MATHEMATICS FOR TTCs

STUDENT'S BOOK

3

YEAR THREE

OPTION:

SOCIAL STUDIES EDUCATION (SSE)

FOREWORD

Dear Student,

Rwanda Education Board (REB) is honoured to present Year 3 Mathematics book for Social Studies Education (SSE) student teachers. This book will serve as a guide to competence-based teaching and learning to ensure consistency and coherence in the learning of the Mathematics. The Rwandan educational philosophy is to ensure that you achieve full potential at every level of education which will prepare you to be well integrated in society and exploit employment opportunities.

The government of Rwanda emphasizes the importance of aligning teaching and learning materials with the syllabus to facilitate your learning process. Many factors influence what you learn, how well you learn and the competences you acquire. Those factors include the relevance of the specific content, the quality of teachers' pedagogical approaches, the assessment strategies and the instructional materials available. In this book, we paid special attention to the activities that facilitate the learning process in which you can develop your ideas and make new discoveries during concrete activities carried out individually or with peers.

In competence-based curriculum, learning is considered as a process of active building and developing knowledge and meanings by the learner where concepts are mainly introduced by an activity, situation or scenario that helps the learner to construct knowledge, develop skills and acquire positive attitudes and values.

For efficiency use of this textbook, your role is to:

- Work on given activities which lead to the development of skills;
- Share relevant information with other learners through presentations, discussions, group work and other active learning techniques such as role play, case studies, investigation and research in the library, on internet or outside;
- Participate and take responsibility for your own learning;
- Draw conclusions based on the findings from the learning activities.

To facilitate you in doing activities, the content of this book is self explanatory so that you can easily use it yourself, acquire and assess your competences. The book is made of units as presented in the syllabus. Each unit has the following structure: the key unit competence is given and it is followed by the introductory activity before the development of mathematical concepts that are connected to real world problems or to other sciences.

The development of each concept has the following points:

- It starts by a learning activity: it is a hand on well set activity to be done by students in order to generate the concept to be learnt;
- Main elements of the content to be emphasized;
- Worked examples; and
- Application activities: those are activities to be done by the user to consolidate competences or to assess the achievement of objectives.

Even though the book has some worked examples, you will succeed on the application activities depending on your ways of reading, questioning, thinking and grappling ideas of calculus not by searching for similar-looking worked out examples.

Furthermore, to succeed in Mathematics, you are asked to keep trying; sometimes you will find concepts that need to be worked at before you completely understand. The only way to really grasp such a concept is to think about it and work-related problems found in other reference books.

I wish to sincerely express my appreciation to the people who contributed towards the development of this book, particularly, REB staff, UR-CE Lecturers and TTC Tutors for their technical support. A word of gratitude goes to Head Teachers and TTCs principals who availed their staff for various activities.

Any comment or contribution would be welcome to the improvement of this text book for the next edition.

Dr. NDAYAMBAJE Irénée

Director General, REB

ACKNOWLEDGEMENT

I wish to express my appreciation to the people who played a major role in the development of this Mathematics book for Year Three student teachers in the option of Social Studies Education (SSE). It would not have been successful without active participation of different education stakeholders.

I owe gratitude to different universities and schools in Rwanda that allowed their staff to work with REB in the in-house textbooks production initiative.

I wish to extend my sincere gratitude to lecturers and TTC tutors whose efforts during writing exercise of this book were very much valuable.

Finally, my word of gratitude goes to the Rwanda Education Board staffs who were involved in the whole process of in-house textbook Elaboration.

Joan MURUNGI

Head of CTLR Department

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UNIT 1

LOGARITHMIC AND EXPONENTIAL FUNCTIONS

Key Unit competence:

Apply logarithmic and exponential functions to model and solve problems about interest rates and population growth.

1.0 Introductory activity

An economist created a business which helped him to make money in an interesting way so that the money he/she earns each day doubles what he/she earned the previous day. If he/she had 200USD on the first day and by taking t as the number of days, discuss the money he/she can have at the t^{th} day through answering the following questions:

a) Draw the table showing the money this economist will have on each day starting from the first to the 10^{th} day.

b) Plot these data in rectangular coordinates

c) Based on the results in a), establish the formula for the economist to find out the money he/she can earn on the n^{th} day. Therefore, if t is the time in days, express the money $F(t)$ for the economist.

d) Now the economist wants to possess the money F under the same conditions, discuss how he/she can know the number of days necessary to get such money from the beginning of the business.

1.1 Logarithmic functions

1.1.1 Definition and domain of definition of Logarithmic functions

Activity 1.1.1

Consider the following real numbers: 50, 100, ½, 0.7, 0.8, -30, -20, -5, 0.9, 10, 20, and 40 taken as values of x

a) Draw and complete the table of values for $\log_{10}(x)$

b) Discuss the value of $\log_{10}(x)$ for $x<0$

c) Discuss the values of $\log_{10}(x)$ for $0<x<1$, $x=1$ and $x>1$

d) Using the findings in a) plot the graph of $\log_{10}(x)$ for $x>0$

e) Explain in your own words the values of x for which $\log_{10}(x)$ is defined (the domain) and output values (the range).

Given the function $y=\log_a x$, it is proven that if $x>0$ and a is a constant $(a>0, a\neq 1)$, then $\log_a x$ is a real number called the "**logarithm of** x **with base** a".

Definition of logarithmic function

For a positive constant a $(a\neq 1)$, we call logarithmic function, the function $f:\mathbb{R}\to\mathbb{R}: x\mapsto \log_a x$. The domain of the logarithm function is the set of positive real numbers and the range is the line of all real numbers.

This means that $dom\ f=\{x\in\mathbb{R}: x>0\}=]0,+\infty[=\mathbb{R}_0^+$ and range $f=\mathbb{R}=]-\infty,+\infty[$.

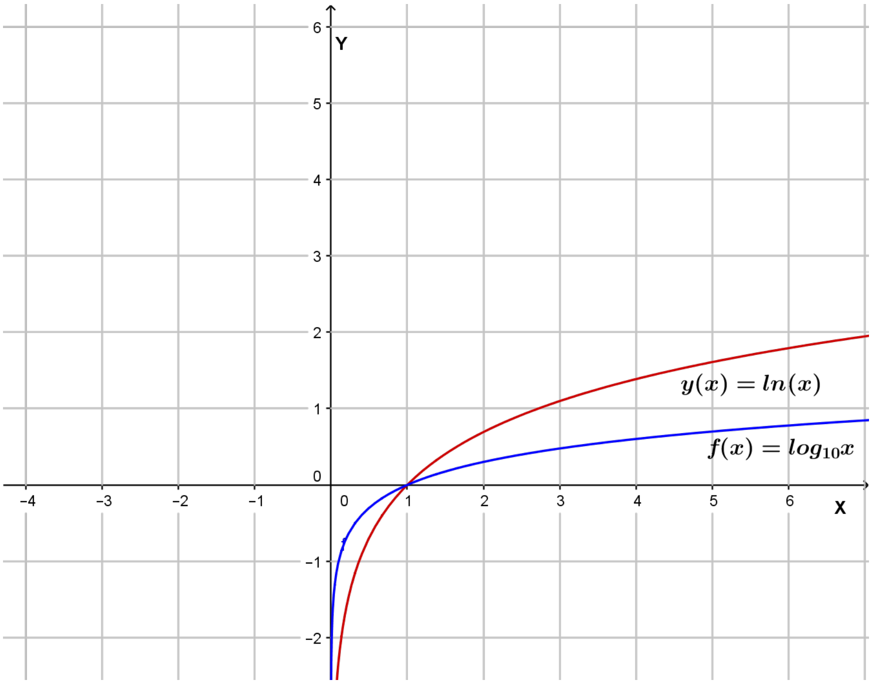
The logarithmic function is neither even nor odd. If $u:\mathbb{R}\to\mathbb{R}: x\mapsto u(x)$ is any other function we can compose u and the logarithmic function as $y=\log_a(u(x))$ defined for x such that $u(x)\geq 0$.

In the expression $y=\log_a x$, y is referred to as the logarithm, a is the base, and x is the argument.

If **the base is 10**, it is not necessary to write the base, and we call it **decimal logarithm** or **common logarithm** or **Brigg's logarithm**. So, the notation will become $y=\log x$. If the base is e (where $e=2.718281828$), we call it **Neperean logarithm** or **natural logarithm**.

The natural logarithm is usually written using the shorthand notation $y=\ln x$ instead of $y=\log_e x$ as we might expect.

Graphs of logarithmic functions $f(x)=\log_{10}(x)$ **and** $y(x)=\ln(x)$



Example

Find the domain and range for the function

a) $f(x)=\log(x-4)$

b) $g(x)=\ln(x+6)$

Solution

a) To find the domain and the range of the function $y=\log(x-4)$, recalling that:

- **Domain:** Includes all values of x for which the function is defined.
- **Range:** Includes all values y for which there is some x such that $y=\log(x-4)$

Because $\log x$ defined only for positive values of x, in this problem $y = \log(x-4)$ is defined if and only if $x-4>0 \Leftrightarrow x>4$ which means that $x \in]4,+\infty[$.

The range of y is still all real number $\mathbb{R}$.

$$Dom\, f = \{x \in \mathbb{R} : x-4>0\} = \{x \in \mathbb{R} : x>4\} =]4,+\infty[.\ Range\, f = \mathbb{R}.$$

b) The function $y = \ln(x+6)$ is defined if and only if $x+6>0 \Leftrightarrow x>-6$ and means that $x \in]-6,+\infty[$ which is the domain. The range is $\mathbb{R}$.

$$Dom\, g = \{x \in \mathbb{R} : x+6>0\} = \{x \in \mathbb{R} : x>-6\} =]-6,+\infty[.\ Range\, g = \mathbb{R}.$$

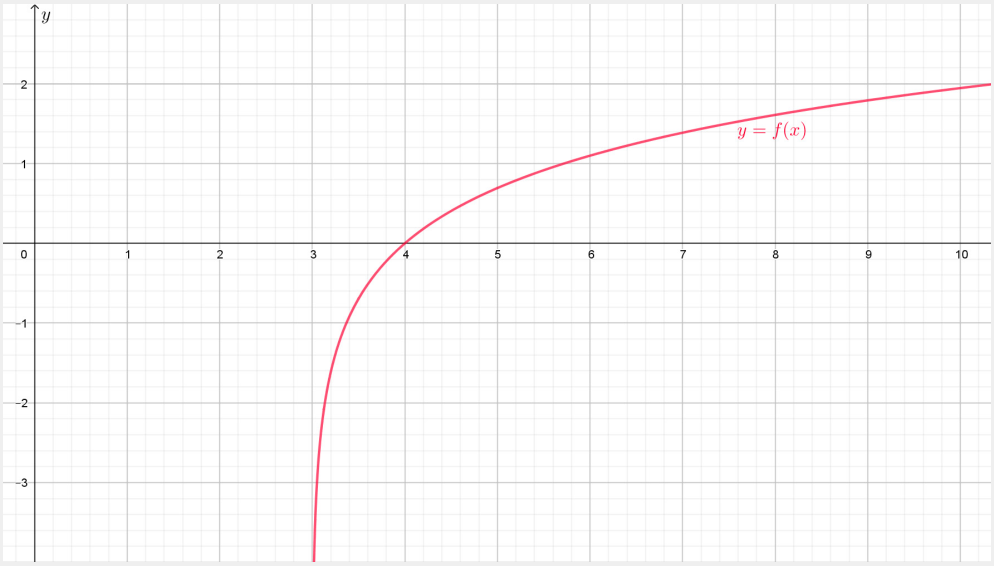
Application activity 1.1.1

1. State the domain and range of the following functions:

   a. $y = \log_3(x-2)+4$

   b. $y = \log_5(8-2x)$

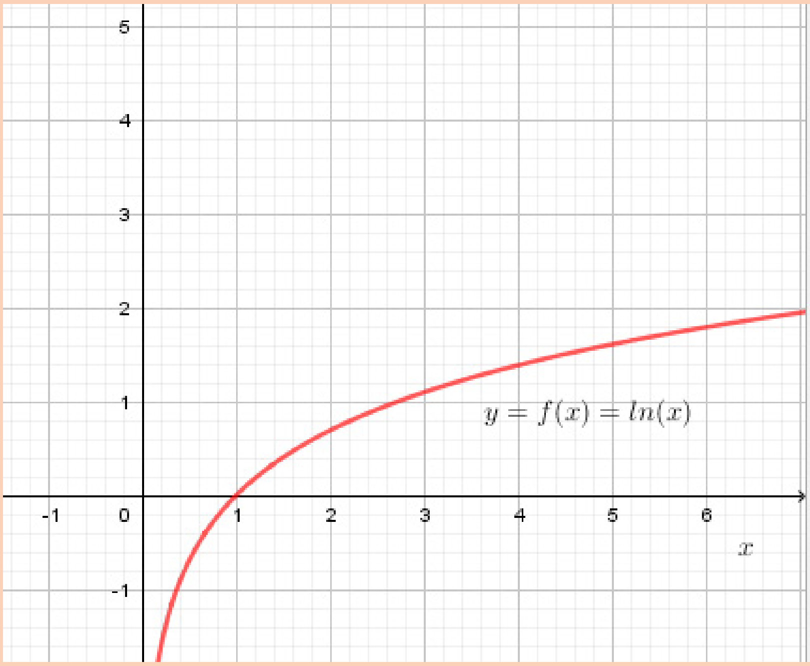
2. Observe the following graph of a given logarithmic function, then state its domain and range. Justify your answers.



1.1.2 Limits of logarithmic functions

Activity 1.1.2

1. The graph below represents natural logarithmic function $f\left(x\right)=\ln x$



Considering the form of this graph and the logarithm of the following numbers in the table below,

x	0.5	0.001	0.001	0.0001	2	100	1001	10000
$\ln x$								

Use a calculator to complete the table and

a) Discuss the values of $\ln x$ when x takes values closer to 0 from the right and deduce $\lim\limits_{x\to 0^+} \ln x$.

b) Discuss the values of $\ln x$ when x takes greater values and conclude about the $\lim\limits_{x\to +\infty} \ln x$.

The limit of any logarithmic function can be determined in the same way as the limit of the natural function. If you feel more comfortable with the natural logarithmic function, use the change of base formula:

$f(x)=\log_a u(x)=\dfrac{\ln(u(x))}{\ln a}$ provided $a>0, a\neq 1$.

Example 1

Determine each of the following limit

a) $\lim\limits_{x\to e}\ln x$ b) $\lim\limits_{x\to 2}(1-\ln x)$ c) $\lim\limits_{x\to+\infty}\log_3\left(\dfrac{x-4}{2x+6}\right)$

Solution

a) $\lim\limits_{x\to e}\ln x=1$

b) $\lim\limits_{x\to 2}(1-\ln x)=1-\ln 2$

c) $\lim\limits_{x\to+\infty}\log_3\left(\dfrac{x-4}{2x+6}\right)=\log_3\dfrac{1}{2}=-\log_3 2$

Since $\lim\limits_{x\to\infty}\dfrac{x-4}{2x+6}=\dfrac{1}{2}$

Alternatively, using natural logarithmic function, we have

$$\lim_{x\to+\infty}\log_3\left(\frac{x-4}{2x+6}\right)=\lim_{x\to+\infty}\frac{\ln\left(\frac{x-4}{2x+6}\right)}{\ln 3}=\frac{1}{\ln 3}\lim_{x\to+\infty}\ln\left(\frac{x-4}{2x+6}\right)=\frac{1}{\ln 3}\times\ln\frac{1}{2}=-\frac{\ln 2}{\ln 3}=-\log_3 2=\log_3\frac{1}{2}$$

Application activity 1.1.2

I. Evaluate the following limits

1) $\lim\limits_{x\to+\infty}\ln(7x^3-x^2+1)$

2) $\lim\limits_{x\to 1^+}\left(\ln\dfrac{1}{x-1}\right)$

3) $\lim\limits_{x\to 2^-}\log_5\left(x^2-5x+6\right)$

4) $\lim\limits_{a\to 4^+}\ln\dfrac{a}{\sqrt{a-4}}$

5) $\lim\limits_{x\to+\infty}\ln\left(x^2-4x+1\right)$

6) $\lim\limits_{x\to+\infty}\dfrac{2+4\log x}{x}$

1.1.3 Applications of Limits of logarithmic functions to continuity and asymptotes

Activity 1.1.3

I. Let us consider the logarithmic function $f:\mathbb{R}^{+}\to\mathbb{R},\ \ y=f(x)=\log_2(x)$

1. Complete the following table:

$x=x_0$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4
$y=\log_2 x$					
$\lim\limits_{x\to x_0}\log_2 x$					

Can you conclude that $\lim\limits_{x\to x_0}\log_2 x=\log_2(x_0)$? What about the continuity of $y=f(x)=\log_2(x)$?

2. By using the information drawn in the above table and the scientific calculator, plot the graph of $y=\log_2(x)$

3. Give any justification that allows you to decide on the continuity of the function

4. Find $\lim\limits_{x\to 0^{+}}\ln x$ and deduce the equation of asymptotes of $f(x)=\ln x$ if any.

II. Observe the graph of the function $p(x)=\dfrac{\ln x}{x}$ and deduce $\lim\limits_{x\to+\infty}\dfrac{\ln x}{x}$, $\lim\limits_{x\to 0^{+}}\dfrac{\ln x}{x}$, $\lim\limits_{x\to 1}\dfrac{\ln x}{x}$ calculate $\lim\limits_{x\to\frac{1}{5}}\left(\dfrac{\ln x}{x}\right)$

The limit $\lim_{x\to 0^+} \ln x = -\infty$ shows that the line OY with equation $x=0$ is the vertical asymptote. This means that as the independent variable x takes values approaching 0 from the right, the graph of the function approaches the line of equation $x=0$ without intercepting. In other words, the dependent variable y takes "bigger and bigger" negative values.

Then, $\lim_{x\to+\infty} \ln x = +\infty$, which implies that there is no horizontal asymptote.

The $\lim_{x\to 0^-} \ln x$ does not exist because values closer to 0 from the left are not included in the domain of the given function.

The graph of the logarithmic function $f:\mathbb{R}^+ \to \mathbb{R}, f(x)=\log_a(x),\ a>1$ has the following characteristics:

- The domain is $]0,+\infty[$ and $f(x)$ is continuous on this interval.
- The range is $\mathbb{R}$
- The graph intersects the $x-axis$ at $(1,0)$
- As $x\to 0^+, y\to -\infty$, so the line of equation $x=0$ (the y- axis) is an asymptote to the curve.
- As x increases, the graph rises more steeply for $x\in[0,1]$ and is flatter for $x\in[1,+\infty[$.
- The logarithmic function is increasing and takes its values (range) from negative infinity to positive infinity.

Example

Let us consider the logarithmic function $y=\log_2(x-3)$

a) What is the equation of the asymptote line?

b) State the domain and range

c) Find the $x-$ intercept.

d) Determine another point through which the graph passes

e) Sketch the graph

Solution

a) The basic graph of $y=\log_2 x$ has been translated 3 units to the right, so the line $L \equiv x=3$

is the vertical asymptote.

b) The function $y=\log_2(x-3)$ is defined for $x-3>0$

So, the domain is $]3,+\infty[$ .The range is $\mathbb{R}$

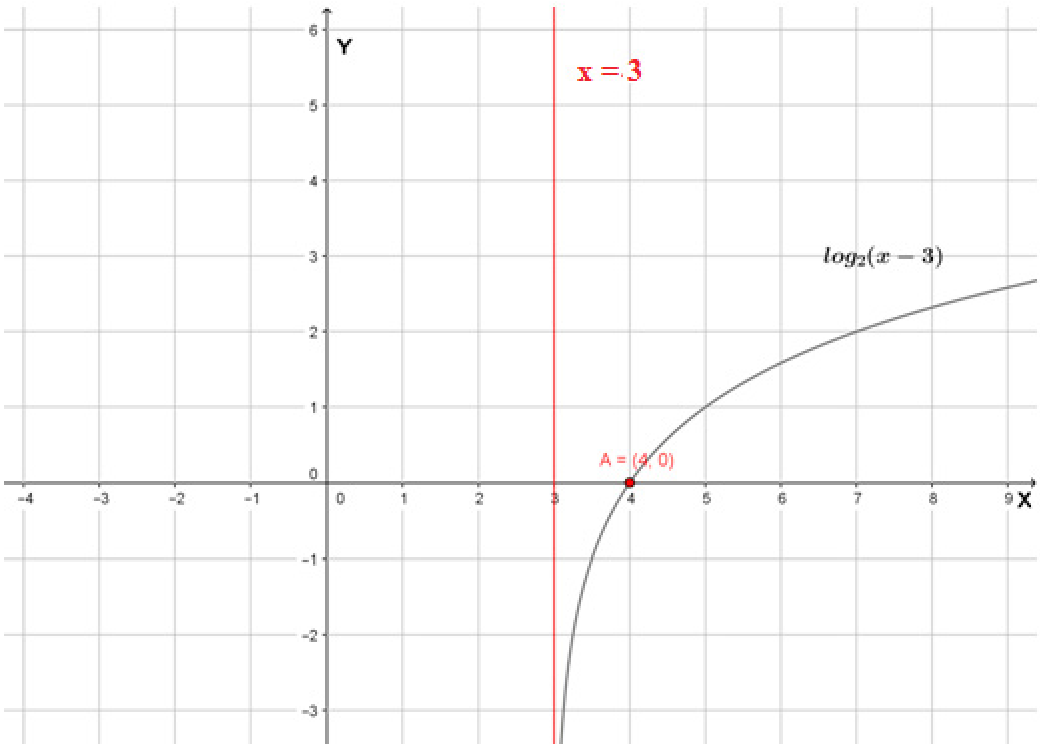
c) The x-intercept is (4, 0) since $\log_2(x-3)=0 \Leftrightarrow x=4$

d) Another point through which the graph passes can be found by allocating an arbitrary value to x in the domain then compute y.

For example, when $x=4, y=\log_2(4-2)=\log_2 2=1$ which gives the point $(4,1)$.

Note that the graph does not intercept *y*-axis because the value 0 for *x* does not belong to the domain of the function.

The graph of $y=f(x)=\log_2(x-3)$



Application activity 1.1.3

1) Given the logarithmic function $y=-1+\ln(x+1)$,

   i) Find equation of asymptote lines (if any)?

   ii) State the domain and range

   iii) Find the $x-$ intercept

   iv) Find the $y-$ intercept

   v) Determine another point belonging to the graph

   vi) Sketch the graph

2) Sketch the graph of the logarithmic function of base a with $0<a<1$. Precise the characteristics of the graph.

1.1.4 Differentiation of a logarithmic functions

Activity 1.1.4

Let $f(x)=\ln x$

a) Find $f(x+h)$ and $f(2+h)$

b) Complete the following table

h	$\frac{\ln(2+h)-\ln 2}{h}$
-0.1	
-0.001	
-0.00001	
0.1	
0.001	
0.00001	

From the results found in the above table approximate the value of $f'(2)=\lim\limits_{h\to 0}\dfrac{\ln(2+h)-\ln 2}{h}$ and deduce the expression of $f'(\mathrm{x})$.

Based on your existing knowledge, provide any interpretation of the number $f'(2)$.

