MATHEMATICS FOR TTCs

STUDENT'S BOOK

YEAR

2

OPTION:

SCIENCE AND MATHEMATICS EDUCATION (SME)

FOREWORD

Dear Student,

Rwanda Education Board (REB) is honored to present Year Two Mathematics book for Sciences and Mathematics Education (SME) Student Teachers which serves as a guide to competence-based teaching and learning to ensure consistency and coherence in the learning of the Mathematics content. The Rwandan educational philosophy is to ensure that you achieve full potential at every level of education which will prepare you to be well integrated in society and exploit employment opportunities.

The government of Rwanda emphasizes the importance of aligning teaching and learning materials with the syllabus to facilitate your learning process. Many factors influence what you learn, how well you learn and the competences you acquire. Those factors include the relevance of the specific content, the quality of teachers' pedagogical approaches, the assessment strategies and the instructional materials available. In this book, we paid special attention to the activities that facilitate the learning process in which you can develop your ideas and make new discoveries during concrete activities carried out individually or with peers.

In competence-based curriculum, learning is considered as a process of active building and developing knowledge and meanings by the learner where concepts are mainly introduced by an activity, situation or scenario that helps the learner to construct knowledge, develop skills and acquire positive attitudes and values.

For efficiency use of this textbook, your role is to:

- Work on given activities which lead to the development of skills;
- Share relevant information with other learners through presentations, discussions, group work and other active learning techniques such as role play, case studies, investigation and research in the library, on internet or outside;
- Participate and take responsibility for your own learning;
- Draw conclusions based on the findings from the learning activities.

To facilitate you in doing activities, the content of this book is self explanatory so that you can easily use it yourself, acquire and assess your competences. The book is made of units as presented in the syllabus. Each unit has the following structure: the key unit competence is given and it is followed by the introductory activity before the development of mathematical concepts that are connected to real world problems or to other sciences.

The development of each concept has the following points:

- It starts by a learning activity: it is a hand on well set activity to be done by students in order to generate the concept to be learnt;
- Main elements of the content to be emphasized;
- Worked examples; and
- Application activities: those are activities to be done by the user to consolidate competences or to assess the achievement of objectives.

Even though the book has some worked examples, you will succeed on the application activities depending on your ways of reading, questioning, thinking and grappling ideas of calculus not by searching for similar-looking worked out examples.

Furthermore, to succeed in Mathematics, you are asked to keep trying; sometimes you will find concepts that need to be worked at before you completely understand. The only way to really grasp such a concept is to think about it and work related problems found in other reference books.

I wish to sincerely express my appreciation to the people who contributed towards the development of this book, particularly, REB staff, UR-CE Lecturers, secondary school teachers and TTC Tutors for their technical support. A word of gratitude goes to Head Teachers and TTCs principals who availed their staff for various activities.

Any comment or contribution would be welcome to the improvement of this text book for the next edition.

Dr. NDAYAMBAJE Irénée

Director General, REB

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I wish to express my appreciation to the people who played a major role in the development of this Mathematics book for Year Two student teachers in Sciences and Mathematics Education (SME). It would not have been successful without active participation of different education stakeholders.

I owe gratitude to different universities and schools in Rwanda that allowed their staff to work with REB in the in-house textbooks production initiative.

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Joan MURUNGI

Head of CTLR Department

TABLE OF CONTENT

UNIT 1

SEQUENCES AND SERIES

Key unit competence: Apply arithmetic and geometric sequences to solve problems in financial mathematics.

1.0. INTRODUCTORY ACTIVITY

Suppose that an insect population is growing in such a way that each new generation is 2 times as large as the previous generation. If there are 126 insects in the first generation, on a piece of paper, write down the number of insects that will be there in second, third, fourth,…n^{th} generation.

How can we name the list of the number of insects for different generations?

1.1 Generalities on sequences

ACTIVITY 1.1

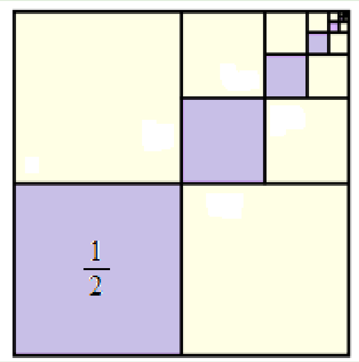
Fold once a square paper, what is the fraction that represents the coloured part?

Fold it twice, what is the fraction that represents the coloured part?

What is the fraction that represents the coloured part if you fold it ten times?

What is the fraction that represents the coloured part if you fold it n times?

Write a list of the fractions obtained starting from the first until the n^{th} fraction.



CONTENT SUMMARY

Let us consider the following list of numbers: $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots \frac{1}{n}, \dots$. The terms of this list are compared to the images of the function $f(x) = \frac{1}{x}$. The list never ends, as the ellipsis indicates. The list is called a sequence, and the numbers in this ordered list are called the terms of the sequence. In dealing with sequences, we usually use subscripted letters, such as u_1 to represent the first term, u_2 for the second term, u_3 for the third term, and so on such as in the sequence $f(n) = u_n = \frac{1}{n}$.

However, in the sequence such as $\{u_n\}: u_n = \sqrt{n-3}$, the first term is u_3 as the previous are not possible, in the sequence $\{u_n\}: u_n = 2n-5$, the first term is u_0.

Definition

A sequence is a function whose domain is a subset of the set of natural numbers.

The terms of a sequence are the range elements of the function. A sequence

is denoted $u_1, u_2, u_3, \dots, u_{n-1}, u_n$ shortly by $\{u_n\}$ or (u_n). We can also write $\{u_1, u_2, u_3, \dots, u_{n-1}, u_n\}$ for a finite sequence whose first term is u_1.

The numbers $u_1, u_2, u_3, \dots, u_{n-1}, u_n$ in a sequence are called **terms of the sequence**. The natural number n is called **term number** and value u_n is called a **general term** of a sequence and the term u_1 is the **initial term** or **the first term**.

If a sequence continues indefinitely, it can be denoted as $\{u_n\}_{n=1}^{\infty}$.

The number of terms of a sequence (possibly infinite) is called the **length of the sequence**.

Notice

- Sometimes, the term number, n, starts from 0. In this case terms of a sequence are $u_0, u_1, u_2, \dots, u_{n-1}, u_n, \dots$ and this sequence is denoted by $\{u_n\}_{n=0}^{+\infty}$. In this case the initial term is u_0.
- A sequence can be finite, like the sequence $2, 4, 8, 16, \dots, 256$.

A finite sequence can be empty. This particular case will not be considered in our study of sequences.

Usually a numerical sequence is given by some formula $u_n = f(n)$, permitting to find any term of the sequence in terms of its number *n*; this formula is called a **general term formula**.

A second way of defining a sequence is to assign a value to the first (or the first few) term(s) and specify the *n*th term by a formula or equation that involves one or more of the terms preceding it. Sequences defined this way are said to be defined **recursively,** and the rule or formula is called a **recursive formula.**

Example: The sequence $u_1 = 1, u_n = n.u_{n-1}$

Infinite and finite sequences:

Consider the sequence of odd numbers less than 11: This is 1, 3,5,7,9. This is a finite sequence as the list is limited and countable. However, the sequence made by all odd numbers is:

$1,3,5,7,9,\ldots 2n+1,\ldots$ This suggests that an i**nfinite sequence** is a sequence whose terms are infinite.

Note that it is not always possible to give the numerical sequence by a general term formula; sometimes a sequence is given by description of its terms.

Examples

1) Numerical sequences:

0,1,2,3,4,5,...a sequence of natural numbers;

0,2,4,6,8,10,... a sequence of even numbers;

$1.4, 1.41, 1.414, 1.4142, \ldots$ *a numerical sequence of approximate, defined more precisely values of* $\sqrt{2}$.

For the last sequence it is impossible to give a general term formula, nevertheless this sequence is described completely.

2) List the first five term of the sequence $\left\{2^n\right\}_{n=1}^{+\infty}$

Solution

Here, we substitute $n=1,2,3,4,5,\ldots$ into the formula 2^n. This gives $2^1,2^2,2^3,2^4,2^5,\ldots$

Or, equivalently, $2,4,8,16,32,\ldots$

3) Express the following sequences in general notation

a) $\frac{1}{2},\frac{2}{3},\frac{3}{4},\frac{4}{5},\ldots$

b) $\frac{1}{2},\frac{1}{4},\frac{1}{8},\frac{1}{16},\ldots$

Solution

a) $\frac{1}{2},\frac{2}{3},\frac{3}{4},\frac{4}{5},\ldots$

Beginning by comparing terms and term numbers:

Term number	1	2	3	4	...
Term	$\frac{1}{2}$	$\frac{2}{3}$	$\frac{3}{4}$	$\frac{4}{5}$	...

In each term, the numerator is the same as the term number, and the denominator is one greater the term number.

Thus, the n^{th} term is $\frac{n}{n+1}$ and the sequence may be written as $\left\{\frac{n}{n+1}\right\}_{n=1}^{+\infty}$.

b) $\frac{1}{2},\frac{1}{4},\frac{1}{8},\frac{1}{16},\ldots$

Beginning by comparing terms and term numbers:

Term number	1	2	3	4	$\cdots$
Term	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{16}$	$\cdots$

Or

Term number	1	2	3	4	$\cdots$
Term	$\frac{1}{2^1}$	$\frac{1}{2^2}$	$\frac{1}{2^3}$	$\frac{1}{2^4}$	$\cdots$

In each term, the denominator is equal to the power of 2 ,where the exponent is the term number. We observe that the n^{th} term is $\frac{1}{2^n}$ and the sequence may be written as $\left\{\frac{1}{2^n}\right\}_{n=1}^{+\infty}$.

4) A sequence is defined by

$$\{u_n\}:\begin{cases} u_0 = 1 \\ u_{n+1} = 3u_n + 2 \end{cases}$$

Determine $u_1,\ u_2$ *and* u_3

Solution

Since $u_0 = 1$ and $u_{n+1} = 3u_n + 2$, replace n by $0,1,2$ to obtain u_1, u_2, u_3 respectively.

$$n = 0, \quad u_{0+1} = u_1 = 3u_0 + 2$$
$$= 3\times 1 + 2$$
$$= 5$$

$$n = 1, \quad u_{1+1} = u_2 = 3u_1 + 2$$
$$= 3\times 5 + 2$$
$$= 17$$

$$n = 2, \quad u_{2+1} = u_3 = 3u_2 + 2$$
$$= 3\times 17 + 2$$
$$= 53$$

Thus,

$$\begin{cases} u_1 = 5 \\ u_2 = 17 \\ u_3 = 53 \end{cases}$$

APPLICATION ACTIVITY 1.1

1. A sequence is given by $\{u_n\}:\begin{cases} u_0 = 1 \\ u_n = \dfrac{2n^2}{n^2+1} \end{cases}$

Determine u_1, u_2 and u_3

2. List the first five terms of the sequence $\left\{\sqrt{n+1}-\sqrt{n}\right\}_{n=1}^{+\infty}$

3. Express the following sequence in general notation

1, 3, 5, 7, 9, 11, ...

1.2 Convergent or divergent sequences

ACTIVITY 1.2

Discuss the value of the general term of each of the following sequences as n tends to $+\infty$ (plus infinity).

1. $\left\{\dfrac{3n^2-1}{n^2}\right\}$

2. $\{n^2\}$

A numerical sequence $\{u_n\}$ is said to be **convergent** if the limit $\lim\limits_{x\to\infty} u_n$ exists and finite whereas if the limit does not exist (or is infinity) the sequence is said to be **divergent.**

A number L *is called* **a limit** of a numerical sequence $\{u_n\}$ if $\lim\limits_{x\to\infty} u_n = L$

In other words, Convergent sequence is when $\lim\limits_{x\to\infty} u_n = L$ while divergent sequence is when $\lim\limits_{x\to\infty} u_n = \infty$ or does not exisit.

Examples

1) Determine whether the sequence $\left\{\dfrac{n}{2n+1}\right\}_{n=1}^{+\infty}$ converges or diverges.

Solution

First we find the limit of this sequence as n tends to infinity

$$\lim_{n\to\infty}\frac{n}{2n+1}=\lim_{n\to\infty}\frac{n}{n\left(2+\frac{1}{n}\right)}=\frac{1}{2}$$

Thus, $\left\{\frac{n}{2n+1}\right\}_{n=1}^{+\infty}$ converges to $\frac{1}{2}$.

2) Determine whether the sequence $\{8-2n\}_{n=1}^{+\infty}$ converges or diverges.

Solution

First we find the limit of this sequence as n tends to infinity

$$\lim_{n\to\infty}(8-2n)=8-2(+\infty)=-\infty$$

Thus, $\{8-2n\}_{n=1}^{+\infty}$ **diverges.**

APPLICATION ACTIVITY 1.2

Which of the sequences converge, and which diverge? Find the limit of each convergent sequence.

1) $\{2+(0.1)^n\}$ 2) $\left\{\frac{1-2n}{1+2n}\right\}$ 3) $\left\{\frac{1-5n^4}{n^4+8n^3}\right\}$

4) $\{-1^n\}$ 5) $\left\{\frac{2n}{\sqrt{3n+1}}\right\}$ 6) $\frac{\sqrt{7n^2+2}}{n^3+8}$

1.3 Monotonic sequences

ACTIVITY 1.3

For each of the following sequences, state whether the terms are in ascending or descending order, both or neither order.

1) 1, 2, 3, 4, 5, 6, ...

2) $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots$

3) 1, -1, 1, -1, 1, ...

4) 2, 2, 2, 2, 2, 2, ...

Content summary

A sequence $\{u_n\}$ is said to be

- **Increasing** or in **ascending** order if $u_1 < u_2 < u_3 < \ldots < u_n < \ldots$
- **Non decreasing** if $u_1 \leq u_2 \leq u_3 \leq \ldots \leq u_n \leq \ldots$
- **decreasing** or in **descending** order if $u_1 > u_2 > u_3 > \ldots > u_n > \ldots$
- **non increasing** $u_1 \geq u_2 \geq u_3 \geq \ldots \geq u_n \geq \ldots$

A sequence that is either non decreasing or non-increasing is called **monotone (or monotonic)**, and a sequence that is increasing or decreasing is called **strictly monotone**. Observe that a strictly monotone sequence is monotone, but not conversely.

In order, for a sequence to be **increasing**, all pairs of successive terms, u_n and u_{n+1}, must satisfy $u_n < u_{n+1}$, or equivalently, $u_n - u_{n+1} < 0$.

More generally, monotonic sequences can be classified as follows:

Difference between successive terms	**Classification**
$u_n - u_{n+1} < 0$	Increasing
$u_n - u_{n+1} > 0$	Decreasing
Otherwise	Non decreasing or Non increasing.

If the terms in the sequence are all positive, then we can divide both sides of the inequality $u_n < u_{n+1}$ by u_n to obtain $1 < \frac{u_{n+1}}{u_n}$ or equivalently $\frac{u_{n+1}}{u_n} > 1$
More, generally, monotonic sequences with positive terms can be classified as follows:

Difference between successive terms	Classification
$\frac{u_{n+1}}{u_n} > 1$	Increasing
$1 > \frac{u_{n+1}}{u_n}$	Decreasing
Otherwise	Non decreasing or Non increasing

Examples

1) Prove that the sequence $\frac{1}{2},\frac{2}{3},\frac{3}{4},\ldots,\frac{n}{n+1},\ldots$ *is an increasing sequence.*

Solution

Here, $u_n=\frac{n}{n+1}$ and

$u_{n+1}=\frac{n+1}{n+2}$

Thus, for $n\geq 1$

$$u_n-u_{n+1}=\frac{n}{n+1}-\frac{n+1}{n+2}$$
$$=\frac{n^2+2n-n^2-2n-1}{(n+1)(n+2)}$$
$$=-\frac{1}{(n+1)(n+2)}<0$$

This proves that the given sequence is **increasing.**

Alternative method,

We can show that the sequence $\frac{1}{2},\frac{2}{3},\frac{3}{4},\ldots,\frac{n}{n+1},\ldots$ *is increasing by examining the ratio of successive terms as follows*

$$\frac{u_{n+1}}{u_n}=\frac{\frac{n+1}{n+2}}{\frac{n}{n+1}}=\frac{n+1}{n+2}\times\frac{n+1}{n}=\frac{n^2+2n+1}{n^2+2n}$$

Since the numerator exceeds the denominator, the ratio exceeds 1, that is

$\frac{u_{n+1}}{u_n}>1$ for $n\geq 1$. This proves that the sequence is increasing.

2) The sequence 4, 4, 4, 4, ... is both no decreasing no increasing.

3) The sequence -2, 2, -2, 2, -2, ... is not monotonic.

APPLICATION ACTIVITY 1.3

Which of the following sequences are in increasing, decreasing, non increasing, non decreasing, not monotonic?

1. 1, 2, 3, ..., n, ...

2. $\left\{\frac{n}{n+1}\right\}$

3. $\left\{\frac{1}{2^n}\right\}$

4. 3, 3, 3, 3, ...

5) 1, -1, 1, -1, ...

1.4 Arithmetic sequences

ACTIVITY 1.4

In each of the following sequences, each term can be found by adding a constant number to the previous. Guess that constant number.

a) Sequence $\{u_n\}$: 5,8,11,14,17,...

b) Sequence $\{v_n\}$: 26,31,36, 41, 46,...

c) Sequence $\{w_n\}$: 20,18,16,14,12, ...

e) $\{3n+2\}$

f) $\{16-6n\}$

Content summary

Let u_1 be an initial term of a sequence. If we add d successively to the initial term to find other terms, the difference between successive terms of a sequence is always the same number and the sequence is called **arithmetic.**

This sequence has the following term u_1, $u_2=u_1+d$, $u_3=u_2+d=u_1+2d$, $u_4=u_3+d=u_1+3d$, $u_n=u_{n-1}+d=u_1+(n-1)d$,...

An **arithmetic sequence** may be defined recursively as $u_n=u_1+(n-1)d$ where

u_1 and d are real numbers. The number u_1 is the first term, and the number d is called the **common difference.**

Examples

The following sequences are arithmetic sequences:

Sequence $\{u_n\}$: $5, 8, 11, 14, 17, \ldots$

Sequence $\{v_n\}$: $26, 31, 36, 41, 46, \ldots$

Sequence $\{w_n\}$: $20, 18, 16, 14, 12, \ldots$

Common difference

The fixed numbers that bind each sequence together are called the **common differences.** Sometimes mathematicians use the letter ***d*** when referring to these types of sequences.

d can be calculated by subtracting any two consecutive terms in an arithmetic sequence. That is $d = u_{n+1} - u_n$ or $d = u_n - u_{n-1}$.

Note

If three consecutive terms are in arithmetic sequence, the double of the middle term is equal to the sum of extreme terms. That is for an arithmetic sequence u_{n-1}, u_n, u_{n+1}, we have $2u_n = u_{n-1} + u_{n+1}$.

Proof: If u_{n-1}, u_n, u_{n+1} form an arithmetic sequence, then $u_{n+1} = u_n + d$ and

$u_{n-1} = u_n - d$ Adding two equations, you get $u_{n+1} + u_{n-1} = 2u_n$.

General term of an arithmetic sequence

The n^{th} term, u_n of an arithmetic sequence $\{u_n\}$ with common difference d and initial term u_1 is given by $u_n = u_1 + (n-1)d$ which is **the general term of an arithmetic sequence.** If the initial term is u_0, then the general term of an arithmetic sequence becomes $u_n = u_0 + nd$.

Generally, if u_p is any p^{th} term of a sequence then the n^{th} term is given by

$$u_n = u_p + (n-p)d$$

Examples

1) $4, 6, 8$ *are three consecutive terms of an arithmetic sequence because*

$$2 \times 6 = 4 + 8 \Leftrightarrow 12 = 12$$

2) If the first and tenth terms of an arithmetic sequence are 3 and 30, respectively, find the fiftieth term of the sequence.

Solution

$u_1 = 3$ and $u_{10} = 30$

But $u_n = u_1 + (n-1)d$,

$u_{10} = u_1 + (10-1)d$

Then $30 = 3 + (10-1)d \Leftrightarrow 30 = 3 + 9d \Rightarrow d = 3$

Now, $u_{50} = u_1 + (50-1)d = 3 + 49 \times 3 = 150$

Thus,

The fiftieth term of the sequence is 150.

3) If the 3rd term and the 8th term of an arithmetic sequence are 5 and 15 respectively, find the common difference.

Solution

$u_3 = 5,\ u_8 = 15$

Using the general formula: $u_n = u_p + (n-p)d$

$u_3 = u_8 + (3-8)d$

$5 = 15 - 5d$

$\Leftrightarrow 5d = 15 - 5$

$\Rightarrow 5d = 10 \Rightarrow d = 2$

The common difference is 2.

4) Consider the sequence $5,\ 8,\ 11,\ 14,\ 17,\ \ldots,\ 47$. Find the number of terms in this sequence

Solution

We see that $u_1 = 5$, $u_n = 47$ and $d = 3$.

But we know that $u_n = u_1 + (n-1)d$. This gives

$$47 = 5 + (n-1)3$$
$$\Leftrightarrow 42 = 3n - 3 \Rightarrow n = 15$$

This means that there are 15 terms in the sequence.

5) Consider the sequence which went from 20 to -26 with -2 as common difference. Find the number of terms.

Solution

We have

$$-26 = 20 + (n-1)(-2)$$
$$\Leftrightarrow -46 = -2n + 2 \Rightarrow n = 24$$

This means that there are 24 terms in the sequence.

6) Show that the following sequence is arithmetic. Find the first term and the common difference. $\{s_n\} = \{3n+5\}$

Solution:

The first term is $s_1 = 8$. The nth term and the $(n-1)$ st term of the sequence $\{s_n\}$ are $s_n = 3n+5 \quad and \quad s_{n-1} = 3(n-1)+5 = 3n+2$

The first term is $s_1 = 8$.

Their difference d is $s_n - s_{n-1} = (3n+5) - (3n+2) = 3$.

Since the difference of any two successive terms is the constant 3, the sequence $\{s_n\}$ is arithmetic and the common difference is 3.

APPLICATION ACTIVITY 1.4

1) If $\frac{1}{a+b}, \frac{1}{a+c}, \frac{1}{b+c}$ are 3 consecutive terms of an arithmetic progression, show that it will be the same for a^2, b^2, c^2.

2) Form an arithmetic progression of three numbers such that the sum of its terms is 30 and the sum of the squares of its terms is 332.

3) Calculate x so that the squares of $1+x,\ q+x,\ and\ q^2+x$ will be three consecutive terms of an arithmetic progression where q is any given number.

1.5. Arithmetic Means of an arithmetic sequence

ACTIVITY 1.5

Suppose that you need to form an arithmetic sequence of 7 terms such that the first term is 2 and the seventh term is 20. Write down that sequence given that those terms are $2, A, B, C, D, E, 20$.

Content summary

If three or more than three numbers form an arithmetic sequence, then all terms lying between the first and the last numbers are called **arithmetic means.** If B is

arithmetic mean between A and C, then $B = \frac{A+C}{2}$.

Let us see how to insert k terms between two terms u_1 and u_n to form an arithmetic sequence:

u_1	...	...	...	...	u_n

The terms to be inserted are called **arithmetic means** between two terms u_1 and u_n .

This requires to form an arithmetic sequence of $n = k + 2$ terms, whose first term is u_1 and last term is u_n.

While there are several methods, we will use our n^{th} term formula $u_n = u_1 + (n-1)d$.

As u_1 and u_n are known, we need to find the common difference d taking $n = k + 2$ where k is the number of terms to be inserted and 2 stands for the first and last terms..

Examples

1) Insert three arithmetic means between 7 and 23.

Solution

Here $k = 3$ and then $n = k + 2 = 5, u_1 = 7$ and $u_n = u_5 = 23$.

Then

$u_5 = u_1 + (5-1)d$

$\Leftrightarrow 23 = 7 + 4d \Rightarrow d = 4$

Now, insert the terms using $d = 4$, the sequence is $7, 11, 15, 19, 23$.

2) Insert five arithmetic means between 2 and 20.

Solution

Here $k = 5$ and then $n = k + 2 = 7$, $u_1 = 2$ and $u_n = u_7 = 20$.

Then

$u_7 = u_1 + (7-1)d$

$\Leftrightarrow 20 = 2 + 6d \Rightarrow d = 3$

Now, insert the terms using $d = 3$, the sequence is $2, 5, 8, 11, 14, 17, 20$.

