm\sqrt{\frac{1-\left(-\frac{7}{25}\right)}{2}} \qquad \Rightarrow \sin\frac{1}{2}A=\pm\sqrt{\frac{1+\frac{7}{25}}{2}}$$

$$\Rightarrow \sin\frac{1}{2}A=\pm\sqrt{\frac{32}{50}}$$

Finally, $\sin\frac{1}{2}A=\pm\sqrt{\frac{16}{25}}=\pm\frac{4}{5}$

$$\cos\frac{1}{2}A=\pm\sqrt{\frac{1+\cos A}{2}} \quad \Rightarrow \cos\frac{1}{2}A=\pm\sqrt{\frac{1-\frac{7}{25}}{2}}$$

$$\Rightarrow \cos\frac{1}{2}A=\pm\sqrt{\frac{18}{50}}$$

Thus, $\cos\frac{1}{2}A=\pm\sqrt{\frac{9}{25}}=\pm\frac{3}{5}$

$$\tan\frac{1}{2}A=\pm\sqrt{\frac{1-\cos A}{1+\cos A}}$$

$$\Rightarrow \tan\frac{1}{2}A=\pm\sqrt{\frac{1+\frac{7}{25}}{1-\frac{7}{25}}} \quad \Rightarrow \tan\frac{1}{2}A=\pm\sqrt{\frac{32}{18}}$$

Finally, $\cos\frac{1}{2}A=\pm\sqrt{\frac{16}{9}}=\pm\frac{4}{3}$.

Application activity 1.3

1. If $\cos A=-\frac{1}{3}$, find the values of $\sin\frac{1}{2}A$, $\cos\frac{1}{2}A$.
2. If $\tan 2A=\frac{7}{24}$, $0<A<\frac{\pi}{4}$, find the value of $\tan A$.
3. Find $\sin x$, $\cos x$ and $\tan x$ if $\cos 2x=\frac{1}{8}$.

1.1.4. Transformation of product into sum

Activity 1.4

From addition and subtraction formulae, evaluate:

1. $\sin(x+y)+\sin(x-y)$
2. $\sin(x+y)-\sin(x-y)$
3. $\cos(x+y)+\cos(x-y)$
4. $\cos(x+y)-\cos(x-y)$

From activity 1.4:

The formulae for transforming product in sum are:

$$\cos x\cos y=\frac{1}{2}\left[\cos(x+y)+\cos(x-y)\right]$$

$$\sin x\sin y=-\frac{1}{2}\left[\cos(x+y)-\cos(x-y)\right]$$

$$\sin x\cos y=\frac{1}{2}\left[\sin(x+y)+\sin(x-y)\right]$$

$$\cos x\sin y=\frac{1}{2}\left[\sin(x+y)-\sin(x-y)\right]$$

Example 1.8

Transform in sum the product $\sin 3x\cos 4x$.

Solution

$$\begin{aligned}\sin 3x\cos 4x&=\frac{1}{2}\left[\sin(3x+4x)+\sin(3x-4x)\right]\\&=\frac{1}{2}\left[\sin 7x+\sin(-x)\right]\\&=\frac{1}{2}\left[\sin 7x-\sin x\right]\end{aligned}$$

Example 1.9

Change the following product into a sum or difference: $\sin 9x\sin 11x$

Solution

$$\sin 9x \sin 11x = -\frac{1}{2}\left[\cos(9x+11x)-\cos(9x-11x)\right]$$
$$= -\frac{1}{2}\left[\cos 20x-\cos(-2x)\right] = -\frac{1}{2}(\cos 20x-\cos 2x)$$
$$= \frac{1}{2}(\cos 2x-\cos 20x)$$

Application activity 1.4

Transform in sum:

1. $\sin x \cos 3x$
2. $\cos 12x \sin 9x$
3. $2\cos\frac{5x}{2}\cos\frac{3x}{2}$

1.1.5. Transformation of sum into product

Activity 1.5

Using the relations $x+y=p$ and $x-y=q$, express each of the formulae for transforming product in sum in function of p and q.

Hint:

$$\begin{cases} x+y=p \\ x-y=q \end{cases} \Rightarrow \begin{cases} x=\dfrac{p+q}{2} \\ y=\dfrac{p-q}{2} \end{cases}$$

From activity 1.5, the formulae for transforming sum in product are:

$$\cos p+\cos q = 2\cos\frac{p+q}{2}\cos\frac{p-q}{2}$$

$$\cos p-\cos q = -2\sin\frac{p+q}{2}\sin\frac{p-q}{2}$$

$$\sin p+\sin q = 2\sin\frac{p+q}{2}\cos\frac{p-q}{2}$$

$$\sin p-\sin q = 2\cos\frac{p+q}{2}\sin\frac{p-q}{2}$$

Example 1.10

Transform in product the sum $\cos 2x - \cos 4x$

Solution

$$\cos 2x - \cos 4x = -2\left(\sin\frac{2x+4x}{2}\sin\frac{2x-4x}{2}\right)$$

Or $\cos 2x - \cos 4x = -2\sin 3x \sin(-x)$.

So, $\cos 2x - \cos 4x = 2\sin 3x \sin x$.

Application activity 1.5

Transform in product:

1. $\cos x + \cos 7x$
2. $\sin 4x - \sin 9x$
3. $\sin 3x + \sin 4x$

1.2. Trigonometric equations

1.2.1. The solution of equations reducible to the form

$$\sin(x+\alpha)=k, \cos(x+\alpha)=k \ \ for\ |k|\le 1 \quad and\ \tan(x+\alpha)=b\ \ for\ b\in\mathbb{R}$$

Activity 1.6

1. Find at least three angles whose sine is $\frac{1}{2}$.
2. Find at least three angles whose cosine is $\frac{\sqrt{2}}{2}$.

The solutions of a trigonometric equation for which $0 \le x \le 2\pi$ are called principle solutions while the expression (involving integer k) of solution containing all values of the unknown angle is called the general solution of the trigonometric equation. When the interval of solution is not given, you are required to find the general solution.

When solving trigonometric equations, the following identities are helpful:

$\sin\alpha = \sin(\alpha + 2k\pi),\ k \in \mathbb{Z}$
$\cos\alpha = \cos(\alpha + 2k\pi),\ k \in \mathbb{Z}$
$\tan\alpha = \tan(\alpha + k\pi),\ k \in \mathbb{Z}$

$\sin\alpha = \sin(\pi - \alpha)$
$\cos\alpha = \cos(-\alpha)$
$\tan\alpha = \tan(\alpha + \pi)$

Example 1.11

Find the principal solutions of the equation:

$$\sin x = \frac{1}{\sqrt{2}}$$

Solution

$\sin x = \frac{1}{\sqrt{2}}$ is positive $\Rightarrow x$ lies in the 1st or 2nd quadrant.

Here $\sin x = \frac{1}{\sqrt{2}} = \sin\frac{\pi}{4}$ or $\sin\left(\pi - \frac{\pi}{4}\right)$

$$\Rightarrow x = \frac{\pi}{4} \text{ or } x = \frac{3\pi}{4}$$

Thus, $x = \frac{\pi}{4}$ or $x = \frac{3\pi}{4}$.

Example 1.12

Solve in the set of real numbers $\cos 2x = -\frac{\sqrt{3}}{2}$.

Solution

$\cos 2x = -\frac{\sqrt{3}}{2}$ is negative $\Rightarrow 2x$ lies in the 2nd or 3rd quadrant.

Here, $\cos 2x = -\cos\frac{\pi}{6} = \cos\left(\pi - \frac{\pi}{6}\right)$ *or* $\cos\left(\pi + \frac{\pi}{6}\right)$

$$\Rightarrow 2x = \frac{5\pi}{6} + 2k\pi \text{ or } \frac{7\pi}{6} + 2k\pi, k \in \mathbb{Z}$$

Example 1.13

Solve $\sin\frac{x}{3} = -\frac{\sqrt{3}}{2}$ for $x \in [0, 2\pi]$

Solution

$\sin\frac{x}{3} = -\frac{\sqrt{3}}{2}$ for $x \in [0, 2\pi]$

$\frac{x}{3} = \begin{cases} -\frac{\pi}{3} + 2k\pi \\ \pi - \left(-\frac{\pi}{3}\right) + 2k\pi \end{cases}$ $\quad \frac{x}{3} = \begin{cases} -\frac{\pi}{3} + 2k\pi \\ \frac{4\pi}{3} + 2k\pi \end{cases}$ $\quad \Rightarrow x = \begin{cases} -\pi + 6k\pi \\ 4\pi + 6k\pi \end{cases}, k \in \mathbb{Z}$

Since we are given the condition $x \in [0, 2\pi]$, we need to substitute k with some integers (…, -2, -1, 0, 1, 2 …). But doing this, no value can be found in the given interval. Thus, there is no solution.

Application activity 1.6

Solve:

1. $\cos 8x = -\frac{1}{2}$
2. $\sin 3x \cos 7x = 0$ for $0 \le x \le \pi$
3. $\sin 3x = \frac{1}{2}$ for $x \in [0, 2\pi]$
4. $6\cos^2\theta + \sin\theta - 5 = 0$ for $0^0 < \theta \le 360^0$

   Hint. Let $t = \sin\theta; \cos^2\theta = 1 - \sin^2\theta$
5. $2\cos^2 x - 5\sin x + 1 = 0$ for $x \in [0, \pi]$
6. $\tan 3x = 1$
7. Using transformation of sum or difference of trigonometric expressions into the product, solve the following equations:

   a) $\sin 3x = \sin x$ b) $\cos 4x = \cos x$

   c) $\sin 4x + \sin 2x = 0$ d) $\cos 3x + \cos x = 0$

1.2.2. Solving the equation of the form $a\sin x + b\cos x = c$

Activity 1.7

Consider the equation $\sqrt{3}\cos x - \sin x = \sqrt{3}$

1. Using t-formulae (Exercise 1.2 Question 4), express the given equation in terms of t where $t = \tan\frac{x}{2}$.
2. Solve the quadratic equation obtained in 1).
3. From t, find the value of x. **Hint**: Remember that $t = \tan\frac{x}{2}$.

Form activity 1.7, one of the methods of solving trigonometric equation of the form $a\sin x + b\cos x = c$, is to use the t-formulae.

Since $\cos x = \frac{1-t^2}{1+t^2}$ and $\sin x = \frac{2t}{1+t^2}$ where $t = \tan\frac{x}{2}$,

$$a\sin x + b\cos x = c \Leftrightarrow \frac{2at}{1+t^2} + b\frac{1-t^2}{1+t^2} = c$$

$$\Leftrightarrow 2at + b - bt^2 = c\left(1+t^2\right)$$

$$\Leftrightarrow (b+c)t^2 - 2at = c - b$$

$$\Leftrightarrow (b+c)t^2 - 2at - c + b = 0$$

which is a quadratic equation in t.

Example 1.14

Solve $\sqrt{3}\sin x + \cos x = 2$

Solution

Using t-formulae, $\cos x = \frac{1-t^2}{1+t^2}$ and $\sin x = \frac{2t}{1+t^2}$ where $t = \tan\frac{x}{2}$,

By substitution,

$$\frac{2\sqrt{3}t}{1+t^2} + \frac{1-t^2}{1+t^2} = 2$$

By cross multiplying and collecting like terms,

$3t^2 - 2\sqrt{3}t + 1 = 0$

By solving, we get:

$\Delta = \left(2\sqrt{3}\right)^2 - 12 = 0 \qquad t_1 = t_2 = \frac{2\sqrt{3}}{6} = \frac{\sqrt{3}}{3}$

$t = \frac{\sqrt{3}}{3} \Leftrightarrow \tan\frac{x}{2} = \frac{\sqrt{3}}{3} \quad \Rightarrow \frac{x}{2} = \frac{\pi}{6} + k\pi, k \in \mathbb{Z} \ (1)$

or $\frac{x}{2} = \left(\pi + \frac{\pi}{6}\right) + k\pi \ (2)$

$For(1),\ x = \frac{\pi}{3} + 2k\pi$

$For(2),\ x = \frac{7\pi}{3} + 2k\pi$

Application activity 1.7

Solve in $\mathbb{R}$

1. $\cos x + \sqrt{3}\sin x = \sqrt{3}$
2. $\cos x + \sin x = \sqrt{2}$
3. $3\sin x + \sqrt{3}\cos x = 3$
4. $3\cos x + 4\sin x = 2$, $0^0 \le x \le 360^0$

1.3. Applications

1.3.1. Simple harmonic motion

Activity 1.8

Discuss how trigonometric theory is used in harmonic motion.

An object that moves on a coordinate axis is in **simple harmonic motion** if its distance from the origin, d, at time t is given by either $d = a\cos\omega t$ or $d = a\sin\omega t$.

The motion has **amplitude** $|a|$, the maximum displacement of the object from its rest position. The period of the motion is $\frac{2\pi}{\omega}$, where $\omega > 0$. The **period** gives the time it takes for the motion to go through one complete cycle.

In describing simple harmonic motion, the equation with the cosine function, $d = a\cos\omega t$, is used if the object is at its greatest distance from rest position, the origin, at $t = 0$. By contrast, the equation with the sine function, $d = a\sin\omega t$, is used if the object is at its rest position, the origin at $t = 0$.

Approximatively "$\pi = 3.14$" and the unit of angle in simple harmonic motion is "radian."

Example 1.15

If the instantaneous voltage in a current is given by the equation $E = 204\sin 3680t$, where E is expressed in volts and t is expressed in seconds, find E if $t = 0.27$ seconds.

Solution

$E = 204\sin 3680t$

$E = 204\sin\left[(3680)(0.27)\right]$

$E = 204\sin 993.6$

$E \approx 154\ volts$

Example 1.16

The horizontal displacement d of the end of a pendulum is $d = K\sin 2\pi t$. Find K if d=12 centimetres and $t = 3.25$ seconds.

Solution

$d = K\sin 2\pi t$

$12 \approx K\sin\left[(2)(3.1415)(3.25)\right]$

$12 \approx K\sin 20.42$

$K \approx \dfrac{12}{\sin 20.42}$

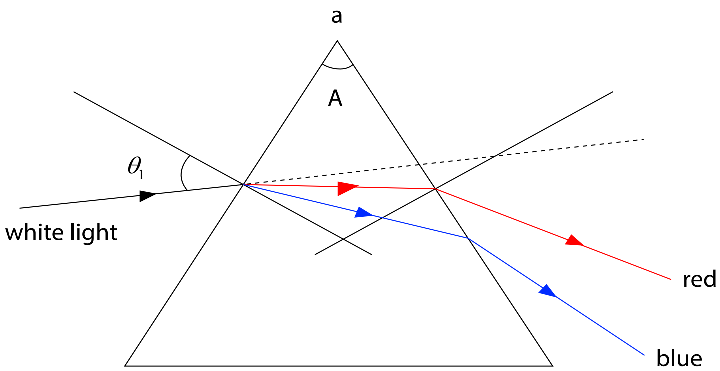
$K \approx 12$

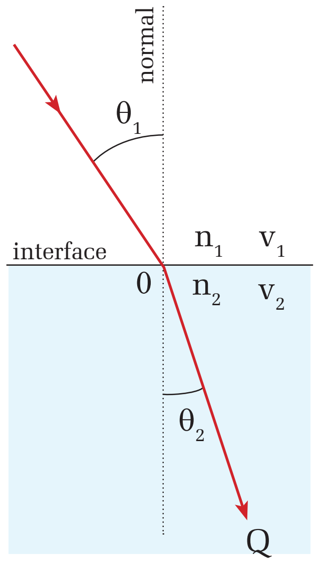
1.3.2. Refraction of light

Activity 1.9

Discuss how trigonometric theory is used in refraction of light.

In optics, light changes speed as it moves from one medium to another (for example, from air into the glass of the prism). This speed change causes the light to be refracted and to enter the new medium at a different angle (Huygens principle).
A prism is a transparent optical element with flat, polished surfaces that refract light.





The degree of bending of the light's path depends on the angle that the incident beam of light makes with the surface, and on the ratio between the refractive indices of the two media (Snell's law).
If the light is traveling from a rarer region (lower *n*) to a denser region (higher *n*), it will bend towards the normal but if it is traveling from a denser region (higher *n*) to a rarer region (lower *n*), it will bend away from the normal.

Snell's law states that: $n_1 \sin\theta_1 = n_2 \sin\theta_2$

Example 1.17

Light travels from air into an optical fiber with an index of refraction of 1.44.

a) In which direction does the light bend?

b) If the angle of incidence on the end of the fiber is 22^0, what is the angle of refraction inside the fiber?

c) Sketch the path of light as it changes media.

Solution

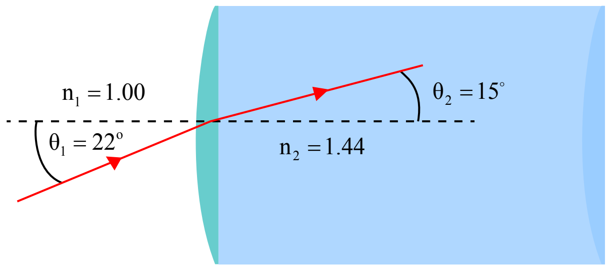
a) Since the light is traveling from a rarer region (lower n) to a denser region (higher n), it will bend towards the normal.

b) We will identify air as medium 1 and the fiber as medium 2. Thus, n_1=1.00 (index of air), $n_2 = 1.44$ and $\theta_1 = 22^0$

$$n_1 \sin\theta_1 = n_2 \sin\theta_2, \ \sin 22 = 1.44 \sin\theta_2, \ \sin\theta_2 = \frac{\sin 22}{1.44}$$

$$\theta_2 = \sin^{-1}(0.26) \Rightarrow \theta_2 = 15^0$$

c) The path of the light is shown in the figure below:



Example 1.18

A ray of light is incident through glass, with refractive index 1.52, on an interface separating glass and water with refractive index 1.32. What is the angle of refraction if the angle of incidence of the ray in glass is 25 °?

Solution

Let the needed angle be t, use Snell's law to write:

$$1.52 \sin 25^0 = 1.32 \sin t$$

$$\Leftrightarrow \sin t = \frac{1.52 \sin 25^0}{1.32}$$

$$t = \sin^{-1}\left(\frac{1.52 \sin 25^0}{1.32}\right)$$

$$\Rightarrow t = 29.1^0$$

1.3.3. 1.3.3 Application in medicine

Activity 1.10

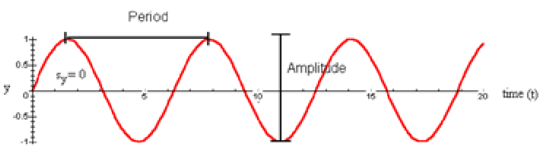
Conduct research in different books of the library or on the internet to discover the application of trigonometry in the field of medicine such as heart's electrical activities, trucking the rhythms of lungs capacity, testing electrical activities of the brain and abnormalities in the brain...

Trigonometric concepts such as sinusoidal waves contribute to various medical testing and interpretation of those test results.

1. Electrocardiography: The measurement of electrical activities in the heart. Through this process, it is possible to determine how long the electrical wave takes to travel from one part of the heart to the next by showing if the electrical activity is normal or slow, fast or irregular.

2. Pulmonary function testing: a spirometer is used to measure the volume of air inhaled and exhaled while breathing by recording the changing volume over time. The output of a spirogram can be quantified using trigonometric equations and generally, it is possible to describe any repeating rhythms by a sine wave:

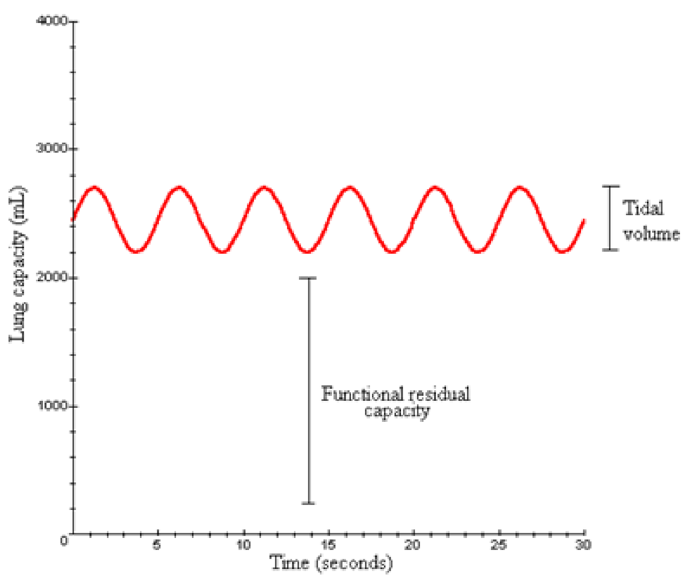
   $y=\left(\frac{A}{2}\right)\sin\left(\frac{2\pi}{p}(\text{t})\right)+s_y$ where y is some property that exhibits a rhythm, ***t*** is time , ***A*** is the amplitude, ***p*** is the period, and sy is a phase shift in the height of ***y*** compared to the simplest sine wave ($y=\sin t$, where s_y =0). For the simplest sine wave, the amplitude is 2 , the period 2π of a wave is the time for completion of one cycle.



In humans, when each breath is completed, the lung still contains a volume of air, called the functional residual capacity (approximately 2200 mL in humans). Each inhalation adds from 500 mL of additional air for normal (resting) breathing. Each exhalation removes approximately the same volume as was inhaled. The volume of air inhaled and exhaled normally is called the tidal volume. It takes approximately 5 seconds to complete a single cycle. With this information, we can develop an equation for lung capacity as a function of time for normal (resting) breathing. Since sy describes the midpoint of the sine wave, we take the functional residual capacity plus 1/2 the amplitude to give us ***sy=2200+250=2450 mL***. Our period is the time for a single cycle, or p=5 seconds. The volume of air inhaled gives the amplitude, or A=1000 mL. We therefore have the following equation for normal breathing:

$$Lung\ \ capacity = \left(\frac{500}{2}\right)\sin\left(\frac{2\pi}{5}(t)\right) + 2450$$

Graph: Rhythms of lungs capacity



Unit Summary

1. The **addition and subtraction formulae:**

$$\cos(x+y)=\cos x\cos y-\sin x\sin y$$

$$\cos(x-y)=\cos x\cos y+\sin x\sin y$$

$$\sin(x+y)=\sin x\cos y+\cos x\sin y$$

$$\sin(x-y)=\sin x\cos y-\cos x\sin y$$

$$\tan(x+y)=\frac{\tan x+\tan y}{1-\tan x\tan y}$$

$$\tan(x-y)=\frac{\tan x-\tan y}{1+\tan x\tan y}$$

$$\cot(x+y)=\frac{\cot x\cot y-1}{\cot y+\cot x}$$

$$\cot(x-y)=\frac{\cot x\cot y+1}{\cot y-\cot x}$$

2. The **double angle formulae:**

$$\sin 2x=2\sin x\cos x$$

$$\cos 2x=\cos^2 x-\sin^2 x$$

$$=1-2\sin^2 x$$

$$=2\cos^2 x-1$$

$$\tan 2x=\frac{2\tan x}{1-\tan^2 x}$$

$$\cot 2x=\frac{\cot^2 x-1}{2\cot x}$$

T- formulae

If $t=\tan\frac{x}{2}$,

a) $\sin x=\frac{2t}{1+t^2}$ b) $\cos x=\frac{1-t^2}{1+t^2}$ c) $\tan x=\frac{2t}{1-t^2}$

3. The **half angle formulae:**

$$\sin\frac{x}{2}=\pm\sqrt{\frac{1-\cos x}{2}} \qquad \cos\frac{x}{2}=\pm\sqrt{\frac{\cos x+1}{2}}$$

$$\tan\frac{x}{2}=\frac{1-\cos x}{\sin x}$$ or

$$\tan\frac{x}{2}=\frac{\sin x}{1+\cos x}$$

4. The **formulae for transforming product in sum:**

$$\cos x\cos y=\frac{1}{2}\left[\cos(x+y)+\cos(x-y)\right]$$

$$\sin x\sin y=-\frac{1}{2}\left[\cos(x+y)-\cos(x-y)\right]$$

$$\sin x\cos y=\frac{1}{2}\left[\sin(x+y)+\sin(x-y)\right]$$

$$\cos x\sin y=\frac{1}{2}\left[\sin(x+y)-\sin(x-y)\right]$$

5. The **formulae for transforming sum in product:**

$$\cos p+\cos q=2\cos\frac{p+q}{2}\cos\frac{p-q}{2}$$

$$\cos p-\cos q=-2\sin\frac{p+q}{2}\sin\frac{p-q}{2}$$

$$\sin p+\sin q=2\sin\frac{p+q}{2}\cos\frac{p-q}{2}$$

$$\sin p-\sin q=2\cos\frac{p+q}{2}\sin\frac{p-q}{2}$$

6. When solving trigonometric equation, transform the sum or difference if any, into product or rearrange and rewrite the given expression using trigonometric identities to remain with a simple equation. Simple equation involves one trigonometric function with one unknown, like $\sin x=\frac{1}{2}$. In solving the equation $a\sin x+b\cos x=c$, use t-formulae; from $t=\tan\frac{x}{2}$, find the value of x.

End of Unit Assessment

1. Express $\tan 4x$ in function of tangent.
2. Express $\cot 4x$ in function of cotangent and tangent.
3. Evaluate $\dfrac{\tan 69^0 + \tan 66^0}{1-\tan 69^0 \tan 66^0}$.
4. If $\tan A = \dfrac{a}{a+1}$ and $\tan B = \dfrac{1}{2a+1}$, prove that $A+B=\dfrac{\pi}{4}$.
5. Find, without using calculator, the exact value of

   a) $2\sin 75^\circ \cos 75^\circ$ b) $\cos^2\left(\dfrac{45^\circ}{2}\right)-\sin^2\left(\dfrac{45^\circ}{2}\right)$

   c) $\dfrac{2\tan\left(\dfrac{135^\circ}{2}\right)}{1-\tan^2\left(\dfrac{135^\circ}{2}\right)}$ d) $1-2\sin^2 15^\circ$

6. If $\cos\theta=\dfrac{3}{5}$ and θ is acute, find the exact value of $\cos 2\theta$.
7. If $\tan\theta=\dfrac{12}{5}$ and θ is acute, find the exact value of $\tan 2\theta$.
8. Transform in product:

   a) $\cos 8x-\cos 9x$ b) $\sin 3x+\sin 11x$

9. Transform in sum

   a) $\sin 4x\cos 11x$ b) $\cos 7x\sin 9x$

10. Solve the following equations:

    a) $2\cos^2 x-5\cos x+2=0$ b) $\sin^2 x-2\cos x+\dfrac{1}{4}=0$

    c) $2\sin^2 x+\sqrt{3}\cos x+1=0$ d) $3\tan^2 x+2\sqrt{3}\tan x-3=0$

11. Solve the following equations:

    a) $\cos 5x+\cos 3x=0$ b) $\sin 8x=\sin 3x$

    c) $\sin 2x+\sin 6x+\sin 4x=0$ d) $\sin mx+\sin nx=0$

12. Solve the following trigonometric equations:

    a) $\cos x+\sin x=1$ b) $\sqrt{3}\cos x-\sin x=1$

    c) $2\sin x+\sqrt{3}\cos x=1+\sin x$ d) $\cos x+\sin x=\sqrt{2}$

Unit 2
Sequences

Introductory activity

Suppose that an insect population is growing in such a way that each new generation is 2 times as large as the previous generation. If there are 126 insects in the first generation, on a piece of paper, write down the number of insects that will be there in second, third, fourth,…nth generation.

Objectives

By the end of this unit, a student will be able to:

- Define a sequence.
- Identify an arithmetic, a harmonic or a geometric sequence.
- Determine n^{th} term and the sum of the first n terms of an arithmetic or geometric sequence.
- Apply the concepts of sequences to solve problems involving arithmetic, harmonic or geometric sequence.
- Determine the convergence or divergence of a sequence.

When a set of numbers follows a pattern where there is a clear rule for finding the next number in the pattern, then we have **'a sequence'**

The following are examples of sequences

a) 1,2,3,4,.. *a sequence of counting numbers*

b) 1, 3, 5, 7, 9, … a sequence of odd numbers

c) 0,2,4,6,8,10,… *a sequence of even numbers*

Each number in the sequence is called a **'term'** of the sequence.

In dealing with sequences, we usually use subscripted letters, such as u_1 to represent the first term, u_2 for the second term, u_3 for the third term and so on. The nth term of a sequence is denoted u_n and the term before u_n is u_{n-1} That is, a sequence is denoted $\{u_1, u_2, u_3, ..., u_{n-1}, u_n\}$ A sequence is a function whose domain is a subset of the set of natural numbers

A sequence is denoted shortly by $\{u_n\}$ or (u_n).

The natural number n is called **term number** and value u_n is called a **general term** of a sequence and the term u_1 is the **initial term** or the **first term**.

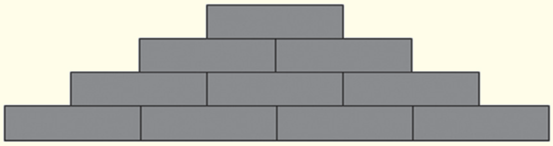
2.1. Arithmetic and harmonic sequences

2.1.1. Definition

Activity 2.1

Suppose that you want to build a tower with blocks.

a) On a piece of paper, draw that tower starting with 15 blocks for the bottom row until you are not able to add another row.
b) How many rows are there?
c) Write down the number of blocks that are in each row (from bottom row to the top row).
d) In the numbers written down each number can be found by adding a constant number to the previous, refer to the following picture and guess that constant number .



Sequences of numbers that follow a pattern of adding a fixed number from one term to the next are called **arithmetic sequences** or **arithmetic progressions**.

A sequence (a_n) is said to be arithmetic if $a_{n+1} - a_n = d,\ n \in \mathbb{N}$ where d is a constant. The constant d is called the **common difference** of the sequence.

Example 2.1

A sequence (a_n) is given by the formula $a_n = 3n + 2,\ n \in \mathbb{N}$. Prove that it is an arithmetic sequence.

Solution

$a_n = 3n + 2 \Rightarrow a_{n+1} = 3(n+1) + 2 = 3n + 5$

$a_{n+1} - a_n = (3n+5) - (3n+2) = 3$ which is a constant.

Hence (a_n) is an arithmetic sequence.

Example 2.2

A sequence (a_n) is given by the formula $a_n = n^2 + 2,\ n \in \mathbb{N}$. Prove that it is not an arithmetic sequence.

Solution

$a_n = n^2 + 2 \Rightarrow a_{n+1} = (n+1)^2 + 21 = n^2 + 2n + 2$

$a_{n+1} - a_n = (n^2 + 2n + 2) - (n^2 + 2) = 2n$ which is not constant.

Hence, (a_n) is not an arithmetic sequence.

Note

If three terms are consecutive terms of an arithmetic sequence, the double of the medium term is equal to the sum of extreme terms. That is for an arithmetic sequence a_{n-1}, a_n, a_{n+1}, we have $2a_n = a_{n-1} + a_{n+1}$.

Example 2.3

Show that $4, 6, 8$ are three consecutive terms of an arithmetic sequence.

Solution

Here $a_{n-1}=4,\ a_n=6,\ a_{n+1}=8$

If a_{n-1}, a_n, a_{n+1} are in arithmetic sequence, we have $2a_n=a_{n-1}+a_{n+1}$

Or $2\times 6=8+4 \Leftrightarrow 12=12$

Thus, $4,6,8$ are three consecutive terms of an arithmetic sequence.

Example 2.4

Find x such that $6,x,12$ are in arithmetic progression.

Solution

If $6,x,12$ are 3 consecutive terms of an arithmetic progression, then

$2x=6+12 \Leftrightarrow 2x=18 \Rightarrow x=9$

Thus, $6,x,12$ are in arithmetic progression, if $x=9$.

Example 2.5

Form an arithmetic progression of three numbers such that the sum of its terms is 30 and the sum of the squares of its terms is 332.

Solution

Let the second term be x. The first term is $x-d$ and the third term is $x+d$ where d is the common difference.

Now, $x-d+x+x+d=30 \Rightarrow 3x=30$ or $x=10$

Also, $(x-d)^2+x^2+(x+d)^2=332$

Or $(10-d)^2+100+(10+d)^2=332$

Or $2d^2=32 \Rightarrow d=\pm 4$

Therefore, the progression is 6, 10, 14 or 14, 10, 6.

Example 2.6

The nth term of the arithmetic sequence is $a_n=2n-1$. Find its 7th term.

Solution

Here, $a_n = 2n-1$, its 7th term is $a_7 = 2\times 7-1=13$

Application activity 2.1

1. The following pairs of numbers are respectively the first term and the common difference of an arithmetic sequence. Find the first 4 terms and the 10th term of each sequence:
   a) $5, 6$ b) $43, -5$
   c) $-7, 4$ d) $-1, -7$
2. Is the sequence 2, 7, 12, 17, 23, 27 arithmetic progression? Why?
3. Determine the common difference of the sequence $\{2n+1\}$.
4. Given that $24,\ 5x+1,\ x^2-1$ are three consecutive terms of an arithmetic progression, find the values of x and the numerical value of the fourth term for each value of x found.
5. The nth term of a sequence is given by $a_n = 2n^2-5n+17$. Show that it is not an arithmetic sequence.

2.1.2. General term of an arithmetic sequence

Activity 2.2

If $\{u_n\}$ is an arithmetic sequence with common difference d and initial term u_1, then

$u_2 = u_1 + d$

$u_3 = u_2 + d = (u_1+d)+d = u_1 + 2d$

Continue in this manner up to u_{10} and conclude that the general formula could be used for u_n.

From activity 2.2, the n^{th} term, u_n, of an arithmetic sequence $\{u_n\}$ with common difference d and initial term u_1 is given by

$$u_n = u_1 + (n-1)d$$

Generally, if u_p is any p^{th} term of a sequence, then the n^{th} term is given by $u_n = u_p + (n-p)d$

Example 2.7

Determine the 25^{th} of an arithmetic progression (sequence) whose 9th term is -6 and common difference $\frac{5}{4}$.

Solution

$a_n = a_1 + (n-1)d \Rightarrow a_9 = a_1 + (9-1)\frac{5}{4}$

$\Leftrightarrow -6 = a_1 + 10 \qquad \Rightarrow a_1 = -16$

From $a_n = a_1 + (n-1)d$, we get $a_{25} = -16 + (25-1)\frac{5}{4} = -16 + 30 = 14$

Hence, the 25^{th} term is 14

Alternative method:

We know that $a_n = a_p + (n-p)d$

Here, $n = 25, p = 9, d = \frac{5}{4}$; then $a_{25} = -6 + (25-9)\frac{5}{4} = -6 + 20 = 14$

Hence, the 25^{th}, the 9^{th} term of arithmetic progression is 499 and 499^{th} term is 9. The term which is equal to zero is

i) 501^{st} ii) 502^{nd}

iii) 508^{th} iv) None of these answers term is 14

Example 2.8

If the 3^{rd} term and the 8^{th} term of an arithmetic sequence are 5 and 15 respectively, find the common difference.

Solution

$u_3 = 5,\ u_8 = 15$

Using the general formula: $u_n = u_p + (n-p)d$

$u_3 = u_8 + (3-8)d$

$5 = 15 - 5d$

$\Leftrightarrow 5d = 15 - 5$

$\Leftrightarrow 5d = 10$

$\Leftrightarrow d = 2$

The common difference is 2.

Example 2.9

Consider the sequence which went from 20 to -26 with -2 as common difference. Find the number of terms.

Solution

We have

$-26 = 20 + (n-1)(-2)$

$\Leftrightarrow -46 = -2n + 2 \Rightarrow n = 24$

This means that there are 24 terms in the sequence.

Example 2.10

A body falls 16 metres in the first second of its motion, 48 metres in the second, 80 metres in the third, 112 metres in the fourth and so on. How far does it fall during the 11^{th} second of its motion?

Solution

The distance through which the body falls in the first, second, third, fourth,…seconds form an arithmetic progression:

$16, 48, 80, 112, \ldots$

Here, $a_1 = 16\, and\ d = 32$.

Distance through which it falls in 11^{th} second is 11^{th} term of the arithmetic progression or a_{11}

$a_{11} = a_1 + 10d = 16 + 320 = 336$.

Distance through which it falls in 11^{th} second is $336m$

Application activity 2.2

1. If the 2nd term and the 6th term of an arithmetic sequence are 4 and 16 respectively, find the common difference.
2. Find the number of terms in the sequence 1, 4, 7, 10, …, 25.
3. Consider the sequence $5, 8, 11, 14, 17, ..., 47$. Find the number of terms in this sequence.
4. The 9th term of arithmetic progression is 499 and 499th term is 9. The term which is equal to zero is
   i) 501st ii) 502nd
   iii) 508th iv) None of these answers
5. Write down the nth term of the arithmetic sequence $\{4,11,18,\ldots\}$. What is the term nearest to 140? Find the least value of n for which the 4th term is greater than 250.

2.1.3. Arithmetic means

Activity 2.3

Suppose that you need to form an arithmetic sequence of 7 terms such that the first term is 2 and the seventh term is 20. Write down that sequence.

If three or more than three numbers are in arithmetic sequence, then all terms lying between the first and the last numbers are called arithmetic means. If B is arithmetic mean between A and C, then $B = \frac{A+C}{2}$.

To insert k terms called **arithmetic means** between two terms u_1 and u_n is to form an arithmetic sequence of $n = k+2$ terms whose the first term is u_1 and the last term is u_n.

As u_1 and u_n are known, we get the common difference d from

$u_n = u_1 + (n-1)d$ taking $n = k+2$ where k is the number of terms to be inserted.

Example 2.11

Insert three arithmetic means between 7 and 23.

Solution

Here, $k = 3$ and then, $n = k+2 = 5$, $u_1 = 7$ and $u_n = u_5 = 23$.

Then

$u_5 = u_1 + (5-1)d$

$\Leftrightarrow 23 = 7 + 4d \Rightarrow d = 4$

Now, inserting the terms using $d = 4$, the sequence is

$7, 11, 15, 19, 23$.

Example 2.12

Insert five arithmetic means between 2 and 20.

Solution

Here, $k = 5$ and then $n = k+2 = 7$, $u_1 = 2$ and $u_n = u_7 = 20$.

Then,

$u_7 = u_1 + (7-1)d$

$\Leftrightarrow 20 = 2 + 6d \Rightarrow d = 3$

Now, insert the terms using $d = 3$, the sequence is

$2, 5, 8, 11, 14, 17, 20$.

Application activity 2.3

1. Insert 4 arithmetic means between -3 and 7.
2. Insert 9 arithmetic means between 2 and 32.
3. Find 12 arithmetic means between $-3\frac{1}{2}$ and $-42\frac{1}{2}$.
4. The difference of two numbers is 4 and arithmetic means between them is 6. Find the numbers.

2.1.4. Sum of arithmetic sequence

Activity 2.4

Consider a finite arithmetic sequence $\{u_n\} = u_1, u_2, u_3, \ldots u_n$ with common difference d. Let s_n denote the sum of these terms. We have;

$$u_1 = u_1$$
$$u_2 = u_1 + d$$
$$\vdots$$
$$u_{n-1} = u_1 + (n-2)d$$
$$u_n = u_1 + (n-1)d$$

Sum up these terms and give the expression of s_n.

For finite arithmetic sequence $\{u_n\} = u_1, u_2, u_3, \ldots u_n$, the sum $\sum_{r=1}^{n} u_r = u_1 + u_2 + u_3 + \ldots + u_n$ is called **the sum of n terms of an arithmetic sequence**.

We denote the sum of the first n terms of the sequence by S_n.

Thus, $S_n = u_1 + u_2 + u_3 + \ldots + u_n = \sum_{r=1}^{n} u_r$

From activity 2.4, the sum of the first n terms of a finite arithmetic sequence with initial term u_1 is given by $S_n = \frac{n}{2}(u_1 + u_n)$.

If the initial term is u_0, the formula becomes $S_n = \frac{(n+1)}{2}(u_0 + u_n)$

Example 2.13

Calculate the sum of first 100 terms of the sequence $2,4,6,8,...$

Solution

We see that the common difference is 2 and the initial term is $u_1 = 2$. We need to find $u_n = u_{100}$.

$$\begin{aligned} u_{100} &= 2 + (100-1)2 \\ &= 2 + 198 \\ &= 200 \end{aligned}$$

Now,

$$\begin{aligned} S_{100} &= \frac{100}{2}(u_1 + u_{100}) \\ &= 50(2+200) \\ &= 10100 \end{aligned}$$

Example 2.14

Find the sum of the first k even integers ($k \neq 0$).

Solution

$u_1 = 2$ and $d = 2$

$$\begin{aligned} u_n &= u_k \\ &= 2 + (k-1)2 \\ &= 2k \end{aligned}$$

$$\begin{aligned} S_n &= S_k \\ &= \frac{k}{2}(2+2k) \\ &= k(k+1) \end{aligned}$$

Application activity 2.4

1. Consider the arithmetic sequence 8, 12, 16, 20, … Find the expression for S_n.
2. Find the sum of the first twenty terms of the sequence 5, 9, 13, …
3. The sum of the terms in the sequence 1, 8, 15, … is 396. How many terms does the sequence contain?
4. Find the sum of 9 terms of the arithmetic series $-12-5+2+...$
5. Find the sum of all the multiple of 11 which are less than 1000.

2.1.5. Harmonic sequences

Activity 2.5

Consider the following arithmetic sequence: 2, 4, 6, 8, 10, 12, 14, 16.

Form another sequence whose terms are the reciprocals of the terms of the given sequence. What can you say about the new sequence?

Harmonic sequence is a sequence of numbers in which the reciprocals of the terms are in arithmetic sequence.

Example 2.15

From the following sequences

i) $\frac{1}{3},\frac{1}{6},\frac{1}{9},\ldots$ ii) $\frac{1}{a_1},\frac{1}{a_1+d},\frac{1}{a_1+2d},\frac{1}{a_1+3d},\ldots$

If you take the reciprocal of each term of the above sequences, you get:

i) 3, 6, 9, ... which is an arithmetic sequence with a common difference of 3.

ii) $a_1, a_1+\text{d}, a_1+2\text{d}, a_1+3\text{d},\ldots$ which is an arithmetic sequence with a common difference of *d*.

Example 2.16

Another example of harmonic sequence is 6, 3, 2. The reciprocals of each term are $\frac{1}{6},\frac{1}{3},\frac{1}{2}$ which is an arithmetic sequence with a common difference of $\frac{1}{6}$.

Remark

To find the term of harmonic sequence, convert the sequence into arithmetic sequence then do the calculations using the arithmetic formulae. Then take the reciprocal of the answer in arithmetic sequence to get the correct term in harmonic sequence.

Example 2.17

Find the 9^{th} term of the harmonic sequence $6, 4, 3, \ldots$

Solution

If sequence $6, 4, 3, \ldots$ is harmonic, the sequence of reciprocals of its terms is arithmetic.

That is, $\frac{1}{6}, \frac{1}{4}, \frac{1}{3}, \ldots$ are in arithmetic sequence ; $a_1 = \frac{1}{6}$ and

$d = \frac{1}{4} - \frac{1}{6} = \frac{3-2}{12} = \frac{1}{12}$.

Let the 9^{th} term of the given harmonic be h_9, thus, $h_9 = \frac{1}{a_9}$.

Or $a_9 = a_1 + (9-1)d = \frac{1}{6} + 8 \times \frac{1}{12} = \frac{1}{6} + \frac{4}{6} = \frac{5}{6}$.

Hence, 9^{th} term of the given harmonic sequence is $\frac{6}{5}$.

Notice

Harmonic means and harmonic series

If three terms a, b, c are in harmonic progression, the middle one is said to be Harmonic mean between the other two and $b = \frac{2ac}{a+c}$.

Example 2.18

Insert 4 harmonic means between $\frac{2}{3}$ and $\frac{6}{19}$.

Solution

Let the four harmonic means be h_1, h_2, h_3, h_4.

Then, $\frac{2}{3}, h_1, h_2, h_3, h_4, \frac{6}{19}$ are in harmonic progression

$\Rightarrow \frac{3}{2}, \frac{1}{h_1}, \frac{1}{h_2}, \frac{1}{h_3}, \frac{1}{h_4}, \frac{19}{6}$ are in arithmetic progression.

where $a_1 = \frac{3}{2}$ and $a_6 = \frac{19}{6}$

$a_6 = \frac{19}{6} \Leftrightarrow a_1 + 5d = \frac{19}{6}$ with d common difference.

$\Rightarrow \frac{3}{2} + 5d = \frac{19}{6} \Leftrightarrow 5d = \frac{19}{6} - \frac{3}{2} \Leftrightarrow 5d = \frac{10}{6} \Rightarrow d = \frac{1}{3}$

$$\Rightarrow \begin{cases} \frac{1}{h_1} = \frac{3}{2} + \frac{1}{3} = \frac{11}{6} \equiv 1^{st} \text{ term of arithmetic Progression} \\ \frac{1}{h_2} = \frac{3}{2} + \frac{2}{3} = \frac{13}{6} \equiv 2^{nd} \text{ term of arithmetic Progression} \\ \frac{1}{h_3} = \frac{3}{2} + \frac{3}{3} = \frac{15}{6} = \frac{2}{5} \equiv 3^{rd} \text{ term of arithmetic Progression} \\ \frac{1}{h_4} = \frac{3}{2} + \frac{4}{3} = \frac{17}{6} \equiv 4^{th} \text{ term of arithmetic Progression} \end{cases}$$

The four harmonic means are $\frac{6}{11}, \frac{6}{13}, \frac{2}{5}, \frac{6}{17}$.

Example 2.19

Find the nth term of the series $2\frac{1}{2} + 1\frac{7}{13} + 1\frac{1}{9} + \frac{20}{23} + \cdots$

Solution

The given series is $\frac{5}{2} + \frac{20}{13} + \frac{10}{9} + \frac{20}{23}, \cdots$

The reciprocal of the terms are $\frac{2}{5}, \frac{13}{20}, \frac{9}{10}, \frac{23}{20}, \cdots$

They are in arithmetic progression, with the first term $\frac{2}{5}$ and the common difference $\frac{13}{20} - \frac{2}{5} = \frac{1}{4}$

The given series in arithmetic progression: nth term of arithmetic progression: $a_n = \frac{2}{5} + (n-1)\frac{1}{4} = \frac{8+5n-5}{20} = \frac{5n+3}{20}$

Hence nth term of the given harmonic progression is $h_n = \frac{1}{a_n}$ or

$h_n = \frac{20}{5n+3}$

The nth term of the series $2\frac{1}{2} + 1\frac{7}{13} + 1\frac{1}{9} + \frac{20}{23} + \cdots$ is $\frac{20}{5n+3}$.

Application activity 2.6

1. Find the 4th and 8th term of the harmonic series $6, 4, 3, \ldots$
2. Insert two harmonic means between 3 and 10.
3. Find the nth term of the harmonic progression, whose first two terms are 6 and 3 respectively.
4. Find the nth term of the harmonic series
   $2\frac{1}{2} + 1\frac{7}{13}, 1\frac{1}{9}, \frac{20}{23}, \ldots$
5. Which term of the series $2 + 1\frac{3}{4} + 1\frac{5}{9} + \ldots$, is $\frac{1}{2}$?

2.2. Geometric sequences

2.2.1. Definition

Activity 2.6

Take a piece of paper which is in a square shape.

1. Cut it into two equal parts.
2. Write down a fraction corresponding to one part according to the original piece of paper.
3. Take one part obtained in step (2) and cut, repeat step (1) and then step 2).
4. Continue until you remain with a small piece of paper that you are not able to cut it into two equal parts.
5. Observe the sequence of numbers you obtained and give the relationship between any two consecutive numbers.

Sequences of numbers that follow a pattern of multiplying a fixed number from one term to the next are called **geometric sequences** or **geometric progression**.

A sequence (u_n) is said to be geometric if $\frac{u_{n+1}}{u_n} = r,\ n \in \mathbb{N}$ where r is a constant. The constant r is called the **common ratio** of the sequence.