The definition of derivative shows that if $y=\ln x$,

$$y'=\lim_{h\to 0}\frac{\ln(x+h)-\ln x}{h}=\lim_{h\to 0}\ln\left(\frac{x+h}{x}\right)^{\frac{1}{h}}$$

$$=\lim_{h\to 0}\ln\left(1+\frac{h}{x}\right)^{\frac{1}{h}}=\ln\lim_{h\to 0}\left(1+\frac{h}{x}\right)^{\frac{1}{h}}=\ln e^{\frac{1}{x}}=\frac{1}{x}$$

Then, the natural logarithmic function $y=\ln x$ is differentiable on $]0,+\infty[$ and

$$\frac{d}{dx}(\ln x)=\frac{1}{x},\quad (x>0).$$

Using the formula of base change,

$\dfrac{d}{dx}(\log_a x)=\dfrac{1}{x.\ln a}$ for any base a provided the conditions are fulfilled.

In the more general form, if $u(x)$ is any differentiable function such that $u(x)>0$, $\dfrac{d}{dx}[\ln u(x)]=\dfrac{1}{u(x)}\times u'(x)=\dfrac{u'(x)}{u(x)}$ and

$$\frac{d}{dx}[\log_a u(x)]=\frac{1}{u(x)}\times\frac{u'(x)}{\ln a}=\frac{u'(x)}{u(x).\ln a}$$

Example

1. Differentiate each of the following functions with respect to x

   a) $f(x)=\ln(x^3+3x-4)$

   b) $f(x)=x^2\ln x$

   c) $y=\log_2(5x^3)$

2. Find the slope of the line tangent to the graph of $y=\log_2(3x+1)$ at $x=1$

Solution

1) Differentiation

a) $\frac{d}{dx}\ln\left(x^3+3x-4\right)=\frac{1}{x^3+3x-4}\left(x^3+3x-4\right)'=\frac{3x^2+3}{x^3+3x-4}$

b) $\frac{d}{dx}\left(x^2\ln x\right)=\ln x\frac{d}{dx}x^2+x^2\frac{d}{dx}\ln x=\ln x\times 2x+x^2\times\frac{1}{x}=2x\ln x+x$

c) $\frac{d}{dx}\log_2\left(5x^3\right)=\frac{d}{dx}\left(\frac{\ln 5x^3}{\ln 2}\right)=\frac{1}{\ln 2}\frac{d}{dx}\left(\ln 5+3\ln x\right)$

$$=\frac{1}{\ln 2}\left(\frac{d}{dx}\ln 5+\frac{d}{dx}(3\ln x)\right)=\frac{1}{\ln 2}\left[0+3\frac{d}{dx}\ln x\right]$$

$$=\frac{1}{\ln 2}\times 3\times\frac{1}{x}=\frac{3}{x\ln 2}$$

2. To find the slope, we must evaluate $\frac{dy}{dx}$ at $x=1$

$$\frac{d}{dx}\log_2(3x+1)=\frac{d}{dx}\left(\frac{\ln(3x+1)}{\ln 2}\right)=\frac{1}{\ln 2}\frac{d}{dx}\left(\ln 3x+1\right)=\frac{3}{(3x+1)\ln 2}$$

By evaluating the derivative at $x=1$, we see that the tangent line to the curve at the point $(1,\log_2 4)=(1,2)$ has the slope

$$\left.\frac{dy}{dx}\right|_{x=1}=\frac{3}{4\ln 2}=\frac{3}{\ln 16}.$$

Application activity 1.1.4

1. Differentiate $y=\ln\sqrt{\frac{1+x}{1-x}}$ with respect to x.
2. An airplane takes off from an airport at sea level. If its altitude (in kilometres) at time t (in minutes) is given by $h=2000\ln\left(t+1\right)$, find the rate of climb at time $t=3\,\text{min}$.

1.1.5 Application of derivative to the variation of a logarithmic function

Activity 1.1.5

Given two functions $f(x)=\ln x$ and $g(x)=\log_{10} x$

1) Compare $f(2)$ and $f(10)$, $g(2)$ and $g(10)$ to verify whether those functions are increasing or decreasing on $[2,10]$.

2) Use the tables of signs for $f'(x)$ and $g'(x)$ to establish the intervals of variation of those functions.

3) Which function f or g is increasing or decreasing faster than another on [2,10]?

4) Evaluate the second derivative $f''(x)$

5) Use the tables of signs for $f''(x)=(f'(x))'$ and discuss the concavity of the function f(x).

The logarithmic function $f(x)=\log_a x$, $a>0, a\neq 1$ varies in the following way:

For $x>0$, if $f(x)=\log_a x$, then

$f'(x)=\dfrac{1}{x\ln a}$. The sign of $f'(x)$ depends therefore on the value of the base a.

a) If $a>1$, $\ln a>0$ then $f'(x)$ is always positive. Thus $f(x)=\log_a x$ is strictly increasing on $\mathbb{R}_0^+$.

Variation table for $y = f(x) = \log_a x$ **for** $a > 1$

x	0		1		a		$+\infty$
y'		$+$	$\frac{1}{\ln a}$	$+$	$\frac{1}{a\ln a}$	$+$	
y		$-\infty$ ↗	0	↗	1	↗	$+\infty$

b) If $0 < a < 1$, $\ln a < 0$. Therefore $f'(x)$ is always negative.

Thus $f(x) = \log_a x$ is strictly decreasing on $\mathbb{R}_0^+$. This implies the absence of extrema (maxima or minima) values.

Table of variation for $y = f(x) = \log_a x$ for $0 < a < 1$

x	0		a		1		$+\infty$
y'		-	$\frac{1}{a\ln a}$	-	$\frac{1}{\ln a}$	-	
y		$+\infty$ ↘	1	↘	0	↘	$-\infty$

Example

Discuss variations of the logarithmic function $f(x) = x - \ln x$.

Solution

$f(x) = x - \ln x$ is defined for all $x > 0$

$$f'(x) = \frac{d}{dx}(x - \ln x) = 1 - \frac{1}{x}$$

$$f'(x)=0 \Leftrightarrow 1-\frac{1}{x}=0$$

$$\Leftrightarrow \frac{1}{x}=1 \Leftrightarrow x=1$$

If $x=1,\ y=f(1)=1-\ln 1=1$, thus$(1,1)$ is a point of the graph.

$$f''(x)=\frac{d}{dx}\left(1-\frac{1}{x}\right)=\frac{1}{x^2}.$$

$$f''(x)=0 \Leftrightarrow 0=\frac{1}{x^2};\quad x\neq 0$$

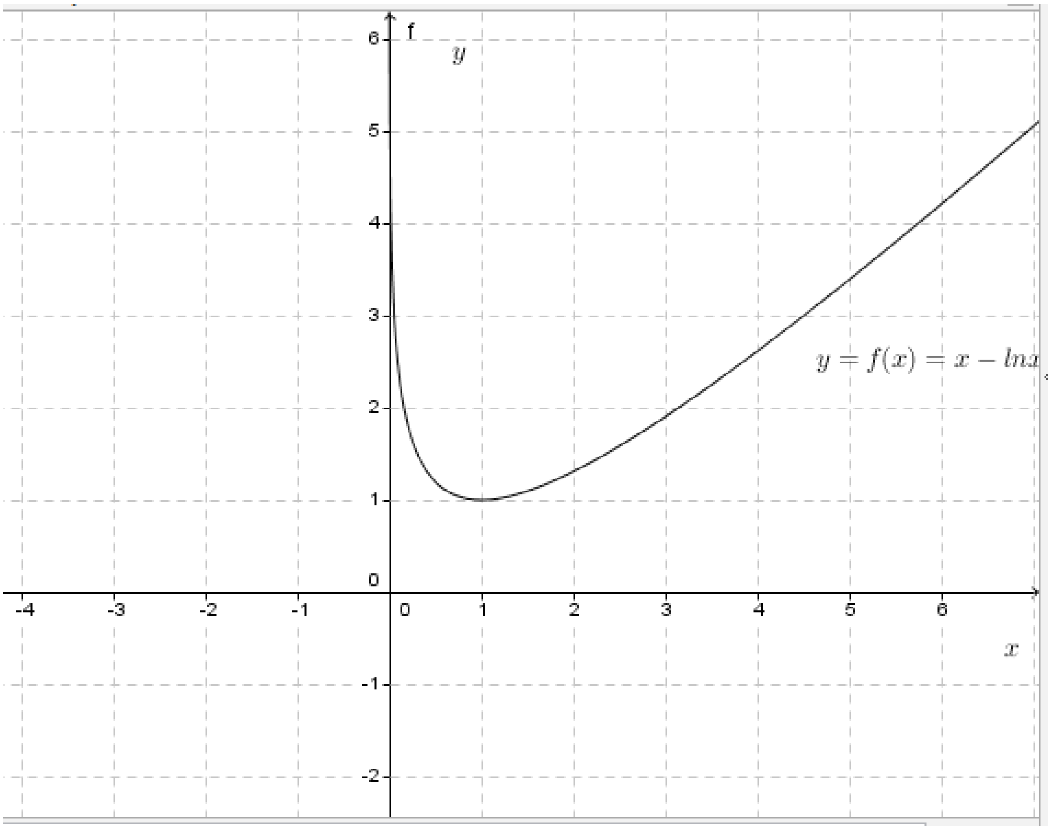
It means that $f''(x)$ is positive.

Variation table of $y=f(x)=x-\ln x$

x	0	1	$+\infty$
y'	-	0 +	
$f''(x)$	+ +	+ +	
y	$+\infty$ $\searrow$	$\nearrow$	$+\infty$

From the table, one can observe that the function is decreasing for values when x lies in] 0, 1] and increasing for x greater than 1. The point (1,1) is minimum or equivalently the function takes the minimum value equal for x=1. The minimum value that is equal to 1 is absolute.

There is **no inflection point,** and the graph **curves up/turns up.**



Limits involving indeterminate forms

1. Evaluate $\lim\limits_{x\to+\infty}\dfrac{\ln x}{x}=0$

Solution

$\lim\limits_{x\to+\infty}\dfrac{\ln x}{x}$ takes indeterminate form $\dfrac{\infty}{\infty}$. Apply Hospital rule:

$$\lim_{x\to+\infty}\frac{\ln x}{x}=\lim_{x\to+\infty}\frac{\frac{1}{x}}{1}=\lim_{x\to+\infty}\frac{1}{x}=0$$

2. Evaluate $\lim\limits_{x\to0^+}\dfrac{\ln(1+x)}{x}$

Solution

$$\lim_{x\to 0^+}\frac{\ln(1+x)}{x} \text{ (indeterminate form } \frac{0}{0}).$$

$$\lim_{x\to 0^+}\frac{\ln(1+x)}{x} = \lim_{x\to 0^+}\frac{1}{1+x} = 1$$

Application of derivative of a logaritmic function in solving various problems

Derivative of logaritmic functions can be used to solve various types of problems. For example in the fields of earthquake measurement, electronics, econonomic and finance.

Example

A cessna plane takes off from airport at sea level and altitude in feet at time t in minutes is given by $h = 2000\ln(t+1)$. Find the rate of climb at time t=3minutes.

Solution

We need to find the first derivative

$$\frac{d}{dt}(2000\ln(t+1))$$

At $t = 3$

We have $\frac{d}{dt}(2000\ln(t+1)) = \frac{2000}{t+1}$

$$v = \frac{2000}{3+1} = \frac{2000}{4} = 500\, feet/\min$$

The required rate of climb is $500/\min$

Application activity 1.1.5

1) Discuss the variations of the function $f(x) = \frac{\ln(x-2)}{x-2}$

1.2 Exponential functions

1.2.1 Definition and domain of definition of Exponential functions

Activity 1.2.1

1. Let f(x) = and $g(x)$ denotes the inverse function of f(x).

i. complete the following table:

x	0	1	e	e^2		
$g(x)=f^{-1}(x)$					3	4

Discuss and find out the set of all values of $g(x)$

2. Consider the function $h(x)=3^x$ and complete the following table

x	-10	-1	0	1	10
$h(x)=3^x$					

Discuss whether $\forall x\in\mathbb{R}, h(x)\in\mathbb{R}$ and deduce the domain of $h(x)$

a) Discuss whether $h(x)$ can be negative or not and deduce the range of $h(x)$

Remember that for $a>0, a\neq 1$ the logarithmic function is defined as $\log:\mathbb{R}_0^+\to\mathbb{R}$ or $x\to y=\log_a x$.

The inverse of logarithmic function is called exponential function and defined as: $\exp_a:\mathbb{R}\to\mathbb{R}_0^+ : x\mapsto y=\exp_a x.$ For simplicity we write $\exp_a x = a^x$.

Therefore $a^x=y$ if and only if $\log_a y = x$.

Obviously, the domain of the exponential function $y=f(x)=a^x$ is $]-\infty+\infty[$ and its range is $]0,+\infty[$. In the expression $a^x=y$, a is the base, *x* the exponent and *y* the exponential of *x* in base *a*.

Generally, if $u(x)$ is a defined function of x, the function $f(x)=a^{u(x)}$ has the range $]0,+\infty[$ and its domain is the domain of $u(x)$.

Similarly to logarithmic function, if the base "*a*" is the number "e", we have exponential function $y = e^x$ as the inverse of natural logarithm $y = \ln x$.

Example 1

Determine the domain and the range of the function

$f(x) = 3^{\sqrt{2x}}$

Solution

Condition for the existence of $\sqrt{2x}$ in $\mathbb{R}$: $x \geq 0$.

Thus, $Domf = [0,+\infty[$ and the range is $[1,+\infty[$

Example 2

Find the domain and the range of $f(x) = 2^{\ln x}$

Solution

Condition: $x > 0$

Thus, $Domf =]0,+\infty[$ and the range is is $]0,+\infty[$.

Example 3

Find the domain and the range of $f(x) = 3^{\frac{x+1}{x-2}}$

Solution

Condition: $x - 2 \neq 0 \Rightarrow x \neq 2$

Thus, $Domf = \mathbb{R} \setminus \{2\}$ and the range is $]0,3[\cup]3,+\infty[$.

Example 4

Find the domain and the range of $f(x) = 4^{\sqrt{x^2-4}}$

Solution

Condition: $x^2 - 4 \geq 0 \Rightarrow x \in]-\infty,-2] \cup [2,+\infty[$

Thus, $Domf =]-\infty,-2] \cup [2,+\infty[$ and the range is $[1,+\infty[$

Example 5

Determine the domain and the range of each of the following functions:

1. $g(x)=e^{\frac{x+2}{x-3}}$
2. $h(x)=e^{\sqrt{x^2-4}}$

Solution

1.Condition for the existence of $\dfrac{x+2}{x-3}$ in $\mathbb{R}$: $x\neq 3$.

Therefore $Dom\ g=\mathbb{R}\setminus\{3\}=]-\infty,3[\cup]3,+\infty[$

and range is $]0,e[\cup]e,+\infty[$

2.Condition $x^2-4\geq 0\Rightarrow x\in]-\infty,-2]\cup[2,+\infty[$.

Thus, $Dom\ h=]-\infty,-2]\cup[2,+\infty[$ and range is $[1,+\infty[$.

Example 6

Find the domain and the range of $f(x)=e^{\sqrt{x}}$

Solution

Condition: $x\geq 0$

Thus, $Domf=[0,+\infty[$ and range is $[1,+\infty[$.

Example 7

Find the domain and the range of $g(x)=e^{\frac{x+1}{x-2}}$

Solution

Condition: $x-2\neq 0\Rightarrow x\neq 2$

Thus, $Domg=\mathbb{R}\setminus\{2\}$ and range is $]0,e[\cup]e,+\infty[$.

Example 8

Find the domain of $h(x)=e^{\sqrt{x^2-1}}$

Solution

Condition: $x^2-1\geq 0\Rightarrow x\in]-\infty,-1]\cup[1,+\infty[$

Thus, $Domh=]-\infty,-1]\cup[1,+\infty[$ and range is $[1,+\infty[$.

Application activity 1.2.1

Discuss and determine the domain and range of the following functions

1) $f(x)=5e^{2x}$ 2) $h(x)=2^{\ln x}$ 3) $f(x)=3^{\frac{x+1}{x-2}}$

1.2.2 Limits of exponential function

Activity 1.2.2

1) You are familiar with the graph of $f(x)=\ln x.$ Explain in your own words how you can obtain the graph of its inverse $y=f^{-1}(x)=e^{x}$.

2) From the graph deduce $\lim\limits_{x\to-\infty} e^{x}$ and $\lim\limits_{x\to+\infty} e^{x}$

3) Discuss $\lim\limits_{x\to-\infty}\left(\frac{1}{2}\right)^{x}$ *and* $\lim\limits_{x\to+\infty}\left(\frac{1}{2}\right)^{x}$.

4) Generalize above results to $\lim\limits_{x\to-\infty} a^{x}$ *and* $\lim\limits_{x\to+\infty} a^{x}$

It is clear that:

$\lim\limits_{x\to-\infty} e^{x}=0$ and $\lim\limits_{x\to+\infty} e^{x}=+\infty$

In general:

If $a>1$, $\lim\limits_{x\to-\infty} a^{x}=0$ *and* $\lim\limits_{x\to+\infty} a^{x}=+\infty$

If $0<a<1,$, $\lim\limits_{x\to-\infty} a^{x}=+\infty$ *and* $\lim\limits_{x\to+\infty} a^{x}=0$

Examples

1) Evaluate

a) $\lim\limits_{x\to\infty} e^{1-4x-5x^{2}}$ b) $\lim\limits_{x\to1}\left(\frac{3}{5}\right)^{\frac{1}{x-1}}$ c) $\lim\limits_{x\to-\infty} 3^{\frac{1}{x}}$ d) $\lim\limits_{x\to1} 3^{\frac{1}{x-1}}$

Solution

a) $\lim\limits_{x\to\infty} e^{1-4x-5x^{2}}$

We know that $\lim\limits_{x\to\infty} 1-4x-5x^{2}=-\infty$

Therefore, as the exponent goes to minus infinity in the limit and so the exponential must go to zero in the limit using the ideas from the previous formula.

Hence, $\lim\limits_{x\to\infty} e^{1-4x-5x^2} = 0$

b) The exponent goes to infinity in the limit and so the exponential will also need to go to zero in the limit since the base is less than 1.

Hence, $\lim\limits_{x\to 1}\left(\frac{3}{5}\right)^{\frac{1}{x-1}} = 0$

c) $\lim\limits_{x\to -\infty} 3^{\frac{1}{x}} = 3^{\lim\limits_{x\to-\infty}\frac{1}{x}} = 3^0 = 1$

d) $\lim\limits_{x\to 1} 3^{\frac{1}{x-1}} = 3^{\lim\limits_{x\to 1}\frac{1}{x-1}} = 3^{\frac{1}{0}}$

Study one side limit:

x	$3^{\frac{1}{x-1}}$
0	0.33
0.2	0.25
0.4	0.16
0.6	0.06
0.8	0.004
0.9	0.00001

x	$3^{\frac{1}{x-1}}$
2	3
1.8	3.948
1.6	6.24
1.4	15.59
1.2	243
1.1	59049

$\lim\limits_{x\to 1^-} 3^{\frac{1}{x-1}} = 0 \text{ and } \lim\limits_{x\to 1^-} 3^{\frac{1}{x-1}} = +\infty$

Hence, $\lim\limits_{x\to 1} 3^{\frac{1}{x-1}}$ does not exist.

Alternatively:

Since, $\lim\limits_{x\to 1^+} \frac{1}{x-1} = +\infty$ and $\lim\limits_{x\to 1^-} \frac{1}{x-1} = -\infty$, apply results on $\lim\limits_{x\to\pm\infty} a^x$ for $a > 1$ to have:

$\lim\limits_{x\to 1^-} 3^{\frac{1}{x-1}} = 0 \text{ and } \lim\limits_{x\to 1^-} 3^{\frac{1}{x-1}} = +\infty$

Hence, $\lim\limits_{x\to 1} 3^{\frac{1}{x-1}}$ does not exist.

2) Consider $f(x)=\dfrac{e^{-3x}-2e^{8x}}{9e^{8x}-7e^{-3x}}$, evaluate each of the following:

$\lim\limits_{x\to-\infty} f(x)$ and $\lim\limits_{x\to+\infty} f(x)$.

Solution

$$\lim_{x\to-\infty} f(x)=\lim_{x\to-\infty}\frac{e^{-3x}-2e^{8x}}{9e^{8x}-7e^{-3x}}=\lim_{x\to-\infty}\frac{e^{-3x}\left(1-2e^{11x}\right)}{e^{-3x}\left(9e^{11x}-7\right)}$$

$$=\lim_{x\to-\infty}\frac{1-2e^{11x}}{9e^{11x}-7}=-\frac{1}{7}$$

$$\lim_{x\to+\infty} f(x)=\lim_{x\to+\infty}\frac{e^{-3x}-2e^{8x}}{9e^{8x}-7e^{-3x}}=\lim_{x\to+\infty}\frac{e^{8x}\left(e^{-11x}-2\right)}{e^{8x}\left(9-7e^{-11x}\right)}=-\frac{2}{9}$$

Application activity 1.2.2

For each given function, evaluate limit at $+\infty$ and $-\infty$

1. $f(x)=e^{8+2x-x^3}$
2. $f(x)=e^{\frac{6x^2+x}{5+3x}}$
3. $f(x)=2e^{6x}-e^{-7x}-10e^{4x}$
4. $f(x)=3e^{-x}-8e^{-5x}-e^{10x}$
5. $f(x)=\dfrac{e^{-3x}-2e^{8x}}{9e^{8x}-7e^{-3x}}$

1.2.3 Application of limits to determine the continuity and asymptotes of exponential functions

Activity 1.2.3

Given the function $f(x) = 2^{(x-2)}$,

a. Find the domain and range of $f(x)$.

b. Determine $\lim\limits_{x\to-\infty} f(x)$ and deduce the equation of horizontal asymptote for the graph.

c. Evaluate the value of $f(x)$ *for* $x=0$ and deduce y- intercept

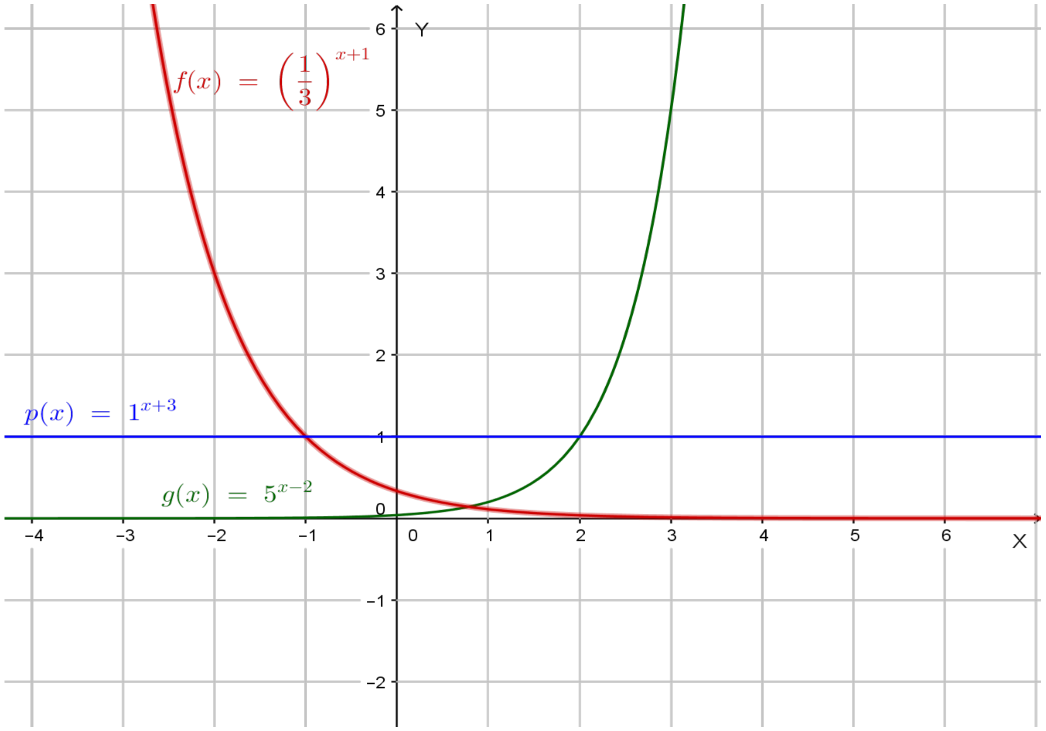
d. Determine $\lim\limits_{x\to\infty} f(x)$ and $\lim\limits_{x\to-\infty} \frac{f(x)}{x}$

e. Evaluate $\lim\limits_{x\to0^+} f(x)$ and $\lim\limits_{x\to0^-} f(x)$. Discuss the continuity of this function at $x=0$.

f. Sketch the graph of $f(x)$

For $a>0, a\neq1$, the exponential function $f(x) = a^x$ is continuous on $\mathbb{R}$ and takes always nonnegative values. Its graphs admits the line of equation y = 0 as horizontal asymptote and intercepts y-axis at (0, 1). The function f is increasing from 0 to $+\infty$ if a is greater than 1 and decreasing from $+\infty$ to 0 if a is smaller than 1. The function is the constant 1 if $a=1$ and its graph is the horizontal line of equation $y=1$.