APPLICATION ACTIVITY 1.5

1. Insert 4 arithmetic means between -3 and 7
2. Insert 9 arithmetic means between 2 and 32
3. Between 3 and 54, n terms have been inserted in such a way that the ratio of 8th mean and $(n-2)^{\text{th}}$ mean is $\frac{3}{5}$. Find the value of n.
4. There are n arithmetic means between 3 and 54 terms such that 8^{th} mean is equal to $(n-1)^{\text{th}}$ mean as 5 to 9. Find the value of n.
5. Find the value of n so that $\frac{a^{n+1} + b^{n+1}}{a^n + b^n}$ will be the arithmetic mean between a and b

1.6 Arithmetic series

ACTIVITY 1.6

Consider the infinite arithmetic sequence $2, 5, 8, 11, 14, ...$

a) what is its first term, the common difference d and the general term?

b) Determine the sum s_6 of the first 6 terms for this sequence taking that for each $u_n = u_1 + (n-1)d$ where for example $u_3 = u_1 + 2d$.

c) Try to generalize your results to determine the sum s_n for the first n terms of the arithmetic sequence $\{u_n\}$.

Content summary

For finite arithmetic sequence $\{u_n\} = u_1, u_2, u_3, ... u_n$, the sum

$$\sum_{r=1}^{n} u_r = u_1 + u_2 + u_3 + ... + u_n$$

is called an **arithmetic series.** We denote the sum of the first *n* terms of the sequence by S_n.

Thus, $S_n = u_1 + u_2 + u_3 + ... + u_n = \sum_{r=1}^{n} u_r$

The sum of first *n* terms of a finite arithmetic sequence with initial term u_1 is given by

$$\begin{aligned} s_n &= u_1 + u_2 + u_3 + ... + u_n \\ &= u_1 + (u_1 + d) + (u_1 + 2d) + ... + (u_1 + (n-1)d) \\ &= \underbrace{(u_1 + u_1 + ... u_1)}_{n \text{ terms}} + (d + 2d + ... + (n-1)d) \\ &= nu_1 + d[1 + 2 + 3 + ... + (n-1)] = nu_1 + d[\frac{n(n-1)}{2}] \end{aligned}$$

$$s_n = nu_1 + \frac{n}{2}(n-1)d = \frac{n}{2}\left[2u_1 + (n-1)d\right]$$

$$= \frac{n}{2}\left[u_1 + u_1 + (n-1)d\right]$$

$$= \frac{n}{2}(u_1 + u_n)$$

$$s_n = \frac{n}{2}(u_1 + u_n)$$

If the initial term is u_0, the formula becomes $S_n = \frac{n+1}{2}(u_0 + u_n)$

Examples

1) Calculate the sum of first 100 terms of the sequence $2, 4, 6, 8, \ldots$

Solution

We see that the common difference is 2 and the initial term is $u_1 = 2$. We need to find $u_n = u_{100}$.

$$u_{100} = 2 + (100-1)2$$
$$= 2 + 198$$
$$= 200$$

Now,

$$S_{100} = \frac{100}{2}(u_1 + u_{100})$$
$$= 50(2 + 200)$$
$$= 10100$$

2) Find the sum of first k positive even integers ($k \neq 0$ *).*

Solution

$u_1 = 2$ and $d = 2$

$$u_n = u_k$$
$$u_k = 2 + (k-1)2$$
$$u_k = 2k$$

$$S_n = S_k$$

$$S_k = \frac{k}{2}(2+2k)$$

$$S_k = k(k+1)$$

3) Find the sum: 60 + 64 + 68 + 72 + ... + 120

Solution:

This is the sum s_n of an arithmetic sequence u_n whose first term is $u_1 = 60$ and whose common difference is $d = 4$. The *n*th term is u_n.

We have $u_n = u_1 + (n-1)d$ and

$$120 = 60 + (n-1)4 \Leftrightarrow 60 = 4(n-1)$$

$$n = 16$$

Now, the sum is $u_{16} = 60 + 64 + 68 + \ldots + 120 = \frac{16}{2}(60+120) = 1440$

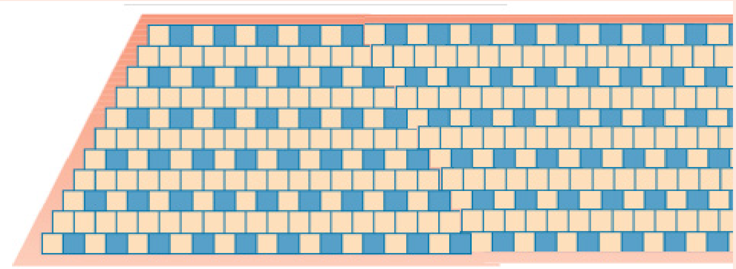
APPLICATION ACTIVITY 1.6

1) Consider the arithmetic sequence 8, 12, 16, 20, ... Find the expression for S_n

2) Sum the first twenty terms of the sequence 5, 9, 13, ...

3) The sum of the terms in the sequence 1, 8, 15, ... is 396. How many terms does the sequence contain?

4) **Practical activity:** A ceramic tile floor is designed in the shape of a trapezoid 10m wide at the base and 5m wide at the top as illustrated on the figure below.



The tiles, 10cm by 10cm, are to be placed so that each successive row contains one less tile than the preceding row. How many tiles will be required?

1.7 Harmonic sequences

ACTIVITY 1.7

Consider the following arithmetic sequence:

2, 4, 6, 8, 10, 12, 14, 16, $\ldots$2n,...

a) Form another sequence whose terms are the reciprocals of the terms of the given sequence.

b) What can you say about the new sequence? What is its first term? The third term and the general term? Is there a relationship between two consecutive terms?

Content summary

Harmonic sequence is a sequence of numbers in which the reciprocals of the terms are in arithmetic sequence. It is of the following form:

$\frac{1}{u}, \frac{1}{u+d}, \frac{1}{u+2d}, \ldots \frac{1}{u+(n-1)d}, \ldots$ where u is not zero, n-1 is a natural number.

Example of harmonic sequence is $\frac{1}{3}, \frac{1}{6}, \frac{1}{9}, \ldots$

If you take the reciprocal of each term from the above harmonic sequence, the sequence will become 3, 6, 9, ... which is an arithmetic sequence with a common difference of 3.

Another example of harmonic sequence is 6, 3, 2. The reciprocals of the terms are $\frac{1}{6}, \frac{1}{3}, \frac{1}{2}$ respectively; this is an arithmetic sequence with a common difference of $\frac{1}{6}$.

Remark

To find the terms of a **harmonic sequence**, convert the sequence into an arithmetic sequence, then do the calculations using the arithmetic formulae. Then take the reciprocals of the terms in arithmetic sequence to get the terms in the harmonic sequence.

Example:

The 2^{nd} term of an harmonic progression is $\frac{1}{6}$ and 6^{th} term is $-\frac{1}{6}$. Find 20^{th} term and n^{th} term.

Solution

In harmonic progression, $h_2 = \frac{1}{6}$ and $h_6 = -\frac{1}{6}$.

Thus, in the corresponding arithmetic progression $a_2 = 6$ and $a_6 = -6$

Or $a_6 = a_2 + 4d \Rightarrow 6 + 4d = -6$ or $d = -3$.

Hence $a_{20} = 6 + 18(-3) = -48 \Rightarrow h_{20} = -\frac{1}{48}$

$$a_n = 6 + (n-2)(-3) = 12 - 3n \Rightarrow h_n = \frac{1}{12 - 3n}$$

Notice: Harmonic Means

If three terms a, b, c are in harmonic progression, the middle one is said to be Harmonic mean between the other two and $b = \frac{2ac}{a+c}$. This can be shown as follow: If a, b, c are in harmonic progression

$$\frac{1}{b} = \left(\frac{1}{a} + \frac{1}{c}\right) : 2 \Leftrightarrow 2ac = bc + ba \Leftrightarrow 2ac = b(a+c) \Rightarrow \mathrm{b} = \frac{2ac}{a+c}$$

Example:

1) Insert 4 harmonic means between $\frac{2}{3}$ and $\frac{6}{19}$

Solution

Let the four harmonic means be h_1, h_2, h_3, h_4.

Then $\frac{2}{3}, h_1, h_2, h_3, h_4, \frac{6}{19}$ are in harmonic progression

$\Rightarrow \frac{3}{2}, \frac{1}{h_1}, \frac{1}{h_2}, \frac{1}{h_3}, \frac{1}{h_4}, \frac{19}{6}$ are in arithmetic progression. where $a_1 = \frac{3}{2}$ and $a_6 = \frac{19}{6}$

$a_6 = \frac{19}{6} \Leftrightarrow a_1 + 5d = \frac{19}{6}$ with d common difference.

$$\Rightarrow \frac{3}{2} + 5d = \frac{19}{6} \Leftrightarrow 5d = \frac{19}{6} - \frac{3}{2} \Leftrightarrow 5d = \frac{10}{6} \Rightarrow d = \frac{1}{3}$$

$$\Rightarrow \begin{cases} \dfrac{1}{h_1}=\dfrac{3}{2}+\dfrac{1}{3}=\dfrac{11}{6}\equiv 1^{st} \text{ term of arithmetic progression} \\ \dfrac{1}{h_2}=\dfrac{3}{2}+\dfrac{2}{3}=\dfrac{13}{6}\equiv 2^{nd} \text{ term of arithmetic progression} \\ \dfrac{1}{h_3}=\dfrac{3}{2}+\dfrac{3}{3}=\dfrac{15}{6}=\dfrac{5}{2}\equiv 3^{rd} \text{ term of arithmetic progression} \\ \dfrac{1}{h_4}=\dfrac{3}{2}+\dfrac{4}{3}=\dfrac{17}{6}\equiv 4^{th} \text{ term of arithmetic progression} \end{cases}$$

The four harmonic means are $\dfrac{6}{11},\dfrac{6}{13},\dfrac{5}{2},\dfrac{6}{17}$

2) Find the n^{th} term of the series $2\dfrac{1}{2}+1\dfrac{7}{13}+1\dfrac{1}{9}+\dfrac{20}{23}+\cdots$

Solution

The given series is $\dfrac{5}{2}+\dfrac{20}{13}+\dfrac{10}{9}+\dfrac{20}{23},\cdots$

The reciprocal of the terms are $\dfrac{2}{5},\dfrac{13}{20},\dfrac{9}{10},\dfrac{23}{20},\cdots$

There are in arithmetic progression, with the first term $\dfrac{2}{5}$ and the common difference $\dfrac{13}{20}-\dfrac{2}{5}=\dfrac{1}{4}$

The given series in arithmetic progression:

n^{th} term of arithmetic progression: $a_n=\dfrac{2}{5}+(n-1)\dfrac{1}{4}=\dfrac{8+5n-5}{20}=\dfrac{5n+3}{20}$

Hence n^{th} term of the given harmonic progression is $h_n=\dfrac{1}{a_n}$ or $h_n=\dfrac{20}{5n+3}$

The n^{th} term of the series $2\dfrac{1}{2}+1\dfrac{7}{13}+1\dfrac{1}{9}+\dfrac{20}{23}+\cdots$ is $\dfrac{20}{5n+3}$

APPLICATION ACTIVITY 1.7

1. Find the 4^{th} **and** 8^{th} **term of the harmonic series** $6,4,3,...$
2. Insert two harmonic means between 3 and 10.
3. If a, b, c are in harmonic progression, show that $\frac{a}{b+c},\frac{b}{c+a},\frac{c}{a+b}$ are also in harmonic progression.
4. Find the term number of harmonic sequence $\frac{\sqrt{5}}{3},\frac{\sqrt{5}}{4},\frac{1}{\sqrt{5}},...,\frac{\sqrt{5}}{13}$
5. The harmonic mean between two numbers is 3 and the arithmetic mean is 4. Find the numbers.
6. Find the n^{th} term of the series $4+4\frac{2}{7}+4\frac{8}{13}+5+\cdots$

1. 8 Generalities on Geometric sequences

ACTIVITY 1.8

Take a piece of paper in a square shape.

1. Cut it into two equal parts.
2. Write down a fraction corresponding to one part according to the original piece of paper
3. Take one part obtained in step 2) and repeat step 1) and then step 2)
4. Continue until you remain with a small piece of paper that you are not able to cut it into two equal parts and write down the sequence of fractions obtained.
5. Observe the sequence of numbers you obtained and give the relationship between any two consecutive numbers.

Content summary

Sequences of numbers that follow a pattern of multiplying a term by a fixed number ***r***, from one term u_1 to the next, are called **geometric sequences**.

The following are examples of geometric sequences:

Sequence $\{u_n\}$: $5,10,20,40,80,...$

Sequence $\{v_n\}$: $2,1,\frac{1}{2},\frac{1}{4},\frac{1}{8},\dots$

Sequence $\{w_n\}$: $1,-2,4,-8,16,\dots$

Common ratio

We can examine these sequences to greater depth, we must know that the fixed numbers that bind each sequence together are called the **common ratios**, denoted by the letter ***r***. This means if u_1 is the first term,

$$u_2 = ru_1; u_3 = r^2u_1; u_4 = r^3u_1; \dots u_n = r^{n-1}u_1; \dots$$

The n^{th} term or the general term of a geometric sequence becomes $u_n = r^{n-1}u_1$. If the first term is u_0, then the general term of an geometric sequence becomes $u_n = r^n u_0$.

The common ration *r* can be calculated by dividing any two consecutive terms in a geometric sequence. That is

$$r = \frac{u_{n+1}}{u_n} \text{ or } r = \frac{u_n}{u_{n-1}} \text{ or } u_n = ru_{n-1}.$$

Generally,

If u_p is the p^{th} term of the sequence, then the n^{th} term is given by $u_n = u_p r^{n-p}$.

Note that if three terms are consecutive terms of a geometric sequence, the square of the middle term is equal to the product of extreme terms. That is for a geometric sequence u_{n-1}, u_n, u_{n+1}, we have $u_n^2 = u_{n-1} \cdot u_{n+1}$.

Examples

1) $6,12,24$ *are consecutive terms of a geometric sequence because*

$$(12)^2 = 6\times 24 \Leftrightarrow 144 = 144$$

Find b such that $8,b,18$ *will be in geometric sequence.*

Solution

$$b^2 = 8\times 18 = 144$$

$$b = \pm\sqrt{144} = \pm 12$$

Thus, $8,12,18$ or $8,-12,18$ are in geometric sequence.

2) The product of three consecutive numbers in geometric progression is 27. The sum of the first two and nine times the third is -79. Find the numbers.

Solution

Let the three terms be $\frac{x}{a}, x, ax$.

The product of the numbers is 27. So $\frac{x}{a}xax = 27 \Rightarrow x^3 = 27 \Rightarrow x = 3$

The sum of the first two and nine times the third is -79:

$$\frac{x}{a} + x + 9ax = -79 \Rightarrow \frac{3}{a} + 3 + 27a = -79$$

$$27a^2 + 82a + 3 = 0 \Rightarrow a = -3 \;\; or \;\; a = -\frac{1}{27}$$

The numbers are: -1, 3, -9 or -81, 3, $-\frac{1}{9}$.

3) If the first and tenth terms of a geometric sequence are 1 and 4, respectively, find the nineteenth term.

Solution

$u_1 = 1$ and $u_{10} = 4$

But $u_n = u_1 r^{n-1}$, then $4 = 1r^9 \Leftrightarrow r = \sqrt[9]{4} \;\; or \;\; r = 4^{\frac{1}{9}}$

Now,

$$\begin{aligned} u_{19} &= u_1 r^{19-1} \\ &= 1\left(4^{\frac{1}{9}}\right)^{18} \\ &= 16 \end{aligned}$$

Thus, the nineteenth term of the sequence is 16.

4) If the 2^{nd} term and the 9^{th} term of a geometric sequence are 2 and $-\frac{1}{64}$ respectively, find the common ratio.

Solution

$$u_2 = 2, \;\; u_9 = -\frac{1}{64}$$

Using the general formula: $u_n = u_p r^{n-p}$

$u_2 = u_9 r^{2-9}$

$2 = -\frac{1}{64} r^{-7}$

$\Leftrightarrow 128 = -\frac{1}{r^7}$

$\Leftrightarrow r^7 = -\frac{1}{128}$

$\Leftrightarrow r = \sqrt[7]{-\frac{1}{128}} \Rightarrow r = -\frac{1}{2}$

The common ratio is $r = -\frac{1}{2}$.

5) Find the number of terms in sequence $2, 4, 8, 16, ..., 256$.

Solution

This sequence is geometric with common ratio 2, $u_1 = 2$ and $u_n = 256$

But $u_n = u_1 r^{n-1}$, then $256 = 2 \times 2^{n-1} \Leftrightarrow 256 = 2^n$ or $2^8 = 2^n \Rightarrow n = 8$.

Thus, the number of terms in the given sequence is 8.

APPLICATION ACTIVITY 1.8

1. If the second and fifth terms of a geometric sequence are 6 and -48, respectively, find the sixteenth term.
2. If the third term and the 8th term of a geometric sequence are $\frac{1}{2}$ and $\frac{1}{128}$ respectively, find the common ratio.
3. The 4th term of a geometric sequence is square of its 2nd term, and the first term is -3. Determine its 7th term.
4. Find the fourth term from the end of geometric sequence $8, 4, 2, \cdots, \frac{1}{128}$
5. The fifth term of a geometric sequence is $\frac{81}{32}$ and the ratio of 3rd and 4th is $\frac{2}{3}$, write the geometric sequence and its 8th term.
6. If p^{th} terms of two sequences $5, 10, 20, \cdots$ and $1280, 640, 320, \cdots$,are equal, find the value of p.

1.9. Geometric Means

ACTIVITY 1.9

Suppose that you need to form a geometric sequence of 6 terms such that the first term is 1 and the sixth term is 243. Given that these terms are $1, A, b, C, D, 243$. Write down that sequence.

Content summary

To insert k terms called **geometric means** between two terms u_1 and u_n is to form a geometric sequence of $n = k+2$ terms whose the first term is u_1 and the last term is u_n.

While there are several methods, we will use our nth term formula $u_n = u_1 r^{n-1}$.

As u_1 and u_n are known, we need to find the common ratio r taking $n = k+2$ where k is the number of terms to be inserted.

Example:

1) Insert three geometric means between 3 and 48.

Solution

Here $k=3$, then $n=5, u_1=3$ and $u_n = u_5 = 48$

$u_5 = u_1 r^{n-1} \Leftrightarrow 48 = 3r^4 \Rightarrow r = 2$

Inserting three terms using common ratio $r=2$ gives $3, 6, 12, 24, 48$

2) Insert 6 geometric means between 1 and $-\frac{1}{128}$.

Solution

Here $k=6$, then $n=8$, $u_1=1$ and $u_n = u_8 = -\frac{1}{128}$

$u_8 = u_1 r^{n-1}$

$\Leftrightarrow -\frac{1}{128} = 1r^7$

$\Leftrightarrow r^7 = -\frac{1}{128}$

$$\Leftrightarrow r^7 = -\frac{1}{(2)^7}$$

$$\Leftrightarrow r = \left[-\frac{1}{(2)^7}\right]^{\frac{1}{7}} = -\frac{1}{2}$$

Inserting 6 terms using common ratio $r = -\frac{1}{2}$ gives

$1, -\frac{1}{2}, \frac{1}{4}, -\frac{1}{8}, \frac{1}{16}, -\frac{1}{32}, \frac{1}{64}, -\frac{1}{128}$.

APPLICATION ACTIVITY 1.9

1. Insert 5 geometric means between $\frac{1}{4}$ and $\frac{1}{256}$.
2. Insert 5 geometric means between 2 and $\frac{2}{729}$
3. Find the geometric mean between

a) 2 and 98

b) $\frac{3}{2}$ and $\frac{27}{2}$

4. Suppose that $4, 36, 324$ are in geometric progression. Insert two more numbers in this sequence so that it again forms a geometric sequence.

5. The arithmetic mean of two numbers is 34 and their geometric mean is 16. Find the numbers.

1. 10. Geometric series

ACTIVITY 1.10

During a competition of student teachers at the district level, 5 first winners were paid an amount of money in the way that the first got 100,000Frw , the second earned the half of this money, the third got the half of the second's money, and so on until the fifth who got the half of the fourth's money.

a) Discuss and calculate the money earned by each student from the second to the fifth.

b) Determine the total amount of money for all the 5 student teachers.

c) Compare the money for the first and the fifth student and discuss the importance of winning at the best place.

d) Try to generalize the situation and guess the money for the student who passed at the n^{th} place if more students were paid. In this case, evaluate the total amount of money for n students.

Content summary

A **Geometric series** is an infinite sum $\sum_{n=1}^{\infty} u_n = u_1 + u_1 r + u_1 r^2 + ... + u_1 r^{n-1} + ...$ of the terms a geometric sequence.

If we have **a finite geometric sequence** $\{u_n\} = u_1, u_2, u_3, ...u_n$, the sum is

$$S_n = \sum_{r=1}^{n} u_r = u_1 + u_1 r + u_1 r^2 + ... + u_1 r^{n-1}.$$

The sum of first *n* terms of a geometric sequence

The sum S_n of the first n terms of a geometric sequence $\{u_n\} = \{u_1 r^{n-1}\}$ is

$$S_n = u_1 + r u_1 + ... + r^{n-1} u_1 \quad (1)$$

Multiply each side by r to obtain $rS_n = r u_1 + r^2 u_1 + ... + r^n u_1 \quad (2)$

Subtracting equation (2) from equation (1) we obtain

$$S_n - rS_n = u_1 - u_1 r^n$$

$$(1-r) S_n = u_1 (1 - r^n)$$

If $r \neq 1$, then we can solve for S_n and find $S_n = u_1 \frac{(1-r^n)}{(1-r)}$

If the initial term u_1 and common ratio r are given, then the sum is $s_n = \frac{u_1(1-r^n)}{1-r}$, with $r \neq 1$

If the initial term is u_0, then the formula is $s_n = \frac{u_0(1-r^{n+1})}{1-r}$ with $r \neq 1$; in this case, there are $n+1$ terms in the geometric sequence.

If $r = 1$, $s_n = nu_1$

Also, the product of first n terms of a geometric sequence with initial term u_1 and common ratio r is given by $P_n = (u_1)^n r^{\frac{n(n-1)}{2}}$

If the initial term is u_0 then $P_n = (u_0)^{n+1} r^{\frac{n}{2}(n+1)}$

Examples

1) *Find the sum of the first 20 terms of a geometric sequence if the first term is 1 and common ratio is 2.*

Solution

Here $u_1 = 1, r = 2, n = 20$

Then,

$$s_{20} = \frac{1(1-2^{20})}{1-2} = \frac{1-2^{20}}{-1} = 1048575$$

2) Consider the sequence $\{u_n\}$ defined by $u_0 = 0$ and $u_{n+1} = u_n + \frac{1}{2^n}$. Consider another sequence $\{v_n\}$ defined by $v_n = u_{n+1} - u_n$.

a) *Show that $\{v_n\}$ is a geometric sequence and find its first term and common ratio.*

b) *Express $\{v_n\}$ in term of n.*

Solution

a) $u_0 = 0, \quad v_0 = u_1 - u_0 = 1$

$$u_1 = u_0 + \frac{1}{2^0} = 1,\ u_2 = u_1 + \frac{1}{2^1} = \frac{3}{2};$$

$$v_1 = u_2 - u_1 = \frac{1}{2},\ v_2 = u_3 - u_2 = \frac{1}{4}$$

$\{v_n\}$ is a geometric sequence if $v_1^2 = v_0 \cdot v_2$.

$v_1^2 = \frac{1}{4}$ and $v_0 \cdot v_2 = \frac{1}{4}$.

Thus, $\{v_n\}$ is a geometric sequence.

First term is $v_0 = 1$

Common ratio is $r = \frac{v_1}{v_0} = \frac{1}{2}$

b)General term

$$v_n = v_0 r^n$$
$$= \frac{1}{2^n}$$

Thus, $\{v_n\}$ is defined by $v_n = \frac{1}{2^n}$

3) Find the product of the first 20 terms of a geometric sequence if the first term is 1 and common ratio is 2.