Example 2.20

The following sequences are examples of geometric sequences:

Sequence $\{u_n\}$: $5, 10, 20, 40, 80, \ldots$

Sequence $\{v_n\}$: $2, 1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots$

Sequence $\{w_n\}$: $1, -2, 4, -8, 16, \ldots$

Note

If three consecutive terms are in a geometric sequence, the square of the middle term is equal to the product of extreme terms. That is, for a geometric sequence u_{n-1}, u_n, u_{n+1}, we have $u_n^2 = u_{n-1} \cdot u_{n+1}$

Example 2.21

Show that $6, 12, 24$ are consecutive terms of a geometric sequence

Solution

$(12)^2 = 6 \times 24 \Leftrightarrow 144 = 144$

Thus, $6, 12, 24$ are consecutive terms of a geometric sequence

Example 2.22

Find b such that $8, b, 18$ will be in geometric sequence.

Solution

$b^2 = 8 \times 18 = 144$ $\qquad b = \pm\sqrt{144} = \pm 12$

Thus, $8, 12, 18$ or $8, -12, 18$ are in geometric sequence.

Example 2.23

The product of three consecutive numbers in geometric progression is 27. The sum of the first two terms and nine times the third is -79. Find the numbers.

Solution

Let the three terms be $\frac{x}{a}, x, ax$.

The product of the numbers is 27. So, $\frac{x}{a} x a x = 27 \Rightarrow x^3 = 27 \Rightarrow x = 3$

The sum of the first two and nine times the third is -79:

$$\frac{x}{a} + x + 9ax = -79 \Rightarrow \frac{3}{a} + 3 + 27a = -79$$

$$27a^2 + 82a + 3 = 0 \Rightarrow a = -3 \text{ or } a = -\frac{1}{27}$$

The numbers are: -1, 3, -9 or -81, 3, $-\frac{1}{9}$.

Application activity 2.6

1. Find x such that 2, x, 18 are in geometric progression.
2. Is the sequence -2, 4, -8, 16, 32, 64 a geometric progression? Why?
3. Determine the common ratio of the sequence $\left\{3(-2)^n\right\}$.

2.2.2. General term of a geometric sequence

Activity 2.7

A person has two parents (father and mother), four grandparents, eight great grandparents etc. Assuming that there are no intermarriages:

1. Find the number of ancestors which the person has up to 6^{th} generation?
2. Find the number of ancestors which the person has up to 8^{th} generation?
3. Find the number of ancestors which the person has up to 10^{th} generation?
4. Find the number of ancestors which the person has up to 12^{th} generation?

Refer to thr table above and find the general formula that should be used for finding the number of ancestors which the person has up to n^{th} generation.

Generation (n)	Number of ancestors	Observation
1^{st}	2	2×1
2^{nd}		
3^{rd}		
4^{th}		
5^{th}		
6^{th}		
7^{th}		
8^{th}		
9^{th}		
10^{th}		
11^{th}		
12^{th}		
⋮		
n		

From activity 2.7, the n^{th} term, u_n, of a geometric sequence $\{u_n\}$ with common ratio r and initial term u_1 is given by $u_n = u_1 r^{n-1}$.

Generally,

If u_p is the p^{th} term of the sequence, then the n^{th} term is given by $u_n = u_p r^{n-p}$.

Example 2.24

If the first and tenth terms of a geometric sequence are 1 and 4, respectively, find the nineteenth term.

Solution

$u_1 = 1$ and $u_{10} = 4$

But $u_n = u_1 r^{n-1}$, then, $4 = 1r^9 \Leftrightarrow r = \sqrt[9]{4}$ or $r = 4^{\frac{1}{9}}$

Now,

$$u_{19} = u_1 r^{19-1}$$
$$= 1\left(4^{\frac{1}{9}}\right)^{18}$$
$$= 16$$

Thus, the nineteenth term of the sequence is 16.

Example 2.25

If the 2[nd] and the 9[th] terms of a geometric sequence are 2 and $-\frac{1}{64}$ respectively, find the common ratio.

Solution

$u_2 = 2,\ u_9 = -\frac{1}{64}$

Using the general formula: $u_n = u_p r^{n-p}$

$u_2 = u_9 r^{2-9}$

$$2=-\frac{1}{64}r^{-7}$$

$$\Leftrightarrow 128=-\frac{1}{r^7}$$

$$\Leftrightarrow r^7=-\frac{1}{128}$$

$$\Leftrightarrow r=\sqrt[7]{-\frac{1}{128}}\Rightarrow r=-\frac{1}{2}$$

The common ratio is $r=-\frac{1}{2}$.

Example 2.26

Find the number of terms in sequence $2,4,8,16,\ldots,256$.

Solution

This sequence is a geometric sequence with common ratio 2, $u_1=2$ and $u_n=256$

But $u_n=u_1r^{n-1}$, then $256=2\times 2^{n-1}\Leftrightarrow 256=2^n$ or $2^8=2^n\Rightarrow n=8$.

Thus, the number of terms in the given sequence is 8.

Application activity 2.7

1. Find the 4th term of the geometric progression whose 5th term is 32 and whose 8th term is 256.
2. If the second and fifth terms of a geometric sequence are 6 and -48, respectively, find the sixteenth term.
3. If the third term and the 8th term of a geometric sequence are $\frac{1}{2}$ and $\frac{1}{128}$ respectively, find the common ratio.

4. The first term of a geometric sequence is 54 and the common ratio is $\frac{2}{3}$ .Find its 10th term.
5. Find the first term to exceed 1000 in each of the following geometric sequences:

   i) $\{16, 24, 36, \ldots\}$ ii) $\{1, 1.3, 1.69, \ldots\}$

2.2.3. Geometric means

Activity 2.8

Suppose that you need to form a geometric sequence of 6 terms such that the first term is 1 and the sixth term is 243. Write down that sequence.

If three or more than three numbers are in geometric sequence, then all terms lying between the first and the last numbers are called geometric means. If G is geometric mean between A and C, then $G = \pm\sqrt{AC}$.

To insert k terms called geometric means between two terms u_1 and u_n is to form a geometric sequence of $n = k + 2$ terms whose first term is u_1 and the last term is u_n.

As u_1 and u_n are known, we find the common ratio, r, from the formula $u_n = u_1 r^{n-1}$ by taking $n = k + 2$ where k is the number of terms to be inserted.

Example 2.27

Insert three geometric means between 3 and 48.

Solution

Here $k = 3$ and then $n = 5$, $u_1 = 3$ and $u_n = u_5 = 48$

$u_5 = u_1 r^{n-1} \Leftrightarrow 48 = 3r^4 \Rightarrow r = 2$

Inserting three terms using common ratio $r = 2$ gives $3, 6, 12, 24, 48$.

Example 2.28

Insert 6 geometric means between 1 and $-\frac{1}{128}$.

Solution

Here $k=6$ and then $n=8$, $u_1=1$ and $u_n=u_8=-\frac{1}{128}$

$u_8=u_1r^{n-1}$

$\Leftrightarrow -\frac{1}{128}=1r^7$

$\Leftrightarrow r^7=-\frac{1}{128}$

$\Leftrightarrow r^7=-\frac{1}{(2)^7}$

$\Leftrightarrow r=\left[-\frac{1}{(2)^7}\right]^{\frac{1}{7}}=-\frac{1}{2}$

Inserting 6 terms using common ratio $r=-\frac{1}{2}$ gives

$1,-\frac{1}{2},\frac{1}{4},-\frac{1}{8},\frac{1}{16},-\frac{1}{32},\frac{1}{64},-\frac{1}{128}$.

Application activity 2.8

1. Insert 5 geometric means between $\frac{1}{4}$ and $\frac{1}{256}$.
2. Insert 5 geometric means between 2 and $\frac{2}{729}$.
3. Find the geometric mean between $\frac{3}{2}$ and $\frac{27}{2}$
4. For the numbers 4 and 9 find:
   a) The arithmetic mean,
   b) geometric mean.
5. Insert three geometric means between $\frac{1}{9}$ and 9

2.2.4. The sum of n terms of a geometric sequence

Activity 2.9

1. Consider a geometric sequence with initial term u_1 and common ratio r.

Let $s_n = u_1 + u_2 + u_3 + \ldots + u_n$

$s_n = u_1 + u_1 r + u_1 r^2 + \ldots + u_1 r^{n-1}$ (1)

- Multiply both sides of (1) by r to obtain relation (2),
- Subtract (2) from (1),
- Give the general formula for S_n.

2. Suppose that we need the product of $u_1, u_2, u_3, \ldots, u_n$.

Then, $P_n = u_1 \times u_2 \times u_3 \times \ldots \times u_n$ or $P_n = u_1 \times u_1 r \times u_1 r^2 \times \ldots \times u_1 r^{n-1}$. Develop this relation and show the general formula that should be used for P_n.

You will need the sum $S_{n-1} = 1 + 2 + \ldots + n - 1$ which is

$$S_{n-1} = \frac{n-1}{2}(1 + n - 1) = \frac{n(n-1)}{2}$$

For finite geometric sequence $\{u_n\} = u_1, u_2, u_3, \ldots u_n$, the sum $\sum_{r=1}^{n} u_r = u_1 + u_2 + u_3 + \ldots + u_n$ is called an **the sum of n terms of a geometric sequence**.

We denote the sum of the first n terms of the sequence by S_n.
Thus, $S_n = u_1 + u_2 + u_3 + \ldots + u_n = \sum_{r=1}^{n} u_r$

From activity 2.9, the sum of first n terms of a geometric sequence with initial term u_1 and common ratio, r, is given by: $s_n = \dfrac{u_1(1 - r^n)}{1 - r}$ with $r \neq 1$.

If the initial term is u_0, then the formula is $s_n = \dfrac{u_0(1 - r^{n+1})}{1 - r}$ with $r \neq 1$.

If $r = 1$, $s_n = nu_1$

Also, the product of the first n terms of a geometric sequence with initial term u_1 and common ratio r is given by $P_n = (u_1)^n r^{\frac{n(n-1)}{2}}$

If the initial term is u_0, then, $P_n = (u_0)^{n+1} r^{\frac{n}{2}(n+1)}$

Example 2.29

Find the sum of the first 20 terms of a geometric sequence if the first term is 1 and common ratio is 2.

Solution

Here, $u_1 = 1, r = 2, n = 20$

Then,

$$s_{20} = \frac{1(1-2^{20})}{1-2} = \frac{1-2^{20}}{-1} = 1048575$$

Example 2.30

A geometric sequence has first term 27 and common ratio $\frac{4}{3}$.

Find the least number of terms the sequence can have if its sum exceeds 550.

Solution

Here $u_1 = 27$ and $r = \frac{4}{3}$

Now suppose that $S_n = 550$ i.e. $\frac{u_1(1-r^n)}{1-r} = 550$

then, $27\frac{1-\left(\frac{4}{3}\right)^n}{1-\frac{4}{3}} = 550 \Leftrightarrow 27\frac{1-\left(\frac{4}{3}\right)^n}{-\frac{1}{3}} = 550$

$$\Leftrightarrow -81\left[1-\left(\frac{4}{3}\right)^n\right]=550 \qquad \Leftrightarrow 1-\left(\frac{4}{3}\right)^n=-\frac{550}{81}$$

$$\Leftrightarrow -\left(\frac{4}{3}\right)^n=-\frac{631}{81} \qquad \Leftrightarrow \left(\frac{4}{3}\right)^n=\frac{631}{81}$$

Taking logarithms $\qquad \Leftrightarrow n\log\left(\frac{4}{3}\right)=\log\frac{631}{81}$

Hence, $\qquad \Leftrightarrow n=\dfrac{\log\frac{631}{81}}{\log\left(\frac{4}{3}\right)}=7.136$

Thus, for $S_n > 550$, we require $n > 7.136$ i.e. $n = 8$.

Example 2.31

Find the product of the first 20 terms of a geometric sequence if the first term is 1 and common ratio is 2.

Solution

$$P_n=(u_1)^n\, r^{\frac{n(n-1)}{2}}$$

Here $u_1 = 1, r = 2, n = 20$,

Thus,

$$P_{20}=(1)^{20}\, 2^{\frac{20(19)}{2}}$$
$$=2^{190}$$

Application activity 2.9

1. Find the sum of the first 8 terms of the geometric sequence 32, -16, 8, …
2. Find the sum of the geometric sequence with the first term 0.99 and the common ration is equal to the first term.
3. Find the first term and the common ratio of the geometric sequence for which $S_n = \frac{5^n - 4^n}{4^{n-1}}$
4. Find the product of the first 10 terms of the sequence in question 1.
5. Find the position of $\frac{1}{4374}$ in the following sequence

   $\frac{1}{2}, \frac{1}{6}, \frac{1}{18}, \frac{1}{54}, \ldots$

2.2.3. Geometric series

Activity 2.10

Consider the infinite geometric series $\sum_{n=1}^{\infty} u_1 r^{n-1}$ where the sum of the first n terms is $S_n = \frac{u_1\left(1-r^n\right)}{1-r}$ $(r \neq 1)$. Evaluate $\lim_{n \to +\infty} \frac{u_1(1-r^n)}{1-r}$ for $-1 < r < 1$.

A geometric series has the form $\sum_{n=1}^{\infty} u_1 r^{n-1}$.

From activity 2.10, the sum to infinity of a geometric series with first term u_1 and the common ratio, r, is $S_\infty = \frac{u_1}{1-r}$ provided $-1 < r < 1$.

Example 2.32

Given the geometric progression 16, 12, 9, …. Find the sum of terms up to infinity.

Solution

Here $u_1 = 16, r = \frac{12}{16} = \frac{3}{4}$

Thus, $-1 < r < 1$ and hence the sum to infinity will exist as

$$S_\infty = \frac{u_1}{1-r} = \frac{16}{1-\frac{3}{4}} = 64$$

The sum to infinity is 64.

Example 2.33

Express the recurring decimal $0.\overline{32}$ as a rational number.

Solution

$0.\overline{32} = \frac{32}{10^2} + \frac{32}{10^4} + \frac{32}{10^6} + \dots$ which is an infinite geometric series with first term $u_1 = 0.32$ and common ratio $r = 0.01$.

Since $-1 < r < 1$, the sum to infinity exist and equal to

$$\frac{u_1}{1-r} = \frac{0.32}{1-0.01} = \frac{0.32}{0.99} = \frac{32}{99}$$

Therefore, $0.\overline{32} = \frac{32}{99}$

Application activity 2.10

1. Consider the infinite geometric series $\sum_{n=1}^{\infty} 10\left(1-\frac{3x}{2}\right)^n$.
   a) For what values of x does a sum to infinity exist?
   b) Find the sum of the series if $x = 1.3$.
2. A ball is dropped from a height of 10 m and after each bounce, returns to a height which is 84% of the previous height. Calculate the total distance travelled by the ball before coming to rest.

2.3. Convergent or divergent sequences

Activity 2.11

Discuss the value of the general term of each of the following sequences as n tends to $+\infty$ (plus infinity) .

1. $\left\{\frac{3n^2-1}{n^2}\right\}$

2. $\left\{n^2\right\}$

A numerical sequence $\{u_n\}$ is said to be convergent if the limit $\lim_{n\to\infty} u_n = L$ exists and finite whereas if the limit does not exist (or is infinity) the sequence is said to be divergent.

A number L is called a limit of a numerical sequence $\{u_n\}$ if $\lim_{n\to\infty} u_n = L$

In other words, Convergent sequence is when $\lim_{n\to\infty} u_n = L$ while divergent sequence is when $\lim_{n\to\infty} u_n = \infty$ or does not exist.

Example 2.34

1. Determine whether the sequence $\left\{\frac{n}{2n+1}\right\}_{n=1}^{+\infty}$ converges or diverges.

Solution

First, we find the limit of this sequence as n tends to infinity

$$\lim_{n\to\infty}\frac{n}{2n+1}=\lim_{n\to\infty}\frac{n}{n\left(2+\frac{1}{n}\right)}=\frac{1}{2}$$

Thus, $\left\{\frac{n}{2n+1}\right\}_{n=1}^{+\infty}$ converges $\frac{1}{2}$.

2. Determine whether the sequence $\{8-2n\}_{n=1}^{+\infty}$ converges or diverges.

Solution

Firstly, we find the limit of this sequence as n tends to infinity

$$\lim_{n\to\infty}(8-2n)=8-2(+\infty)=-\infty$$

Thus, $\{8-2n\}_{n=1}^{+\infty}$ diverges.

Application activity 2.11

Which of the sequences converge, and which diverge? Find the limit of each convergent sequence.

1) $\left\{2+(0.1)^n\right\}$ 2) $\left\{\frac{1-2n}{1+2n}\right\}$ 3) $\left\{\frac{1-5n^4}{n^4+8n^3}\right\}$

4) $\left\{-1^n\right\}$ 5) $\left\{\frac{2n}{\sqrt{3n+1}}\right\}$ 6) $\frac{\sqrt{7n^2+2}}{n^3+8}$

2.4. Applications of sequences in real life

Activity 2.12

Discuss how sequences are used in real life problems.

There are many applications of sequences. Sequences are useful in our daily lives as well as in higher mathematics. For example; the interest portion of monthly payments made to pay off an automobile or home loan, and the list of maximum daily temperatures in one area for a month are sequences. Sequences are used in calculating interest, population growth, half-life and decay in radioactivity, in construction, repeated drug dosage …

Example 2.35 **Construction**

A child building a tower with blocks uses 15 for the bottom row. Each row has 2 fewer blocks than the previous row. Suppose that there are 8 rows in the tower.

a) How many blocks are used for the top row?

b) What is the total number of blocks in the tower?

Solution

a) The number of blocks in each row forms an arithmetic sequence with $u_1 = 15$ and $d = -2$

b) $n = 8,\ u_8 = u_1 + (8-1)(-2)$. There is just one block in the top row.

Here, we must find the sum of the terms of the arithmetic sequence formed with $u_1 = 15, n = 8, u_8 = 1$

$$S_8 = \frac{8}{2}(15+1) = 64$$

There are 64 blocks in the tower.

Example 2.36 **Population growth**

An insect population is growing in such a way that each new generation is 1.5 times as large as the previous generation. Suppose there are 100 insects in the first generation,

a) How many will there be in the fifth generation?

b) What will be the total number of insects in the five generations?

Solution

a) The population can be written as a geometric sequence with $u_1 = 100$ as the first-generation population and common ratio $r = 1.5$. Then, the fifth generation population will be $u_5 = 100(1.5)^{5-1} = 506.25$. In the fifth generation, the population will number about 506 insects.

b) The sum of the first five terms using the formula for the sum of the first n terms of a geometric sequence.

$$S_5 = \frac{100\left(1-(1.5)^5\right)}{1-1.5} = 1318.75$$

The total population for the five generations will be about 1319 insects.

Another application of sequences is their use in compound interest and simple interest.

The compound interest formula:

$A = P\left(1+\frac{r}{k}\right)^{kt}$ with P = principle, t = time in years,

r = annual rate, and k = number of periods per year.

The simple interest formula:

$I = \mathrm{P}rt$ with I = total interest, P = principle, r = annual rate, and t = time in years.

Example 2.37 Compound interest 1

If Linda deposits $1300 in a bank at 7% interest compounded annually, how much will be in the bank 17 years later?

Solution

$P = 1300,\ r = 7\% = 0.07,\ k = 1$

$$A = 1300\left(1+\frac{0.07}{1}\right)^{1\times 17} = 4106.46$$

The account will contain $4,106.46.

Example 2.38 Compound interest 2

Find the accumulated value of $15,000 at 5% per year for 18 years using simple interest.

Solution

$P = 15000,\ r = 0.05, t = 18$

$$I = 15000(0.05)(18)$$
$$= 13500$$

A total of $13,500 in interest will be earned.

Hence, the accumulated value in the account will be 13,500 + 15,000 = $28,500.

Example 2.39 **Radioactivity**

A mine worker discovers an ore sample containing 500 mg of radioactive material. It is discovered that the radioactive material has a half life of 1 day. Find the amount of radioactive material in the sample at the beginning of the 7^{th} day.

Solution

Half life of one day means that half of the amount remains after 1 day.

Beginning of 1^{st} day: 500 mg	Beginning of 2^{nd} day: 250 mg	Beginning of 3^{rd} day: 125 mg	...
End of 1^{st} day: 250 mg	End of 2^{nd} day: 125 mg	End of 3^{rd} day: 62.5 mg	...

Decide to either work with the "beginning" of each day, or the "end" of each day, as each can yield the answer. Only the starting value and number of terms will differ. We will use "beginning":

$$u_n = u_1 r^{n-1}$$

$$u_8 = 500\left(\frac{1}{2}\right)^{7-1} = 7.8125\,mg$$

Example 2.40 **Repeated drug dosage**

Malaria is a parasitic infection transmitted by mosquito bites, mainly in tropical areas of the world. The drug quinine is the active ingredient and it is still used today. Suppose a person is given a 50-mg dose of quinine at the same time every day for the prevention of malaria. After the first dose, the person has 50 mg of quinine in the body. What about after the second dose?

Each day, the person's body metabolizes some of the quinine so that, after one day, 23% of the original amount remains. After the second dose, the amount of quinine in the body is the amount from the second dose (50 mg) plus the remnants of the first dose (that is, 50·0.23 = 11.5 mg) for a total of 61.5 mg. Let Q_n represent the quantity, in mg, of quinine in the body right after the n^{th} dose. Then

Q_1 = First dose = 50.

Q_2 = Second dose + Remnants of first dose = 50 + 50(0.23) = 61.5.

Q_3 = Third dose + Remnants of previous doses = 50 + 61.5(0.23) = 64.145.

Notice that we can multiply out the expression for Q_3 to show the contributions of the first and second dose separately:

$Q_3 = 50 + 61.5(0.23) = 50 + (50 + 50(0.23))\,(0.23)$

$Q_3 = 50 + 50(0.23) + 50(0.23)2,$

so we have

Q_3 = Third dose + Remnants of second dose + Remnants of first dose.

The multiplied-out form of Q_3 enables us to guess formulas for later values of Q_n:

$Q_4 = 50 + 50(0.23) + 50(0.23)^2 + 50(0.23)^3 = 64.753.$

$Q_5 = 50 + 50(0.23) + 50(0.23)^2 + 50(0.23)^3 + 50(0.23)^4 = 64.893.$

$Q_6 = 50 + 50(0.23) + 50(0.23)^2 + 50(0.23)^3 + 50(0.23)^4 + 50(0.23)^5 = 64.925. . . .$

$Q_{10} = 50 + 50(0.23) + 50(0.23)^2 + \cdots + 50(0.23)^8 + 50(0.23)^9 = 64.935.$

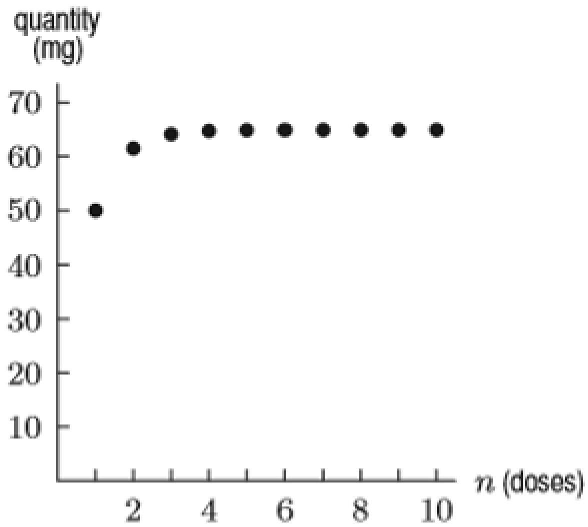
The values of Q_6 and Q_{10} suggest that the quantity is stabilizing at around 64.9 mg.

$$Q_n = Q_1 + Q_1 r + Q_1 r^2 + Q_1 r^3 + \ldots\ldots\ldots\ldots Q_1 r^{n-1}$$

$$Q_n = Q_1 \frac{(1-r^n)}{1-r}, \textit{provided that } (r \neq 1)$$

Remember that ***n*** is the number of terms in the sum $\mathbf{Q_n}$ and ***r*** is the common ratio

See Figure below showing the quantity of quinine levels off:



Application activity 2.12

1. Each morning, a patient receives a 25 mg injection of an anti-inflammatory drug, and 40% of the drug remains in the body after 24 hours. Find the quantity in the body: (a) Right after the 3rd. injection. (b) Right after the 6 th injection. (c) In the long run, right after an injection.
2. A smoker inhales 0.4 mg of nicotine from a cigarette. After one hour, 71% of the nicotine remains in the body. If a person smokes one cigarette every hour beginning at 7 am, how much nicotine is in the body right after the 11 pm cigarette?

Unit Summary

1. Numbers in sequence are denoted $u_1, u_2, u_3, ..., u_{n-1}, u_n, ...$ and shortly $\{u_n\}$.
2. The natural number n is called term number and value u_n is called a general term of a sequence and the term u_1 is the initial term.
3. Sequences of numbers that follow a pattern of adding a fixed number from one term to the next are called arithmetic sequences or arithmetic progressions.
4. For an arithmetic sequence u_{n-1}, u_n, u_{n+1}, we have $2u_n = u_{n-1} + u_{n+1}$.
5. If u_p is any p^{th} term of a sequence, then the nth term is given by $u_n = u_p + (n-p)d$
6. The sum of first n terms of a finite arithmetic sequence with initial term u_1 is given by $s_n = \frac{n}{2}[u_1 + u_n]$.
7. Harmonic sequence is a sequence of numbers in which the reciprocals of the terms are in arithmetic sequence: its nth term is $h_n = \dfrac{1}{u_p + (n-p)d}$ where $u_p + (n-p)d$ is n^{th} term of arithmetic sequence.
8. Sequences of numbers that follow a pattern of multiplying a fixed number from one term to the next are called geometric sequences or geometric progression.
9. For a geometric sequence u_{n-1}, u_n, u_{n+1}, we have $u_n^2 = u_{n-1} \cdot u_{n+1}$
10. The n^{th} term, u_n, of a geometric sequence $\{u_n\}$ with common ratio r and initial term u_1 is given by $u_n = u_1 r^{n-1}$
11. The sum of first n terms of a geometric sequence with initial term u_1 and common ratio r is given by:

    $$s_n = \frac{u_1(1-r^n)}{1-r} \text{ with } r \neq 1.$$

12.

13. Also, the product of first n terms of a geometric sequence with initial term u_1 and common ratio r is given by $P_n = (u_1)^n r^{\frac{n(n-1)}{2}}$.

14. For the formula $s_n = \frac{u_1(1-r^n)}{1-r}$
If $-1 < r < 1$, $S_\infty = \frac{u_1}{1-r}$.

15. Sequences are used in calculating interest, population growth, half-life and decay in radioactivity.

End of Unit Assessment

1. Find the 20th term of the following arithmetic progressions and calculate the sum of first 20 terms:

   a) 2, 6, 10, 14, …

   b) -5, -3.5, -2, -0.5, …

2. Find the nth term of the following arithmetic progression and calculate the sum of first n terms:

   a) 4, 6, 8, 10, …

   b) 17, 14, 11, 8, …

   c) 1, $\frac{n-1}{n}, \frac{n-2}{n}, \frac{n-3}{n}, \ldots$

3. In an arithmetic progression, we have:

   a) $u_1 = 4, d = 2, n = 8$; find u_n and sum of terms.

   b) $d = 4, u_n = 39, n = 10$; find u_1 and sum of terms.

   c) $u_1 = 3, u_n = 21, S_n = 120$; find n and d.

   d) $u_n = 199, n = 100, S_n = 10000$; find u_1 and d.

4. Form an arithmetic progression such that the 4th term and 12th term are 40 and 42 respectively.

5. In an arithmetic progression, the sum of the 8th and 14th terms is 50. The 5th term is equal to 13. Find that progression.

6. Insert 8 arithmetic means between -2 and $\frac{1}{4}$.

7. Find x consecutive integer numbers known that the first number is 8 and their sum is x^3.

8. The sum of 3 consecutive terms in arithmetic progression is 33 and their product is 1287. What are those numbers?

9. In a geometric progression, we have:

   a) $u_1 = 3, r = 4, n = 5$; find u_n and sum of terms.

   b) $u_n = \frac{3}{64}, u_1 = 12, n = 9$; find r and sum of terms.

10. In a geometric progression, the first and the third terms are 8 and 18 respectively. Find the 5^{th} term.
11. In a geometric progression, the first term is 32 and the product of the 3^{rd} and the 6^{th} terms is 17496. Find the 8^{th} term.
12. Insert 3 geometric terms between 2 and 8.
13. The sum of 3 numbers forming a geometric progression is 21 and the sum of their squares is 189. Find those numbers.
14. In a geometric progression with 5 terms, the common ratio is equal to the $\frac{1}{4}$ of the first term, and the sum of the first two terms is 24. Find the 5^{th} term.
15. Calculate the numbers x, y, z known that x, y, z form an arithmetic progression, y, x, z form a geometric progression and the product xyz is equal to 216.
16. The sum of three numbers that form arithmetic progression is 51, and the difference between the squares of the greatest and the least is 408. Find the numbers.
17. The sum of four numbers that form an arithmetic progression is 38, and the sum of their squares is 406. Find the numbers.
18. The sum of five numbers that form an arithmetic progression is 10, and the product of the first, third and fifth is -64. Find the numbers.
19. The fourth, seventh and sixteenth terms of an arithmetic progression are in geometric progression. If the first six terms of the arithmetic progression have a sum of 12, find the common difference of the arithmetic progression and the common ratio of the geometric progression.

20. The third, fifth and seventeenth terms of an arithmetic progression are in geometric progression. Find the common ratio of the geometric progression.
21. A mathematical child negotiates a new pocket money deal with her unsuspecting father in which he/she receives 1 pound on the first day of the month, 2 pounds on the second day, 4 pounds on the third day, 8 pounds on the fourth day, 16 pounds on the fifth day, … until the end of the month. How much would the child receive during the course of a month of 30 days? (Give your answer to the nearest million pounds.)
22. Find the common ratio of a geometric progression that has a first term of 5 and sum to infinity of 15.
23. The sum of the first two terms of a geometric progression is 9 and the sum to infinity is 25. If the common ratio is positive, find the common ratio and the first term.
24. A culture of bacteria doubles every 2 hours. If there are 500 bacteria at the beginning, how many bacteria will there be after 24 hours?
25. You complain that the hot tub in your hotel suite is not hot enough. The hotel tells you that they will increase the temperature by 10% each hour. If the current temperature of the hot tub is 75^{o} F, what will be the temperature of the hot tub after 3 hours, to the nearest tenth of a degree?
26. The sum of the interior angles of a triangle is 180^{o}, of a quadrilateral is 360^{o} and of a pentagon is 540^{o}. Assuming this pattern continues, find the sum of the interior angles of a dodecagon (12 sides).

Unit 3

Logarithmic and Exponential Equations

Introductory activity

A pharmacist created a business which helped him to make money in an interesting way so that the money he/she earns each day doubles what he/she earned the previous day. If he/she had 200 USD on the first day and by taking t as the number of days, discuss the money he/she can have at the t^{th} day through answering the following questions:

a) Draw the table showing the money this pharmacist will have on each day starting from the first to the 10^{th} day.

b) Plot these data in rectangular coordinates

c) Based on the results in a), establish the formula for the economist to find out the money he/she can earn on the n^{th} day. Therefore, if t is the time in days, express the money $F(t)$ for the pharmacist.

d) Now the pharmacist wants to possess the money F under the same conditions, discuss how he/she can know the number of days necessary to get such money from the beginning of the business.

Objectives

By the end of this unit, a student will be able to:

- solve exponential equations.
- solve logarithmic equations.
- apply exponential and logarithmic equations in real life problems.

3.1. Introduction to Exponential and logarithmic functions

Activity 3.1

Draw the graph of $y = 2^x$
for $-2 \le x \le 3$
In the same plane, sketch the graph of $y = 2$ *and* $y = -3$.
How many times does the horizontal line cross the curve of $y = 2^x$? How can you conclude?
Reflect $y = 2^x$ on the line $y = x$ and name the new curve $g(x)$.

Remember that only a one to one function is invertible.

To find the inverse of the function $y = a^x$, where a is a positive real number different from 1, we make x the subject of the formula by introducing a new function called **logarithm** and write $x = \log_a y$ which is read "x is ***logarithm of*** y ***in base*** a".

The graph of a logarithmic function is found by reflecting the graph of the corresponding exponential function in the line $y = x$. Thus, from activity 1, the curve of $y = 2^x$ and $g(x)$ are inverse to each other. Thus, $g(x) = \log_2 x$.

Since $g(x) = \log_2 x$ is the inverse of $y = 2^x$, the curve of $g(x) = \log_2 x$ is the image of the curve of $y = 2^x$ with respect to the **first bisector**, $y = x$. Then, the coordinates of the points for $y = 2^x$ are reversed to obtain the coordinates of the points for $g(x) = \log_2 x$.

Note that the words **power, index, exponent** and **logarithm** are synonymous; they are four different words to describe exactly the same thing.

power
exponent
index
logarithm → c

$a^c = b$ ← number $a, b \in \mathbb{R}^+$ and $a \ne 1$

Base → a

Example 3.1

In the same Cartesian plane, sketch the curve of the function $f(x)=3^x$ for $-2\leq x\leq 2$ and its inverse $f^{-1}(x)$ with the first bisector.

Solution

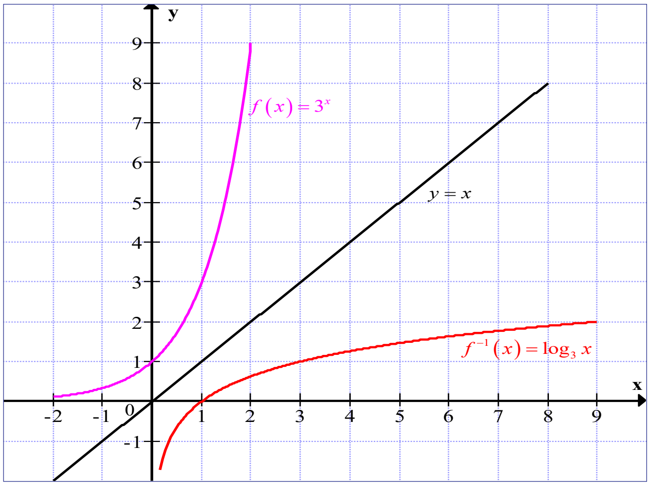
Table of coordinates of $f(x)=3^x$

x	-2	-1.6	-1.2	-0.8	-0.4	0	0.4	0.8	1.2	1.6	2
y	0.1	0.2	0.3	0.4	0.6	1.0	1.6	2.4	3.7	5.8	9.0

Table of coordinates of $f^{-1}(x)$

x	0.1	0.2	0.3	0.4	0.6	1.0	1.6	2.4	3.7	5.8	9.0
y	-2	-1.6	-1.2	-0.8	-0.4	0	0.4	0.8	1.2	1.6	2

Curve



Application activity 3.1

Sketch the following functions in Cartesian plane with their inverses:

1. $y=\left(\frac{1}{2}\right)^x$, $-3\leq x\leq 3$
2. $y=\left(\frac{1}{3}\right)^x$, $-2\leq x\leq 2$

3.2. Exponential and logarithmic equations

Each exponential expression has a corresponding logarithmic expression.

The relationship is $b = a^c \Leftrightarrow c = \log_a b$. Thus, we may write $b = a^{\log_a b}$.

For example, $100 = 10^2 \Leftrightarrow 100 = 10^{\log_{10} 100} \Rightarrow \log_{10} 100 = 2$

$$81 = 3^4 \Leftrightarrow 81 = 3^{\log_3 81} \Rightarrow \log_3 81 = 4$$

There are two common bases for logarithms, 10 and e

$\left(e \text{ is irrational number and } e \approx 2.718281828, \text{which we will prove in senior 6 that can be expressed as } e = \lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x \right)$

You should find an "ln" button on your calculator which will evaluate logarithms to base e and "log" button to evaluate logarithms to base 10.

Activity 3.2

Let $p = \log_a x$, and $q = \log_a y$, where $a > 0$ *and* $a \neq 1$. Remember that these two statements can be written as $x = a^p$ and $y = a^q$.

From product rule of exponent, express $\log_a xy$ in terms of $\log_a x$ *and* $\log_a y$.

HINT: $b = m^c \Leftrightarrow \log_m b = c$

Hence, prove that $\log_a\left(\frac{x}{y}\right) = \log_a x - \log_a y$

Basic rules for exponents

For $a > 0$ *and* $a \neq 1, m, n \in IR$

1. $a^m \times a^n = a^{m+n}$
2. $a^m : a^n = a^{m-n}$
3. $\left(a^m\right)^n = a^{mn}$
4. $a^{-n} = \frac{1}{a^n}$
5. $a^{\frac{1}{n}} = \sqrt[n]{a}$
6. $a^{\frac{m}{n}} = \sqrt[n]{a^m}$
7. $a^{\log_a b} = b$

Basic rules for logarithms

$\forall x, y \in]0,+\infty[, a \in]0,+\infty[\setminus \{1\}$:

a) $\log_a xy = \log_a x + \log_a y$ b) $\log_a \frac{1}{y} = -\log_a y$

c) $\log_a \frac{x}{y} = \log_a x - \log_a y$ d) $\log_a x^r = r\log_a x$

Example 3.2

Write $2^6 = 64$ in logarithmic form.

Solution

$2^6 = 64 \Leftrightarrow 2^6 = 2^{\log_2 64} \Rightarrow \log_2 64 = 6$

Example 3.3

Write $\log_m b = c$ in exponential form.

Solution

$\log_m b = c \Rightarrow b = m^c$

Example 3.4

Find x if $\log_2 32 = x$

Solution

$\log_2 32 = x \Rightarrow 32 = 2^x$

But $32 = 2^5$. So $32 = 2^x \Leftrightarrow 2^5 = 2^x \Rightarrow x = 5$

Example 3.5

Find the numerical value of $\log_3 \sqrt[3]{9}$.

Solution

Let $y=\log_3 \sqrt[3]{9}$, then $3^y=\sqrt[3]{9}$

$\Leftrightarrow 3^y=9^{\frac{1}{3}} \quad \Leftrightarrow 3^y=3^{2\left(\frac{1}{3}\right)} \quad \Leftrightarrow 3^y=3^{\frac{2}{3}} \quad \Rightarrow y=\frac{2}{3}$

Hence, $\log_3 \sqrt[3]{9}=\frac{2}{3}$

Application activity 3.2

1. Prove basic rules for exponents:

   a) $a^m \times a^n = a^{m+n}$ b) $a^m : a^n = a^{m-n}$ c) $\left(a^m\right)^n = a^{mn}$

   d) $a^{-n}=\frac{1}{a^n}$ e) $a^{\frac{1}{n}}=\sqrt[n]{a}$ f) $a^{\frac{m}{n}}=\sqrt[n]{a^m}$

2. Write each of the following in logarithmic form:

   a) $4^3=64$ b) $2^{-3}=\frac{1}{8}$ c) $\left(\frac{1}{2}\right)^x=y$

   d) $p^3=q$ e) $8^x=0.5$ f) $5^{-p}=q$

3. Find the exact value of x, showing your working:

   a) $\log_2 8=x$ b) $\log_x 125=3$ c) $\log_x 64=0.5$

   d) $\log_4 64=x$ e) $\log_9 x=3\frac{1}{2}$ f) $\log_2\left(\frac{1}{2}\right)=x$

4. Find the numerical value of each of the following:

   a) $\log_3 243$ b) $\log_5 \sqrt{125}$ c) $\log_5 0.008$

   d) $\log_5\left(\frac{1}{125}\right)$ e) $\log_{64} 4$ f) $\log_3 3$

   g) $\log_a a$ h) $\log_a 1$

Activity 3.3

Prove each of the following logarithmic laws

1. $\log_a\left(m^p\right)=p\log_a m$ The Power Law

2. $\log_a b=\frac{\log_c b}{\log_c a}$ The Change of Base Law

The change of base rule is very useful since all logarithmic calculations are performed either in base 10 or in base e.

- $\log_{10} x$ is usually written $\log x$ which is called decimal (or common) logarithm.

  $\log x \Leftrightarrow$ *the power to which 10 must be raised to produce* ***x***.

- $\log_e x$ is usually written as $\ln x$ which is called natural logarithm.

Thus, $\ln x \Leftrightarrow$ *the power to which* ***e*** *must be raised to produce* ***x***.

Generally,

$\log_a x \Leftrightarrow$ *the power to which* ***a*** *must be raised to produce* ***x***.

Example 3.6

Calculate to 3 significant figures, the value of $\log_2 10$.

Solution

$$\log_2 10 = \frac{\log 10}{\log 2} = \frac{1}{0.30103} = 3.322\,(3\,s.f.)$$

Or

$$\log_2 10 = \frac{\ln 10}{\ln 2} = \frac{2.302585}{0.693147} = 3.322\,(3\,s.f.)$$

Example 3.7

If $y = 2x^3$, find a linear expression connecting $\log x$ and $\log y$.

Solution

Introducing $\log$ on both sides of $y = 2x^3$ yields

$$\log y = \log 2x^3 \Leftrightarrow \log y = \log 2 + \log x^3$$
$$\Leftrightarrow \log y = \log 2 + 3\log x$$

Example 3.8

Express $\log_a \frac{x^3}{y^2 z}$ in terms of $\log_a x$, $\log_a y$ *and* $\log_a z$

Solution

$$\log_a \frac{x^3}{y^2 z} = \log_a x^3 - \log_a y^2 z$$

$$\Leftrightarrow \log_a \frac{x^3}{y^2 z} = 3\log_a x - \left(\log_a y^2 + \log_a z\right)$$

$$\Leftrightarrow \log_a \frac{x^3}{y^2 z} = 3\log_a x - 2\log_a y - \log_a z$$

Example 3.9

Write an expression equivalent to $\log y = 3 - 2\log x$ without using logarithms.

$$\log y = 3 - 2\log x$$

$$\Leftrightarrow \log y = \log 1000 - \log x^2$$

$$\Leftrightarrow \log y = \log \frac{1000}{x^2}$$

$$\Rightarrow y = \frac{1000}{x^2}$$

Or $\log y = 3 - 2\log x$

$$\Rightarrow y = 10^{3-2\log x} \quad as \log_a b = c \Leftrightarrow b = a^c$$

$$\Leftrightarrow y = 10^{3-\log x^2}$$

$$\Leftrightarrow y = \frac{10^3}{10^{\log x^2}}$$

$$\Rightarrow y = \frac{1000}{x^2} \qquad since\ \ b = a^{\log_a b}$$

Example 3.10

Solve the equation $2^{3x} = 3^{2x-1}$.

Solution

$2^{3x} = 3^{2x-1}$ taking logarithms of both sides and applying logarithmic laws give;

$$3x\log 2 = (2x-1)\log 3 \Leftrightarrow 3x\log 2 = 2x\log 3 - \log 3$$
$$\Leftrightarrow 3x\log 2 - 2x\log 3 = \log 3$$
$$\Leftrightarrow x(3\log 2 - 2\log 3) = \log 3$$
$$\Leftrightarrow x = \frac{\log 3}{3\log 2 - 2\log 3}$$
$$\Rightarrow x = 9.327$$

Example 3.11

Solve the equation $2(5^{2x}) - 5^x = 6$.

Solution

Let $y = 5^x, with\ y > 0$.

Then $2y^2 - y = 6$

Or $2y^2 - y - 6 = 0$

$(2y+3)(y-2) = 0$

$\Rightarrow y = -1\frac{1}{2}$ *excluded* since $y = 5^x$ *must be positive*

Or $y = 2$

So $y = 2$ gives $5^x = 2 \Rightarrow x = \log_5 2 = \frac{\log 2}{\log 5} = 0.431$

Application activity 3.3

1. Given that $\log_m x = p$, express each of the following in terms of p;

   a) $\log_m(x^4)$ b) $\log_m\left(\frac{1}{x^2}\right)$ c) $\log_m(mx)$

2. Simplify;

   a) $e^{2-\ln x}$ b) $e^{\ln x^2 - 2\ln y}$ c) $\ln\left(x^3 e^{(-x^2)}\right)$

3. Solve the equations

   a) $2^x = 5$ b) $3^x = 23$

   c) $4^{3x+1} = 34$ d) $e^{4x} - 13e^{2x} + 36 = 0$

3.3. Applications

Exponential growth

Activity 3.4

In a laboratory, for experiment, we start with 2 cells in a dish. The number of cells in the dish doubles every 30 minutes.

a) How many cells are in the dish after four hours?

b) After what time are there 2^{13} cells in the dish?

c) After $10\frac{1}{2}$ hours, there are 2^{22} cells in the dish and an experiment fluid is added which eliminates half of the cells. How many cells are left?

A population whose rate of increase is proportional to the size of the population at any time obeys a law of the form $P = Ae^{kt}$. This is known as **exponential growth**.

Example 3.12

According to United Nations data, the world population at the beginning of 1975 was approximately 4 billion and growing at rate of about 2% per year. Assuming an exponential growth model, estimate the world population at the beginning of the year 2020.

Solution

Let t be time (in years) elapsed from the beginning of 1975 and $P(t)$ be world population in billions.

Since the beginning of 1975 corresponds to $t=0$, it follows from the given data that $P_0 = P(0) = 4(billions)$.

Since the growth rate is 2% $(k=0.02)$, it follows that the world population at time will be $P(t) = P_0e^{kt} = 4e^{0.02t}$.

Since the beginning of the year 2020 corresponds to $t=45$ (2020-1975=45), it follows that the world population by the year 2020 will be $P(45) = 4e^{0.02(45)}\,(billlion)$

Or $P(45) = 4e^{0.9}\,(billion)$

$= 4(2.459603)(billion)$

$= 9.838412(billion)$

Which is a population of approximately 9.8 billion.

Exponential decay

Activity 3.5

The amount, $A(t)\,gram$, of radioactive material in a sample after t years is given by $A(t) = 80\left(2^{-\frac{t}{100}}\right)$.

a) Find the amount of material in the original sample.

b) Calculate the half-life of the material (the half-life is the time taken for half of the original material to decay).

c) Calculate the time taken for the material to decay $1\,gram$.

A population whose rate of decrease is proportional to the size of the population at any time obeys a law of the forms $P = Ae^{-kt}$ .The negative sign on exponent indicates that the population is decreasing. This is known as **exponential decay**.

If a quantity has an exponential growth model, then the time required for it to double in size is called the **doubling time**. Similarly, if a quantity has an exponential decay model, then the time required for it to reduce in value by half is called the **halving time.** For radioactive elements, halving time is called **half-life.**

Doubling time and halving time

Activity 3.6

Show that the doubling time (T) for a quantity with an exponential growth model $(k > 0)$ depends only on the growth rate not on the amount present initially and is $T = \frac{1}{k}\ln 2$.

Doubling and halving times depend only on the growth rate and not on the amount present initially.

Doubling time for a quantity with an exponential growth model $(k > 0)$ is $T = \frac{1}{k}\ln 2$ and halving time for a quantity with an exponential decay model $(k < 0)$ is $T = -\frac{1}{k}\ln 2$.

Example 3.13

The radioactive element carbon-14 has a half-life of 5,750 years. If 100 grams of this element are present initially, how much will be left after 1,000 years?

Solution

As $T = -\frac{1}{k}\ln 2$, the decay constant is

$$k = -\frac{1}{T}\ln 2$$

$$= -\frac{1}{5750}\ln 2$$

$$= -\frac{1}{5750}0.693147181$$

$$=-0.000120547$$

$$\simeq -0.00012$$

Radioactive decay obeys a law of the forms $P(t) = P_0 e^{-kt}$.

Thus, if we take $t = 0$ to be the present time, then $P_0 = P(0) = 100$, thus, the amount of carbon-14 after 1,000 years will be

$$P(1000) = 100e^{-0.00012(1000)}$$
$$= 100e^{-0.12}$$
$$\simeq 100(0.88692)$$
$$\simeq 88.692$$

Thus, about 88.692grams of carbon-14 will remain.