Graphs of $g\left(x\right)=5^{x-2}$ **,** $f\left(x\right)=\left(\dfrac{1}{3}\right)^{x+1}$ **and** $p\left(x\right)=1^{x+3}$



Example

Let $f\left(x\right)=3^{x+1}-1$.

Find the domain, range and equation of the horizontal asymptote of the graph of f. Precise intercepts (if any) of the graph with axes. .

Solution

The domain of f is the set of all real numbers since the expression $x+1$ is defined for all real values. To find the range of f, we start with the fact that $3^{(x+1)} > 0$ as exponential function.

Then, subtract 1 to both sides to get $3^{x+1}-1>-1$.

Therefore, for any value of *x*, $f(x)>-1$. In other words, the range of *f* is $]-1,\infty[$

As x decreases without bound, $f(x)=3^{x+1}-1$ approaches -1, in other

words $\lim\limits_{x\to-\infty} f\left(x\right)=-1$. Thus, the graph of f has horizontal asymptote the line of equation $y=-1$. To find the x intercept we need to solve the equation $f\ (x)\ =\ 0$.

This is $3^{(x+1)}-1=0$.

Solving yields to $x=-1$. The x- intercept is the point (-1, 0).

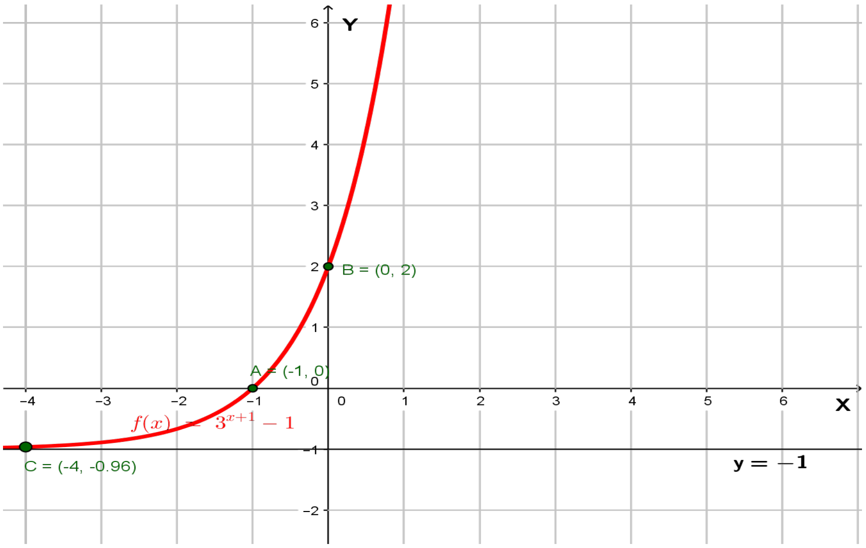
The y- intercept is given by (0, f (0)) = (0, $3^{(0+1)}$ - 1) = (0, 2).

Extra points:

(-2, f (-2)) = (-2, $3^{(-2+1)}$ - 1) = (2, -4/3)

(-4, f (-4)) = (-4, $3^{(-4+1)}$ - 1) = (-4, -26/27)

We can now use all the above information to plot $f(x)=3^{(x+1)}-1$:



Application activity 1.2.3

Given the function $f(x)=2^x+1$

a) Determine domain and range.

b) Write the equation of horizontal asymptote of the graph.

c) Find the x and y intercepts of the graph if there are any.

d) Sketch the graph of $y=f(x)=2^x+1$

1.2.4 Differentiation of Exponential functions

Activity 1.2.4

Given functions $f(x)=e^x$ and $g(x)=2^x$

1. Determine the inverse of $f(x)$ and $g(x)$.
2. Use the derivative of logarithmic functions $p(x)=\ln x$ and $k(x)=\log_2 x$, then apply the rule of differentiating inverse functions to find the derivative of $f(x)=e^x$ and $g(x)=2^x$

The derivative of $f(x)=e^x$ is noted by $\frac{d(e^x)}{dx}=e^x$ or $f^{'}(x)=e^x$.

If u is a function of x, the derivative of $y=e^{u(x)}$ is

$$y'=d\left(\frac{e^{u(x)}}{dx}\right)=e^{u(x)}\frac{du(x)}{dx}=u'e^{u(x)}.$$

Thus, $\left(e^u\right)'=u'e^u$.

The derivative of $g(x)=a^x$ is $g'(x)=a^x\ln a$.

Therefore, if u is a function of x, the derivative of $g(x)=a^{u(x)}$ is

$g'(x)=u'(x)a^{u(x)}\ln a$

Example

Find the derivative of the following functions

1) $f(x)=e^{5x^2}$ 2) $y=e^{4x}$

Solution:

1) $f'(x)=(5x^2)'e^{x^2}=10xe^{5x^2}$ 2) $y'=4e^{4x}$

Application activity 1.2.4

1. Given the function $f(x) = 4^x$

Find $f'(x)$ the derivative function of $f(x)$

2. Find the derivative of each of the following function

(a) $f(x) = 10^{3x}$ (b) $f(x) = xe^{x^2+1}$ (c) $f(x) = \frac{3^{4x+2}}{x}$

1.2.5 Application of derivative to determine the continuity and the variation of exponential functions

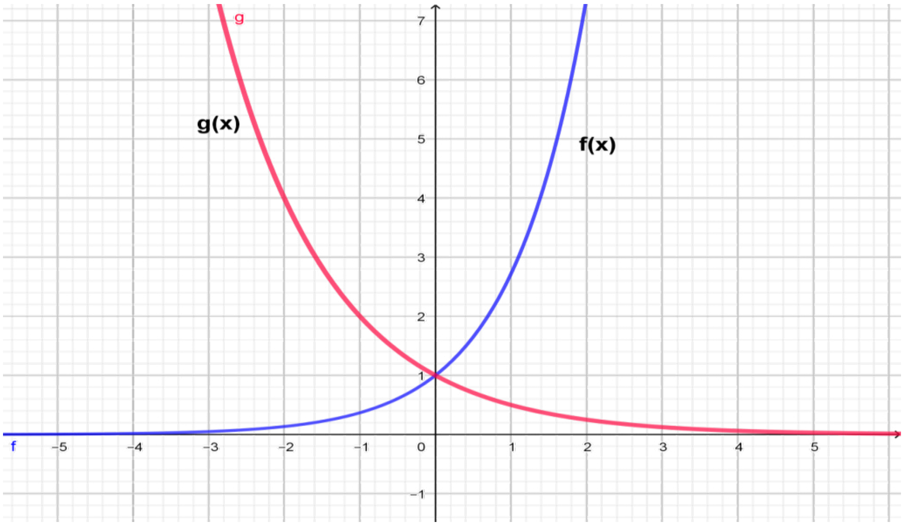
Activity 1.2.5

Given two functions $f(x) = 2^x$ and $g(x) = 0.5^x$

1) Compare $f(1)$ and $f(10)$ to verify whether the function $f(x)$ is increasing or decreasing on the interval $[1,10]$;
2) Compare $g(1)$ and $g(10)$ to verify whether the function $g(x)$ is increasing or decreasing on the interval $[1,10]$;
3) Use derivatives $f'(x)$ and $g'(x)$ to discuss the variation of each function;
4) Plot the graphs of $f(x)$ and $g(x)$.
5) Explain in your own words the variations of exponential function.

a) Variations of exponential functions

The function $g(x) = a^x, a > 1$ defined on $\mathbb{R}$ is always increasing. When $0 < a < 1$, the function $g(x) = a^x$ is always decreasing. This means the exponential functions $g(x) = a^x$ does not have extremum (maximum or minimum); this means that the function increases or decreases "indefinitely".



Example

Given the function $f(x) = xe^x$

i) Find the derivative of $f(x) = xe^x$

ii) Solve $f'(x) = 0$

iii) Discuss extrema of the function.

iv) Establish the table of sign of $f'(x)$ and variations of $f(x)$

v) Plot the graph of the function.

Solution

i. The domain of the function is $\mathbb{R}$.

The derivative of $f(x) = xe^x$ is defined by

$f'(x) = e^x + xe^x = (1 + x)e^x$

ii. $f'(x) = 0$ if $x = -1$

iii) Sign table for $f(x)$

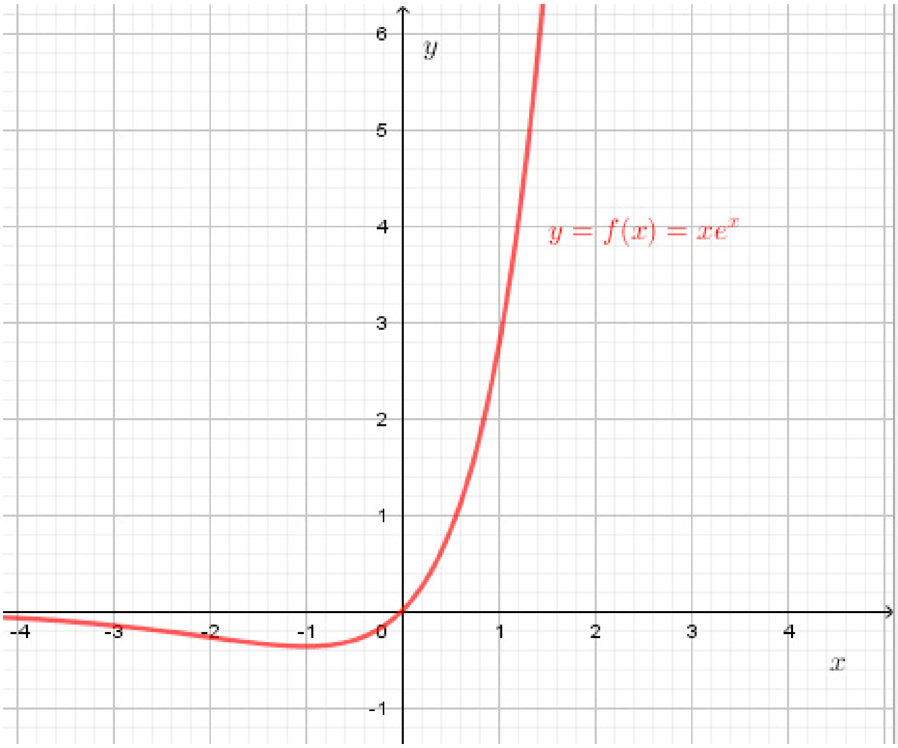
There is need to find limit of the function at the boundaries of the domain:
$\lim\limits_{x \to -\infty} xe^x = 0$ and $\lim\limits_{x \to +\infty} xe^x = +\infty$.
The limit at $-\infty$ tells us that line of equation $y = 0$ is an horizontal asymptote when x is taking "indefinitely" negative values.

X	$-\infty$		-1		0		$+\infty$
$f'(x)$	$-$		0	$+$		$+$	
$f(x)$	0	$\searrow$	$-1/e$	$\nearrow$	0	$\nearrow$	$+\infty$

Therefore, $f(x)$ is decreasing from 0 to $-\frac{1}{e}$ on the interval $]-\infty,-1]$ and increases from $-\frac{1}{e}$ to $+\infty$ on $[-1,+\infty[$. The function has minimum (absolute) equal to $-\frac{1}{e}$ when $x=-1$.

Note: Using a calculator, one can complete a table of additional values of $f(x)$ which facilitate to plot the graph.

v. Graph of $f(x)=xe^x$



b) Application of derivatives to remove indeterminate form $0^0, 1^\infty$ *and* ∞^0

These indeterminate forms are found in functions of the form $y=[f(x)]^{g(x)}$.

To remove these indeterminate forms we change the function in the form

$$y=\left[f(x)\right]^{g(x)}=e^{g(x)\ln f(x)}$$

Also $\lim\limits_{x\to k} e^{f(x)\ln g(x)}=e^{\lim\limits_{x\to k} f(x)\ln g(x)}$

Example

a) Show that $\lim\limits_{x\to 0^+} x^x=1$

Solution

$$\lim_{x\to 0^+} x^x=0^0 \quad (IF)$$

$$\lim_{x\to 0^+} x^x=\lim_{x\to 0^+} e^{x\ln x}=e^{\lim\limits_{x\to 0^+} x\ln x}$$

$$\lim_{x\to 0^+} x\ln x \ (0\times\infty \ IF)$$

$$\lim_{x\to 0^+} x\ln x=\lim_{x\to 0^+}\frac{\ln x}{\frac{1}{x}}\left(\frac{\infty}{\infty} IF\right)$$

$$\lim_{x\to 0^+}\frac{\ln x}{\frac{1}{x}}=\lim_{x\to 0^+}\frac{\frac{1}{x}}{-\frac{1}{x^2}}=0 \text{ (Hospital rule).}$$

Finally, $\lim\limits_{x\to 0^+} x^x=1$

b) Show that $\lim\limits_{x\to+\infty}\left(1+\frac{1}{x}\right)^x=e$

Solution

$$\lim_{x\to+\infty}\left(1+\frac{1}{x}\right)^x \quad \left(1^\infty \quad IF\right)$$

$$\lim_{x\to+\infty}\left(1+\frac{1}{x}\right)^x=\lim_{x\to+\infty} e^{x\ln\left(1+\frac{1}{x}\right)}=e^{\lim\limits_{x\to+\infty} x\ln\left(1+\frac{1}{x}\right)}$$

But,

$$\lim_{x\to+\infty} x\ln\left(1+\frac{1}{x}\right)=\lim_{x\to+\infty}\frac{\ln\left(1+\frac{1}{x}\right)}{\frac{1}{x}}\quad\left(\frac{0}{0}\,IF\right)$$

$$\lim_{x\to+\infty}\frac{\ln\left(1+\frac{1}{x}\right)}{\frac{1}{x}}=\lim_{x\to+\infty}\frac{-\frac{1}{x^2}}{\left(1+\frac{1}{x}\right)\left(-\frac{1}{x^2}\right)}$$ (Hospital rule)

$$=\lim_{x\to+\infty}\frac{1}{1+\frac{1}{x}}=1$$

Thus, $\lim\limits_{x\to+\infty}\left(1+\frac{1}{x}\right)^{x}=e^{1}=e$

c) Show that $\lim\limits_{x\to-1}\left(\frac{1}{x+1}\right)^{x+1}=1.$

Solution

$$\lim_{x\to-1}\left(\frac{1}{x+1}\right)^{x+1}\quad(\infty^{0}IF)$$

$$\Rightarrow\lim_{x\to-1}\left(\frac{1}{x+1}\right)^{x+1}=\lim_{x\to-1}e^{(x+1)\ln\left(\frac{1}{x+1}\right)}=e^{\lim\limits_{x\to-1}(x+1)\ln\left(\frac{1}{x+1}\right)}$$

Or $\lim\limits_{x\to-1}(x+1)\ln\left(\frac{1}{x+1}\right)=\lim\limits_{x\to-1}\frac{\ln\left(\frac{1}{x+1}\right)}{\frac{1}{x+1}}=\lim\limits_{x\to-1}\frac{-\frac{1}{x+1}}{-\frac{1}{(x+1)^2}}=0$ (Hospital rule).

Finally, $\lim\limits_{x\to-1}\left(\frac{1}{x+1}\right)^{x+1}=e^{0}=1$

Application activity 1.2.5

1) Given the function $f(x) = xe^{x^2}$

   a. Find the first derivative of $f(x)$

   b. Establish the table of variation of $f(x)$ and deduce whether $f(x)$ is increasing or decreasing and write down the interval where the function is increasing or decreasing

2) Evaluate the following limits

   a) $\lim\limits_{x \to \infty} \frac{e^x + 1}{e^x - 2}$ b) $\lim\limits_{x \to -\infty} xe^x$

1.3 Applications of logarithmic and exponential functions in real life

Logarithmic and exponential functions are very essential in pure sciences, social sciences and real life situations. They are used by bank officers to deal with interests on loans they provide to clients. Economists and demographists use such functions to estimate the number of population after a certain period and many researchers use them to model certain natural phenomena. In the following paragraphs, we are going to develop some of these applications.

1.3.1 Interest rate problems

Activity 1.3.1

An amount of 2 000 dollars is invested at a bank that pays an interest rate of 10% compounded once annually. Find the total amount at the end of t years by proceeding as follows:

Complete the table below:

At the end of	The total amount
The first year	2000+0.1(2000) =2000(1+0.1)
The second year	2000(1+0.1)+0.1[2000(1+0.1)] =2000(1+0.1)^2
The third year	2000(1+0.1)… + …= 2000(1+01)3

The fourth year	…
The fifth year	…
...	…
The t^{th} year	…

If a principal P (the money you put in) is invested at the bank on interest rate r for a period of t years, then the amount A (how much you make) of the investment can be calculated by the following generalised formula of the interest rate problems:

- $A = P(1+r)$ Simple interest for one year
- $A = P\left(1+\frac{r}{n}\right)^{nt}$ Interest compounded n times per year
- $A = Pe^{rt}$ Interest compounded continuously.

Compound interest

a.1 Discrete compound interest

If you put money in a savings account then the bank will pay you interest (a percentage of your account balance) at the end of each time of period, typically one month or one day.

For example if the time of period is one month this process is called **Monthly compounding.** The term compounding refers to the fact that interest is added to your account each month. If it is one day is called daily compounding. The exponential model that describes this situation is called discrete compounding interest formula.

$A = P_0\left(1+\frac{r}{n}\right)^{nt}$ Where A is total amount at the end of periods of time, P_0 is the principal amount, n is the number of times that the interest is compounded, r is the interest rate per period, t is the time.

Examples:

1. If the principal money is \$100/ the annual interest rate is 5% and the interest is compounded daily. What will be the balance after ten years?

Solution:

Let $P_0 = 100, r = 5\% = 0.05, n = 1year = 365days\ and\ t = 10$

$$P(t)=100\left(1+\frac{0.05}{365}\right)^{365\times10}$$
$$=164.87$$

After 10 years balance will be \$164.87

2. An amount of 500 000 FRW is invested at a bank that pays an interest rate of 12% compounded annually.

   a) How much will the owner have at the end of 10 years, in each of the following cases?

   The interest rate is compounded:

   (i). once a year.

   (ii). Twice a year

   b) What type of interest rate among the two would the client prefer? Explain why.

Solution

a) (i). For once a year, at the end of 10 years the owner will have

$$A=P(1+r)^t=500000(1+0.12)^{10}$$
$$=500000(1.12)^{10}=1552924.10Frw$$

(ii). For twice a year, at the end of 10 years the owner will have

$$A=P(1+\frac{r}{2})^{2t}=500000(1+\frac{0.12}{2})^{2(10)}$$
$$=500000(1.06)^{20}=1603567Frw$$

b) Since $1603567>1552924.10$, the client will prefer compounding many times per year as it results in more money.

a.2 continuous compound interest

If we start with discrete compound interest formula and let the number of times compounded per year approaches ∞, then we end up with what is known as **continuous compounding** then the balance at time t years is given by $A=P_0e^{rt}$ where P_0 is the principal amount, r is annual interest and time t years.

Example

If the principal money is \$10,000 the annual interest rate is 5% and the interest is is compounded continuously. What will be the balance after 40 years?

Solution:

$P_0 = 10,000$, $r = 5\% = 0.05$, $t = 40$

$P(40) = 10000e^{0.05\times 40} = 73890.56$

The balance after 40 years is \$73890.56.

Application activity 1.3.1

Your aunt would like to invests 300 000 FRW at a bank. The Bank I pays an interest rate of 10% compounded once annually. The Bank II pays an interest rate of 9.8% compounded continuously. Your aunt will withdraw the money plus interest after 10 years.

At which bank do you advice your Aunt to invest her money so as to get much money at the end of 10 years?

1.3.2 Mortgage amount

Activity 1.3.2

1) Go to conduct a research in the library, on internet or conduct a conversation with a bank officer to write down the meaning of the following when you get a loan from the bank:

   i. the periodic payment P

   ii. the annual interest rate r

   iii. the mortgage amount M

   iv. the number t of years to cover the mortgage

   v. the number n of payments per year.

   vi. Among all these elements/components, what is the most useful for the client to be informed about by the bank once he/she is given the mortgage loan ?

2) Your elder brother is newly employed at a company and earns 500 000 FRW per month. He would like to know if he can afford monthly payments on a mortgage of 20 000 000 FRW with an interest rate of 6% that runs for 20 years. Given that the quantities above are governed by the relation

$$P=\frac{\frac{rM}{n}}{1-\left(1+\frac{r}{n}\right)^{-nt}}$$

Show your brother that he can afford the monthly payments by determining the following:

i. the monthly payment, that will be retained at the bank

ii. The balance that your brother can withdraw each month from the bank

iii. How much interest your brother will pay to the bank by the end of 20 years

When a person gets a loan (mortgage) from the bank, the mortgage amount ***M***, the number of payments or the number ***t*** of years to cover the mortgage, the amount of the payment ***P***, how often the payment is made or the number ***n*** of payments per year, and the interest rate ***r,*** it is proved that all the 5 components are related by the following formula:

$$P=\frac{\frac{rM}{n}}{1-\left(1+\frac{r}{n}\right)^{-nt}}$$

The payment P required to pay off a loan of M Francs borrowed for n payment periods at a rate of interest i per payment period is

$$P=M\left[\frac{i}{1-(1+i)^{-n}}\right] \text{ where } i=\frac{r}{n}.$$

Example

1) A business woman wants to apply for a mortgage of 75 000 US dollars with an interest of 8% that runs for 20 years. How much interest she will pay over the 20 years?