Solution

$$P_n = (u_1)^n r^{\frac{n(n-1)}{2}}$$

Here $u_1 = 1, r = 2, n = 20$,

Then,

$$P_{20} = (1)^{20}\, 2^{\frac{20(19)}{2}} = 2^{190}$$

APPLICATION ACTIVITY 1.10

1. Find the sum of the first 8 terms of the geometric sequence 32, -16, 8, ...

2. Find the sum of the geometric sequence with the first term 0.99 and the common ratio is equal to the first term.

3. Find the first term and the common ratio of the geometric sequence for which

$$S_n = \frac{5^n - 4^n}{4^{n-1}}$$

4. Find the product of the first 10 terms of the sequence in question 1.

5. Alexis wants to begin saving money for school. He decides to deposit \$500 at the beginning of each quarter (January1, April 1, July1, and October1) in a saving account which pays an annual percentage of 6% compounded quarterly. The interest for each quarter is posted on the last day of the quarter. Determine Alexis's account balance at the end of one year.

1.11 Infinity geometric series

ACTIVITY 1.11

Given that the sum of n terms of a geometric series $\{u_n\} = \left\{5\left(\frac{1}{2}\right)^{n-1}\right\}$ is given by:

$$S_n = 5\frac{\left(1-\left(\frac{1}{2}\right)^n\right)}{\left(1-\left(\frac{1}{2}\right)\right)},$$

a) evaluate $\lim\limits_{n\to\infty} S_n = \lim\limits_{n\to\infty}\left[5\frac{\left(1-\left(\frac{1}{2}\right)^n\right)}{\left(1-\left(\frac{1}{2}\right)\right)}\right]$

b) Extend your results considering the infinite geometric series $\sum\limits_{n=1}^{\infty} u_1 r^{n-1}$ and

determine $\lim\limits_{n\to\infty} S_n = \lim\limits_{n\to\infty} u_1 \frac{(1-r^n)}{(1-r)}, \quad -1 < r < 1.$

Content summary

The sum of a geometric series with first term u_1 and the common ratio r is the limit of this geometric sequence as n approaches ∞.

As $S_n = u_1 \frac{(1-r^n)}{(1-r)}$, this limit is $\lim_{n\to\infty} u_1 \frac{(1-r^n)}{(1-r)} = \lim_{n\to\infty} u_1 \frac{1}{1-r} - \lim_{n\to\infty} u_1 \frac{r^n}{1-r}$.

This limit is a real number if $|r|<1$ as in this case $|r^n|$ approaches o when $n \to \infty$. Therefore, $\lim_{n\to\infty} u_1 \frac{(1-r^n)}{(1-r)} = \lim_{n\to\infty} u_1 \frac{1}{1-r} - \lim_{n\to\infty} u_1 \frac{r^n}{1-r} = u_1 \frac{1}{1-r} - 0$

Therefore, the sum of our geometric sequence becomes:

$S_\infty = \frac{u_1}{1-r}$ provided $-1<r<1$

As a conclusion, if $|r|<1$, *the infinite geometric series* $\sum_{n=1}^{\infty} u_1 r^{n-1}$ *converges. Its sum is* $\sum_{n=1}^{\infty} u_1 r^{n-1} = \frac{u_1}{1-r}$.

Examples:

1) Given the geometric progression 16, 12, 9, ... Find the sum of terms up to infinity.

Solution

Here $u_1 = 16, r = \frac{12}{16} = \frac{3}{4}$

Thus $-1<r<1$ and hence the sum to infinity will exists

$$S_\infty = \frac{u_1}{1-r} = \frac{16}{1-\frac{3}{4}} = 64$$

The sum to infinity is 64.

2) Express the recurring decimal $0.\overline{32}$ as a rational number.

Solution

$0.\overline{32} = \frac{32}{10^2} + \frac{32}{10^4} + \frac{32}{10^6} + \ldots$ which is an infinite geometric series with first term $u_1 = 0.32$ and common ratio is $r = 0.01$.

Since $-1<r<1$, the sum to infinity exist and equal to $\frac{u_1}{1-r} = \frac{0.32}{1-0.01} = \frac{0.32}{0.99} = \frac{32}{99}$

Therefore, $0.\overline{32} = \frac{32}{99}$.

APPLICATION ACTIVITY 1.11

1) Consider the infinite geometric series $\sum_{n=1}^{\infty} 10\left(1-\frac{3x}{2}\right)^{n}$

   a. For what values of x does a sum to infinity exist?

   b. Find the sum of the series if $x=1.3$

2) Show that the repeating decimal 0.9999.... equals 1.

3) evaluate if the geometric series $\sum_{n=1}^{\infty} 2\left(\frac{2}{3}\right)^{n-1}=2+\frac{4}{3}+\frac{8}{9}+\ldots$ converges or diverges. If it converges, find its sum.

1.12 Application of sequences in real life

ACTIVITY 1.12

Carry out a research in the library or on internet and find out at least 3 problems or scenarios of the real life where sequences and series are applied.

Content summary

There are many applications of sequences. Sequences are useful in our daily lives as well as in higher mathematics. For example; the monthly payments made to pay off an automobile or home loan with interest portion, the list of maximum daily temperatures in one area for a month are sequences. Sequences are used in calculating interest, population growth, half-life and decay in radioactivity ...

In economics and Finance, sequences and series can be used for example in solving problems related to:

a) Final sum, the initial sum, the time period and the interest rate for an investment.

The amount A after t years due to a principal P invested at an annual interest rate r compounded n times per year is $A=P.\left(1+\frac{r}{n}\right)^{n.t}$.

(i) If the compounding takes place Annually, n=1 and $A = P.\left(1+\frac{r}{1}\right)^{t}$

(ii) If the compounding takes place Monthly, n=12 and $A = P.\left(1+\frac{r}{12}\right)^{12t}$

(iii) If the compounding takes place Daily, n=365 and $A = P.\left(1+\frac{r}{365}\right)^{365t}$

In each case, **the Interest due** is $A - P$.

Note that: From the compound interest formula $A = P\left(1+\frac{r}{n}\right)^{nt}$ If we let $k = \frac{n}{r}$,then $n = kr$ and $nt = krt$, and we may write the formula as

$$A = P\left(1+\frac{1}{k}\right)^{krt} = P\left[\left(1+\frac{1}{k}\right)^{k}\right]^{rt}.$$

For continuously compounded interest, we may let n (the number of interest period per year) increase without bound towards infinity ($n \to \infty$), equivalently, by $k \to \infty$.

Using the definition of e, we see that $P\left[\left(1+\frac{1}{k}\right)^{k}\right]^{rt} \to P[e]^{rt}$ as $k \to \infty$. This result gives us the following formula: $A = Pe^{rt}$ Where P=Principal or initial value at t = 0 ; r is the Interest rate expressed as a decimal; r is the number of years P is invested; A is the amount after t years.

The amount A after t years due to a principal P invested at an annual interest rate r compounded continuously is $A = P.\mathrm{e}^{rt}$.

b) Annual Equivalent Rate (AER) for part year investments and the nominal annual rate of return.

For loan repayments the annual equivalent rate is usually referred to as the annual percentage rate (APR). If you take out a bank loan you will usually be quoted an APR even though you will be asked to make monthly repayments.

The **corresponding AER for any given monthly rate of interest** i_m can be found using the formula $AER = (i + i_m)^{12} - 1$.

The APR on loans is the same thing as the annual equivalent rate and so the same formula applies.

The relationship between the daily interest rate i_d on a deposit account and the AER can be formulated as $AER = (i + i_d)^{365} - 1$.

c) The Present value of investment

The present value P of A dollars to be received after t years, assuming a per annum interest rate r compounded n times per year, is $P = A.\left(1+\frac{r}{n}\right)^{-n.t}$.

If the interest is compounded continuously, $P = A.\mathrm{e}^{-n.t}$

d) Monthly repayments and the Annual Percentage Rate (APR) for a loans.

Suppose P is the deposit made at the end of each payment period for an annuity paying an interest rate of i per payment period. The amount A of the annuity after n deposits is

$$A = P.\left[\frac{(1+i)^n - 1}{i}\right]$$

Examples

1) A child building a tower with blocks uses 15 for the bottom row. Each row has 2 fewer blocks than the previous row. Suppose that there are 8 rows in the tower.

a) *How many blocks are used for the top row?*

b) *What is the total number of blocks in the tower?*

Solution

a) The number of blocks in each row forms an arithmetic sequence with $u_1 = 15$ and $d = -2$

$n = 8,\ u_8 = u_1 + (8-1)(-2)$. There is just one block in the top row.

b) Here we must find the sum of the terms of the arithmetic sequence formed with $u_1 = 15, n = 8, u_8 = 1$

$$S_8 = \frac{8}{2}(15+1) = 64$$

There are 64 blocks in the tower.

2) An insect population is growing in such a way that each new generation is 1.5 times as large as the previous generation. Suppose there are 100 insects in the first generation.

a) *How many will there be in the fifth generation?*

b) *What will be the total number of insects in the five generations?*

Solution

a) *The population can be written as a geometric sequence with* $u_1 = 100$ *as the first-generation population and common ratio* $r = 1.5$. *Then the fifth-generation population will be* $u_5 = 100(1.5)^{5-1} = 506.25$. *In the fifth-generation, the population will number about 506 insects.*

b) *The sum of the first five terms using the formula for the sum of the first n terms of a geometric sequence.*

$$S_5 = \frac{100\left(1-(1.5)^5\right)}{1-1.5} = 1318.75$$

The total population for the five generations will be about 1319 insects.

3) Find the accumulated value of \$15,000 at 5% per year for 18 years using simple interest.

Solution

$P = 15000,\ r = 0.05, t = 18$

$I = 15000(0.05)(18)$

$= 13500$

A total of \$13,500 in interest will be earned.

Hence, the accumulated value in the account will be 13,500 + 15,000 = \$28,500.

***4)** Suppose 20,000Frw is deposit in a bank account that pays interest at a rate of 8% per year compounded Continuously. Determine the balance in the account in 5 years.*

Solution: Applying the formula for Continuously compound interest with $P = 20,000;\ r = 0.08;\ and\ t = 5$, we have

$A = Pe^{rt} = 20,000e^{0.08(5)} = 20,000\,e^{0.4} = 29,836.49Frw$.

5) Find the amount of an annuity after 5 deposits if a deposit of \$100 is made each year, at 4% compounded annually. How much interest is earned?

Solution

The deposit is $P = \$100$. The number of deposits is $n = 5$ and the interest per payment period is $i = 0.04$. Using Formula , the amount A after 5 deposits is

$$A = P.\left[\frac{(1+i)^n - 1}{i}\right] = 100\left[\frac{\left((1+0.04)^5 - 1\right)}{0.04}\right] = 541.63$$

The interest earned is the amount after 5 deposits less the 5 annual payments of \$100 each:

Interest earned $= A - 500 = 543.63 - 500 = 41.63$. It is \$41.63.

6) Mary decides to put aside \$100 every month in a credit union that pays 5% compounded monthly. After making 8 deposits, how much money does Mary have?

Solution:

This is an annuity with $P = \$100, n = 8$ deposits , and interest $i = \frac{0.05}{12}$ per payment period. Using the formula, the amount A after 8 deposits is

$$A = P.\left[\frac{(1+i)^n - 1}{i}\right] = 100\left[\frac{\left(\left(1+\frac{0.05}{12}\right)^5 - 1\right)}{\frac{0.05}{12}}\right] = 811.76$$

Mary has \$811.76 after making 8 deposits.

7) To save for her daughter's college education, Martha decides to put \$50 aside every month in a bank guaranteed-interest account paying 4% interest compounded monthly.

She begins this savings program when her daughter is 3 years old. How much will she have saved by the time she makes the 180th deposit? How old is her daughter at this time?

Solution:

This is an annuity with $P = \$50$, $n = 180$ deposits, and $i = \dfrac{0.04}{12}$. The amount A saved is

$$A = P.\left[\frac{(1+i)^n - 1}{i}\right] = 50\left[\frac{\left(\left(1+\dfrac{0.04}{12}\right)^{180} - 1\right)}{\dfrac{0.04}{12}}\right] = 12{,}304.52$$

Since there are 12 deposits per year, when the 180th deposit is made $\dfrac{180}{12} = 15$ have passed and Martha's daughter is 18 years old.

APPLICATION ACTIVITY 1.12

1) If Linda deposits \$1300 in a bank at 7% interest compounded annually, how much will be in the bank 17 years later?

2) The population of a city in 1970 was 153,800. Assuming that the population increases continuously at a rate of 5% per year, predict the population of the city in the year 2000.

3) To save for retirement, Manasseh, at age 35, decides to place 2000Frw into an Individual Retirement Account (IRA) each year for the next 30 years. What will the value of the IRA be when Joe makes his 30th deposit? Assume that the rate of return of the IRA is 4% per annum compounded annually.

4) A private school leader received permission to issue 4,000,000Frw in bonds to build a new high school. The leader is required to make payments every 6 months into a sinking fund paying 4% compounded semiannually. At the end of 12 years the bond obligation will be retired. What should each payment be?

1.13. END UNIT ASSESSMENT

1) Find first four terms of the sequence

a) $\left\{\frac{1-n}{n^2}\right\}$ **b)** $\left\{\frac{(-1)^{n+1}}{2n-1}\right\}$ **c)** $\left\{2+(-1)^n\right\}$

2) Find the formula for the n^{th} term of the sequence

a) 1, -1, 1, -1, 1, …

b) 0, 3, 8, 15, 24, …

c) 1, 5, 9, 13, 17, …

3) Which of the following sequences converge, and which diverge? Find the limit of each convergent sequence.

a) $\left\{\sqrt{\frac{2n}{n+1}}\right\}$ b) $\left\{\frac{n}{2^n}\right\}$ c) $\left\{8^{\frac{1}{n}}\right\}$

4) A child with mathematical mid, negotiates a new pocket money deal with her unsuspecting father in which she receives 1 pound on the first day of the month, 2 pounds on the second day, 4 pounds on the third day, 8 pounds on the fourth day, 16 pounds on the fifth day, ... until the end of the month. How much would the child receive during the course of a month of 30 days? (Give your answer to the nearest million pounds).

5) A culture of bacteria doubles every 2 hours. If there are 500 bacteria at the beginning, how many bacteria will there be after 24 hours?

6) You complain that the hot tub in your hotel suite is not hot enough. The hotel tells you that they will increase the temperature by 10% each hour. If the current temperature of the hot tub is 75º F, what will be the temperature of the hot tub after 3 hours, to the *nearest tenth* of a degree?

7) The sum of the interior angles of a triangle is 180º, of a quadrilateral is 360º and of a pentagon is 540º. Assuming this pattern continues, find the sum of the interior angles of a dodecagon (12 sides).

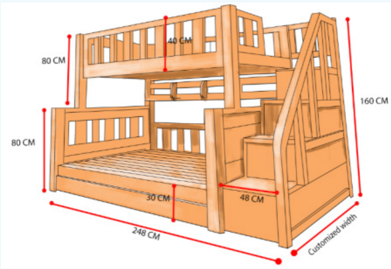
UNIT 2

POINTS, STRAIGHT LINES AND PLANES IN 3D.

Key Unit Competence: Extend understanding and use geometric presentations of lines and planes, locate points and determine equation of a line and plane in 3D.

2.0. INTRODUCTORY ACTIVITY

Observe handcraft objects, building and bed on the diagram below



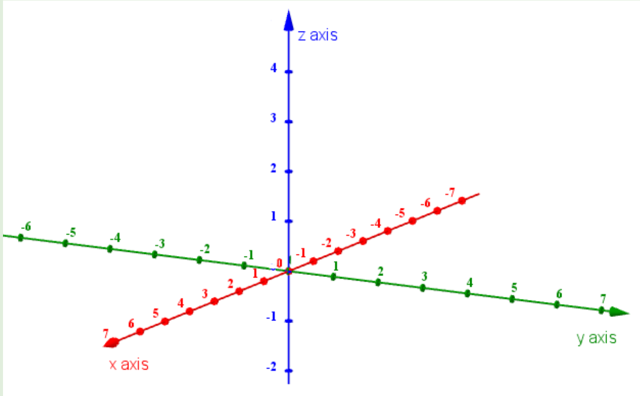
Write down what can illustrate points, lines, angles, planes and shapes. In each case explain your answer

2.1. Points in 3D

2.1.1. Cartesian coordinates of a point and its representation in 3D

ACTIVITY 2.1.1

Consider the point $A(2,3,5)$. Copy the following figure on a piece of paper:

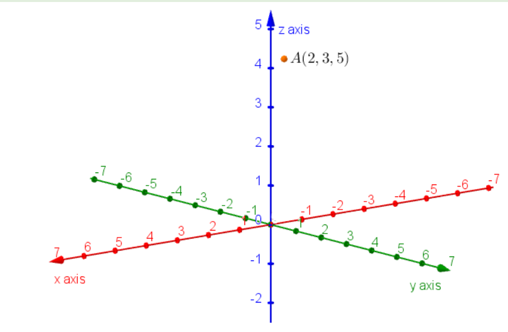


a. Through the point 2 units from the origin, on the x -axis, draw the line parallel to the *y-axis*

b. Through the point 3 units from the origin, on the y -axis, draw the line parallel to the *x-axis* and label P the intersection of the line in (a) and in (b)

c. From point P draw the line parallel to the *z-axis* , at 5 units from P; this is point A

Check your answer against the following:



d. Use the procedure above to plot point $B(-3,1,2)$

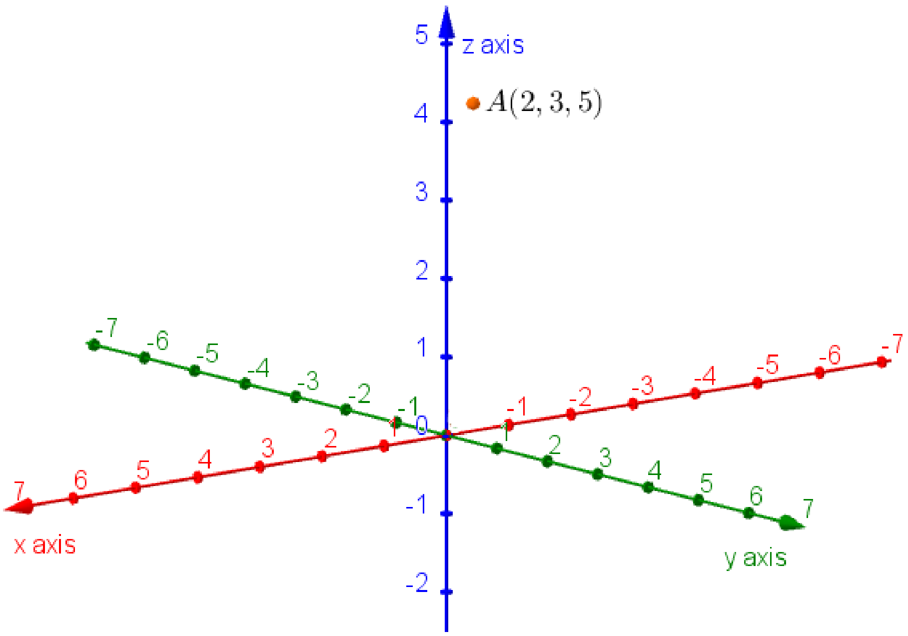
Content summary

Given a set of three axes (OX, OY and OZ) intersecting at the origin O, any two of the axes being perpendicular, any point M in the space can be characterised by an ordered triple of real numbers, called Cartesian coordinates of point M, and denoted $M(x, y, z)$. x, y, z are respectively the signed distances to move from the origin in the directions OX, OY and OZ, respectively.

Example

Represent on the Cartesian plane points $A(1,-4,5)$ and $B(5,2,1)$

Solution



APPLICATION ACTIVITY 2.1.1

Represent the following points in space

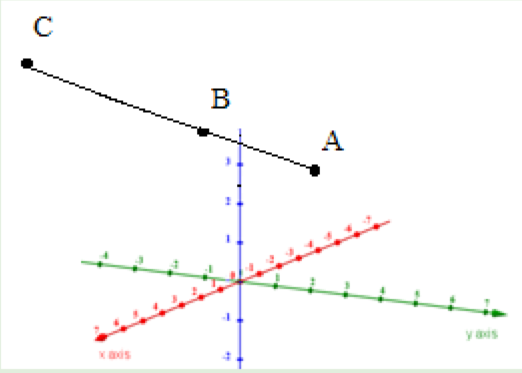
$A(1,1,1), B(-1,2,3), C(3,4,1)$

$D(-2,1,2), E(3,2,1), F(-2,0,1)$

2.1.2. Distance between two points

ACTIVITY 2.1.2

Consider the diagram below:



Compare the position of point B and the position of point C with respect to point A .

Which mathematical concept is used for this comparison?

Content summary

Given points $A(x_A, y_A, z_A)$ and $B(x_B, y_B, z_B)$, the distance between them is the positive real number denoted and defined by

$$d(A,B) = \left\|\overrightarrow{AB}\right\| = \sqrt{(x_B - x_A)^2 + (y_B - y_A)^2 + (z_B - z_A)^2}$$

Example

Calculate the distance between points $A(1,-4,5)$ and $B(5,2,1)$

Solution

$$\sqrt{(5-1)^2 + (2+4)^2 + (1-5)^2} = \sqrt{68}$$

APPLICATION ACTIVITY 2.1.2

Consider the triangle with vertices at points $A(1,-1,2); \mathrm{B}(3,-1,3)$ and $C(1,1,1)$.

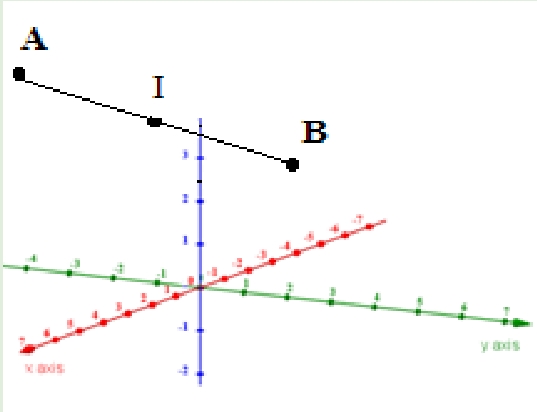
a. Find the lengths of sides AB and AC

b. Hence, determine the type of triangle ABC.

2.1.3. Midpoint of a line segment

ACTIVITY 2.1.3

Consider the diagram below:



Describe the position of point I with respect to points A and B if $A(2,-3,5)$ and $B(0,1,-1)$

Content summary

Given that $A(x_A, y_A, z_A)$ and $B(x_B, y_B, z_B)$, to find the coordinates (x_I, y_I, z_I) in terms of the coordinates of A and B, we proceed as follows:

$$I\left(\frac{x_A + x_B}{2}, \frac{y_A + y_B}{2}, \frac{z_A + z_B}{2}\right)$$

Example

Find the coordinates of the midpoint of the line joining $(1,2,3)$ *and* $(3,2,1)$

Solution

The coordinates of the midpoint of the line joining $(1,2,3)$ *and* $(3,2,1)$ *is given by* $\left(\frac{1+3}{2}, \frac{2+2}{2}, \frac{3+1}{2}\right)$ Which is $(2,2,2)$.

APPLICATION ACTIVITY 2.1.3

Find the coordinates of the midpoint of the line joining segment $[AB]$ if A and B are respectively:

a. $(1,3,6)$ and $(-1,4,5)$

b. $(11,2,4)$ and $(1,3,-5)$

c. $(-9,8,2)$ and $(2,3,8)$

2.2. Straight lines in 3D

2.2.1. Vectors in 3D

ACTIVITY 2.2.1

a. Differentiate between "Scalars" and "vectors"

b. State whether each of the following is a scalar or a vector:

  i. Velocity

  ii. Force

  iii. Mass

c. Consider points $A(2,0,-1); B(1,11,-2)$ and $C(-2,2,1)$.

  i. Calculate the dot product $\overrightarrow{CA}\cdot\overrightarrow{CB}$ and determine the type of triangle ABC

  ii. Calculate the cross product $\overrightarrow{CA}\wedge\overrightarrow{CB}$ and the area of the triangle ABC

Content summary

Taking Cartesian axes x, y and z, any point M in three-dimensional space can be represented by giving x, y and z coordinates. Denoting unit vectors along these axes by $\vec{i}$, $\vec{j}$ and $\vec{k}$, respectively, we can write the vector from O to M(x, y, z) as $\overrightarrow{OM} = x\,\vec{i} + y\,\vec{j} + z\,\vec{k} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$. In this case x, y, z are the components of the vector

$\vec{OM}$ in space.