Example 3.14

Magnitudes of **earthquakes** are measured using the **Richter scale**. On this scale, the magnitude R of an earthquake is given by $R = \log\left(\frac{I}{I_0}\right)$ where I_0 is a fixed standard intensity used for comparison, and I is the intensity of earthquakes being measured.

a) Show that if an earthquake measures $R = 3$ on Richter scale, then its intensity is 1,000 times the standard, that is , $I = 1,000I_0$.

b) The San Francisco earthquake of 1906 registered $R = 8.2$ on Richter scale. Express its intensity in terms of the standard intensity.

c) How many times more intense is an earthquake measuring $R = 8$ than on measuring $R = 4$?

Solution

a) If an earthquake measures $R = 3$ on Richter scale,

then $\log\left(\frac{I}{I_0}\right) = 3$

$$\Rightarrow \frac{I}{I_0} = 10^3$$
$$\Leftrightarrow I = 10^3 I_0$$
$$\Leftrightarrow \text{I=1000 I}_0$$

Therefore, intensity is 1,000 times the standard, that is,

$I = 1,000 I_0$.

b) The San Francisco earthquake of 1906 registered

$R = 8.2$ on Richter scale. It means that $\log\left(\frac{I}{I_0}\right) = 8.2$

or $\frac{I}{I_0} = 10^{8.2} \Leftrightarrow I = 10^{8.2} I_0$ expresses its intensity in terms of the standard intensity.

c) Let E_1, E_2 be earthquakes measuring $R = 8$ and $R = 4$ respectively.

For $E_1 : R = 8 \Rightarrow \frac{I}{I_0} = 10^8 \Leftrightarrow I = 10^8 I_0$;

For $E_2 : R = 4 \Rightarrow \frac{I}{I_0} = 10^4 \Leftrightarrow I = 10^4 I_0$;

Intensity of E_1 is $I_1 = 10^8 I_0$ (1)

Intensity of E_2 is $I_2 = 10^4 I_0$ (2)

The ratio of two above equations yields

$$\frac{I_1}{I_2} = \frac{10^8 I_0}{10^4 I_0} = 10^4 \Rightarrow I_1 = 10^4 I_2 \Leftrightarrow I_1 = 10,000 I_2$$

An earthquake measuring $R = 8$ is 10,000 times more intense than one measuring $R = 4$.

Example 3.15

Jack operates an account with a certain bank which pays a compound interest rate of 13.5% per annum. He opened the account at the beginning of the year with 500,000 Frw and deposits the same amount of money at the beginning of every year. Calculate how much he will receive at the end of 9 years. After how long will the money have accumulated to Frw 3.32 million?

Solution

The compound interest formula:

The 1^{st} deposit will be

$$500,000+\frac{500,000\times 13.5}{100}=500,000\left(1+\frac{13.5}{100}\right)$$

Or

$$500,000+\frac{500,000\times 13.5}{100}=500,000\times 1.135$$

The 2nd deposit will grow to $500,000\times(1.135)^2$

The 3rd deposit will grow to $500,000\times(1.135)^3$

The nth deposit will grow to $500,000\times(1.135)^n$

So the 9th deposit will grow to $500,000\times(1.135)^9$

The total sum

$$500,000\times(1.135)+500,000\times(1.135)^2+500,000\times(1.135)^3+\cdots+500,000\times(1.135)^9$$

$$=500,000\left[1.135+(1.135)^2+(1.135)^3+\cdots+(1.135)^9\right]$$

From $S_n=u_1\left(\frac{1-r^n}{1-r}\right)$, we get

$$S_9=500,000\left[1.135\left(\frac{1-(1.135)^9}{1-1.135}\right)\right]$$

Or $S_9=\frac{-500,000\times 1.135\times 2.125811278}{-0.135}$

Or $S_9=8,936,281$

Finding how long it will take the money to accumulate to 3,320,000 Frw

$$S_n=3,320,000$$

$$\Rightarrow 500,000\left[1.135\left(\frac{1-(1.135)^n}{1-1.135}\right)\right]=3,320,000$$

$$\Rightarrow \frac{1-(1.135)^n}{1-1.135}=\frac{3,320,000}{500,000\times 1.135}\Leftrightarrow \frac{1-(1.135)^n}{-0.135}=\frac{3,320,000}{500,000\times 1.135}$$

$\Leftrightarrow 1-(1.135)^n = -\frac{332\times 0.135}{50\times 1.135} \Leftrightarrow (1.135)^n - 1 = \frac{332\times 0.135}{50\times 1.135}$

$\Leftrightarrow (1.135)^n - 1 = 0.7897$

$(1.135)^n = 0.7897 + 1$

$(1.135)^n = 1.7897$

Introducing logarithm to the base 10 on both sides gives

$n\log(1.135) = \log(1.7897)$

$n = \frac{\log(1.7897)}{\log(1.135)}$

$n \approx 4.6$

Hence it will take 4.6 years for the amount to accumulate to 3.32 million Frw.

Example 3.16

A man deposits 800,000 Frw into his savings account on which interest is 15% per annum. If he makes no withdrawals, after how many years will his balance be 8 million Frw?

Solution

Here, the interest rate will be compound such that amount is $P\left(1+\frac{r}{100}\right)^n$, where n=period of time.

$8,000,000 = 800,000\left(1+\frac{15}{100}\right)^n$

$10 = (1+0.15)^n$

$10 = (1.15)^n$

$$\log 10 = \log(1.15)^n$$
$$1 = n\log(1.15)$$
$$n = \frac{1}{\log(1.15)}$$
$$n \approx 16.5 \ \ years$$

Example 3.17

Cholera, an intestinal disease, is caused by a cholera bacterium that multiplies exponentially by cell division as modelled by $N=N_0\, e^{1.386t}$, where N is the number of bacteria present after t hours and N_0 is the number of bacteria present at t = 0. If we start with 1 bacterium, how many bacteria will be present in

a) 5 hours?

b) 12 hours?

Solution

For t = 5

$N= N_0 e^{1.386t}$

$N_0 = 1$ and t=5

$N=e^{1.386*5}$

$N=e^{6.93}$

N=1,022.493

After t =5, there will be 1,022.493 bacteria

for t =12 hours

$N=N_0e^{1.386t}$

$N_0 = 1$ and t=12

$N=e^{1.386*12}$

$N = e^{16.632}$

N=16,718,057.823

After t =12, there will be **16,718,057.823** bacteria

Application activity 3.4

1. A certain cell culture grows at a rate proportional to the number of cells present. If the culture contains 500 cells initially 800 after $24h$, how many cells will be there after a further $12h$?
2. A radioactive substance decays at a rate proportional to the amount present. If 30% of such a substance decays in 15 years, what is the half-life of the substance?
3. A colony of bacteria is grown under ideal conditions in laboratory so that the population increases exponentially with time. At the end of three hours there are 10,000 bacteria. At the end of 5 hours there are 40,000. How many bacteria were present initially?
4. A radioactive material has a half-life of 1,200 years.
   a) What percentage of the original radioactivity of a sample is left 10 years?
   b) How many years are required to reduce the radioactivity by 10%?
5. Scientists who do carbon-14 dating use a figure of 5700 years for its half-life. Find the age of a sample in which 10% of the radioactive nuclei originally present have decayed.
6. How much money needs to be invested today at a nominal rate of 4% compounded continuously, in order that it should grow to Frw 10,000 in 7 days?
7. The number of people cured is proportional to the number y that is infected with the disease.
   a) Suppose that in the course of any given year the number of cases of disease is reduced by 20 %. If there are 10,000 cases today, how many years will it take to reduce the number to 1000?
   b) Suppose that in any given year the number of cases can be reduced by 25% instead of 20% .
      (i) How long will it take to reduce the number of cases to 1000?
      (ii) How long will it take to eradicate the disease, that is, to reduce the number of cases to less than 1?

Unit Summary

1. To find the inverse of the function $y = a^x$, we write $x = \log_a y$.
2. The graph of a logarithmic function is found by reflecting the graph of the corresponding exponential function in the line $y = x$.
3. Each exponential expression has a corresponding logarithmic expression. The relationship is $b = a^c \Leftrightarrow c = \log_a b$. Thus, we may write $b = a^{\log_a b}$.
4. **Basic rules for exponents**

For $a > 0$ *and* $a \neq 1,\ m, n \in \mathbb{R}$

a) $a^m \times a^n = a^{m+n}$

b) $a^m : a^n = a^{m-n}$

c) $\left(a^m\right)^n = a^{mn}$

d) $a^{-n} = \frac{1}{a^n}$

e) $a^{\frac{1}{n}} = \sqrt[n]{a}$

f) $a^{\frac{m}{n}} = \sqrt[n]{a^m}$

g) $a^{\log_a b} = b$

5. **Basic rules for logarithms**

$\forall x, y \in \left]0, +\infty\right[, a \in \left]0, +\infty\right[\setminus \{1\}:$

a) $\log_a xy = \log_a x + \log_a y$

b) $\log_a \frac{1}{y} = -\log_a y$

c) $\log_a \frac{x}{y} = \log_a x - \log_a y$

d) $\log_a x^r = r \log_a x$

e) $\log_a b = \frac{\log_c b}{\log_c a}$

6. Exponential and logarithmic functions are used in population growth, half life, and decay in radioactivity. Logarithmic functions are also used to find interest rate problems.

End of Unit Assessment

In number 1-6, find the numerical value

1. $\log_2 32$
2. $\log_4 8$
3. $\log_6 7$
4. $\log_5 \sqrt{125}$
5. $\log_5 0.008$
6. $\log_9 10$

In number 7-12, solve for x

7. $\log_3 x = 4$
8. $\ln(x-2)(x-1) = \ln(2x+8)$
9. $\ln(x-2) + \ln(x-1) = \ln(2x+8)$
10. $\log_x 5 = \log_5 x$
11. $144^x = \sqrt{12}$
12. $9^x - 2\times 3^{x+1} = 27$
13. The population of a country grows according to the law $P = Ae^{0.06t}$ where P million is the population at time t years and A is a constant. Given that at time $t=0$, the population is 27.3 million, calculate the population when;
    a) $t=10$
    b) $t=15$
    c) $t=25$
14. The population of a city, $P(n)$, n years after the population was p is given by $P(n) = p\left(e^{\frac{n}{30}}\right)$. Find:
    a) The time taken for the population to double.
    b) The time taken for the population to reach 1 million from an original population of 10000.
15. The speed $V(t)$, of a certain chemical reaction at $t\,^{\circ}C$ is given by $V(t) = V(0)\times 5^{\frac{t}{30}}$. At what temperature will the speed of reaction be twice that at $0\,^{\circ}C$?
16. The speed, $S(t)\,ms^{-1}$, at which a man falls t seconds after jumping from a plane is given by $S(t) = 48\left(1-2^{-0.3t}\right)$. After how long is the man falling at;
    a) $24\,ms^{-1}$
    b) $30\,ms^{-1}$
    c) $45\,ms^{-1}$?

Unit 4

Trigonometric Functions and their Inverses

Introductory activity

Given the function $y = f(x) = \cos x$,

a) Complete the table of values of $y = f(x)$ for $-2\pi \le x \le 2\pi$

b) use the values obtained from a) to draw the graph for $y = f(x)$,

c) Find the values of x for which $f(x) = 0$. They are $f^{-1}(0)$.

1. You studied trigonometry in previous levels, give two examples of applications of trigonometric functions in real life.

Objectives:

By the end of this unit, a student will be able to:

- find the domain and range of trigonometric functions and their inverses.
- study the parity of trigonometric functions.
- study the periodicity of trigonometric functions.
- evaluate limits of trigonometric functions.
- differentiate trigonometric functions and their inverses.

4.1. Generalities on trigonometric functions and their inverses

4.1.1. Domain and range of six trigonometric functions

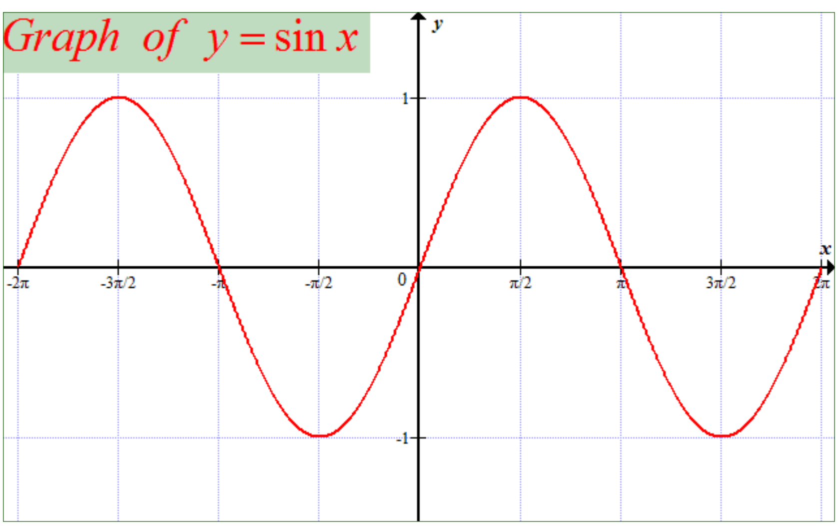
Activity 4.1

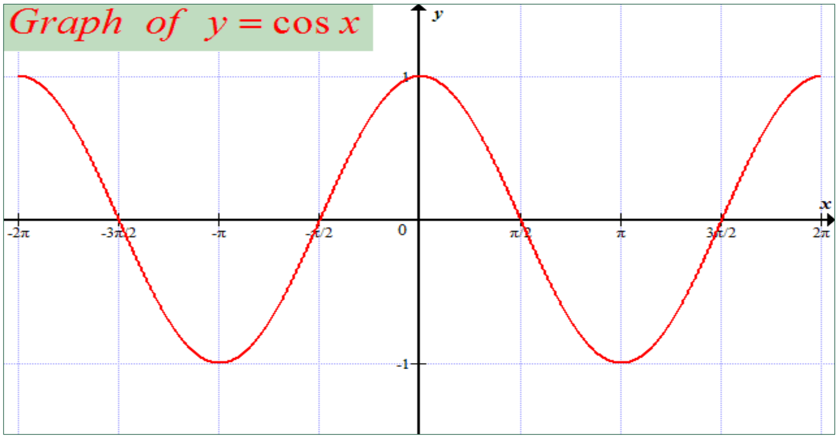
State the values of x where the following functions are not defined:

1. $y = \sin x$ 2. $y = \cos x$ 3. $y = \tan x$ 4. $y = \cot x$

Cosine and sine

$\sin x$ and $\cos x$ are functions which are defined for all positive and negative values of x even for $x = 0$. Thus, the domain of $\sin x$ and $\cos x$ is the set of real numbers. The range of $\sin x$ and $\cos x$ is $[-1,1]$.

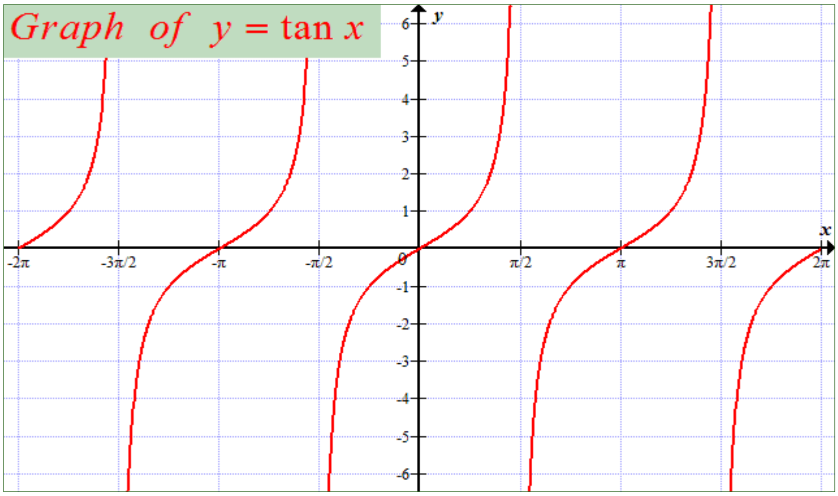


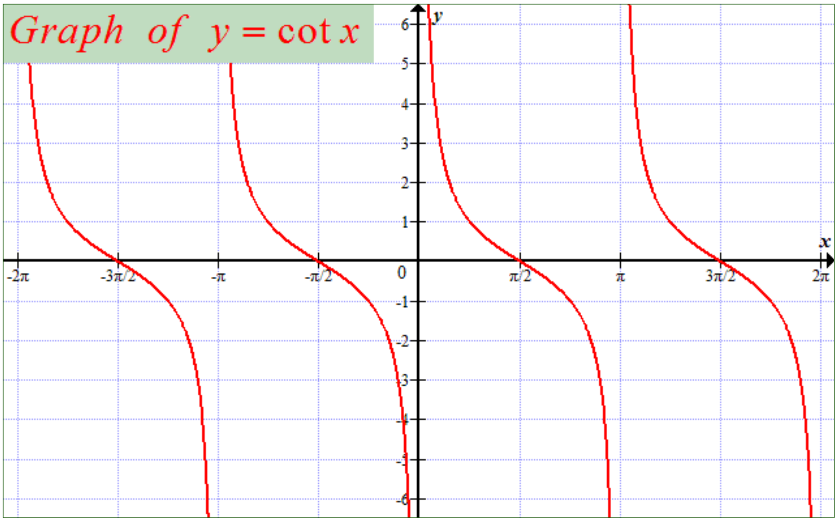


Tangent and cotangent

Function $\tan x$ is not defined for $x = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \ldots$. Generally, $\tan x$ is not defined for $x = \frac{\pi}{2} + k\pi,\ k \in \mathbb{Z}$. Thus, domain of $\tan x$ is $\mathbb{R} \setminus \left\{\frac{\pi}{2} + k\pi\right\}, k \in \mathbb{Z}$. The range of $\tan x$ is the set of real numbers.

Function $\cot x$ is not defined for $x = 0, \pm\pi, \pm 2\pi, \ldots$. Generally, $\cot x$ is not defined for $x = k\pi,\ k \in \mathbb{Z}$. Thus, domain of $\cot x$ is $\mathbb{R} \setminus \{k\pi\}, k \in \mathbb{Z}$. The range of $\cot x$ is the set of real numbers.





Secant and cosecant

Function $\sec x$ is not defined for $x = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \dots$.

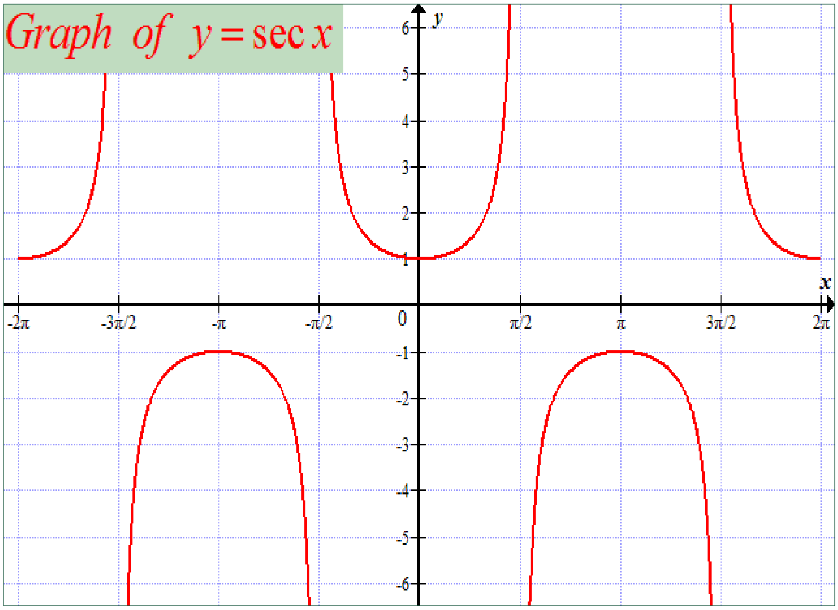
Generally, $\sec x$ is not defined for $x = \frac{\pi}{2} + k\pi,\ k \in \mathbb{Z}$.

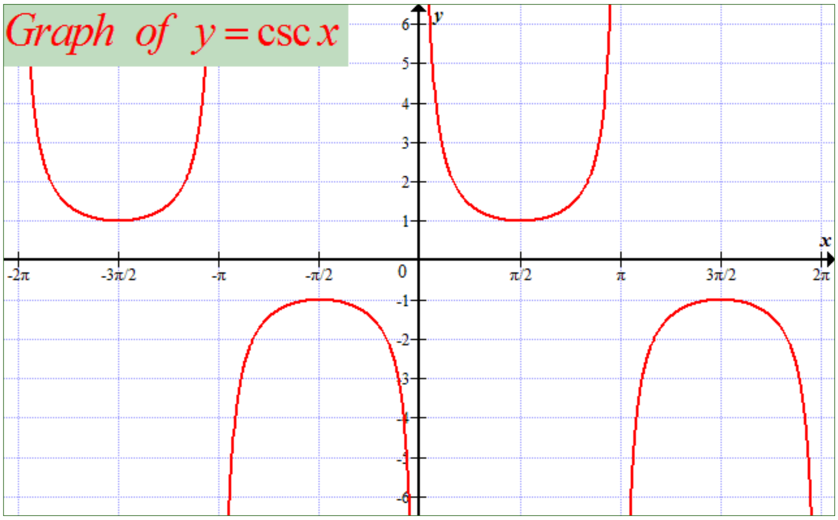
Thus, similar to tangent, domain of $\sec x$ is $\mathbb{R} \setminus \left\{\frac{\pi}{2} + k\pi\right\}, k \in \mathbb{Z}$. Since

$\sec x = \frac{1}{\cos x}$ and range of cosine is $[-1,1]$, $\frac{1}{\cos x}$ will vary from negative infinity to -1 or from 1 to positive infinity. Thus, the range of $\sec x$ is $]-\infty,-1] \cup [1,+\infty[$

Function $\csc x$ is not defined for $x = 0, \pm\pi, \pm 2\pi, \dots$. Generally, $\csc x$ is not defined for $x = k\pi,\ k \in \mathbb{Z}$. Thus, similar to cotangent, domain of $\csc x$ is $\mathbb{R} \setminus \{k\pi\}, k \in \mathbb{Z}$.

Since $\csc x = \frac{1}{\sin x}$ and range of sine is $[-1,1]$, $\frac{1}{\sin x}$ will vary from negative infinity to -1 or from 1 to positive infinity. Thus, the range of $\csc x$ is $]-\infty,-1] \cup [1,+\infty[$





Application activity 4.1

Find the domain of definition for each of the following functions:

1. $f(x) = \sin x + \cos x$
2. $f(x) = \sin \dfrac{1}{x}$
3. $f(x) = \cos\left(\dfrac{x+1}{x}\right)$

4.1.2. Domain and range of inverses of trigonometric functions

Activity 4.2

Use properties of inverse functions and state the values of x where the following functions are not defined:

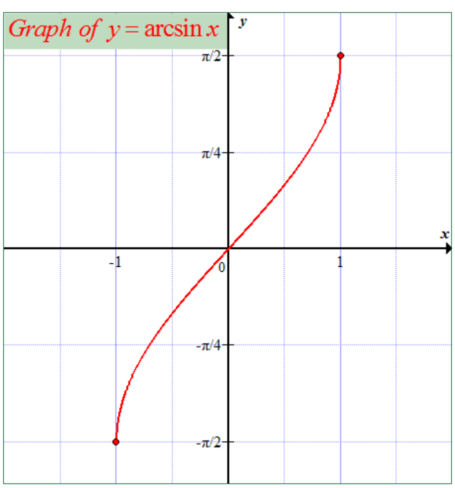
1. $y = \sin^{-1} x$ 2. $y = \cos^{-1} x$ 3. $y = \tan^{-1} x$

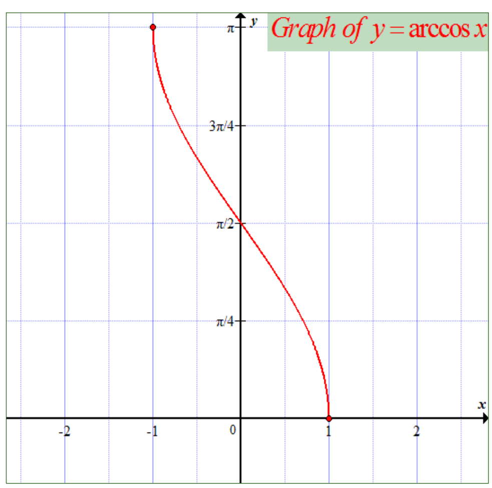
Inverse sine and inverse cosine

$\sin x$ and $\cos x$ are defined on the entire interval $(-\infty, +\infty)$. They have the inverses called inverse sine and inverse cosine denoted by $\sin^{-1} x$ and $\cos^{-1} x$ respectively.

Note that the symbols $\sin^{-1} x$ and $\cos^{-1} x$ are never used to denote $\frac{1}{\sin x}$ and $\frac{1}{\cos x}$ respectively. If desired, $\frac{1}{\sin x}$ and $\frac{1}{\cos x}$ can be written as $(\sin x)^{-1}$ and $(\cos x)^{-1}$ (or $\csc x$ and $\sec x$) respectively.

In older literature, $\sin^{-1} x$ and $\cos^{-1} x$ are called **arcsine** of x and **arccosine** of x and they are denoted by $arc\sin x$ and $arc\cos x$ respectively.





Remark

The inverses of the trigonometric functions are not functions, they are relations. The reason they are not functions is that for a given value of x, there is an infinite number of angles at which the trigonometric functions take on the value of x. Thus, the range of the inverses of the trigonometric functions must be restricted to make them functions. Without these restricted ranges, they are known as the inverse trigonometric relations.

To define $\sin^{-1} x$ and $\cos^{-1} x$, we restrict the domain of $\sin x$ and $\cos x$ to the interval $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$ and $[0,\pi]$ respectively to obtain a one-to-one function.

There are other ways to restrict the domain of $\sin x$ and $\cos x$ to obtain one-to-one functions, we might have required that $\frac{3\pi}{2}\le x\le\frac{5\pi}{2}$ and $\pi\le x\le 2\pi$ (or $\frac{-5\pi}{2}\le x\le\frac{-3\pi}{2}$ and $-2\pi\le x\le-\pi$) respectively.

Because $\sin x$ (restricted) and $\sin^{-1} x$; $\cos x$ (restricted) and $\cos^{-1} x$ are inverses to each other, it follows that:

- $\sin^{-1}(\sin y)=y$ if $-\frac{\pi}{2}\le y\le\frac{\pi}{2}$; $\sin\left(\sin^{-1} x\right)=x$ if $-1\le x\le 1$.
- $\cos^{-1}(\cos y)=y$ if $0\le y\le\pi$; $\cos\left(\cos^{-1} x\right)=x$ if $-1\le x\le 1$.

From these relations, we obtain the following important result:

Theorem 4.1

- If $-1\le x\le 1$ and $-\frac{\pi}{2}\le y\le\frac{\pi}{2}$, then $y=\sin^{-1} x$ and $\sin y=x$ are equivalent.
- If $-1\le x\le 1$ and $0\le y\le\pi$, then $y=\cos^{-1} x$ and $\cos y=x$ are equivalent.

Example 4.1

Find;

a) $\sin^{-1}\left(\frac{1}{2}\right)$ b) $\sin^{-1}\left(-\frac{1}{\sqrt{2}}\right)$

Solution

a) Let $y=\sin^{-1}\left(\frac{1}{2}\right)$. From Theorem 4.1, this equation is equivalent to $\sin y=\frac{1}{2}$, $-\frac{\pi}{2}\le y\le\frac{\pi}{2}$. The only value satisfying these conditions is $y=\frac{\pi}{6}$, so $\sin^{-1}\left(\frac{1}{2}\right)=\frac{\pi}{6}$

b) Let $y=\sin^{-1}\left(-\frac{1}{\sqrt{2}}\right)$.

From Theorem 4.1, this equation is equivalent to $\sin y=-\frac{1}{\sqrt{2}}$; $-\frac{\pi}{2}\le y\le\frac{\pi}{2}$. The only value satisfying these conditions is $y=-\frac{\pi}{4}$, so $\sin^{-1}\left(-\frac{1}{\sqrt{2}}\right)=-\frac{\pi}{4}$.

Example 4.2

Simplify the function $\cos\left(\sin^{-1}x\right)$.

Solution

The idea is to express cosine in terms of sine in order to take advantage of the simplification $\sin^{-1}\left(\sin x\right)=x$.

Thus, we start by the identity $\cos^2\theta=1-\sin^2\theta$ and substitute $\theta=\sin^{-1}x$ to obtain $\cos^2\left(\sin^{-1}x\right)=1-\sin^2\left(\sin^{-1}x\right)$

Or by taking square root $\left|\cos\left(\sin^{-1}x\right)\right|=\sqrt{1-\sin^2\left(\sin^{-1}x\right)}$

Or $\left|\cos\left(\sin^{-1}x\right)\right|=\sqrt{1-x^2}$

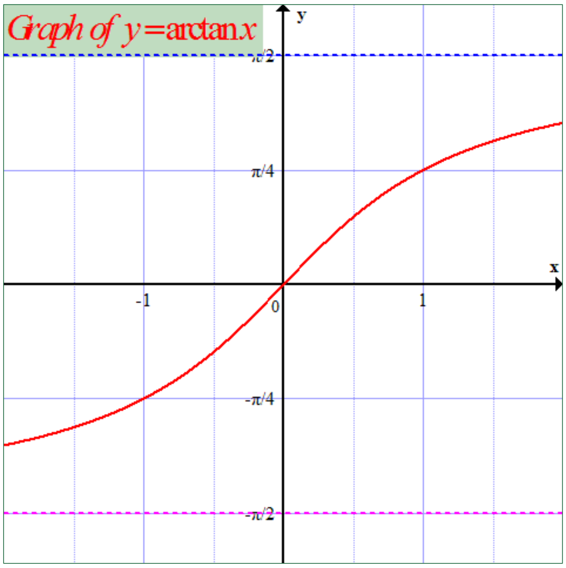
Since $-\frac{\pi}{2}\le\sin^{-1}x\le\frac{\pi}{2}$, it follows that $\cos\left(\sin^{-1}x\right)$ is non-negative.

Thus, we can drop the absolute value and write $\cos\left(\sin^{-1}x\right)=\sqrt{1-x^2}$

Inverse tangent

Tangent of x, denoted $\tan x$, is a function which is defined for all positive and negative values of x except $\pm 90^0, \pm 270^0, \ldots$. The range of $\tan x$ is $(-\infty, +\infty)$. It has the inverse called **inverse tangent** and is denoted by $\tan^{-1}x$.

To define $\tan^{-1}x$, we restrict the domain of $\tan x$ to $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.



Because $\tan x$ (restricted) and $\tan^{-1}x$ are inverse to each other, it follows that:

- $\tan^{-1}(\tan y)=y$ if $-\frac{\pi}{2}<y<\frac{\pi}{2}$
- $\tan\left(\tan^{-1}x\right)=x$ if $-\infty<x<+\infty$

From these relations, we obtain the following important result:

Theorem 4.2

- If $-\infty<x<+\infty$ and $-\frac{\pi}{2}<y<\frac{\pi}{2}$, then $y=\tan^{-1}x$ and

$\tan y = x$ are equivalent.

Example 4.3

Simplify the function $\sec^2\left(\tan^{-1} x\right)$.

Solution

The idea is to express secant in terms of $\tan x$ to take the advantage of simplification $\tan\left(\tan^{-1} x\right) = x$.

Let $\theta = \tan^{-1} x$ in the identity $\sec^2\theta = 1 + \tan^2\theta$, we get

$\sec^2\left(\tan^{-1} x\right) = 1 + \tan^2\left(\tan^{-1} x\right) = 1 + x^2$

Thus, $\sec^2\left(\tan^{-1} x\right) = 1 + x^2$

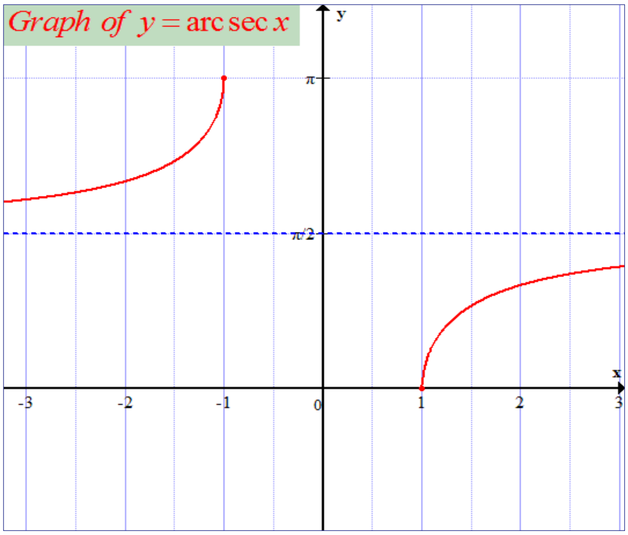
Inverse secant

The inverse secant, denoted $\sec^{-1} x$, is defined to be the inverse of restricted secant function.

$f(x) = \sec x$, $0 \le x < \frac{\pi}{2}$ *or* $\frac{\pi}{2} < x \le \pi$.

If we let $y = \sec^{-1} x$, then we find that $x \le -1$ or $x \ge 1$ and $0 \le y < \frac{\pi}{2}$ or $\frac{\pi}{2} < y \le \pi$.

Thus, the domain of $\sec^{-1} x$ is $(-\infty, -1] \cup [1, +\infty)$ and the range is $\left[0, \frac{\pi}{2}\right[\cup \left]\frac{\pi}{2}, \pi\right]$



Theorem 4.3

If $x \le 1$ or $x \ge 1$ and if $0 \le y < \frac{\pi}{2}$ *or* $\frac{\pi}{2} < y \le \pi$, then $y = \sec^{-1} x$ and $\sec y = x$ are equivalent statements.

Example 4.4

Simplify $\tan^2\left(\sec^{-1} x\right)$.

Solution

We know that $\sec^2\theta = 1 + \tan^2\theta$, then $\tan^2\theta = \sec^2\theta - 1$

Putting $\theta = \sec^{-1} x$, we have $\tan^2\left(\sec^{-1} x\right) = \sec^2\left(\sec^{-1} x\right) - 1 = x^2 - 1$

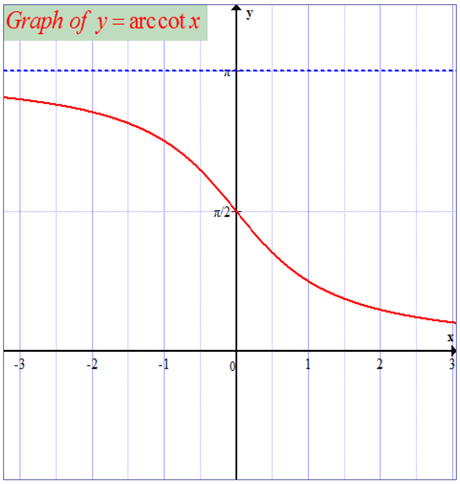
Thus, $\tan^2\left(\sec^{-1} x\right) = x^2 - 1$

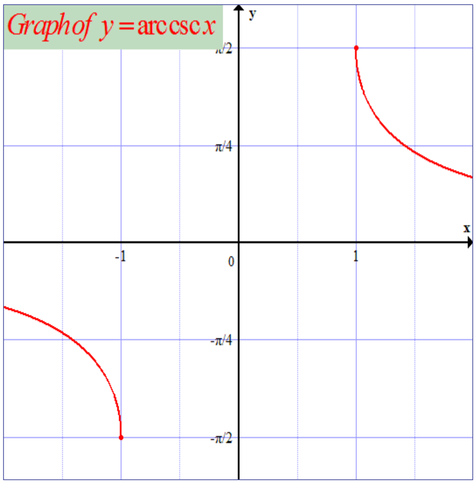
Inverse cotangent and inverse cosecant

We will summarise their properties briefly

$y = \cot^{-1} x$ is equivalent to $x = \cot y$ if $0 < y < \pi$ and $-\infty < x < +\infty$

$y = \csc^{-1} x$ is equivalent to $x = \csc y$ if $-\frac{\pi}{2} \le y < 0$ or $0 < y \le \frac{\pi}{2}$ and $|x| \ge 1$





Notice

If α and β are acute complementary angles, then from basic trigonometry, $\sin\alpha$ and $\cos\beta$ are equal. Let us write $x = \sin\alpha = \cos\beta$ so that $\alpha = \sin^{-1} x$ and $\beta = \cos^{-1} x$.

Since $\alpha + \beta = \frac{\pi}{2}$, we obtain the identity $\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}$.
Similarly, we can obtain the identities

$\tan^{-1} x + \cot^{-1} x = \frac{\pi}{2}$ and $\sec^{-1} x + \csc^{-1} x = \frac{\pi}{2}$

Remark

$\sin^{-1}(-x) = -\sin^{-1} x$

$\tan^{-1}(-x) = -\tan^{-1} x$

$\sec^{-1}(-x) = \pi + \sec^{-1} x,\ if\ x \geq 1$

Example 4.5

For which values of x is true that:

a) $\tan^{-1}(\tan x) = x$ b) $\tan(\tan^{-1} x) = x$

c) $\csc^{-1}(\csc x) = x$ d) $\csc(\csc^{-1} x) = x$

Solution

The values of x are:

a) $-\frac{\pi}{2}<x<\frac{\pi}{2}$ b) $-\infty<x<+\infty$

c) $-\frac{\pi}{2}\le x<0$ or $0<x\le\frac{\pi}{2}$ d) $|x|\ge 1$

Application activity 4.2

Find the domain of definition of the following functions:

1. $f(x)=\frac{1}{x}+\sin^{-1}2x$
2. $f(x)=\cos^{-1}x+\tan^{-1}x$
3. $f(x)=\cos^{-1}\frac{\sqrt{x}}{x}$
4. $f(x)=\sin^{-1}\frac{1}{x}$

4.1.3. Parity of trigonometric functions

Activity 4.3

For the function:

1. $f(x)=\frac{\sin x}{x}$, find $f(-x), -f(x)$ and compare the two results to $f(x)$.
2. $g(x)=\frac{\cos x}{x}$, find $g(-x), -g(x)$ and compare the two results to $g(x)$.

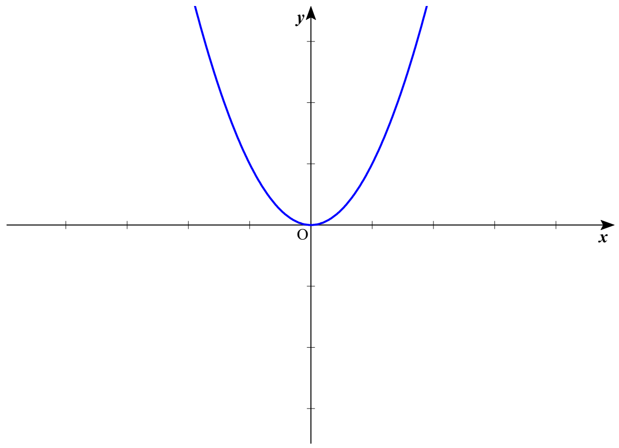
Even function

A function $f(x)$ is said to be even if the following conditions are satisfied:

- $\forall x\in Domf, -x\in Domf$
- $f(-x)=f(x)$

The graph of such function is **symmetric about the vertical axis.**

i.e $x=0$



Example 4.6

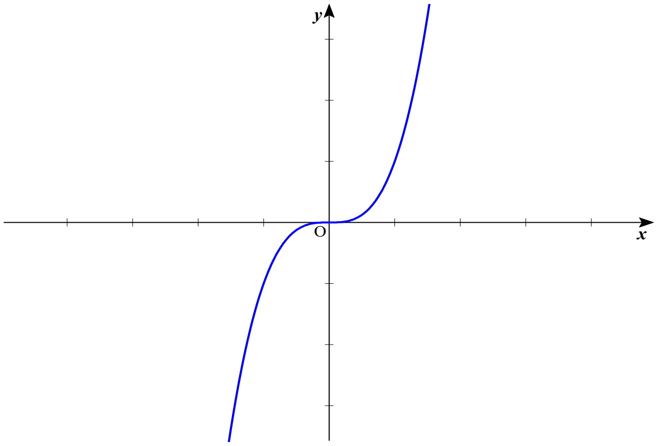
The function $\cos x$ is an even function since $\forall x \in \mathbb{R}, -x \in \mathbb{R}$ and $f(-x) = \cos(-x) = \cos x = f(x)$

Odd function

A function $f(x)$ is said to be odd if the following conditions are satisfied:

- $\forall x \in Domf, -x \in Domf$
- $f(-x) = -f(x)$

The graph of such function looks the same when rotated through half a revolution about 0. This is called **rotational symmetry**.



Example 4.7

The function $\sin x$ is an odd function since $\forall x \in \mathbb{R}, -x \in \mathbb{R}$ and $f(-x) = \sin(-x) = -\sin x = -f(x)$

Application activity 4.3

Study the parity of the following functions:

1. $f(x) = \dfrac{x^2}{\cos x}$
2. $f(x) = x + \sin 4x$
3. $f(x) = \sqrt[3]{x} + \sin x$
4. $f(x) = \dfrac{\tan x}{x+1}$

4.1.4. Period of trigonometric functions

Activity 4.4

What would be the value(s) of P to make the following relations true?

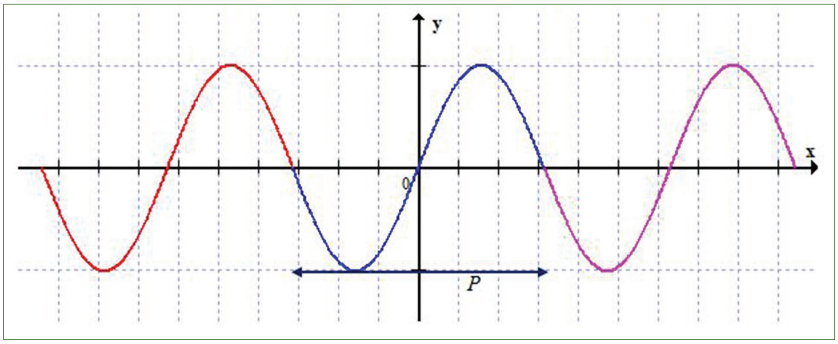
1. $\sin(x+P) = \sin x$
2. $\cos(x+P) = \cos x$
3. $\tan(x+P) = \tan x$

A function *f* is called **periodic** if there is a positive number P such that $f(x+P) = f(x)$ $(*)$ whenever x and $x+P$ lie in the domain of *f*.

We call P a **period** of the function. The smallest positive period is called the **fundamental period** (also **primitive period**, **basic period**, or **prime** period) of *f*.

A function with period P repeats on intervals of length P, and these intervals are referred to as **periods**.

Geometrically, a periodic function can be defined as a function whose graph exhibits translational symmetry. Specifically, a function is periodic with period P if its graph is invariant under translation in the *x-direction* by a distance of P.



The most important examples of periodic functions are the trigonometric functions.

Any function which is not periodic is called **aperiodic.**

Example 4.8

a) For the sine and cosine functions, 2π is the period since $\sin(x+2\pi)=\sin x$ and $\cos(x+2\pi)=\cos x$.

Also $4\pi, 6\pi, 8\pi, \ldots,$ are periods for sine and cosine functions since

$\sin(x+4\pi)=\sin x, \sin(x+6\pi)=\sin x, \sin(x+8\pi)=\sin x, \ldots$ and $\cos(x+4\pi)=\cos x, \cos(x+6\pi)=\cos x, \cos(x+8\pi)=\cos x, \ldots$.

The fundamental period of sine and cosine functions is 2π.

b) For tangent and cotangent functions, π is a period since $\tan(x+\pi)=\tan x$ and $\cot(x+\pi)=\cot x$. Also, $2\pi, 3\pi, 4\pi, \ldots$ are periods, but π is the fundamental period.

Or using definition, and solving for P;

For $\sin x$, we have $\sin(x+P)=\sin x$
$\Leftrightarrow x+P=x+2k\pi,\ k$ integer

$\Leftrightarrow P=2k\pi$. Since we need the smallest positive period, we take $k=1$

Thus, $P=2\pi$

For $\cos x$, we have $\cos(x+P)=\cos x \Leftrightarrow x+P=x+2k\pi, k\in\mathbb{Z}$

$\Leftrightarrow P=2k\pi$. Since we need the smallest positive period, we take $k=1$

Thus, $P=2\pi$

For $\tan x$, we have $\tan(x+P)=\tan x \Leftrightarrow x+P=x+k\pi, k\in\mathbb{Z}$

$\Leftrightarrow P=k\pi$. Since we need the smallest positive period we take $k=1$

Thus, $P=\pi$

Example 4.9

For $\sin 3x$ and $\cos 3x$ functions, the fundamental period is $\frac{2\pi}{3}$ since $\sin\left[3\left(x+\frac{2\pi}{3}\right)\right]=\sin(3x+2\pi)=\sin 3x$ and

$\cos\left[3\left(x+\frac{2\pi}{3}\right)\right]=\cos(3x+2\pi)=\cos 3x$.

Theorem 4.4

If $a\neq 0$ and $b\neq 0$, then the functions $a\sin bx$ and $a\cos bx$ have fundamental period $\frac{2\pi}{|b|}$ and their graphs oscillate between $-a$ and a. The number $|a|$ is called the amplitude of the function.

Example 4.10

Find the fundamental period of $f(x)=2\sin 6x$ and $g(x)=4\cos 3x$

Solution

For $f(x)$, we have $2\sin 6(x+P)=2\sin 6x$

$\Leftrightarrow 6x+6P=6x+2k\pi, k\in\mathbb{Z}$

$\Leftrightarrow 6P = 2k\pi$.

Since we need the smallest positive period, we take $k = 1$

Thus, $P = \dfrac{\pi}{3}$

For $g(x)$, we have $4\cos 3(x+P) = 4\sin 3x$

$\Leftrightarrow 3x + 3P = 3x + 2k\pi, k \in \mathbb{Z}$

$\Leftrightarrow 3P = 2k\pi$.

Since we need the smallest positive period, we take $k = 1$

Thus, $P = \dfrac{2\pi}{3}$

Application activity 4.4

Find the fundamental period of the following functions:

1. $f(x) = \sin 2x$
2. $f(x) = \cos\left(\dfrac{2x}{3}\right)$
3. $g(x) = \tan 3x$
4. $h(t) = 2\sin t$
5. $f(t) = \sin(wt + \varphi)$
6. $f(x) = \tan(2x+3)$

Combining periodic functions

Activity 4.5

Find the Lowest Common Multiple of:

1. π and 2π
2. $\dfrac{\pi}{2}$ and π

We have seen that *sine* and *cosine* are both periodic and have the same period. When we add them up, subtract them, multiply them, etc we get functions that are also periodic.

To see this, let us assume that $f(x+kP) = f(x)$ is true for all real x, k integers.

Simply multiplying each side by some constant does not change the equation and adding or subtracting some constant to each side does not change periodicity.

If we have two functions $f(x)$ and $g(x)$ with the same period, P say, we can throw them together any way we want.

Let $h(x) = f(x) + g(x)$, at any value a of x:

$h(a) = f(a) + g(a) = c$

$h(a + kP) = f(a + kP) + g(a + kP) = c$

Thus, $h(a) = h(a + kP)$

This is different for functions that don't have the same fundamental period.

Let us say that we have two periodic functions $f(x)$ and $g(x)$ with period P and Q respectively:

$f(x + kP) = f(x)$ is true for all real x, k integers.

$g(x + kQ) = g(x)$ is true for all real x, k integers.

Now, we cannot construct that nice $h(x)$ as we did before because we have different periods.

Consider the following case:

If a function repeats every 2 units, then it will also repeat every 6 units. So if we have one function with fundamental period 2, and another function with fundamental period 8, we have got no problem because 8 is a multiple of 2, and both functions will cycle every 8 units.

So, if we can patch up the periods to be the same,we know that if we combine them, we will get a function with the patched up period.