Solution

Substituting for *M=75 000, r =0.08, t=20,n=12* in the equation

$$P=\frac{\frac{rM}{n}}{1-\left(1+\frac{r}{n}\right)^{-nt}}, \text{ we have } P=\frac{\frac{(0.08)(75000)}{12}}{1-\left(1+\frac{0.08}{12}\right)^{-(12)(20)}}=627.33$$

Each month she will be paying 627.33 US dollars.

The total amount she will pay is $627.33\times12\times20$ US dollars $=150559.2$ US dollars

The interest will be $(150559.2-75000)$ US dollars $=75559.2$ US dollars.

2) What monthly payment is necessary to pay off a loan of $800 at 10% per annum.

(a) In 2 years? (b) In 3 years?

(c) What total amount is paid out for each loan?

Solution

(a) For the 2-year loan, $M=\$800,\quad n=24,\quad and\ i=\frac{0.10}{12}$. The monthly payment P is

$$P=M\left[\frac{i}{1-(1+i)^{-n}}\right]=800\left[\frac{\frac{0.10}{12}}{1-\left(1+\frac{0.10}{12}\right)^{-24}}\right]=36.92$$

(b) For the 3-year loan, $M=\$800,\quad n=36,\quad and\ i=\frac{0.10}{12}$. The monthly payment P is

$$P=M\left[\frac{i}{1-(1+i)^{-n}}\right]=800\left[\frac{\frac{0.10}{12}}{1-\left(1+\frac{0.10}{12}\right)^{-36}}\right]=25.81$$

(c) For the 2-year loan, the total amount paid out is $(36.92)(24)=\$886.08$;

For the3-year loan, the total amount paid out is $(25.81)(36)=\$929.16$.

3) A certain family has just purchased a \$300,000 house and has made a down payment of \$60,000. It can amortize the balance $(\$3000,000-\$60,000)$ at 6% for 30 years.

(a) What are the monthly payments?

(b) What is their total interest payment?

(c) After 20 years, what equity does it have in the house (that is, what is the sum of the down payment and the amount paid on the loan)?

Solution

(a) The monthly payment *P* needed to pay off the loan of \$240,000 at 6% for 30 years

(360 months) is

$$P = M\left[\frac{i}{1-(1+i)^{-n}}\right] = \$240,000\left[\frac{\frac{0.06}{12}}{1-\left(1+\frac{0.06}{12}\right)^{-360}}\right] = \$1438.92$$

(b) The total paid out for the loan is $(\$1438.92)(360) = \$518,011.20$.

The interest on this loan amounts to $\$518,011.20 - \$240,000 = \$278,011.20$.

(c) After 20 years (240 months) there remains 10 years (or 120 months) of payments.

The present value of the loan is the present value of a monthly payment of \$1438.92 for 120 months at 6%, namely,

$$M = P\left[\frac{1-(1+i)^{-n}}{i}\right] = \$1438.92\left[\frac{1-\left(1+\frac{0.06}{12}\right)^{-120}}{1-\frac{0.06}{12}}\right] = \$129,608.49$$

The amount paid on the loan is

$$(\textit{Original loan amount}) - (\textit{Present value}) = \$240,000 - \$129,608.49 = \$110,391.51$$

The equity after 20 years is

$$(\textit{Down payment}) + (\textit{Amount paid on loan}) = \$60,000 + \$110,391.51 = \$170,391.51$$

NOTE:

This equity does not include any appreciation in the value of the house over the 20 year period.

The table below gives a partial schedule of payments for the loan. It is interesting to observe how slowly the amount paid on the loan increases early in the payment schedule, with very little of the payment used to reduce principal, and how quickly the amount paid on the loan increases during the last 5 years.

Payment Number	Monthly Payment	Principal	Interest	Amount Paid on Loan
1	$1,438.92	$ 238.92	$1,200.00	$ 238.92
60	$1,438.92	$ 320.67	$1,118.26	$ 16,669.54
120	$1,438.92	$ 432.53	$1,006.39	$ 39,154.26
180	$1,438.92	$ 583.42	$ 855.50	$ 69,482.77
240	$1,438.92	$ 786.94	$ 651.98	$110,391.39
300	$1,438.92	$1,061.47	$ 377.45	$165,570.99
360	$1,438.92	$1,431.76	$ 7.16	$240,000.00

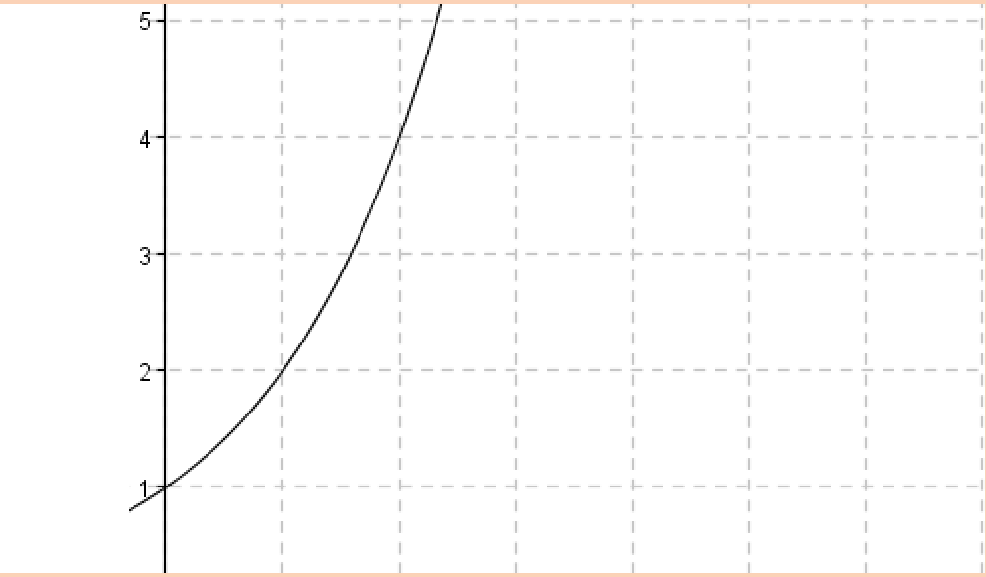
Application activity 1.3.2

A bank can offer a mortgage at 10 % interest rate to be paid back with monthly payments for 20 years. After analysis, a potential borrower finds that she can afford monthly payment of 200000FRW. How much of mortgage can she ask for?

1.3.3 Population growth problems

Activity 1.3.3

Analyse the graph below showing the number of cells recorded by a student in a biology laboratory of his/her school during an experiment as function of time t.



a) Complete the table below:

Time t(minutes)	0	1	2	3	4			
Number of cells	…	…	…					

b) Given that $N(t) = N_0 e^{kt}$, where N (t) is the quantity at time t, N0 is the initial quantity and k is a positive constant, what is the value of N_0? Predict the number of cells after 5 minutes.

c) What happens to the number of cells as the time becomes larger and larger? Is the number of cells growing or not? Explain your answer

If P_0 is the population at the beginning of a certain period and $r\%$ is the constant rate of growth per period, the population for n periods will be $P_n = P_0(1+r)^n$.

This is similar to the final value (F) of an initial investment (A) deposited for t discrete time periods at an interest rate of $i\%$ which is calculated using the formula $F = A(1+i)^t$

To derive a formula that will give the final sum accumulated after a period of continuous growth, we first assume that growth occurs at several discrete time intervals throughout a year. We also assume that A is the initial sum, r is the nominal annual rate of growth, n is the number of times per year that increments are accumulated and y is the final value. This means that after t years of growth the final sum will be:

$$y = A\left(1+\frac{r}{n}\right)^{t}$$

Growth becomes continuous as the number of times per year that increments in growth are accumulated increases towards infinity. When $n \to \infty$, we get

$$\lim_{x\to\infty} y = \lim_{x\to\infty} A\left(1+\frac{r}{n}\right)^{t} = Ae^{rt}.$$

This is similar to $N(t) = N_0 e^{kt}$ where A and N_0, r and k take respectively the same meanings.

Therefore, the final value $A(t)$ of any variable growing continuously at a known annual rate r from a given original value A_0 is given by the following formula $A(t) = A_0 e^{rt}$

Example

1. The number of bacteria in a culture increases according to an equation of the type $N(t) = N_0 e^{kt}$ Given that the number of bacteria triples in 2 hours,

   a. find an equation free of N_0 and solve the equation for k

   b. How long would it take for the number of bacteria to be 5 times the initial number?

Solution

a) $N(2) = 3N_0 \Leftrightarrow 3N_0 = N_0 e^{k(2)} \Leftrightarrow e^{2k} = 3 \Rightarrow 2k = \ln 3 \Leftrightarrow k = \dfrac{\ln 3}{2} = 0.5493$

b) $5N_0 = N_0 e^{0.5493t} \Leftrightarrow e^{0.5493t} = 5 \Rightarrow t = \dfrac{\ln 5}{0.5493} \approx 2.93.$

It will take 2.93 hours for the number of bacteria to be 5 times the initial number.

2. Population in a developing country is growing continuously at an annual rate of 3%. If the population is now 4.5 million, what will it be in 15 years' time?

Solution

The final value of the population (in millions) is found by using the formula $y = Ae^{rt}$ and substituting the given numbers: initial value A = 4.5; rate of growth r = 3% = 0.03; number of time period's t = 15, giving $y = 4.5e^{0.03(15)} = 7.0574048\text{ million}$

Application activity 1.3.3

1. The population of a city increases according to the law of uninhibited growth. If the population doubles in 5 years and the current population is one million, what will be the size of the population in ten years from now?
2. A country economy is forecast to grow continuously at an annual rate of 2.5%. If its Gross National product (GNP) is currently56 billions of USD, what will the forecast for GNP be after 1.75 years (at the end of the third quarter the year after Next)?
3. One town of a given country had a population of 11,000 in 2000 and 13,000 in 2017.

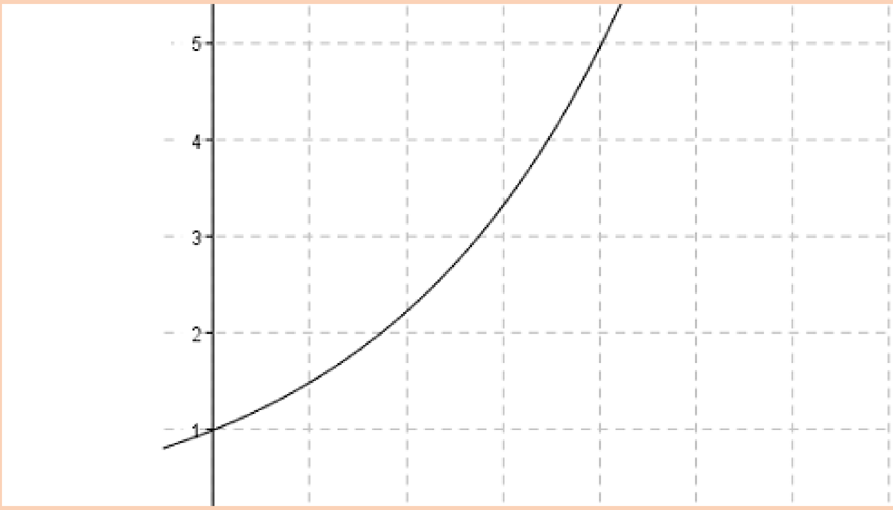
Assuming an exponential growth model, determine the constant rate of growth per year

1.3.4 Problems about alcohol and risk of car accident

Activity 1.3.4

1. a) Discuss the danger of excess for alcohol taken by the drivers.

   b) The following graph shows the risk of a car accident with respect to the driver's blood concentration of alcohol:



   i. What is the risk when there is no alcohol in the blood? Why is that risk not 0?

   ii. Comment on the variation of the risk with respect to the concentration of alcohol in the driver's blood.

   c. Write down approximately the type of the equation that can be used to model the risk.

Science shows that the concentration of alcohol in a person's blood is measurable. Recent medical research suggests that the risk R (given as a present) of having an accident while driving can be modelled by an equation of the type $R(x)=R_0e^{kx}$ where x is the variable concentration of alcohol in the blood and k is a constant.

Example

The risk R of having an accident while driving is modelled by the equation $R(x)=2e^{kx}$, where x is the concentration of alcohol in the driver's blood.

Suppose that a concentration of alcohol in the blood of 0.06 results in 4% risk (R=4) of an accident.

a. Find the value of the constant k in the equation $R(x) = 2e^{kx}$

b. Using this value of k, what is the risk if the concentration of alcohol is 0.08?

c. Using the same value of k, what concentration of alcohol corresponds to a risk of 100%?

d. If the law stipulates that anyone with a risk of having an accident of 10% or more should not drive, at what concentration of alcohol should the driver be arrested and charged?

Solution:

a. From the equation $R(x) = R_0 e^{kx}$, substituting,

$$4 = 2e^{k(0.06)} \Leftrightarrow e^{0.06k} = 2 \Rightarrow 0.06k = \ln 2 \Leftrightarrow k = \frac{\ln 2}{0.06} = 11.552453.$$

b. The equation becomes $R(\text{x}) = 2e^{11.552453x}$. Now,

$R(0.08) = 2e^{(11.552453)(0.08)} = 5.0$

Therefore, the risk is 5%

c. $100 = 2e^{11.552453x} \Leftrightarrow e^{11.552453x} = 50 \Rightarrow x = \frac{\ln 50}{11.552453} = 0.33$.

Application activity 1.3.4

Suppose that the risk R of having accident while driving a car is modelled by the equation $\text{R}(\text{x}) = 4e^{kx}$. Suppose the concentration of alcohol of 0.05 results in 8% of risk of accident, what is the risk if the concentration is 0.18 and what concentration yields to 100% of risk of accident.

1.4 End unit assessment

1. Determine the domain and range of the following functions

a. $f(x)=\log_2(3x-2)$

b. $f(x)=\ln(x^2-1)$

c. $f(x)=2e^{3x+1}$

d. $f(t)=4^{\sqrt{3t+1}}$

2. Evaluate each of the following limits and give the equation of the asymptotes if any

a) $\lim_{x\to 0^+}\frac{\ln x}{x}$

b) $\lim_{x\to +\infty}(3+x^2\ln x)$

3. Differentiate with respect to x the following functions

a) $f(x)=\log_2\sqrt{\frac{x^2-4}{x+2}}$

b) $h(x)=\frac{1}{3}(4^{2x+5})$

4. Carry out a research in the library or on internet and explain at least 5 applications of logarithmic or exponential functions in other human sciences.

5. The population of the world in 1995 was 5.7 billion, and the estimated relative growth rate is 2% per year. If the population continues to grow at this rate, when will it reach 114 billion?

6. Discuss how this unit inspired you in relation of learning other subjects or to your future. If no inspiration at all, explain why.

UNIT 2

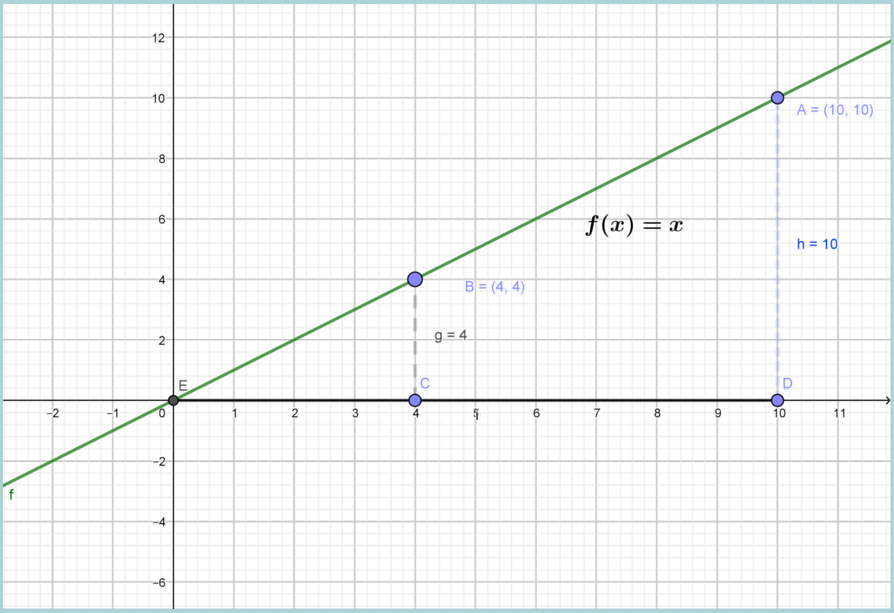
INTEGRATION

Key Unit competence:

Use integration as the inverse of differentiation to solve problems related to marginal and total cost.

2.0 Introductory activity

Two groups of students were asked to calculate the area of a quadrilateral field BCDA shown in the following figure:



The first group calculated the difference of the area for two triangles EDA and ECB

$A_1 = area(\Delta EDA) - area(\Delta ECB)$, The second group with high critical thinking skills used a function $F(x)$ that was differentiated to find $f(x) = x$ (which means $F'(x) = f(x)$ and the x-coordinate d of D and the x- coordinate c of C in the following way: $A_2 = F(d) - F(c)$.

1. Determine the area A_1 found by the first group.
2. Discuss and determine the function $F(x)$ used by the second group. What is the name of $F(x)$ if you relate it with $f(x)$?
3. Determine A_2 the area found by the second group using $F(x)$
4. Compare A_1 and A_2. Discuss if it is possible to find the area bounded by a function $f(x)$, the x-axis and lines with equation $x = x_1$ and $x = x_2$?

2.1 Indefinite integral

2.1.1 Increment and differential of a function

Activity2.1.1

The total consumption of a company is modelled by the function $y = f(x) = 4 + 0.5x + 0.1\sqrt{x}$, where x is the total disposable income (one unit representing 10^6 Rwandan Francs). If $x = 24$ with a maximum error of 0.2

a. What is the consumption of the company at $x = 2$ and at $x = 10$?

b. If Δx is the increment of x from 2 to 10, what is the corresponding increment Δy of the consumption of the company? Represent graphically this situation

c. Discuss the increment of f if x changes from x_0 to x_1 where $x_1 > x_0$

d. Given that the variation of f when x changes from x_0 *to* $x_0 + \Delta x$ is $\Delta y = f'(x_0)\Delta x$ determine the limit of Δy *as* Δx becomes very small

e. Represent graphically the increment on x and the increment on f showing x_0 and x_1 and compare Δy and its limit when $\Delta x \to 0$

Let be given a function $y = f(x)$ continuous on a certain real interval. When the variable x change from x to $x+h$ within the interval, $f(x)$ changes from $f(x)$ to $f(x+h)$. The variation in x is $\Delta x = h$ while the corresponding variation in y becomes $\Delta y = f(x+\Delta x) - f(x)$.

The increment of $y = f(x)$ is $\Delta y = f(x+\Delta x) - f(x)$.

When Δx becomes very small, the change in y can be approximated by the differential of y, that is, $\Delta y \approx dy \text{ and } \Delta x = dx$.

The rate of change $\frac{\Delta y}{\Delta x} = \frac{f(x+h) - f(x)}{h}$ means that $\Delta y = f'(x)\Delta x$

The increment of $y = f(x)$ is $\Delta y = f(x+\Delta x) - f(x)$ and can be approximated by $\Delta y = f'(x)\Delta x$

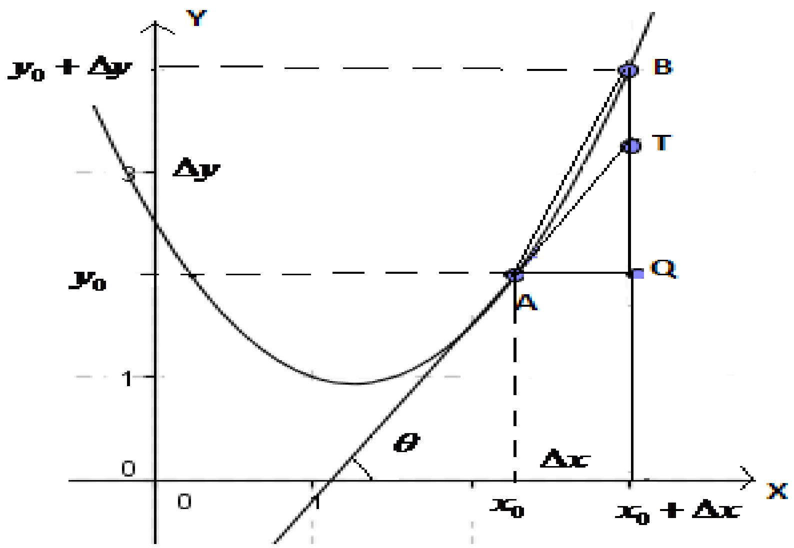
Therefore, $dy = f'(x)dx$.

The differential of a function $f(x)$ is the approximated increment of that function when the variation in x becomes very small. It is given by $dy = f'(x)dx$ .f ′ (x) {\display style f'(x)}

Geometrically, the ratio $\frac{\Delta y}{\Delta x} = \frac{f(x_0+\Delta x) - f(x_0)}{\Delta x}$ represents the slope of the line AB passing through $A(x_0, f(x_0)) \text{ and } B(x_0+\Delta x, f(x_0+\Delta x))$ as illustrated in Figure bellow. When the change in x becomes smaller and smaller, that is Δx approaches 0, the line L becomes the tangent line (T) to the graph at the point $(x_0, f(x_0))$.

This means that the ratio $\frac{\Delta y}{\Delta x}$ becomes the slope of this tangent or equivalently $\lim\limits_{x \to 0} \frac{\Delta y}{\Delta x} = f'(x_0)$

Figure 2.1: differential and increment of a function $y = f(x)$ when x varies to $x + \Delta x$



The derivative of function $y = f(x)$ at x_0 is the slope of the geometric tangent line to the graph of the function at the point A; it is such that $y' = \tan\theta = \dfrac{QT}{AQ} = \dfrac{QT}{dx}$.

In addition, the graph shows that dy is the change in y along the tangent line, while Δy is the resulting change in y along the curve of the function.