Given two points $A(x_A, y_A, z_A)$ and $B(x_B, y_B, z_B)$, the vector $\overrightarrow{AB}$ is given by the vector

$$\overrightarrow{OB}-\overrightarrow{OA}=\left(x_B\vec{i}+y_B\vec{j}+z_B\vec{k}\right)-\left(x_A\vec{i}+y_A\vec{j}+z_A\vec{k}\right)=(x_B-x_A)\vec{i}+(y_B-y_A)\vec{j}+(z_B-z_A)\vec{k}$$

Therefore, the position vector of $\overrightarrow{AB}$ is of the following form:

$$\overrightarrow{AB}=(x_B-x_A)\vec{i}+(y_B-y_A)\vec{j}+(z_B-z_A)\vec{k}=\begin{pmatrix} x_B-x_A \\ y_B-y_A \\ z_B-z_A \end{pmatrix}.$$

The following operations can be performed with vectors: addition of vectors, multiplication of a vector by a real number, the dot product of two vectors and the cross product of two vectors.

If $\vec{u}=\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ and $\vec{u'}=\begin{pmatrix} a' \\ b' \\ c' \end{pmatrix}$, then:

The dot product of two vectors

The dot product of two vectors $\vec{u}$ and $\vec{u'}$ is given by $\vec{u}\cdot\vec{u'}=aa'+bb'+cc'$

Indeed, $\vec{u}=a\vec{i}+b\vec{j}+c\vec{k}$ and $\vec{u}'=a'\vec{i}+b'\vec{j}+c'\vec{k}$. Given that $\vec{i}.\vec{i}=\vec{j}.\vec{j}=\vec{k}.\vec{k}=1$

and $\vec{i}.\vec{j}=\vec{j}.\vec{k}=\vec{i}.\vec{k}=0$, then

$$\vec{u}.\vec{u}'=\left(a\vec{i}+b\vec{j}+c\vec{k}\right).\left(a'\vec{i}+b'\vec{j}+c'\vec{k}\right)=aa'+bb'+cc'.$$

The magnitude of vector

The magnitude of vector $\vec{u}$ is the positive real number denoted and defined by $\|\vec{u}\|=\sqrt{\vec{u}\cdot\vec{u}}$.

If $\vec{u}=\begin{pmatrix} a \\ b \\ c \end{pmatrix}$, $\vec{u}.\vec{u}=\left(\vec{u}\right)^2$. However,

$$\vec{u}.\vec{u}=\left(a\vec{i}+b\vec{j}+c\vec{k}\right).\left(a\vec{i}+b\vec{j}+c\vec{k}\right)=a^2+b^2+c^2$$

Therefore, $\left\|\vec{u}\right\|=\sqrt{a^2+b^2+c^2}$;

Note: $\vec{u}$ is a unit vector if and only if $\left\|\vec{u}\right\|=1$.

More other, the dot product of $\vec{u}$ and $\vec{u'}$ can be obtained by $\vec{u}.\vec{u}'=\left\|\vec{u}\right\|.\left\|\vec{u}'\right\|\cos\theta$ where θ is the angle between $\vec{u}$ and $\vec{u}'$.

Example:

If $\vec{u}=3\vec{i}+\vec{j}-\vec{k}$ and $\vec{v}=2\vec{i}+\vec{j}=2\vec{k}$ determine the angle between these vectors.

Solution:

$$\vec{u}.\vec{v}=3.2+1.1+(-1).2=5$$

Furthermore, from the definition of the scalar product $\vec{u}.\vec{v}=\left\|\vec{u}\right\|.\left\|\vec{v}\right\|\cos\theta$. Now,

$$\left\|\vec{u}\right\|=\sqrt{9+1+1}=\sqrt{11}\ ,\ \left\|\vec{v}\right\|=\sqrt{4+1+4}=3$$

Therefore, $\cos\theta=\dfrac{\vec{u}.\vec{v}}{\left\|\vec{u}\right\|.\left\|\vec{v}\right\|}=\dfrac{5}{3\sqrt{11}}$.

This angle is approximately $\theta=59.8\text{degrees or }1.04\text{ radians}$.

The cross product or vector product of two vectors

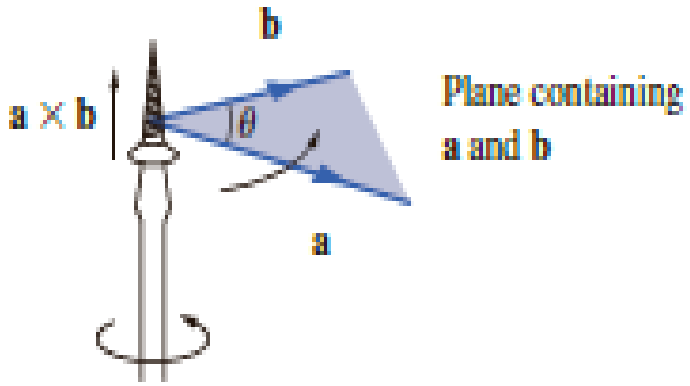
The cross product of two vectors $\vec{u}$ and $\vec{u'}$ is a vector $\vec{u}\wedge\vec{u}'$ perpendicular to the plane made by these vectors and given by $\vec{u}\wedge\vec{u}'=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ a & b & c\\ a' & b' & c'\end{vmatrix}$ which gives:

$$\vec{u}\wedge\vec{u'}=\begin{vmatrix}b & c\\ b' & c'\end{vmatrix}\vec{i}-\begin{vmatrix}a & c\\ a' & c'\end{vmatrix}\vec{j}+\begin{vmatrix}a & b\\ a' & b'\end{vmatrix}\vec{k},$$

where $\vec{i},\vec{j},\vec{k}$ are unit vectors in the directions OX,OY and OZ, respectively.

The result of finding the vector product of $\vec{u}$ and $\vec{v}$ is a vector of length or magnitude $\left\|\vec{u}\right\| . \left\|\vec{v}\right\| \sin\theta$, where θ is an acute angle between $\vec{u}$ and $\vec{v}$. The direction of this vector is such that it is perpendicular $\vec{u}$ and $\vec{v}$, and so it is perpendicular to the plane containing $\vec{u}$ and $\vec{v}$. There are, however, two possible directions for this vector, but it is conventional to choose the one associated with the application of the right-handed screw rule.

Imagine turning a right-handed screw in the sense from $\vec{u}$ towards $\vec{v}$. A right-handed screw is one which, when turned clockwise, enters the material into which it is being screwed. The direction in which the screw advances is the direction of the required vector product.



$\vec{a}\times\vec{b}$ is perpendicular to the plane containing $\vec{a}$ and $\vec{b}$. The right-handed screw rule allows the direction of $\vec{a}\times\vec{b}$ to be found(see the figure above).

The cross product and the vector product verify the following properties:

- Vectors $\vec{u}$ and $\vec{v}$ are orthogonal (perpendicular) if and only if $\vec{u}\cdot\vec{v}=0$
- Vector $\vec{u}\wedge\vec{v}$ is perpendicular to both $\vec{u}$ and $\vec{v}$
- The area of triangle with vertices at points A, B and C is $\frac{1}{2}\left\|\overrightarrow{AB}\wedge\overrightarrow{AC}\right\|$

Example 2.2.1

Given vector $\vec{F} = 2\vec{i} + 7\vec{j} - 3\vec{k}$ *and points* $A(1,2,3)$ *and* $B(-9,-3,-3)$, *find:*

a. The components of vector $\overrightarrow{AB}$

b. The dot product $\vec{F} \bullet \overrightarrow{AB}$

c. The cross product $\vec{F} \wedge \overrightarrow{AB}$

Solution

a. $\overrightarrow{AB} = \begin{pmatrix} -9-1 \\ -3-2 \\ -3-3 \end{pmatrix} = \begin{pmatrix} -10 \\ -5 \\ -6 \end{pmatrix}$

b. $\vec{F} \cdot \overrightarrow{AB} = 2(-10) + 7(-5) - 3(-3) = -37$

c. $\vec{F} \wedge \overrightarrow{AB} = \begin{vmatrix} 7 & -3 \\ -5 & -6 \end{vmatrix}\vec{i} - \begin{vmatrix} 2 & -3 \\ -10 & -6 \end{vmatrix}\vec{j} + \begin{vmatrix} 2 & 7 \\ -10 & -5 \end{vmatrix}\vec{k} = -57\vec{i} + 42\vec{j} + 60\vec{k}$

APPLICATION ACTIVITY 2.2.1

a. The moment, at point B, of force $\vec{F}$ applied at point A, is the cross product $\vec{F} \wedge \overrightarrow{AB}$ Calculate the moment, at point $(1,-2,2)$, of force $\vec{F} = 3\vec{i} - 2\vec{j} + \vec{k}$ Newton, applied at the origin O

b. The work done by a force $\vec{F}$ Newton acting on an object and moving it from point A to point B, the distance being measured in meters, is given by the dot product $\vec{F} \bullet \overrightarrow{AB}$ joules.

Force $\vec{F} = 2\vec{i} + 7\vec{j} - 3\vec{k}$ Newton, acting on an object, moves it from point $A(1,2,3)$ to point $B(-9,-3,-3)$ along a straight line segment joining these

points; the distance is measured in meters. Find the work done by the force.

c. Consider points $A(2,0,-1); B(1,11,-2)$ and $C(-2,2,1)$.

i. Calculate the dot product $\overrightarrow{CA} \cdot \overrightarrow{CB}$ and determine the type of triangle ABC

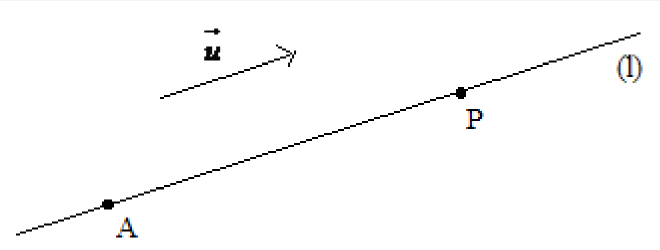
ii. Calculate the cross product $\overrightarrow{CA} \wedge \overrightarrow{CB}$ and the area of the triangle ABC

2.2.2. Equations of a straight line in 3D

ACTIVITY 2.2.2

a. How is a straight line determined in a three-dimensional space?

b. Line (l) passes through point A and is parallel to the vector $\vec{u}$



Express, mathematically, the relationship between vectors $\vec{u}$ and $\overrightarrow{AP}$

c. Given that $P(x,y,z)$, $A(x_0,y_0,z_0)$ and $\vec{u}=\begin{pmatrix}a\\b\\c\end{pmatrix}$, express the relationship found

in part (b) in three different ways, use $\vec{i},\vec{j},\vec{k}$ as unit vectors in the directions OX,OY and OZ, respectively.

Content summary

In a three-dimensional space, a straight line is determined by a point A through which it passes and a vector $\vec{u}$ parallel to the line. Vector $\vec{u}$ is the **direction** vector of the line.

Line (l) through point A and with direction vector $\vec{u}$ is the set of all points P in the space such that $\overrightarrow{AP}$ is a scalar multiple of $\vec{u}$, that is $\overrightarrow{AP}=t\vec{u}$, for some scalar t.

The way the equality $\overrightarrow{AP}=t\vec{u}$ is transformed gives rise to different equations:

- **Vector equation:** $\overrightarrow{OP}=\overrightarrow{OA}+t\vec{u}$ or $(x,y,z)=(x_0,y_0,z_0)+t(a,b,c)$,

  where t is a parameter.

- **The parametric equations:** $\begin{cases}x=x_0+at\\y=y_0+bt\\z=z_0+ct\end{cases}$

- The **Cartesian equations** (or **symmetric equations**):

  $\frac{x-x_0}{a}=\frac{y-y_0}{b}=\frac{z-z_0}{c}$,where $abc\neq 0$

Example

a. Find the vector, parametric and symmetric equations of the line (l) *passing through the point* $A(3,-2,4)$ *with direction vector* $\vec{u}=(2,3,5)$.

b. Find vector, parametric and symmetric equations of the line (r) *passing through the points* $A(3,-2,5)$ *and* $B(1,4,-2)$.

Solution

a. Let $P(x,y,z)$ be any point on the line, then:

Vector equation: $(l):\overrightarrow{OP}=3\vec{i}-2\vec{j}+4\vec{k}+t(2\vec{i}+3\vec{j}+5\vec{k})$, where t is a parameter.

Parametric equations: (l) : $\begin{cases} x=3+2t \\ y=-2+3t \\ z=4+5t \end{cases}$

Symmetric equations: $(l):\dfrac{x-3}{2}=\dfrac{y+2}{3}=\dfrac{z-4}{5}$

b. First we find the direction vector, which is $\overrightarrow{AB}=(-2,6,-7)$

Vector equation: $\overrightarrow{OP}=3\vec{i}-2\vec{j}+5\vec{k}+t(-2\vec{i}+6\vec{j}-7\vec{k})$

Parametric equations: $(r):\begin{cases} x=3-2t \\ y=-2+6t \\ z=5-7t \end{cases}$

Symmetric equations: $(r):\dfrac{x-3}{-2}=\dfrac{y+2}{6}=\dfrac{z-5}{-7}$

APPLICATION ACTIVITY 2.2.2

Find vector, parametric and symmetric equations of the line (l) passing through points A and B in each of the following cases:

a. $A(2,1,4)$ and $B(3,1,1)$.

b. $A(1,1,3)$ and $B(2,5,4)$.

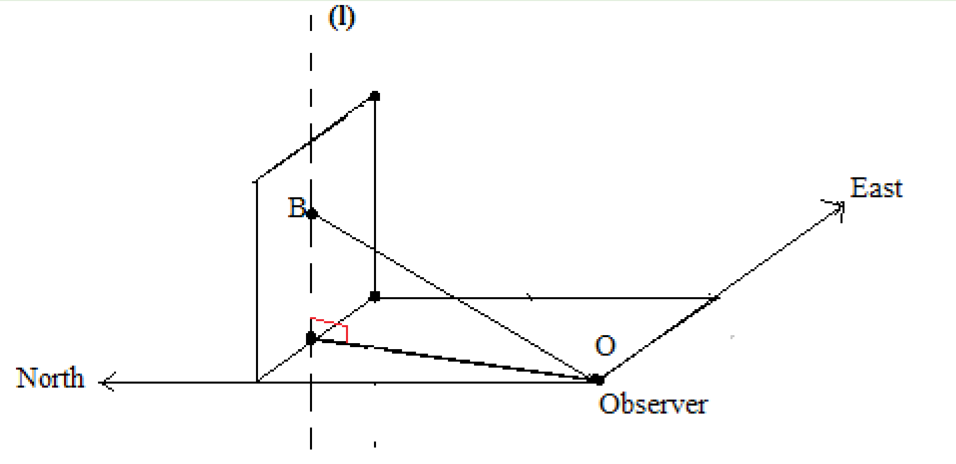
c. $A(2,1,4)$ and $B(6,3,2)$.

d. $A(1,1,1)$ and $B(4,5,6)$.

2.2.3. Distance from a point to a line

ACTIVITY 2.2.3

Consider the diagram below to answer the questions:



a. Label A the intersection point of the plane through point O and perpendicular to line (l) and plot it on the diagram?

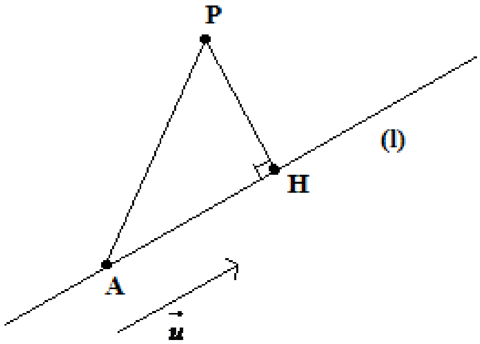
b. What is the distance from point O to line (l)?

c. Write down a direction vector of line (l)

d. In the right angled triangle OAB, let θ be the angle between the direction vector of line (l) and $\overrightarrow{BO}$, express the distance from point O to line (l) in terms of $\sin\theta$. Multiply and divide the side containing $\sin\theta$ by the magnitude of the direction vector. Hence, write down the formula for the distance from a point to a line.

Content summary

Let (l) be the straight line through point A and with direction vector $\vec{u}$.



The distance from point P to line (l) is the distance from P to H, where h is the foot of the perpendicular to (l) through P.

It is found by the formula

$$\left\|\overrightarrow{PH}\right\| = \frac{\left\|\vec{u} \wedge \overrightarrow{AP}\right\|}{\left\|\vec{u}\right\|}.$$

Example

Find the distance from the point $Q(1,3,-2)$ *to the line given by the parametric equations:*

$$\begin{cases} x = 2 + t \\ y = -1 - t \\ z = 3 + 2t \end{cases}$$

Solution

From the parametric equations we know the direction vector, $\vec{u} = (1,-1,2)$ and if we let $t = 0$ a point A on the line is $A(2,-1,3)$

Thus $\overrightarrow{AQ} = \begin{pmatrix} 2-1 \\ -1-3 \\ 3+2 \end{pmatrix} = \begin{pmatrix} 1 \\ -4 \\ 5 \end{pmatrix}$

Find the cross product is $\vec{u} \wedge \overrightarrow{AQ} = -3\vec{i} + 3\vec{j} + 3\vec{k}$; its magnitude is $\left\|\vec{u} \wedge \overrightarrow{AQ}\right\| = 3\sqrt{3}$

The distance is $\frac{\left\|\vec{u} \wedge \overrightarrow{AQ}\right\|}{\left\|\vec{u}\right\|} = \frac{3\sqrt{3}}{\sqrt{6}} = \frac{3\sqrt{2}}{2}$ units

APPLICATION ACTIVITY 2.2.3

Find the distance from the point A to the line (l), in each of the following cases:

a. $A(0,0,12)$; $(l): \begin{cases} x = 4t \\ y = -2t \\ z = 2t \end{cases}$

b. $A(2,1,3); (l): \vec{r} = 2\vec{i} + \vec{j} + 3\vec{k} + \lambda(\vec{i} + 3\vec{j})$

c. $A(3,-1,4); (l): \begin{cases} x = 4 - t \\ y = 3 + 2t \\ z = -5 + 3t \end{cases}$

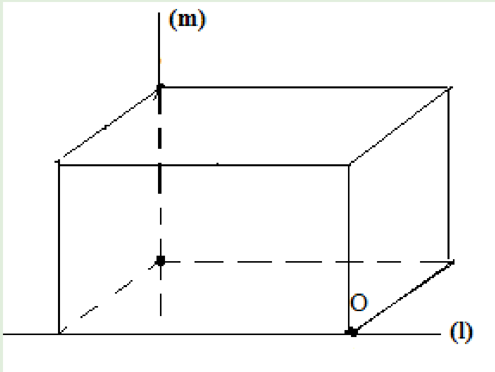
d. $A(1,3,-2); (l): \frac{x-2}{3} = \frac{y+1}{1} = \frac{z-1}{-2}$

e. $A(1,2,3); (l): \frac{x-2}{2} = \frac{y-3}{2} = z - 4$

2.2.4. Distance between two straight lines

ACTIVITY 2.2.4

Use a real box and consider the diagram below to answer the questions:



a) Copy the cuboid and shade the face illustrating the plane containing line (l) and perpendicular to line (m).

b) Plot the intersection point of line (m) and the plane shaded in part (a) and label it A

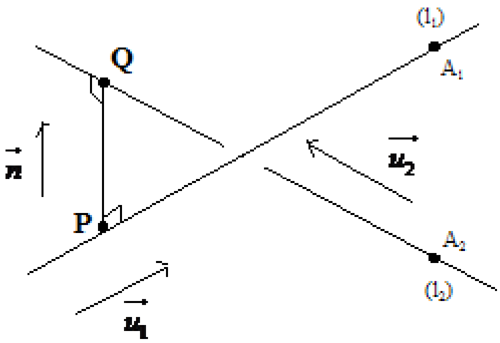
c) Shade on the cuboid (in different color) the face illustrating the plane containing line(m) and perpendicular to line (l)

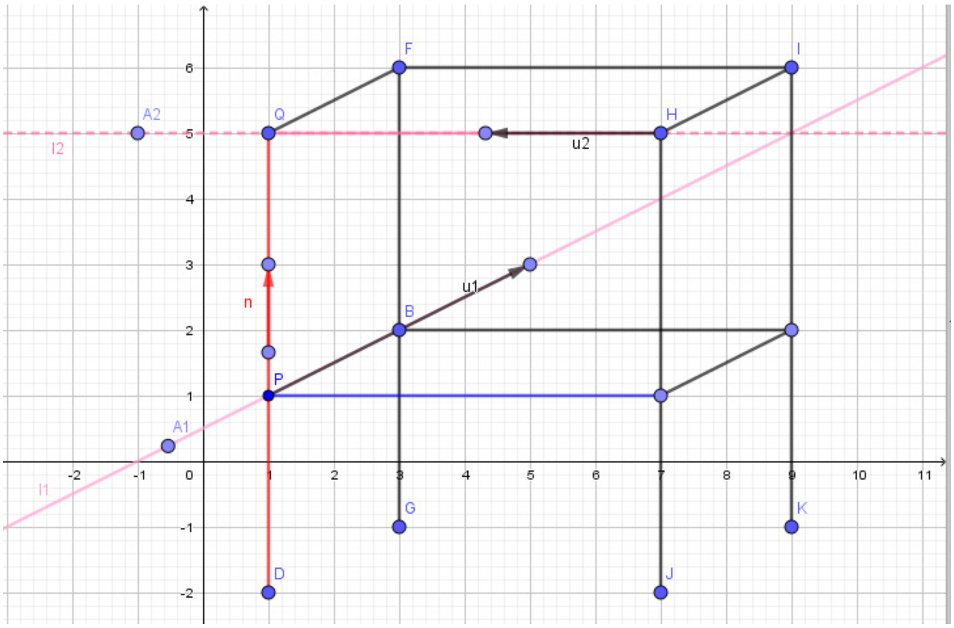
d) Plot the intersection point of line (l) and the plane shaded in part(c) and label it B

e) What is the distance between lines (l) and (m)?

Content summary

Let (l_1) be the line through point A_1 and with direction vector $\overrightarrow{u_1}$, and (l_2) the line through point A_2 and with direction vector $\overrightarrow{u_2}$.





If P and Q are points, one on each line, which are the closest together, then the line (PQ) is perpendicular to both lines (l_1) and (l_2), and hence parallel to

$\vec{n} = \vec{u_1} \wedge \vec{u_2}$

The required distance is then $\left\|\overrightarrow{PQ}\right\| = \dfrac{\left|\overrightarrow{A_1A_2}\cdot(\vec{u_1}\wedge\vec{u_2})\right|}{\left\|\vec{u_1}\wedge\vec{u_2}\right\|}$

In particular, if lines (l_1) and (l_2) are coincident or secant the distance between them is zero.

If lines (l_1) and (l_2) are parallel and distinct, then the distance between them is the distance from any point on any of the two lines to the other line.

Example

Find the distance between the lines (l_1) and (l_2) in each of the following cases:

a. $(l_1): \vec{r} = 5\vec{i} + 3\vec{j} + t(2\vec{i} - \vec{j})$ and $(l_2): \vec{R} = 2\vec{i} + 9\vec{k} + s(\vec{j} - \vec{k})$

b. $(l_1): \dfrac{x-5}{3} = \dfrac{y-7}{-16} = \dfrac{z-3}{7}$ and $(l_2): \dfrac{x-9}{3} = \dfrac{y-13}{8} = \dfrac{z-15}{-5}$

Solution

a. (l_1) passes through point $A_1(5,3,0)$ and has direction vector $\vec{u_1} = \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}$.