What we have to do is to find the Lowest Common Multiple (LCM) of two periods.

What about if one function has period 4 and another has period 5? We can see that in 20 units, both will cycle, so they are fine.

Example 4.11

Find the fundamental period of the function

$$f(x) = \tan\left(\frac{x+1}{2}\right)\sin\left(\frac{2x+1}{5}\right)$$

Solution

For $\tan\left(\frac{x+1}{2}\right)$, $P_1 = 2\pi$

For $\sin\left(\frac{2x+1}{5}\right)$, $P_2 = 5\pi$

$LCM(2\pi, 5\pi) = 10\pi$, $P = 10\pi$

Another important case is where the periods are fractions.

Suppose that we have function $f(x)$ with period $\frac{13}{12}$ and another function $g(x)$ with period $\frac{2}{21}$. What we need are two numbers of periods that we can multiply by the periods to get some common, patched up period.

First, we can simplify the problem by multiplying each period by its denominator to find whole number periods. So, we know that $f(x)$ has period of 13 (in 12 fundamental periods) and $g(x)$ a period of 2 (in 21 fundamental periods).

Now we can simply do what we did before and multiply both periods to find a period for the new combination function. So the combination function has a period of 26.

This suggests the following theorem:

Theorem 4.5

If two periodic functions have rational periods, then any addition or multiplication combination of those functions (not composition) will also be periodic.

Also if $f(x)$ is a periodic function and $g(x)$ is not a periodic function, then $g(f(x))$ is periodic and $f(g(x))$ is not.

Example 4.12

Find the fundamental period of the function $f(x)=\dfrac{\sin 3x}{\tan 7x}$

Solution

For $\sin 3x$, $P_1=\dfrac{2\pi}{3}$

For $\tan 7x$, $P_2=\dfrac{\pi}{7}$

P_1 is 2π in 3 fundamental periods

P_2 is π in 7 fundamental periods

But 2π is a multiple of π

Thus, $P=2\pi$

Example 4.13

Find the fundamental period of the function $f(x)=\sin x+\sin 4x$

Solution

For $\sin x, P_1=2\pi$

For $\sin 4x, P_2=\dfrac{\pi}{2}$

$P=LCM\left(2\pi,\dfrac{\pi}{2}\right)$ $\quad P=2\pi$

Application activity 4.5

Find the fundamental period of the following functions:

1. $f(x)=3\sin 2x-\tan 5x$
2. $f(x)=\sqrt{2}\sin 4x+\sin 5x$
3. $f(x)=\cos x-\tan 2x$
4. $f(x)=\cos\sqrt{3}x+\sin 6x$

4.2. Limits of trigonometric functions and their inverses

4.2.1. Limits of trigonometric functions

Activity 4.6

A. Evaluate

1) $\lim_{x\to 0}\sin x$ 2) $\lim_{x\to 0}x\sin x$ 3) $\lim_{x\to 0}\cos x$

4) $\lim_{x\to 0}\frac{1}{x}$ 5) $\lim_{x\to 0}\frac{\cos x}{x}$

B. Consider the function $f(x)=\frac{\sin x}{x}$ where x is in radians.

1) Use a calculator to complete the following tables;

x	$\frac{\sin x}{x}$
1	
0.9	
0.8	
0.7	
0.6	
0.5	
0.4	
0.3	
0.2	
0.1	
0.01	
0.001	
0.0001	

x	$\frac{\sin x}{x}$
-1	
-0.9	
-0.8	
-0.7	
-0.6	
-0.5	
-0.4	
-0.3	
-0.2	
-0.1	
-0.01	
-0.001	
-0.0001	

3) From results in 1), what is the limit of $\frac{\sin x}{x}$ as x approaches 0 from the right side.

4) From results in 1), what is the limit of $\frac{\sin x}{x}$ as x approaches 0 from the left side.

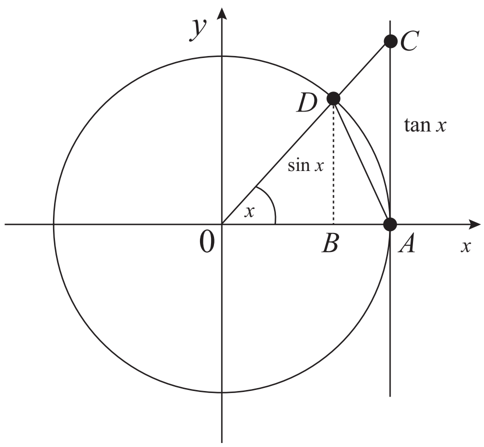
5) What can you say about $\lim_{x\to 0}\frac{\sin x}{x}$?

From activity 4.6,

$$\lim_{x\to 0}\frac{\sin x}{x}=1$$

There is another way to prove this.

Let us consider the trigonometric circle below:



Assume $0<x<\frac{\pi}{2}$. Since the circle is trigonometric the radius $\overline{OA}=1$.

The area of the triangle OAD is $\frac{1}{2}\overline{OA}\sin x=\frac{1}{2}\sin x$

The area of the sector OAD:

Recall that the area of a sector with subtended angle measuring x radian is $A=\frac{\theta}{2}r^2$, where r is the radius. The subtended angle of sector OAD is x and radius is *r*, the area of the sector OAD is $\frac{1}{2}\overline{OA}^2x=\frac{1}{2}x$

The area of the triangle OAC is $\frac{1}{2}\overline{OA}\tan x=\frac{1}{2}\tan x$

From the figure we see that

$$area\ of\ \Delta OAD\leq area\ of\ sector\ OAD\leq area\ of\ \Delta OAC$$

Or $\frac{1}{2}\sin x\leq\frac{1}{2}x\leq\frac{1}{2}\tan x\Leftrightarrow\sin x\leq x\leq\frac{\sin x}{\cos x}$

Dividing by $\sin x$, we get $1\leq\frac{x}{\sin x}\leq\frac{1}{\cos x}$

Taking the inverse [remember that when taking the inverse the order of the inequality must be changed], we get $\cos x \le \frac{\sin x}{x} \le 1$

Taking limit as x approaches 0, we get $1 \le \lim_{x\to 0}\frac{\sin x}{x} \le 1$

Using Squeeze theorem, since $\cos x \le \frac{\sin x}{x} \le 1$ and

$\lim_{x\to 0}\cos x = \lim_{x\to 0} 1 = 1$ then $\lim_{x\to 0}\frac{\sin x}{x} = 1$

Thus, $\lim_{x\to 0}\frac{\sin x}{x} = 1$

This result will help us find limit of some other trigonometric functions.

Example 4.14

Evaluate $\lim_{x\to 0}\frac{1-\cos x}{x}$

Solution

$\lim_{x\to 0}\frac{1-\cos x}{x} = \frac{0}{0}$ Indeterminate case (I.C)

$$\lim_{x\to 0}\frac{1-\cos x}{x} = \lim_{x\to 0}\frac{2\sin\frac{x}{2}\sin\frac{x}{2}}{x} \qquad \left[\textit{Since } 1-\cos x = 2\sin^2\frac{x}{2}\right]$$

$$= \lim_{x\to 0}\frac{2\sin\frac{x}{2}}{x}\lim_{x\to 0}\sin\frac{x}{2}$$

$$= \lim_{x\to 0}\frac{2\sin\frac{x}{2}}{2\frac{x}{2}}\lim_{x\to 0}\sin\frac{x}{2}$$

$$= \lim_{x\to 0}\frac{\sin\frac{x}{2}}{\frac{x}{2}}\lim_{x\to 0}\sin\frac{x}{2}$$

$$= 1\times 0$$

$$= 0$$

Thus, $\lim_{x \to 0} \frac{1 - \cos x}{x} = 0$

Example 4.15

Evaluate $\lim_{x \to 0} \frac{x}{\sin x} = 1$

Solution

$\lim_{x \to 0} \frac{x}{\sin x} = \frac{0}{0}$ I.C

$$\lim_{x \to 0} \frac{x}{\sin x} = \lim_{x \to 0} \frac{x}{x \frac{\sin x}{x}}$$

$$= \lim_{x \to 0} \frac{1}{\frac{\sin x}{x}} = \frac{\lim_{x \to 0} 1}{\lim_{x \to 0} \frac{\sin x}{x}}$$

$$= \frac{1}{1} = 1$$

Thus, $\lim_{x \to 0} \frac{x}{\sin x} = 1$

Example 4.16

Evaluate $\lim_{x \to 0} \frac{\tan x}{x} = 1$

Solution

$\lim_{x \to 0} \frac{\tan x}{x} = \frac{0}{0}$ I.C

$$\lim_{x \to 0} \frac{\tan x}{x} = \lim_{x \to 0} \frac{\frac{\sin x}{\cos x}}{x}$$

$$= \lim_{x \to 0} \frac{\sin x}{x \cos x}$$

$$= \lim_{x \to 0} \frac{\sin x}{x} \frac{1}{\cos x}$$

$$= \lim_{x \to 0} \frac{\sin x}{x} \lim_{x \to 0} \frac{1}{\cos x}$$

$$= 1 \times 1 = 1$$

Thus, $\lim_{x \to 0} \frac{\tan x}{x} = 1$

Example 4.17

Evaluate $\lim_{x \to 0} \frac{\cot x}{x}$

Solution

$$\lim_{x \to 0} \frac{\cot x}{x} = \frac{\infty}{0} = \infty$$

Or

$$\lim_{x \to 0} \frac{\cot x}{x} = \lim_{x \to 0} \cot x \lim_{x \to 0} \frac{1}{x} = \infty \times \infty = \infty$$

Left and right hand limits: $\frac{\cot x}{x} = \frac{\frac{\cos x}{\sin x}}{x} = \frac{\cos x}{x \sin x}$

x	$-\frac{\pi}{2}$	0	$\frac{\pi}{2}$
$\cos x$	+	1	+
x	-	0	+
$\sin x$	-	0	+
$x \sin x$	+	0	+
$\frac{\cos x}{x \sin x}$	+	|| ∞	+

Thus, $\lim_{x \to 0^+} \frac{\cot x}{x} = +\infty$ and $\lim_{x \to 0^-} \frac{\cot x}{x} = +\infty$ and hence $\lim_{x \to 0} \frac{\cot x}{x}$ **does not exist.**

When finding limits of trigonometric functions, sometimes we need to change the variable.

Example 4.18

Evaluate $\lim_{x\to\frac{\pi}{3}}\frac{1-2\cos x}{x-\frac{\pi}{3}}$

Solution

$$\lim_{x\to\frac{\pi}{3}}\frac{1-2\cos x}{x-\frac{\pi}{3}}=\frac{1-2\times\frac{1}{2}}{\frac{\pi}{3}-\frac{\pi}{3}}=\frac{0}{0}\qquad (\text{I.C.})$$

Let $x-\frac{\pi}{3}=t\Rightarrow x=t+\frac{\pi}{3}$. If $x\to\frac{\pi}{3}, t\to 0$

$$\lim_{t\to 0}\frac{1-2\cos\left(t+\frac{\pi}{3}\right)}{t}=\lim_{t\to 0}\frac{1-2\left(\cos t\cos\frac{\pi}{3}-\sin t\sin\frac{\pi}{3}\right)}{t}$$

$$=\lim_{t\to 0}\frac{1-2\left(\frac{1}{2}\cos t-\frac{\sqrt{3}}{2}\sin t\right)}{t}$$

$$=\lim_{t\to 0}\frac{1-\cos t+\sqrt{3}\sin t}{t}$$

$$=\lim_{t\to 0}\left(\frac{1-\cos t}{t}+\frac{\sqrt{3}\sin t}{t}\right)$$

$$=\lim_{t\to 0}\frac{1-\cos t}{t}+\lim_{t\to 0}\frac{\sqrt{3}\sin t}{t}$$

$$=\lim_{t\to 0}\frac{1-\cos t}{t}+\sqrt{3}\lim_{t\to 0}\frac{\sin t}{t}$$

$$=0+\sqrt{3}\times 1$$

$$=\sqrt{3}$$

Example 4.19

Evaluate $\lim_{x\to 3}\frac{x-3}{\sin \pi x}$

Solution

$\lim_{x\to 3}\frac{x-3}{\sin \pi x}=\frac{0}{0}$ I.C

Let $t=x-3 \Rightarrow x=t+3$. If $x\to 3,\ t\to 0$

$$\lim_{x\to 3}\frac{x-3}{\sin \pi x}=\lim_{t\to 0}\frac{t}{\sin\left(\pi t+3\pi\right)}$$

$$=\lim_{t\to 0}\frac{t}{\sin \pi t\cos 3\pi+\cos \pi t\sin 3\pi}$$

$$=\lim_{t\to 0}\frac{t}{-\sin \pi t} \qquad \left[\textit{Since}\quad \cos 3\pi=-1,\ \sin 3\pi=0\right]$$

$$=-\lim_{t\to 0}\frac{\frac{t}{\pi t}}{\frac{\sin \pi t}{\pi t}}=-\lim_{t\to 0}\frac{\frac{1}{\pi}}{\frac{\sin \pi t}{\pi t}}=-\frac{1}{\pi}$$

Application activity 4.6

Find the limit of the following functions:

1. $\lim_{\theta\to\frac{\pi}{4}}\left(\theta\tan\theta\right)$
2. $\lim_{x\to\frac{\pi}{2}}\frac{\cos^2 x}{1-\sin x}$
3. $\lim_{t\to 0}\frac{1-\cos t}{\sin t}$
4. $\lim_{t\to 0}\frac{\sin 3t}{t}$

4.2.2. Limits of inverse trigonometric functions

Activity 4.7

1. Find the exact value of:

   a) $\cos^{-1}(-1)$ b) $\tan^{-1}(-1)$

   c) $\csc^{-1}(1)$ d) $\cos^{-1}\left(\frac{1}{2}\right)$

2. Evaluate the following limits;

   a) $\lim_{x\to 1}\cos^{-1}(1-2x^2)$ b) $\lim_{x\to 0}\tan^{-1}(x-1)$

   c) $\lim_{x\to 1}\cos^{-1}\frac{x-1}{1-x^2}$ d) $\lim_{x\to 1}\frac{x^2-1}{\sin^{-1}(1-x)}$

We can also evaluate the limits of inverse trigonometric functions. We find the numerical value of the given function at given value and see if the result is indeterminate case or not. One of the methods used to remove indeterminate case is l'Hôpital's rule:

Recall that if $\lim_{x\to a}\frac{f(x)}{g(x)}=\frac{0}{0}$ or $\lim_{x\to a}\frac{f(x)}{g(x)}=\frac{\infty}{\infty}$, we remove this indeterminate case by differentiating function $f(x)$ and $g(x)$ and then evaluate the limit. That is if $\lim_{x\to a}\frac{f(x)}{g(x)}=\frac{0}{0}$ or $\lim_{x\to a}\frac{f(x)}{g(x)}=\frac{\infty}{\infty}$ then we evaluate $\lim_{x\to a}\frac{f'(x)}{g'(x)}$.

We do this until the indeterminate case is removed. Other methods used to remove indeterminate cases are also applied.

Example 4.20

Evaluate $\lim_{x\to +\infty}\cos^{-1}\left(\frac{1}{x}\right)$

Solution

$$\lim_{x\to+\infty}\cos^{-1}\left(\frac{1}{x}\right)=\cos^{-1}\left(\frac{1}{+\infty}\right)$$
$$=\cos^{-1}(0)$$
$$=\frac{\pi}{2}$$

Example 4.21

Evaluate $\lim_{x\to+\infty}\cos^{-1}\left(\frac{x-1}{2x}\right)$

Solution

$$\lim_{x\to+\infty}\cos^{-1}\left(\frac{x-1}{2x}\right)=\cos^{-1}\left(\frac{\infty}{\infty}\right)\ \text{I.C}$$

Remove this indeterminate case by l'Hôpital's rule

$$\lim_{x\to+\infty}\cos^{-1}\left(\frac{x-1}{2x}\right)=\lim_{x\to+\infty}\cos^{-1}\left(\frac{1}{2}\right)$$
$$=\frac{\pi}{3}$$

Application activity 4.7

Evaluate the following limits

1. $\lim_{x\to-2}\cos^{-1}\frac{x+1}{\sqrt{x^2-2}}$
2. $\lim_{x\to4}\sec^{-1}\sqrt{\frac{x}{x-1}}$
3. $\lim_{x\to-1}\sec^{-1}\frac{x-1}{\sqrt{1+x^2}}$
4. $\lim_{x\to-1}\tan^{-1}\frac{1-x^2}{2x+2}$

4.3. Differentiation of trigonometric functions and their inverses

4.3.1. Derivative of sine and cosine

Activity 4.8

1. Using definition of derivative, find the derivative of $\sin x$.
2. Use result in 1) and relation $\cos x = \sin\left(\frac{\pi}{2} - x\right)$ to find the derivative of $\cos x$.

The functions $f(x) = \sin x$ and $f(x) = \cos x$ are differentiable on the set of real numbers. In addition, from activity 4.8,

$(\sin x)' = \cos x$ and $(\cos x)' = -\sin x$.

After differentiation of composite functions, if u is another function, then $(\sin u)' = u'\cos u$ and $(\cos u)' = -u'\sin u$

Example 4.22

Find the derivative of $f(x) = \sin(3x^2 + 4)$

Solution

$$f'(x) = (3x^2 + 4)'\cos(3x^2 + 4)$$
$$= 6x\cos(3x^2 + 4)$$

Example 4.23

Find the derivative of $f(x) = \cos(3x)$

Solution

$$f'(x) = -(3x)'\sin(3x)$$
$$= -3\sin(3x)$$

Application activity 4.8

Find the derivative of the following functions:

1. $f(x)=\sin(x^2+3)$
2. $f(x)=\sin^3(x^2+4)$
3. $f(x)=\cos 3x^2$
4. $f(x)=\cos^3 2x$

4.3.2. Derivative of tangent and cotangent

Activity 4.9

1. Use rule for derivative of a quotient and the relation $\tan x=\frac{\sin x}{\cos x}$ to find the derivative of $\tan x$.
2. Use result in 1) and relation $\cot x=\tan\left(\frac{\pi}{2}-x\right)$ to find the derivative of $\cot x$.

The function $f(x)=\tan x$ is differentiable on $\mathbb{R}\setminus\left\{\frac{\pi}{2}+k\pi\right\}\ k\in\mathbb{Z}$ and the function $f(x)=\cot x$ is differentiable on $\mathbb{R}\setminus\{k\pi\}\ k\in\mathbb{Z}$. In addition, from activity 4.9,

$\forall x\neq\frac{\pi}{2}+k\pi, k\in\mathbb{Z}$

$$(\tan x)'=\frac{1}{\cos^2 x}$$
$$=\sec^2 x$$
$$=1+\tan^2 x$$

Thus, $(\tan x)'=1+\tan^2 x$

If u is another, function then,

$$(\tan u)'=\frac{u'}{\cos^2 u}$$
$$=u'\sec^2 u$$
$$=u'(1+\tan^2 u)$$

Thus, $(\tan u)'=u'(1+\tan^2 u)$

$\forall x\neq k\pi, k\in\mathbb{Z}$

$$(\cot x)'=\frac{-1}{\sin^2 x}$$
$$=-\csc^2 x$$
$$=-(1+\cot^2 x)$$

Thus, $(\cot x)'=-(1+\cot^2 x)$

If u is another function, then,

$$(\cot u)'=\frac{-u'}{\sin^2 u}$$
$$=-u'\csc^2 u$$
$$=-u'(1+\cot^2 u)$$

Thus, $(\cot u)'=-u'(1+\cot^2 u)$

Example 4.24

Find the derivative of $f(x)=x^2\tan x$.

Solution

$$f'(x)=\left(x^2\right)'\tan x+x^2(\tan x)'$$
$$=2x\tan x+x^2\sec^2 x$$

Example 4.25

Find the derivative $f(x)=\cot x^2$.

Solution

$$f'(x)=-\left(x^2\right)'\csc^2 x^2$$
$$=-2x\csc^2 x^2$$

Application activity 4.9

Find the derivative of the following functions:

1. $f(x)=x\tan x$
2. $f(x)=\tan(3x+2)$
3. $f(x)=\cot\left(x^2-5\right)$
4. $f(x)=\sin x\cot 4x$

4.3.3. Derivative of secant and cosecant

Activity 4.10

1. Use rule for derivative of reciprocal of a function and relation $\sec x=\dfrac{1}{\cos x}$ to find the derivative of $\sec x$.
2. Use rule for derivative of reciprocal of a function and relation $\csc x=\dfrac{1}{\sin x}$ to find the derivative of csc*x*.

From activity 4.10,

$(\sec x)'=\sec x\tan x$ and $(\csc x)'=-\csc x\cot x$

If u is another function, then

$(\sec u)'=u'\sec u\tan u$ and $(\csc u)'=-u'\csc u\cot u$

Example 4.26

Find the derivative of $f(x)=\sec(2x+1)$.

Solution

$f'(x)=2\sec(2x+1)\tan(2x+1)$

Example 4.27

Find the derivative of $f(x)=\csc(x^2+1)$.

Solution

$f'(x)=-2x\csc(x^2+1)\cot(x^2+1)$

Application activity 4.10

Find the derivative of the following functions:

1. $f(x)=\sec(3x+2)$
2. $f(\theta)=\theta^3\csc 2\theta$
3. $f(x)=\sec^4 3x$

4.3.4. Derivative of inverse sine and inverse cosine

Activity 4.11

1. We know that $f(x)=\sin^{-1}x$ for $x\in[-1,1]$ and $x=\sin y$ for $y\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$ where $y=f(x)$. Use rule for derivative of composite functions to find the derivative of $\sin^{-1}x$, the inverse of sine function.
2. We also know that $f(x)=\cos^{-1}x$ for $x\in[-1,1]$ and $x=\cos y$ for $y\in[0,\pi]$ where $y=f(x)$. Use rule for derivative of composite functions to find the derivative of $\cos^{-1}x$, the inverse of cosine function.

From activity 4.11,

$\forall\ x \in \left]-1,1\right[$,

$\left(\sin^{-1} x\right)' = \dfrac{1}{\sqrt{1-x^2}}$ and $\left(\cos^{-1} x\right)' = \dfrac{-1}{\sqrt{1-x^2}}$

If u is another function, then

$\left(\sin^{-1} u\right)' = \dfrac{u'}{\sqrt{1-u^2}}$ and $\left(\cos^{-1} u\right)' = \dfrac{-u'}{\sqrt{1-u^2}}$

Example 4.28

Find the derivative of $f(x) = \sin^{-1} x^3$.

Solution

$$f'(x) = \frac{\left(x^3\right)'}{\sqrt{1-\left(x^3\right)^2}} = \frac{3x^2}{\sqrt{1-x^6}}$$

Example 4.29

Find the derivative of $f(x) = \cos^{-1}(2x+1)$.

Solution

$$f'(x) = \frac{-(2x+1)'}{\sqrt{1-(2x+1)^2}}$$

$$= \frac{-2}{\sqrt{1-4x^2-4x-1}}$$

$$= \frac{-1}{\sqrt{-x^2-x}}$$

Example 4.30

Find the derivative of $y = \sin^{-1}\left(1-x^2\right)$.

Solution

$$y' = \frac{-2x}{\sqrt{1-\left(1-x^2\right)^2}} = \frac{-2x}{\sqrt{-x^4+2x^2}}$$

Example 4.31

Find the derivative of $y = 3\cos^{-1}\left(x^2+0.5\right)$.

Solution

$$y' = 3\frac{-2x}{\sqrt{1-\left(x^2+0.5\right)^2}} = \frac{-6x}{\sqrt{0.75-x^2-x^4}}$$

Example 4.32

Find the derivative of $y = \left(x^2+1\right)\sin^{-1}4x$.

Solution

$$y' = (2x)\sin^{-1}4x + \left(x^2+1\right)\frac{4}{\sqrt{1-\left(4x^2\right)^2}} = \frac{4\left(x^2+1\right)}{\sqrt{1-16x^2}} + 2x\sin^{-1}4x$$

Application activity 4.11

Find the derivative of the following functions:

1. $f(x) = \cos^{-1}\frac{1}{x}$
2. $f(x) = \cos^{-1}x^2$
3. $f(x) = \sin^{-1}(1-x)$
4. $f(x) = \sin^{-1}\sqrt{2x}$

4.3.5. Derivative of inverse tangent and inverse cotangent

Activity 4.12

1. We know that $f(x)=\tan^{-1}x$ for $x\in\mathbb{R}$ and $x=\tan y$ for $y\in\left]-\frac{\pi}{2},\frac{\pi}{2}\right[$ where $y=f(x)$. Use rule for derivative of composite functions to find the derivative of $\tan^{-1}x$; the inverse of tangent function.
2. We also know that $f(x)=\cot^{-1}x$ for $x\in\mathbb{R}$ and $x=\cot y$ for $y\in\left]0,\pi\right[$ where $y=f(x)$. Use rule for derivative of composite functions to find the derivative of $\cot x$; the inverse of cotangent function.

From activity 4.12,

$$\left(\tan^{-1}x\right)'=\frac{1}{1+x^2} \text{ and } \left(\cot^{-1}x\right)'=\frac{-1}{1+x^2}$$

If u is another function, then

$$\left(\tan^{-1}u\right)'=\frac{u'}{1+u^2}, \text{ and } \left(\cot^{-1}u\right)'=\frac{-u'}{1+u^2}$$

Example 4.33

Find the derivative of $f(x)=\left(\tan^{-1}2x\right)^4$.

Solution

$$f'(x)=4\left(\tan^{-1}2x\right)^3\left(\tan^{-1}2x\right)'$$

$$=4\left(\tan^{-1}2x\right)^3\left(\frac{2}{1+4x^2}\right)$$

$$=\frac{8\left(\tan^{-1}2x\right)^3}{1+4x^2}$$

Example 4.34

Find the derivative of $f(x)=2\cot^{-1}3x$.

Solution

$$f'(x)=\frac{-2(3x)'}{1+(3x)^2}=\frac{-6}{1+9x^2}$$

Application activity 4.12

Find the derivative of the following functions:

1. $f(x)=\cot^{-1}\sqrt{x}$
2. $f(x)=\cos^{-1}\frac{1}{x}-\cot^{-1}x$
3. $f(x)=\cot^{-1}\sqrt{x-1}$

Derivative of inverse secant and inverse cosecant

Activity 4.13

1. We know that $f(x)=\sec^{-1}x$ for $x\le -1$ or $x\ge 1$ and $x=\sec y$ for $y\in[0,\pi], y\neq\frac{\pi}{2}$ where $y=f(x)$. Use rule for derivative of composite functions to find the derivative of $\sec^{-1}x$; the inverse of secant function.
2. We know that $f(x)=\csc^{-1}x$ for $x\le -1$ or $x\ge 1$ and $x=\csc y$ for $y\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right], y\neq 0$ where $y=f(x)$. Use rule for derivative of composite functions to find the derivative of, $\csc^{-1}x$; the inverse of cosecant function.

From activity 4.13,

$$\left(\sec^{-1}x\right)'=\frac{1}{x\sqrt{x^2-1}} \text{ and } \left(\csc^{-1}x\right)'=\frac{-1}{x\sqrt{x^2-1}}$$

If u is another function, then

$$\left(\sec^{-1}u\right)'=\frac{u'}{u\sqrt{u^2-1}} \text{ and } \left(\csc^{-1}u\right)'=\frac{-u'}{u\sqrt{u^2-1}}$$

Example 4.35

Find the derivative of $f(x)=\sec^{-1}2x$.

Solution

$$f'(x)=\frac{2}{2x\sqrt{4x^2-1}}$$

Example 4.36

Find the derivative of $f(x)=\csc^{-1}\sqrt{x}$.

Solution

$$f'(x)=\frac{-\frac{1}{2\sqrt{x}}}{\sqrt{x}\sqrt{\left(\sqrt{x}\right)^2-1}}=\frac{-1}{2x\sqrt{x-1}}$$

Application activity 4.13

Find the derivative of the following functions:

1. $f(x)=\sec^{-1}(2x+1)$
2. $f(x)=\sec^{-1}5x$
3. $f(x)=\csc^{-1}(x^2+1),\ x>0$

4.3.6. Successive derivatives

Activity 4.14

Consider the function $g(x)=\sin(4x)$. Find

1. $g'(x)$
2. the derivative of the function obtained in 1.
3. the derivative of the function obtained in 2.
4. the derivative of the function obtained in 3.
5. the derivative of the function obtained in 4.

We have seen that the derivative of a function of x is in general also a function of x. This new function may also be differentiable, in which case the derivative of the first derivative is called the **second derivative** of the original function.

Similarly, the derivative of the second derivative is called the **third derivative** and so on.

The successive derivatives of a function f are **higher order derivatives** of the same function.

We denote higher order derivatives of the same function as follows:

The second derivative is:

$$\frac{d}{dx}\left(\frac{dy}{dx}\right)=\frac{d^2y}{dx^2}=f''(x)=y''$$

The third derivative is:

$$\frac{d}{dx}\left(\frac{d^2y}{dx^2}\right)=\frac{d^3y}{dx^3}=f'''(x)=y'''$$

And the nth derivative is:

$$\frac{d}{dx}\left(\frac{d^{n-1}y}{dx^{n-1}}\right)=\frac{d^ny}{dx^n}=f^{(n)}(x)=y^{(n)}$$

Example 4.37

Find the nth derivative of $y=\sin x$

Solution

$$y'=\cos x=\sin\left(x+\frac{\pi}{2}\right)$$

$$y''=-\sin x=\sin\left(x+\frac{2\pi}{2}\right)$$

$$y'''=-\cos x=\sin\left(x+\frac{3\pi}{2}\right)$$

$\vdots$

$$y^{(n)}=\sin\left(x+\frac{n\pi}{2}\right)$$

Thus, if $y=\sin x$, $y^{(n)}=\sin\left(x+\frac{n\pi}{2}\right)$

Example 4.38

Find the n[th] derivative of $y=\cos x$

Solution

$$y'=-\sin x=\cos\left(x+\frac{\pi}{2}\right)$$

$$y''=-\cos x=\cos\left(x+\frac{2\pi}{2}\right)$$

$$y'''=\sin x=\sin\left(x+\frac{3\pi}{2}\right)$$

$$\vdots$$

$$y^{(n)}=\cos\left(x+\frac{n\pi}{2}\right)$$

Thus, if $y=\cos x$ $y^{(n)}=\cos\left(x+\frac{n\pi}{2}\right)$

Application activity 4.14

1. Find the first, second and third derivatives of

   a) $y=\tan x$ b) $y=\sec x$

   c) $y=\cos\left(x^2\right)$ d) $y=\frac{\sin x}{x}$

2. Find the n^{th} derivative of

   a) $f\left(x\right)=\cos^2 x$ b) $f\left(x\right)=\sin 2x\cos 3x$

4.4. Applications

Simple harmonic motion

Activity 4.15

Discuss how differentiation of trigonometric functions is used to find the velocity, acceleration and jerk of a moving object knowing the function representing its position.

In simple harmonic motion (SHM), the displacement $x(t)$ of a particle from its equilibrium position is defined by the equation $x = x_m \cos(\omega t + \phi)$ in which x_m is the **amplitude** of the displacement, the quantity $(\omega t + \phi)$ is **phase** of the motion, and ϕ is the **phase constant**. The **angular frequency** ω is related to the period and the frequency of the motion by

$$\omega = \frac{2\pi}{T} = 2\pi f .$$

The motion of an object or weight bobbing freely up down with no resistance on the end of a spring is an example of simple harmonic motion. The motion is periodic and repeats indefinitely, so we represent it using trigonometric functions. If we have the function representing the position, say $S(t)$, then,

- The velocity of the object is $v = \frac{ds}{dt}$.
- The acceleration of the object is $a = \frac{d^2s}{dt^2}$.
- The jerk of the object is $j = \frac{d^3s}{dt^3}$.

Example 4.39

A weight hanging from a spring is stretched down 5 units beyond its rest position and released at time $t = 0$ to bob up down. Its position at any later time t is $s = 5\cos t$. What are its velocity, acceleration and jerk at time t?

Solution

Position: $s = 5\cos t$

Velocity (derivative of function representing the position):

$$v = \frac{ds}{dt} = \frac{d}{dt}(5\cos t) = -5\sin t$$

Acceleration (derivative of function representing the velocity):

$$a = \frac{dv}{dt} = \frac{d}{dt}(-5\sin t) = -5\cos t$$

Jerk (derivative of function representing the acceleration):

$$j = \frac{da}{dt} = \frac{d}{dt}(-5\cos t) = 5\sin t$$

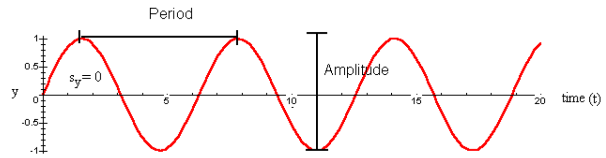
Pulmonary function testing

In addition to that, sine waves are also used in pulmonary function testing (we can describe any repeating rhythm by a sine wave):

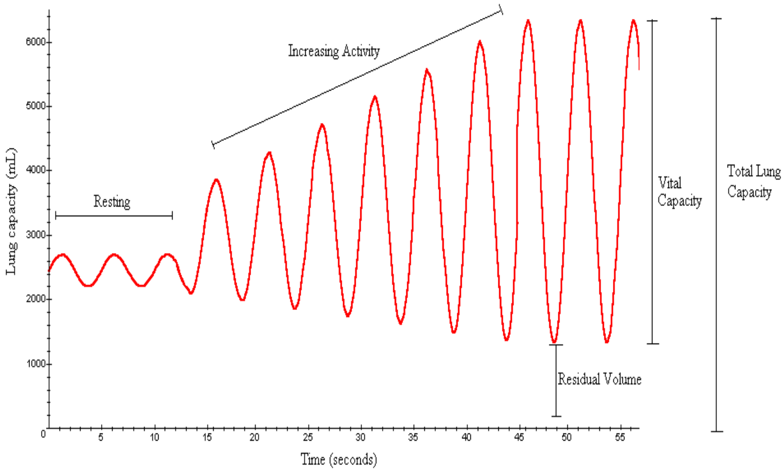
$y = \left(\frac{A}{2}\right)\sin\left(\frac{2\pi}{p}(t)\right) + s_y$, where $\boldsymbol{y}$ is some property that exhibits a rhythm, $\boldsymbol{t}$ is time, $\boldsymbol{A}$ is the amplitude, $\boldsymbol{p}$ is the period, and $\boldsymbol{s_y}$ is a phase shift in the height of $\boldsymbol{y}$ compared to the simplest sine wave ($y = \sin t$, where $\boldsymbol{s_y} = 0$).

We therefore have the following equation for normal breathing:

$$Lung\ \ capacity = \left(\frac{500}{2}\right)\sin\left(\frac{2\pi}{5}(t)\right) + 2450$$



As the degree of activity increases, the volume inhaled and exhaled increases from 500 up to 3000 and the functional residual volume decreases as greater amounts of air are exhaled. In our equation for lung capacity, this is reflected as an increase in amplitude (A) and a decrease in functional residual capacity.



The sinusoidal wave figure above shows the lungs capacity with respect to time. It describes the rhythm of the lungs in normal breathing, the rhythm of the lungs when there is increasing of the activity, the residual volume when lungs increase their activity, the vital capacity and total lung capacity.

Application activity 4.15

1. A body oscillates with simple harmonic motion according to the equation $x = 6\cos\left(3\pi t + \frac{\pi}{3}\right)$ $(x\,in\,metre)$

At time $t = 2s$, what are

a) the displacement b) the velocity
c) the acceleration d) the phase of motion
e) the frequency f) the period of the motion.

2. An object oscillates with simple harmonic motion along the $x - axis$. Its displacement from the origin varies in metre with time according to the equation $x = 4\cos\left(\pi t + \frac{\pi}{4}\right)$ where t is in seconds and the angles

in radians.

a) Determine the amplitude, frequency, period of motion and angular frequency.
b) Calculate the velocity and acceleration of the object at any time.
c) Find displacement, velocity and acceleration at $t = 1$
d) Determine the maximum speed and maximum acceleration

Unit Summary

1. Domain and range of trigonometric functions

Function	Domain	Range
$y=\sin x$	$\mathbb{R}$	$-1\le y\le 1$
$y=\cos x$	$\mathbb{R}$	$-1\le y\le 1$
$y=\tan x$	$\mathbb{R}\setminus\left\{\frac{\pi}{2}+k\pi\right\},\ k\in\mathbb{Z}$	$\mathbb{R}$
$y=\csc x$	$\mathbb{R}\setminus\{k\pi\},k\in\mathbb{Z}$	$y\le -1\ or\ y\ge 1$
$y=\sec x$	$\mathbb{R}\setminus\left\{\frac{\pi}{2}+k\pi\right\},k\in\mathbb{Z}$	$y\le -1\ or\ y\ge 1$
$y=\cot x$	$\mathbb{R}\setminus\{k\pi\},k\in\mathbb{Z}$	$\mathbb{R}$

2. Domain and range of inverses of trigonometric functions

Function	Domain	Range
$y=\sin^{-1}x$	$-1\le x\le 1$	$-\frac{\pi}{2}\le y\le\frac{\pi}{2}$
$y=\cos^{-1}x$	$-1\le x\le 1$	$0\le y\le\pi$
$y=\tan^{-1}x$	$\mathbb{R}$	$-\frac{\pi}{2}< y<\frac{\pi}{2}$
$y=\csc^{-1}x$	$x\le -1\ or\ x\ge 1$	$-\frac{\pi}{2}\le y\le\frac{\pi}{2}, y\ne 0$
$y=\sec^{-1}x$	$x\le -1\ or\ x\ge 1$	$0\le y\le\pi, y\ne\frac{\pi}{2}$
$y=\cot^{-1}x$	$\mathbb{R}$	$0< y<\pi$

3. A function $f(x)$ is said to be even if the following conditions are satisfied:

   - $\forall x\in Domf, -x\in Domf$
   - $f(-x)=f(x)$

The graph of such function is **symmetric about the vertical axis.** i.e $x=0$

4. A function $f(x)$ is said to be odd if the following conditions are satisfied:

   - $\forall x \in Domf, -x \in Domf$
   - $f(-x) = -f(x)$

The graph of such a function looks the same when rotated through half a revolution about 0. This is called **rotational symmetry**.

5. A function *f* is called **periodic** if there is a positive number P such that $f(x+P) = f(x)$ whenever x and $x+P$ lie in the domain of *f*. We call P a **period** of the function.

6. When finding limit of trigonometric functions, we use the result saying that $\lim_{x\to 0}\frac{\sin x}{x} = 1$

7. Derivative of trigonometric functions and their inverses:

$(\sin u)' = u'\cos u$, $(\cos u)' = -u'\sin u$

$$(\tan u)' = \frac{u'}{\cos^2 u} = u'\sec^2 u = u'(1+\tan^2 u)$$

$$(\cot u)' = \frac{-u'}{\sin^2 u} = -u'\csc^2 u = -u'(1+\cot^2 u)$$

$(\sec u)' = u'\sec u \tan u$, $(\csc u)' = -u'\csc u \cot u$

$$(\sin^{-1} u)' = \frac{u'}{\sqrt{1-u^2}}, \quad (\cos^{-1} u)' = \frac{-u'}{\sqrt{1-u^2}}$$

$$(\tan^{-1} u)' = \frac{u'}{1+u^2}, \quad (\cot^{-1} u)' = \frac{-u'}{1+u^2}$$

$$(\sec^{-1} u)' = \frac{u'}{u\sqrt{u^2-1}}, \quad (\csc^{-1} u)' = \frac{-u'}{u\sqrt{u^2-1}}$$

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End of Unit Assessment

1. State whether each of the following functions is periodic. If the function is periodic, give its fundamental period:

   a) $f(x)=\sin 3x$ b) $f(x)=1+\tan x$

   c) $f(x)=\cos(x+1)$ d) $f(x)=\cos(x^2)$

   e) $f(x)=\cos^2 x$ f) $f(x)=x+\sin x$

2. Study the parity of the following functions:

   a) $f(x)=\cos x+\sin x$ b) $f(x)=\dfrac{\sin x}{x^2+1}$

   c) $f(x)=\dfrac{\sin x}{x^2+1}$ d) $f(x)=\dfrac{x+\sin x}{x^2}$

3. Find the limit of the following functions:

   a) $\lim\limits_{x\to\frac{\pi}{4}}(1+\cot x)$ b) $\lim\limits_{x\to 0}\dfrac{1-\cos^3 x}{\sin^2 x}$

   c) $\lim\limits_{x\to 0}\dfrac{1+\sin x}{1+\cos x}$ d) $\lim\limits_{x\to\frac{\pi}{4}}\dfrac{\csc^2 x-2}{\cot x-1}$

   e) $\lim\limits_{x\to 0}\dfrac{\sin 7x}{x}$ f) $\lim\limits_{x\to 0}\dfrac{\sin 8x}{\sin 5x}$

   g) $\lim\limits_{x\to 0}\dfrac{\sin^2 \frac{x}{2}}{4x^2}$ h) $\lim\limits_{x\to 0}\dfrac{\sin^2(-11x)}{\tan 9x}$

   i) $\lim\limits_{x\to 0}\dfrac{\sin 3x\sin 5x}{7x^2}$ j) $\lim\limits_{x\to 3}\dfrac{\sin(x^2-3x)}{x^2-9}$

   k) $\lim\limits_{x\to 1}\dfrac{(x^2-x)\sin(x-1)}{x^2-2x+1}$ l) $\lim\limits_{x\to a}\dfrac{\sin x-\sin a}{x-a}$

   m) $\lim\limits_{x\to 0}\dfrac{\sin(a+x)+\sin(a-x)-2\sin a}{x\sin x}$

   n) $\lim\limits_{x\to 0}\dfrac{\sec x-1}{x^2}$ o) $\lim\limits_{x\to 0}\dfrac{3\sin x-\sin 3x}{x^3}$

   p) $\lim\limits_{x\to 0}x^2\left(\sin\dfrac{1}{x}\right)\csc x$ q) $\lim\limits_{x\to 0}\dfrac{\sec 9x-\sec 7x}{\sec 5x-\sec 3x}$

r) $\lim_{x\to\frac{\pi}{2}} \frac{\sec x - \tan x}{\frac{\pi}{2} - x}$

s) $\lim_{x\to 0} \frac{x^2 + \sin 3x}{2x + \tan 2x}$

t) $\lim_{\theta\to\frac{\pi}{2}} \frac{\sec\theta - \tan\theta}{\frac{\pi}{2} - \theta}$

u) $\lim_{x\to 1} \frac{1 + \cos \pi x}{(1-x)^2}$

v) $\lim_{x\to\pi} \frac{\sqrt{5 + \cos x} - 2}{\pi - x}$

w) $\lim_{x\to\frac{\pi}{4}} \frac{\sqrt{2} - \cos x - \sin x}{(4x - x)^2}$

4. Find the first derivative of the following functions:

a) $f(x) = 3\sec x - 10\,\text{cotx}$

b) $f(x) = 3x^{-4} - x^2 \tan x$

c) $y = 5\sin x \cos x + 4\csc x$

d) $P(t) = \frac{\sin t}{3 - 2\cos t}$

e) $y = 4\cos(6x^2 + 5)$

f) $y = 3\sin^3(2x^4 + 1)$

g) $y = (x - \cos^2 x)^4$

h) $y = \frac{2x + 3}{\sin 4x}$

i) $y = x\sqrt{1 - x^2} + \cos^{-1} x$

j) $y = \sec^{-1} \frac{1}{x}$

k) $y = \csc^{-1} \frac{x}{2}$

l) $y = \sqrt{x^2 - 1} - \sec^{-1} x$

m) $y = x\sin^{-1} x + \sqrt{1 - x^2}$

5. Suppose that the amount of money in a bank account is given by $P(t) = 500 + 100\cos t - 150\sin t$ where t is in years. During the first 10 years in which the account is open, when is the amount of money in the account increasing?

6. Evaluate the following limits

a) $\lim_{x\to 2} \tan^{-1} \frac{1 - 2x}{\sqrt{x + 1}}$

b) $\lim_{x\to t} \sin(\cos^{-1} x)$

c) $\lim_{x\to 1} \sin^{-1} \frac{1 + x}{2x}$

d) $\lim_{x\to 0^-} \frac{\tan^{-1} \frac{1}{x}}{\cos^{-1} x}$

e) $\lim_{x\to 0} \cos^{-1}\left(\frac{\sqrt{x+1} - 1}{x}\right)$

f) $\lim_{x\to -1} \tan^{-1}\left(\frac{1 - x^2}{2x + 2}\right)$

Unit 5

Vector Space of Real Numbers

Introductory activity

A vector space (also called a linear space) is a collection of objects called vectors, which may be added together and multiplied by numbers, called scalars in this context.

To put it really simple, vectors are basically all about directions and magnitudes. These are critical in basically all situations.

In physics, vectors are often used to describe forces, and forces add as vectors do.

a) Discuss the properties of addition of vectors.

b) What happens when a vector is multiplied by a scalar(real number)?

c) Give at least 3 examples of vectors in real life.

Objectives

By the end of this unit, a student will be able to:

- define and apply different operations on vectors.
- Calculate the scalar and vector product of two vectors.
- calculate the angle between two vectors.
- apply and transfer the skills of vectors to other area of knowledge.

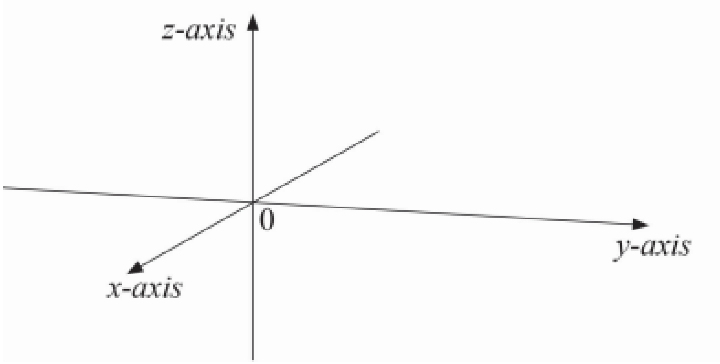
5.1. Vectors and operations in R3

Activity 5.1

1. In space, from the point A (3, 2, 2), along x-axis measure OM = 3 units, measure MN = 2 units parallel to y-axis, then measure NP = 2 units parallel to z-axis. OM, MN and NP are coordinates of point A in space.
2. In the same space, present the point B (1,3,2) and then join points A and B with arrow from A to B.
3. Find B-A

Position of points and vectors in 3 dimensions

In plane, the position of a point is determined by two coordinates x and y obtained with reference to two straight lines ($x-axis$ and $y-axis$ respectively) intersecting at right angle. The position of point in space is, however, determined by three coordinates x, y, z , obtained with reference to three straight lines ($x-axis$, $y-axis$ and $z-axis$ respectively) intersecting at right angles.



Meaning and components of a vector in 3 dimension

A vector is a directed line segment. That is to say, a vector has a given **length (magnitude)** and a given **direction**. The vector joining point A and point B is denoted by $\overrightarrow{AB}$ and its components are found by subtracting the coordinates of point A from the coordinates of point B.

For example, the components of vector $\overrightarrow{AB}$ defined by two points $A(a_1, a_2, a_3)$ and $B(b_1, b_2, b_3)$ are given by $(b_1, b_2, b_3) - (a_1, a_2, a_3)$.

Example 5.1

Given the points $A(3,-2,5)$ *and* $B(-1,3,2)$. Find vectors $\overrightarrow{AB}$ and $\overrightarrow{BA}$

Solution

$\overrightarrow{AB} = (-4,5,-3)$ and $\overrightarrow{BA} = (4,-5,3)$

Then a vector in space may be described by an ordered triple of coordinates (a, b, c).

The point A is called the **initial point** or **tail** of $\overrightarrow{AB}$ and B is called the **terminal point** or **tip**. If the initial point is fixed, the vector is called a **bound** or **localised vector**. All other vectors are called **free vectors**. The set of vectors of space is denoted by V.