Given the functions f and g, it is easy to show that when f and g are differentiable functions on an interval I of $\mathbb{R}$, and $k \in \mathbb{R}$, the following properties hold:

a) $d(f+g) = df + dg$

b) $d(f.g) = gdf + fdg$

c) $d(kf) = kdf$

d) $d\left(\dfrac{f}{g}\right) = \dfrac{gdf - fdg}{g^2}$

e) $d(g \circ f) = g'(u)df$ where $g(u) = g\left[f(x)\right]$

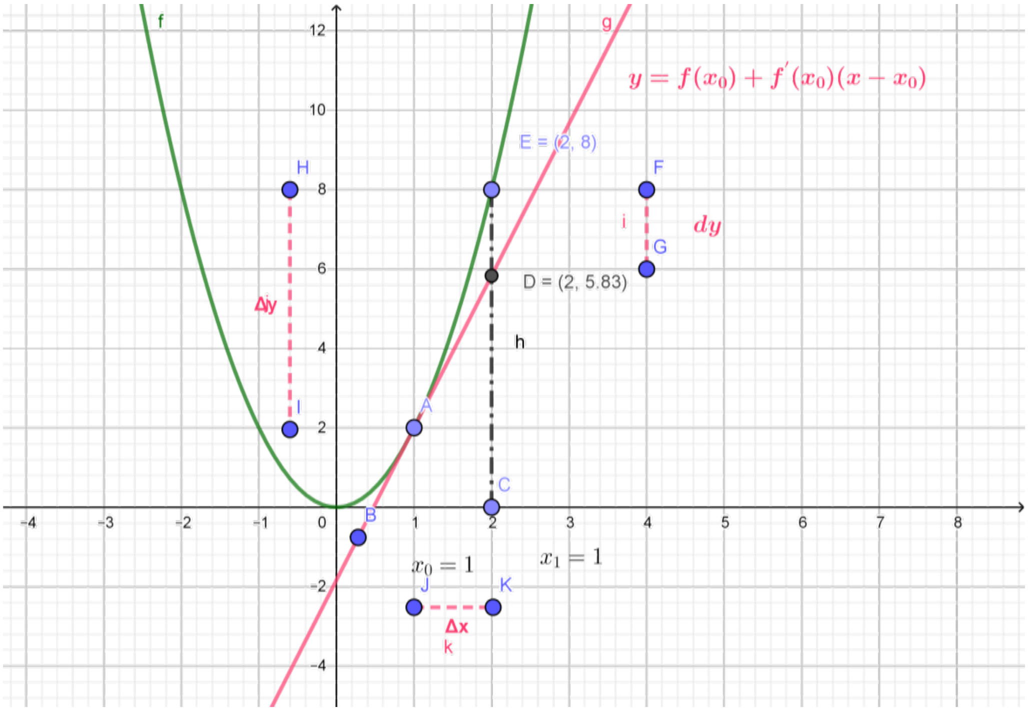
f) $d\left(\sqrt{f}\right) = \dfrac{df}{2\sqrt{f}}$

Example

1) Consider the function $y = f(x) = 2x^2$. Illustrate the increment of y when x increase from 1 to 2

Solution

1. Figure 2.2: illustration of increment of the function $y = f(x) = 2x^2$



This graph shows that from the point $x = x_0 = 1$ *to* $x = x_1 = 2$ where the increment is $\Delta x = 2 - 1 = 1$ the function $y = f(x) = 2x^2$ varies from $y_0 = f(1) = 2$ to $y_0 = f(2) = 8$. That means the increment of the function $\Delta y = 8 - 2 = 6$ which is different to the differential dy measured from the tangent to the graph as it illustrated on the above figure 2.2

Application of differentials on approximation

It was highlighted above that if the variable in x *changes* by δx instead of Δx, then the change for the value of the function $y = f(x)$ is Δy which is approximately $\Delta y \approx f'(x)\,\mathrm{d}x$.

Examples

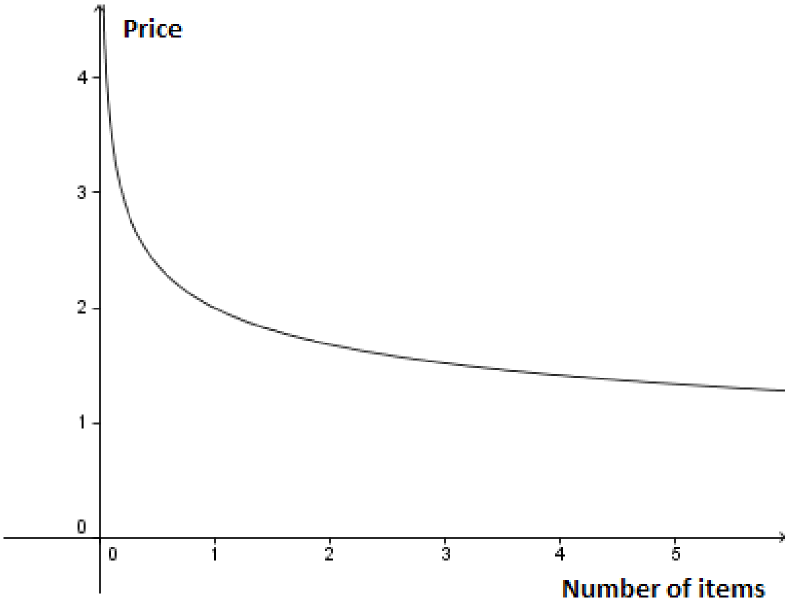
1) The demand function of an item is modeled by the equation $y=\frac{2}{\sqrt[4]{x}}$, where x the number of units is demanded and y is the price in thousands of Frw. Given that $x=16$, with a maximum error of 2, use differentials to approximate the maximum error in y and interpret your result.

Solution

$dy=\frac{-dx}{2\sqrt[4]{x^5}}$. For $x=16$ and $dx=2$, we have $dy=\frac{-2}{2\sqrt[4]{16^5}}=-\frac{1}{32}=-0.03125$

For 16 units demanded ($x=16$), with an error of 2, the corresponding price is $y=\frac{2}{\sqrt[4]{16}}=1$ thousand (1 000Frw), with an approximate error of 31Frw or ($0.03125\times 1000=31.25Frw$).

Figure 5.3: The price in thousands of Frw as function of the number of units demanded



X	0.5	1	2	3	4	....
Y	2.4	2	1.7	1.5	1.4	....

One can observe that as the number of demanded units increases, the price decreases.

2) By approximately what percentage does the area of a circle increase if the radius increases by: 2% *?*

Solution

The area A of a circle is given in terms of the radius r by $A = \pi r^2$

$$\Delta A = dA = \frac{dA}{dr} = 2\pi r dr$$

We divide this approximation by $A = \pi r^2$ to get an approximation that links the relative changes in A and r:

$$\frac{\Delta A}{A} \approx \frac{dA}{A} = \frac{2\pi r dr}{\pi r^2} = 2\frac{dr}{r}$$

If r increases by 2%, then $dr = \frac{2r}{100}$, so $\frac{\Delta A}{A} \approx 2 \times \frac{2}{100} = \frac{4}{100}$

Thus, A increases by approximately 4%.

Application of differentials on the calculation of error

It was proved that the increment Δy is given by $\Delta y \approx f'(x)\,\mathrm{d}x$. When small error dx is made on the variable x, the error made on the value of the function $f(x)$ is Δy.

Examples

1) While measuring the diameter of a circular garden, Felix made an error of 0.02m. Given that he obtained 15m of diameter, estimate the error made on the area of this garden.

Solution:

Let x be the diameter of the garden,

The error made on the diameter is dx.

The area of the garden is $A = \frac{\pi x^2}{4}$. The error made on the area of this garden is

$$dA = \frac{\pi x}{2} dx.$$

Then, for $x = 15m$ and $dx = 0.02$, we have

$$dA = \frac{\pi x}{2} dx = \frac{\pi.15.(0.02)}{2} m^2 = 0.47 m^2$$

2) The deflection at the centre of a road of length l and diameter d supported at its ends and loaded at the centre with a weight w varies as wl^3d^{-4}. What is the percentage increase in the deflection corresponding to the percentage increase in $w, l \; and \; d$ of $3, 2 \text{ and } 1$ respectively?

Solution

Let the deflection of the road at the centre be D

$$D = \frac{kwl^3}{d^4}$$

$$\ln D = \ln \frac{kwl^3}{d^4}$$

$$\Rightarrow \ln D = \ln k + \ln w + 3\ln l - 4\ln d$$

$$\Rightarrow \frac{\Delta D}{D} = \frac{\Delta w}{w} + 3\frac{\Delta l}{l} - 4\frac{\Delta d}{d}$$

$$\Rightarrow 100\frac{\Delta D}{D} = 100\frac{\Delta w}{w} + 3\times 100\frac{\Delta l}{l} - 4\times 100\frac{\Delta d}{d}$$

$$\Rightarrow 3 + 3\times 2 - 4\times 1 = 5\%$$

3) While determining the volume of a cube of 7dm of edge, Anathalia committed an error of $0.03dm^3$. What is the corresponding error made on the edge?

Solution:

Let x be the edge of the cube and V its volume.

We have $x = 7dm$ and $dV = 0.03dm^3$. However $V = x^3$ and $dV = 3x^2dx$.

Therefore, $0.03 = 3x^2dx \Leftrightarrow 0.03 = 3.49dx$

$$\Leftrightarrow dx = \frac{0.03}{3.49}$$

$$dx = 0.00002$$

The error made on the edge for the cube is $0.00002dm$.

Application activity 2.1.1

1) Find the differential of each of the following function:

a. $f(x) = x^2 e^x$ b. $f(x) = \frac{\ln x}{x}$

2) A company designed a tank in the shape of a cube. It claims that the side measures 4 meters, with an error of 0.02. Approximate, in liters, the capacity of the container and use differentials to approximate the error on the measurement of the volume.

2.1.2 Definition of Indefinite integrals

Activity 2.1.2

Suppose that three caterpillars are moving on a straight line with constant velocity $v = 2m\min^{-1}$ (in meters per min).

1) Write down the position of each caterpillar at time t if their respective initial positions are:

i) 1 meter ii) 2 meters iii) 4 meters.

2) If $e(t)$ is the position in function of time, draw the graph of $e(t)$ for the third caterpillar and verify whether or not $e'(t) = v(t)$ where $v(t)$ is the velocity.

3) In the same way:

i) Find a function $F(x)$ whose derivative is $f(x) = 2x$, that is, $F'(x) = f(x) = 2x$

ii) Discuss the number of possibilities for $F(x)$ which are there and the relationship among them.

iii) How do functions $F(x)$ differ?

Anti-derivatives

Let $y = f(x)$ be a continuous function of variable x. An anti-derivative of $f(x)$ is any function $F(x)$ such that $F'(x) = f(x)$. A function has infinitely many anti-derivatives, all of them differing by an additive constant. It means that if $F(x)$ is an anti-derivative of $f(x)$, $F(x) + c$ (where c is an arbitrary constant) is also an anti-derivative of function $f(x)$.

Example

Given the function $f(x) = x\ln x - x$,

a) Find the derivative of $f(x)$

b) From the answer in (a), deduce the anti-derivative of $g(x) = \ln x$ whose graph passes through point $(e,1)$. Plot the graph of the function g and its anti-derivative on the same rectangular coordinate.

Solution:

a) $f'(x) = (x\ln x - x)' = \ln x$

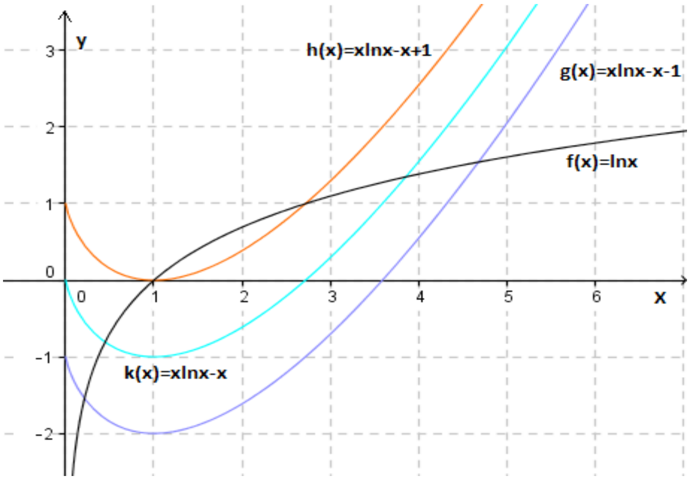
b) The anti-derivatives of $g(x) = \ln x$ are of the type $F(x) = x\ln x - x + c$

$F(e) = e\ln e - e + c = 1 \Leftrightarrow c = 1$

Therefore, the required anti-derivative is $F(x) = x\ln x - x + 1$

The figure below shows function $g(x) = \ln x$ and three of its anti-derivatives.

Figure 5.2: Graph of the function $\mathrm{f}(x) = \ln x$ and 3 of its anti-derivatives



Indefinite integrals

Let $y = f(x)$ be a continuous function of variable x. The **indefinite integral** of $f(x)$ is the set of all its anti-derivatives.

If $F(x)$ is any anti-derivative of function $f(x)$, then the indefinite integral of $f(x)$ is denoted and defined as follows:

$\int f(x)dx = F(x)+c$ where c, an arbitrary constant is called the **constant of integration**.

Thus, $\int f(x)dx = F(x)+c$ if and only if $[F(x)+c]' = F'(x) = f(x)$.

The process of finding the indefinite integral of a function is called **integration.**

The symbol $\int$ is the sign of integration while $f(x)$ is **the integrand.** Note that the integrand is a differential, dx shows that one is integrating with respect to variable x.

Example

Evaluate the following indefinite integrals:

a) $\int 5dx$ b) $\int e^t dt$ c) $\int \frac{1}{x}dx \text{ where } x>0$

Solution:

a) $\int 5dx = 5x+c$

b) $\int e^t dt = e^t + c$

c) $\int \frac{1}{x}dx = \ln x + c$ for $x>0$.

Application activity 2.1.2

Evaluate the following integrals

a) $\int xdx$ b) $\int 3xdx$ c) $\int x^2 dx$

2.1.3 Properties of Indefinite integrals

Activity 2.1.3

Let $f(x)=5$ and $g(x)=\frac{1}{x}$

a) Determine $\int f(x)dx$ and $\int g(x)dx$

b) Evaluate $\int (f+g)(x)dx$

c) Compare $\int (f+g)(x)dx$ and $\int f(x)dx + \int g(x)dx$

Let $y = f(x)$ and $y = g(x)$ be continuous functions and k a constant. Integration obeys the following properties:

$\int kf(x)dx = k\int f(x)dx$: The integral of the product of a constant by a function is equal to the product of the constant by the integral of the function.

$\int [f(x) \pm g(x)]dx = \int f(x)dx \pm \int g(x)dx$: The indefinite integral of the algebraic sum or difference of two functions is equal to the algebraic sum or difference of the indefinite integrals of those functions.

The derivative of the indefinite integral is equal to the function to be integrated.

$$\frac{d}{dx}\int f(x)dx = f(x)$$

Examples

1. Determine whether the following is correct or not. In any case explain your answer:

   $$\int xe^x dx = x\int e^x dx = xe^x + c$$

2. Evaluate:

   a) $\int (3x^2 + 4x - 5)dx$ b) $\int 8e^{-2x}dx$ c) $\int (3^{2x} - \frac{1}{x})dx,\ where\ x > 0$

Solution:

1. $\int xe^x dx = x\int e^x dx = xe^x + c$ is not correct because the variable x is considered as a constant

2. a) $\int (3x^2 + 4x - 5)dx = x^3 + 2x^2 - 5x + c$

   b) $\int 8e^{-2x}dx = -4e^{-2x} + c$

   c) $\int (3^{2x} - \frac{1}{x})dx = \frac{1}{2\ln 3}e^{2x} - \ln x + c$

Note that: integration is a process which is the inverse of differentiation. In differentiation, we are given a function and we are required to find its derivative or differential coeffient. In integration, we are to find a function whose differential coeffient is given.

The process of finding a function is colled integration and it reverses the operation of differentiation.

If the function $F(x)$ is an antiderivative of the function f(x) , then the expression $F(x)+C$ is called the indefinte integration of $f(x)$ and is usually denoted by $\int f(x)\,dx$

$\int \ldots dx$ means the ntegral of…with respect to x. For indefinite integrals, we get a complete integral by adding an unkown constant.

Application activity 2.1.3

1. Evaluate:

   a. $\int (x^3+3\sqrt{x}-7)dx$ b. $\int (4x-12x^2+8x-9)dx$ c. $\int (\frac{1}{x^2}+e^{-x}-\frac{2}{x})dx$

2. A student calculated $\int \frac{x^3-2}{x^3}dx$ as follows: $\frac{\int (x^3-2)dx}{\int x^3dx}=\frac{\frac{1}{4}x^4-2x}{\frac{1}{4}x^4}+c$, which is not correct.

   Show the mistake and suggest the correct working step and solution.

3. Function $y=f(x)$ is such that $\frac{dy}{dx}=\frac{x^3-5}{x^2}$. Find the expression of $y=f(x)$

   if $f(1)=\frac{1}{2}$

4. In Economics, if $f(x)$ is the total cost of producing x units of a certain item, then the marginal cost is the derivative, with respect to x, of the total cost.

   Given that the marginal cost is $M(x)=1+50x-4x^2$, graph $f(x)$ and $M(x)$ on the same diagram.

2.2 Techniques of integration

2.2.1 Basic integration formulae

Activity 2.2.1

Discuss how to calculate $\int 5x^2dx$ and $\int \frac{2x+1}{x^2+x+4}dx$

What formula that can be used to find these integrals?

Given any anti-derivative F of a function f, every possible anti-derivative of f can be written in the form of $F(x)+c$, where c is any constant

This means that when you remember formulae used to differentiate some functions, it is easy to determine integrals. Roughly speaking, the integration is backward of the differentiation.

List of some basic integration formula

1. If k is constant, $\int kdx = kx + c$
2. $\int x^n dx = \frac{1}{n+1}x^{n+1} + c$,where $n \neq -1$, n is a constant
3. If $b \neq -1$, and u a differentiable function , $\int u^b du = \frac{u^{b+1}}{b+1} + c$
4. By definition, $\int x^{-1}dx = \int \frac{1}{x}dx = \ln|x| + c$ for x nonzero
5. $\int e^x dx = e^x + c$, the integral of exponential function of base e
6. If $a > 0$ *and* $a \neq 1$, $\int a^x dx = \frac{a^x}{\ln a} + c$
7. $\int a^{nx+b} dx = \frac{1}{n}\frac{a^{nx+b}}{\ln a} + c$
8. $\int \frac{f'(x)}{f(x)}dx = \ln|f(x)| + c$
9. If $a \neq 0$, $\frac{1}{a}\int \frac{dx}{ax+b} = \frac{1}{a}\ln|ax+b| + c$
10. If $a \neq 0$ *and* $n \neq -1$, $\int (ax+b)^n dx = \frac{(ax+b)^{n+1}}{a(n+1)} + c$

11. $\int \frac{1}{(ax+b)^n}dx = \frac{1}{a(-n+1)(ax+b)^{n-1}} + c$, where $n \neq -1$

12. $\int \frac{dx}{x^2-a^2} = \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right| + c$

13. $\int \frac{dx}{a^2-x^2} = \frac{1}{2a}\ln\left|\frac{a+x}{a-x}\right| + c$

14. $\int \frac{dx}{\sqrt{x^2 \pm a^2}} = \ln\left|x+\sqrt{x^2 \pm a^2}\right| + c$

15. $\int f'(x)\left[f(x)\right]^n dx = \frac{1}{n+1}\left[f(x)\right]^{n+1} + c$, where $n \neq -1$

Application activity 7.1

Compute the following integrals:

1. $\int e^{3x+1}dx$
2. $\int 3^x dx$
3. $\int (8-x^5)dx$
4. $\int \frac{dx}{\sqrt{x^2+9}}$

2.2.2 Integration by change of variables

Activity 2.2.2

1. Using the basic integration formula, integrate the following:

i) $\int x^5 dx$ ii) $\int 2x\left(x^2+4\right)^5 dx$

Explain the problems faced when integrating (ii) above if any

2. Let $u = x^2+4$ what is the derivative of u? Deduce dx in function of u and discuss how to determine $\int 2x(x^2+4)^5 dx$ using expression of u.

Some functions could be difficult to integrate by using anti derivatives and basic integration formula immediately. To overcome this problem, other techniques such as change of variable or integration by substitution could be used. It is the method in which the original variables are expressed as functions of other variables.

Generally if we cannot integrate $\int h(x)dx$ directly, it is possible to find a new variable u and function $f(u)$ for which $\int h(x)dx = \int f(u(x))dx = \int f(u)du$

The process of integration by changing variable or integration by substitution can be described as follows:

If we have to integrate the following,

$\int f(u)u'dx = F(\text{u}) + c$

1. Change variable and differentiate. For example, set t then $dt = udx$.

2. Find out the value u and du by substituting these values into the integral and get

$$\int f(t)u\frac{dt}{u} = \int f(t)dt$$

3. Integrate $\int f(t)dt$ by using anti-derivative method and basic immediate integration formula to get $\int f(t)dt = F(t) + c$

4. Return to the initial variable by replacing variable t by variable u, to get $f(t) + c = f(u) + c$

Examples

Evaluate the following integrals:

1. $\int (2x^2 - 5)xdx$ 2) $\int \frac{x}{\sqrt{x-2}}dx$ 3) $\int \frac{\ln x}{x}dx$

Solution:

1. $\int (2x^2 - 5)xdx$

Suppose, $u = 2x^2 - 5$, then $du = 4xdx \Leftrightarrow \frac{1}{4}du = xdx$.

Hence $\int \left(2x^2 - 5\right)xdx = \frac{1}{4}\int udu = \frac{1}{4}\frac{u^2}{2} + C = \frac{1}{8}u^2 + C$.

Substituting u by $2x^2 - 5$ we get

$$\int \left(2x^2 - 5\right)xdx = \frac{1}{8}(2x^2 - 5)^2 + C$$

2. $\int \frac{x}{\sqrt{x-2}}dx$

Suppose $u = x-2$

$u+2 = x \Rightarrow du = dx$

$$\int \frac{x}{\sqrt{x-2}}dx = \int \frac{(u+2)}{\sqrt{u}}du \Rightarrow \int \frac{(u+2)}{u^{\frac{1}{2}}}du = \frac{2}{3}u^{\frac{3}{2}} + 4u^{\frac{1}{2}} + c$$

Replacing u by $x-2$, we get

$$\int \frac{x}{\sqrt{x-2}}dx = \frac{2}{3}(x-2)^{\frac{3}{2}} + 4(x-2)^{\frac{1}{2}} + c$$

3. $\int \frac{\ln x}{x}dx$

Let $u = \ln x$. Then, $du = \frac{1}{x}dx$.

So, $\int \frac{\ln x}{x}dx = \int u du = \frac{1}{2}u^2 + c$

Substituting u by $\ln x$ yields

$$\int \frac{\ln x}{x}dx = \frac{1}{2}\ln^2 x + c$$

Application activity 2.2.2

Determine the following integrals:

1. $\int xe^{x^2} dx$
2. $\int \frac{dx}{(1-2x)^2}$
3. $\int \frac{x+x^2}{(-3x^2+4-2x^3)^2}dx$
4. $\int \frac{x}{(1-2x^2)^{\frac{1}{3}}}dx$
5. $\int x\sqrt{-1+x^2}\,dx$

2.2.3 Integration by Parts

Activity 7.2.1

Use the integration by changing variable to integrate the following:

1. $\int 3x^2(x^3+1)dx$
2. $\int xe^x dx$
   i. Was it easy for you to integrate (2) using changing variable methods?
   ii. Let $u = x \text{ and } dv = e^x dx$. Find v
   iii. Compare $\int xe^x dx$ and $\int udv$
   iv. Determine $\int udv = uv - \int vdu$ and deduce $\int xe^x dx$

When we integrate, we can find some functions which can't be integrated immediately by using integration by changing variable method. To overcome that problem you should use integration by parts or partial integration technique. In this, you have to find the integral of a product of two functions in terms of the integral of their derivative and anti-derivative.