(l_2) passes through point $A_2(2,0,9)$ and has direction vector $\vec{u_2} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$

The distance between these lines is $\left\|\overrightarrow{PQ}\right\| = \dfrac{\left|\overrightarrow{A_1A_2}\cdot(\vec{u_1}\wedge\vec{u_2})\right|}{\left\|\vec{u_1}\wedge\vec{u_2}\right\|}$, where

$\overrightarrow{A_1A_2} = \begin{pmatrix} -3 \\ -3 \\ 9 \end{pmatrix}$, $\vec{u_1}\wedge\vec{u_2} = \begin{vmatrix} -1 & 0 \\ 1 & -1 \end{vmatrix}\vec{i} - \begin{vmatrix} 2 & 0 \\ 0 & -1 \end{vmatrix}\vec{j} + \begin{vmatrix} 2 & -1 \\ 0 & 1 \end{vmatrix}\vec{k} = \vec{i} + 2\vec{j} + 2\vec{k}$;

$\overrightarrow{A_1A_2}\cdot(\vec{u_1}\wedge\vec{u_2}) = -3(1)-3(2)+9(2)=9$ and $\left\|\vec{u_1}\wedge\vec{u_2}\right\| = \sqrt{1^2+2^2+2^2} = 3$

It follows that the distance between the two lines is

$$\left\|\overrightarrow{PQ}\right\| = \frac{\left|\overrightarrow{A_1A_2}\cdot(\vec{u_1}\wedge\vec{u_2})\right|}{\left\|\vec{u_1}\wedge\vec{u_2}\right\|} = \frac{|9|}{3} = 3$$

b. (l_1) passes through point $A_1(5,7,3)$ and has direction vector $\overrightarrow{u_1}=\begin{pmatrix}3\\-16\\7\end{pmatrix}$.

(l_2) passes through point $A_2(9,13,15)$ and has direction vector $\overrightarrow{u_2}=\begin{pmatrix}3\\8\\-5\end{pmatrix}$

The distance between these lines is $\left\|\overrightarrow{PQ}\right\|=\dfrac{\left|\overrightarrow{A_1A_2}\cdot(\overrightarrow{u_1}\wedge\overrightarrow{u_2})\right|}{\left\|\overrightarrow{u_1}\wedge\overrightarrow{u_2}\right\|}$, where

$$\overrightarrow{A_1A_2}=\begin{pmatrix}4\\6\\12\end{pmatrix},\ \overrightarrow{u_1}\wedge\overrightarrow{u_2}=\begin{vmatrix}-16 & 7\\8 & -5\end{vmatrix}\vec{i}-\begin{vmatrix}3 & 7\\3 & -5\end{vmatrix}\vec{j}+\begin{vmatrix}3 & -16\\3 & 8\end{vmatrix}\vec{k}=14\vec{i}+36\vec{j}+72\vec{k}$$

; $\overrightarrow{A_1A_2}\cdot(\overrightarrow{u_1}\wedge\overrightarrow{u_2})=4(14)+6(36)+12(72)=1136$ and

$\left\|\overrightarrow{u_1}\wedge\overrightarrow{u_2}\right\|=\sqrt{14^2+36^2+72^2}=\sqrt{6676}$

It follows that the distance between the two lines is

$$\left\|\overrightarrow{PQ}\right\|=\frac{\left|\overrightarrow{A_1A_2}\cdot(\overrightarrow{u_1}\wedge\overrightarrow{u_2})\right|}{\left\|\overrightarrow{u_1}\wedge\overrightarrow{u_2}\right\|}=\frac{|1136|}{\sqrt{6676}}=13.9$$

APPLICATION ACTIVITY 2.2.4

Find the distance between the lines

1. $L_1\equiv\begin{cases}x=1+4t\\y=5-4t\\z=-1+5t\end{cases}$ $\quad L_2\equiv\begin{cases}x=2+8t\\y=4-3t\\z=5+t\end{cases}$

2. $L_1\equiv\begin{cases}x=3+2t\\y=-2t\\z=4-t\end{cases}$ $\quad L_2\equiv\begin{cases}x=4-t\\y=3+5t\\z=2-t\end{cases}$

3. $L\equiv\dfrac{x+8}{2}=\dfrac{y-10}{3}=\dfrac{z-6}{1}$ $\quad M\equiv\dfrac{x-1}{-1}=\dfrac{y-1}{2}=\dfrac{z-1}{4}$

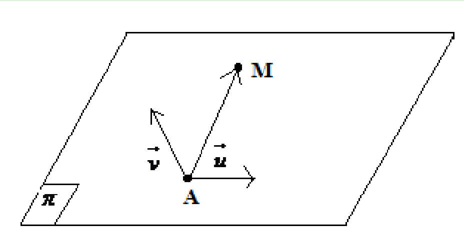
1.3 Planes in 3D

2.3.1. Equations of a plane in 3D

ACTIVITY 2.3.1

a. How is a plane determined

b. Plane π passes through point A and is parallel to both vectors $\vec{u}$ and $\vec{v}$ of different directions.



Express, mathematically, the relationship between vectors $\vec{u}$, $\vec{v}$ and $\overrightarrow{AM}$

c. Given that $M(x, y, z)$, $A(x_0, y_0, z_0)$, $\vec{u} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ and $\vec{v} = \begin{pmatrix} a' \\ b' \\ c' \end{pmatrix}$, express the relationship found in part (b) in three different ways, use $\vec{i}, \vec{j}, \vec{k}$ as unit vectors in the directions OX,OY and OZ, respectively.

Content summary

In a three-dimensional space, a plane π is determined in six different ways that can be brought to one:

By a point A through which π passes and two vectors $\vec{u}$ and $\vec{v}$ of different directions, both parallel to the plane. Vectors $\vec{u}$ and $\vec{v}$ is the **direction** vectors of the plane.

Plane π through point A and with direction vectors $\vec{u}$ and $\vec{v}$ is the set of all points M in the space such that $\overrightarrow{AM} = t\vec{u} + s\vec{v}$, for some scalars t and s.

The way the equality $\overrightarrow{AM} = t\vec{u} + s\vec{v}$ is transformed gives rise to different equations:

- **Vector equation:** $\overrightarrow{OM} = \overrightarrow{OA} + t\vec{u} + s\vec{v}$ or

  $(x, y, z) = (x_0, y_0, z_0) + t(a, b, c) + s(a', b', c')$,

  where *t and s* are parameters.

- **The parametric equations:** $\begin{cases} x = x_0 + at + a's \\ y = y_0 + bt + b's \\ z = z_0 + ct + c's \end{cases}$

- **The Cartesian equation:** $n_1(x - x_0) + n_2(y - y_0) + n_3(z - z_0) = 0$ where

  $\vec{n} = \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix} = \vec{u} \wedge \vec{v}$

Example

a. Find the vector, parametric and symmetric equations of the plane, α, passing through the point $A(2,7,-1)$ with direction vectors $\vec{u} = (3,1,1)$ and $\vec{v}(-1,-2,-3)$.

b. Find the vector, parametric and Cartesian equation of plane β containing points $A(3,-2,-1)$ and $B(4,2,7)$ with direction vector $\vec{u} = (1,1,3)$.

c. Find vector, parametric and Cartesian equation of plane β passing through points $A(1,3,5)$, $B(-2,5,4)$ and $C(3,-6,-5)$.

Solution

a. Let $P(x, y, z)$ represents any point of plane α, *t* and *s* be the parameters.

The vector equation: $x\vec{i} + y\vec{j} + z\vec{k} = 2\vec{i} + 7\vec{j} - \vec{k} + t(3\vec{i} + \vec{j} + \vec{k}) + s(-\vec{i} - 2\vec{j} - 3\vec{k})$

Parametric equations: $\begin{cases} x = 2 + 3t - s \\ y = 7 + t - 2s \\ z = -1 + t - 3s \end{cases}$

Cartesian equation:

$$\vec{u} \wedge \vec{v} = \begin{vmatrix} 1 & 1 \\ -2 & -3 \end{vmatrix}\vec{i} - \begin{vmatrix} 3 & 1 \\ -1 & -3 \end{vmatrix}\vec{j} + \begin{vmatrix} 3 & 1 \\ -1 & -2 \end{vmatrix}\vec{k} = -\vec{i} + 8\vec{j} - 5\vec{k}$$

The Cartesian equation is $-1(x-2) + 8(y-7) - 5(z+1) = 0$

Which is equivalent to $\alpha \equiv x-8y+5z+59=0$

b. Let $X(x,y,z)$ be any point on the plane

Direction vectors are $\overrightarrow{AB}=(1,4,8)$ and $\vec{u}=(1,1,3)$.

The vector equation: $\vec{r}=3\vec{i}-2\vec{j}-\vec{k}+t(\vec{i}+4\vec{j}+8\vec{k})+s(\vec{i}+\vec{j}+3\vec{k})$, *where t* and s are parameters

Parametric equations: $\begin{cases} x=3+t+s \\ y=-2+4t+s \\ z=-1+8t+3s \end{cases}$

Cartesian equation:

$$\vec{u}\wedge\vec{v}=\begin{vmatrix}4 & 8\\1 & 3\end{vmatrix}\vec{i}-\begin{vmatrix}1 & 8\\1 & 3\end{vmatrix}\vec{j}+\begin{vmatrix}1 & 4\\1 & 1\end{vmatrix}\vec{k}=4\vec{i}+5\vec{j}-3\vec{k}$$

The Cartesian equation is $4(x-3)+5(y+2)-3(z+1)=0$

Which is equivalent to $4x+5y-3z-5=0$

c. Let A be the starting point. Then the two direction vectors are $\overrightarrow{AB}=(-3,2,-1)$ and $\overrightarrow{AC}=(2,-9,-10)$.

Let $X(x,y,z)$ represents any point on this plane, then

Vector equation : $\vec{r}=\vec{i}+3\vec{j}+5\vec{k}+t(-3\vec{i}+2\vec{j}-\vec{k})+s(2\vec{i}-9\vec{j}-10\vec{k})$,where t and s are parameters.

Parametric equations: $\begin{cases} x=1-3t+2s \\ y=3+2t-9s \\ z=5-t-10s \end{cases}$

Cartesian equation: $29x+32y-23z-10=0$

APPLICATION ACTIVITY 2.3.1

Find vector, parametric and Cartesian equation of plane β passing through points

a. $A(2,4,1)$, $B(1,3,-1)$ and $C(2,1,3)$.

b. $A(1,1,1)$, $B(4,-2,1)$ and $C(-2,4,3)$.

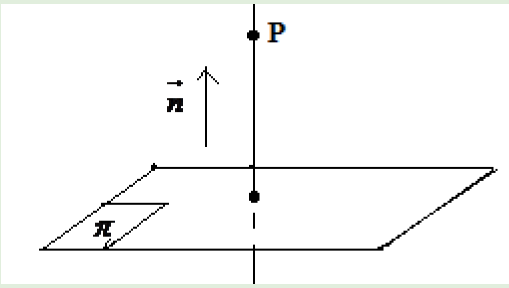
c. $A(3,6,0)$, $B(1,0,1)$ and $C(5,1,7)$.

d. $A(4,3,8)$, $B(-4,1,1)$ and $C(-2,8,6)$.

2.3.2. Distance from a point to a plane

ACTIVITY 2.3.2

Consider the diagram below to answer the questions:



a. Copy the diagram and plot the intersection point of the line (l) perpendicular to plane π through point P and label it H.

b. What is the distance from point P to plane π ?

c. Express mathematically the relationship between the direction vector $\vec{n}$ of line (l) and vector $\overrightarrow{HP}$ Hence, find the parametric equations of line (l) through point $P(x_0, y_0, z_0)$ and with direction vector $\vec{n} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$.

d. Use the values of x, y and z from parametric equations of line (p) in the Cartesian equation of plane π: $ax+by+cz+d=0$, to find the value of the parameter.

e. Find the expression of the distance from point $P(x_0, y_0, z_0)$ to plane: $ax+by+cz+d=0$

Content summary

The distance from point $P(x_0, y_0, z_0)$ to plane $\pi:\ ax+by+cz+d=0$ is given by

$$\frac{|ax_0+by_0+cz_0+d|}{\sqrt{a^2+b^2+c^2}}$$

In particular:

- If point P lies in the plane π, then the distance from P to the plane is zero.
- If two planes are parallel (their Cartesian equations can be expressed as $ax+by+cz+d=0; ax+by+cz+d'=0$), then to find the distance between them, it is sufficient to find the distance from a point in one of the two planes to the other plane.
- If a straight line is parallel to a plane, the direction vector $\vec{w}$ of the line can be expressed as $\vec{w}=t\vec{u}+s\vec{v}$ for some real values t and s, where $\vec{u}$ and $\vec{v}$ are direction vectors of the plane. Alternatively the dot product of $\vec{w}\cdot(\vec{u}\wedge\vec{v})=\vec{w}\cdot\vec{n}=0$ .In this case, to find the distance from the line to the plane, it is sufficient to find the distance from a point on the line to the plane.

Example

a. Find the distance from the point $P(2,4,7)$ to the plane $\alpha \equiv 3x+5y-6z=18$.

b. Show that planes $\pi_1: 3x-y+2z-6=0$ and $\pi_2: 6x-2y+4z+4=0$ are parallel and find the distance between them

c. Show that line $(l): \frac{x-2}{3}=\frac{y-1}{-2}=\frac{z-1}{5}$ is parallel to plane $\pi: x-y-z+2=0$ and find the distance from the line to the plane

Solution

a. The distance is $\frac{|3(2)+5(4)-6(7)-18|}{\sqrt{3^2+5^2+(-6)^2}}=\frac{34}{\sqrt{70}}$

b. The equation $6x-2y+4z+4=0$ can be re-organized as $3x-y+2z+2=0$, which shows that the two planes are parallel. A point in plane $6x-2y+4z+4=0$ is p(0,0,-1).The distance between the planes is $\frac{|3(0)-0+2(-1)-6|}{\sqrt{b^2-4ac}}=\frac{8}{\sqrt{14}}$

c. The direction vector of the line is $\vec{u}=\begin{pmatrix}3\\-2\\5\end{pmatrix}$ and a vector normal to the plane is $\vec{n}=\begin{pmatrix}1\\-1\\-1\end{pmatrix}$

We have $\vec{u}\cdot\vec{n}=3(1)-2(-1)+5(-1)=0$.Therefore; the line is parallel to the plane.

The distance from the line to the plane is $\frac{|2-1-1+2|}{\sqrt{1+1+1}}=\frac{2}{\sqrt{3}}=\frac{2\sqrt{3}}{3}$

APPLICATION ACTIVITY 2.3.2

a. Find the distance from the point $(2,-3,4)$ to the plane $x+2y+2z=13$

b. Find the distance from the point $(0,1,1)$ to the plane $4y+3z=-12$

c. Find the distance from the point $(0,-1,0)$ to the plane $2x+y+2z=4$

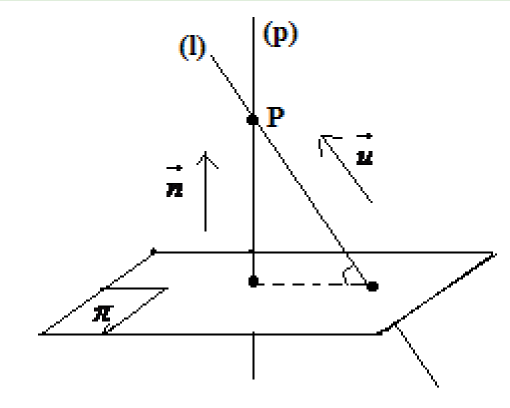
d. Find the shortest distance between the planes $x+2y+6z=1$ and $x+2y+6z=10$

e. Find the shortest distance between the planes $-2x+y+z=0$ and $6x-3y-3z-5=0$

2.3.3. Angles

ACTIVITY 2.3.3

Consider the diagram below to answer the questions:



a. Copy the diagram and plot the intersection point of line (l) and plane π and label it A

b. Plot the intersection point of the line (p), perpendicular to plane π through point P ,on line (l), and plane π, and label it H.

c. How is triangle AHP? Label θ the acute angle between line (l) and line (AH). θ is the angle between the line (l) and the plane π.

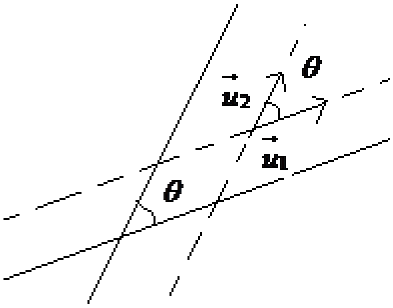
d. Express mathematically the relationship between the direction vector $\vec{n}$ of line (p,) direction vector $\vec{u}$ of line (l) and angle θ

e. Given that (l) passes through $P(x_0, y_0, z_0)$ and has direction vector $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$, and $n_1x + n_2y + n_3z + d = 0$, find the value of $\sin\theta$.

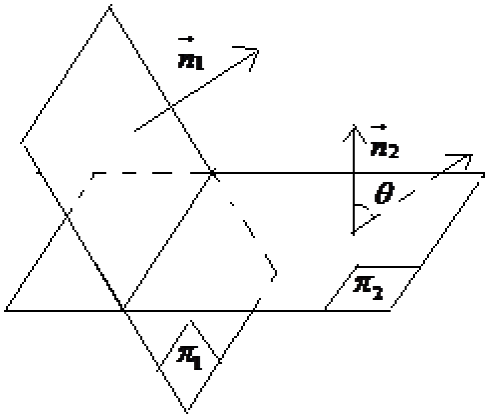
Content summary

The angle θ between the line with direction vector $\vec{u}$ and plane π with vector $\vec{n}$ normal to the plane is such that $\sin\theta = \frac{\vec{n}\cdot\vec{u}}{\|\vec{n}\|\|\vec{u}\|}$

The angle θ between two lines (l_1) and (l_2), with direction vectors $\vec{u}_1$ and $\vec{u}_2$, respectively, is such that $\cos\theta = \frac{\vec{u}_1 \cdot \vec{u}_2}{\|\vec{u}_1\|\|\vec{u}_2\|}$



The angle θ between two planes π_1 and π_2, with vectors $\vec{n}_1$ and $\vec{n}_2$ normal respectively to planes π_1 and π_2 is such that $\cos\theta = \frac{\vec{n}_1 \cdot \vec{n}_2}{\|\vec{n}_1\|\|\vec{n}_2\|}$



Example

a. Find the angle between the plane $x+y+z=4$ and the line

$$\begin{cases} x = 1 + r \\ y = 1 + 2r \\ z = 1 + 3r \end{cases}$$

b. Find the angle between the planes $x+y+z=4$ and $x+2y+3z=5$.

c. Find the angle between the lines $(l_1): \frac{x+1}{2} = \frac{y-1}{1} = \frac{z-1}{2}$ *and* $(l_2): \begin{cases} x = t \\ y = -5 - t \\ z = -1 \end{cases}$

Solution

a. $\vec{n}=(1,1,1)$ is normal to the plane and $\vec{u}=(1,2,3)$ is the direction vector of the line.

Let θ be the angle between that plane and that line. Then, $\sin\theta=\frac{\vec{n}\cdot\vec{u}}{\|\vec{n}\|\cdot\|\vec{u}\|}=\frac{6}{\sqrt{42}}$

$\theta = Arc\sin(\frac{6}{\sqrt{42}}) = 67.8^0$

b. Vectors normal to the planes are $\overrightarrow{n_1}=\begin{pmatrix}1\\1\\1\end{pmatrix}$ and $\overrightarrow{n_2}=\begin{pmatrix}1\\2\\3\end{pmatrix}$, respectively.

The angle θ between the planes is such that $\cos\theta=\frac{6}{\sqrt{42}}$; $\theta = Arc\cos\frac{6}{\sqrt{42}}$;

$\theta = 22.2^0$

c. The direction vectors of the lines are $\overrightarrow{u_1}=\begin{pmatrix}2\\1\\2\end{pmatrix}$ and $\overrightarrow{n_2}=\begin{pmatrix}1\\-1\\0\end{pmatrix}$

The angle θ between the lines is such that $\cos\theta=\frac{1}{3\sqrt{2}}=\frac{\sqrt{2}}{6}$; $\theta = Arc\cos\frac{\sqrt{2}}{6}$

APPLICATION ACTIVITY 2.3.3

1.Find the angle between the plane $8x+5y+9z=10$ and the line

$$\begin{cases}x=2-2t\\ y=1+4t\\ z=1+t\end{cases}$$

2.Find the angle between the planes $x+2y-6z=10$ and $2x-3y+4z=-15$

3.Find the angle between the line $\frac{x-2}{-1}=\frac{y-1}{1}=\frac{z-1}{2}$ and the plane $2x+y-z-1=0$

4.Find the angle between the planes $2x+2y+2z=3$, $2x-2y-z=5$

5.Find the angle between the planes $2x+2y-z=3$, $x+2y+z=2$

6.Find the angle between the planes $x+2y-2z=5$, $6x-3y+2z=8$

7.Determine the angle between the line $\frac{x-1}{2}=\frac{y+1}{1}=\frac{z}{2}$ and the plane $x+y-1=0$

2.4 END UNIT ASSESSMENT

1. Determine the vector equation of the line which is parallel to the vector $\vec{u}=2\vec{i}+3\vec{j}-\vec{k}$ and which passes through the point $A(1,1,1)$.

2. Find the vector equation of the line which passes through the point $A(-1,2,1)$ and which is parallel to the vector $\vec{u}=(1,2,3)$.

3. Show that the point with position vector $4\vec{i}-\vec{j}+12\vec{k}$ lies on the line with vector equation $x\vec{i}+y\vec{j}+z\vec{k}=2\vec{i}+3\vec{j}+4\vec{k}+r(\vec{i}-2\vec{j}+4\vec{k})$

4. If the point $A(a,b,3)$, lies on the line

$$L\equiv\begin{cases}x=2+r\\y=4+r\\z=-1+r\end{cases}$$

Find the value of a and b.

and the plane

$3x+y+z=7$

5. Given points $A(2,-1,1)$ and $B(5,2,-2)$. Find

   a. $\overrightarrow{AB}$

   b. The vector equation of the line that passes through A and B.

   c. Obtain the equation of the plane passing through the point $(-2,-2,2)$ and containing the line joining the points $(1,1,1)$ and $(1,-1,2)$.

6. Calculate the angle between the planes

$2x+3y+z=10$ and $x-2y-3z+12=0$

7. Find the angle between the lines:

a. $\frac{x}{2}=-(y+1)=\frac{z+1}{-2}$ and $x-6=y=\frac{1}{2}(z-1)$

b. $\vec{r}=\vec{i}-\vec{j}+t(\vec{i}+\vec{j}-\vec{k})$ and $\begin{cases} x=-2+3t \\ y=6t \\ z=1+2t \end{cases}$

8. Find the distance from point $P(1,5,-4)$ to plane $\pi: 3x-y+2z-6=0$

9. Find the Cartesian equations of the line with vector equations

a. $(x,y,z)=(2,3,-1)+r(2,3,1)$

b. $(x,y,z)=(3,-1,2)+r(3,2,-4)$

c. $(x,y,z)=(2,1,1)+r(2,-1,-1)$

10. Find the vector equation of the line with parametric equations

$$\begin{cases} x=2+3r \\ y=5-2r \\ z=4-r \end{cases}$$

11. Find the vector equations of the lines with the following symmetric equations

a. $\frac{x-2}{3}=\frac{y-2}{2}=\frac{z+1}{4}$

b. $x-3=\frac{y+2}{4}=\frac{z-3}{-1}$

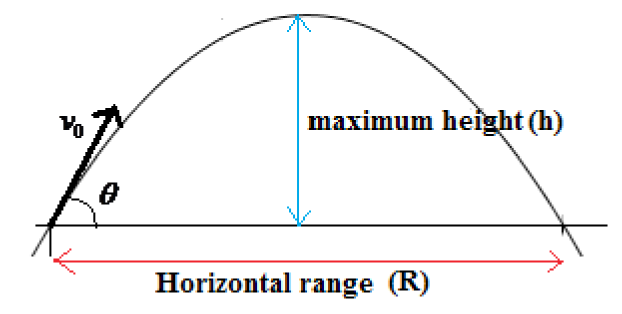
UNIT 3

TRIGONOMETRIC EQUATIONS

Key Unit Competence: Solve trigonometric equations and related problems using trigonometric identities and transformation formulas.