A vector is entirely determined by only one of its couples or by only one of its representatives. Let the point 0 be fixed, as common origin of all representatives. This point 0 will be called the origin of the space E and define a bijection of the set of points of the space E on the set V of vectors of the space E.

The set of vectors of the space E with origin 0 is denoted by E_0 and

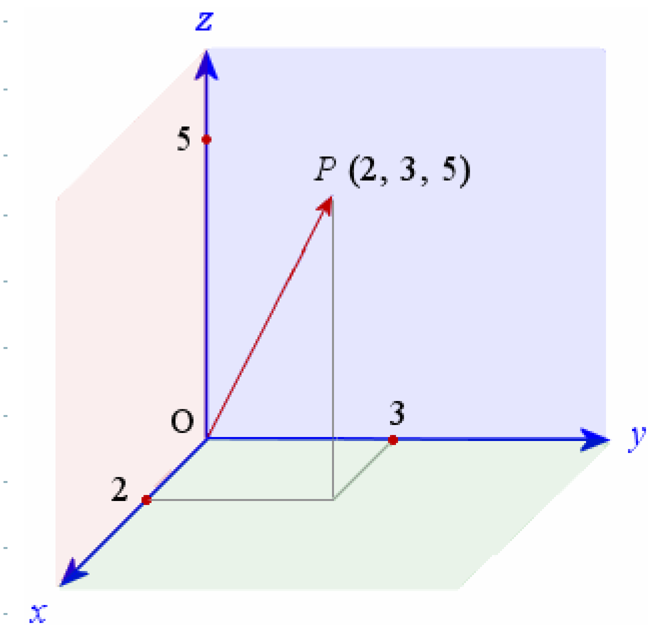
$$E_0 = \{\overrightarrow{OA} : a \in E\} \quad .$$

The position vector in 3 dimension

The vector $\overrightarrow{OP}$ joining the origin, 0, to the point P is called the **position vector** of P with respect to 0, or simply **the position vector of** P. We sometimes denote the position vector of P by $\vec{P}$.

That is $\overrightarrow{OP} = \vec{P}$.

Example: The vector $\overrightarrow{OP} = \vec{P}$ is described by an ordered triple of coordinates (2, 3, 5) as shown by the figure .



The zero vector (o,o,o) is denoted by $\vec{0}$

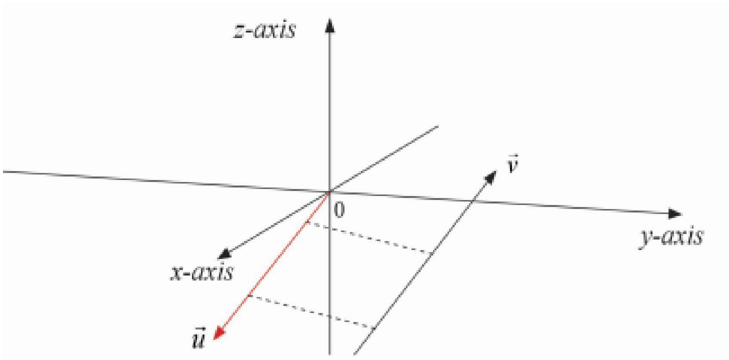
Parallel vectors in 3 dimensions

Two vectors are parallel if and only if

a) they have the same direction, or

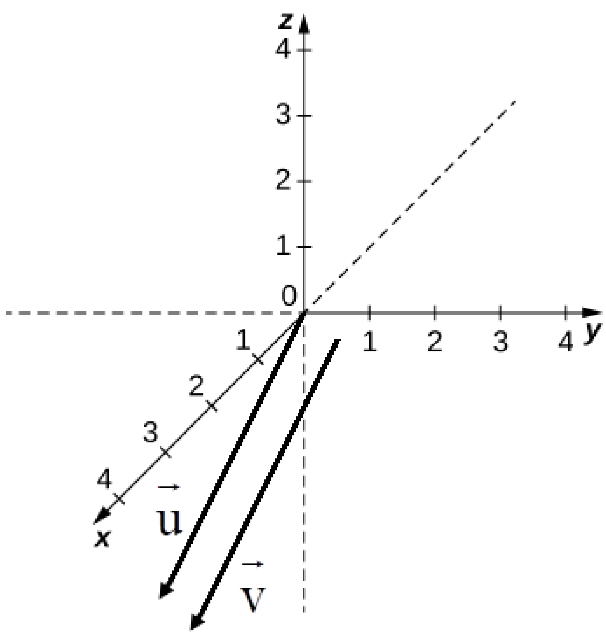
b) they have opposite directions.

Thus, two vectors are parallel if and only if one can be expressed as a scalar multiple of the other. i.e. if vector $\vec{U}$ is parallel to vector $\vec{V}$, then $\vec{U} = r\vec{V}$ or $\vec{V} = s\vec{U}$ for real numbers $\boldsymbol{r}$ and $\boldsymbol{s}$. In this case, we write $\vec{U} \parallel \vec{V}$



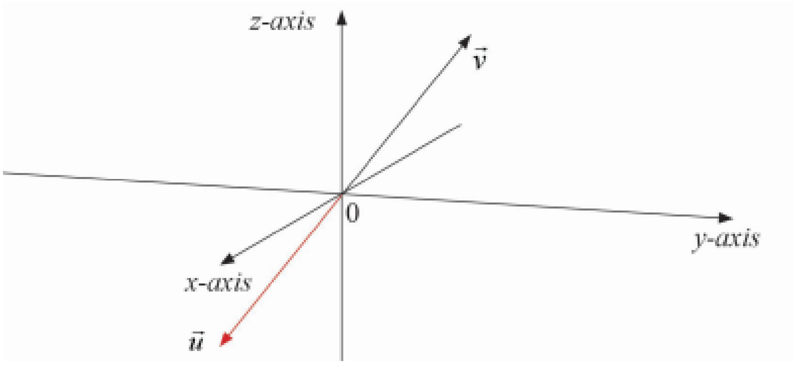
Equal vectors in 3 dimensions

Two vectors are equal if they have the same length and the same direction. If $\vec{U}$ is equal to $\vec{V}$, we write $\vec{U} = \vec{V}$.



Opposite vectors in 3 dimensions

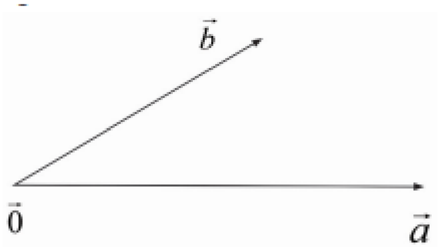
Two vectors are opposite if the coordinates of one vector are additive inverse of the coordinates of the other. That is, if $\vec{U}$ and $\vec{V}$ are opposite then $\vec{U} = -\vec{V}$. Then, $\vec{U} + \vec{V} = \vec{0}$



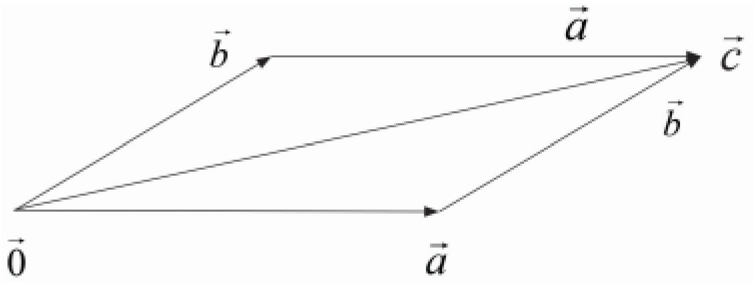
Operations on vectors in 3 dimensions

Sum of two vectors

Two non-parallel (or opposite) vectors of the same origin (means that their tails are together) determine one and only one plane in space.

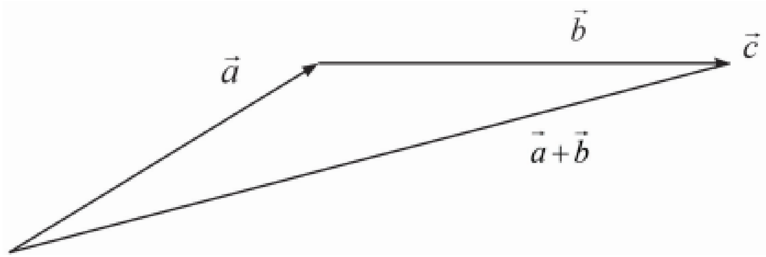


The addition of two vectors of the same origin is done by means of parallelogram technique or by using algebraic method.



$\vec{a}+\vec{b}=(a_1+b_1, a_2+b_2, a_3+b_3)$.

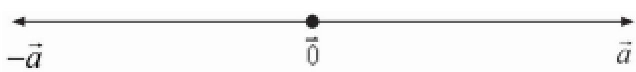
If the tails are not together, and the tail of $\vec{b}$ is joined to the tip of $\vec{a}$, then the sum $\vec{a}+\vec{b}$ is the vector joining the tail of $\vec{a}$ and the tip of $\vec{b}$.



Particular cases

1. If two vectors are **parallel**, to find the sum; the second is newly replaced by equal vector but having its origin at the end of the first one

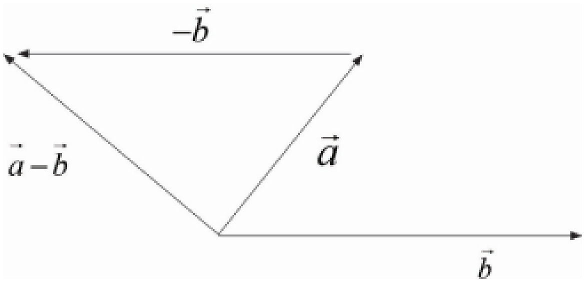
1. If two vectors are **opposite**, their sum is zero vector. The opposite of the vector $\vec{a}$ is denoted by $-\vec{a}$.

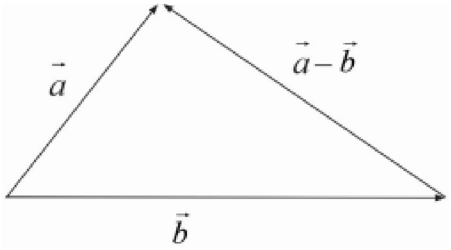


$\vec{a}+(-\vec{a})=\vec{0}$

From the addition of vectors, we define the **subtraction of vectors** as $\vec{a}-\vec{b}=\vec{a}+(-\vec{b})$.

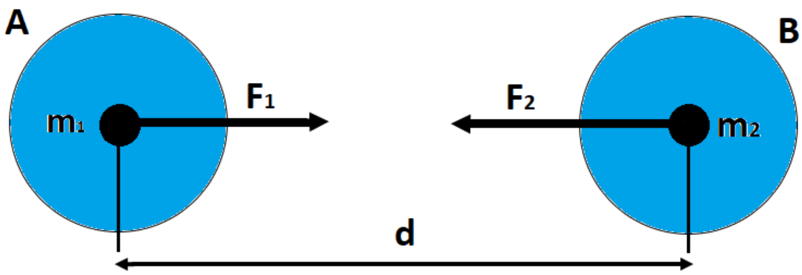
If $\vec{a}=(a_1, a_2, a_3)$ and $\vec{b}=(b_1, b_2, b_3)$ then $\vec{a}-\vec{b}=(a_1-b_1, a_2-b_2, a_3-b_3)$.





Example of vectors:

1) **Gravitational force as opposite vectors**



From the graph, the two objects A and B exerts forces in **opposite directions to attract each other (from Newton's Law)**

m_1 is the mass of the first object

m_2 is the mass of the second object

d is the distance from the centre of A to the centre of B

F_1 is the force exerted by A

F_2 is the force exerted by B

The gravitation force $\overrightarrow{F_G} = G\dfrac{m_1 \times m_2}{d^2}$ **unit vector,** where $G = 6.67 \times 10^{-11} Nm/kg^2$ is the gravitation constant.

2) **The weight** $\vec{F} = m.\vec{g} = m.(9.81)\vec{j}$, the force on each object of mass m which is oriented towards the center of earth, where $g = 9.81m/s^2$ and $\vec{j}$ is the unit vector towards the centre of earth.

Example 5.2

Calculate the force of gravity between block $B_1(20kg)$ and block $B_2(30kg)$ if the distance between them is 2m.

Solution

$m_1 = 20kg$

$m_2 = 30kg$

$d = 2m$

$G = 6.67\times10^{-11} Nm/kg^2$

$$F_G = G\frac{m_1\times m_2}{d^2}$$

$$F_G = 6.67\times10^{-11} N(m/kg)^2 \frac{20kg\times30kg}{(2m)^2}$$

$F_G = 66.7\times10^{-10} N\times15$

$F_G = 66.7\times10^{-10} N\times15$

$F_G = 1{,}000.05\times10^{-10}$

$F_G = 10.005\times10^{-8} N$

Properties of vectors under addition

1. The addition defined above verifies the closure property in E_0. That is, $\forall\vec{a},\vec{\mathrm{b}}\in E_0, \vec{a}+\vec{b}\in E_0$
2. It is commutative. That is, $\forall\vec{a},\vec{b}\in E_0, \vec{a}+\vec{b}=\vec{b}+\vec{a}\in E_0$

3. It is associative. That is, $\forall \vec{a}, \vec{b}, \vec{c} \in E_0, \left(\vec{a}+\vec{b}\right)+\vec{c}=\vec{a}+\left(\vec{b}+\vec{c}\right) \in E_0$
4. The identity element is zero vector. That is, $\forall \vec{a} \in E_0, \vec{0} \in E_0 : \vec{a}+\vec{0}=\vec{0}+\vec{a}=\vec{a}$
5. The symmetric element is the opposite of a vector. That is, $\forall \vec{a} \in E_0, \exists -\vec{a} \in E_0 : \vec{a}+(-\vec{a})=\vec{0}$

Scalar multiplication

The definition of scalar multiplication in space E_0 is the same as in plane. The multiplication of a vector $\vec{a}$ with a real number α is defined by $\alpha(\vec{a})$. **It change the magnitude of the vector.**

If $\vec{a}=(a_1, a_2, a_3)$, $\alpha\vec{a}=(\alpha\,\mathrm{a}_1, \alpha\,\mathrm{a}_2, \alpha\,\mathrm{a}_3)$

Note

If the real number α is positive, the resulting vector has the same direction as $\vec{a}$ and if it is negative the resulting vector has the opposite direction to that of $\vec{a}$

Properties of scalar multiplication

- **Associative property,** $\forall \vec{a} \in E_0,\ r, s \in \mathbb{R}; (rs)\vec{a}=r(s\vec{a})$
- **Distributive property with respect to addition of vectors**
  $\forall \vec{a}, \vec{b} \in E_0, r \in \mathbb{R}, r(\vec{a}+\vec{b})=r\vec{a}+r\vec{b}$
- **Distributive property with respect to addition of real numbers**
  $\forall \vec{a} \in E_0,\ r, s \in \mathbb{R}\ (r+s)\vec{a}=(r\vec{a})+(s\vec{a})$

- **1 is the identity for scalar multiplication**

  $\forall \vec{a} \in E_0, 1 \times \vec{a} = \vec{a}$

Application activity 5.1

Given points $A(6,0,-3)$ and $B(3,-3,0)$; vectors

$\vec{u} = (3,4,6)$ *and* $\vec{v} = (1,1,1)$.

Find;

1) Vector $\overrightarrow{AB}$
2) Sum $\overrightarrow{AB} + \vec{u} - \vec{v}$
3) Sum $2\overrightarrow{AB} - 3\vec{u} + \vec{v}$
4) Sum $4\vec{u} - \overrightarrow{AB} + 2\vec{v}$

5.2. Scalar product of two vectors

Activity 5.2

Use the formula $(a,b,c)\cdot(d,e,f) = ad + be + cf$ to find:

a) $(3,1,4)\cdot(-4,3,1)$ b) $(4,2,-2)\cdot(1,5,6)$

The scalar product or dot product (or sometimes inner product) is an algebraic operation that takes two coordinate vectors and returns a single number.

Algebraically, the scalar product of vectors $\vec{u} = (a_1, a_2, a_3)$ and $\vec{v} = (b_1,\ b_2,\ b_3)$ of space is defined by $\vec{u} \cdot \vec{v} = a_1b_1 + a_2b_2 + a_3b_3$.

Properties of scalar product

$\forall \vec{u}, \vec{v} \in E_0$

a) If $\vec{u} = \vec{0}$ or $\vec{v} = \vec{0}$, then $\vec{u} \cdot \vec{v} = \vec{0}$

b) If $\vec{u} \parallel \vec{v}$ and $\vec{u}, \vec{v}$ have same direction, then $\vec{u} \cdot \vec{v} > 0$

c) If $\vec{u} \parallel \vec{v}$ and $\vec{u}, \vec{v}$ have opposite direction, then $\vec{u} \cdot \vec{v} < 0$

d) If $\vec{u} \perp \vec{v}$, then $\vec{u} \cdot \vec{v} = 0$

e) $\forall \vec{u}, \vec{v} \in E_0$, $\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$

f) $\forall \vec{u}, \vec{v}, \vec{w} \in E_0$, $\vec{u}\left(a\vec{v} + b\vec{w}\right) = a\vec{v}.\vec{u} + b\vec{u}.\vec{w}$,
$\left(a\vec{u} + b\vec{v}\right) \cdot \vec{w} = a\vec{u} \cdot \vec{w} + b\vec{v} \cdot \vec{w}$

g) $\forall \vec{u} \in E_0 \setminus \left\{\vec{0}\right\}$, $\vec{u} \cdot \vec{u} > 0$

We define the square of $\vec{u}$ to be $\vec{u} \cdot \vec{u} = \left(\vec{u}\right)^2$

Example 5.3

Find the scalar product of vectors $(3,2,4)$ and $(-1,4,2)$.

Solution

The scalar product is

$$\begin{aligned}(3,2,4)\cdot(-1,4,2) &= 3\times(-1)+2\times4+4\times2\\ &= -3+8+8\\ &= 13\end{aligned}$$

Example 5.4

The scalar product of $\vec{u} = (2,3,4)$ and $\vec{v} = (1,-2,2)$ is
$\vec{u} \cdot \vec{v} = 2 - 6 + 8 = 4$.

The square of $\vec{u} = (2,3,4)$ is $\vec{u} \cdot \vec{u} = 4 + 9 + 16 = 29$.

Application activity 5.2

Find the scalar product $\vec{u} \cdot \vec{v}$ if;

1. $\vec{u} = (-3,2,6)$ and $\vec{v} = (2,1,5)$
2. $\vec{u} = (-2,1,2)$ and $\vec{v} = (4,-3,1)$
3. $\vec{u} = (12,21,-5)$ and $\vec{v} = (-20,5,12)$
4. $\vec{u} = (2,0,-5)$ and $\vec{v} = (0,11,25)$

5.3. Magnitude (or norm or length) of a vector

Activity 5.3

Use the formula $\|(a,\text{b},\text{c})\| = \sqrt{a^2+b^2+c^2}$ to find;

a) $\|(3,4,5)\|$

b) $\|(-3,6,1)\|$

The magnitude of the vector $\vec{u}$ denoted by $\|\vec{u}\|$ is defined as its length and is the square root of its square. Thus, if $\vec{u} = (a,b,c)$ then $\|\vec{u}\| = \sqrt{a^2+b^2+c^2}$.

Note

The notation of absolute value $|\ |$ is also used for the magnitude of a vector. That is, the magnitude of a vector $\vec{u}$ is also denoted by $|\vec{u}|$.

Properties of magnitude of a vector

a) $\forall \vec{u} \in E_0$, if $\vec{u} = \vec{0}$ then $\|\vec{u}\| = 0$

b) $\forall \vec{u} \in E_0, k \in \mathbb{R}$, $\|k\vec{u}\| = \|k\|\|\vec{u}\|$

c) Distance between two points: If A and B are two points, we can form a vector $\overrightarrow{AB}$ and the distance between these two points denoted as $d(A,B)$ is given by $\|\overrightarrow{AB}\|$. Thus, if $A(a_1,a_2,a_3)$ and $B(b_1,b_2,b_3)$ then

$$d(A,B) = \|\overrightarrow{AB}\| = \sqrt{(b_1-a_1)^2+(b_2-a_2)^2+(b_3-a_3)^2}$$

d) Consider two vectors $\vec{u}$ and $\vec{v}$ on the same line:

If they have the same direction then $\vec{u}\cdot\vec{v} = \|\vec{u}\|\|\vec{v}\|$

If they have the opposite direction then $\vec{u}\cdot\vec{v} = -\|\vec{u}\|\|\vec{v}\|$.

e) Let θ be the angle between two vectors $\vec{u}$ and $\vec{v}$.

If θ is an obtuse angle then the scalar product $\vec{u} \cdot \vec{v}$ is negative.

If θ is an acute angle then the scalar product $\vec{u} \cdot \vec{v}$ is positive.

f) **Unit vector**: A vector $\vec{u}$ is said to be **unit vector** if and only if its magnitude is 1. That is $\left\|\vec{u}\right\| = 1$.

g) **Normalised vector**: The normalised vector of a vector is a vector in the same direction but with magnitude 1. It is also called the unit vector. Given a vector $\vec{v}$, the **normalised vector** parallel to $\vec{v}$ and with same direction is given by $\frac{\vec{v}}{\left\|\vec{v}\right\|}$.

Remark

A vector is said to be a **normal vector** or simply the **normal** to a surface if it is perpendicular to that surface. Often, the normal unit vector is desired, which is sometimes known as the **unit normal.**

The terms normal vector and normalised vector should not be confused, especially since unit norm vectors might be called normalised normal vectors without redundancy.

Example 5.5

Find the magnitude of $\vec{u} = (3, 6, 8)$.

Solution

The magnitude is $\left\|\vec{u}\right\| = \sqrt{3^2 + 6^2 + 8^2} = \sqrt{109}$

Example 5.6

Find the distance between $A(1, -1, 3)$ and $B(2, 4, 5)$.

Solution

The distance is

$$d(A, B) = \sqrt{(2-1)^2 + (4+1)^2 + (5-3)^2} = \sqrt{1 + 25 + 4} = \sqrt{30}$$

Example 5.7

Find the normalised vector parallel to $\vec{v}=(2,4,4)$ and with the same direction.

Solution

The needed vector is given by

$$\vec{e}=\frac{\vec{v}}{\|\vec{v}\|}=\frac{1}{\|\vec{v}\|}\vec{v}=\frac{1}{\sqrt{2^2+4^2+4^2}}\vec{v}=\frac{1}{\sqrt{4+16+16}}(2,4,4) \text{ which is } \vec{e}=\left(\frac{1}{3},\frac{2}{3},\frac{2}{3}\right)$$

Application activity 5.3

Find the magnitude of:

1. $\vec{u}=(-3,2,6)$
2. $\vec{v}=(4,-3,1)$
3. $\vec{u}=(12,21,-5)$
4. $\vec{v}=(0,11,25)$

5.4. Angle between two vectors

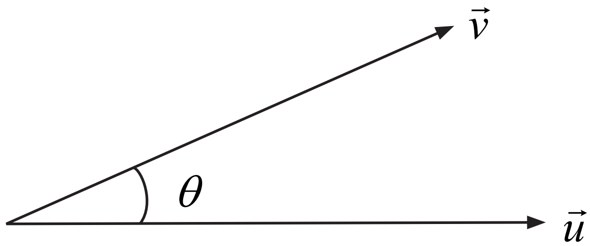
Activity 5.4

Consider two vectors $\vec{u}=(-1,-1,-1)$ and $\vec{v}=(7,7,7)$.

1. Find the scalar product $\vec{u}\cdot\vec{v}$.
2. Find the product $\|\vec{u}\|\|\vec{v}\|$.
3. Evaluate $\cos^{-1}\left(\dfrac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\|\vec{v}\|}\right)$.

Consider two non zero vectors $\vec{u}$ and $\vec{v}$. Geometrically, the scalar product of $\vec{u}$ and $\vec{v}$ is the product of their magnitudes and the cosine of the angle between them. That is, the scalar product of vectors $\vec{u}$ and $\vec{v}$ is also defined to be $\vec{u}\cdot\vec{v}=\|\vec{u}\|\cdot\|\vec{v}\|\cdot\cos\left(\vec{u},\vec{v}\right)$.

From this definition, we can write $\cos(\vec{u},\vec{v})=\dfrac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\cdot\|\vec{v}\|}$.



Note

When we are calculating the angle between two vectors, we calculate the smallest positive angle (the acute angle).

Properties

- If the two vectors are perpendicular, their scalar product is zero which means that the angle between them is $\frac{\pi}{2}$ (if the second is upward) or $-\frac{\pi}{2}$ (if the second is downward). Thus, if $\vec{u}\perp\vec{v}$ then, $\vec{u}\cdot\vec{v}=0$.
- If the two vectors are parallel, then, $\vec{u}\cdot\vec{v}=\|\vec{u}\|\|\vec{v}\|$ or $\vec{u}\cdot\vec{v}=-\|\vec{u}\|\|\vec{v}\|$, which means that the angle between them is 0 (if they have the same direction) or π (if they have the opposite direction).

Example 5.8

Find the angle between vectors $\vec{u}=(-2,1,2)$ and $\vec{v}=(4,-3,1)$ to nearest degree.

Solution

Let θ be the angle between the two vectors, then $\cos\theta=\dfrac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\|\vec{v}\|}$

$$\cos\theta = \frac{-9}{3\sqrt{26}} = \frac{-3}{\sqrt{26}} \Rightarrow \theta = \cos^{-1}\left(\frac{-3}{\sqrt{26}}\right) \approx 126^0$$

The obtained angle is not acute, then the required angle is $180^0 - 126^0 = 54^0$

Therefore, the angle between vectors $\vec{u} = (-2,1,2)$ and $\vec{v} = (4,-3,1)$ is 54^0.

Example 5.9

Consider the vector $\vec{u} = (3,8,1)$. What is the measure of the angle between this vector and z-axis of coordinates system?

$\vec{u} = (3,8,1)$

Solution

Take the normal vector on z-axis, $\vec{e} = (0,0,1)$

We need $\theta = \angle(\vec{u}, \vec{e})$.

$$\cos\theta = \frac{\vec{u}\cdot\vec{e}}{\|\vec{u}\|\|\vec{e}\|} = \frac{1}{\sqrt{9+64+1}\cdot\sqrt{1}} = \frac{1}{\sqrt{74}}$$

$$\cos\theta = \frac{1}{\sqrt{74}} \Leftrightarrow \theta = \arccos\frac{1}{\sqrt{74}} = 83.3^\circ$$

Notice

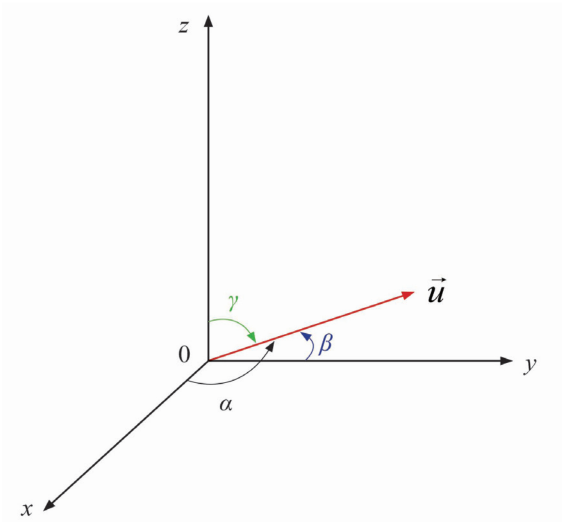
Direction cosine

Direction cosine (or directional cosine) of a vector is the angles between the vector and the three coordinates axes. Or equivalently, it is the component contributions of the basis to the unit vector.

The direction cosines of the vector $\vec{v} = (x, y, z)$ are

$\cos\alpha = \frac{x}{\sqrt{x^2+y^2+z^2}}$, $\cos\beta = \frac{y}{\sqrt{x^2+y^2+z^2}}$ and

$$\cos\gamma = \frac{z}{\sqrt{x^2+y^2+z^2}}.$$



Note that the sum of squares of direction cosines of a vector is 1.

In fact,

$$\cos^2\alpha+\cos^2\beta+\cos^2\gamma = \left(\frac{x}{\sqrt{x^2+y^2+z^2}}\right)^2+\left(\frac{y}{\sqrt{x^2+y^2+z^2}}\right)^2+\left(\frac{z}{\sqrt{x^2+y^2+z^2}}\right)^2$$

$$\cos^2\alpha+\cos^2\beta+\cos^2\gamma = \frac{x^2}{x^2+y^2+z^2}+\frac{y^2}{x^2+y^2+z^2}+\frac{z^2}{x^2+y^2+z^2}=\frac{x^2+y^2+z^2}{x^2+y^2+z^2}=1$$

Thus, $\cos^2\alpha+\cos^2\beta+\cos^2\gamma=1$

Example 5.10

Determine the direction cosines of the vector with components $(1,2,-3)$.

$$\cos\alpha = \frac{1}{\sqrt{1^2+2^2+(-3)^2}} = \frac{1}{\sqrt{14}}$$

$$\cos\beta = \frac{2}{\sqrt{1^2+2^2+(-3)^2}} = \frac{2}{\sqrt{14}}$$

$$\cos\gamma = \frac{-3}{\sqrt{1^2+2^2+(-3)^2}} = \frac{-3}{\sqrt{14}}$$

Application activity 5.4

1. Find the angle formed by the vectors:
   a) $\vec{u}=(-3,2,6)$ and $\vec{v}=(2,1,5)$
   b) $\vec{u}=(1,3,4)$ and $\vec{u}=(6,4,2)$
   c) $\vec{u}=(12,21,-5)$ and $\vec{v}=(-20,5,12)$
   d) $\vec{u}=(2,0,-5)$ and $\vec{v}=(0,11,25)$
2. Find the direction cosines of the vector:
   a) $\vec{u}=(2,3,4)$ b) $\vec{v}=(12,-3,0)$
   c) $\vec{u}=(1,-2,-14)$ d) $\vec{v}=(22,0,0)$

5.5. Vector product

Activity 5.5

1. Consider vectors $\vec{u}=(4,2,1)$ and $\vec{v}=(-2,4,2)$. Find vector $\vec{w}$ that is perpendicular to both $\vec{u}$ and $\vec{v}$.
2. Calculate the determinant

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 4 & 2 & 1 \\ -2 & 4 & 2 \end{vmatrix}$$

3. Comment on results in 1 and 2.

The vector product (or cross product or Gibbs vector product) is a binary operation on two vectors in three-dimensional space. It results in a vector which is perpendicular to both of the vectors being multiplied and therefore normal to the plane containing them.

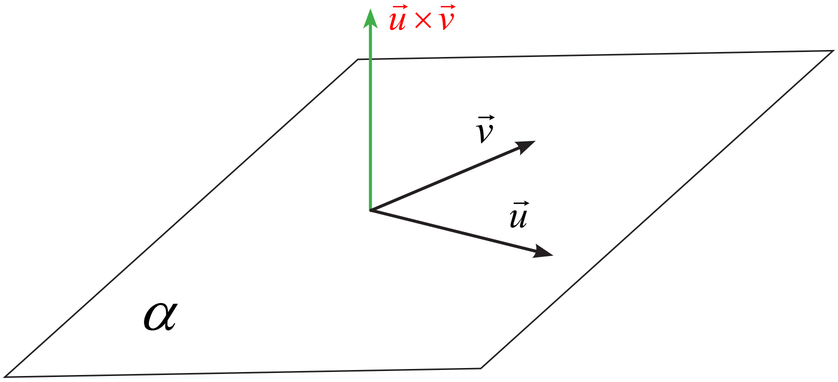
Consider $\left\{\vec{i}=(1,0,0),\vec{j}=(0,1,0),\vec{k}=(0,0,1)\right\}$, a positive orthonormal basis of E_0 and two linearly independent vectors $\vec{u}=(a_1,a_2,a_3)$ and $\vec{v}=(b_1,b_2,b_3)$.

The vector product of $\vec{u}$ and $\vec{v}$ is denoted $\vec{u}\times\vec{v}$. From activity 5.4,

$$\vec{u}\times\vec{v}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\end{vmatrix}=\begin{vmatrix}a_2 & a_3\\ b_2 & b_3\end{vmatrix}\vec{i}-\begin{vmatrix}a_1 & a_3\\ b_1 & b_3\end{vmatrix}\vec{j}+\begin{vmatrix}a_1 & a_2\\ b_1 & b_2\end{vmatrix}\vec{k}$$

Or

$$\vec{u}\times\vec{v}=\left(\begin{vmatrix}a_2 & a_3\\ b_2 & b_3\end{vmatrix},-\begin{vmatrix}a_1 & a_3\\ b_1 & b_3\end{vmatrix},\begin{vmatrix}a_1 & a_2\\ b_1 & b_2\end{vmatrix}\right)$$



Example 5.11

Find the vector product of $\vec{u}=(1,3,-3)$ and $\vec{v}=(4,3,1)$.

Solution

$$\vec{u}\times\vec{v}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ 1 & 3 & -3\\ 4 & 3 & 1\end{vmatrix}=\begin{vmatrix}3 & -3\\ 3 & 1\end{vmatrix}\vec{i}-\begin{vmatrix}1 & -3\\ 4 & 1\end{vmatrix}\vec{j}+\begin{vmatrix}1 & 3\\ 4 & 3\end{vmatrix}\vec{k}=12\vec{i}-13\vec{i}-9\vec{k}$$

Or

$$\vec{u}\times\vec{v}=12\vec{i}-13\vec{i}-9\vec{k}$$

Example 5.12

Find the vector product of $\vec{u}=(2,3,5)$ and $\vec{v}=(-2,5,6)$.

Solution

$$\vec{u}\times\vec{v}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ 2 & 3 & 5\\ -2 & 5 & 6\end{vmatrix}=\begin{vmatrix}3 & 5\\ 5 & 6\end{vmatrix}\vec{i}-\begin{vmatrix}2 & 5\\ -2 & 6\end{vmatrix}\vec{j}+\begin{vmatrix}2 & 3\\ -2 & 5\end{vmatrix}\vec{k}=-7\vec{i}-22\vec{j}+16\vec{k}$$

Or

$$\vec{u}\times\vec{v}=(-7,-22,+16)$$

Properties of vector product

1. If $\vec{w}$ is vector product $\vec{u}$ and $\vec{v}$, then, $\vec{w}\perp\vec{u}$ and $\vec{w}\perp\vec{v}$.
2. The vector product is anti-symmetric: $\vec{u}\times\vec{v}=-\vec{v}\times\vec{u}$.
3. If $\vec{u}=\vec{0}$ and $\vec{v}=\vec{0}$ then $\vec{u}\times\vec{v}=\vec{0}$.
4. If two vectors are linearly dependent then their vector product is a zero vector.
5. If $\vec{u}\times\vec{v}=\vec{0}$ then $\vec{u}=\vec{0}$ or $\vec{v}=\vec{0}$.
6. The vector product is bilinear:
   $\vec{u}\times\left(r\vec{v}+s\vec{w}\right)=r\left(\vec{u}\times\vec{v}\right)+s\left(\vec{u}\times\vec{w}\right)$
   $\left(r\vec{u}+s\vec{v}\right)\times\vec{w}=r\left(\vec{u}\times\vec{w}\right)+s\left(\vec{v}\times\vec{w}\right)$.
7. The vector product is not associative:
   $\left(\vec{u}\times\vec{v}\right)\times\vec{w}\neq\vec{u}\times\left(\vec{v}\times\vec{w}\right)$.

Application activity 5.5

Calculate the vector product $\vec{u}\times\vec{v}$ of the following vectors:

1. $\vec{u}=(11,2,-1)$, $\vec{v}=(-1,0,1)$
2. $\vec{u}=(0,0,2)$, $\vec{v}=(-1,1,0)$
3. $\vec{u}=(1,1,-1)$, $\vec{v}=(9,1,1)$
4. $\vec{u}=(6,3,1)$, $\vec{v}=(1,1,0)$
5. $\vec{u}=(5,1,2)$, $\vec{v}=(-1,0,2)$

5.6. Mixed product

Activity 5.6

1. Find the determinant $\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$

Where $\vec{i}=(1,0,0)$, $\vec{j}=(0,1,0)$, $\vec{k}=(0,0,1)$.

2. Find the scalar product of vector $\vec{u}=(c_1,c_2,c_3)$ and vector obtained in 1).

The mixed product (also called the **scalar triple product** or **box product** or **compound product**) of three vectors is a scalar which numerically equals the vector product multiplied by a vector as the dot product.

Then the mixed product of the vectors $\vec{u}=(a_1,a_2,a_3)$, $\vec{v}=(b_1,b_2,b_3)$ and $\vec{w}=(c_1,c_2,c_3)$ is equal to the dot product of the first vector by the vector product of the other two. It is denoted by $\left[\vec{u},\vec{v},\vec{w}\right]$.

Thus, $\left[\vec{u},\vec{v},\vec{w}\right]=\vec{u}\cdot\left(\vec{v}\times\vec{w}\right)$.

From activity 5.5,

$$\vec{u}\cdot\left(\vec{v}\times\vec{w}\right)=a_1\begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix}-a_2\begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix}+a_3\begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

this product is equivalent to the development of a determinant whose columns are the coordinates of these vectors with respect to an orthonormal basis.

That is,

$$\vec{u}\cdot\left(\vec{v}\times\vec{w}\right)=\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Mixed product properties

1. The mixed product does not change if the orders of its factors are circularly rotated, but changes sign if they are transposed. That is,

$\vec{u}\cdot(\vec{v}\times\vec{w})=\vec{w}\cdot(\vec{u}\times\vec{v})=\vec{v}\cdot(\vec{w}\times\vec{u})$ and

$\vec{u}\cdot(\vec{v}\times\vec{w})=-\vec{w}\cdot(\vec{v}\times\vec{u})=-\vec{v}\cdot(\vec{u}\times\vec{w})$.

2. If three vectors are linearly dependent, the mixed product is zero.

Example 5.13

Calculate the mixed product $\vec{u}\cdot(\vec{v}\times\vec{w})$ of the following vectors: $\vec{u}=(2,-1,3)$, $\vec{v}=(0,2,-5)$ and $\vec{w}=(1,-1,-2)$.

$$\vec{v}\times\vec{w}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ 0 & 2 & -5\\ 1 & -1 & -2\end{vmatrix}=-9\vec{i}-5\vec{j}-2\vec{k}=(-9,-5,-2)$$

$$\vec{u}\cdot(\vec{v}\times\vec{w})=(2,-1,3)\cdot(-9,-5,-2)=-18+5-6=-19$$

Or $\vec{u}\cdot(\vec{v}\times\vec{w})=\begin{vmatrix}2 & -1 & 3\\ 0 & 2 & -5\\ 1 & -1 & -2\end{vmatrix}=-8+0+5-6-10-0=-19$

Application activity 5.6

Calculate the mixed product $\vec{u}\cdot(\vec{v}\times\vec{w})$ of the following vectors:

1. $\vec{u}=(11,2,-1)$, $\vec{v}=(-1,0,1)$ and $\vec{w}=(1,2,1)$
2. $\vec{u}=(0,0,2)$, $\vec{v}=(-1,1,0)$ and $\vec{w}=(3,1,-1)$
3. $\vec{u}=(1,1,-1)$, $\vec{v}=(9,1,1)$ and $\vec{w}=(1,6,3)$
4. $\vec{u}=(6,3,1)$, $\vec{v}=(1,1,0)$ and $\vec{w}=(3,2,6)$
5. $\vec{u}=(5,1,2)$, $\vec{v}=(-1,0,2)$ and $\vec{w}=(2,1,1)$

5.7. Applications

5.7.1. Work done as scalar product

Activity 5.7

From the definition of work done by a force on a body, if a constant force F acting on a particle displaces from A to B, express the work done in function of vectors $\vec{F}$ and $\overrightarrow{AB}$.

From activity 5.6, if a constant force F acting on a particle displaces it from A to B, the work done is given by $work\ done = \vec{F}\cdot\overrightarrow{AB}$

Example 5.14

Constant forces $\vec{P} = 2\vec{i} - 5\vec{j} + 6\vec{k}$ and $\vec{Q} = -\vec{i} + 2\vec{j} - \vec{k}$ act on a particle. Determine the work done when the particle is displaced from A to B, the position vectors of A and B being $4\vec{i} - 3\vec{j} + 2\vec{k}$ and $6\vec{i} + \vec{j} - 3\vec{k}$ respectively.

Solution

Total force: $\left(2\vec{i} - 5\vec{j} + 6\vec{k}\right) + \left(-\vec{i} + 2\vec{j} - \vec{k}\right) = \vec{i} - 3\vec{j} + 5\vec{k}$

Displacement: $\left(6\vec{i} + \vec{j} - 3\vec{k}\right) - \left(4\vec{i} - 3\vec{j} - 2\vec{k}\right) = 2\vec{i} + 4\vec{j} - \vec{k}$

Work done: $\left(\vec{i} - 3\vec{j} + 5\vec{k}\right)\left(2\vec{i} + 4\vec{j} - \vec{k}\right) = 2 - 12 - 5 = -15$

Work done is 15 unit of work.

Example 5.15

Forces of magnitudes 5 and 3 units acting in the direction $6\vec{i} + 2\vec{j} + 3\vec{k}$ and $3\vec{i} - 2\vec{j} + 6\vec{k}$ respectively act on a particle which is displaced from the point $(2,2,-1)$ to $(4,3,1)$. Find the work done by the forces.

Solution

First force of magnitude 5 units, acting in the direction $6\vec{i} + 2\vec{j} + 3\vec{k}$

is $5\frac{6\vec{i}+2\vec{j}+3\vec{k}}{\sqrt{6^2+2^2+3^2}}=\frac{5}{7}\left(6\vec{i}+2\vec{j}+3\vec{k}\right)$

Second force of magnitude 3 units, acting in the direction $3\vec{i}-2\vec{j}+6\vec{k}$ is $3\frac{3\vec{i}-2\vec{j}+6\vec{k}}{\sqrt{3^2+(-2)^2+6^2}}=\frac{3}{7}\left(3\vec{i}-2\vec{j}+6\vec{k}\right)$

Resulting force is $\frac{5}{7}\left(6\vec{i}+2\vec{j}+3\vec{k}\right)+\frac{3}{7}\left(3\vec{i}-2\vec{j}+6\vec{k}\right)=\frac{1}{7}\left(39\vec{i}+4\vec{j}+33\vec{k}\right)$

Displaced from the point $(2,2,-1)$ to $(4,3,1)$ is

$\left(4\vec{i}+3\vec{j}+\vec{k}\right)-\left(2\vec{i}+2\vec{j}-\vec{k}\right)=2\vec{i}+\vec{j}+2\vec{k}$

Work done: $\frac{1}{7}\left(39\vec{i}+4\vec{j}+33\vec{k}\right)\cdot\left(2\vec{i}+\vec{j}+2\vec{k}\right)=\frac{1}{7}(78+4+66)=\frac{148}{7}$ units

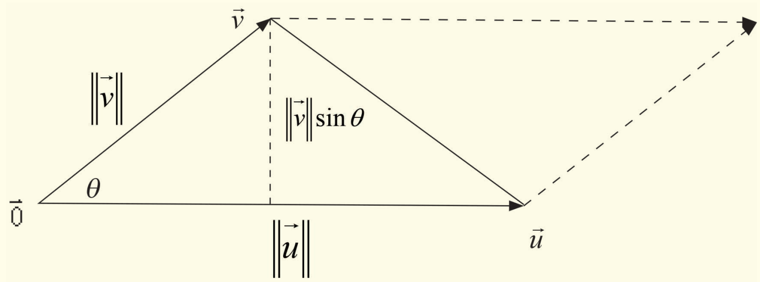
Application activity 5.7

1. A particle acted on by constant forces $2\vec{i}+\vec{j}-\vec{k}$, $\vec{i}-2\vec{j}+3\vec{k}$ and $3\vec{i}+\vec{j}+5\vec{k}$ is displaced from the point $\vec{i}+2\vec{j}+3\vec{k}$ to the point $6\vec{i}+3\vec{j}+\vec{k}$. Find the work done.
2. Constant forces $12\vec{i}-15\vec{j}+6\vec{k}$, $\vec{i}+2\vec{j}-2\vec{k}$ and $2\vec{i}+8\vec{j}+\vec{k}$ act on a point P which is displaced from the position $2\vec{i}-3\vec{j}+\vec{k}$ to the position $4\vec{i}+2\vec{j}+\vec{k}$. Find the total work done.
3. The point of application of force $(-2,4,7)$ is displaced from the point $(3,-5,1)$ to the point $(5,9,7)$. But the force is suddenly halved when the point of application moves half the distance. Find the work done.
4. A force of magnitude 6 units acting parallel to $2\vec{i}-2\vec{j}+\vec{k}$ displaces the point of application from $(1,2,3)$ to $(5,3,7)$. Find the work done.

5.7.2. Area of a parallelogram

Activity 5.8

Consider the following figure



Write down the formula for area of this parallelogram in terms of $\|\vec{u}\|$, $\|\vec{v}\|$ and $\sin\theta$ and give its equivalent relation using vector product.

Geometrically, the magnitude of the vector product of two vectors is the product of their magnitudes and the sine of the angle between them.

From activity 5.7, the area of a parallelogram with vectors $\vec{u}$ and $\vec{v}$ as two adjacent sides is given by $S_{\square} = \|\vec{u} \times \vec{v}\|$.

Thus, the magnitude of the vector product of two vectors $\vec{u}$ and $\vec{v}$ represents the area of a parallelogram with vectors $\vec{u}$ and $\vec{v}$ as two adjacent sides.

Example 5.16

Find the area of parallelogram with vectors $\vec{u} = (3,0,4)$ and $\vec{v} = (3,2,1)$ as two adjacent sides.

Solution

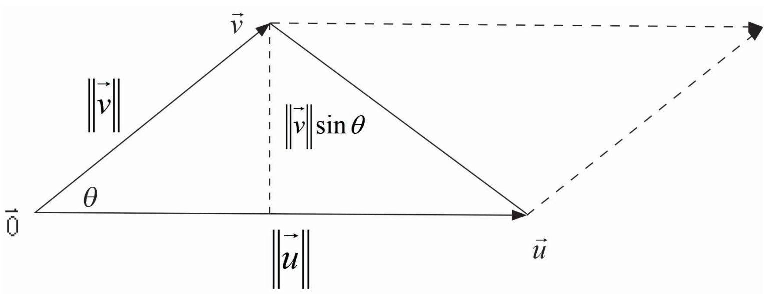
$$\vec{u} \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & 0 & 4 \\ 3 & 2 & 1 \end{vmatrix} = \begin{vmatrix} 0 & 4 \\ 2 & 1 \end{vmatrix}\vec{i} - \begin{vmatrix} 3 & 4 \\ 3 & 1 \end{vmatrix}\vec{j} + \begin{vmatrix} 3 & 0 \\ 3 & 2 \end{vmatrix}\vec{k} = -8\vec{i} + 9\vec{j} + 6\vec{k}$$

$$S_{\square} = \|\vec{u} \times \vec{v}\| = \|-8\vec{i} + 9\vec{j} + 6\vec{k}\| = \sqrt{64+81+36} = \sqrt{181} \text{ sq. units}$$

Notice: Area of a triangle

Since the area of the parallelogram is twice the area of the triangle, we may use the vector product to find the area of triangle.

Thus, the area of triangle with vectors $\vec{u}$ and $\vec{v}$ as two sides is $S_{\triangle} = \frac{1}{2}\left\|\vec{u}\times\vec{v}\right\|$.



Consider the triangle ABC whose vertices are points $A(a_1,a_2,a_3), B(b_1,b_2,b_3)$ and $C(c_1,c_2,c_3)$. Letting A to be the starting point, we can form two vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$ and the area of this triangle is $S_{\triangle} = \frac{1}{2}\left\|\overrightarrow{AB}\times\overrightarrow{AC}\right\|$.

Example 5.17

Find the area of triangle with vectors $\vec{u}=(3,0,4)$ and $\vec{v}=(3,2,1)$ as two sides.