If u and v are two functions of x, the product rule for differentiation can be used to integrate the product udv or vdu in the following way. Since $d(uv) = udv + vdu$

$$\text{It comes that } \int d(uv) = \int udv + \int vdu$$

this leads to: $uv = \int udv + \int vdu$

Thus $\int udv = uv - \int vdu$. **Or** $\int vdu = uv - \int udv$

When using integration by parts, keep in mind that you are splitting up the integrand into two parts. One of these parts, corresponding to u will be easy to differentiate, and the other, corresponding to dv, will be easy to integrate

Examples

Calculate the following integral

1) $\int \ln x dx$

2) $\int xe^{3x} dx$

Solution

1. $\int \ln x dx \quad choose \begin{cases} u = \ln x \\ dv = dx \end{cases} then \begin{cases} du = \dfrac{1}{x} \\ v = x \end{cases}$

Using the integration by parts rule, we get

$$\int udv = uv - \int vdu = x\ln x - \int \cancel{x}\frac{dx}{\cancel{x}} = x\ln x - x + c$$

2. $\int xe^{3x}dx$

$$let \begin{cases} u = x \\ dv = e^{3x} \end{cases} then \begin{cases} du = dx \\ v = \dfrac{1}{3}e^{3x} \end{cases}$$

$$\int udv = uv - \int vdu = \frac{1}{3}xe^{3x} - \int \frac{1}{3}e^{3x}dx$$

$$= \frac{xe^{3x}}{3} - \frac{1}{3}\int e^{3x}dx$$

$$= \frac{xe^{3x}}{3} - \frac{1}{3}\frac{e^{3x}}{3} = \frac{xe^{3x}}{3} - \frac{1}{9}e^{3x} + c$$

Then, $\int xe^{3x}dx = \dfrac{xe^{3x}}{3} - \dfrac{1}{9}e^{3x} + c$

Application activity 2.2.3

Compute the following integrals using integration by parts

1. $\int 3x^2e^{-x}dx$
2. $\int x^2 \ln x dx$
3. $\int x\sqrt{x+5}dx$

2.3 Definite integral

2.3.1 Definition of definite integral

Activity 2.3

A learner in Senior Six is preparing to sit an end year exam of Mathematics. He/she draws on the same axes the linear function defined by $f(x)=2x, y=0$, and two vertical lines, $x=0$, and $x=4$

a) Draw the shape obtained and prove that it is in the form of a triangle.

b) By using the formula for the area of a triangle, calculate the area enclosed by the functions $y=2x, y=0, x=0$ and $x=4$

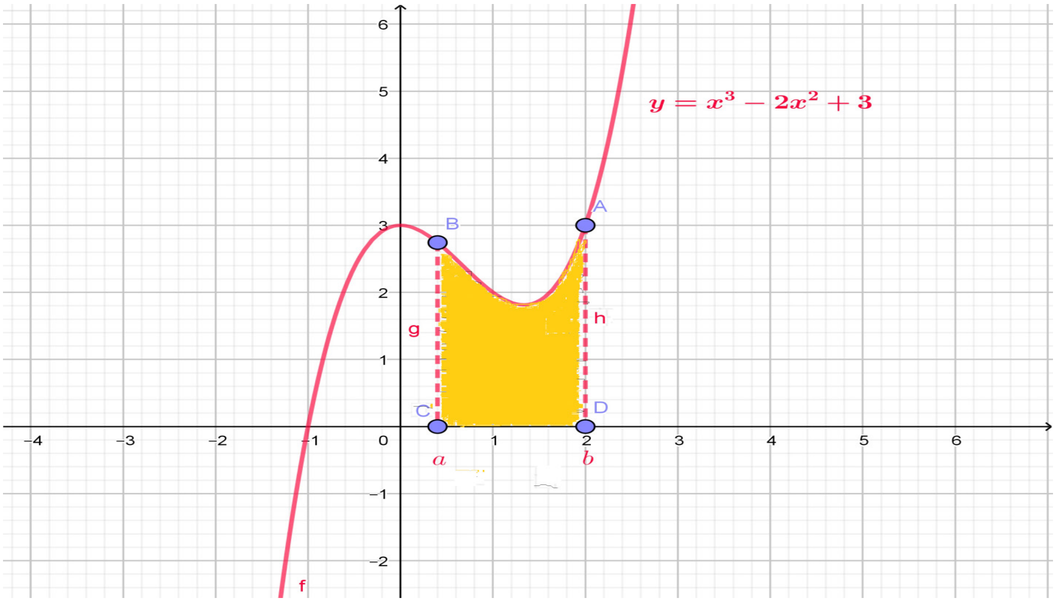
c) Consider the function $F(x)$ as an anti- derivative of $f(x)=2x$. Find $F(x)$ and carry out $F(4)-F(0)$

Let f be a continuous function defined on a closed interval $[a,b]$ and F be an anti-derivative of f for any anti-derivative $F(x)$ of $f(x)$ on $[a,b]$ the difference $F(b)-F(a)$ has a unique value. This value is defined as a definite integral of $f(x)$ for $a \le x \le b$. We write, $\int_a^b f(x)dx = F(b)-F(a)$. Thus, if $F(x)$ is an anti-derivative of $f(x)$ then,

$$\int_a^b f(x)dx = [F(x)+c]_a^b = [(F(b)+c)-(F(a)+c)] = [F(b)+\mathrm{c}-\mathrm{F(a)}-\mathrm{c}] = F(b)-F(a)$$

$\int_a^b f(x)dx$ is read as the "integral from a to b of $f(x)$, a is called **lower limit** and b is called **upper limit**. The interval $[a,b]$ is called **the range of integration**. Geometrically, the definite integral $\int_a^b f(x)dx$ is the area of the region enclosed by the curve $y=f(x)$, the vertical lines $x=a, x=b$ and the $x-axis$ as illustrated in the following figure.

Figure : Definite integral of a function $f(x)$ on a given interval $[a,b]$



The area of coloured region is given by $\int_a^b (x^3 - 2x^2 + 3)dx$. If measurement units are provided for axes, then the area of the region is the product of this definite integral and the area of square unit.

Fundamental theorem of integral calculus:

Let $F(x)$ and $f(x)$ be functions defined on an interval $[a,b]$. If $f(x)$ is continuous and $F'(x) = f(x)$, *then* $\int_a^b f(x)dx = [F(x)]_a^b = F(b) - F(a)$.

Example

$$\int_1^2 x^2 dx$$

Solution:

$$\int_1^2 x^2 dx = \left[\frac{x^3}{3}\right]_1^2 = \frac{8}{3} - \frac{1}{3} = \frac{7}{3}$$

Application activity 2.3.1

Calculate

1. $\int_0^3 xdx$

2. $\int_0^3 (x+1)dx$

2.3.2 Properties of definite integral

Activity 2.3.2

Given that $f(x)=(x^3+3)$ and $g(x)=-2x^2$,

1) Evaluate a) $\int_1^2 f(x)dx$ b) $\int_1^2 g(x)dx$ and c) $\int_1^2 (f+g)(x)dx$

2) Compare $\int_1^2 (f+g)(x)dx$ and $\int_1^2 f(x)dx+\int_1^2 g(x)dx$ and conclude on

how to find. $\int_a^b (f+g)(x)dx$

If $f(x)$ *and* $g(x)$ are continuous functions on a closed interval $[a,b]$ then:

1. $\int_a^b 0dx=0$

2. $\int_a^b f(x)dx=-\int_b^a f(x)dx$ (Permutation of bounds)

3. $\int_a^b [\alpha f(x)\pm\beta g(x)]dx=\alpha\int_a^b f(x)dx\pm\beta\int_a^b g(x)dx, \alpha \text{ and } \beta\in\mathbb{R}\ (linearity)$

4. $\int_a^b f(x)dx=0\ (bounds\ are\ equal, a=b)$

5. $\int_a^b f(x)dx=\int_a^c f(x)dx+\int_c^b f(x)dx \text{ with } a<c<b$ (Chasles relation)

6. $\forall x\in[a,b], f(x)\le g(x)\Rightarrow\int_a^b f(x)dx\le\int_a^b g(x)dx$ it follows that

$fx(x)\ge 0\Rightarrow\int_a^b f(x)\ge 0$ and

$\left|\int_a^b f(x)dx\right|\le\int_a^b |f(x)|dx$ (Positivity)

7. $\int_{-a}^{a} f(x)dx = \begin{cases} 2\int_{0}^{a} f(x)\,dx, \text{ if } f(x) \text{ is even function} \\ 0, \text{ if } f(x) \text{ is odd function} \end{cases}$

Remark: $\int_{a}^{a} f(x)dx = 0$ **and** $\int_{a}^{a} 0dx = 0$

Example

1. Calculate the definite integral: $\int_{1}^{2} x^3 dx$

Solution

Fist we calculate $\int_{1}^{2} x^3 dx = \left[\frac{1}{4}x^4\right]_1^2$

Then, $\int_{1}^{2} x^3 dx = \frac{1}{4}\left[x^4\right]_1^2 = \frac{1}{4}\left(2^4 - 1^4\right) = \frac{15}{4}$

Therefore, $\int_{1}^{2} x^3 dx = \frac{15}{4}$

2. $\int_{1}^{4} (e^x - 2\sqrt{x})dx$

$$\int_{1}^{4} (e^x - 2\sqrt{x})dx = \int_{1}^{4} e^x dx - 2\int_{1}^{4} \sqrt{x}dx = \left[e^x\right]_1^4 - 2\left[\frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1}\right]_1^4$$

$$= (e^4 - e^1) - 2\left(\frac{4^{\frac{3}{2}}}{\frac{3}{2}} - \frac{1^{\frac{3}{2}}}{\frac{3}{2}}\right)$$

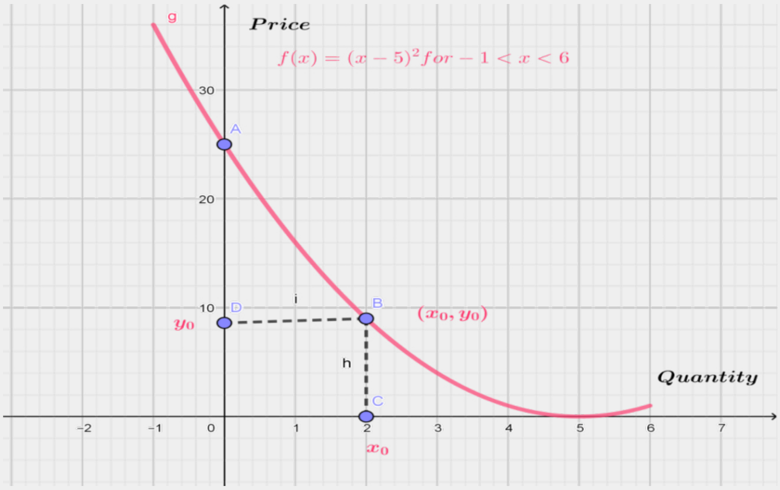
$$= e^4 - e - \frac{28}{3}$$

Application activity 2.3.2

Evaluate each of the following definite integrals

1. $\int_{1}^{2}(4x^2-3x)dx$

2. In business and economics it is known that when $f(x)$ is the demand function (the quantities of a commodity that would be purchased at various prices), the consumer's surplus (total consumer gain) given by $\int_{0}^{x_0} f(x)dx - x_0 y_0$ is represented by the area below the demand curve and above the line $y = y_0$ where y_0 is the market price corresponding to the market demand x_0 as shown in the figure below.



a) Find the consumer's surplus for $x_0 = 3$, if the demand function is $f(x) = y = 30 - 2x - x^2$

b) Plot the demand function $y = f(x) = 30 - 2x - x^2$

2.3.3 Techniques of integration

Activity 2.3.3

1. Consider the continuous function $f(x)=e^{x^2}$ on a closed interval $[a,b]$

   i. Let $t=x^2$, determine the value of t when $x=0$ and when $x=2$

   ii. Determine the value of dx in function of dt

   iii. Evaluate the integral $\int_a^b 2xe^{x^2}dx$ using expression of t, considering the results found in i) and ii).

   iv. Explain what happens to the boundaries of the integral when you apply the substitution method

2. Evaluate $\int_1^e x^2 \ln x dx$

Many times, some functions cannot be integrated directly.

In that case we have to adopt other techniques in finding the integrals. The fundamental theorem in calculus tells us that computing definite integral of $f(x)$ requires determining its anti-derivative, therefore the techniques used in determining indefinite integrals are also used in computing definite integrals.

a) Integration by substitution

The method in which we change the variable to some other variable is called "**Integration by substitution".** When definite integral is to be found by substitution then change the lower and upper limits of integration. If substitution is $\varphi(x)$ and lower limit of integration is a and upper limit is b then new lower and upper limits will be $\varphi(\text{a})$ and $\varphi(\text{b})$ respectively.

Example

1. $\int_0^2 x\sqrt{5-x^2}dx$

2. $\int_0^3 6xe^{x^2+1}dx$

Solution

1. $\int_0^2 x\sqrt{5-x^2}dx$ Let $5-x^2=t, then -2xdx=dt, or\ xdx=-\frac{1}{2}dt$

   When $x=0, t=5, when\ x=2, t=5-4=1$

$$\int_0^2 x\sqrt{5-x^2}\,dx=\int_1^5\sqrt{t}\,\frac{dt}{2}=\frac{1}{2}\int_1^5\sqrt{t}\,dt$$

$$=\frac{1}{2}\int_1^5 t^{1/2}dt=\frac{1}{2}\left[\frac{t^{\frac{1}{2}+1}}{\frac{1}{2}+1}\right]_1^5=\frac{1}{2}\left[\frac{t^{\frac{3}{2}}}{\frac{3}{2}}\right]_1^5=\frac{1}{2}\times\frac{2}{3}\left[5^{\frac{3}{2}}-1^{\frac{3}{2}}\right]=\frac{1}{3}\left(\sqrt{125}-1\right)$$

2. $\int_0^3 6xe^{x^2+1}dx$ Let $x^2+1=t$, then $2xdx=dt\ or\ xdx=\frac{1}{2}dt$

When $x=0, t=1\ and\ when\ x=3, t=10$

$$\int_0^3 6xe^{x^2+1}dx=\int_1^{10}6e^t\frac{dt}{2}=3\int_1^{10}e^t dt=3\left[e^t\right]_1^{10}=3(e^{10}-e^1)=3(e^{10}-e)$$

b) Integration by parts

To compute the definite integral of the form $\int_a^b f(x)g(x)dx$ using integration by parts, simply we suppose that $u=f(x)\ and\ dv=g(x)dx$. Then $du=f'(x)dx$ and $v=G(x)$, anti-derivative of $g(x)$ so that the integration by parts becomes:
$\int_a^b udv=[uv]_a^b-\int_a^b vdu$

Example

Evaluate the following definite integral: $\int_0^3 xe^x dx$

Solution

Let $\begin{cases} u=x & du=dx \\ dv=e^x dx & v=\int e^x dx=e^x+c \end{cases}$

Applying the integration parts formula $I=[uv]_a^b-\int_a^b vdu$ yields to

$$I=xe^x\Big|_0^3-\int_0^3 e^x dx=xe^x\Big|_0^3-e^x\Big|_0^3=\left[3e^3-0e^0\right]-\left[e^3-e^0\right]=3e^3-e^3+1=2e^3+1$$

Application activity 2.3.3

Evaluate the following definite integrals by using integration by parts

1. $\int_0^1 \ln(1+x)dx$

2. $\int_1^2 x^2 \ln x dx$

3. $\int_0^2 3x^2 e^{-x} dx$

2.4 Applications of integrals

2.4.1 Applications of definite integrals on marginal and total cost, consumer and producer surplus

Activity 2.4.1

At a certain factory, the marginal cost is $3(q-4)^2$ dollars per unit when the level of production is q units. By how much will the total manufacturing cost increase if the level of production is raised from 6 units to 10 units.

a) Determination of cost function in Economics

In economics, the marginal function is obtained by differentiating the total function. Now, when marginal function is given and initial values are given, the total function can be obtained using integration.

If C denotes the total cost and $M(x)=\frac{dC}{dx}$ is the marginal cost, we can write $C=C(x)=\int M(x)dx+K$, where the constant of integration K represents the fixed cost.

Example

The marginal cost function of manufacturing $x\ units$ of a product is $5-16x+3x^2$ (FRW). Find the total cost of producing 5 up to 20 items.

Solution

$$C=\int_5^{20}(5-16x+3x^2)dx=\left[5x-16\frac{x^2}{2}+3\frac{x^3}{3}\right]_5^{20}$$

$$=\left[5(20)-8(20^2)+20^3\right]-\left[5(5)-8(25)+5^3\right]$$

$$=(100-3200+8000)-(25-200+125)$$

$$=4900-(-50)$$

$$=4950$$

The required cost is 4950 FRW

b) Marginal cost and change in total cost

Suppose $C(q)$ represents the cost of producing q items. The derivative, $C'(q)$ is the marginal cost. Since marginal cost $C'(q)$ is the rate of change of the cost function with respect to quantity, by the fundamental theorem, the integral $\int_a^b C'(q)dq$ represent the total change in the cost function between $q=a$ and $q=b$. In other words, the integral gives the amount it costs to increase production from a units to b units.

The cost of producing 0 units is the fixed cost $C(0)$. The area under marginal cost curve between $q=0$ *and* $q=b$ is the total increase in cost between a producing of 0 and a production of b. This is called Total variable cost. Adding this to fixed cost gives the total cost to produce b units.

Cost to increase production from a units to b units is given

$$C(b)-C(a)=\int_a^b C'(q)dq$$

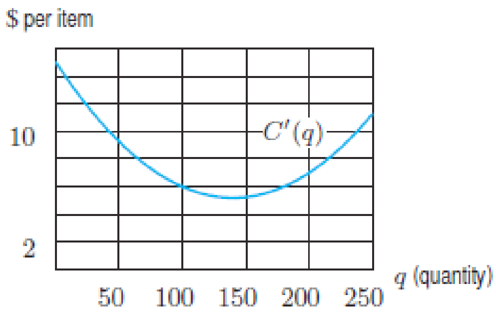
$$\textit{Total variable cost to produce b units}=\int_0^b C'(q)dq$$

$$\textit{Total cost of producing b units}=\textit{Fixed cost}+\textit{Total variable cost}$$

$$=\mathrm{C}(0)+\int_0^b C'(\mathrm{q})\,\mathrm{dq}$$

Example

The following is a marginal cost curve



If the fixed cost is $\$1000$, estimate the total cost of producing 250 items.

Solution

The total cost of production is fixed cost + Variable cost. The variable cost of producing 250 items is represented by the area under the marginal cost curve.

The area in the figure between $q = 0$ and $q = 250$ is about 20 grid squares.

Each grid square has area (2dollars/ item)(50items) =100 dollars, so

$$Total\ \text{variable cost} = \int_0^{250} C'(q)dq \approx 20(100) = 2000$$

The total cost to produce 250 items is given by:

$Total\ \text{cost} = \text{Fixed cost} + \text{Total variable cost}$

$\approx \$1000 + \$2000 = \$3000$

c) Definite integrals of marginal revenue functions

Consider the following phenomenon on MR and TR

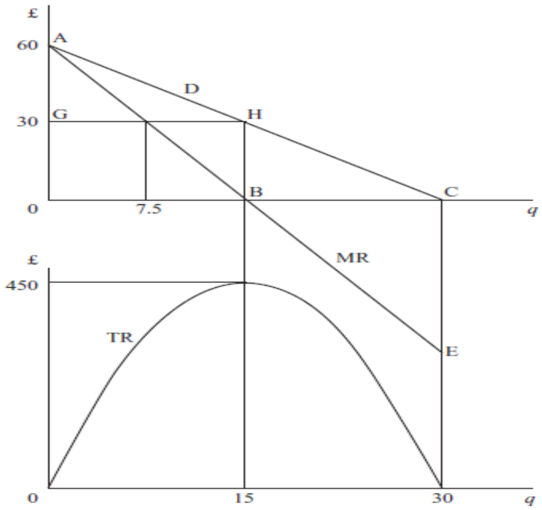


Figure 2.6.1

This phenomenon is illustrated in the above figure 2.6.1 shows the linear demand schedule $p = 60 - 2q$ and the linear marginal revenue schedule $MR = 60 - 4q$. The corresponding total Revenue schedule $TR = 60q - 2q^2$ is shown in the lower part of the diagram. Total revenue is at its maximum when $MR = 60 - 4q = 0, q = 15$

Therefore, $p = 60 - 2(15) = 30$ and so the maximum value of TR is $pq = 450$

Given the linear demand and marginal revenue schedules we can see that TR rises from 0 to 450 when q increases from 0 to 15, and then falls back again to zero when q increases from 15 to 30. These changes in TR correspond to the values of the definite integrals over these quantity ranges and are represented by the area between the MR schedule and the quantity axis. When q is 15, TR will be equal

to the area 0AB which is $\int_0^{15} MRdq = \int_0^{15} (60 - 4q)dq = \left[60q - 2q^2\right]_0^{15} = 900 - 450$

The change in TR when q increases from 15 to 30 will be the 'negative' area BCE which lies above the MR schedule and below the quantity axis. This will be equal to

$$\int_{15}^{30} MRdq = \int_{15}^{30} (60 - 4q)dq = \left[60q - 2q^2\right]_{15}^{30}$$

$$= (1800 - 1800) - (900 - 450) = -450$$

This checks with our initial assessment. Total revenue rises by 450 and then falls by the same amount. Finally, let us see what happens when we look at the definite integral of the MR function over the entire output range 0–30. This will be $\int_0^{30} MRdq = \int_0^{30} (60 - 4q)dq = \left[60q - 2q^2\right]_0^{30} = 1800 - 1800 = 0$

The negative area BCE has exactly cancelled out the positive area 0AB, giving zero TR when

q is 30, which is correct.

d) Consumer and Producer Surplus

Notice that at equilibrium, a number of consumers have bought the item at a lower price than they would have been willing to pay. (For example, there are some consumers who would have been willing to pay prices up to P_1.) Similarly, there are some suppliers who would have been willing to produce the item at a lower price (down to P_0, in fact). We define the following terms:

- The **consumer surplus** measures the consumers' gain from trade. It is the total amount gained by consumers by buying the item at the current price rather than at the price they would have been willing to pay.

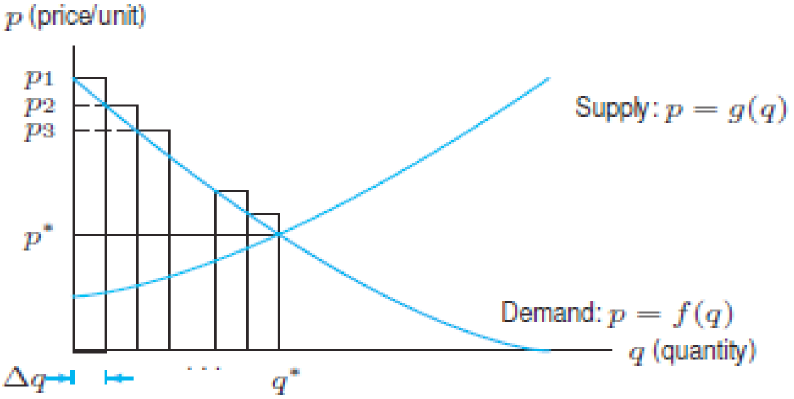
▪ The **producer surplus** measures the suppliers' gain from trade. It is the total amount gained by producers by selling at the current price, rather than at the price they would have been willing to accept.