3.0. INTRODUCTORY ACTIVITY

Consider the diagram below to answer the questions:



The horizontal range of a projectile that is launched with an initial speed v_0 at an angle θ with the horizontal is given by $R = \frac{v_0^2 \sin 2\theta}{g}$, where g is the acceleration due to gravity(≈ 9.8 m/s2)

a. A projectile is launched with a speed of 9.8 m/s and it reaches a horizontal range of 4.9 meters? Use an equation to express this statement and find the angle θ

b. Use $\theta = 30^0$ to determine whether $\sin 2\theta = 2\sin\theta$ or not.

c. The maximum height attained by the projectile is given by

$$h = \frac{v^2 \sin^2 \theta}{2g}.$$

Determine the height attained by the particle for the angle obtained in part (a)

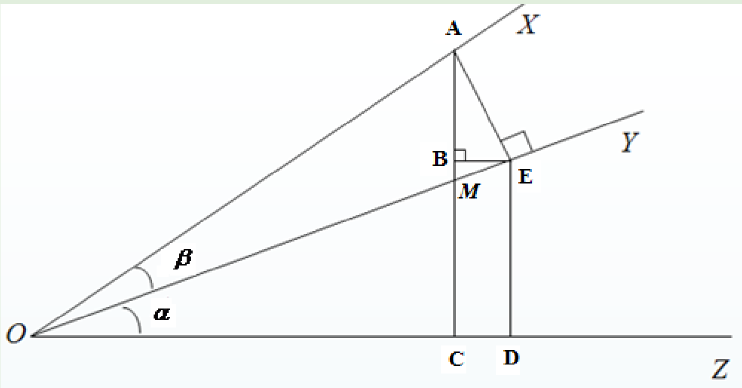
In this unit we will be introduced to modelling real life problems by trigonometric equations and solving them by different methods, including trigonometric transformations

3.1 Transformation formulas

3.1.1 Addition and subtraction formulas

ACTIVITY 3.1.1

Consider the diagram below to answer the questions:



a. in the right angled triangle ODE, $\sin\alpha = \frac{DE}{OE}$ and $\cos\alpha = \frac{OD}{OE}$

Complete:

i.In the right angled triangle OEA, $\sin\beta = \ldots$ and $\cos\beta = \ldots$

ii. In the right angled triangle OCA, $\sin(\alpha+\beta) = \ldots$ and $\cos(\alpha+\beta) = \ldots$

b. Complete:

i) Since lines (BE) and (OD) are parallel and line (OE) is a transversal,

$M\hat{E}B = \ldots$, $B\hat{E}A = \ldots$ and $B\hat{A}E = \ldots$

ii. In the right angled triangle EBA,the ratio $\frac{BA}{EA} = \ldots$

c. Answer by true or false:

- $CA = CB + BA$;
- $CB = DE$;
- $\frac{DE}{OA} = \frac{DE}{OE}\cdot\frac{OE}{OA}$
- $\frac{BA}{OA} = \frac{EA}{OA}\cdot\frac{BA}{EA}$

d. Express $\sin(\alpha+\beta)$ in terms of $\sin\alpha, \cos\alpha, \sin\beta, \cos\beta$

e. Use a method similar to the one above to derive the expression of $\cos(\alpha+\beta)$ in terms of $\sin\alpha, \cos\alpha, \sin\beta, \cos\beta$

f. Hence obtain an expression for $\sin(\alpha-\beta)$, $\cos(\alpha-\beta)$ in terms of

$\sin\alpha, \cos\alpha, \sin\beta, \cos\beta$.

g. Obtain an expression for $\tan(\alpha+\beta)$, $\tan(\alpha-\beta)$ in terms of $\tan\alpha, \tan\beta$

h. True or false:

In general, i. $\sin(\alpha+\beta)=\sin\alpha+\sin\beta$

ii. $\cos(\alpha+\beta)=\cos\alpha+\cos\beta$

iii. $\tan(\alpha+\beta)=\tan\alpha+\tan\beta$

Content summary

If α and β are the measures of angles, then the following are the identities used to express trigonometric ratios of compound angles $\alpha+\beta$ and $\alpha-\beta$ in terms of the trigonometric ratios of simple angles α and β:

- **Addition formulas:**

$$\sin(\alpha+\beta)=\sin\alpha\cos\beta+\sin\beta\cos\alpha$$

$$\cos(\alpha+\beta)=\cos\alpha\cos\beta-\sin\alpha\sin\beta$$

$$\tan(\alpha+\beta)=\frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}$$

- **Subtraction formulas:**

$$\sin(\alpha-\beta)=\sin\alpha\cos\beta-\sin\beta\cos\alpha$$

$$\cos(\alpha-\beta)=\cos\alpha\cos\beta+\sin\alpha\sin\beta$$

$$\tan(\alpha-\beta)=\frac{\tan\alpha-\tan\beta}{1+\tan\alpha\tan\beta}$$

Example

Use addition and subtraction formula to find, without using a calculator, the value, in rational form of: a. $\cos 75^0$

b. $\sin\frac{\pi}{12}$

c. $\tan\frac{5\pi}{3}$

Solution

$$a)cos75^0 = \cos\left(45^0 + 30^0\right)$$

$$= \cos 45^0 \cos 30^0 - \sin 45^0 \sin 30^0$$

$$= \frac{\sqrt{2}}{2}\frac{\sqrt{3}}{2} - \frac{\sqrt{2}}{2}\frac{1}{2}$$

$$= \frac{\sqrt{6}-\sqrt{2}}{4}$$

$$b)\sin\frac{\pi}{12} = \sin\left(\frac{\pi}{4}-\frac{\pi}{6}\right)$$

$$= \sin\frac{\pi}{4}\cos\frac{\pi}{6} - \cos\frac{\pi}{4}\sin\frac{\pi}{6}$$

$$= \frac{\sqrt{2}}{2}\frac{\sqrt{3}}{2} - \frac{\sqrt{2}}{2}\frac{1}{2} = \frac{\sqrt{6}-\sqrt{2}}{4}$$

$$\text{b)}\tan\frac{5\pi}{3} = \tan\left(2\pi - \frac{\pi}{3}\right)$$

$$= \frac{\tan 2\pi - \tan\frac{\pi}{3}}{1 + \tan 2\pi \tan\frac{\pi}{3}}$$

$$= \frac{0-\sqrt{3}}{1+0} = -\sqrt{3}$$

APPLICATION ACTIVITY 3.1.1

1. Simplify $2\sin\theta\sin 4\theta + 2\cos\theta\cos 4\theta$

2. Use addition and subtraction formulas to find:

a. $\sin 75^0$ b. $\cos\frac{13\pi}{6}$ c. $\tan 330^0$

3. Prove that $\frac{\cos 11^0 + \sin 11^0}{\cos 11^0 - \sin 11^0} = \tan 56^0$

4. Evaluate :

a. $\tan 75^0$ b. $\sin 15^0$

c. $\sin 47^0\cos 13^0 + \cos 47^0\sin 13^0$ d. $\cos 70^0\cos 10^0 + \sin 70^0\sin 10^0$

5. Given that $\sin A\cos B = 0.75$ and $\sin B\cos A = 0.25$,

a. Write down the value of $\sin(A+B)$ and the value of $\sin(A-B)$

b. Find acute angles A and B

3.1.2. Double angle formulas

ACTIVITY 3.1.2

a. In each of the formulas below, substitute θ for both α and β, and simplify where possible:

i. $\sin(\alpha+\beta) = \sin\alpha\cos\beta + \sin\beta\cos\alpha$

ii. $\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$

iii. $\tan(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1-\tan\alpha\tan\beta}$

b. Find the expressions for $\sin 2\theta, \cos 2\theta$ and $\tan 2\theta$ in terms of $\sin\theta$, $\cos\theta$ and $\tan\theta$

c. True or false:

In general, i. $\sin 2\theta = 2\sin\theta$

ii. $\cos 2\theta = 2\cos\theta$

iii. $\tan 2\theta = 2\tan\theta$

Content summary

If θ is the measure of an angle, then the following are the identities used to express trigonometric ratios of double angle 2θ in terms of the trigonometric ratios of simple angle θ:

Double angle formulas:

$\sin 2\theta = 2\sin\theta\cos\theta$

$\cos 2\theta = \cos^2\theta - \sin^2\theta$

$\tan 2\theta = \dfrac{2\tan\theta}{1-\tan^2\theta}$

In the expression $\cos 2\theta = \cos^2\theta - \sin^2\theta$, substituting $\sin^2\theta$ by $1-\cos^2\theta$, we get $\cos 2\theta = 2\cos^2\theta - 1$

In the expression $\cos 2\theta = \cos^2\theta - \sin^2\theta$, substituting $\cos^2\theta$ by $1-\sin^2\theta$, we get $\cos 2\theta = 1 - 2\sin^2\theta$

Example

a. Express $\cos 4x$ in function of $\sin x$ only

b. Given that $\tan x = \dfrac{a}{b}$ and $\pi \le x \le \dfrac{3\pi}{2}$, evaluate :

i. $\sin 2x$

ii. $\tan 2x$

Solution

a. $\cos 4x = \cos 2(2x) = 1 - 2\sin^2 2x$

$= 1 - 2(2\sin x\cos x)^2$

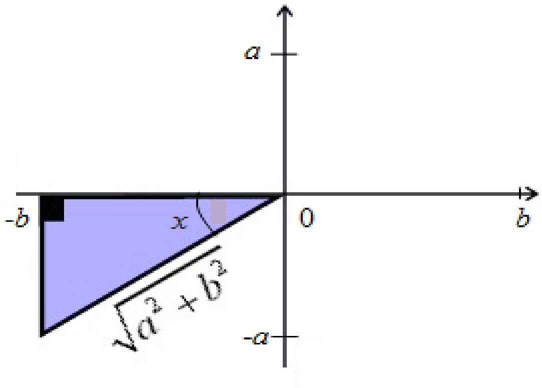
$= 1 - 2(4\sin^2 x\cos^2 x)$

$= 1 - 8\sin^2 x\cos^2 x$

$= 1 - 8\sin^2 x(1-\sin^2 x)$

$= 1 - 8\sin^2 x + 8\sin^4 x$

b . The given information is shown on the diagram below.



i. $\sin 2x = 2\sin x \cos x$

$$= 2 \cdot \frac{-a}{\sqrt{a^2+b^2}} \cdot \frac{-b}{\sqrt{a^2+b^2}}$$

$$= \frac{2ab}{a^2+b^2}$$

ii. $\tan 2x = \dfrac{2\tan x}{1-\tan^2 x}$

$$= \frac{2\frac{a}{b}}{1-\left(\frac{a}{b}\right)^2}$$

$$= \frac{2ab}{b^2-a^2}$$

APPLICATION ACTIVITY 3.1.2

1.Express $\sin 4x$ in terms of $\sin x$ and $\cos x$

2.Express $\cos 8x$ in terms of $\sin x$

3.Find, without using a calculator, the exact value of $2\sin 15^0 \cos 15^0$

4.If $\sin A = \frac{\sqrt{5}}{5}$, find $\sin 2A, \cos 2A$ and $\tan 2A$ given that:

a. A is acute
b. A is obtuse

3.1.3. Half angle formulas

ACTIVITY 3.1.3

a. In each of the formulas below, substitute θ for $\frac{\gamma}{2}$, and simplify where possible

$$\cos 2\theta = 1 - 2\sin^2 \theta$$

$$\cos 2\theta = 2\cos^2 \theta - 1$$

b. Make $\sin\frac{\gamma}{2}$,and $\cos\frac{\gamma}{2}$ the subject of the formula and obtain the expression for $\tan\frac{\gamma}{2}$

c. Find the expressions for $\sin\frac{\gamma}{2}$, $\cos\frac{\gamma}{2}$ and $\tan\frac{\gamma}{2}$ in terms of $\sin\gamma$, $\cos\gamma$ and $\tan\gamma$

d. Say if the given equations are True or false:

In general, i. $\sin\frac{\gamma}{2} = \frac{1}{2}\sin\gamma$

ii. $\cos\frac{\gamma}{2} = \frac{1}{2}\cos\gamma$

iii. $\tan\frac{\gamma}{2} = \frac{1}{2}\tan\gamma$

Content summary

If θ is the measure of an angle, then the following are the identities used to express trigonometric ratios of half angle $\frac{\theta}{2}$ in terms of the trigonometric ratios of simple angle θ:

Half angle formulas:

$$\sin\frac{\theta}{2} = \pm\sqrt{\frac{1-\cos\theta}{2}}$$

$$\cos\frac{\theta}{2} = \pm\sqrt{\frac{1+\cos\theta}{2}}$$

$$\tan\frac{\theta}{2} = \pm\sqrt{\frac{1-\cos\theta}{1+\cos\theta}}\text{, where } \cos\theta \neq -1$$

The sign is chosen taking into account the quadrant in which $\frac{\theta}{2}$ lies.

Example

a. Using the half angle formula, find the exact value of $\cos 15^0$

b. Given that $\sin\theta = \frac{24}{25}$ and $180^0 < \theta < 270^0$, find $\sin\frac{\theta}{2}$

c. Use the change of variable $\tan\frac{x}{2} = t$ to express $\sin x$, $\cos x$ and $\tan x$ in terms of t

Solution

a. 15^0 is in first quadrant, then $\cos 15^0$ must be positive

$$\cos 15^0 = \cos\left(\frac{1}{2}\left(30^0\right)\right)$$

$$= \sqrt{\frac{1+\cos 30^0}{2}}$$

$$= \sqrt{\frac{1+\frac{\sqrt{3}}{2}}{2}}$$

$$= \frac{\sqrt{2+\sqrt{3}}}{2}$$

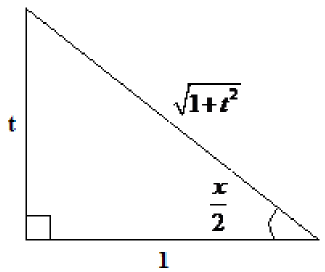
b. $\cos^2\theta = 1-\sin^2\theta = 1-(\frac{24}{25})^2 = \frac{49}{625}$; $\cos\theta = -\frac{7}{25}$, since , for $180^0 < \theta < 270^0$,

$\cos\theta < 0$

Now, $180^0 < \theta < 270^0$ implies that $90^0 < \frac{\theta}{2} < 135^0$; $\sin\frac{\theta}{2} > 0$

$$\sin\frac{\theta}{2} = \sqrt{\frac{1-\cos\theta}{2}} = \sqrt{\frac{1-(-\frac{7}{25})}{2}} = \frac{4}{5}$$

c. Consider the diagram below:



$$\tan\frac{x}{2} = \frac{opp}{adj} = \frac{t}{1}; \sin\frac{x}{2} = \frac{opp}{hyp} = \frac{t}{\sqrt{1+t^2}}; \cos\frac{x}{2} = \frac{adj}{hyp} = \frac{1}{\sqrt{1+t^2}}$$

From $\sin x = 2\sin\frac{x}{2}\cos\frac{x}{2}$ and $\cos x = \cos^2\frac{x}{2} - \sin^2\frac{x}{2}$, we have:

$$\sin x = 2(\frac{t}{\sqrt{1+t^2}})(\frac{1}{\sqrt{1+t^2}}) = \frac{2t}{1+t^2}$$

$$\cos x = (\frac{1}{\sqrt{1+t^2}})^2 - (\frac{t}{\sqrt{1+t^2}})^2 = \frac{1-t^2}{1+t^2}$$

$$\tan x = \frac{\sin x}{\cos x} = \frac{2t}{1-t^2}$$

APPLICATION ACTIVITY 3.1.3

1.Use the change of variable $\tan\frac{x}{2}=t$ to simplify completely the expression:

$$\frac{\sin x+\cos x+1}{\text{sinx}-\cos x+1}$$

2.Given that $3\sin x+4\cos x=5$, find the values of $\cos x$ and $\sin x$

3.If $\cos A=-\frac{7}{25}$, find the values of $\sin\frac{1}{2}A$, $\cos\frac{1}{2}A$ and $\tan\frac{1}{2}A$

4.If $\tan 2A=\frac{7}{24}$, $0<A<\frac{\pi}{4}$, find the value of $\tan A$

5.Find the value of $\sin 22\frac{1}{2}^{0}$, $\cos 22\frac{1}{2}^{0}$ and $\tan 22\frac{1}{2}^{0}$

3.1.4. Transformation of product to sum (or difference)

ACTIVITY 3.1.4

a. Evaluate each of the following, simplify where possible:

i. $\sin(\alpha+\beta)+\sin(\alpha-\beta)$

ii. $\sin(\alpha+\beta)-\sin(\alpha-\beta)$

iii. $\cos(\alpha+\beta)+\cos(\alpha-\beta)$

iv. $\cos(\alpha+\beta)-\cos(\alpha-\beta)$

b. Hence, express each of the following product as a sum or difference of trigonometric expressions:

i. $\sin\alpha\cos\beta$

ii. $\sin\alpha\sin\beta$

iii. $\cos\alpha\cos\beta$

c. Say if the given equations are True or false:

In general, i. $\sin\alpha\sin\beta=\sin^2\alpha\beta$

ii. $\cos\alpha\cos\beta=\cos^2\alpha\beta$

Content summary

If α and β are the measures of angle, then the following are the identities used to express a product of trigonometric ratios as sum or difference of trigonometric ratios

Product to sum formulas:

- $\cos\alpha\cos\beta = \frac{1}{2}[\cos(\alpha+\beta)+\cos(\alpha-\beta)]$
- $\sin\alpha\sin\beta = -\frac{1}{2}[\cos(\alpha+\beta)-\cos(\alpha-\beta)]$
- $\sin\alpha\cos\beta = \frac{1}{2}[\sin(\alpha+\beta)+\sin(\alpha-\beta)]$

Example

Express each of the following product as a sum or difference of sines or cosines:

i. $\sin 3x\cos 5x$

ii. $\sin 6x\sin 4x$

iii. $\cos 2x\cos 4x$

Solution

i. $\sin 3x\cos 5x = \frac{1}{2}[\sin(3x+5x)+\sin(3x-5x)]$

$$= \frac{1}{2}(\sin 8x - \sin 2x)$$

$$= \frac{1}{2}\sin 8x - \frac{1}{2}\sin 2x$$

ii. $\sin 6x\sin 4x = -\frac{1}{2}[\cos(6x+4x)-\cos(6x-4x)]$

$$= -\frac{1}{2}(\cos 10x \text{-} \cos 2x)$$

$$= -\frac{1}{2}(\cos 10x + \frac{1}{2}\cos 2x)$$

iii. $\cos 2x\cos 4x = \frac{1}{2}[\cos(2x+4x)+\cos(2x\text{-}4x)]$

$= \frac{1}{2}\ (\cos 6x + \cos 2x)$

$= \frac{1}{2}\cos 6x + \frac{1}{2}\cos 2x)$

APPLICATION ACTIVITY 3.1.4

1.Find, without using a calculator, the exact value of :

$\sin\frac{\pi}{12}\sin\frac{5\pi}{12}$

$\cos\frac{5\pi}{12}\sin\frac{11\pi}{12}$

$\cos\frac{\pi}{12}\cos\frac{7\pi}{12}$

2. Express each of the following product as a sum or difference of sines or cosines:

i. $\sin x + \sin 3x$

ii. $\cos 6x + \cos 4x$

iii. $\cos 2x + \sin 4x$

iv. $\sin x\cos 3x$

v. $\cos\frac{5x}{2}\cos\frac{3x}{2}$

3.1.5. Transformation of sum (or difference) to product

ACTIVITY 3.1.5

a. Evaluate each of the following, simplify where possible:

i. $\sin(\alpha+\beta)+\sin(\alpha-\beta)$

ii. $\sin(\alpha+\beta)-\sin(\alpha-\beta)$

iii. $\cos(\alpha+\beta)+\cos(\alpha-\beta)$

iv. $\cos(\alpha+\beta)-\cos(\alpha-\beta)$

b. Let $\begin{cases}\alpha+\beta=P\\ \alpha-\beta=Q\end{cases}$.Express α and β in terms of P and Q.Hence, express each of the following sums or differences as a product of sines or cosines

i. $\sin P+\sin Q$

ii. $\cos P+\cos Q$

iii. $\cos P-\cos Q$

c. Say if the given inequalities areTrue or false:

In general, i. $\sin\alpha+\sin\beta=\sin(\alpha+\beta)$

ii. $\cos\alpha+\cos\beta=\cos(\alpha+\beta)$

iii. $\sin\alpha-\sin\beta=\sin(\alpha-\beta)$

iv. $\cos\alpha-\cos\beta=\cos(\alpha-\beta)$

Content summary

If α and β are the measures of angle, then the following are the identities used to express a sum or a difference of sinus or cosines as product of sinus or cosines

Sum (or difference) to product formulas:

- $\sin P+\sin Q=2\sin\frac{P+Q}{2}\cos\frac{P-Q}{2}$
- $\cos P+\cos Q=2\cos\frac{P+Q}{2}\cos\frac{P-Q}{2}$

- $\cos P - \cos Q = -2\sin\frac{P+Q}{2}\sin\frac{P-Q}{2}$

Example

a. Transform each of the following to a product:

i. $\sin 3x + \sin 4x$

ii. $\cos 3x + \cos 2x$

iii. $\cos 6x - \cos 2x$

iv. $\sin 5x - \sin 3x$

b. Show that $\frac{\sin 4x + \sin 2x}{\cos 4x + \cos 2x} = \tan 3x$

Solution

a. i. $\sin 3x + \sin 4x = 2\sin\frac{3x+4x}{2}\cos\frac{3x-4x}{2}$

$$= 2\sin\frac{7x}{2}\cos\frac{x}{2}$$

ii. $\cos 3x + \cos 2x = 2\cos\frac{3x+2x}{2}\cos\frac{3x-2x}{2}$

$$= 2\cos\frac{5x}{2}\cos\frac{x}{2}$$

iii. $\cos 6x - \cos 2x = -2\sin\frac{6x+2x}{2}\sin\frac{6x-2x}{2}$

$$= -2\sin 4x \sin 2x$$

iv. $\sin 5x - \sin 3x = \sin 5x + \sin(-3x) = 2\sin\frac{5x-3x}{2}\cos\frac{5x+3x}{2}$

$$= 2\sin x\cos 4x$$

b. $\frac{\sin 4x + \sin 2x}{\cos 4x + \cos 2x} = \frac{2\sin 3x\cos x}{2\cos 3x\cos x}$

$$= \frac{\sin 3x}{\cos 3x}$$

$$= \tan 3x$$

APPLICATION ACTIVITY 3.1.5

1. Show that $\dfrac{\sin 3x+\sin 5x}{\cos 5x-\cos 3x}=-\cot x$
2. factorize completely $\cos x-\cos 3x-\sin 2x$
3. Simplify completely, assume the denominator not zero:

   $\dfrac{\sin 3x-\sin 2x+\sin x}{\cos 3x-\cos 2x+\cos x}$
4. Transform each of the following to a product:

   i. $\sin 3\mathrm{x}+\sin\mathrm{x}$

   ii. $\cos\mathrm{x}+\cos 7\mathrm{x}$

   iii. $\cos 2\mathrm{x}-\cos 4\mathrm{x}$

   iv. $\sin 4\mathrm{x}-\sin 9\mathrm{x}$

5.Show that:

   a. $\dfrac{\sin\mathrm{A}+\sin\mathrm{B}}{\cos\mathrm{A}+\cos\mathrm{B}}=\tan\dfrac{A+B}{2}$

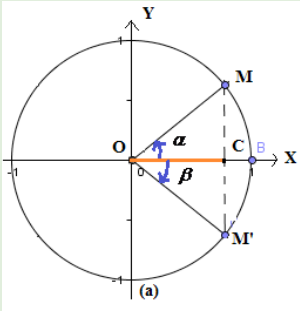
   b. $\dfrac{\cos 2B-\mathrm{s}\cos 2\mathrm{A}}{\sin 2A+\sin 2B}=\tan(A-B)$

3. 2. Trigonometric equations

3.2.1. Solving simple trigonometric equations

ACTIVITY 3.2.1

1.Consider the diagram below to answer the questions:



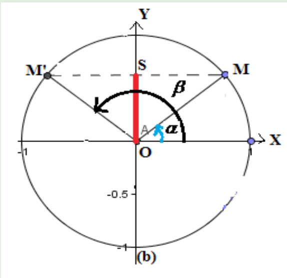
a. Compare $\cos\alpha$ and $\cos\beta$

b. How are the angles α and β ? Express β in terms of α

c. Write down the general expression of angles co terminal to α , and the general expression of angles co terminal to β ,

d. Discuss , with respect to parameter ***a*** the general solution of the trigonometric equation $\cos x = a$

2. Consider the diagram below to answer the questions:



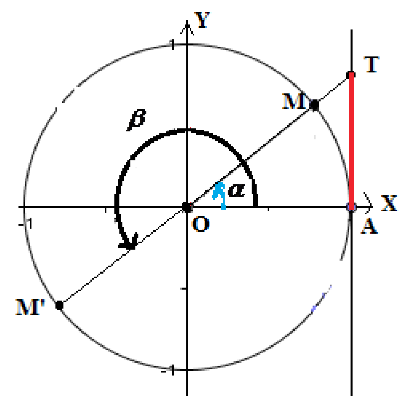
a. Compare $\sin\alpha$ and $\sin\beta$

b. How are the angles α and β ? Express β in terms of α

c. Write down the general expression of angles co terminal to α , and the general expression of angles co terminal to β ,

d. Discuss , with respect to parameter ***a*** the general solution of the trigonometric equation $\sin x = a$

3. Consider the diagrams below to answer the questions:



(c)

a. Compare $\tan\alpha$ and $\tan\beta$

b. How is β obtained from angle α ? Express β in terms of α

c. Write down the general expression of angle β in terms of α ,

d. Write down the general solution of the trigonometric equation

$\tan x = a$

Content summary

Simple trigonometric equations are of the types:

$\cos X = a$; $\sin X = a$, where $-1 \le a \le 1$,and $\tan X = a$, where a is any real number and X is expressed in radians.