Solution

$$\vec{u}\times\vec{v} = \begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ 3 & 0 & 4\\ 3 & 2 & 1\end{vmatrix} = \begin{vmatrix}0 & 4\\ 2 & 1\end{vmatrix}\vec{i} - \begin{vmatrix}3 & 4\\ 3 & 1\end{vmatrix}\vec{j} + \begin{vmatrix}3 & 0\\ 3 & 2\end{vmatrix}\vec{k} = -8\vec{i}+9\vec{j}+6\vec{k}$$

$$S_{\triangle} = \frac{1}{2}\left\|\vec{u}\times\vec{v}\right\| = \frac{1}{2}\left\|-8\vec{i}+9\vec{j}+6\vec{k}\right\| = \frac{1}{2}\sqrt{64+81+36} = \frac{1}{2}\sqrt{181}$$ Sq units.

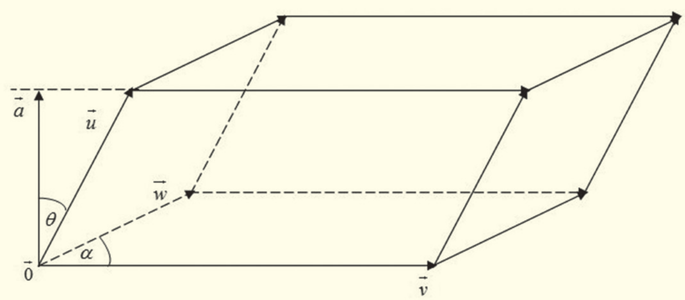
Application activity 5.8

1. Find the area of a parallelogram with vectors;
   a) $\vec{u}=(1,-2,-14)$ and $\vec{v}=(22,0,0)$ as two adjacent sides.
   b) $\vec{u}=(21,4,-2)$ and $\vec{v}=(0,-1,0)$ as two adjacent sides.
2. Find the area of triangle with vectors $\vec{u}=(1,0,0)$ and $\vec{v}=(3,3,3)$ as two sides.
3. Find the area of triangle formed by the points whose position vectors are $3\vec{i}+\vec{j}$, $5\vec{i}+2\vec{j}+\vec{k}$, $\vec{i}-2\vec{j}+3\vec{k}$.
4. The vertices of a triangle are $(1,1,1)$, $(0,1,2)$ and $(3,2,1)$. Find the area of the triangle.

5.7.3. Volume of a parallelepiped

Activity 5.9

Consider the following figure.



Write down the formula for volume of this parallelepiped in terms of $\|\vec{u}\|$, $\|\vec{v}\|$, $\|\vec{w}\|$, $\cos\theta$ and $\sin\alpha$ and give its equivalent relation using mixed product.

Geometrically, the magnitude of the mixed product represents the volume of the parallelepiped whose edges are three vectors that meet in the same vertex.

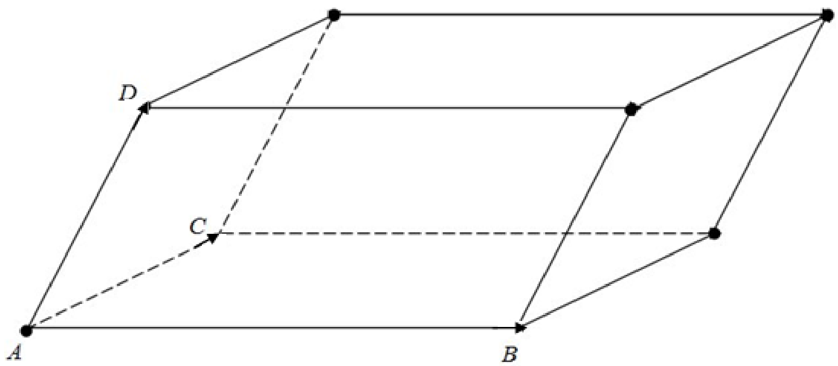
From activity 5.8, for a parallelepiped which has vectors $\vec{u}$, $\vec{v}$ and $\vec{w}$ as three concurrent edges, where $\vec{v}$ and $\vec{w}$ define its base, the volume is given by $V=\left|\vec{u}\cdot\left(\vec{v}\times\vec{w}\right)\right|$.

Remember that if $\vec{u}=(a_1,a_2,a_3)$, $\vec{v}=(b_1,b_2,b_3)$ and $\vec{w}=(c_1,c_2,c_3)$,

then, $\vec{u}\cdot(\vec{v}\times\vec{w})=\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$.

If the parallelepiped is defined by four points $A(a_1,a_2,a_3)$, $B(b_1,b_2,b_3)$, $C(c_1,c_2,c_3)$ and $D(d_1,d_2,d_3)$, its volume is

$V=\left|\overrightarrow{AD}\cdot(\overrightarrow{AB}\times\overrightarrow{AC})\right|$



Example 5.18

Find the volume of the parallelepiped formed by the vectors:

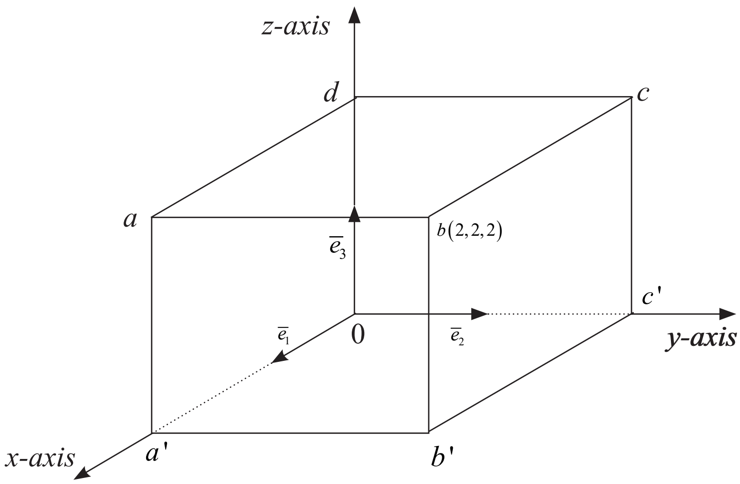
$\vec{u}=(3,-2,5)$, $\vec{v}=(2,2,-1)$ and $\vec{w}=(-4,3,2)$.

Solution

$$V=\vec{u}\cdot(\vec{v}\times\vec{w})=\begin{vmatrix} 3 & -2 & 5 \\ 2 & 2 & -1 \\ -4 & 3 & 2 \end{vmatrix}=12+30-8+40+9+8=91 \textit{ cube units}$$

Example 5.19

Consider the following cube with vertices $a,b,c,d,a',b',0,c'$



a) From coordinates of the vertex b, find the coordinates of other vertices.

b) Calculate the area of triangle $a'bc'$.

c) Calculate the volume of this cube.

Solution

a) First: The vertices a', c', d are intercepts of coordinate axes.

a' is x-axis intercept. It has the form $a'(m,0,0)$.

c' is y-axis intercept. It has the form $c'(0,n,0)$.

d is z-axis intercept. It has the form $d(0,0,k)$.

Second: Considering the *xy*-plane and the given figure, vertex b is 2 units upwards, meaning that the vertices a, c *and* d are also 2 units upward since the figure is a cube. Then the *z-coordinate* of a, c *and* d are the same and equal to 2.

Third: Considering the *xz*-plane and the given figure, vertex b is 2 units in direction of *y-positive*, meaning that the vertices b', c *and* c' are also 2 units in direction of *y-positive* since the figure is a cube. Then the *y-coordinate* of b', c *and* c' are the same and equal to 2.

Fourth: Considering the *yz*-plane and the given figure, vertex b is 2 units in direction of *x-positive*, meaning that the vertices a, a' *and* b' are also 2 units in direction of *x-positive* since the figure is a cube. Then the *x-coordinate* of a, a' *and* b' are the same and equal to 2.

Fifth: Vertex a lies on *xz plane*, thus its *y-coordinate* is zero. Vertex c lies on *yz plane*, thus its *x-coordinate* is zero. Vertex b' lies on *xy plane*, thus its *z-coordinate* is zero.

Combining the above results we get:

$a(2,0,2)$, $a'(2,0,0)$, $b'(2,2,0)$, $c(0,2,2)$, $c'(0,2,0)$ and $d(0,0,2)$

b) Area of triangle $a'bc'$.

This triangle is built from vectors $\overrightarrow{a'b}$ and $\overrightarrow{a'c'}$. The area is given by $\frac{1}{2}\left\|\overrightarrow{a'b}\times\overrightarrow{a'c'}\right\|$.

$\overrightarrow{a'b}=(0,2,2)$, $\overrightarrow{a'c'}=(-2,2,0)$

$$\overrightarrow{a'b}\times\overrightarrow{a'c'}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ 0 & 2 & 2\\ -2 & 2 & 0\end{vmatrix}=-4\vec{i}-4\vec{j}+4\vec{k}$$

The area is $\frac{1}{2}\left\|\overrightarrow{a'b}\times\overrightarrow{a'c'}\right\|=\frac{1}{2}\sqrt{16+16+16}=2\sqrt{3}$ Sq. units.

c) The volume of the given cube is:

$$V=\left|\vec{d}\cdot\left(\vec{a}'\times\vec{c}'\right)\right|$$

$$V=\begin{vmatrix}0 & 0 & 2\\ 2 & 0 & 0\\ 0 & 2 & 0\end{vmatrix}=0+8+0-0-0-0=8 \text{ cube units.}$$

Remark

A parallelepiped is a **prism (or polyhedron) which has a parallelogram as its base**.

Example 5.20 **The volume of an automatic glassware washer**

Calculate the volume of an automatic glassware washer in the form of a cuboid formed by the following vectors:

$\vec{u}=(5,1,0)$, $\vec{v}=(-5,4,1)$, $\vec{w}=(5,7,6)$

Solution:

$\vec{u}=(5,1,0)$, $\vec{v}=(-5,4,1)$, $\vec{w}=(5,7,6)$

The volume is

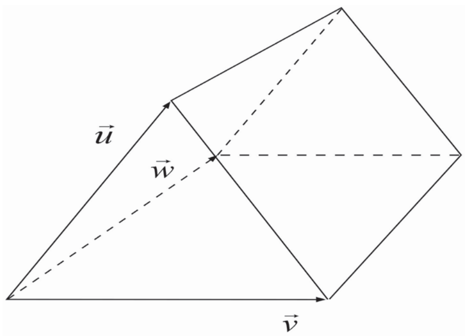
$$\begin{vmatrix} 5 & -5 & 5 \\ 1 & 4 & 7 \\ 0 & 1 & 6 \end{vmatrix} = (120-0+5-0-35+30) = 155-35 = 120 \textit{ cube units}$$

The volume of the automatic glassware washer is 120 *cube units*

Notice

Volume of a triangular prism

The parallelepiped can be split into 2 triangular prisms of equal volume. Since the volume of a parallelepiped is the magnitude of the mixed product, then the volume of a triangular prism is equal to ½ of the magnitude of the mixed product.



Thus, the volume of a triangular prism which has vectors $\vec{u}=(a_1,a_2,a_3)$, $\vec{v}=(b_1,b_2,b_3)$ and $\vec{w}=(c_1,c_2,c_3)$ as three concurrent edges, where $\vec{v}$ and $\vec{w}$ define its base, is given by

$$V=\frac{1}{2}\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Example 5.21

Find the volume of a triangular prism whose vertices are the points $A(1,2,1)$, $B(2,4,0)$, $C(-1,2,1)$ and $D(2,-2,2)$.

Solution

$\overrightarrow{AB}=(1,2,-1)$ $\overrightarrow{AC}=(-2,0,0)$ $\overrightarrow{AD}=(1,-4,1)$

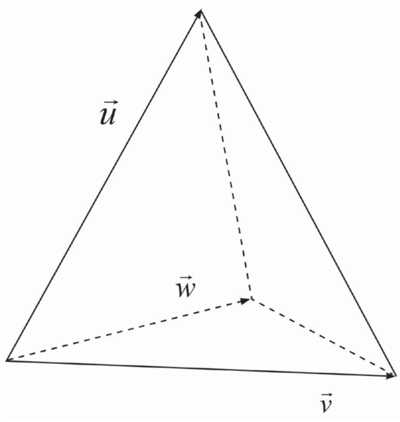
The volume is

$$V=\frac{1}{2}\begin{vmatrix}1 & 2 & -1\\ -2 & 0 & 0\\ 1 & -4 & 1\end{vmatrix}=\frac{1}{2}(0-8+0-0-0+4)=-2$$

We need to take absolute value. Thus, the volume is $V=2$ *cubic units*.

Volume of a tetrahedron

The parallelepiped can be split into 6 tetrahedra of equal volume. Since the volume of a parallelepiped is the magnitude of the mixed product, then the volume of a tetrahedron is equal to $\frac{1}{6}$ of the magnitude of the mixed product.



Thus, the volume of a tetrahedron which has vectors $\vec{u}=(a_1,a_2,a_3)$, $\vec{v}=(b_1,b_2,b_3)$ and $\vec{w}=(c_1,c_2,c_3)$, as three concurrent edges, where $\vec{v}$ and $\vec{w}$ define its base, is given by

$$V=\frac{1}{6}\begin{vmatrix}a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\\ c_1 & c_2 & c_3\end{vmatrix}$$

Remark

A tetrahedron is also called **triangular pyramid**.

Example 5.22

Find the volume of the tetrahedron whose vertices are the points $A(3,2,1)$, $B(1,2,4)$, $C(4,0,3)$ and $D(1,1,7)$.

Solution

$\overrightarrow{AB}=(-2,0,3)$ $\qquad$ $\overrightarrow{AC}=(1,-2,2)$ $\qquad$ $\overrightarrow{AD}=(-2,-1,6)$

The volume is

$$V=\frac{1}{6}\begin{vmatrix}-2 & 0 & 3\\ 1 & -2 & 2\\ -2 & -1 & 6\end{vmatrix}=\frac{1}{6}(24-3+0-12-4-0)=\frac{5}{6}\ cube\ units$$

Application activity 5.9

1. Find the volume of a triangular prism whose vertices are the points;
   a) $A(1,2,1)$, $B(0,-2,4)$, $C(1,1,1)$ and $D(1,6,4)$.
   b) $A(-1,3,1)$, $B(0,-1,0)$, $C(3,1,2)$ and $D(1,2,4)$.
2. Find the volume of the tetrahedron whose vertices are the points;
   a) $A(3,1,4)$, $B(1,0,0)$, $C(3,4,1)$ and $D(1,0,2)$.
   b) $A(-1,-2,1)$, $B(-5,2,3)$, $C(1,1,1)$ and $D(1,1,0)$.
3. Find the volume of the parallelepiped with adjacent sides $\overrightarrow{OA}=3\vec{i}-\vec{j}$, $\overrightarrow{OB}=\vec{j}+2\vec{k}$, $\overrightarrow{OC}=\vec{i}+5\vec{j}+4\vec{k}$ extending from origin of coordinates.
4. Find the volume of the tetrahedron whose vertices are the points $A(2,-1,-3)$, $B(4,1,3)$, $C(3,2,-1)$ and $D(1,4,2)$

Unit Summary

1. The scalar product of vectors $\vec{u}=(a_1,a_2,a_3)$ and $\vec{v}=(b_1,\ b_2,\ b_3)$ of space is defined by $\vec{u}\cdot\vec{v}=a_1b_1+a_2b_2+a_3b_3$

2. If $\vec{u}=(a,b,c)$ then, $\|\vec{u}\|=\sqrt{a^2+b^2+c^2}$.

3. If $A(a_1,a_2,a_3)$ and $B(b_1,b_2,b_3)$ then,
$$d(A,B)=\|\overrightarrow{AB}\|=\sqrt{(b_1-a_1)^2+(b_2-a_2)^2+(b_3-a_3)^2}$$

4. The scalar product of vectors $\vec{u}$ and $\vec{v}$ is also defined to be
$\vec{u}\cdot\vec{v}=\|\vec{u}\|\cdot\|\vec{v}\|\cdot\cos(\vec{u},\vec{v})$.

5. If a constant force F acting on a particular particle displaces it from A to B, the work done is given by
$work\ done=\vec{F}\cdot\overrightarrow{AB}$.

6. The vector product of $\vec{u}$ and $\vec{v}$ is denoted $\vec{u}\times\vec{v}$ and defined by
$$\vec{u}\times\vec{v}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\end{vmatrix}=\begin{vmatrix}a_2 & a_3\\ b_2 & b_3\end{vmatrix}\vec{i}-\begin{vmatrix}a_1 & a_3\\ b_1 & b_3\end{vmatrix}\vec{j}+\begin{vmatrix}a_1 & a_2\\ b_1 & b_2\end{vmatrix}\vec{k}$$

7. The magnitude of the vector product of two vectors $\vec{u}$ and $\vec{v}$ represents the area of a parallelogram with vectors $\vec{u}$ and $\vec{v}$ as two adjacent sides.

8. The area of triangle with vectors $\vec{u}$ and $\vec{v}$ as two sides is $S_{\triangle}=\frac{1}{2}\|\vec{u}\times\vec{v}\|$.

9. The mixed product of the vectors $\vec{u}=(a_1,a_2,a_3)$, $\vec{v}=(b_1,b_2,b_3)$ and $\vec{w}=(c_1,c_2,c_3)$ is denoted and defined by
$[\vec{u},\vec{v},\vec{w}]=\vec{u}.(\vec{v}\times\vec{w})$.

10. The volume of a parallelepiped which has vectors $\vec{u}$, $\vec{v}$ and $\vec{w}$ as three concurrent edges, where $\vec{v}$ and $\vec{w}$ define its base, is given by $V = \left|\vec{u} \cdot \left(\vec{v} \times \vec{w}\right)\right|$.
11. The volume of a triangular prism which has vectors $\vec{u}$, $\vec{v}$ and $\vec{w}$ as three concurrent edges, where $\vec{v}$ and $\vec{w}$ define its base, is given by $V = \frac{1}{2}\left|\vec{u} \cdot \left(\vec{v} \times \vec{w}\right)\right|$.
12. The volume of a tetrahedron which has vectors $\vec{u}$, $\vec{v}$ and $\vec{w}$ as three concurrent edges, where $\vec{v}$ and $\vec{w}$ define its base, is given by $V = \frac{1}{6}\left|\vec{u} \cdot \left(\vec{v} \times \vec{w}\right)\right|$.

End of Unit Assessment

1. Find each of the following vector product;

   a) $\vec{i}\times\vec{i}$ b) $\vec{i}\times\vec{j}$ c) $\vec{i}\times\vec{k}$

   d) $\vec{j}\times\vec{k}$ e) $\vec{i}\times\left(\vec{j}\times\vec{k}\right)$ f) $\left(\vec{i}\times\vec{j}\right)\times\vec{k}$

2. The vectors $\vec{a}$ and $\vec{b}$ are two sides of a parallelogram in each of the following. Calculate the area of each parallelogram;

   a) $\vec{a}=3\vec{i}+\vec{j},\vec{b}=-3\vec{i}-2\vec{j}+2\vec{k}$

   b) $\vec{a}=4\vec{i}-\vec{j}+3\vec{k},\vec{b}=8\vec{i}+3\vec{j}+\vec{k}$

   c) $\vec{a}=2\vec{i}-2\vec{j}+\vec{k},\vec{b}=\vec{i}-5\vec{k}$

   d) $\vec{a}=2\vec{i}+3\vec{j}-5\vec{k},\vec{b}=\vec{i}+5\vec{j}-6\vec{k}$

3. Let $\vec{u}=(2,-1,3),\vec{v}=(0,1,7)$ and $\vec{w}=(1,4,5)$. Find:

   a) $\vec{u}\cdot\vec{v}$ b) $\vec{u}\cdot\vec{w}$ c) $\vec{v}\cdot\vec{w}$

4. Let $\vec{u}=(2,-1,3),\vec{v}=(0,1,7)$ and $\vec{w}=(1,4,5)$. Find:

   a) $\vec{u}\times\left(\vec{v}\times\vec{w}\right)$ b) $\left(\vec{u}\times\vec{v}\right)\times\vec{w}$ c) $\vec{u}\times\left(\vec{v}-2\vec{w}\right)$

   d) $\left(\vec{u}\times\vec{v}\right)-2\vec{w}$ e) $\left(\vec{u}\times\vec{v}\right)\times\left(\vec{v}\times\vec{w}\right)$ f) $\left(\vec{v}\times\vec{w}\right)\times\left(\vec{u}\times\vec{v}\right)$

5. Find the area of the triangle having vertices P, q and R;

   a) $P(1,5,-2),Q(0,0,0),R(3,5,1)$

   b) $P(2,0,-3),Q(1,4,5),R(7,2,9)$

6. What is wrong with expression $\vec{u}\times\vec{v}\times\vec{w}$?

7. Find the volume of the parallelepiped with sides $\vec{a},\vec{b}$ and $\vec{c}$;

   a) $\vec{a}=(2,-6,2),\vec{b}=(0,4,-2),\vec{c}=(2,2,-4)$

   b) $\vec{a}=3\vec{i}+\vec{j}+2\vec{k},\vec{b}=4\vec{i}+5\vec{j}+\vec{k},\vec{c}=\vec{i}+2\vec{j}+4\vec{k}$

8. Consider the parallelepiped with sides
$\vec{a} = 3\vec{i} + 2\vec{j} + \vec{k}$, $\vec{b} = \vec{i} + \vec{j} + 2\vec{k}$ and $\vec{c} = \vec{i} + 3\vec{j} + 3\vec{k}$
   a) Find the volume.
   b) Find the area of the face determined by $\vec{a}$ and $\vec{c}$.
9. Find the area of the triangle whose vertices are;
   a) $(2,1,3),(3,0,2),(4,1,2)$
   b) $(a,0,0),(0,b,0),(0,0,c)$
10. Find the area of the triangle whose vertices are;
$(0,0,0),(x_1,y_1,z_1)$ and (x_2,y_2,z_2)
11. Find the volume of the tetrahedron whose vertices are;
$(0,1,2),(3,0,1),(4,3,6),(2,3,2)$.
12. Calculate the angle between the vectors $\vec{u} = (2,4,5)$ and $\vec{v} = (-6,4,-3)$.
13. Estimate the gravitational force between two sum of wrestlers, with masses 220kg and 240kg, when they are embraced and their centers are 1.2m apart

Unit 6

Matrices and Determinant of Order 3

Introductory activity

A pharmacist sold 3 types of medicines to patients in three consecutive months as follows:

First month: 7 boxes of type I, 2 boxes of type II, 9 boxes of type III.

Second month: 5 boxes of type I, 4 boxes of type II, 6 boxes of types III

Third month: 8 boxes of type I, 7 boxes of type II, 9 boxes of type III

a) Write a matrix A representing the distribution of boxes in three months in the form $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$.

For the form a_{ij}, i stands for months and j stands for type

b) If there is a number **DET A** such that

$$Det\,A = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{31}a_{22}a_{13} - a_{32}a_{23}a_{11} - a_{33}a_{21}a_{12}$$

Calculate DET A from (a)

c) Give 3 examples of real-life problems where matrices are applied.

Objectives

By the end of this unit, a student will be able to:

- define and give example of matrix of order three.
- perform different operations on matrices of order three.
- find matrix representation of a linear transformation
- find the determinant of order three.
- find the inverse of matrix of order three.
- solve system of three linear equations by matrix inverse method.

6.1. Square matrices of order three

6.1.1. Definitions

Activity 6.1

Consider the transformation

$f(x,y,z)=(-12x-5z,3x-2y+z,6x+2y)$

Rewrite this transformation in the form

$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

where a,b,c,d,e,f,g,h *and* i are constant.

A square matrix is formed by the same number of rows and columns.

Square matrix of order three has the form

$$M=\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

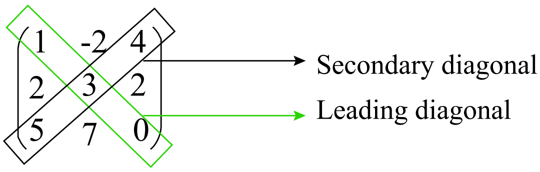
The elements of the form $\left(a_{ij}\right)$, where the two subscripts i and j

are equal, constitute the **principal diagonal** (or **leading diagonal** or **main diagonal** or **major diagonal** or **primary diagonal**).

The **secondary diagonal** (or **minor diagonal** or **antidiagonal** or **counterdiagonal**) is formed by the elements with $i + j = n + 1$.

Example 6.1

Matrix of order three



6.1.2. Types of matrices

Upper triangular matrix

In an upper triangular matrix, the elements located below the leading diagonal are zeros.

Example 6.2

$$M = \begin{pmatrix} 1 & 1 & 3 \\ 0 & 3 & 1 \\ 0 & 0 & 2 \end{pmatrix}$$

Lower triangular matrix

In a lower triangular matrix, the elements above the leading diagonal are zeros.

Example 6.3

$$N = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 2 & 0 \\ 3 & 2 & -2 \end{pmatrix}$$

Diagonal matrix

In a diagonal matrix, all the elements above and below the leading diagonal are zeros.

Example 6.4

$$M = \begin{pmatrix} 11 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 4 \end{pmatrix}$$

Scalar matrix

A scalar matrix is a diagonal matrix in which the leading diagonal elements are equal.

Example 6.5

$$M = \begin{pmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{pmatrix}$$

Identity matrix or unity matrix

An identity matrix (denoted by I) is a diagonal matrix in which the leading diagonal elements are equal to 1.

Example 6.6

Identity matrix of order three

$$I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Equality of matrices

Two matrices are equal if the elements of the two matrices that occupy the same position are equal.

If $\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & a_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix}$,

then $\begin{matrix} a_{11} = b_{11},\ a_{12} = b_{12},\ a_{13} = b_{13} \\ a_{21} = b_{21},\ a_{22} = b_{22},\ a_{23} = b_{23} \\ a_{31} = b_{31},\ a_{32} = b_{32},\ a_{33} = b_{33} \end{matrix}$

Application activity 6.1

1. Give five examples of
   a) matrices of order three
   b) upper triangular matrices of order three
   c) diagonal matrices of order three

2. Given matrices

$$A=\begin{pmatrix}1 & 2x-7 & x^2-7x+10\\ 2 & 5 & y+1\\ -2 & y^2-4 & 4\end{pmatrix},\quad B=\begin{pmatrix}1 & x-2 & 0\\ 2 & 5 & -1\\ -2 & 0 & 4\end{pmatrix},$$

if $A=B$ find the value(s) of x and y.

6.1.3. Operations on matrices

Activity 6.2

Consider the matrices $A=\begin{pmatrix}1 & 2 & 1\\ 0 & 1 & 1\\ 3 & 0 & -1\end{pmatrix}$, $B=\begin{pmatrix}2 & 0 & -1\\ 0 & 3 & 2\\ 1 & 0 & 1\end{pmatrix}$, and $C=\begin{pmatrix}1 & 2 & -1\\ 5 & 1 & 2\\ 1 & 4 & 0\end{pmatrix}$ find;

1. $A+3B$
2. $2A-B$
3. $A+(-A)$
4. $A+B$ and $B+A$. From the results, give your comment.
5. $A+(B+C)$ and $(A+B)+C$. Give your comment.
6. Interchange/switch the rows and columns of matrix A, B and C.

Adding matrices

When adding two matrices of the same dimension, the resultant matrix's elements are obtained by adding the elements that occupy the same position.

If $A=\begin{pmatrix}a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{pmatrix}$ and $B=\begin{pmatrix}b_{11} & b_{12} & b_{13}\\ b_{21} & b_{22} & b_{23}\\ b_{31} & b_{32} & b_{33}\end{pmatrix}$, then

$$A+B=\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}+\begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix}=\begin{pmatrix} a_{11}+b_{11} & a_{12}+b_{12} & a_{13}+b_{13} \\ a_{21}+b_{21} & a_{22}+b_{22} & a_{23}+b_{23} \\ a_{31}+b_{31} & a_{32}+b_{32} & a_{33}+b_{33} \end{pmatrix}$$

$$A-B=\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}-\begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix}=\begin{pmatrix} a_{11}-b_{11} & a_{12}-b_{12} & a_{13}-b_{13} \\ a_{21}-b_{21} & a_{22}-b_{22} & a_{23}-b_{23} \\ a_{31}-b_{31} & a_{32}-b_{32} & a_{33}-b_{33} \end{pmatrix}$$

Example 6.7

Considering the matrices $A=\begin{pmatrix} 2 & 0 & 1 \\ 3 & 0 & 0 \\ 5 & 1 & 1 \end{pmatrix}$ and $B=\begin{pmatrix} 1 & 0 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 0 \end{pmatrix}$, find $A+B$ and $A-B$.

Solution

$$A+B=\begin{pmatrix} 2+1 & 0+0 & 1+1 \\ 3+1 & 0+2 & 0+1 \\ 5+1 & 1+1 & 1+0 \end{pmatrix}=\begin{pmatrix} 3 & 0 & 2 \\ 4 & 2 & 1 \\ 6 & 2 & 1 \end{pmatrix}$$

$$A-B=\begin{pmatrix} 2-1 & 0-0 & 1-1 \\ 3-1 & 0-2 & 0-1 \\ 5-1 & 1-1 & 1-0 \end{pmatrix}=\begin{pmatrix} 1 & 0 & 0 \\ 2 & -2 & -1 \\ 4 & 0 & 1 \end{pmatrix}$$

Properties

1. **Closure property**

   The sum of two matrices of order three is another matrix of order three.

2. **Associative property**

   $A + (B + C) = (A + B) + C$

3. **Additive identity**

   $A + 0 = A$, where 0 is the zero-matrix of the same dimension.

4. **Additive inverse**

   $A + (-A) = O$

   The opposite matrix of A is $-A$.

5. **Commutative property**

   $A + B = B + A$

6.1.4. Scalar multiplication

Given a matrix, $A = \left(a_{ij}\right)$, and a real number, $k \in IR$, the product of a real number by a matrix is a matrix of the same dimension as **A**, and each element is multiplied by *k*.

$k \cdot A = \left(k\, a_{ij}\right)$

If $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$, then $kA = \begin{pmatrix} ka_{11} & ka_{12} & ka_{13} \\ ka_{21} & ka_{22} & ka_{23} \\ ka_{31} & ka_{32} & ka_{33} \end{pmatrix}$

Example 6.8

Consider the matrix $A = \begin{pmatrix} 2 & 0 & 1 \\ 3 & 0 & 0 \\ 5 & 1 & 1 \end{pmatrix}$, find $2A$.

Solution

$$2A = 2\begin{pmatrix} 2 & 0 & 1 \\ 3 & 0 & 0 \\ 5 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 4 & 0 & 2 \\ 6 & 0 & 0 \\ 10 & 2 & 2 \end{pmatrix}$$

Properties

1. $\alpha\left(\beta A\right) = \left(\alpha\beta\right)A, \quad A \in M_{m\times n}, \alpha, \beta \in IR$.
2. $\alpha\left(A+B\right) = \alpha A + \alpha B, \quad A, B \in M_{m\times n}, \alpha \in IR$.
3. $\left(\alpha+\beta\right)A = \alpha A + \beta A, \quad A \in M_{m\times n}, \alpha, \beta \in IR$.
4. $1A = A, \quad A \in M_{m\times n}$.

Application activity 6.2

If $A = \begin{pmatrix} 3 & 2 & 1 \\ 0 & -1 & 1 \\ 1 & 0 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 0 & -1 \\ 4 & 3 & 2 \\ 1 & -1 & 1 \end{pmatrix}$, and $C = \begin{pmatrix} 1 & 2 & -1 \\ 3 & 0 & 1 \\ 1 & 2 & 0 \end{pmatrix}$.

Evaluate;

1. $A+2B$
2. $A+B-C$
3. $3A-4B+C$

Transpose matrix

Activity 6.3

Consider the matrices $A = \begin{pmatrix} 3 & 3 & 1 \\ 1 & -1 & 1 \\ 2 & -2 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 1 & -1 \\ 1 & -1 & 1 \\ -2 & -1 & 0 \end{pmatrix}$ find;

1. Interchange/switch the rows and columns of matrix A and B.
2. Add two matrices obtained in 1.
3. Add matrices A and B.
4. Interchange/switch the rows and columns of matrix obtained in 3.
5. What can you say about result in 2 and 4?
6. Interchange/switch the rows and columns of matrix A twice. What can you conclude?

Given matrix A, the transpose of matrix A, denoted A^t, is another matrix where the elements in the columns and rows have switched. In other words, the rows become the columns and the columns become the rows.

If $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$, then $A^t = \begin{pmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{pmatrix}$

Example 6.9

$A = \begin{pmatrix} 1 & 3 & 6 \\ 0 & 2 & 0 \\ 3 & 5 & 8 \end{pmatrix}$ $\qquad$ $A^t = \begin{pmatrix} 1 & 0 & 3 \\ 3 & 2 & 5 \\ 6 & 0 & 8 \end{pmatrix}$

Properties of transpose of matrices

Let A, B be matrices of order three

1. $(A^t)^t = A$
2. $(A + B)^t = A^t + B^t$
3. $(\alpha \times A)^t = \alpha \times A^t,\ \alpha \in \mathbb{R}$
4. $(A \times B)^t = B^t \times A^t$

Application activity 6.3

Consider matrices $A=\begin{pmatrix}0&1&2\\1&0&1\\3&-2&1\end{pmatrix}$ and $B=\begin{pmatrix}1&1&0\\-4&1&3\\0&2&2\end{pmatrix}$.

Evaluate;

1. $(A+B)^t$ 2. $3A^t+B$ 3. $(-3B+4A)^t$

4. Find the value of x in $M=\begin{pmatrix}1&2&x^2\\4&1&0\\1&x+3&8\end{pmatrix}$ if $M^t=\begin{pmatrix}1&4&1\\2&1&1\\4&0&8\end{pmatrix}$

Multiplying matrices

Activity 6.4

Consider the matrices $P=\begin{pmatrix}-1&1&1\\2&1&2\\0&3&-1\end{pmatrix}$ and $Q=\begin{pmatrix}1&1&-1\\1&2&3\\-1&0&2\end{pmatrix}$.

Given that

$$A\times B=\begin{pmatrix}a_{11}&a_{12}&a_{13}\\a_{21}&a_{22}&a_{23}\\a_{31}&a_{32}&a_{33}\end{pmatrix}\times\begin{pmatrix}b_{11}&b_{12}&b_{13}\\b_{21}&b_{22}&b_{23}\\b_{31}&b_{32}&b_{33}\end{pmatrix}$$

$$=\begin{pmatrix}a_{11}b_{11}+a_{12}b_{21}+a_{13}b_{31}&a_{11}b_{12}+a_{12}b_{22}+a_{13}b_{32}&a_{11}b_{13}+a_{12}b_{23}+a_{13}b_{33}\\a_{21}b_{11}+a_{22}b_{21}+a_{23}b_{31}&a_{21}b_{12}+a_{22}b_{22}+a_{23}b_{32}&a_{21}b_{13}+a_{22}b_{23}+a_{23}b_{33}\\a_{31}b_{11}+a_{32}b_{21}+a_{33}b_{31}&a_{31}b_{12}+a_{32}b_{22}+a_{33}b_{32}&a_{31}b_{13}+a_{32}b_{23}+a_{33}b_{33}\end{pmatrix}$$

Find P.Q

Two matrices A and B can be multiplied together if and only if the number of columns of A is equal to the number of rows of B.

$$M_{m\times n}\times M_{n\times p}=M_{m\times p}$$

The element, c_{ij}, of the product matrix is obtained by multiplying every element in row **i** of matrix A by each element of column **j** of matrix B and then adding them together. This method of multiplication is called **ROCO** (row, column).

If $A=\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$ and $B=\begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix}$, then

$$A\times B=\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}\times\begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix}$$

$$=\begin{pmatrix} a_{11}b_{11}+a_{12}b_{21}+a_{13}b_{31} & a_{11}b_{12}+a_{12}b_{22}+a_{13}b_{32} & a_{11}b_{13}+a_{12}b_{23}+a_{13}b_{33} \\ a_{21}b_{11}+a_{22}b_{21}+a_{23}b_{31} & a_{21}b_{12}+a_{22}b_{22}+a_{23}b_{32} & a_{21}b_{13}+a_{22}b_{23}+a_{23}b_{33} \\ a_{31}b_{11}+a_{32}b_{21}+a_{33}b_{31} & a_{31}b_{12}+a_{32}b_{22}+a_{33}b_{32} & a_{31}b_{13}+a_{32}b_{23}+a_{33}b_{33} \end{pmatrix}$$

Example 6.10

Consider matrices $A=\begin{pmatrix} 2 & 0 & 1 \\ 3 & 0 & 0 \\ 5 & 1 & 1 \end{pmatrix}$ and $B=\begin{pmatrix} 1 & 0 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 0 \end{pmatrix}$, find $A\times B$.

Solution

$$A\times B=\begin{pmatrix} 2 & 0 & 1 \\ 3 & 0 & 0 \\ 5 & 1 & 1 \end{pmatrix}\times\begin{pmatrix} 1 & 0 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

$$=\begin{pmatrix} 2\times1+0\times1+1\times1 & 2\times0+0\times2+1\times1 & 2\times1+0\times1+1\times0 \\ 3\times1+0\times1+0\times1 & 3\times0+0\times2+0\times1 & 3\times1+0\times1+0\times0 \\ 5\times1+1\times1+1\times1 & 5\times0+1\times2+1\times1 & 5\times1+1\times1+1\times0 \end{pmatrix}$$

$$=\begin{pmatrix} 3 & 1 & 2 \\ 3 & 0 & 3 \\ 7 & 3 & 6 \end{pmatrix}$$

Application activity 6.4

If $A=\begin{pmatrix}1 & -2 & 6\\ 2 & 2 & -5\\ 0 & 4 & 8\end{pmatrix}$, $B=\begin{pmatrix}0 & 8 & 9\\ 14 & 4 & 6\\ 0 & 6 & 7\end{pmatrix}$ and $C=\begin{pmatrix}13 & -2 & 0\\ 10 & 0 & 3\\ 9 & 1 & -5\end{pmatrix}$.

Evaluate;

1) $A\times B$ 2) $A\times C$ 3) $B\times C$

Properties of matrices multiplication

Activity 6.5

Consider the matrices $A=\begin{pmatrix}3 & 0 & 1\\ 0 & -1 & 1\\ -2 & 3 & 2\end{pmatrix}$ $B=\begin{pmatrix}0 & 1 & -1\\ 1 & 1 & -2\\ -1 & 0 & 1\end{pmatrix}$

and $C=\begin{pmatrix}1 & 1 & -1\\ 1 & -2 & 1\\ -1 & 1 & 0\end{pmatrix}$ find:

1. $A\times B$ and $B\times A$
2. $(A\times B)^t$ and $B^t\times A^t$
3. $A\times(B\times C)$ and $(A\times B)\times C$
4. $A\times(B+C)$ and $A\times B+A\times C$

Comment on your results.

Let A, B, C be matrices of order three

1. **Associative**

   $A\times(B\times C)=(A\times B)\times C$

2. **Multiplicative identity**

   $A\times I=A$, where I is the identity matrix with the same order as matrix A.

3. **Not commutative**

   $A\times B=B\times A$

4. **Distributive**

   $A\times(B+C)=A\times B+A\times C$

5. $(A \times B)^t = B^t \times A^t$

Example 6.11

Given the matrices:

$$A = \begin{pmatrix} 2 & 0 & 1 \\ 3 & 0 & 0 \\ 5 & 1 & 1 \end{pmatrix} \text{ and } B = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

Find;

a) The product $A \times B$

b) The product $B \times A$

Solution

a)

$$A \times B = \begin{pmatrix} 2 & 0 & 1 \\ 3 & 0 & 0 \\ 5 & 1 & 1 \end{pmatrix} \times \begin{pmatrix} 1 & 0 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 2\times1+0\times1+1\times1 & 2\times0+0\times2+1\times1 & 2\times1+0\times1+1\times0 \\ 3\times1+0\times1+0\times1 & 3\times0+0\times2+0\times1 & 3\times1+0\times1+0\times0 \\ 5\times1+1\times1+1\times1 & 5\times0+1\times2+1\times1 & 5\times1+1\times1+1\times0 \end{pmatrix}$$

$$= \begin{pmatrix} 3 & 1 & 2 \\ 3 & 0 & 3 \\ 7 & 3 & 6 \end{pmatrix}$$

b)

$$B \times A = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 0 \end{pmatrix} \times \begin{pmatrix} 2 & 0 & 1 \\ 3 & 0 & 0 \\ 5 & 1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1\times2+0\times3+1\times5 & 1\times0+0\times0+1\times1 & 1\times1+0\times0+1\times1 \\ 1\times2+2\times3+1\times5 & 1\times0+2\times0+1\times1 & 1\times1+2\times0+1\times1 \\ 1\times2+1\times3+0\times5 & 1\times0+1\times0+0\times1 & 1\times1+1\times0+0\times1 \end{pmatrix}$$

$$= \begin{pmatrix} 7 & 1 & 2 \\ 13 & 1 & 2 \\ 5 & 0 & 1 \end{pmatrix}$$

Notice

- If $AB = 0$, it does not necessarily follow that $A = 0$ or $B = 0$.
- **Commuting matrices in multiplication**

In general, the multiplication of matrices is not commutative, i.e, $AB \neq BA$, but we can have the case where two matrices A and B satisfy $AB = BA$. In this case, A and B are said to be **commuting**.

Example 6.12

Show that matrices $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 2 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & 4 & 0 \\ 3 & 1 & 0 \\ -1 & -4 & 1 \end{pmatrix}$ commute in multiplication.

Solution

$$AB = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 2 \end{pmatrix}\begin{pmatrix} 2 & 4 & 0 \\ 3 & 1 & 0 \\ -1 & -4 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 4 & 0 \\ 3 & 1 & 0 \\ 0 & -4 & 2 \end{pmatrix}$$

$$BA = \begin{pmatrix} 2 & 4 & 0 \\ 3 & 1 & 0 \\ -1 & -4 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 2 \end{pmatrix} = \begin{pmatrix} 2 & 4 & 0 \\ 3 & 1 & 0 \\ 0 & -4 & 2 \end{pmatrix}$$

$$\Rightarrow AB = BA$$

- **Trace of matrix**

The sum of the entries on the leading diagonal of a square matrix, A, is known as the **trace** of that matrix, denoted $tr(A)$.

Example 6.13

1. Trace of $\begin{pmatrix} 1 & -2 & 4 \\ 2 & 3 & 2 \\ 5 & 7 & 2 \end{pmatrix} = 1 + 3 + 2 = 6$
2. Trace of $\begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix} = 1 + 1 = 2$

Properties of trace of matrix

1. $tr(A+B)=tr(A)+tr(B)$
2. $tr(\alpha A)=\alpha\, tr(A)$
3. $tr(A)=tr(A)^t$
4. $tr(AB)=tr(BA)$
5. $tr(ABC)=tr(BCA)=tr(CAB)$, cyclic property.
6. $tr(ABC)\neq tr(ACB)$, arbitrary permutations are not allowed.

Application activity 6.5

Consider the matrices $A=\begin{pmatrix}1&0&1\\1&-1&1\\0&0&0\end{pmatrix}$ $B=\begin{pmatrix}-2&0&1\\1&0&2\\-1&0&1\end{pmatrix}$ and $C=\begin{pmatrix}1&0&-1\\-2&2&0\\-1&1&0\end{pmatrix}$ find;

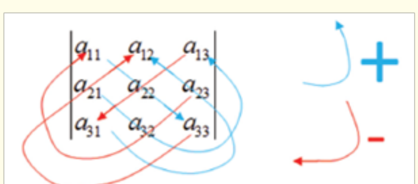
1. $A\times B$ and $B\times A$
2. $A\times(B\times C)$ and $(A\times B)\times C$
3. $A\times(B+C)$ and $A\times B+A\times C$
4. $tr(AB)$

6.2. Determinants of order three

6.2.1. Determinant

Activity 6.6

Observe the following notation of Matrix A and the corresponding explanation:



The arrows and signs mean that

$DetA=a_{11}a_{22}a_{33}+a_{21}a_{32}a_{13}+a_{31}a_{23}a_{12}-a_{13}a_{22}a_{31}-a_{23}a_{32}a_{11}-a_{33}a_{21}a_{12}$.

Refer to your observation, use the same signs and evaluate detP and detQ where

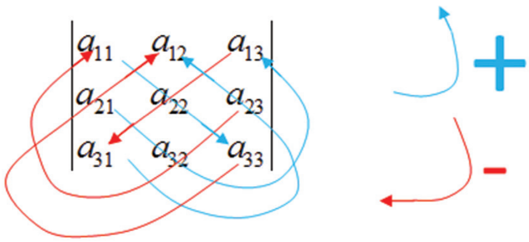
$$P=\begin{pmatrix}1 & -4 & 2\\ 3 & 6 & 1\\ 5 & 0 & 1\end{pmatrix} \text{ and } Q=\begin{pmatrix}10 & 2 & 4\\ -6 & 5 & 2\\ 0 & 3 & 1\end{pmatrix}$$

The determinant of a matrix $A=\begin{pmatrix}a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{pmatrix}$ is noted by

$\det A=\begin{vmatrix}a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{vmatrix}$ and calculated using the rule of SARRUS.

The terms with a **positive sign** are formed by the elements of the **principal diagonal** and those of the **parallel diagonals** with its corresponding **opposite vertex**.

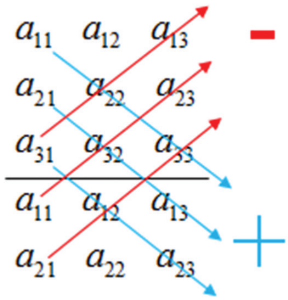
The terms with a **negative sign** are formed by the elements of the **secondary diagonal** and those of the **parallel diagonals** with its corresponding **opposite vertex**.



$$DetA=a_{11}a_{22}a_{33}+a_{21}a_{32}a_{13}+a_{31}a_{23}a_{12}-a_{13}a_{22}a_{31}-a_{23}a_{32}a_{11}-a_{33}a_{21}a_{12}$$

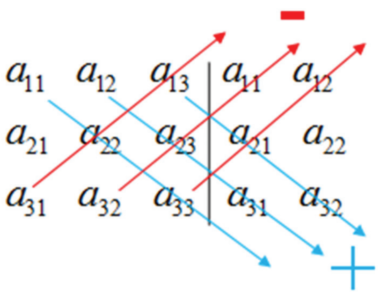
Or we can work out as follows:

To calculate the 3x3 determinant, we rewrite the first two rows below the determinant (or first two columns to the right of the determinant).



$$DetA = a_{11}a_{22}a_{33} + a_{21}a_{32}a_{13} + a_{31}a_{23}a_{12} - a_{13}a_{22}a_{31} - a_{23}a_{32}a_{11} - a_{33}a_{21}a_{12}$$

Or



$$\det = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{31}a_{22}a_{13} - a_{32}a_{23}a_{11} - a_{33}a_{21}a_{12}$$

As multiplication of real numbers is commutative, the three are the same.

Example 6.14

$$\begin{vmatrix} 3 & 2 & 1 \\ 0 & 2 & -5 \\ -2 & 1 & 4 \end{vmatrix} = 3\times2\times4+0\times1\times1+(-2)\times(-5)\times2-1\times2\times(-2)-(-5)\times1\times3-4\times0\times2$$

$$= 24+0+20+4+15-0$$

$$= 63$$

General rule for nxn matrices (minor and cofactor)

General method of finding the determinant of matrix with nxn dimension $(2\times2,\ 3\times3,\ 4\times4,\ 5\times5,\ldots)$ is the use of cofactors.

Minor

An element, a_{ij}, to the value of the determinant of order $n-1$, obtained by deleting the row i and the column j in the matrix is called a minor.

$$\begin{vmatrix} 1 & 2 & 1 \\ 2 & [5] & 4 \\ 3 & 6 & 2 \end{vmatrix} \to \begin{vmatrix} 1 & 1 \\ 3 & 2 \end{vmatrix}$$

Cofactor

The cofactor of the element a_{ij} is its minor prefixing:

The **+** sign if **i+j** is **even**.