In the absence of price controls, the current price is assumed to be the equilibrium price.

Both consumers and producers are richer for having traded. The consumer and producer surplus

Measure how much richer they are.

Suppose that all consumers buy the good at the maximum price they are willing to pay. Subdivide the interval from 0 to q^* into intervals of length Δq. The figure bellow shows that a quantity Δq of items are sold at a price of about P_1 another Δq are sold for a slightly lower price of about P_2, the next Δq for a price of about P_3, and so on.



Thus, the consumers' total expenditure is about $p_1\Delta q + p_2\Delta q + p_3\Delta q + ... = \sum p_i\Delta q$

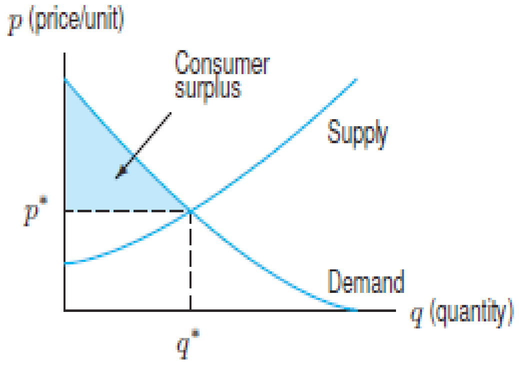
If the demand curve has equation $P = f(q)$, and if all consumers who were willing to pay more than P^* paid as much as they were willing, then as $\Delta q \to 0$, we would have

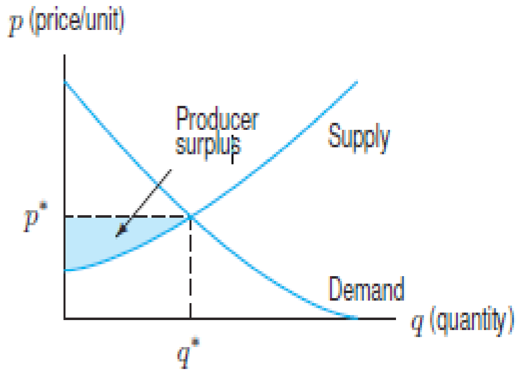
$$consumer\ expenditure = \int_0^{q^*} f(q)dq = area\ under\ demand\ curve\ from\ 0\ to\ q^*$$

Consumer surplus is the difference between the total consumer expenditure if all consumers pay the maximum they are willing to pay and the actual consumer expenditure if all consumers pay the current price.

a. Consumer surplus

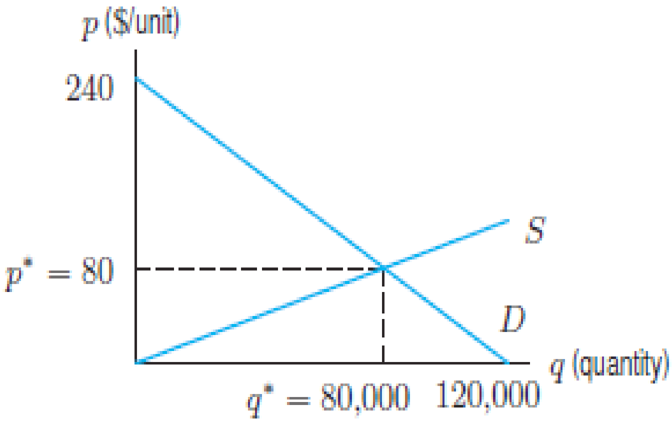
b. Producer surplus





Example

The following is the supply and demand curves for a product



(a) What are the equilibrium price and quantity?

(b) At the equilibrium price, calculate and interpret the consumer and producer surplus.

Solutions

a) The equilibrium price is $P^* = \$80$ and the equilibrium quantity is

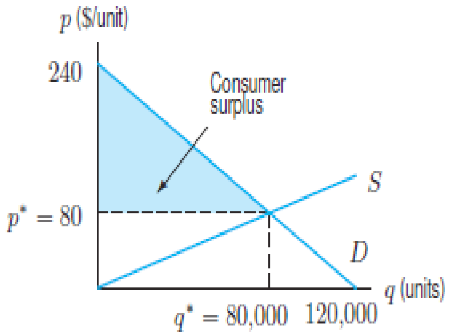
$q^* = 80,000 units$

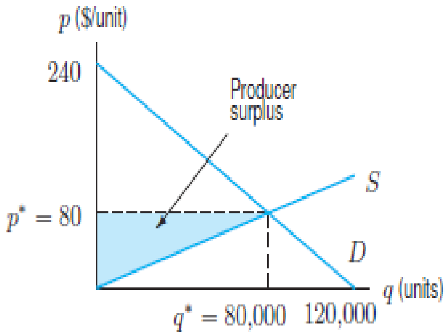
b) The consumer surplus is the area under the demand curve and the above line $P = 80$

We have consumer surplus equal to

$$Area\ of\ triangle = \frac{1}{2} Base \times Height = \frac{1}{2} 80,000 \times 160 = \$6,400,000$$

This tells us that consumer gain $6,400,000 in buying goods at the equilibrium price instead of at the price they would have been willing to pay.





The producer surplus is the area above the supply curve and below the line $p = 80$

Producer surplus equal to

$$Area\ of\ triangle = \frac{1}{2} Base \times Height = \frac{1}{2} 80,000 \times 80 = \$3,200,000$$

So, producers gain $3,200,000 by supplying goods at the equilibrium price instead of the price at which they would have been willing to provide the goods.

Application activity 2.4.1

1. For the non-linear demand function $p = 1800 - 0.6q^2$ and the corresponding marginal Revenue function $MR = 1800 - 1.8q^2$, use definite integrals to find:

   (i) TR When q is10

   (ii) the change in TR when q increases from 10 to 20

   (iii) Consumer surplus when q is 10.

2. The marginal profit for a product is model by $\frac{dP}{dx} = 40 - 3\sqrt{x}$ where P is the profit and x the sales. Find the change in profit when sales increase from 100 to 121 units.

2.4.2 Applications of definite integrals: Present, Future Values of an Income Stream

Activity 2.4.2

Suppose you want to have $\$50,000$ in 8 years' time in a bank account earning 2% interest, compounded continuously.

(a) If you make one lump sum deposit now, how much should you deposit?

(b) If you deposit money continuously throughout the 8 year period, at what rate should you deposit it?

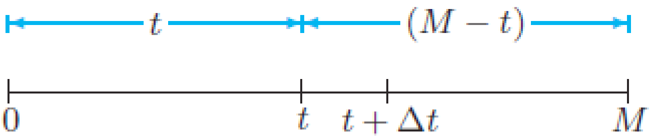
Present and Future Values of an Income Stream

Just as we can find the present and future values of a single payment, so we can find the present and future values of a stream of payments. As before, the future value represents the total amount of money that you would have if you deposited an income stream into a bank account as you receive it and let it earn interest until that future date. The present value represents the amount of money you would have to deposit today (in an interest-bearing bank account) in order to match what you would get from the income stream by that future date.

When we are working with a continuous income stream, we will assume that interest is compounded continuously. If the interest rate is r, the present value P of a deposit B made t years in the future is $P = Be^{-rt}$

Suppose that we want to calculate the present value of the income stream described by a rate of

$S(t)$ Dollars per year, and that we are interested in the period from now until M years in the future. In order to use what we know about single deposits to calculate the present value of an income stream, we divide the stream into many small deposits, and imagine each deposited at one instant. Dividing the interval $0 \le t \le M$ into subintervals of length Δt



Assuming Δt is small, the rate, $S(t)$ at which deposits are being made does not vary much within one subinterval. Thus, between t and $t+\Delta t$:

$$Amount\ paid \approx Rate\ of\ deposits \times Time$$

$$= S(t)\Delta t\ dollars$$

The deposit of $S(t)\Delta t$ is made t years in the future. Thus, assuming a continuous interest rate

$present\ value \approx S(t)\Delta t e^{-rt}$. Deposit in interval t to $t+\Delta t$

$$Total\ present\ value \approx \sum S(t)e^{-rt}\Delta t$$

In the limit as $\Delta t \to 0$, we get the following integral:

$$\Pr esent\ value = \int_0^M S(t)e^{-rt}dt$$

$$Future\ value = \Pr esent\ value \times e^{rM}$$

Example

Find the present and future values of a constant income stream of \$1000 per year over a period of

20 years, assuming an interest rate of 6% compounded continuously.

Solution

$S(t)=1000\ and\ r=0.06$, we have $Present\ value=\int_0^{20}1000e^{-0.06t}dt=\$11,647$

We can get the future value B from the present value P, using $B=Pe^{rt}$. So,

$Future\ value=11,647e^{0.06(20)}=\$38,669$

Application activity 2.4.2

1. Find the present and future values of an income stream of $1,000 a year for 20 years. The interest rate is 6%, compounded continuously.

2.4.3 Application of integrals on the population growth rates

Activity 2.4.3

The number of individuals (population) P present at a given time t is a function of time. The rate of change (in time) of this population $\frac{dP}{dt}$ is proportional to the

population P present with a constant k of proportionality which is expressed as

$\frac{dP}{P}=kdt$.

Given that the population of a town was 11,500,00 at initial time $t=0$,

a) Find the population of this town after 5 years if the constant $k=5\%$.

b) Plot the related graph showing the variation of the population of that town in function of time and give your interpretation in your own words.

c) What are pieces of advice would you provide to policy makers of that town?

Content summary

The growth of a population is usually modeled by an equation of the form $\frac{dP}{P}=Kdt$ where P represents the number of individuals on a given time t. The constant K is a positive constant when the population grows and negative when the population decreases.

Integration of each side gives: $\int \frac{dP}{P} = \int K dt$

Which implies that $\ln P = Kt + c$ and $P = ce^{Kt}$

If the initial population at time $t = 0$ is P_0, then $P_0 = ce^0 = c$.

Therefore, we have $P = P_0 e^{Kt}$

This means that the variation of a population from P_0 is modelled by an exponential function: $P(t) = P_0 e^{Kt}$ where P_0 is the initial population at time $t = 0$, K the annual growth rate or the annual decay rate.

Example

1) Consider the population P of a region where there is no immigration or emigration. The rate at which the population is growing is often proportional to the size of the population. This means larger populations grow faster, as we expect since there are more people to have babies.

If the population has a continuous growth rate of 2% per unit time, what is its population at any time t?

Solution

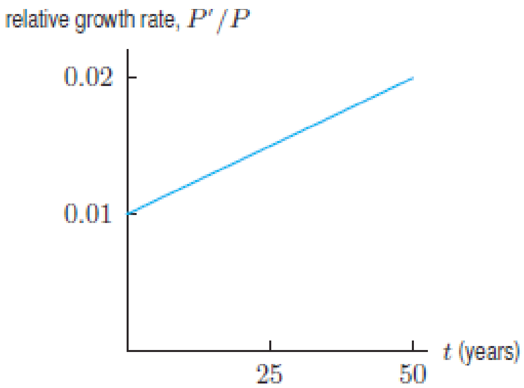
We know that $\frac{dP}{dt} = KP$

$\int \frac{dP}{P} = k \int dt \Rightarrow \ln P = Kt + c \Rightarrow P = ce^{Kt}$ **for** $K = 0.02$ **and has the general solution** $P = ce^{0.02t}$

If the initial population at time $t = 0$ is P_0 , then $P_0 = ce^{(0.02)(0)} = c$

So, $P_0 = c$ and we have $P = P_0 e^{0.02t}$

2) The relative rate of growth $P'(t)/P(t)$ of a population $P(t)$ over a 50 years period is given by

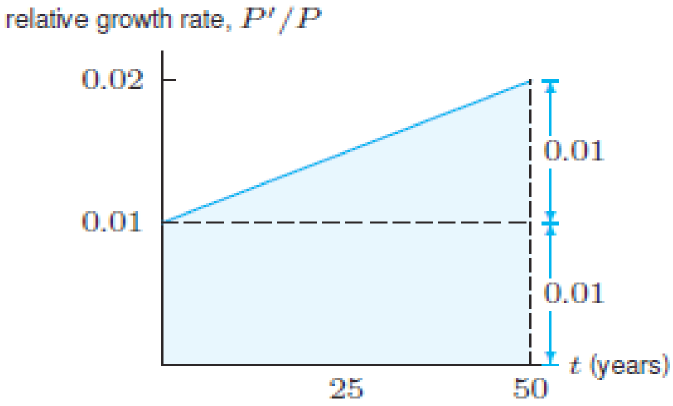


By what factor did the population increase during the period?

Solution

We have $\ln\left(\frac{P(50)}{P(0)}\right) = \int_0^{50} \frac{P'(t)}{P(t)} dt$

This integral equals the area under the graph of $P'(t)/P(t)$ between $t=0$ and $t=50$



The area of rectangle and triangle gives $Area = 50(0.01) + \frac{1}{2} \times 50(0.01) = 0.75$

Thus, $\ln\left(\frac{P(50)}{P(0)}\right) = \int_0^{50} \frac{P'(t)}{P(t)} dt$

$$\ln\left(\frac{P(50)}{P(0)}\right) = \int_0^{50} \frac{P'(t)}{P(t)} dt = \ln\left(\frac{P(50)}{P(0)}\right) = 0.75$$

$$\ln\left(\frac{P(50)}{P(0)}\right) = 0.75$$

$$\frac{P(50)}{P(0)} = e^{0.75} = 2.1$$

$$P(50) = 2.1P(0)$$

The population more than doubled during the 50 years, increasing by a factor of about 2.1.

Note: If everything else remains constant, the relative growth rate increases if either the relative birth rate increases or the relative death rate decreases. Even if the relative birth rate decreases, we can still see an increase in the relative growth rate if the relative death rate decreases faster. This is the case with the population of the world today.

The difference between the birth rate and the death rate is an important variable.

Application activity 2.4.3

1) The population of a given city is double in 20 years. We suppose that the rate of increasing is proportional to the number of population. In how many time the population will be three times, if on $t=0$ we have the population P_0.

2) In laboratory, it is observed that the population of bacteria is increasing from 1000 to 3000 during 10 hours. If the rate of increasing of bacteria is supposed to be proportional to the present number of bacteria on the time t, find the number of bacteria after 5 hours.

2.5 End unit assessment

1. Calculate the following integrals

a. $\int (9x^7 + \frac{1}{x-1} - \frac{1}{2}e^x)dx$ b. $\int \frac{x}{\sqrt{x+3}}dx$

2. Discuss and solve the following problems

a) The marginal cost function of producing x units of soft drink is given by the function

$$MC = \frac{x}{\sqrt{x^2+1600}}$$

Given that the fixed cost is $500 Frw$

Determine

i. The total cost function

ii. An average cost function

b) Consider the function f defined by $f(x) = 4 - \sqrt{x}$

– Plot the graph of $y = f(x)$ showing the intercepts with the coordinate axes.

- On the diagram, shade the area which is bounded by the curve and the coordinate axes.
- Express the shaded area in terms of a definite integral

3. Discuss how this unit inspired you in relation of learning other subjects or to your future. If no inspiration at all, explain why.

UNIT 3

ORDINARY DIFFERENTIAL EQUATIONS

Key Unit competence:

Use ordinary differential equations of first order to model and solve related problems in Economics

3.0 Introductory activity

A quantity $y(t)$ is said to have an exponential growth model if it increases at a rate that is proportional to the amount of the quantity present, and it is said to have an exponential decay model if it decreases at a rate that is proportional to the amount of the quantity present.

Thus, for an exponential growth model, the quantity $y(t)$ satisfies an equation of the form $\frac{dy}{dt}=ky$ (k is a non-negative constant called annual growth rate).

Given that $\frac{dy}{dt}=ky$ can be written as $\frac{dy}{y}=kdt$, solve this equation and apply the answer $y(t)$ obtained in the following problem:

The size of the resident Rwandan population in 2018 is estimated to 12,089,721 with a growth rate of about 2.37% comparatively to year 2017 (www.statistics.gov.rw/publication/demographic-dividend)

Assuming an exponential growth model and constant growth rate,

1. Estimate the national population at the beginning of the year 2020, 2030, 2040 and 2050
2. Discuss your observations on the behaviour of the national population along these 4 years
3. What are pieces of advice would provide to policy makers?
4. Draw a graph representing your observations mentioned in 2

3.1 Definition and classification of ordinary differential equation (ODE)

Activity 3.1

For each of the following equations, form a differential equation by using derivatives in eliminating arbitrary constants k and b.

Discuss and write down the highest order of the derivative that occurs in the obtained equation:

1. $y = 4kx$
2. $y = kx + bx$

A differential equation is any equation which contains derivatives of the unknown function; it shows the relationship between an **independent variable** *x*, a **dependent variable** y (unknown) and one or more differential coefficients of *y* with respect to *x*.

An **ordinary differential equation** (ODE) for a **dependent variable** *y* (unknown) in terms of an **independent variable** *x* is any equation which involves first or higher order derivatives of *y* with respect to *x*, and possibly *x* and *y*.

The general ordinary differential equation of the n^{th} order is

$$F(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, ..., \frac{d^n}{dx^n}) = 0$$

Or $F(x, y, y', y'', ..., y^n) = 0$

The general differential equation of the 1st order is $F(x, y, \frac{dy}{dx}) = 0 \; or \; \frac{dy}{dx} = f(x, y)$

Order of a differential equation: Differential equations are classified according to the highest derivative which occurs in them.

The **order** of a differential equation is the highest derivative present in the differential equation.

The **degree** of an ordinary differential equation is the algebraic degree of its highest ordered derivative after simplification.

Examples

1) $(y')^2+2x+y'''=0$ *or* $2\left(\frac{dy}{dx}\right)^2+2x+\frac{d^3y}{dx^3}=0$, the order is 3 and the degree is 1

2) $y''+2y'+x^2=0$, the order is 2 and the degree is 1

3) $y''+2x+y^2=0$, the order is 2 and the degree is 1

4) $\frac{dy}{dx}=x^2$, the order is 1 and degree is 1

5) $\left(\frac{dy}{dx}\right)^2+y=x$, Order 1 and degree 2

Application activity 3.1

Discuss and state the order and the degree of each of the following differential equations. Explain your answer

a. $\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^4-4x+y=1$

b. $(y'')^3+(y')-2y=x$

c. $:^2\left(\frac{d^2y}{dx^2}\right)^4+y\left(\frac{dy}{dx}\right)+y^4=($

3.2 Differential equations with separable variables

Activity 3.2

1. Consider $4y'-2x=0$,
   a. Solve the equation for y' and integrate both sides to deduce the value of the dependent variable y .
   b. What can you say if you were given $4\frac{dy}{dx}-2x=0$?
   c. check whether y is solution of the given equation.
2. Apply the technique so-called separation of variables used in (1) to solve the following:

   a. $x\frac{dy}{dx}=1$ b. $x\frac{dy}{dx}=1$

3. Discuss how to solve $f(y)\frac{dy}{dx} = g(x)$

A **separable differential equation** is an equation of the form $\frac{dy}{dx} = f(x)h(y)$

These are called separable variables because the expression for $\frac{dy}{dx}$ or y' can be separated into a product of separate functions of x and y alone. This means that they can be rewritten so that all terms involving y are on one side of the equation and all terms involving x are on the other.

That is: $\frac{dy}{h(y)} = f(x)dx$

Hence, solving the equation requires simply integrating both sides with respect to their respective variables;

$$\int \frac{dy}{h(y)} = \int f(x)dx + c$$

Of course the left-hand side is now an integral with respect to *y*, the right-hand side with respect to *x*. Note that we only need one arbitrary constant.

In particular if $h(y) = m$ (a constant), the differential equation of the form $\frac{dy}{dx} = mf(x)$ is solved by direct integration. That is:

$$dy = mf(x)dx \Leftrightarrow \quad y = m\int f(x)dx + c\,.$$

Similarly, equation of the form $\frac{dy}{dx} = mf(y)$ is solved by direct integration:

$$\frac{dy}{f(y)} = mdx \quad \Leftrightarrow \int \frac{dy}{f(y)} = mx + c$$

A **solution** to a differential equation on an interval $\alpha < x < \beta$ is any function which satisfies the differential equation in question on that interval. It is important to note that solutions are often accompanied by intervals and these intervals can impart some important information about the solution.

Example

Show that $y=\frac{x}{2}\left(\frac{x^2}{6}+1\right)$ is a solution of $\frac{dy}{dx}=\frac{x^2+2}{4}$ on $]-\infty,+\infty[$.

Solution

Given that $y=\frac{x}{2}\left(\frac{x^2}{6}+1\right)=\frac{x^3+6x}{12}$, we have $\frac{dy}{dx}=\frac{3x^2+6}{12}=\frac{x^2+2}{4}$. Therefore,

$$y=\frac{x}{2}\left(\frac{x^2}{6}+1\right) \text{ is solution of } \frac{dy}{dx}=\frac{x^2+2}{4} \text{ on }]-\infty,+\infty[.$$

It is easily checked that for any constant c, $y=\frac{x}{2}\left(\frac{x^2}{6}+1\right)+c$ is also a solution to the equation called general solution of the given equation.

A **general solution** to a given differential equation is the most general form that the solution can take and doesn't take any initial conditions into account. In this way, there are an infinite number of solutions to a differential equation depending on the value of the constant; it is better (especially in applied problems) to precise conditions which lead to a particular solution.

Initial Conditions are conditions or set of conditions imposed to the general solution that will allow us to determine one **particular solution** also called **actual solution** that we are looking for.

In other words, initial conditions are values of the solution and/or its derivative(s) at specific points which help to determine values of arbitrary constants that appear in the general solution. Since the number of arbitrary constants in general solution to a given differential equation is equal to the order of the ODE, it follows that it requires *n* conditions to determine values for all *n* arbitrary constants in the general solution of an n^{th}-order differential equation (one condition for each constant). For a first order equation, the single arbitrary constant can be determined by specifying the value of the unknown function *y(x)* at an arbitrary *x*-value x_0, say $y(x_0)=y_0$.

Geometrically, initial condition of a first order differential equation ($\frac{dy}{dx}=f(x,y)$) enables us to identify a specific function $(y=y(x))$ whose curve passes through the point (x_0,y_0) and the slope is $f(x_0,y_0)$.

A differential equation along which an appropriate number of initial conditions are given is called **Initial Value Problem** (or **IVP**). Therefore the **actual solution or particular solution** to a differential equation is the specific solution that not only satisfies the differential equation, but also satisfies the given initial condition(s).