We solve such equations in the following ways:

- $\cos X = a$

  i. Determine a particular angle Y such that $\cos Y = a$

  ii. The equation becomes $\cos X = \cos Y$

  iii. Then $X = Y + 2k\pi$ or $X = -Y + 2k\pi$, where k is any integer.

  iv. Solve for the unknown and write down the **general solutions**

- $\sin X = a$

  i. Determine a particular angle Y such that $\sin Y = a$

  ii. The equation becomes $\sin X = \sin Y$

  iii. Then $X = Y + 2k\pi$ or $X = \pi - Y + 2k\pi$, where k is any integer.

  iv. Solve for the unknown and write down the **general solutions**

- $\tan X = a$

  i. Determine a particular angle Y such that $\tan Y = a$

  ii. The equation becomes $\tan X = \tan Y$

  iii. Then $X = Y + k\pi$, where k is any integer.

  iv. Solve for the unknown and write down the **general solutions**

If X is expressed in degrees, then

For, $\cos X = a$,the general solution is $X = Y + 360^0 k$ or $X = -Y + 360^0 k$, where k is any integer.

For, $\sin X = a$, the general solution is $X = Y + 360^0 k$ or $X = 180^0 - Y + 360^0 k$,

where k is any integer.

For, $\tan X = a$, the general solution is $X = Y + 180^0 k$,where k is any integer.

Note: In the same expression, do not mix the units.

The solution within a specific range is called a **particular solution**; it is obtained from the general solution by giving to k all possible integral values such that the resulting value of the unknown falls in the given range.

A particular solution x is said to be **principal value** if and only if x lies in the interval shown on the table below;

Equation	Principal value
$\cos X = a$	$0 \le X \le \pi$
$\sin X = a$	$-\frac{\pi}{2} \le X \le \frac{\pi}{2}$
$\tan X = a$	$-\frac{\pi}{2} \le X \le \frac{\pi}{2}$

Example

a. Find the general solution of each of the following equations:

i. $\cos 3x = \frac{1}{2}$

ii. $2\sin\frac{x}{2} - \sqrt{2} = 0$

iii. $\tan 3\text{x} = \tan(\text{x} + \frac{\pi}{4})$

iv. $\sin x - \sin(3x - \frac{\pi}{3}) = 0$

b. Find the particular solution of each of the following equations in the corresponding interval:

i. $2\cos x + \sqrt{3} = 0 ; [0, 2\pi]$

ii. $2\sin 2x - 1 = 0;\ [0, 2\pi]$

Solution

a. i. $\cos 3x = \frac{1}{2} = \cos\frac{\pi}{6}$

$\Leftrightarrow 3x = \frac{\pi}{6} + 2k\pi$ or $3x = -\frac{\pi}{6} + 2k\pi$,where $k \in \mathbb{Z}$

$\Leftrightarrow x = \frac{\pi}{18} + \frac{2k\pi}{3}$ or $x = -\frac{\pi}{18} + \frac{2k\pi}{3}$:general solution

ii. $2\sin\frac{x}{2} - \sqrt{2} = 0 \Leftrightarrow \sin\frac{x}{2} = \frac{\sqrt{2}}{2} = \sin\frac{\pi}{4}$

$\Leftrightarrow \frac{x}{2} = \frac{\pi}{4} + 2k\pi$ or $\frac{x}{2} = \pi - \frac{\pi}{4} + 2k\pi$, where $k \in \mathbb{Z}$

$\Leftrightarrow x = \frac{\pi}{2} + 4k\pi$ or $x = \frac{3\pi}{2} + 4k\pi$

iii. $\tan 3\text{x} = \tan(\text{x} + \frac{\pi}{4}) \Leftrightarrow 3\text{x} = \text{x} + \frac{\pi}{4} + k\pi$, where $k \in \mathbb{Z}$

$\Leftrightarrow x = \frac{\pi}{8} + \frac{k\pi}{2}$: general solution

iv. $\sin x - \sin(3x - \frac{\pi}{3}) = 0 \Leftrightarrow \sin(3x - \frac{\pi}{3}) = \sin x$

$$\Leftrightarrow 3x - \frac{\pi}{3} = x + 2k\pi \text{ or } 3x - \frac{\pi}{3} = \pi - x + 2k\pi$$

(1) For $3x - \frac{\pi}{3} = x + 2k\pi$, we have: $2x = \frac{\pi}{3} + 2k\pi \Leftrightarrow x = \frac{\pi}{6} + k\pi$

(2) For $3x - \frac{\pi}{3} = \pi - x + 2k\pi$, we have: $4x = \frac{4\pi}{3} + 2k\pi \Leftrightarrow x = \frac{\pi}{3} + \frac{k\pi}{2}$

The solution set is $S = \left\{ \left\{ \frac{\pi}{6} + k\pi; k \in \mathbb{Z} \right\} \cup \left\{ \frac{\pi}{3} + \frac{k\pi}{2}; k \in \mathbb{Z} \right\} \right\}$

b. i. $2\cos x + \sqrt{3} = 0 \Leftrightarrow \cos x = -\frac{\sqrt{3}}{2} = -\cos\frac{\pi}{6} = \cos(\pi - \frac{\pi}{6}) = \cos\frac{5\pi}{6}$
$\Leftrightarrow x = \frac{5\pi}{6} + 2k\pi$ or $x = -\frac{5\pi}{6} + 2k\pi$, where $k \in \mathbb{Z}$

(1) $0 \le x = \frac{5\pi}{6} + 2k\pi \le 2\pi \Leftrightarrow k = 0$;it follows that $x_1 = \frac{5\pi}{6}$

(2) $0 \le x = -\frac{5\pi}{6} + 2k\pi \le 0 \Leftrightarrow k = 1$;it follows that $x_2 = \frac{7\pi}{6}$

The solution set is $S = \left\{ \frac{5\pi}{6}; \frac{7\pi}{6} \right\}$.

ii. $2\sin 2x - 1 = 0 \Leftrightarrow \sin 2x = \frac{1}{2} = \sin\frac{\pi}{6}$

$$\Leftrightarrow 2x = \frac{\pi}{6} + 2k\pi \text{ or } 2x = \frac{5\pi}{6} + 2k\pi$$

$$\Leftrightarrow x = \frac{\pi}{12} + k\pi \text{ or } x = \frac{5\pi}{12} + k\pi$$

(1) $0 \le \frac{\pi}{12} + k\pi \le 2\pi \Leftrightarrow k = 0$ or $k = 1$.Therefore, $x_1 = \frac{\pi}{12}$; $x_2 = \frac{13\pi}{12}$

(2) $0 \le \frac{5\pi}{12} + k\pi \le 2\pi \Leftrightarrow k = 0$ or $k = 1$.Therefore, $x_3 = \frac{5\pi}{12}$; $x_4 = \frac{17\pi}{12}$

The solution set is $S=\left\{\frac{\pi}{12};\frac{5\pi}{12};\frac{13\pi}{12};\frac{7\pi}{12}\right\}$

APPLICATION ACTIVITY 3.2.1

1.Find the principal solutions of the following equations:

a. $\sin x=\frac{\sqrt{3}}{2}$ b. $\tan x=-\sqrt{3}$ c. $\cot x=\frac{1}{\sqrt{3}}$

d. $\sqrt{3}\tan x=1$ e. $2\sin x-\sqrt{3}=0$

2.Find the general solutions of the following equations:

a. $\sqrt{2}-\sec 3x=0$ b. $3\cot 2x-\sqrt{3}=0$ c. $\sin 3x=\sin(\frac{\pi}{6}-x)$

d. $2\cos(4x-\frac{\pi}{5})-\sqrt{3}=0$ e. $\cos(2x+\frac{\pi}{3})-\cos(5x-\frac{\pi}{4})=0$

f. $\sqrt{3}\sec x+2=0$ g. $\tan x+\sqrt{3}=0$

3.2.2. Solving trigonometric equations by using trigonometric formulas

ACTIVITY 3.2.2

1. Angle β is such that $\cos 2\beta=\sin 3\beta$

   a. Complete $2\beta+3\beta=5\beta=...$

   b. Find acute angle β

   c. Determine the general solution of the equation $\cos 2\beta=\sin 3\beta$

2.Consider the trigonometric equation $5\cos x-2\sin x-2=0$

   a. Use the change of variable $\tan\frac{x}{2}=t$ to express the equation in terms of t.

   b. Solve for t the equation obtained in part (a)

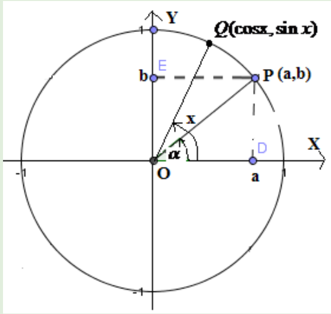
   c. Find the general solution of the equation $5\cos x-2\sin x-2=0$

3. Consider the trigonometric equation $\cos 2\theta + 3 = 5\cos\theta$

a. Express $\cos 2\theta$ in terms of $\cos\theta$

b. Determine the general solution of the equation $\cos 2\theta + 3 = 5\cos\theta$

4.Use the diagram below to answer the questions that follow:



a. In the right angled triangle ODP, write down the expression of $\cos\alpha$ and $\sin\alpha$

b. We have: $\overrightarrow{OP} = \begin{pmatrix} a \\ b \end{pmatrix}$ and $\overrightarrow{OQ} = \begin{pmatrix} \cos x \\ \sin x \end{pmatrix}$.Determine the dot product $\overrightarrow{OP} \cdot \overrightarrow{OQ}$ and the magnitude of vectors $\overrightarrow{OP} = \begin{pmatrix} a \\ b \end{pmatrix}$ and $\overrightarrow{OQ} = \begin{pmatrix} \cos x \\ \sin x \end{pmatrix}$

c. Use part (a) to determine $\cos(x - \alpha)$

d. Hence, find another expression for $a\cos x + b\sin x$ and solve the equation

$a\cos x + b\sin x = c$

Content summary

- Algebraic and trigonometric transformations can be used to express a trigonometric equation as a product of simple trigonometric factors, and by using the product rule the solutions of the proposed equation are found.
- To solve trigonometric equations involving different trigonometric functions, express all the trigonometric functions in terms of only one trigonometric function, avoiding radical expressions. Sometimes, this may lead to a trigonometric equation, quadratic in form, which is easy to solve.
- Sometimes, it may be necessary to change the variable in order to transform the equation to a form easy to solve.

The equation $a\cos x + b\sin x = c$ can be brought to a simple case by expressing it as $\sqrt{a^2+b^2}\cos(x-\alpha) = c$, where $\cos\alpha = \frac{a}{\sqrt{a^2+b^2}}$ and $\sin\alpha = \frac{b}{\sqrt{a^2+b^2}}$

Example

Find the general solution of each of the following equations:

a. $4\cos^3\frac{x}{2} - 3\cos\frac{x}{2} = 0$

b. $3\cos x + 3 = 2\sin^2 x$

c. $\cos x + \sin x - 1 = 0$

Solution

a. $\cos\frac{x}{2} = t$ .The equation becomes $4t^3 - 3t = 0$, which is equivalent to

$t(2t-\sqrt{3})(2t+\sqrt{3}) = 0$

Then $t = 0$ or $t = \frac{\sqrt{3}}{2}$ or $t = -\frac{\sqrt{3}}{2}$

(1) $\cos\frac{x}{2} = 0 = \cos\frac{\pi}{2}$

$$\Leftrightarrow \frac{x}{2} = \frac{\pi}{2} + 2k\pi \text{ or } \frac{x}{2} = -\frac{\pi}{2} + 2k\pi$$

$$\Leftrightarrow x = \pi + 4k\pi \text{ or } x = -\pi + 4k\pi \text{ ,where } k \in \mathbb{Z}$$

(2) $\cos\frac{x}{2} = \frac{\sqrt{3}}{2} = \cos\frac{\pi}{6}$

$$\Leftrightarrow \frac{x}{2} = \frac{\pi}{6} + 2k\pi \text{ or } \frac{x}{2} = -\frac{\pi}{6} + 2k\pi$$

$$\Leftrightarrow x = \frac{\pi}{3} + 4k\pi \text{ or } x = -\frac{\pi}{3} + 4k\pi$$

,where $k \in \mathbb{Z}$

(3) $\cos\frac{x}{2}=-\frac{\sqrt{3}}{2}=-\cos\frac{\pi}{6}=\cos(\pi-\frac{\pi}{6})=\cos\frac{5\pi}{6}$

$\Leftrightarrow \frac{x}{2}=\frac{5\pi}{6}+2k\pi$ or $\frac{x}{2}=-\frac{5\pi}{6}+2k\pi$

$\Leftrightarrow x=\frac{5\pi}{3}+4k\pi$ or $x=-\frac{5\pi}{3}+4k\pi$

where $k\in\mathbb{Z}$

b. $3\cos x+3=2\sin^2 x \Leftrightarrow 3\cos x+3=2(1-\cos^2 x) \Leftrightarrow 2\cos^2 x+3\cos x+1=0$

$\Leftrightarrow \cos x=-\frac{1}{2}$ or $\cos x=-1$

$\Leftrightarrow x=\frac{2\pi}{3}+2k\pi$ or $x=-\frac{2\pi}{3}+2k\pi$ or $x=\pi+2k\pi$, where $k\in\mathbb{Z}$

c. The equation $\cos x+\sin x=1$ can be expressed as $\sqrt{2}\cos(x-\frac{\pi}{4})=1$

or $\cos(x-\frac{\pi}{4})=\frac{\sqrt{2}}{2}=\cos\frac{\pi}{4}$. This is equivalent to: $x-\frac{\pi}{4}=\frac{\pi}{4}+2k\pi$ or

$x-\frac{\pi}{4}=-\frac{\pi}{4}+2k\pi$

Therefore, $x=\frac{\pi}{2}+2k\pi$ or $x=2k\pi$, where $k\in\mathbb{Z}$

APPLICATION ACTIVITY 3.2.2

1.Find the particular solutions of the following equations:

a. $\sqrt{3}\sin\frac{x}{2}+\cos\frac{x}{2}+\sqrt{3}=0$, where $0<x<2\pi$

b. $\tan^2 x=2\cos^2 x$, where $-\frac{\pi}{2}\le x\le\frac{\pi}{2}$

c. $\sin\theta\cos\theta=-\frac{1}{2}$, where $0<x<2\pi$

d. $\cos\theta\cos 30^\circ-\sin\theta\sin 30^\circ=\frac{1}{2}$ for $-180^\circ<\theta<180^\circ$

e. $\sin^2 x+\sin x\cos x=0$ for $0^0\le x\le 360^0$

2. Find the general solution of the equation:

a. $\cos\left(\theta+60^0\right)=\sin\theta$

b. $\sin(x\text{-}30^0)\sin(x+30^0)=\frac{1}{2}$

c. $2\sin^2 x\tan x\text{-}\tan x=0$

d. $\sqrt{3}\tan x=2\sin x$

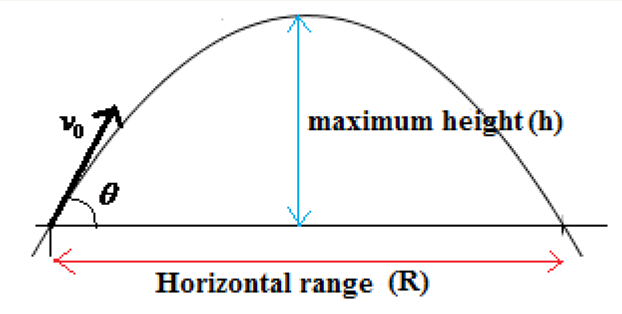
e. $\cot^2 3x-1=0$

f. $(1+\tan 2x)(1-\cot x)=0$

3. 3. Applications of trigonometric equations

ACTIVITY 3.3

Consider the diagram below to answer the questions:



The horizontal range of a projectile that is launched with an initial speed v_0 at an angle θ with the horizontal is given by $R=\frac{v_0{}^2\sin 2\theta}{g}$, where g is the acceleration due to gravity $\left(\approx 9.8m/s^2\right)$.

A projectile is launched with a speed of 9.8 m/s and it reaches a horizontal range of 4.9 meters? Use an equation to express this statement and find the angle θ

b. The maximum height attained by the projectile is given by $h=\frac{v^2\sin^2\theta}{2g}$

Determine the height attained by the particle for the angle obtained in part (a)

Content summary

Different problems from real life can be modelled by trigonometric equations.

These problems include land surveying and measuring, projectiles, simple harmonic motion, refraction of light, alternating current, and so forth.

Example 1

Points A, B and C are the corners of a right triangular field, where the angle at vertex B is 90^0 and AC=60 meters. Point D lies on side AC and point E lies on side AB. A straight path joins D to E so that, in triangle ADE, the angle at vertex D is 90^0, and AD=BE=10 meters.

Let in triangle ABC, the angle at vertex A be denoted by x.

a. Express, in terms of x, the lengths AE and AB

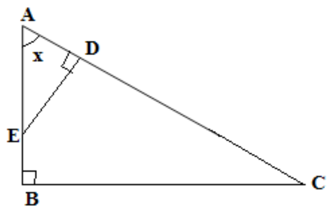
b. Show that $6\cos^2 x - \cos x - 1 = 0$ .Then solve the equation

$6\cos^2 x - \cos x - 1 = 0$, where $0^0 < x < 360^0$

c. Calculate the lengths, in meters, of paths ED and BD, to one decimal point.

Solution

a.



In triangle ADE,

$$\cos x = \frac{AD}{AE}$$

$$AE = \frac{AD}{\cos x} = \frac{10}{\cos x}$$

In triangle ABC,

$$\cos x = \frac{AB}{AC}$$

$$AB = AC\cos x = 60\cos x$$

b. AB=AE+EB Then:

$$60\cos x = \frac{10}{\cos x} + 10$$
$$\Leftrightarrow 60\cos x = \frac{10 + 10\cos x}{\cos x};$$
$$\Leftrightarrow 60\cos^2 x = 10\cos x + 10;$$
$$\Leftrightarrow 60\cos^2 x - 10\cos x - 10 = 0$$
$$\Leftrightarrow 6\cos^2 x - \cos x - 1 = 0$$

The equation is equivalent to $(2\cos x - 1)(3\cos x + 1) = 0$

The solutions are $x = 60^0; x = 300^0; x = 109.5^0; x = 250.5^0$

c. $DE^2 = AE^2 - AD^2 = \frac{100}{\cos^2 x} - 100 = \frac{100}{(\frac{1}{2})^2} - 100 = 300$

Therefore, $DE = \sqrt{300} = 10\sqrt{3} = 17.3$ meters

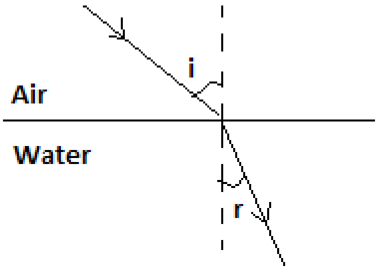
In triangle BDE, from the cosine rule, $BD^2=BE^2+DE^2-2BE.DE\cos 150^0$

$$= 10^2 + (10\sqrt{3})^2 - 2(10)(10\sqrt{3})(-\frac{\sqrt{3}}{2}) = 700$$

It follows that: $BD = \sqrt{700} = 10\sqrt{7} = 26.5$ meters

Example 2

In Physics, SNELL's law states that $\frac{\sin i}{\sin r} = n$, where i is the angle of incidence, r is the angle of refraction and n is the refractive index.



Suppose that light is incident in air at 50^0 on a plane water surface. The refractive index for water is 1.33. Calculate the angle of refraction

Solution

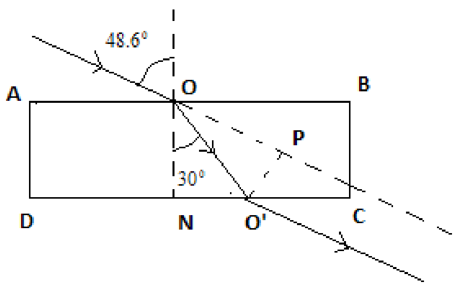
$$\frac{\sin 50^0}{\sin r} = 1.3$$

$$\sin r = \frac{\sin 50^0}{1.3} = 0.57597$$

$r = 30^0$

Example 3

On the figure below, ABCD is a rectangle such that AD=10 cm, O'P is perpendicular to OP, NO is perpendicular to DC



Find:

a. The measures of angles $N\hat{O}P$ and $O'\hat{O}P$

b. The lengths of segments O'O and O'P

c. Calculate the lateral displacement of a ray of light penetrating a rectangular plate of thickness 10 cm at an incident angle of 48.6^0; the ray is refracted at an angle of 30^0 from the normal at the point of incidence.

Solution

a. $N\hat{O}P = 48.6^0$;

$O'\hat{O}P = 48.6^0 - 30^0 = 18.6^0$

b. $O'O = \dfrac{10}{\cos 30^0} = 11.547\text{ cm}$

$O'P = O'O \sin O'\hat{O}P = 11.547 \sin 18.6^0 = 3.683\text{cm}$

c. The lateral displacement is 3.683 cm.

Example 4

The electric current I, in amperes, flowing through a circuit at time t seconds is given by $I = 220 \sin\left(30\pi t + \frac{\pi}{6}\right)$, where $t \geq 0$.Find:

a. The initial current

b. After what time, from the initial time, the current will be 220 Amperes

Solution

a. The initial current is when t=0.Then $I = 220 \sin\frac{\pi}{6} = 110\, Amperes$

b. Say after t seconds, the current will be 220 Amperes .Then

$220 \sin(30\pi t + \frac{\pi}{6}) = 220$

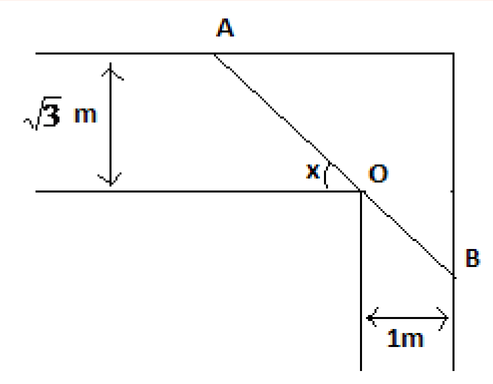
This is equivalent to $\sin\left(30\pi t + \frac{\pi}{6}\right) = 1 = \sin\frac{\pi}{2}$

$30\pi t + \frac{\pi}{6} = \frac{\pi}{2} + 2k\pi$ or $30\pi t + \frac{\pi}{6} = -\frac{\pi}{2} + 2k\pi$

The smallest positive value of t verifying this equation is $\frac{1}{90}$ seconds

APPLICATION ACTIVITY 3.3

1. A path, $\sqrt{3}$ meters wide, turns at right angle and its width is now 1 meter only (see the figure below)



A line passes through point O (the inner corner of the path) and makes an angle x with one of the walls and cuts the other two walls at points A and B.

a. Express OA, OB and AB in terms of x.

b. Let $f(x)$ =AB.Show that $f(x)=\dfrac{4\cos(x-30^0)}{\sin 2x}$

c. Find the value of x such that:

i. AB=4

ii. OA=OB

2. A wheel has horizontal axis. The diameter of the wheel is 16 meters and its center is 9 meters above the ground. Bikino rides on the big wheel and starts timing when she reaches the highest point. The wheel completes one revolution every 20 seconds. The height h above the ground the girl reaches at time t seconds is given by h=a+bcos(18t), where 18t is expressed in degrees t seconds after timing starts.

i. Find a and b

ii. Find the time when Bikino is 13 meters above the ground (the time between the first half minute)

3.4 END UNIT ASSESSMENT

1. a. Given that $\tan A$ and $\tan B$ are the roots of the equation ,

$$x^2+px+q=0$$

Express $\tan(A+B)$ in terms of p and q

b. Given that $\sin A=-\frac{5}{13}, \pi<A<\frac{3\pi}{2}$ and $\cos B=-\frac{3}{5}, \frac{\pi}{2}<B<\pi$, find the value

of $\tan(A-B)$

2. a. Given that $\cos\theta=\frac{3}{5}$ and θ is acute, find the exact value of $\cos 2\theta$

b. If $\tan\theta=\frac{12}{5}$ and θ is acute, find the exact value of $\tan 2\theta$

3. a. Show that $\dfrac{\sin 2\theta}{1+\cos 2\theta} = \tan\theta$

b. Hence, find the value of $\cot\dfrac{\pi}{8}$ in the form $a + b\sqrt{2}$, where a and b are integers

4. Acute angle x is such that $\sin x = \dfrac{\sqrt{6}-\sqrt{2}}{4}$. Find $\cos 2x$, hence, find the value of x, where $0 < x < \dfrac{\pi}{2}$

5. Express $\sin^2 t - 2(1-\cos t)$ in terms of $\sin\dfrac{t}{2}$

6. a. If $\tan 2A = \dfrac{7}{24}$, where $0 < A < \dfrac{\pi}{4}$, find the value of $\tan A$

b. If $\cos A = -\dfrac{7}{25}$, find $\sin\dfrac{A}{2}, \cos\dfrac{A}{2}$ and $\tan\dfrac{A}{2}$.