The **−** sign if **i+j** is **odd**.

$$\begin{vmatrix} 1 & 2 & 1 \\ [2] & 5 & 4 \\ 3 & 6 & 2 \end{vmatrix} \to -\begin{vmatrix} 2 & 1 \\ 6 & 2 \end{vmatrix}$$

The value of a determinant is equal to the sum of the products of the elements of a line (row or column) by its corresponding cofactors:

$$|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

Example 6.15

$$\begin{vmatrix} 3 & 2 & 1 \\ 0 & 2 & -5 \\ -2 & 1 & 4 \end{vmatrix} = 3\begin{vmatrix} 2 & -5 \\ 1 & 4 \end{vmatrix} - 2\begin{vmatrix} 0 & -5 \\ -2 & 4 \end{vmatrix} + 1\begin{vmatrix} 0 & 2 \\ -2 & 1 \end{vmatrix}$$

$$= 3(8+5) - 2(0-10) + 1(0+4)$$

$$= 39 + 20 + 4$$

$$= 63$$

Note that we choose only one line (row or column).

Application activity 6.6

Find the determinants of the following matrices:

1. $A = \begin{pmatrix} 1 & -1 & 1 \\ -2 & 3 & 0 \\ -3 & 1 & 2 \end{pmatrix}$ 2. $B = \begin{pmatrix} 1 & 2 & -1 \\ -1 & 1 & 1 \\ 1 & -3 & 1 \end{pmatrix}$ 3. $C = \begin{pmatrix} 1 & 0 & 1 \\ -1 & 0 & 1 \\ -1 & -4 & 0 \end{pmatrix}$

Properties of a determinant

Activity 6.7

Consider the matrices $A=\begin{pmatrix}1&0&1\\1&-1&1\\0&0&0\end{pmatrix}$, $B=\begin{pmatrix}-2&1&-1\\1&3&4\\-1&4&3\end{pmatrix}$,

$C=\begin{pmatrix}1&0&-1\\0&2&1\\0&0&3\end{pmatrix}$ and $D=\begin{pmatrix}1&3&-1\\-1&2&2\\0&1&-1\end{pmatrix}$ find:

1. $|A|$ and $|B|$
2. $|C\cdot D|$ and $|C|\cdot|D|$. How can you conclude?
3. Product of leading diagonal elements of matrix C and $|C|$. How can you conclude?

1. The determinant of matrix A and its transpose A^t are equal.

$|A^t| = |A|$

Example 6.16

$$|A|=\begin{vmatrix}2&3&0\\3&2&7\\2&1&6\end{vmatrix} \qquad |A^t|=\begin{vmatrix}2&3&2\\3&2&1\\0&7&6\end{vmatrix}, \qquad |A|=|A^t|=-2$$

2. $|A| = 0$ if:

- It has two equal lines

Example 6.17

$$|A|=\begin{vmatrix}2&3&2\\3&2&3\\2&3&2\end{vmatrix}=0$$

- All elements of a line are zero.

Example 6.18

$$|A| = \begin{vmatrix} 2 & 3 & 2 \\ 3 & 2 & 3 \\ 0 & 0 & 0 \end{vmatrix} = 0$$

- The elements of a line are a linear combination of the others.

Example 6.19

$$|A| = \begin{vmatrix} 2 & 3 & 2 \\ 1 & 2 & 4 \\ 3 & 5 & 6 \end{vmatrix} = 0$$

$r_3 = r_1 + r_2$

3. A triangular matrix determinant is the product of the leading diagonal elements.

Example 6.20

$$|A| = \begin{vmatrix} 2 & 0 & 0 \\ 1 & 2 & 0 \\ 3 & 5 & 6 \end{vmatrix} = 2 \times 2 \times 6 = 24$$

4. If a determinant switches two parallel lines its determinant changes sign.

Example 6.21

$$|A| = \begin{vmatrix} 2 & 1 & 2 \\ 1 & 2 & 0 \\ 3 & 5 & 6 \end{vmatrix} = -\begin{vmatrix} 1 & 2 & 0 \\ 2 & 1 & 2 \\ 3 & 5 & 6 \end{vmatrix}$$

5. If the elements of a line are added to the elements of another parallel line previously multiplied by a real number, the value of the determinant is unchanged.

Example 6.22

$$\begin{vmatrix} 2 & 1 & 2 \\ 1 & 2 & 0 \\ 3 & 5 & 6 \end{vmatrix} = 16 \qquad c_3 = 2c_1 + c_2 + c_3 \qquad \begin{vmatrix} 2 & 1 & 7 \\ 1 & 2 & 4 \\ 3 & 5 & 17 \end{vmatrix} = 16$$

6. If a determinant is multiplied by a real number, any line can be multiplied by the above mentioned number, but only one.

Example 6.23

$$2 \times \begin{vmatrix} 2 & 1 & 2 \\ 1 & 2 & 0 \\ 3 & 5 & 6 \end{vmatrix} = \begin{vmatrix} 2\times 2 & 1 & 2 \\ 2\times 1 & 2 & 0 \\ 2\times 3 & 5 & 6 \end{vmatrix} = \begin{vmatrix} 4 & 1 & 2 \\ 2 & 2 & 0 \\ 6 & 5 & 6 \end{vmatrix} = 32 \; 2 \times \begin{vmatrix} 2 & 1 & 2 \\ 1 & 2 & 0 \\ 3 & 5 & 6 \end{vmatrix} = 2 \times 16 = 32$$

7. If all the elements of a line are formed by two addends, the above mentioned determinant decomposes in the sum of two determinants.

Example 6.24

$$\begin{vmatrix} 2 & 1 & 2 \\ a+b & a+c & a+d \\ 3 & 5 & 6 \end{vmatrix} = \begin{vmatrix} 2 & 1 & 2 \\ a & a & a \\ 3 & 5 & 6 \end{vmatrix} + \begin{vmatrix} 2 & 1 & 2 \\ b & c & d \\ 3 & 5 & 6 \end{vmatrix}$$

Example 6.25

$$\begin{vmatrix} 1 & 2 & 1 \\ 7 & 8 & 9 \\ 3 & 2 & 3 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 1 \\ 2 & 4 & 8 \\ 3 & 2 & 3 \end{vmatrix} + \begin{vmatrix} 1 & 2 & 1 \\ 5 & 4 & 1 \\ 3 & 2 & 3 \end{vmatrix}$$

$8 = 24 - 16$

$8 = 8$

8. The determinant of a product equals the product of the determinants.

$|A \times B| = |A| \times |B|$

Example 6.26

Let $A=\begin{pmatrix}1&2&1\\2&4&8\\3&2&3\end{pmatrix}$, $B=\begin{pmatrix}3&-1&1\\0&4&1\\3&2&2\end{pmatrix}$

$A\times B=\begin{pmatrix}6&9&5\\30&30&22\\18&11&11\end{pmatrix}$ $\quad |A\times B|=\begin{vmatrix}6&9&5\\30&30&22\\18&11&11\end{vmatrix}=72$

$|A|=\begin{vmatrix}1&2&1\\2&4&8\\3&2&3\end{vmatrix}=24,\ |B|=\begin{vmatrix}3&-1&1\\0&4&1\\3&2&2\end{vmatrix}=3$

$|A|\times|B|=24\times3=72$

Application activity 6.7

Find the determinants of the following matrices:

1. $A=\begin{pmatrix}12&0&1\\34&0&2\\-3&0&3\end{pmatrix}$ 2. $B=\begin{pmatrix}1&4&5\\2&6&8\\3&2&5\end{pmatrix}$ 3. $C=\begin{pmatrix}6&7&6\\2&4&8\\1&3&9\end{pmatrix}$

6.2.2. Matrix inverse

Activity 6.8

Consider the matrix $A=\begin{pmatrix}1&1&1\\2&1&-1\\3&2&1\end{pmatrix}$

1. Calculate the determinant $|A|$ of A.
2. Replace every element in matrix A by its cofactor to find a new matrix called cofactor matrix.
3. Find the transpose of the cofactor matrix.
4. Multiply the inverse value of determinant obtained in 1 by the matrix obtained in 3.
5. Multiply matrix A by matrix obtained in 4. Discuss your result.

Calculating matrix inverse of matrix A, is to find matrix A^{-1} such that,

$$A \cdot A^{-1} = A^{-1} \cdot A = I$$

Where I is identity matrix.

From activity 6.8, the matrix inverse of matrix A is equal to the its determinant multiplied by the adjugate matrix.

$$A^{-1} = \frac{1}{|A|}.adj(A)$$

Where $adj(A)$ is the **adjoint** matrix which is the transpose of the cofactor matrix. The cofactor matrix is found by replacing every element in matrix A by its cofactor.

Example 6.27

Find the inverse of the following matrix:

$$A = \begin{pmatrix} 2 & 0 & 1 \\ 3 & 0 & 0 \\ 5 & 1 & 1 \end{pmatrix}$$

Solution

We find its inverse as follows:

a) $|A| = 3$

b) Cofactor of each element:

$$C(2) = \begin{vmatrix} 0 & 0 \\ 1 & 1 \end{vmatrix} \qquad c(0) = -\begin{vmatrix} 3 & 0 \\ 5 & 1 \end{vmatrix} = -3 \qquad c(1) = \begin{vmatrix} 3 & 0 \\ 5 & 1 \end{vmatrix} = 3$$

$$C(3) = \begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix} \qquad c(0) = \begin{vmatrix} 2 & 1 \\ 5 & 1 \end{vmatrix} = -3 \qquad c(0) = -\begin{vmatrix} 2 & 0 \\ 5 & 1 \end{vmatrix} = -2$$

$$C(5) = \begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix} \qquad c(1) = -\begin{vmatrix} 2 & 1 \\ 3 & 0 \end{vmatrix} = 3 \qquad c(1) = \begin{vmatrix} 2 & 0 \\ 3 & 0 \end{vmatrix} = 0$$

The cofactor matrix is

$$\begin{pmatrix} 0 & -3 & 3 \\ 1 & -3 & -2 \\ 0 & 3 & 0 \end{pmatrix}, \text{ and then } adj(A) = \begin{pmatrix} 0 & 1 & 0 \\ -3 & -3 & 3 \\ 3 & -2 & 0 \end{pmatrix}$$

The matrix inverse of A is $A^{-1}=\frac{1}{|A|}adj(A)=\frac{1}{3}\begin{pmatrix}0 & 1 & 0\\ -3 & -3 & 3\\ 3 & -2 & 0\end{pmatrix}$

Therefore, $A^{-1}=\begin{pmatrix}0 & \frac{1}{3} & 0\\ -1 & -1 & 1\\ 1 & \frac{-2}{3} & 0\end{pmatrix}$

Example 6.28

Find the inverse of the following matrix:

$$A=\begin{pmatrix}2 & 3 & -1\\ 1 & 0 & 2\\ 3 & 3 & 1\end{pmatrix}$$

Solution

$|A|=0$

Since the determinant is zero, the given matrix has no inverse.

Properties of the Inverse Matrix

1. $(A\cdot B)^{-1}=B^{-1}\cdot A^{-1}$
2. $\left(A^{-1}\right)^{-1}=A$
3. $(\alpha\cdot A)^{-1}=\alpha^{-1}\cdot A^{-1}$
4. $\left(A^{t}\right)^{-1}=\left(A^{-1}\right)^{t}$

Application activity 6.8

Find the inverse of the following matrices:

1. $A=\begin{pmatrix}1 & 4 & 5\\ 2 & 6 & 8\\ 3 & 2 & 5\end{pmatrix}$
2. $B=\begin{pmatrix}1 & 4 & 5\\ 2 & 6 & 8\\ 3 & 2 & 5\end{pmatrix}$
3. $C=\begin{pmatrix}6 & 7 & 6\\ 2 & 4 & 8\\ 1 & 3 & 9\end{pmatrix}$
4. $D=\begin{pmatrix}-3 & 5 & 1\\ 2 & 10 & 1\\ 1 & 8 & 1\end{pmatrix}$

6.3. Application

6.3.1. System of 3 linear equations

Activity 6.9

Consider the following system of 3 linear equations in 3 unknowns.

$$\begin{cases} a_{11}x + a_{12}y + a_{13}z = c_1 \\ a_{21}x + a_{22}y + a_{23}z = c_2 \\ a_{31}x + a_{32}y + a_{33}z = c_3 \end{cases}$$

1. Rewrite this system in matrix form.
2. If we premultiply (multiply to the left) both sides of the equality obtained in 1) by $\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}^{-1}$, what will be the new equality?

From activity 6.9, the solution of the following system of 3 linear equations in 3 unknowns.

$$\begin{cases} a_{11}x + a_{12}y + a_{13}z = c_1 \\ a_{21}x + a_{22}y + a_{23}z = c_2 \\ a_{31}x + a_{32}y + a_{33}z = c_3 \end{cases}$$

is $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = A^{-1}B$, provided that A^{-1} exists.

where

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \text{ and } B = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$$

Notice

- If at least, one of c_i is different from zero, the system is said to be **non-homogeneous** and if all c_i are zero the system is said to be **homogeneous**.
- The set of values of x, y, z that satisfy all the equations of system (1) is called **solution of the system**.
- For the homogeneous system, the solution $x = y = z = 0$ is **called trivial solution**. Other solutions are **non-trivial solutions**.
- Non- homogeneous system cannot have a trivial solution as at least one of x, y, z is not zero.
- It is not allowed to use $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = BA^{-1}$

Alternative method: Cramer's rule

Consider the system

$$\begin{cases} a_{11}x + a_{12}y + a_{13}z = c_1 \\ a_{21}x + a_{22}y + a_{23}z = c_2 \\ a_{31}x + a_{32}y + a_{33}z = c_3 \end{cases}$$

We use Cramer's rule as follows:

$$\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \qquad \Delta_x = \begin{vmatrix} c_1 & a_{12} & a_{13} \\ c_2 & a_{22} & a_{23} \\ c_3 & a_{32} & a_{33} \end{vmatrix}$$

$$\Delta_y = \begin{vmatrix} a_{11} & c_1 & a_{13} \\ a_{21} & c_2 & a_{23} \\ a_{31} & c_3 & a_{33} \end{vmatrix} \qquad \Delta_z = \begin{vmatrix} a_{11} & a_{12} & c_1 \\ a_{21} & a_{22} & c_2 \\ a_{31} & a_{32} & c_3 \end{vmatrix}$$

$x = \frac{\Delta_x}{\Delta}$, $y = \frac{\Delta_y}{\Delta}$ and $z = \frac{\Delta_z}{\Delta}$

Recall that

- The solution $\frac{b}{0}, b \neq 0$ means impossible.
- The solution $\frac{0}{0}$ means indeterminate.

Example 6.29

Solve

$$\begin{cases} x + y + z = 6 \\ 2x + y - z = 1 \\ 3x + 2y + z = 10 \end{cases}$$

Solution

$$\begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ 3 & 2 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 6 \\ 1 \\ 10 \end{pmatrix}$$

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ 3 & 2 & 1 \end{pmatrix}, \ X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \ B = \begin{pmatrix} 6 \\ 1 \\ 10 \end{pmatrix}$$

We find the inverse of A. A is invertible if its determinant is not zero.

$\det(A) = \begin{vmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ 3 & 2 & 1 \end{vmatrix} = 1 + 4 - 3 - 3 + 2 - 2 = -1 \neq 0$, then A has inverse.

We have seen that the adjugate matrix and determinant of a matrix are used to find its inverse.

Let use another useful method.

We have $A=\begin{pmatrix}1 & 1 & 1\\ 2 & 1 & -1\\ 3 & 2 & 1\end{pmatrix}$, to find its inverse, suppose that its inverse is given by

$$A^{-1}=\begin{pmatrix}a & d & g\\ b & e & h\\ c & f & i\end{pmatrix}$$

We know that $AA^{-1}=I$, then,

$$\begin{pmatrix}1 & 1 & 1\\ 2 & 1 & -1\\ 3 & 2 & 1\end{pmatrix}\begin{pmatrix}a & d & g\\ b & e & h\\ c & f & i\end{pmatrix}=\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}\Rightarrow\begin{cases}\begin{cases}a+b+c=1\\ 2a+b-c=0\\ 3a+2b+c=0\end{cases} & (1)\\ \begin{cases}d+e+f=0\\ 2d+e-f=1\\ 3d+2e+f=0\end{cases} & (2)\\ \begin{cases}g+h+i=0\\ 2g+h-i=0\\ 3g+2h+i=1\end{cases} & (3)\end{cases}$$

We solve these three systems to find value of a, b, c, d, e, f, g, h, and i.

$$\begin{cases}a+b+c=1\\ 2a+b-c=0\\ 3a+2b+c=0\end{cases}\quad (1)\Rightarrow\begin{cases}a=-3\\ b=5\\ c=-1\end{cases}$$

$$\begin{cases}d+e+f=0\\ 2d+e-f=1\\ 3d+2e+f=0\end{cases}\quad (2)\Rightarrow\begin{cases}d=-1\\ e=2\\ f=-1\end{cases}$$

$$\begin{cases}g+h+i=0\\ 2g+h-i=0\\ 3g+2h+i=1\end{cases}\quad (3)\Rightarrow\begin{cases}g=2\\ h=-3\\ i=1\end{cases}$$

Then,

$$A^{-1} = \begin{pmatrix} -3 & -1 & 2 \\ 5 & 2 & -3 \\ -1 & -1 & 1 \end{pmatrix}$$

$$x = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -3 & -1 & 2 \\ 5 & 2 & -3 \\ -1 & -1 & 1 \end{pmatrix}\begin{pmatrix} 6 \\ 1 \\ 10 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

Therefore, $S = \{(1,2,3)\}$

Alternative method

$$\Delta = \begin{vmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ 3 & 2 & 1 \end{vmatrix} = -1 \qquad \Delta_x = \begin{vmatrix} 6 & 1 & 1 \\ 1 & 1 & -1 \\ 10 & 2 & 1 \end{vmatrix} = -1$$

$$\Delta_y = \begin{vmatrix} 1 & 6 & 1 \\ 2 & 1 & -1 \\ 3 & 10 & 1 \end{vmatrix} = -2 \qquad \Delta_z = \begin{vmatrix} 1 & 1 & 6 \\ 2 & 1 & 1 \\ 3 & 2 & 10 \end{vmatrix} = -3$$

$$x = \frac{\Delta_x}{\Delta} = \frac{-1}{-1} = 1, \quad y = \frac{\Delta_y}{\Delta} = \frac{-2}{-1} = 2, \quad z = \frac{\Delta_z}{\Delta} = \frac{-3}{-1} = 3$$

Therefore, $S = \{(1,2,3)\}$

Example 6.30

A dietitian at Hospital wants a patient to have a meal that has 65 grams of protein, 95 grams of carbohydrates, and 905 milligrams of calcium. The hospital food service tells the dietitian that the dinner for today is chicken , baked potatoes, and 2% milk. Each serving of chicken has 30 grams of protein, 35 grams of carbohydrates, and 200 milligrams of calcium. Each serving of baked potatoes contains 4 grams of protein, 33 grams of carbohydrates, and 10 milligrams of calcium. Each glass of 2% milk contains 9 grams of protein, 13 grams of carbohydrates, and 300 milligrams of calcium. How many servings of each food should the dietitian provide for the patient?

Solution:

Let $C, \text{B}, and\ M$ represent the number of serving of chicken, baked potatoes, and milk respectively.

$$\begin{cases} 30C + \ 4B + \ \ \ 9M = \ \ 65\ protein\ equation \\ 35C + 33B + \ \ 13M = \ \ 95\ carbohydrates\ equation \\ 200C + 10B + 300M = 905\ calcium\ equation \end{cases}$$

$$\Delta = \begin{vmatrix} 30 & 4 & 9 \\ 35 & 33 & 13 \\ 200 & 10 & 300 \end{vmatrix} = 205250$$

$$\Delta_C = \begin{vmatrix} 65 & 4 & 9 \\ 95 & 33 & 13 \\ 905 & 10 & 300 \end{vmatrix} = 307875 \Leftrightarrow C = \frac{307875}{205250} = \frac{3}{2}$$

$$\Delta_B = \begin{vmatrix} 30 & 65 & 9 \\ 35 & 95 & 13 \\ 200 & 905 & 300 \end{vmatrix} = 102625 \Leftrightarrow B = \frac{102625}{205250} = \frac{1}{2}$$

$$\Delta_M = \begin{vmatrix} 30 & 4 & 65 \\ 35 & 33 & 95 \\ 200 & 10 & 905 \end{vmatrix} = 410500 \Leftrightarrow M = \frac{410500}{205250} = 2$$

Therefore, the will be 2 glasses of milk, $\frac{1}{2}$ baked potato and $\frac{3}{2}$ of chicken for each serving.

Example 6.31

Food perfect corporation manufactures three models of the perfect food processor. Each Model X processor requires 30 minutes of electrical assembly, 40 minutes of mechanical assembly and 30 minutes of testing. Each Model Y processor requires 20 minutes of electrical assembly, 50 minutes of mechanical assembly and 30 minutes of testing. Each Model Z processor requires 30 minutes of electrical assembly, 30 minutes of mechanical assembly and 20 minutes of testing. If 2500 minutes of electrical assembly, 3500 minutes of mechanical assembly and 2400 minutes of testing are used in one day; how many of each model will be produced?

Solution: The following table summarizes the given information:

	Model			Time used
	X	Y	Z	
Electrical assembly	30	20	30	2500
Mechanical assembly	40	50	30	3500
Testing	30	30	20	2400

Let assign variables to represent the unknowns:

$x = \text{the number of Model } X \text{ produced}$

$y = \text{the number of Model } Y \text{ produced}$

$z = \text{the number of Model } Z \text{ produced}$

Based on the table , we obtain the following system of equations:

$$\begin{cases} 30x + 20y + 30z = 2500 \\ 40x + 50y + 30z = 3500 \\ 30x + 30y + 20z = 2400 \end{cases}$$

Divide each equation by 10

$$\begin{cases} 3x + 2y + 3z = 250 \\ 4x + 5y + 3z = 350 \\ 3x + 3y + 2z = 240 \end{cases}$$

Write *the system by use of matrices*

$$\begin{pmatrix} 3 & 2 & 3 \\ 4 & 5 & 3 \\ 3 & 3 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 250 \\ 350 \\ 240 \end{pmatrix} \Leftrightarrow A = \begin{pmatrix} 3 & 2 & 3 \\ 4 & 5 & 3 \\ 3 & 3 & 2 \end{pmatrix}$$

find the det A

$$DetA = \begin{vmatrix} 3 & 2 & 3 \\ 4 & 5 & 3 \\ 3 & 3 & 2 \end{vmatrix} = -4$$

$$D_x = \begin{vmatrix} 250 & 2 & 3 \\ 350 & 5 & 3 \\ 240 & 3 & 2 \end{vmatrix} = -160 \Leftrightarrow x = \frac{D_x}{D} = \frac{-160}{-4} = 40$$

$$D_y = \begin{vmatrix} 3 & 250 & 3 \\ 4 & 350 & 3 \\ 3 & 240 & 2 \end{vmatrix} = -80 \Leftrightarrow y = \frac{D_y}{D} = \frac{-80}{-4} = 20$$

$$D_z = \begin{vmatrix} 3 & 2 & 250 \\ 4 & 5 & 350 \\ 3 & 3 & 240 \end{vmatrix} = -120 \Leftrightarrow z = \frac{D_z}{D} = \frac{-120}{-4} = 30$$

$$S = \{(40, 20, 30)\}$$

In one day , Food Perfect Corporation produce

$40\,Model\,X$, $20\,Model\,Y$ *and* $30\,Model\,Z$ perfect processors

Application activity 6.9

1) Use matrix inverse method to solve the following systems:

1. $\begin{cases} 3x + y + z = 0 \\ 2x - y + 2x = 0 \\ 7x + y - 3z = 0 \end{cases}$

2. $\begin{cases} 4x + y - z = 1 \\ x - 3y + z = 2 \\ 5x - 2y = 4 \end{cases}$

3. $\begin{cases} x + y - z = 3 \\ 3x - y + z = 1 \\ -2x + y + z = 0 \end{cases}$

2) A hospital dietician is planning a meal consisting of three foods whose ingredients are summarized as follows:

	Chicken breast	Potato	Spinach
Grams of protein	24	4	5
Grams of carbohydrates	0	26	7
Grams of fat	1.5	0	0.5

Determine the number of servings of each food needed to create a meal containing 38 grams of protein, 40 grams of carbohydrates and 2.5 grams of fat.

Unit Summary

1. Square matrix of order three has the form

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

2. In an upper triangular matrix, the elements located below the leading diagonal are zeros.
3. In a lower triangular matrix, the elements above the leading diagonal are zeros.
4. In a diagonal matrix, all the elements above and below the leading diagonal are zeros.
5. A scalar matrix is a diagonal matrix in which the leading diagonal elements are equal.
6. An identity matrix (denoted by **I**) is a diagonal matrix in which the leading diagonal elements are equal to 1.
7. If $\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & a_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix}$, then

$a_{11} = b_{11}$, $a_{12} = b_{12}$, $a_{13} = b_{13}$
$a_{21} = b_{21}$, $a_{22} = b_{22}$, $a_{23} = b_{23}$
$a_{31} = b_{31}$, $a_{32} = b_{32}$, $a_{33} = b_{33}$

8. If $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$ and $B = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix}$, then

$$A + B = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} + \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix} = \begin{pmatrix} a_{11}+b_{11} & a_{12}+b_{12} & a_{13}+b_{13} \\ a_{21}+b_{21} & a_{22}+b_{22} & a_{23}+b_{23} \\ a_{31}+b_{31} & a_{32}+b_{32} & a_{33}+b_{33} \end{pmatrix}$$

$$A - B = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} - \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix} = \begin{pmatrix} a_{11}-b_{11} & a_{12}-b_{12} & a_{13}-b_{13} \\ a_{21}-b_{21} & a_{22}-b_{22} & a_{23}-b_{23} \\ a_{31}-b_{31} & a_{32}-b_{32} & a_{33}-b_{33} \end{pmatrix}$$

9. If $A=\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$, then $kA=\begin{pmatrix} ka_{11} & ka_{12} & ka_{13} \\ ka_{21} & ka_{22} & ka_{23} \\ ka_{31} & ka_{32} & ka_{33} \end{pmatrix}$

10. If $A=\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$, then $A^t=\begin{pmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{pmatrix}$

11. If $A=\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$ and $B=\begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix}$, then

$$A\times B=\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}\times\begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix}$$

$$=\begin{pmatrix} a_{11}b_{11}+a_{12}b_{21}+a_{13}b_{31} & a_{11}b_{12}+a_{12}b_{22}+a_{13}b_{32} & a_{11}b_{13}+a_{12}b_{23}+a_{13}b_{33} \\ a_{21}b_{11}+a_{22}b_{21}+a_{23}b_{31} & a_{21}b_{12}+a_{22}b_{22}+a_{23}b_{32} & a_{21}b_{13}+a_{22}b_{23}+a_{23}b_{33} \\ a_{31}b_{11}+a_{32}b_{21}+a_{33}b_{31} & a_{31}b_{12}+a_{32}b_{22}+a_{33}b_{32} & a_{31}b_{13}+a_{32}b_{23}+a_{33}b_{33} \end{pmatrix}$$

12. The sum of the entries on the leading diagonal of a square matrix, A, is known as the **trace** of that matrix, denoted $tr(A)$.

13. Consider an arbitrary 3x3 matrix, $A=\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$.

The determinant of A is defined as follows:

$$|A|=\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}=a_{11}a_{22}a_{33}+a_{12}a_{23}a_{31}+a_{13}a_{21}a_{32}-a_{13}a_{22}a_{31}-a_{12}a_{21}a_{33}-a_{11}a_{23}a_{32}$$

14. Steps to calculate the inverse matrix:

a) Calculate the determinant of A, $|A|$. If the determinant is zero the matrix has no inverse.

b) Find the cofactor matrix which is found by replacing every element in matrix A by its cofactor.

c) Find the **adjoint** matrix, denoted $adj(A)$, which is the transpose of the cofactor matrix.

d) The matrix inverse is equal to the inverse value of its determinant multiplied by the adjugate matrix.

15. Consider the following system

$$\begin{cases} a_{11}x + a_{12}y + a_{13}z = c_1 \\ a_{21}x + a_{22}y + a_{23}z = c_2 \\ a_{31}x + a_{32}y + a_{33}z = c_3 \end{cases} \qquad (1)$$

The system (1) can be written in the form

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$$

and the solution of system (1) is given by

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}^{-1} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$$, provided that

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}^{-1}$$ exists.

Or we can use Cramer's rule as follows:

$$\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \qquad \Delta_x = \begin{vmatrix} c_1 & a_{12} & a_{13} \\ c_2 & a_{22} & a_{23} \\ c_3 & a_{32} & a_{33} \end{vmatrix}$$

$$\Delta_y = \begin{vmatrix} a_{11} & c_1 & a_{13} \\ a_{21} & c_2 & a_{23} \\ a_{31} & c_3 & a_{33} \end{vmatrix} \qquad \Delta_z = \begin{vmatrix} a_{11} & a_{12} & c_1 \\ a_{21} & a_{22} & c_2 \\ a_{31} & a_{32} & c_3 \end{vmatrix}$$

$x = \frac{\Delta_x}{\Delta}$, $y = \frac{\Delta_y}{\Delta}$ and $z = \frac{\Delta_z}{\Delta}$

End of Unit Assessment

1. If $A=\begin{pmatrix}3 & -1 & 3\\ 1 & 0 & -6\\ 0 & -4 & 2\end{pmatrix}$, $B=\begin{pmatrix}10 & 2 & 3\\ 1 & -4 & 6\\ 0 & 6 & 4\end{pmatrix}$ and $C=\begin{pmatrix}11 & 12 & 0\\ 1 & 0 & 8\\ 2 & 0 & 7\end{pmatrix}$

   Evaluate:

   a) $A-B$ b) $A+B-2C$

   c) $2A-B+C$ d) $A\times B$

   e) $A\times C$ f) $B\times C$

2. Find the inverse of:

   a) $A=\begin{pmatrix}1 & 0 & 0\\ 1 & 1 & 0\\ 1 & 1 & 1\end{pmatrix}$ b) $B=\begin{pmatrix}2 & -2 & 0\\ 1 & 3 & 4\\ 3 & 1 & 4\end{pmatrix}$

   c) $C=\begin{pmatrix}5 & 0 & 1\\ 2 & 3 & 7\\ 1 & 8 & 4\end{pmatrix}$

3. Using matrix inverse method, solve

$A\times X + 2\times B = 3\times C$ if

$A=\begin{pmatrix}1 & 0 & 0\\ 1 & 1 & 0\\ 1 & 1 & 1\end{pmatrix}$ $B=\begin{pmatrix}0 & 1 & 1\\ 1 & 0 & 0\\ 0 & 0 & 1\end{pmatrix}$ $C=\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 1 & 0 & 1\end{pmatrix}$

4. Use matrix inverse method to solve:

   a) $\begin{cases}x+3y+3z=0\\ 3x+4y-z=0\\ -3x-9y+z=0\end{cases}$ b) $\begin{cases}x+y+z=3\\ 2x-y=1\\ 4x+y-z=4\end{cases}$

   c) $\begin{cases}-x+y-z=-4\\ 3x+10y+z=10\\ x-y-z=2\end{cases}$

Unit 7

Bivariate Statistics

Introductory activity

In Kabeza village, after her 9 observations about farming, UMULISA saw that in every house observed, where there are a number x of cows there are also y domestic ducks, and then she got the following results of (x,y) pairs: (1,4), (2,8), (3,2), (4,12), (5,10), (6,14), (7,16), (8,6), (9,18)

a) Represent this information graphically in a $(x, y) - coordinates$.

b) Chose two points, find the equation of a line joining them and draw it in the same graph. How are the positions of remaining points vis-a -vis this line?

c) According to your observation from (a), explain in your own words if there is any relationship between the variation of the number x of cows and the number y of domestic ducks.

Until now, we know how to determine the measures of central tendency in one variable. In this unit, we will use those measures in two quantitative variables known as double series. In statistics, double series includes technique of analyzing data in two variables, when we focus on the relationship between a dependent variable-y and an independent variable-x. The linear regression method will be used in this unit. The estimation target is a function of the independent variable called the regression function which will be a function of a straight line.

Objectives

By the end of this unit, a student will be able to:

- find measures of variability in two quantitative variables.
- draw the scatter diagram of given statistical series in two quantitative variables.
- determine the linear regression line of a given series,
- calculate a linear coefficient of correlation of a given double series and interpret it.

7.1. Covariance

Activity 7.1

Complete the following table

i	x_i	y_i	$x_i-\bar{x}$	$y_i-\bar{y}$	$(x_i-\bar{x})(y_i-\bar{y})$
1	3	6			
2	1	1			
3	4	3			
4	3	8			
5	2	7			
6	2	8			
	$\sum_{i=1}^{6} x_i = \ldots$	$\sum_{i=1}^{6} y_i = \ldots$			$\sum_{i=1}^{6}(x_i-\bar{x})(y_i-\bar{y}) = \ldots$
	$\bar{x} = \ldots..$	$\bar{y} = \ldots..$			

"What can you get from the following expressions if you divide each one by the total frequency?"

1. $\sum_{i=1}^{k}(x_i-\bar{x})(x_i-\bar{x})$ 2. $\sum_{i=1}^{k}(x_i-\bar{x})(y_i-\bar{y})$

In case of two variables, say x and y, there is another important result called covariance of x and y, denoted $\operatorname{cov}(x,y)$, which is a measure of how these two variables change together.

The **covariance of variables** x **and** y is a measure of how these two variables change together. If the greater values of one variable mainly correspond with the greater values of the other variable, and the same holds for the smaller values, i.e. the variables tend to show similar behavior, the covariance is positive. In the opposite case, when the greater values of one variable mainly correspond to the smaller values of the other, i.e. the variables tend to show opposite behavior, the covariance is negative. If covariance is zero the variables are said to be **uncorrelated,** meaning that there is no linear relationship between them.

Therefore, the sign of covariance shows the tendency in the linear relationship between the variables. The magnitude of covariance is not easy to interpret.

Covariance of variables x and y, where the summation of frequencies $\sum_{i=1}^{k} f_i = n$ are equal for both variables, is defined to be

$$\operatorname{cov}(x,y)=\frac{1}{n}\sum_{i=1}^{k} f_i(x_i-\bar{x})(y_i-\bar{y})$$

Developing this formula we have:

$$\operatorname{cov}(x,y)=\frac{1}{n}\sum_{i=1}^{k} f_i\left(x_i y_i - x_i\bar{y} - \bar{x}y_i + \bar{x}\,\bar{y}\right)$$

$$=\frac{1}{n}\sum_{i=1}^{k} f_i x_i y_i - \frac{1}{n}\sum_{i=1}^{k} f_i x_i\,\bar{y} - \frac{1}{n}\sum_{i=1}^{k} f_i\bar{x}y_i + \frac{1}{n}\sum_{i=1}^{k} f_i\bar{x}\,\bar{y}$$

$$=\frac{1}{n}\sum_{i=1}^{k} f_i x_i y_i - \frac{1}{n}\bar{y}\sum_{i=1}^{k} f_i x_i - \frac{1}{n}\bar{x}\sum_{i=1}^{k} f_i y_i + \bar{x}\,\bar{y}\frac{1}{n}\sum_{i=1}^{k} f_i \qquad \left[\frac{1}{n}\sum_{i=1}^{k} f_i = \frac{1}{n}\times n = 1\right]$$

$$=\frac{1}{n}\sum_{i=1}^{k} f_i x_i y_i - \bar{x}\,\bar{y} - \bar{x}\,\bar{y} + \bar{x}\,\bar{y}$$

$$=\frac{1}{n}\sum_{i=1}^{k} f_i x_i y_i - \bar{x}\,\bar{y}$$

Thus, the covariance is also given by:

$$\operatorname{cov}(x,y)=\frac{1}{n}\sum_{i=1}^{k} f_i x_i y_i - \bar{x}\,\bar{y}$$

Example 7.1

Find the covariance of x and y in the following data sets

x	3	6	4	3	3	2
y	5	3	6	1	7	2

Solution

We have:

x_i	y_i	$x_i-\bar{x}$	$y_i-\bar{y}$	$(x_i-\bar{x})(y_i-\bar{y})$
3	5	-0.5	1	-0.5
6	3	2.5	-1	-2.5
4	6	0.5	2	1
3	1	-0.5	-3	1.5
3	7	-0.5	3	-1.5
2	2	-1.5	-2	3
$\sum_{i=1}^{6} x_i = 21$	$\sum_{i=1}^{6} y_i = 24$			$\sum_{i=1}^{6}(x_i-\bar{x})(y_i-\bar{y})=1$
$\bar{x}=\frac{21}{6}=3.5$	$\bar{y}=\frac{24}{6}=4$			

$$\operatorname{cov}(x,y)=\frac{1}{6}\sum_{i=1}^{6} f_i\left(x_i-\bar{x}\right)\left(y_i-\bar{y}\right)$$
$$=\frac{1}{6}(1)$$
$$=\frac{1}{6}$$

Example 7.2

Find the covariance of the following distribution:

y \ x	0	2	4
1	2	1	3
2	1	4	2
3	2	5	0

Solution

Convert the double entry into a simple table and compute the arithmetic means:

x_i	y_i	f_i	$x_i f_i$	$y_i f_i$	$x_i y_i f_i$
0	1	2	0	2	0
0	2	1	0	2	0
0	3	2	0	6	0
2	1	1	2	2	2
2	2	4	8	8	16
2	3	5	10	15	30
4	1	3	12	3	12
4	2	2	8	4	16
4	3	0	0	0	0
		$\sum_{i=1}^{9} f_i=20$	$\sum_{i=1}^{9} x_i f_i=40$	$\sum_{i=1}^{9} y_i f_i=41$	$\sum_{i=1}^{9} x_i y_i f_i=76$

$$\operatorname{cov}(x,y)=\frac{1}{n}\sum_{i=1}^{n} f_i x_i y_i-\overline{x}\,\overline{y}$$

$$\overline{x}=\frac{1}{n}\sum_{i=1}^{n} f_i x_i$$

$$\overline{y}=\frac{1}{n}\sum_{i=1}^{n} f_i y_i$$

$$\overline{x}=\frac{1}{n}\sum_{i=1}^{n} f_i x_i=\frac{40}{20}=2$$

$$\overline{y}=\frac{1}{n}\sum_{i=1}^{n} f_i y_i=\frac{41}{20}=2.05$$

$$\begin{aligned}\operatorname{cov}(x,y)&=\frac{1}{n}\sum_{i=1}^{n} f_i x_i y_i-\overline{x}\,\overline{y}\\&=\frac{76}{20}-(2\times 2.05)\\&=3.8-4.10\\&=-0.3\end{aligned}$$

Alternative method

$$\overline{x}=\frac{40}{20}=2,\quad \overline{y}=\frac{41}{20}=2.05$$

$$\operatorname{cov}(x,y)=\frac{76}{20}-2\times 2.05=-0.3$$

y \ x	0	2	4	Total
1	2	1	3	6
2	1	4	2	7
3	2	5	0	7
Total	5	10	5	20

$$\overline{x}=\frac{1}{20}(0\times 5+2\times 10+4\times 5)=\frac{40}{20}=2$$

$$\overline{y}=\frac{1}{20}(1\times 6+2\times 7+3\times 7)=\frac{41}{20}=2.05$$

$$\text{cov}(x,y)=\frac{1}{20}\begin{pmatrix}0\times1\times2+0\times2\times1+0\times3\times2+2\times1\times1+2\times2\times4\\+2\times3\times5+4\times1\times3+4\times2\times2+4\times3\times0\end{pmatrix}-2\times2.05$$

$$=\frac{1}{20}(0+0+0+2+16+30+12+16+0)-4.1$$

$$=\frac{76}{20}-4.1$$

$$=-0.3$$

Application activity 7.1

1. Find the covariance of x and y in following data sets:

x	3	5	6	8	9	11
y	2	3	4	6	5	8

2. The scores of 12 students in their mathematics and physics classes are:

Mathematics	2	3	4	4	5	6	6	7	7	8	10	10
Physics	1	3	2	4	4	4	6	4	6	7	9	10

Find the covariance of the distribution

3. The values of two variables x and y are distributed according to the following table:

y \ x	100	50	25
14	1	1	0
18	2	3	0
22	0	1	2

Calculate the covariance.

7.2. Regression lines

We use the regression line to **predict** a value of y for any given value of x and vice versa. The "best" line would make the best predictions: the observed y-values should stray as little as possible from the line.

This straight line is the regression line from which we can adjust its algebraic expressions and it is written as $y = ax + b$.

Activity 7.2

The regression line y on x has the form $y = ax + b$. We need the distance from this line to each point of the given data to be small, so that the sum of the square of such distances to be very small. That is, $D = \sum\left[y - (ax + b)\right]^2$ or $D = \sum(y - ax - b)^2$ (1) is minimum.

1. Differentiate relation (1) with respect to b. In this case, y, x and a will be considered as constants.
2. Equate relation obtained in 1) to zero, divide each side by n and give the value of b.
3. Take the value of b obtained in 2) and put it in relation obtained in 1). Differentiate the obtained relation with respect to a, equate it to zero and divide both sides by n to find the value of a.
4. Using the relations: The variance for variable x is $\sigma_x^2 = \frac{1}{n}\sum\left(x - \overline{x}\right)^2$ and the variance for variable y is $\sigma_y^2 = \frac{1}{n}\sum\left(y - \overline{y}\right)^2$ and the covariance of these two variables is $\text{cov}(x, y) = \frac{1}{n}\sum\left(x - \overline{x}\right)\left(y - \overline{y}\right)$, give the simplified expression equal to a.
5. Put the value of b obtained in 2) and the value of a obtained in 4) in relation $y = ax + b$ and give the expression of regression line y on x

From activity 7.2, the regression line y on x is written as

$$y = \frac{\text{cov}(x, y)}{\sigma_x^2}x + \left(\overline{y} - \frac{\text{cov}(x, y)}{\sigma_x^2}\overline{x}\right)$$

We may write

$$L_{y/x} \equiv y - \bar{y} = \frac{\text{cov}(x,y)}{\sigma_x^2}(x - \bar{x})$$

Note that the regression line x on y is $x = cy + d$ given by

$$x - \bar{x} = \frac{\text{cov}(x,y)}{\sigma_y^2}(y - \bar{y})$$

This line is written as

$$L_{x/y} \equiv x - \bar{x} = \frac{\text{cov}(x,y)}{\sigma_y^2}(y - \bar{y})$$

Shortcut method to find regression lines

To abbreviate the calculations, the two regression lines can be determined as follows:

a) Relation y-x is $L_{y/x} \equiv y = ax + b$ and the values of a and b are found by solving the simultaneous equations:

$$\begin{cases} \sum_{i=1}^{k} f_i y_i = a \sum_{i=1}^{k} f_i x_i + b\,n \\ \sum_{i=1}^{k} f_i x_i y_i = a \sum_{i=1}^{k} f_i x_i^2 + b \sum_{i=1}^{k} f_i x_i \end{cases}$$

These equations are called the normal equations for y on x.

b) Relation x-y is $L_{x/y} \equiv x = cy + d$ and the values of c and d are found by solving the simultaneous equations:

$$\begin{cases} \sum_{i=1}^{k} f_i x_i = c \sum_{i=1}^{k} f_i y_i + d\,n \\ \sum_{i=1}^{k} f_i x_i y_i = c \sum_{i=1}^{k} f_i y_i^2 + d \sum_{i=1}^{k} f_i y_i \end{cases}$$

These equations are called the normal equations for x on y.

Example 7.3

Find the regression line of y on x for the following data and estimate the value of y for $x=4, x=7, x=16$ and the value of x for $y=7, y=9, y=16$.

x	3	5	6	8	9	11
y	2	3	4	6	5	8

Solution

x	y	$x-\bar{x}$	$y-\bar{y}$	$(x-\bar{x})^2$	$(y-\bar{y})^2$	$(x-\bar{x})(y-\bar{y})$
3	2	-4	-2.6	16	6.76	10.4
5	3	-2	-1.6	4	2.56	3.2
6	4	-1	-0.6	1	0.36	0.6
8	6	1	1.4	1	1.96	1.4
9	5	2	0.4	4	0.16	0.8
11	8	4	3.4	16	11.56	13.6
$\sum_{i=1}^{6} x_i = 42$	$\sum_{i=1}^{6} y_i = 28$			$\sum_{i=1}^{6}(x_i-\bar{x})^2 = 42$	$\sum_{i=1}^{6}(y_i-\bar{y})^2 = 23.36$	$\sum_{i=1}^{6}(x_i-\bar{x})(y_i-\bar{y}) = 30$

$$\bar{x} = \frac{42}{6} = 7, \quad \bar{y} = \frac{28}{6} = 4.7$$

$$\operatorname{cov}(x, y) = \frac{1}{n}\sum(x-\bar{x})(y-\bar{y}) = \frac{30}{6} = 5$$

$$\sigma_x^2 = \frac{42}{6} = 7, \quad \sigma_y^2 = \frac{23.36}{6} = 3.89$$

$$L_{y/x} \equiv y - \bar{y} = \frac{\operatorname{cov}(x, y)}{\sigma_x^2}(x-\bar{x})$$

$$L_{y/x} \equiv y - 4.7 = \frac{5}{7}(x-7)$$

Finally, the line of y on x is

$$L_{y/x} \equiv y = \frac{5}{7}x - 0.3$$

And

$$L_{x/y} \equiv x - \bar{x} = \frac{\operatorname{cov}(x, y)}{\sigma_y^2}(y-\bar{y})$$

$$L_{x/y} \equiv x - 7 = \frac{5}{3.89}(y-4.7)$$

Finally, the line of x on y is

$$L_{x/y} \equiv y = 1.3x + 1$$

Alternative method

x	y	x^2	y^2	xy
3	2	9	4	6
5	3	25	9	15
6	4	36	16	24
8	6	64	36	48
9	5	81	25	45
11	8	121	64	88
$\sum_{i=1}^{6} x_i = 42$	$\sum_{i=1}^{6} y_i = 28$	$\sum_{i=1}^{6} x_i^2 = 336$	$\sum_{i=1}^{6} y_i^2 = 154$	$\sum_{i=1}^{6} x_i y_i = 226$

$$L_{y/x} \equiv y = ax + b$$

$$\begin{cases} \sum_{i=1}^{k} f_i y_i = a\sum_{i=1}^{k} f_i x_i + b\,n \\ \sum_{i=1}^{k} f_i x_i y_i = a\sum_{i=1}^{k} f_i x_i^2 + b\sum_{i=1}^{k} f_i x_i \end{cases}$$

$$\begin{cases} 28 = 42a + 6b \\ 226 = 336a + 42b \end{cases} \Leftrightarrow \begin{cases} a = \dfrac{5}{7} \\ b = -0.3 \end{cases}$$

Thus, the line of y on x is

$$L_{y/x} \equiv y = \frac{5}{7}x - 0.3$$

If

$x = 4 \Rightarrow y = 2.5$

$x = 7 \Rightarrow y = 4.7$

$x = 16 \Rightarrow y = 11.1$

$$L_{x/y} \equiv x = cy + d$$

$$\begin{cases} \sum_{i=1}^{k} f_i x_i = c\sum_{i=1}^{k} f_i y_i + d\,n \\ \sum_{i=1}^{k} f_i x_i y_i = c\sum_{i=1}^{k} f_i y_i^2 + d\sum_{i=1}^{k} f_i y_i \end{cases}$$

$$\begin{cases} 42 = 28c + 6d \\ 226 = 154c + 28d \end{cases} \Leftrightarrow \begin{cases} c = 1.3 \\ d = 1 \end{cases}$$

Thus, the line of x on y is

$L_{x/y} \equiv x = 1.3y + 1$

If

$y = 7 \Rightarrow x = 10.1$

$y = 9 \Rightarrow x = 12.7$

$y = 16 \Rightarrow x = 21.8$

Application activity 7.2

1. Consider the following table:

x	y
60	3.1
61	3.6
62	3.8
63	4
65	4.1

a) Find the regression line of y on x.

b) Calculate the approximate y value for the variable $x = 64$.