Examples

1. Solve

(a) $\frac{dy}{dx} = xy$

(b) $\frac{dy}{dx} = \frac{x^2+1}{4}$

2. Solve the IVP: $\frac{dy}{dx} = \frac{y}{x-3}$ with $y(0) = -3$

Solution

1) (a) $\frac{dy}{dx} = xy$ we separate to give $\frac{dy}{y} = xdx$

so, $\int \frac{dy}{y} = \int xdx + c$ and integrating both sides with respect to their respective variables gives

$$\ln y = \frac{x^2}{2} + c$$

$y = e^{\ln y} = e^{\frac{x^2}{2}+c}$ Then, $y = e^{\frac{x^2}{2}+c} \Rightarrow y = e^c \times e^{\frac{x^2}{2}}$

(b) $\frac{dy}{dx} = \frac{x^2+1}{4} \Leftrightarrow 4dy = \left(x^2+1\right)dx$

$$\int 4dy = \int (x^2+1)dx \Rightarrow 4y = \frac{x^3}{3} + x + c \Rightarrow y = \frac{x^3}{12} + \frac{x}{4} + k \quad \text{(where we set } k = \frac{c}{4}).$$

2. IVP: $\frac{dy}{dx} = \frac{y}{x-3}; y(0) = -3$

$$\frac{dy}{dx} = \frac{y}{x-3} \Rightarrow \frac{dy}{y} = \frac{dx}{x-3} \Rightarrow \int \frac{dy}{y} = \int \frac{dx}{x-3}$$

Simple integrations yields to:

$\ln y = \ln(x-3)+c$. For simplicity and aesthetic purpose, set $c=\ln k$. Thus $\ln y = \ln(x-3)+\ln k$ or equivalently

$y = k(x-3)$ (General solution).

Let's apply initial condition to the general solution.

If $y(0)=-3$ it follows that $-3=k(0-3)$ giving $k=1$. Therefore, $y=x-3$ is the required particular solution that represents equation of the unique line passing through (0,-3) and whose slope is $\frac{y_0}{x_0-3}=\frac{-3}{0-3}=1$.

Application activity 3.2

1. Determine the general solution for $x\frac{dy}{dx}=2-4x^3$

2. Solve the following initial value problem: $(x+1)\frac{dy}{dx}=x$, $y(0)=0$

3. (a) The graph of a differentiable function *y* = *y(x)* passes through the point (0, 1) and at every point *P (x, y)* on the graph the tangent line is perpendicular to the line through P and the origin. Find an initial-value problem whose solution is *y(x).*

   (b) Explain why the differential equation in part (a) is separable. Solve the initial-value problem using either separation of variables and describe the curve

4. Determine the particular solution of $(y^2-1)\frac{dy}{dt}=3y$ given that $y=1$ when $t=2\frac{1}{6}$

5. Determine the particular solution of $xy=(1+x^2)\frac{dy}{dx}$ given that $y=1$ when $x=0$.

3.3 Linear differential equations

Activity 3.3

Consider the equation $\frac{dy}{dx}+2xy=x$ (1)

Assume that there exists a function $I(x)$ called an **integrating factor** that must help us to solve the equation (1).

1) Compute $I(x)=e^{\int 2xdx}$. For the time being, set the integration constant to 0.

2) Multiply both sides in the differential equation (1) by $I(x)$ and verify that the left side becomes the product rule $(I(x).y(x))'$ and write it as such.

3) Integrate both sides, make sure you properly deal with the constant of integration

4) Solve for the function $y(x)$

5) Verify if the value of $y(x)$ obtained in 4) is solution of (1).

An ordinary differential equation (ODE) in which the only power to which *y* or any of its derivatives occurs is zero or one is called a **linear ODE**. Any other ODE is said to be **nonlinear**.

Thus, if p and q are functions in x or constants the general linear equation of first order can take the form $\frac{dy}{dx}+py=q$ (2)

There exist a "magical" function $I(x)$ called **integrating factor** that helps to solve the equation (2).

The solution process for a first order linear differential equation is as follows:

a) Determine an **integrating factor** $I(x)=e^{\int pdx}$ taking the integrating constant $c=0$.

b) Multiply both sides in the differential equation (2) by $I(x)$ and verify that the left side becomes the product rule $(I(x).y(x))'$ and write it as such.

c) Integrate both sides, make sure you properly deal with the constant of integration

d) Solve for the solution $y(x)=\dfrac{\int I(x)q(x)+C}{I(x)}$

This process can be simplified by letting $y=uv$ where u and v are functions in x to be determined in the following ways:

$v=e^{-\int pdx}$ by taking the constant $c=0$ and $u=\int q e^{\int pdx}\,dx$

Therefore, the solution of the equation $\dfrac{dy}{dx}+py=q$ becomes $y=uv$ where $u=\int q e^{\int pdx}\,dx$ and $v=e^{-\int pdx}$.

Example

1. State the order and degree of each ODE, and state which are linear or non linear:

   i) $\dfrac{dy}{dx}+y=x$

   ii) $x''+3t^2=0$

   iii) $R\dfrac{dq}{dt}+\dfrac{q}{C}=3$

2. Use integrating factor to solve $y'-x^2y=x^2$.

Solution

1. i) $\dfrac{dy}{dx}+y=x$ is a linear differential equation in y of first order and its degree is 1

ii) $x''+3t^2=0$ is linear differential equation of second order in x and the degree is 1.

iii) $R\dfrac{dq}{dt}+\dfrac{q}{C}=3$ is linear differential equation of first order in q and its degree is 1.

iv) $\dfrac{d^3y}{dx^3}+\dfrac{d^2y}{dx^2}-2\dfrac{dy}{dx}x\sin 2x=y+4x^4$ is linear differential equation of third order in y and its degree is 1.

2) $y'-x^2y=x^2$

The equation becomes $\dfrac{dy}{dx}-x^2y=-1$, $p=-x^2$, $q=x^2$

Therefore, $I(t)=e^{\int -x^2 dx}=e^{-\frac{x^3}{3}}$

$$y(x)=\frac{\int I(x)q(x)dx+C}{I(x)}=\frac{\int e^{-\frac{x^3}{3}}(x^2)dx+C}{e^{-\frac{x^3}{3}}}$$

$$\frac{\int e^{-\frac{x^3}{3}}.(x^2)dx+c}{e^{-\frac{x^3}{3}}}=\frac{-e^{-\frac{x^3}{3}}+c}{e^{-\frac{x^3}{3}}}=-1+ce^{\frac{x^3}{3}}$$

$y(x)=-1+ce^{\frac{x^3}{3}}$ Where c is a constant.

Application activity 3.3

Determine the general solution of the following equations

a) $y'+\frac{y}{x}=1$

b) $y'+xy=x$

c) $y'+\frac{y}{x}=x$

d) $y'+2y=e^x$

e) $y'-2x\,y=e^{x^2}$

3.4 Application of ordinary differential equation (ODE)

3.4.1 Differential equation solutions to predict values in basic market and macroeconomic models

Activity 3.4.1

If the differential equation to be solved is $\frac{dy}{dt}=5y$ then one possible solution is $y=e^{5t}$ as gives $\frac{dy}{dt}=5e^{5t}=5y$. Use the definite solution to predict what will be y_3 for $t=3$

The exponential function helps us to derive the solution to a differential equation.

If $y = e^t \ then \ \frac{dy}{dt} = e^t$ thus, using the chain rule for differentiation, for any constant b,

If $y = e^t \ then \ \frac{dy}{dt} = be^{bt}$. Therefore, if the differential equation to be solved has no constant term and has the format $\frac{dy}{dt} = by$ then a possible solution is $y = e^{bt}$ because this would give $\frac{dy}{dt} = be^{bt} = by$

For example, if the differential equation to be solved is $\frac{dy}{dt} = 5y$ then one possible solution is $y = e^{5t}$ as gives $\frac{dy}{dt} = 5e^{5t} = 5y$

However, there are other possible solutions. For example

If $y = 3e^{5t} \ then \ \frac{dy}{dt} = 5(3e^{5t}) = 5y$

If $y = 7e^{5t} \ then \ \frac{dy}{dt} = 5(7e^{5t}) = 5y$

In fact, we can multiply the original solution of e^{5t} by any constant parameter and still get the same solution after differentiation. Therefore, for any differential equation in the format $\frac{dy}{dt} = by$ the general solution specified as $y = Ae^{bt}$ where A is an arbitrary constant. This must be so since $\frac{dy}{dt} = bAe^{bt} = by$

The actual value of A can be found if the value for y is known for a specific value of t. This will enable us to find the **definite solution**. This is easiest to evaluate when the value of y is known for $t = 0$ as any number taken to the power zero is the number itself. For example $\frac{dy}{dt} = 5y$ will be $y_t = Ae^{5t}$ where y has been given the subscript t to denote the time period that it corresponds to. If it is known that when $t=0$ then $y_0=12$ then by substituting these values into (1) we get $y_0=12=Ae^0$ as we know that $e^0=1$ then $12=A$, substituting this value into **general solution** we get **definite solution** $y_t=12e^{5t}$. ***This definite solution can now be used to predict*** y_t ***for any value*** t***.*** For example when $t=3$ then, $y_3=12e^{5(3)}=12e^{15}=12(3,269,017.4)=39,228,208$

Differential equation solutions and growth rates

The solutions to these differential equations give final values after continuous growth for a given time period. This is because what we have done this time is to derive the relationship between y and t, starting from the knowledge that $\frac{dy}{dt}=ry$ it means that the rate of increase of y (over time) depends on the growth rate r and the specific value of y. This can be a difficult point to grasp, because there are actually two rates involved and it is easy to confuse them.

(i) $\frac{dy}{dt}$ Is the rate of increase of y with respect to time t (but over a specified time period it will be a quantity of y rather than a ratio)

(ii) r It is the rate of increase of y with respect to its own current value

When y increases in magnitude over time, larger and larger increases in the value of y each time period will be necessary to maintain the same proportional rate of growth r. In other words, the value of $\frac{dy}{dt}$ must get bigger as t increases.

Application activity 3.4.1

Solve the differential equation $\frac{dy}{dt}=1.5y$ if the value of y is 34 when $t=0$ and then use the solution to predict the value of y when $t=7$.

3.4.2 Stability of economic models where growth is continuous.

Activity 3.4.2

Assume that in a perfectly competitive market the speed with which price *P* adjusts towards its equilibrium value depends on how much excess demand there is. Given that the rate of change of the price $P(t)$ of a product at time t is proportional to the difference of the demand and the supply for the commodity $(Q_d - Q_s)$

1) Write a differential equation modelling the rate of change of the price if the constant of proportionality $k = 0.08$ is in proportion to excess demand

2) Assuming that $Q_d = 280 - 4P(t)$ and $Q_s = -35 + 8P(t)$ solve the equation obtained in (1)

3) Determine and plot $P(t)$ at the time t if the price is currently 19.

4) Compare the price at $t = 1$ and the price as t gets larger i.e. $\lim_{t \to \infty} P(t)$

In a perfectly competitive market the speed with which price *P* adjusts towards its equilibrium value depends on how much excess demand there is. This is quite a reasonable proposition. If consumers wish to purchase a lot more produce than suppliers are willing to sell at the current price, then there will be great pressure for price to rise, but if there is only a slight shortfall then price adjustment may be sluggish. If excess demand is negative this means that quantity supplied exceeds quantity demanded, in which case price would tend to fall.

To derive the differential equation that describes this process, assume that the demand and supply functions are Q_d and Q_s

$Q_d = a + bP \text{ and } Q_s = c + dP$

With the parameters $a, d > 0 \text{ and } b, c < 0$

If r represents the rate of adjustment of P in proportion to excess demand then we can write $\frac{dP}{dt} = r(Q_d - Q_s)$

Substituting the demand and supply functions for Q_d *and* Q_s gives

$$\frac{dP}{dt} = r\left[(a+bP)-(c+dP)\right]$$

$$= r(a-c+bP-dP)$$

$$= r(\mathrm{b}-\mathrm{d})\mathrm{P}+r(-\mathrm{c})$$

As r, a, b, c and d d are all constant parameters this is effectively a first-order linear differential equation with one term in *P* plus a constant term. This format is similar to the ones in the previous examples, except that it is *P* that changes over time rather than *y*, and so the same method of solution can be employed.

Example

A perfectly competitive market has the demand and supply functions

$Q_d = 170-8P$ *and* $Q_s = -10+4P$ When the market is out of equilibrium the rate of adjustment of price is a function of excess demand such that $\frac{dP}{dt} = 0.5(Q_d - Q_s)$.

In the initial time period price P_0 is 10, which is not its equilibrium value. Derive a function for P in terms of t, and comment on the stability of this market.

Solution:

Substituting the functions for Q_d and Q_s into the rate of price change functions gives

$\frac{dP}{dt} = 0.5\left[(170-8P)-(-10+4P)\right] = 0.5(-8-4)P+0.5(170+10)$ Which simplifies to

$$\frac{dP}{dt} = -6P+90$$

To solve this linear first order differential equation we first consider the reduced equation without the constant term. $\frac{dP}{dt} = -6P$

The complementary function that is the solution this equation will be $P_t = Ae^{-6t}$

The particular solution is found by assuming P is equal to constant K so that

$\frac{dP}{dt} = -6K+90 = 0, K = 15$. ***This is the market equilibrium price***

Then, $P_t = Ae^{-6t} + 15$

The value of A can be determined by putting the initial value of 10 for P_0 into the general solution. Thus,

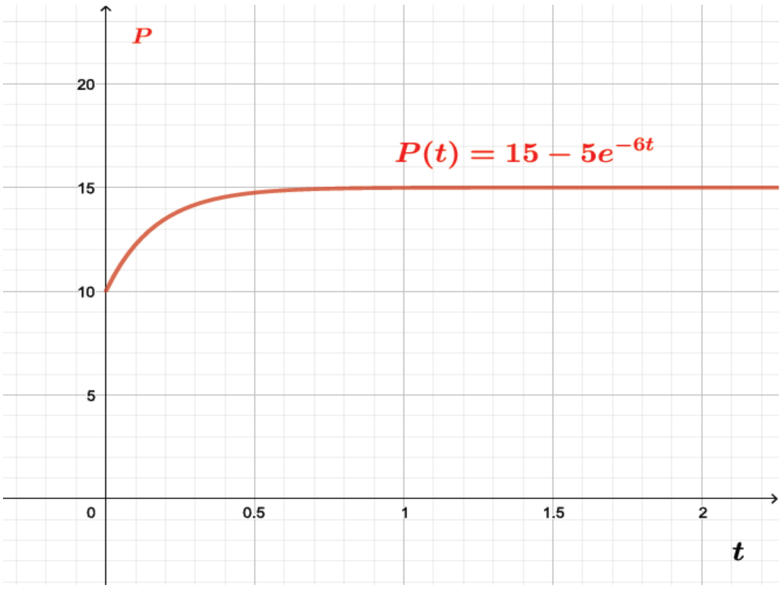
$$P_0 = 10 = Ae^0 + 15 \Leftrightarrow -5 = A$$

Using this value in general solution gives the definite solution to this differential equation, which is $P_t = -5e^{-6t} + 15$

Considering the initial condition, $P(0) = 10$ we find $A = -5$

Therefore, $P(t) = -5e^{-6t} + 15$

Figure: Graph of $P(t) = -5e^{-6t} + 15$



The coefficient of t in this exponential function is negative number. This means that the complementary function, will closer to zero as t gets larger and so P_t will converge on its equilibrium value of 15. T***herefore, this market is stable.***

Application activity 3.4.2

If the demand and supply functions in a competitive market are

$Q_d = 50 - 0.2P$ and $Q_s = -10 + 0.3P$ And the rate of adjustment of price when

the market is out of equilibrium is $\frac{dP}{dt} = 0.4(Q_d - Q_s)$. Derive and solve the relevant difference equation to get a function for P in terms of t, given that price is 100 in time period $t = 0$. Comment on the stability of this market

3.4.3 Continuously compounded interest

Activity 3.4.3

A bank account earns interest continuously at a rate of 6% of the current balance per year. Assume that the initial deposit is $4000 and that no other deposits or withdrawals are made.

(a) Write a differential equation satisfied by the balance in the account.

(b) Solve the differential equation obtained in (a)

Let s be the initial sum of money and r represents an interest rate. We can model the growth of an initial deposit with respect to the interest rate r with differential equations. If t represents time, then the rate of change of the initial deposit is $\frac{dS}{dt}$ and assuming that the initial deposit is compounded continuously, then we have that:

$\frac{dS}{dt} = rs$, We can further set up an initial value problem to this differential equation. Suppose that the initial deposit is S_0 then $S(0) = S_0$. The solution to this initial value with the differential equation and initial condition will give us a function S which give us amount in the individuals account at time t:

$$\frac{dS}{dt} = rS, \frac{1}{S}dS = rdt$$

$$\int \frac{1}{S}dS = \int rdt$$

$$\ln S = rt + k$$

$$S = ce^{rt}$$

Using the initial condition that $S(0)=S_0$ and we have that $c=S_0$. Therefore the solution to this initial value is $S(t)=S_0e^{rt}$ where S_0 is the principal invested, r is rate and $S(t)$ tells us the amount at any time t.

Example

A bank account earns interest continuously at a rate of 5% of the current balance per year. Assume that the initial deposit is \$1000 and that no other deposits or withdrawals are made.

(a) Write a differential equation satisfied by the balance in the account.

(b) Solve the differential equation and graph the solution.

Solution

(a) We are looking for S, the balance in the account in dollars, as a function of t, time in years.

Interest is being added continuously to the account at a rate of 5% of the balance at that moment,

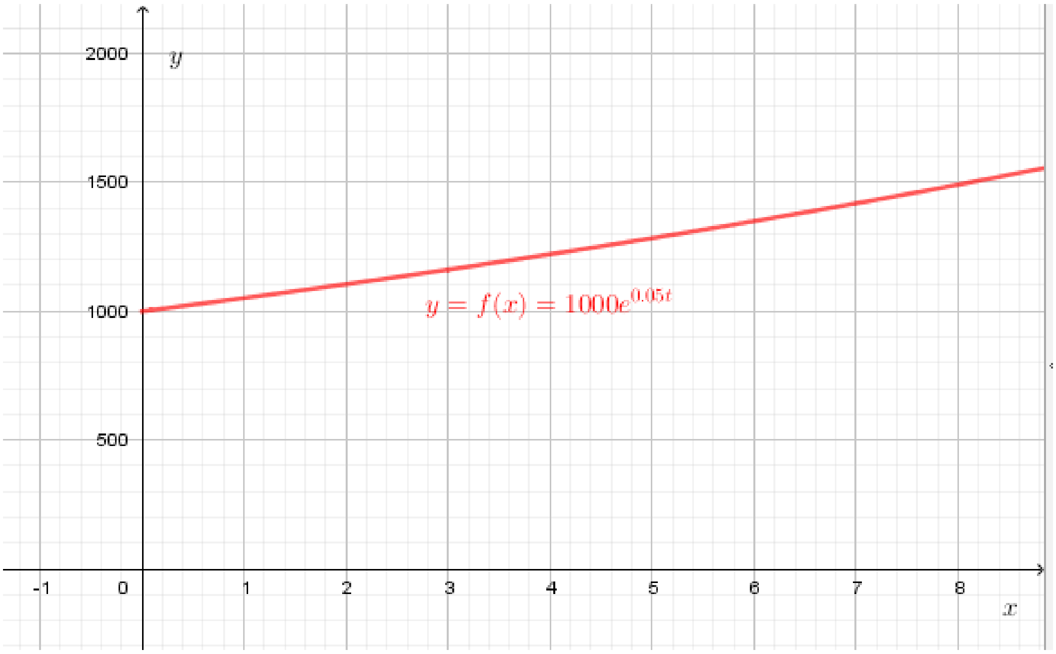
So, Rate at which balance is increasing 5%=0.05 of Current balance.

Thus, a differential equation that describes the process is $\frac{dS}{dt}=0.05S$

It does not involve the \$1000, the initial condition, because the initial deposit does not affect the process by which interest is earned.

(b) Since $S_0=1000$ is the initial value of S, the solution to this differential equation is $S=S_0e^{0.05t}=1000e^{0.05t}$

Graph of $1000e^{0.05t}$



Application activity 3.4.3

1. A principal of $\$1000$ is invested at a constant annual rate of 8%. Interest earned is compounded continuously. Find the accrued amount after 25 years.

2. (a) From the equation $\frac{dP}{dt} = rP$; Deduce the general formula for the principal $P(t)$ when interest is compounded continuously at an annual interest rate r assuming the initial principal is P_0.

   (b) Use the obtained formula in (a) to calculate the accrued amount after investment for 20 years where the interest is 4% compounded continuously and the principal is $500.

3.4.4 Differential equations and the quantity of a drug in the body

Activity 3.4.4

To combat the infection to a human body, appropriate dose of medicine is essential. Because the amount of the drug in the human body decreases with time, the medicine must be given in multiple doses. The rate $\frac{dQ}{dt}$ at which the level of the drug in a patient's blood decays is proportional to the quantity Q of the drug left in the body. If initially, that is, at $t=0$ a patient is given an initial dose Q_0,

1) Establish an equation for modelling the situation
2) Solve the obtained equation and find the quantity of drug $Q(t)$ left in the body at the time t
3) Draw $Q(t)$ and interpret the graph given that the drug provided was 100mg at $t=0$
4) Discuss what happens when the patient does not respect the dose of medicine as prescribed by the Doctor

The rate at which a drug leaves a patient's body is proportional to the quantity of the drug left in the body. If we let Q represent the quantity of drug left, then

$$\frac{dQ}{dt}=-kQ$$

The negative sign indicates that the quantity of drug in the body is decreasing.

The solution to this differential equation is $Q=Q_0e^{-kt}$ and the quantity decreases exponentially.

The constant k depends on the drug and Q_0 is the amount of drug in the body at time zero. Sometimes physicians convey information about the relative decay rate with a *half -life*, which is the time it takes for Q to decrease by a factor of 1/2.

Example

A patient having major surgery is given the antibiotic vancomycin (an antibiotic used to treat a number of bacterial infections) intravenously at a rate of $85\,mg\ per\ hour$. The rate at which the drug is excreted from the body is proportional to the quantity present, with proportionality constant 0.1 if time is in hours. Write a differential equation for the quantity, Q in mg, of vancomycin in the body after t hours.

Solution:

The quantity of vancomycin, Q, is increasing at a constant rate of $85\,mg\ per\ hour$ and is decreasing at a rate of 0.1 times Q. The administration of $85\,mg\ per\ hour$ makes a positive contribution to the rate of change $\frac{dQ}{dt}$. The excretion at a rate of $0.1Q$ makes a negative contribution to $\frac{dQ}{dt}$. Putting these together, we have:

$rate\ of\ change\ of\ a\ quantity = rate\ in - rate\ out$

So, $\frac{dQ}{dt} = 85 - 0.1Q$.

Application activity 3.4.4

Valproic acid is a drug used to control epilepsy; its half-life ($Q = \frac{1}{2}Q_0$) in the human body is about 15 hours.

a) Use the half-life condition to find the constant K in the differential equation

b) At what time will 10% of the original dose remain?

3.5 End unit assessment

1. Given the differential equation $\frac{y}{^2}+\frac{dy}{dx}-6y=0,\ y(0)=10, y'(0)=$

   What is the order and the degree of the equation?

2. If the demand and supply functions in a competitive market are

   $Q_d = 35 - 5P$ and $Q_s = -23 + 6P$ and the rate of adjustment of price when the market is out of equilibrium is $\frac{dp}{dt} = 0.2(Q_d - Q_s)$,

   a) Solve the relevant differential equation to get a function for P in terms of t given that the price is 100 in time period 0.

   b) Plot the graph of P and comment on the stability of this market.

3. Discuss how this unit inspired you in relation to learning other subjects or to your future. If no inspiration at all, explain why.

4. Solve the differential equation $\frac{dy}{dt} = 1.5$ if the value of y is 34 when $t = 0$ and then use the definite solution to predict the value of y when $t = 7$

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