2. The values of two variables x and y are distributed according to the following table:

y \ x	100	50	25
14	1	1	0
18	2	3	0
22	0	1	2

Find the regression lines.

7.3. Coefficient of correlation

Pearson's coefficient of correlation or product moment coefficient of correlation

Activity 7.3

Consider the following table:

x	y
3	6
5	9
7	12
3	10
2	7
6	8

1. Find the standard deviations σ_x, σ_y.
2. Find covariance $\text{cov}(x, y)$.
3. Calculate the ratio $\frac{\text{cov}(x,y)}{\sigma_x \sigma_y}$.

The **Pearson's coefficient** of correlation also called **product moment coefficient** of correlation or simply **coefficient of correlation**, denoted by *r*, is a measure of the strength of linear relationship between two variables.

The coefficient of correlation between two variables x and y is given by:

$$r = \frac{\text{cov}(x,y)}{\sigma_x \sigma_y}$$

Where,

$\text{cov}(x,y)$ is covariance of x and y

σ_x is the standard deviation for x

σ_y is the standard deviation for y

Properties of the coefficient of correlation

a) The coefficient of correlation does not change the measurement scale. That is, if the height is expressed in metres or feet, the coefficient of correlation does not change.

b) The sign of the coefficient of correlation is the same as the covariance.

c) The square of the coefficient of correlation is equal to the product of angular coefficients (slopes) of two regression lines.

In fact, $r = \frac{\text{cov}(x, y)}{\sigma_x \sigma_y}$. Squaring both sides gives

$$r^2 = \left[\frac{\text{cov}(x, y)}{\sigma_x \sigma_y}\right]^2$$

$$= \frac{\text{cov}^2(x, y)}{\sigma_x^2 \sigma_y^2}$$

$$= \frac{\text{cov}(x, y)}{\sigma_x^2} \times \frac{\text{cov}(x, y)}{\sigma_y^2}$$

d) If the coefficient of correlation is known, it can be used to find the angular coefficients of two regression lines.

We know that the angular coefficient of the regression line y on x is $\frac{\text{cov}(x, y)}{\sigma_x^2}$. From this, we have, $\frac{\text{cov}(x, y)}{\sigma_x^2} = \frac{\text{cov}(x, y)}{\sigma_x \sigma_x} \times \frac{\sigma_y}{\sigma_y}$

$$= \frac{\text{cov}(x, y)}{\sigma_x \sigma_y} \times \frac{\sigma_y}{\sigma_x}$$

$$= r\frac{\sigma_y}{\sigma_x}$$

We know that the angular coefficient of the regression line x on y is $\frac{\text{cov}(x, y)}{\sigma_y^2}$. From this, we have, $\frac{\text{cov}(x, y)}{\sigma_y^2} = \frac{\text{cov}(x, y)}{\sigma_y \sigma_y} \times \frac{\sigma_x}{\sigma_x}$

$$= \frac{\text{cov}(x, y)}{\sigma_x \sigma_y} \times \frac{\sigma_x}{\sigma_y}$$

$$= r\frac{\sigma_x}{\sigma_y}$$

Thus, the angular coefficient of the regression line y on x is given by $r\frac{\sigma_y}{\sigma_x}$ and the angular coefficient of the regression line y on x is given by $r\frac{\sigma_x}{\sigma_y}$.

e) Cauchy Inequality: $\text{cov}^2(x, y) \le \sigma_x^2\sigma_y^2$

In fact, $r = \frac{\text{cov}(x, y)}{\sigma_x\sigma_y} \Leftrightarrow \text{cov}(x, y) = r\sigma_x\sigma_y$.

Squaring both sides gives $\text{cov}^2(x, y) = r^2\sigma_x^2\sigma_y^2$

Or $\text{cov}^2(x, y) \le \sigma_x^2\sigma_y^2$

f) The coefficient of correlation takes value ranging between -1 and +1. That is, $-1 \le r \le 1$

In fact, from Cauchy Inequality, we have,

$\text{cov}^2(x, y) \le \sigma_x^2\sigma_y^2$

$$\Leftrightarrow \frac{\text{cov}^2(x, y)}{\sigma_x^2\sigma_y^2} \le 1 \quad \Leftrightarrow \left[\frac{\text{cov}(x, y)}{\sigma_x\sigma_y}\right]^2 \le 1 \quad \Leftrightarrow r^2 \le 1$$

Taking square roots both sides,

$\Leftrightarrow \sqrt{r^2} \le 1$

$\Leftrightarrow |r| \le 1$ since $\sqrt{x^2} = |x|$

$|r| \le 1$ is equivalent to $-1 \le r \le 1$.

Thus, $-1 \le r \le 1$

g) If the linear coefficient of correlation takes values closer to **−1**, the **correlation** is **strong and negative**, and will become stronger the closer r approaches −1.

h) If the linear coefficient of correlation takes values close to **1,** the **correlation** is **strong and positive**, and will become stronger the closer r approaches 1.

i) If the linear coefficient of correlation takes values close to **0**, the **correlation** is **weak**.

j) If r = 1 or r = −1, there is **perfect correlation** and the line on the scatter plot is increasing or decreasing respectively.

k) If r = 0, there is **no linear correlation**.

Example 7.4

Considering Example 7.3, we have seen that

$$\operatorname{cov}(x,y)=5$$

$$\sigma_x^2=\frac{42}{6}=7,\ \sigma_y^2=\frac{23.36}{6}=3.89$$

Then the Pearson's coefficient of correlation is

$$r=\frac{\operatorname{cov}(x,y)}{\sigma_x\sigma_y},\ r=\frac{5}{\sqrt{7}\sqrt{3.89}}=\frac{5}{\sqrt{27.23}}=0.96$$

Then there is a very strong positive linear relationship between two variables.

We have also seen that the two regression lines are

$$L_{y/x}\equiv y=\frac{5}{7}x-0.3$$

$$L_{x/y}\equiv x=1.3y+1$$

Their slopes are $\alpha=\frac{5}{7}$ and $\beta=1.3$

We see that $r^2=(0.96)^2=0.92$. On the other hand,

$$\alpha.\beta=\frac{5}{7}\times 1.3=0.92.$$

Thus, $r^2=\alpha.\beta$

Example 7.5

A test is made over 200 families on number of children x and number of beds y per family. Results are collected in the table below:

x \ y	0	1	2	3	4	5	6	7	8	9	10
1	0	2	7	5	2	0	0	0	0	0	0
2	2	2	10	8	15	1	0	0	0	0	0
3	1	3	5	6	8	6	1	0	0	0	0
4	0	2	8	2	6	12	10	8	0	0	0
5	0	1	0	2	5	6	10	5	7	3	3
6	0	0	0	2	4	5	5	2	3	3	2

a) What is the average number for children and beds per a family?

b) Find the regression line of y on x.

c) Can we confirm that there is a high linear correlation between the number of children and number of beds per family?

Solution

a) Average number of children per family:

$$\bar{x} = \frac{1}{n}\sum_{i=1}^{k} f_i x_i, \quad \bar{y} = \frac{1}{n}\sum_{i=1}^{k} f_i y_i$$

x \ y	0	1	2	3	4	5	6	7	8	9	10	Total
1	0	2	7	5	2	0	0	0	0	0	0	16
2	2	2	10	8	15	1	0	0	0	0	0	38
3	1	3	5	6	8	6	1	0	0	0	0	30
4	0	2	8	2	6	12	10	8	0	0	0	48
5	0	1	0	2	5	6	10	5	7	3	3	42
6	0	0	0	2	4	5	5	2	3	3	2	26
Total	3	10	30	25	40	30	26	15	10	6	5	200

$$\bar{x}=\frac{1}{200}(3\times0+10\times1+30\times2+25\times3+40\times4+30\times5+26\times6+15\times7+10\times8+6\times9+5\times10)$$
$$=\frac{900}{200}=4.5$$

Or there are about 5 children per family.

Average number of beds per family:

$$\bar{y}=\frac{1}{200}(16\times1+38\times2+30\times3+48\times4+42\times5+26\times6)$$
$$=\frac{740}{200}=3.7$$

Or there are about 4 beds per family.

b) The equation of regression line of y on x is given by equation

$$y-\bar{y}=\frac{\operatorname{cov}(x,y)}{\sigma_x^2}(x-\bar{x})$$

where $\bar{y}=3.7$ and $\bar{x}=4.5$

$$\sigma_x^2=\frac{1}{n}\sum_{i=1}^{k}x_i^2f_i-\left(\bar{x}\right)^2$$
$$=\frac{1}{200}\begin{pmatrix}3\times0^2+10\times1^2+30\times2^2+25\times3^2+40\times4^2+30\times5^2\\+26\times6^2+15\times7^2+10\times8^2+6\times9^2+5\times10^2\end{pmatrix}-(4.5)^2$$
$$=\frac{5042}{200}-20.25$$
$$=4.96$$

$$\operatorname{cov}(x,y)=\frac{1}{n}\sum_{i=1}^{k}f_ix_iy_i-\bar{x}\,\bar{y}$$
$$=\frac{1}{200}\begin{pmatrix}2+14+8+4+40+180+10+9+30+54+56+90+18+8\\+64+24+96+240+224+5+30+100+150+300+175\\+280+135+150+36+96+150+84+144+162+120\end{pmatrix}-4.5\times3.7$$
$$=\frac{3751}{200}-16.65$$
$$=18.7555-16.65$$
$$=2.105$$

The required equation of regression of y on x is

$$y-3.7=\frac{2.105}{4.96}(x-4.5)$$

Or

$$y=0.4x+1.8$$

c) Coefficient of correlation is given by $\frac{\operatorname{cov}(x,y)}{\sigma_x\sigma_y}$

$$\sigma_y^2=\frac{1}{n}\sum_{i=1}^{k}f_iy_i^2-(\overline{y})^2$$

$$=\frac{1}{200}(16+38\times4+48\times16+42\times25+26\times36)-(3.7)^2$$

$$=15.96-13.69$$

$$=2.27$$

Therefore, the coefficient of correlation is

$$r=\frac{2.105}{\sqrt{4.96}\sqrt{2.27}}\approx0.63$$

There is a high linear correlation.

Notice

Spearman's coefficient of rank correlation

A Spearman coefficient of rank correlation or **Spearman's rho** is a measure of statistical dependence between two variables. It assesses how well the relationship between two variables can be described using a monotonic function.

The **Spearman's coefficient of rank correlation** is denoted and defined by $\rho=1-\frac{6\sum_{i=1}^{k}d_i^2}{n(n^2-1)}$

Where, d refers to the difference of ranks between paired items in two series and n is the number of observations.

It is much easier to calculate the Spearman's coefficient of rank correlation than to calculate the Pearson's coefficient of correlation as there is far less working involved. However, in general, the Pearson's coefficient of correlation is a more accurate measure of correlation.

Method of ranking

Ranking can be done in ascending or descending order.

Example 7.6

Suppose that we have the marks, x, of seven students in this order:

12, 18, 10, 13, 15, 16, 9

We assign the rank 1, 2, 3, 4, 5, 7 such that the smallest value of x will be ranked 1.

That is

x	12	18	10	13	15	16	9
$Rank(x)$	3	7	2	4	5	6	1

If we have two or more equal values, we proceed as follows:

Consider the following series:

x	66	65	66	67	66	64	68	68

To assign the rank to this series, we do the following:

$x = 64$ will take rank 1, since it is the smallest value of x.

$x = 65$ will be ranked 2.

$x = 66$ appears 3 times, since the previous value was ranked 2 here 66 would be ranked 3, another 66 would be ranked 4 and another 5 but since there are three 66's, we need to find the average of those ranks which is $\frac{3+4+5}{3} = 4$ so that each 66 will be ranked 4.

$x = 67$ will be ranked 6 since we are on the 6th position.

$x = 68$ appears 2 times, since the previous value was ranked 6 here 68 would be ranked 7, and another 66 would be ranked 8 but since there are two 68's, we need to find the average of those ranks which is $\frac{7+8}{2} = 7.5$ so that each 68 will be ranked 7.5.

Thus, we have the following:

x	66	65	66	67	66	64	68	68
$Rank(x)$	4	2	4	6	4	1	7.5	7.5

Example 7.7

Compute the Spearman's coefficient of rank correlation for the data given in Example 7.3.

x	y	$Rank(x)$	$Rank(y)$	$Rank(x)-Rank(y)=d$	d^2
3	2	1	1	0	0
5	3	2	2	0	0
6	4	3	3	0	0
8	6	4	5	-1	1
9	5	5	4	1	2
11	8	6	6	0	0
					$\sum_{i=1}^{6} d_i^2 = 3$

Then the *Spearman's coefficient of rank correlation* is

$$\rho = 1 - \frac{6\sum_{i=1}^{6} d_i^2}{n(n^2-1)}$$

$$= 1 - \frac{6\times 3}{6(36-1)}$$

$$= 1 - \frac{18}{210}$$

$$= 0.91$$

Example 7.8

Calculate the Spearman's coefficient of rank correlation for the series:

x	12	8	16	12	7	10	12	16	12	9
y	6	5	7	7	4	6	8	13	10	10

x	y	$Rank(x)$	$Rank(y)$	$Rank(x)-Rank(y)=d$	d^2
12	6	6.5	3.5	3	9
8	5	2	2	0	0
16	7	9.5	5.5	4	16
12	7	6.5	5.5	1	1
7	4	1	1	0	0
10	6	4	3.5	0.5	0.25
12	8	6.5	7	0.5	0.25
16	13	9.5	10	0.5	0.25
12	10	6.5	8.5	2	4
9	10	3	8.5	5.5	30.25
					$\sum_{i=1}^{10} d_i^2 = 61$

Then

$$\rho = 1 - \frac{6\times 61}{10(100-1)}$$

$$\Leftrightarrow \rho = 1 - \frac{366}{990}$$

$$\Leftrightarrow \rho = \frac{990-366}{990}$$

Or

$$\rho = 0.63$$

Application activity 7.3

1. Calculate the coefficient of correlation of the following distribution:

x	1	8	7	12	11	13	11	20
y	3	4	6	10	11	12	14	21

2. The scores of 12 students in their mathematics and physics classes are:

Mathematics	2	3	4	4	5	6	6	7	7	8	10	10
Physics	1	3	2	4	4	4	6	4	6	7	9	10

Find the coefficient of correlation distribution and interpret it.

3. The values of the two variables x and y are distributed according to the following table:

y \ x	0	2	4
1	2	1	3
2	1	4	2
3	2	5	0

Calculate the coefficient of correlation.

7.4. Applications

Activity 7.4

Discuss how statistics, especially bivariate statistics, can be used in our daily life.

Bivariate statistics can help in prediction of a value for one variable if we know the value of the other.

Example 7.9

One measure of personal fitness is the time taken for an individual's pulse rate to return to normal after strenuous exercise, the greater the fitness, the shorter the time. Following a short program of strenuous exercise, Norman recorded his pulse rates P at time t minutes after he had stopped exercising.

Norman's results are given in the table below;

t	0.5	1.0	1.5	2.0	3.0	4.0	5.0
P	125	113	102	94	81	83	71

Estimate Norman's pulse rate 2.5 minutes after stopping the exercise programme.

Solution

t	P	t^2	P^2	tP
0.5	125	0.25	15625	62.5
1	113	1	12769	113
1.5	102	2.25	10404	153
2	94	4	8836	188
3	81	9	6561	243
4	83	16	6889	332
5	71	25	5041	355
$\sum_{i=1}^{7} t_i = 17$	$\sum_{i=1}^{7} P_i = 669$	$\sum_{i=1}^{7} t_i^2 = 57.5$	$\sum_{i=1}^{7} P_i^2 = 66125$	$\sum_{i=1}^{7} t_i P_i = 1446.5$

We need the line $P = at + b$

Use the formula

$$\begin{cases} \sum_{i=1}^{7} P_i = a\sum_{i=1}^{7} t_i + bn \\ \sum_{i=1}^{7} t_i P_i = a\sum_{i=1}^{7} t_i^2 + b\sum_{i=1}^{7} t_i \end{cases}$$

We have

$$\begin{cases} 669 = 17a + 7b \\ 1446.5 = 57.5a + 17b \end{cases}$$

Solving, we have

$$\begin{cases} a = -11 \\ b = 122.3 \end{cases}$$

Then, $P = -11t + 122.3$

So,

Norman's pulse rate 2.5 minutes after stopping the exercise program is estimated to be $P = -11(2.5) + 122.3$ or 94.8.

Example 7.10

A student found the following data for life expectancy, X years, and the Gross Domestic Production per head in Y dollars, in six countries in South Asia in 1988.

Country	X	Y
Afghanistan	42	143
Bangladesh	50	179
Bhutan	47	197
India	58	335
Pakistan	57	384
Sri Lanka	73	423

$\left[n = 6, \sum x = 327, \sum y = 1661, \sum x^2 = 18415, \sum y^2 = 529909, \sum xy = 96412\right]$

a) It is required to estimate the value of X for Nepal, where the value of Y is 450.

   i) Find the equation for a suitable line of regression. Simplify your answer as far as possible, giving the constants correct to three significant figures

   ii) Use your equation to obtain the required estimate

b) Use your equation to estimate the value of x for North Korea, where the value of Y was 858. Comment on your answer.

Solution

a) i) Neither variable has been controlled in the given data and since you are required to estimate the life expectancy, X years, when the GDP per head, Y dollars is 160 dollars, it is sensible to use the regression line of X on Y

The regression line of X on Y has equation

$x = c + dy$, *where* $c = \overline{x} - d\overline{y}$ *and* $d = \frac{S_{xy}}{S_{yy}}$

$\overline{x} = \frac{\sum x}{n} = \frac{327}{6}$ *and* $\overline{y} = \frac{\sum y}{n} = \frac{1661}{6}$

$s_{xy} = \frac{1}{n}\sum xy - \overline{x}\,\overline{y} = \frac{1}{6}\times 96412 - \frac{327}{6}\times\frac{1661}{6} = 981.25$

$s_{yy} = \frac{1}{n}\sum y^2 - \overline{y}^2 = \frac{1}{6}\times 529909 - \left(\frac{1661}{6}\right)^2 = 11681.47...$

$d = \frac{s_{xy}}{s_{yy}} = \frac{981.25}{11681.47} = 0.08400...$

or

$S_{xy} = \sum xy - \frac{\sum x \sum y}{n} = 96412 - \frac{327\times 1661}{6} = 5887.5$

$S_{yy} = \sum y^2 - \frac{\left(\sum y\right)^2}{n} = 529909 - \frac{1661^2}{6} = 70088.83$

$d = \frac{S_{xy}}{S_{yy}} = \frac{5887.5}{70088.83} = 0.08400...$

Then *calculate* $c = \overline{x} - d\overline{y}$

$= \frac{327}{6} - 0.08400\times\frac{1661}{6}$

$= 31.24$

Equation of the regression line of x on y is $x = 31.2 + 0.0840y$

ii) When

$y = 160, x = 31.2 + 0.0840\times 160 = 45$

The estimated value of the life expectancy in Nepal is 45 years

b) From the equation , when $y = 858$

$x = 31.2 + 0.0840\times 858 = 103$

This would give the life expectancy of 103 year in North Korea, which is clearly not sensible. The value of x is a long way outside the data , and should not be used to estimate a value of x

Unit Summary

1. The **covariance of variables** x **and** y is a measure of how these two variables change together. It is defined as $\operatorname{cov}(x, y)=\frac{1}{n}\sum_{i=1}^{k} f_i x_i y_i-\bar{x}\,\bar{y}$
2. The regression line y on x is $L_{y/x} \equiv y-\bar{y}=\frac{\operatorname{cov}(x, y)}{\sigma_x^2}(x-\bar{x})$.
3. The regression line x on y is $L_{x/y} \equiv x-\bar{x}=\frac{\operatorname{cov}(x, y)}{\sigma_y^2}(y-\bar{y})$
4. The coefficient of correlation between two variables x and y is given by

$$r=\frac{\operatorname{cov}(x, y)}{\sigma_x \sigma_y}.$$

5. The Spearman's coefficient of rank correlation is given by

$$\rho=1-\frac{6\sum_{i=1}^{k} d_i^2}{n\left(n^2-1\right)}$$

Where, d refers to the difference of ranks between paired items in two series.

End of Unit Assessment

1. For each set of data, find:

   a) equation of the regression line of y on x.

   b) equation of the regression line of x on y.

Data set 1

x	3	7	9	11	14	14	15	21	22	23	26
y	5	12	5	12	10	17	23	16	10	10	25

Data set 2

x	1	5	5	5	6	7.5	7.5	7.5	10	11	12.5	14	14.5
y	85	82	85	89	78	66	77	81	70	74	65	69	63

2. The following is a summary of the results of given two variables:

$$\sum_{i=1}^{k} f_i x_i = 500, \sum_{i=1}^{k} f_i y_i = 300, \sum_{i=1}^{k} f_i x_i^2 = 27818, \sum_{i=1}^{k} f_i x_i y_i = 16837, \sum_{i=1}^{k} f_i y_i^2 = 10462$$

Find the equation of regression line of y on x.

Estimate the value of y for $x = 60$.

3. Compute the coefficient of correlation for the following series:

x	80	45	55	56	58	60	65	68	70	75	85
y	81	56	50	48	60	62	64	65	70	74	90

4. The following results were obtained from lineups in Mathematics and Physics examinations:

	Mathematics (x)	Physics (y)
Mean	475	39.5
Standard deviation	16.8	10.8

$r = 0.95$

Find both equations of the regression lines. Also estimate the value of y for $x = 30$.

5. The following results were obtained from records of age (x) and systolic blood pressure (y) of a group of 10 men:

	(x)	(y)
Mean	53	142
Variance	130	165

$$\sum_{i=1}^{k} f_i\left(x_i-\bar{x}\right)\left(y_i-\bar{y}\right)=1220$$

Find both equations of the regression lines. Also estimate the blood pressure of a man whose age is 45.

6. For a given set of data:

$$\sum_{i=1}^{k} f_i x_i=15,\ \sum_{i=1}^{k} f_i y_i=43,\ \sum_{i=1}^{k} f_i x_i^2=55,\sum_{i=1}^{k} f_i x_i y_i=145,\ \sum_{i=1}^{k} f_i y_i^2=397,\ \sum_{i=1}^{k} f_i=5$$

Find the equations of the regression lines y on x, and x on y.

7. For a set of 20 pairs of observations of the variables x and y, it is known that $\sum_{i=1}^{k} f_i x_i=250,\ \sum_{i=1}^{k} f_i y_i=140$, and that the regression line of y on x passes through $(15,10)$.

Find the equation of that regression line and use it to estimate y when $x=10$.

8. The gradient of the regression line x on y is -0.2 and the line passes through $(0,3)$. If the equation of the line is $x=c+dy$, find the value of c and d and sketch the line on a diagram.

9. The heights h, in cm, and weights w, in kg, of 10 people are measured. It is found that

$$\sum_{i=1}^{k} f_i h_i=1710,\ \sum_{i=1}^{k} f_i w_i=760,\ \sum_{i=1}^{k} f_i h_i^2=293162,\sum_{i=1}^{k} f_i h_i w_i=130628,\ \sum_{i=1}^{k} f_i w_i^2=59390$$

a) Calculate the coefficient of correlation between the value of h and w.

b) What is the equation of the regression line of w on h.

10. The regression equations are $7x-16y+9=0$ and $5y-4x-3=0$. Find $\bar{x}$, $\bar{y}$ and r.

11. The regression equations are $7x-16y+9=0$ and $5y-4x-3=0$. Find $\overline{x}$, $\overline{y}$ and r.

12. If two regression coefficients are 0.8 and 0.2, what would be the value of coefficient of correlation?

13. For a given set of data:

$$\sum_{i=1}^{k} f_i x_i = 680,\ \sum_{i=1}^{k} f_i y_i = 996,\ \sum_{i=1}^{k} f_i x_i^2 = 20154,\ \sum_{i=1}^{k} f_i x_i y_i = 24844,\ \sum_{i=1}^{k} f_i y_i^2 = 34670,\ \sum_{i=1}^{k} f_i = 30$$

Find the coefficient of correlation.

14. For a set of data, the equations of the regression lines are

$y=0.648x+2.64$ and $x=0.917y-1.91$

Find the coefficient of correlation.

15. For a set of data, the equations of the regression lines are

$y=-0.219x+20.8$ and

$x=-0.785y+16.2$

Find the coefficient of correlation.

16. For a set of data, the equations of the regression lines are $y=1.3x+0.4$ and $x=0.7y-0.1$

Find;

a) the coefficient of correlation.

b) $\overline{x}$ and $\overline{y}$.

17. In a partially destroyed laboratory record of an analysis of correlation data, the following results only are legible: Variance of x is 9

Equations of regression lines: $8x-10y+66=0$ and $40x-18y-214=0$

What were:

a) the mean values of x and y

b) the standard deviation of y, and

c) the coefficient of correlation between x and y.

18. The following equations of regression lines and variance are obtained from a correlation table:

$20x-9y-107=0$,

$4x-5y+33=0$,

variance of x is 9.

Find;

a) the mean value of x and y.

b) the standard deviation of y.

19. The table below shows the marks awarded to six students in a competition:

Student	A	B	C	D	E	F
Judge 1	6.8	7.3	8.1	9.8	7.1	9.2
Judge 2	7.8	9.4	7.9	9.6	8.9	6.9

Calculate a coefficient of rank correlation.

20. At the end of a season, a league of eight hockey clubs produced the following table showing the position of each club in the league and the average attendances (in hundreds) at home matches:

Club	Position	Average attendance
A	1	27
B	2	29
C	3	9
D	4	16
E	5	24
F	6	15
G	7	12
H	8	22

a) Calculate the Spearman's coefficient of rank correlation between position in the league and average attendance.

b) Comment on your results

21. A company is to replace its fleet of cars. Eight possible models are considered and the transport manager is asked to rank them, from 1 to 8, in order of preference. A saleswoman is asked to use each type of car for a week and grade them according to their suitability for the job (A-very suitable to E-unsuitable).

The price is also recorded:

Model	Transport manager's ranking	Saleswoman's grade	Price (£10s)
S	5	B	611
T	1	B+	811
U	7	D-	591
V	2	C	792
W	8	B+	520
X	6	D	573

Y	4	C+	683
Z	3	A-	716

a) Calculate the Spearman's coefficient of rank correlation between:
   (i) price and transport manager's rankings,
   (ii) price and saleswoman's grades.

b) Based on the result of a, state, giving a reason, whether it would be necessary to use all three different methods of assessing the cars.

c) A new employee is asked to collect further data and to do some calculations. He produces the following results:
   The coefficient of correlation between:
   (i) price and boot capacity is 1.2;
   (ii) maximum speed and fuel consumption in miles per gallons is -0.7;
   (iii) price and engine capacity is -0.9.
   For each of his results, say giving a reason, whether you think it is reasonable.

d) Suggest two sets of circumstances where Spearman's coefficient of rank correlation would be preferred to the Pearson's coefficient of correlation as a measure of association.

22. The scores obtained by a group of students in tests that measure verbal ability (x) and abstract reasoning (y) are represented in the following table:

y \ x	20	30	40	50
(25-35)	6	4	0	0
(35-45)	3	6	1	0
(45-55)	0	2	5	3
(55-65)	0	1	2	7

a) Is there a correlation between the two variables?

b) According to the data, if one of these students obtained a score of 70 points in abstract reasoning, what would be the estimated score in verbal ability?

23. To test the effect of a new drug twelve patients were examined before the drug was administered and given an initial score (I) depending on the severity of various symptoms. After taking the drug they were examined again and given a final score (F). A decrease in score represented an improvement. The scores for the twelve patients are given in the table below:

	Score	
Patients	Initial (I)	Final (F)
1	61	49
2	23	12
3	8	3
4	14	4
5	42	28
6	34	27
7	32	20
8	31	20
9	41	34
10	25	15
11	20	16
12	50	40

Calculate the equation of the line of regression of F on I

a) On the average what improvement would you expect for a patient whose initial score was 30?

b) On the average what improvement would you expect for a patient whose initial score was 30?

Unit 8

Conditional Probability and Bayes Theorem

Introductory activity

A box contains 3 red pens and 4 blue pens. One pen is taken from the box and is not replaced. Another pen is taken from the box. Let A be the event "the first pen is red" and B be the event "the second pen is blue."

Is the occurrence of event B affected by the occurrence of event A? Explain.

Give more other examples of real life problems involving probability.

Some academic fields based on the probability theory are statistics, communication theory, computer performance evaluation, signal and image processing, game theory...

In medical decision-making, clinical estimate of probability strongly affects the physician's belief as to whether or not a patient has a disease, and this belief, in turn, determines actions: to rule out, to treat, or to do more tests, doctors may use conditional probability to calculate the probability that a particular patient has a disease, given the presence of a particular set of symptoms....

Some applications of the probability theory are character recognition, speech recognition, opinion survey, missile control and seismic analysis, etc.

In addition, the game of chance formed the foundations of probability theory

Objectives

By the end of this unit, a student will be able to:

- use tree diagram to find probability of events.
- find probability of independent events.
- find probability of one event given that the other event has occurred.
- use and apply Bayes theorem.

8.1. Independent events

Activity 8.1

A bag contains books of two different subjects. One book is selected from the bag and is replaced. A book is also selected in the bag. Is the occurrence of the event for the second selection affected by the event for the first selection? Explain.

If probability of event B is not affected by the occurrence of event A, events A and B are said to be **independent** and $P(A \cap B) = P(A) \times P(B)$

This rule is the simplest form of the **multiplication law** of probability.

Example 8.1

A die is thrown twice. Find the probability of obtaining a 4 on the first throw and an odd number on the second throw.

Solution

Let A be the event: "a 4 is obtained on the first throw", then $P(A) = \frac{1}{6}$. That is $A = \{4\}$.

B be the event: "an odd number is obtained on the second throw". That is $B = \{1,3,5\}$.

Since the result on the second throw is not affected by the result on the first throw, A and B are independent events.

There are 3 odd numbers, then

$$P(B)=\frac{3}{6}=\frac{1}{2}$$

Therefore,

$$P(A\cap B)=P(A)P(B)$$
$$=\frac{1}{6}\times\frac{1}{2}$$
$$=\frac{1}{12}$$

Example 8.2

A factory runs two machines. The first machine operates for 80% of the time while the second machine operates for 60% of the time and at least one machine operates for 92% of the time. Do these two machines operate independently?

Solution

Let the first machine be M_1 and the second machine be M_2, then $P(M_1)=80\%=0.8$, $P(M_2)=60\%=0.6$ and $P(M_1\cup M_2)=92\%=0.92$

Now,

$$P(M_1\cup M_2)=P(M_1)+P(M_2)-P(M_1\cap M_2)$$

$$P(M_1\cap M_2)=P(M_1)+P(M_2)-P(M_1\cup M_2)$$
$$=0.8+0.6-0.92$$
$$=0.48$$
$$=0.8\times 0.6$$
$$=P(M_1)\times P(M_2)$$

Thus, the two machines operate independently.

Example 8.3

A coin is weighted so that heads is three times as likely to appear as tails. Find $P(H)$ and $P(T)$.

Solution

Let $P(T)=p_1$, then $P(H)=3p_1$.

But $P(H)+P(T)=1$

Therefore $p_1+3p_1=1 \Leftrightarrow 4p_1=1 \Rightarrow p_1=\frac{1}{4}$

Thus, $P(H)=\frac{3}{4}$ and $P(T)=\frac{1}{4}$.

Application activity 8.1

1. A dresser drawer contains one pair of socks with each of the following colors: blue, brown, red, white and black. Each pair is folded together in a matching set.

   You reach into the sock drawer and choose a pair of socks without looking. You replace this pair and then choose another pair of socks. What is the probability that you will choose the red pair of socks both times?
2. A coin is tossed and a single 6-sided die is rolled. Find the probability of landing on the head side of the coin and rolling a 3 on the die.

8.2. Conditional probability

Activity 8.2

A bag contains books of two different subjects. One book is selected from the bag and is not replaced. Another book is selected in the bag. Is the occurrence of the event for the second selection affected by the event for the first selection? Explain.

When the outcome or occurrence of the first event affects the outcome or occurrence of the second event in such a way that the probability sample space is changed, the events are said to be **dependent**.

The probability of an event B given that event A has occurred is called the **conditional probability** of B given A and is written $P(B|A)$.

In this case, $P(B|A)$ is the probability that B occurs considering A as the sample space, and since the subset of A in which B occurs is $A \cap B$, then

$$P(B|A) = \frac{P(B \cap A)}{P(A)}.$$

From this result, we have general statement of the multiplication law: $P(A \cap B) = P(A) \times P(B|A)$.

This shows us that the probability that two events will both occur is the product of the probability that one will occur and the conditional probability that the other will occur given that the first has occurred. We can also write $P(A \cap B) = P(B) \times P(A|B)$. Since A and B are interchangeable.

If A and B are **independent**, then the probability of B is not affected by the occurrence of A and so $P(B|A) = P(B)$ giving $P(A \cap B) = P(A) \times P(B)$

Example 8.4

A die is tossed. Find the probability that the number obtained is a 4 given that the number is greater than 2.

Solution

Let A be the event: "the number is a 4", then $A = \{4\}$.

B be the event: "the number is greater than 2", then $B = \{3, 4, 5, 6\}$

and $P(B)=\frac{4}{6}=\frac{2}{3}$

But $A\cap B=\{4\}$ and $P(A\cap B)=\frac{1}{6}$

Therefore,

$$P(A|B)=\frac{P(A\cap B)}{P(B)}$$

$$P(A|B)=\frac{\frac{1}{6}}{\frac{2}{3}} \qquad P(A|B)=\frac{1}{6}\times\frac{3}{2}=\frac{1}{4}$$

Example 8.5

At a middle school, 18% of all students play football and basketball, and 32% of all students play football. What is the probability that a student who plays football also plays basketball?

Solution

Let A be a set of students who play football and B a set of students who play basketball; then the set of students who play both games is $A\cap B$. We have $P(A)=32\%=0.32$, $P(A\cap B)=18\%=0.18$. We need the probability of B known that A has occurred.

Therefore,

$$P(B|A)=\frac{P(A\cap B)}{P(A)}$$

$$=\frac{0.18}{0.32}$$

$$=0.5625$$

$$=56\%$$

Notice: contingency table

Contingency table (or Two-Way table) provides a different way of calculating probabilities. The table helps in determining conditional probabilities quite easily. The table displays sample values in relation to two different variables that may be dependent or contingent on one another.

Below, the contingency table shows the favorite leisure activities for 50 adults, 20 men and 30 women. Because entries in the table are frequency counts, the table is a frequency table.

	Dance	Sports	TV	Total
Men	2	10	8	20
Women	16	6	8	30
Total	18	16	16	50

Entries in the total row and total column are called marginal frequencies or the marginal distribution. Entries in the body of the table are called joint frequencies.

Example 8.6

Suppose a study of speeding violations and drivers who use car phones produced the following fictional data:

	Speeding violation in the last year	No speeding violation in the last year	Total
Car phone user	25	280	305
Not a car phone user	45	405	450
Total	70	685	755

Calculate the following probabilities using the table:

a) P(person is a car phone user).

b) P(person had no violation in the last year).

c) P(person had no violation in the last year AND was a car phone user).

d) P(person is a car phone user OR person had no violation in the last year).

e) P(person is a car phone user GIVEN person had a violation in the last year).

f) P(person had no violation last year GIVEN person was not a car phone user).

Solution

a) $P(\text{person is a car phone user}) = \dfrac{\text{number of car phone users}}{\text{total number in study}} = \dfrac{305}{755}$

b) $P(\text{person had no violation in the last year}) = \dfrac{\text{number that had no violation}}{\text{total number in study}} = \dfrac{685}{755}$

c) $P(\text{person had no violation in the last year AND was a car phone user}) = \dfrac{280}{755}$

d) P(person is a car phone user OR person had no violation in the last year)

$$= \left(\frac{305}{755} + \frac{685}{755}\right) - \frac{280}{755} = \frac{710}{755}$$

e) The sample space is reduced to the number of persons who had a violation. Then,

$$P(\text{person is a car phone user GIVEN person had a violation in the last year}) = \frac{25}{70}$$

f) The sample space is reduced to the number of persons who were not car phone users. Then,

$$P(\text{person had no violation last year GIVEN person was not a car phone user}) = \frac{405}{450}$$

Application activity 8.2

1. A coin is tossed twice in succession. Let A be the event that the first toss is heads and let B be the event that the second toss is heads. Find $P(B|A)$
2. Calculate the probability of a 6 being rolled by a die if it is already known that the result is even.
3. A jar contains black and white marbles. Two marbles are chosen without replacement. The probability of selecting a black marble and then a white marble is 0.34, and the probability of selecting a black marble on the first draw is 0.47. What is the probability of selecting a white marble on the second draw, given that the first marble drawn was black?

8.3. Bayes theorem and its applications

Activity 8.3

Suppose that entire output of a factory is produced on three machines. Let B_1 denote the event that a randomly chosen item was made by machine 1, B_2 denote the event that a randomly chosen item was made by machine 2 and B_3 denote the event that a randomly chosen item was made by machine 3. Let A denote the event that a randomly chosen item is defective.

1. Use conditional probability formula and give the relation that should be used to find the probability P(A) that the chosen item is defective given that it is made by machine 1 or machine 2 or machine 3.
2. If we need the probability that the chosen item is produced by machine 1 given that it is found to be defective, i.e $P(B_1|A)$, give the formula for this conditional probability. Recall that $P(B_i \cap A)$ can be written as $P(A|B_i)P(B_i)$. Do the same if the item is produced by machine 2 and by machine 3. Give the general formula if the item is produced by machine *i* (*i* from 1 to 3)

From activity 8.3

Let $B_1, B_2, B_3, \ldots, B_n$ be incompatible and exhaustive events and let A be an arbitrary event.

We have:

$$P(B_i \mid A) = \frac{P(B_i \cap A)}{P(A)} = \frac{P(A \mid B_i)P(B_i)}{\sum_{i=1}^{n} P(A \mid B_i)P(B_i)}$$

This formula is called Bayes' formula.

Remark

We also have (Bayes' rule)

$$P(B \mid A) = \frac{P(A \mid B)P(B)}{P(A)}$$

Example 8.7

Suppose that machines M_1, M_2, and M_3 produce, respectively, 500, 1000, and 1500 parts per day, of which 5%, 6%, and 7% are defective. A part produced by one of these machines is taken at random, at the end of a given workday, and it is found to be defective. What is the probability that it was produced by machine M_3?

Solution

Let A_i be the event "the part taken at random was produced by machine M_i," for i $=1, 2, 3$; and let D be "the part taken at random is defective."

Using Bayes' formula, we see

$$P(A_3 | D) = \frac{P(D | A_3)P(A_3)}{\sum_{i=1}^{3} P(D | A_i)P(A_i)}$$

$$= \frac{(0.07)\left(\frac{1500}{3000}\right)}{(0.05)\left(\frac{1}{6}\right) + (0.06)\left(\frac{1}{3}\right) + (0.07)\left(\frac{1}{2}\right)}$$

$$= \frac{105}{190}$$

$$= \frac{21}{38}$$

Example 8.8

Two machines A and B produce 60% and 40% respectively of total output of a factory. Of the parts produced by machine A, 3% are defective and of the parts produced by machine B, 5% are defective. A part is selected at random from a day's production and found to be defective. What is the probability that it came from machine A?

Solution

Let E be the event that the part came from machine A,

C be the event that the part came from machine B and

D be the event that the part is defective.

We require $P(E | D)$.

Now, $P(E) \times P(D | E) = 0.6 \times 0.03 = 0.018$ and

$$P(D) = P(E \cap D) + P(C \cap D)$$

$$= 0.018 + 0.4 \times 0.05$$

$$= 0.038$$

Therefore, the required probability is $\frac{0.018}{0.038} = \frac{9}{19}$

Example 8.9

Consider the population of patients, seen in an emergency ward, suffering from one of three diseases: acute appendicitis (AA), acute pancreatitis (AP), or non-specified abdominal pain (NSAP). The physician attending the patients is faced with the decision of whether to operate immediately or to perform a clinical test in an effort to distinguish between three diagnostic alternatives. The clinical test is to see if the patient demonstrates rebound tenderness /painfulness. It is elicited by pressing down slowly on the patient's abdomen and then suddenly releasing the pressure. In the presence of peritoneal irritation, release is accompanied by a brief episode of sharp pain which is localised to the site of irritation. Studies indicate that 80% of patients with AA, 15% of patients with AP and 20% of patients with NSAP manifest rebound tenderness (Staniland el al., 1972). Further, suppose that prevalence rates for these three conditions are: 30% for AA, 5% for AP and 65% for NSAP. Let D_1, D_2 and D_3 stand for three disease states corresponding to AA, AP and NSAP, and let S represent the event that the patient shows the sign of rebound tenderness / painfulness. Thus, the following probabilities are determined:

$$P(S/D_1)=0.80 \qquad P(D_1)=0.30$$
$$P(S/D_2)=0.15 \quad \text{and} \quad P(D_2)=0.05$$
$$P(S/D_3)=0.20 \qquad P(D_3)=0.65$$

Now the likelihood for each diagnosis in the presence of rebound tenderness is easily calculated Using the laws of addition and multiplication of probabilities and the notion of conditional probability or Bayes' theorem. Thus, the posterior probabilities for AA, AP and NSAP are:

$$P(D_1/S)=\frac{P(S/D_1)P(D_1)}{\sum_{i=1}^{3}P(S/D_i)P(D_i)}$$
$$=\frac{(0.80)(0.30)}{\left[(0.80)(0.30)+(0.15)(0.05)+(0.20)(0.65)\right]}$$
$$=\frac{0.2410}{0.3775}$$
$$=0.64$$

$$P(D_2/S)=\frac{(0.15)(0.05)}{[(0.80)(0.30)+(0.15)(0.05)+(0.20)(0.65)]}$$
$$=\frac{0.0075}{0.3775}$$
$$=0.02$$

$$P(D_3/S)=\frac{(0.20)(0.65)}{[(0.80)(0.30)+(0.15)(0.05)+(0.20)(0.65)]}$$
$$=\frac{0.13}{0.3775}$$
$$=0.34$$

Thus, the most likely diagnosis in the presence of rebound tenderness/ painfulness is acute appendicitis. Since surgery is generally not required for either pancreatitis or NSAP, but is indicated for appendicitis, it is very useful to know the probability that the patient showing rebound tenderness has appendicitis as compared to the probability that s/he has one of the other two conditions.

Application activity 8.3

1. The probabilities of the weather being fine, raining or snowing are respectively $\frac{1}{2}, \frac{1}{3}$ and $\frac{1}{6}$. The probabilities that a student arrives on time for school under each of these conditions are $\frac{3}{4}, \frac{2}{5}$ and $\frac{3}{10}$ respectively. What is the probability that:
   a) the student arrives at school on time on any given day,
   b) if the student is late, it was raining?
2. 20% of a company's employees are engineers and 20% are economists. 75% of the engineers and 50% of the economists hold a managerial position, while only 20% of non-engineers and non-economists have a similar position. What is the probability that an employee selected at random will be both an engineer and a manager?

3. The probability of having an accident in a factory that triggers an alarm is 0.1. The probability of it sounding after the event of an incident is 0.97 and the probability of it sounding after no incident has occurred is 0.02. In an event where the alarm has been triggered, what is the probability that there has been no accident?

Unit Summary

1. A tree diagram is a means which can be used to show the probabilities of certain outcomes occurring when two or more trials take place in succession. The outcome is written at the end of the branch and the fraction on the branch gives the probability of the outcome occurring. For each trial, the number of branches is equal to the number of possible outcomes of that trial. In the diagram, there are two possible outcomes, A and B, of each trial.
2. Events A and B are said to be **independent** if and only if $P(A \cap B) = P(A) \times P(B)$
3. The probability of an event B given that event A has occurred is called the **conditional probability** of B given A and is written $P(B \mid A)$. In this case, $P(B \mid A)$ is the probability that B occurs considering A as the sample space, and since the subset of A in which B occurs is $A \cap B$, then $P(B \mid A) = \dfrac{P(B \cap A)}{P(A)}$.
4. Let $B_1, B_2, B_3, \ldots, B_n$ be incompatible and exhaustive events and A an arbitrary event.
   The Bayes' formula says that
   $$P(B_i \mid A) = \frac{P(B_i \cap A)}{P(A)} = \frac{P(A \mid B_i)P(B_i)}{\sum_{i=1}^{n} P(A \mid B_i)P(B_i)}$$

End of Unit Assessment

1. The probability that it is Friday and that a student is absent is 0.03. Since there are 5 school days in a week, the probability that it is Friday is 0.2. What is the probability that a student is absent given that today is Friday?
2. At Kennedy Middle School, the probability that a student takes Technology and Spanish is 0.087. The probability that a student takes Technology is 0.68. What is the probability that a student takes Spanish given that the student is taking Technology?
3. A car dealership is giving away a trip to Rome to one of their 120 best customers. In this group, 65 are women, 80 are married and 45 married women. If the winner is married, what is the probability that it is a woman?
4. A card is chosen at random from a deck of 52 cards. It is then replaced and a second card is chosen. What is the probability of choosing a jack and then an eight?
5. A jar contains 3 red, 5 green, 2 blue and 6 yellow marbles. A marble is chosen at random from the jar. After replacing it, a second marble is chosen. What is the probability of choosing a green and then a yellow marble?
6. A school survey found that 9 out of 10 students like pizza. If three students are chosen at random with replacement, what is the probability that all three students like pizza?
7. A nationwide survey found that 72% of people in the United States like pizza. If 3 people are selected at random, what is the probability that all three like pizza?

8. A bag contains 2 red, 3 green and 2 blue balls. Two balls are drawn at random. What is the probability that none of the balls drawn is blue?

9. For married couples living in a certain suburb, the probability that the husband will vote on a bond referendum is 0.21, the probability that his wife will vote in the referendum is 0.28, and the probability that both the husband and wife will vote is 0.15. What is the probability that:

   a) at least one member of a married couple will vote?

   b) a wife will vote, given that her husband will vote?

   c) a husband will vote, given that his wife does not vote?

10. In 1970, 11% of Americans completed four years of college, 43% of them were women. In 1990, 22% of Americans completed four years of college; 53% of them were women. (Time, Jan.19, 1996).

   a) Given that a person completed four years of college in 1970, what is the probability that the person was a woman?

   b) What is the probability that a woman would finish four years of college in 1990?

   c) What is the probability that in 1990 a man would not finish college?

11. If the probability is 0.1 that a person will make a mistake on his or her state income tax return, find the probability that:

   a) four totally unrelated persons each make a mistake

   b) Mr. Jones and Ms. Clark both make a mistake and Mr. Roberts and Ms. Williams do not make a mistake.

12. The probability that a patient recovers from a delicate heart operation is 0.8. What is the probability that:
    a) exactly 2 of the next 3 patients who have this operation will survive?
    b) all of the next 3 patients who have this operation survive?
13. In a certain federal prison, it is known that 2/3 of the inmates are under 25 years of age. It is also known that 3/5 of the inmates are male and that 5/8 of the inmates are female or 25 years of age or older. What is the probability that a prisoner selected at random from this prison is female and at least 25 years old?
14. A certain federal agency employs three consulting firms (A, B and C) with probabilities 0.4, 0.35, 0.25, respectively. From past experience it is known that the probabilities of cost overrun for the firms are 0.05, 0.03, and 0.15 respectively. Suppose a cost overrun is experienced by the agency.
    a) What is the probability that the consulting firm involved is company C?
    b) What is the probability that it is company A?
15. In a certain college, 5% of the men and 1% of the women are taller than 180 cm. Also, 60% of the students are women. If a student is selected at random and found to be taller than 180 cm, what is the probability that this student is a woman?

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