7. Transform in sum:

a. $\sin 4x \cos 11x$

b. $\cos 7x \sin 9x$

8. Transform in product:

a. $\cos 8x - \cos 9x$

b. $\sin 3x + \sin 11x$

9. Simplify the expression $\dfrac{\sin A + 2\sin 3A + \sin 5A}{\sin 3A + 2\sin 5A + \sin 7A}$

10. Solve the following equations:

a. $(2\sin x - 1)(\tan x - 1) = 0$ for $0 \le x \le \pi$

b. $\cos 2x \cos x - \sin 2x \sin x = 0$ for $0 \le x \le 2\pi$

c. $\cos\left(4x - \dfrac{\pi}{5}\right) = \dfrac{\sqrt{3}}{2}$, where $0 \le x \le 2\pi$

d. $\cos 2x - \cos\left(\frac{\pi}{5} - x\right) = 0$, where $0 \leq x \leq 2\pi$

11. Solve $3\sin x + \sqrt{3}\cos x = 3$

12. The temperature in an office is modeled by the function $y = 19 + 6\sin\frac{\pi(X-11)}{12}$

, where y is the temperature, in ${}^{0}C$, and x is the time, in hours past midnight.

a. What is the temperature in the office at 9 a. m?

b. What are the maximum and minimum temperatures in the office?

UNIT 4

BIVARIATE STATISTICS

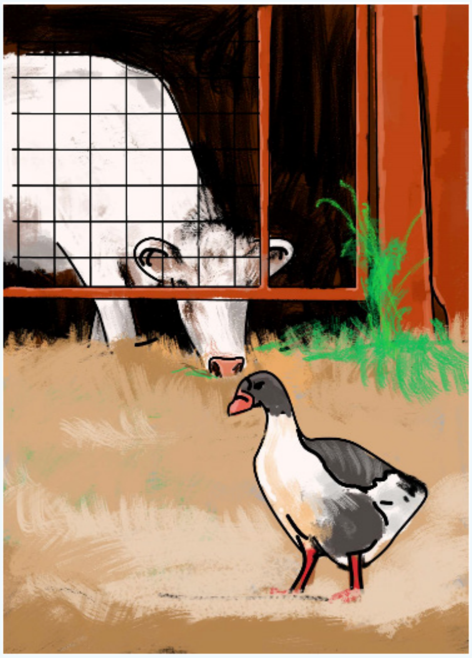
Key unit Competence: Extend understanding, analysis and interpretation of bivariate data to correlation coefficients and regression lines

4.0. INTRODUCTORY ACTIVITY

In Kabeza village, after her 9 observations about farming, UMULISA saw that in every house observed, where there is a cow (X) if there is also domestic duck (Y), then she got the following results:

$(1,4),(2,8),(3,4),(4,12),(5,10)$

$(6,14),(7,16),(8,6),(9,18)$



a) Represent this information graphically in $(x,y)-coordinates$.

b) Find the equation of line joining any two point of the graph and guess the name of this line.

c) According to your observation from (a), explain in your own words if there is any relationship between Cows (X) and domestic duck (Y).

4.1 Bivariate data, scatter diagram and types of correlation

ACTIVITY 4.1

Consider the situation in which the mass, y (g), of a chemical is related to the time, x minutes, for which the chemical reaction has been taking place ,according to the table.

Time, x min	5	7	12	16	20
Mass, y g	4	12	18	21	24

a) Plot the above information in (x, y) coordinates.

b) Explain in your own words the relationship between x *and* y

Content summary

In statistics, **bivariate** or **double series** includes technique of analyzing data in two variables, when focus on the relationship between a dependent variable-y and an independent variable-x.

For example, between age and weight, weight and height, years of education and salary, amount of daily exercise and cholesterol level, etc. As with data for a single variable, we can describe bivariate data both graphically and numerically. In both cases we will be primarily concerned with determining whether there is a *linear* relationship between the two variables under consideration or not.

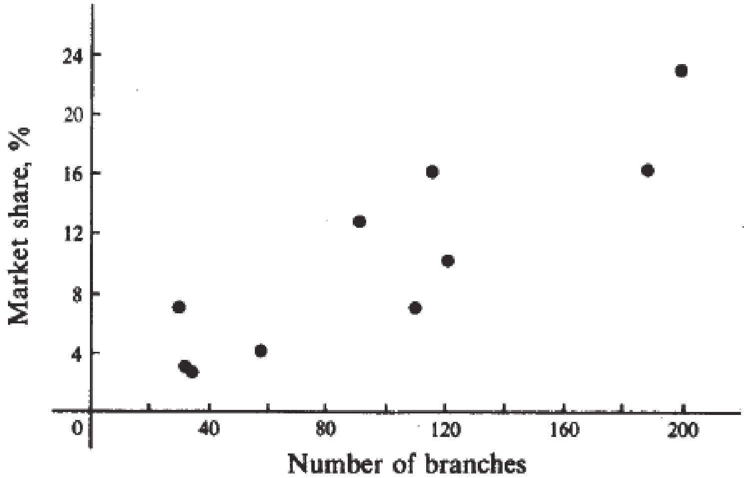
It should be kept in mind that a statistical relationship between two variables does not necessarily imply a *causal* relationship between them. For example, a strong relationship between weight and height does not imply that either variable causes the other.

Scatter plots or Scatter diagram and types of correlation

Consider the following data which relate *x*, the respective number of branches that 10 different banks have in a given common market, with *y*, the corresponding market share of total deposits held by the banks:

x	198	186	116	89	120	109	28	58	34	31
y	22.7	16.6	15.9	12.5	10.2	6.8	6.8	4.0	2.7	2.8

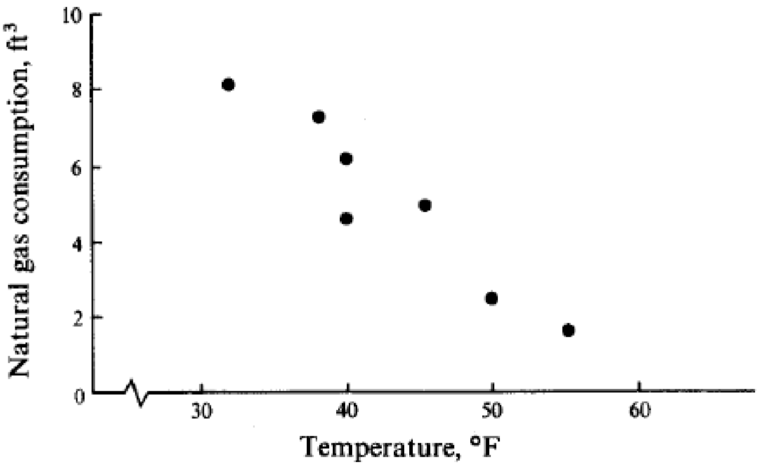
If each point (x, y) of the data is plotted in an x, y *coordinate* plane, **the *scatter plot or Scatter diagram*** is obtained.



The scatter plot or scatter diagram (in the figure above) indicates that, roughly speaking, the market share increases as the number of branches increases. We say that x and y have a ***positive correlation*.**

On the other hand, consider the data below, which relate average daily temperature x, in degrees Fahrenheit, and daily natural gas consumption y, in cubic metre.

x, °F	50	45	40	38	32	40	55
y, ft^3	2.5	5.0	6.2	7.4	8.3	4.7	1.8

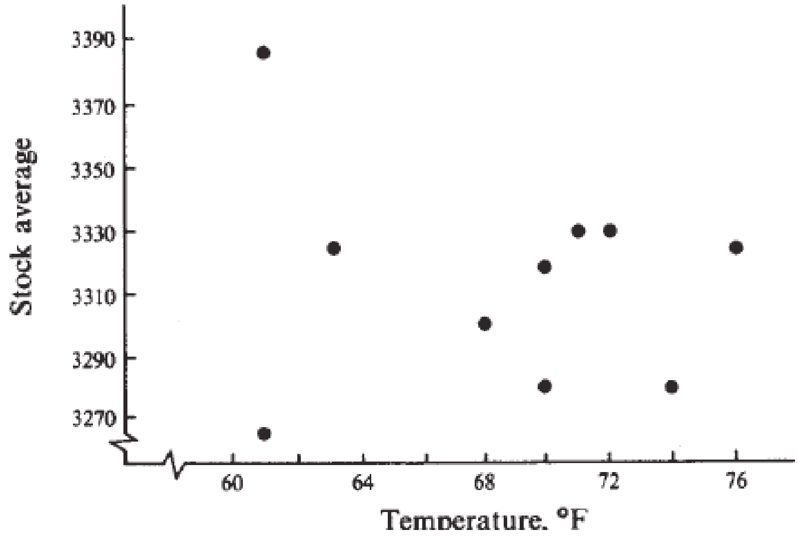


We see that y tends to decrease as x *increases*. Here, x and y have a ***negative correlation***

Finally, consider the data items (x, y) below, which relate daily temperature x over a 10-day period to the Dow Jones stock average y.

(63, 3385); (72, 3330); (76, 3325); (70, 3320); (71, 3330); (65, 3325); (70, 3280); (74, 3280)

(68, 3300); (61, 3265).



There is no apparent relationship between x and y (*no correlation* or *Weak correlation*).

APPLICATION ACTIVITY 4.1

One measure of personal fitness is the time taken for an individual's pulse rate to return to normal after strenuous exercise, the greater the fitness, the shorter the time. Following a short program of strenuous exercise Norman recorded his pulse rates P at time t minutes after he had stopped exercising. Norman's results are given in the table below.

t	0.5	1.0	1.5	2.0	3.0	4.0	5.0
P	125	113	102	94	81	83	71

a) Draw a scatter diagram to represent this information in

(x, y) *coordinates*

b) Explain the relationship between Norman's pulse P and time t.

4.2 Covariance

ACTIVITY 4.2

Complete the following table

i	x_i	y_i	$x_i - \bar{x}$	$y_i - \bar{y}$	$(x_i - \bar{x})(y_i - \bar{y})$
1	3	6			
2	5	9			
3	7	12			
4	3	10			
5	2	7			
6	6	8			

$\sum_{i=1}^{6} x_i = \ldots$ $\quad \sum_{i=1}^{6} y_i = \ldots$ $\quad \sum_{i=1}^{6} (x_i - \bar{x})(y_i - \bar{y}) = \ldots$

$\bar{x} = \ldots\ldots$ $\quad \bar{y} = \ldots\ldots$

What can you get from the following expressions :

1) $\sum_{i=1}^{k} (x_i - \bar{x})(x_i - \bar{x})$

2) $\sum_{i=1}^{k} (x_i - \bar{x})(y_i - \bar{y})$

Content summary

In case of two variables, say x and y, there is another important result called **covariance of x and y**, denoted $\text{cov}(x, y)$.

The **covariance of variables x and y** is a measure of how these two variables change together. If the greater values of one variable mainly correspond with the greater values of the other variable, and the same holds for the smaller values, i.e. the variables tend to show similar behavior, the covariance is positive. In the opposite case, when the greater values of one variable mainly correspond to the smaller values of the other, i.e. the variables tend to show opposite behavior, the covariance is negative. If covariance is zero the variables are said to be **uncorrelated,** it means that there is no linear relationship between them.

Therefore, the sign of covariance shows the tendency in the linear relationship between the variables. The magnitude of covariance is not easy to interpret.

Covariance of variables x and y, where the summation of frequencies $\sum_{i=1}^{k} f_i = n$ are equal for both variables, is defined to be

$$\operatorname{cov}(x, y) = \frac{1}{n}\sum_{i=1}^{k} f_i\left(x_i - \bar{x}\right)\left(y_i - \bar{y}\right)$$

Developing this formula we have

$$\begin{aligned}
\operatorname{cov}(x, y) &= \frac{1}{n}\sum_{i=1}^{k} f_i\left(x_i y_i - x_i \bar{y} - \bar{x} y_i + \bar{x}\,\bar{y}\right)\\
&= \frac{1}{n}\sum_{i=1}^{k} f_i x_i y_i - \frac{1}{n}\sum_{i=1}^{k} f_i x_i \bar{y} - \frac{1}{n}\sum_{i=1}^{k} f_i \bar{x} y_i + \frac{1}{n}\sum_{i=1}^{k} f_i \bar{x}\,\bar{y}\\
&= \frac{1}{n}\sum_{i=1}^{k} f_i x_i y_i - \frac{1}{n}\bar{y}\sum_{i=1}^{k} f_i x_i - \frac{1}{n}\bar{x}\sum_{i=1}^{k} f_i y_i + \bar{x}\,\bar{y}\frac{1}{n}\sum_{i=1}^{k} f_i \qquad \left[\frac{1}{n}\sum_{i=1}^{k} f_i = \frac{1}{n}\times n = 1\right]\\
&= \frac{1}{n}\sum_{i=1}^{k} f_i x_i y_i - \bar{x}\,\bar{y} - \bar{x}\,\bar{y} + \bar{x}\,\bar{y}\\
&= \frac{1}{n}\sum_{i=1}^{k} f_i x_i y_i - \bar{x}\,\bar{y}
\end{aligned}$$

Thus, the covariance is also given by

$$\operatorname{cov}(x, y) = \frac{1}{n}\sum_{i=1}^{k} f_i x_i y_i - \bar{x}\,\bar{y}$$

Examples

1) Find the covariance of x and y in following data sets

x	3	5	6	8	9	11
y	2	3	4	6	5	8

Solution

We have

x_i	y_i	$x_i - \bar{x}$	$y_i - \bar{y}$	$(x_i - \bar{x})(y_i - \bar{y})$
3	2	-4	-2.6	10.4
5	3	-2	-1.6	3.2
6	4	-1	-0.6	0.6
8	6	1	1.4	1.4
9	5	2	0.4	0.8
11	8	4	3.4	13.6
$\sum_{i=1}^{6} x_i = 42$	$\sum_{i=1}^{6} y_i = 28$			$\sum_{i=1}^{6} (x_i - \bar{x})(y_i - \bar{y}) = 30$
$\bar{x} = 7$	$\bar{y} = 4.6$			

Thus,

$$\operatorname{cov}(x, y) = \frac{1}{6} \sum_{i=1}^{6} f_i (x_i - \bar{x})(y_i - \bar{y})$$

$$= \frac{1}{6}(30)$$

$$= 5$$

2) Find the covariance of the following distribution

$y \backslash x$	0	2	4
1	2	1	3
2	1	4	2
3	2	5	0

Solution

Convert the double entry into a simple table and compute the arithmetic means

x_i	y_i	f_i	$x_i f_i$	$y_i f_i$	$x_i y_i f_i$
0	1	2	0	2	0
0	2	1	0	2	0
0	3	2	0	6	0
2	1	1	2	1	2
2	2	4	8	8	16
2	3	5	10	15	30
4	1	3	12	3	12
4	2	2	8	4	16
4	3	0	0	0	0
		$\sum_{i=1}^{9} f_i = 20$	$\sum_{i=1}^{9} x_i f_i = 40$	$\sum_{i=1}^{9} y_i f_i = 41$	$\sum_{i=1}^{9} x_i y_i f_i = 76$

$$\bar{x} = \frac{40}{20} = 2, \quad \bar{y} = \frac{41}{20} = 2.05$$

$$\operatorname{cov}(x, y) = \frac{76}{20} - 2 \times 2.05 = -0.3$$

Alternative method

$$\operatorname{cov}(x, y) = \frac{1}{n} \sum_{i=1}^{k} f_i x_i y_i - \bar{x}\,\bar{y}$$

$$\bar{x} = \frac{1}{n} \sum_{i=1}^{k} x_i f_i, \quad \bar{y} = \frac{1}{n} \sum_{i=1}^{k} y_i f_i$$

y \ x	0	2	4	Total
1	2	1	3	6
2	1	4	2	7
3	2	5	0	7
Total	5	10	5	20

$$\bar{x}=\frac{1}{20}(0\times5+2\times10+4\times5)$$
$$=\frac{40}{20}=2$$

$$\bar{y}=\frac{1}{20}(1\times6+2\times7+3\times7)$$
$$=\frac{41}{20}=2.05$$

$$\operatorname{cov}(x,y)=\frac{1}{20}\begin{pmatrix}0\times1\times2+0\times2\times1+0\times3\times2+2\times1\times1+2\times2\times4\\+2\times3\times5+4\times1\times3+4\times2\times2+4\times3\times0\end{pmatrix}-2\times2.05$$
$$=\frac{1}{20}(0+0+0+2+16+30+12+16+0)-4.1$$

$$=\frac{76}{20}-4.1\ =-0.3$$

APPLICATION ACTIVITY 4.2

1. The scores of 12 students in their mathematics and physics classes are

Mathematics	2	3	4	4	5	6	6	7	7	8	10	10
Physics	1	3	2	4	4	4	6	4	6	7	9	10

Find the covariance of the distribution

2. The values of two variables x and y are distributed according to the following table

y \ x	100	50	25
14	1	1	0
18	2	3	0
22	0	1	2

Calculate the covariance

4.3 Coefficient of correlation

ACTIVITY 4.3

Consider the following table

x	y
3	6
5	9
7	12
3	10
2	7
6	8

Find σ_x, σ_y

1. Find $\operatorname{cov}(x, y)$

2. Calculate the ratio $\dfrac{Cov(x, y)}{\sigma_x . \sigma_y}$

Content summary

The **Pearson's coefficient of correlation** (or **Product moment coefficient of correlation** or simply **coefficient of correlation**), denoted by r, is a measure of the strength of linear relationship between two variables.

The coefficient of correlation between two variables x and y is given by

$$r = \frac{\operatorname{cov}(x, y)}{\sigma_x \sigma_y}$$

Where, $\operatorname{cov}(x, y)$ is covariance of x and y

σ_x is the standard deviation for x

σ_y is the standard deviation for y

Properties of the coefficient of correlation

a) The coefficient of correlation does not change the measurement scale. That is, if the height is expressed in meters or feet, the coefficient of correlation does not change.

b) The sign of the coefficient of correlation is the same as the covariance.

c) The square of the coefficient of correlation is equal to the product of the gradient of the regression line of y on x, and the gradient of the regression line of x on y.

In fact, $r = \frac{\operatorname{cov}(x,y)}{\sigma_x \sigma_y}$. Squaring both sides gives

$$r^2 = \left[\frac{\operatorname{cov}(x,y)}{\sigma_x \sigma_y}\right]^2$$

$$= \frac{\operatorname{cov}^2(x,y)}{\sigma_x^2 \sigma_y^2}$$

$$= \frac{\operatorname{cov}(x,y)}{\sigma_x^2} \times \frac{\operatorname{cov}(x,y)}{\sigma_y^2}$$

$= ac$, where $y = ax + b$ is the equation of the regression line of y on x, and

$x = cy + d$ is the equation of the regression line of x on y

d) If the coefficient of correlation is known, it can be used to find the gradients or slopes of two regression lines.

We know that the gradient of the regression line y on x is $\frac{\operatorname{cov}(x,y)}{\sigma_x^2}$.

From this we have, $\frac{\operatorname{cov}(x,y)}{\sigma_x^2} = \frac{\operatorname{cov}(x,y)}{\sigma_x \sigma_x} \times \frac{\sigma_y}{\sigma_y}$

$$= \frac{\operatorname{cov}(x,y)}{\sigma_x \sigma_y} \times \frac{\sigma_y}{\sigma_x}$$

$$= r\frac{\sigma_y}{\sigma_x}$$

We know that the gradient of the regression line x on y is

$\frac{\operatorname{cov}(x,y)}{\sigma_y^2}$. From this we have, $\frac{\operatorname{cov}(x,y)}{\sigma_y^2} = \frac{\operatorname{cov}(x,y)}{\sigma_y \sigma_y} \times \frac{\sigma_x}{\sigma_x}$

$$= \frac{\operatorname{cov}(x,y)}{\sigma_x \sigma_y} \times \frac{\sigma_x}{\sigma_y}$$

$$= r\frac{\sigma_x}{\sigma_y}$$

Thus, the gradient of the regression line y on x is given by $r\frac{\sigma_y}{\sigma_x}$ and the gradient of the regression line y on x is given by $r\frac{\sigma_x}{\sigma_y}$.

e) Cauchy Inequality: $\operatorname{cov}^2(x,y) \leq \sigma_x^2\sigma_y^2$

In fact, $r = \frac{\operatorname{cov}(x,y)}{\sigma_x\sigma_y} \Leftrightarrow \operatorname{cov}(x,y) = r\sigma_x\sigma_y$.

Squaring both sides gives $\operatorname{cov}^2(x,y) = r^2\sigma_x^2\sigma_y^2$

Or $\operatorname{cov}^2(x,y) \leq \sigma_x^2\sigma_y^2$

f) The coefficient of correlation takes value ranging between -1 and +1. That is $-1 \leq r \leq 1$

In fact, from Cauchy Inequality we have,

$$\operatorname{cov}^2(x,y) \leq \sigma_x^2\sigma_y^2$$

$$\Leftrightarrow \frac{\operatorname{cov}^2(x,y)}{\sigma_x^2\sigma_y^2} \leq 1$$

$$\Leftrightarrow \left[\frac{\operatorname{cov}(x,y)}{\sigma_x\sigma_y}\right]^2 \leq 1$$

$$\Leftrightarrow r^2 \leq 1.$$

Taking square roots both side

$$\Leftrightarrow \sqrt{r^2} \leq 1$$

$\Leftrightarrow |r| \leq 1$ since $\sqrt{x^2} = |x|$

$|r| \leq 1$ is equivalent to $-1 \leq r \leq 1$.

Thus, $-1 \leq r \leq 1$.

g) If the linear coefficient of correlation takes values closer to **−1**, the **correlation** is **strong and negative**, and will become stronger the closer ***r*** approaches −1.

h) If the linear coefficient of correlation takes values close to **1** the **correlation** is **strong and positive**, and will become stronger the closer ***r*** approaches 1

i) If the linear coefficient of correlation takes values close to **0**, the **correlation** is **weak**.

j) If $r = 1$ or $r = -1$, there is **perfect correlation** and the line on the scatter plot is increasing or decreasing respectively.

k) If $r = 0$, there is **no linear correlation**.

Examples:

1) A test is made over 200 families on number of children (x) and number of beds y per family. Results are collected in the table below

2)

y \ x	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
1	0	2	7	5	2	0	0	0	0	0	0
2	2	2	10	8	15	1	0	0	0	0	0
3	1	3	5	6	8	6	1	0	0	0	0
4	0	2	8	2	6	12	10	8	0	0	0
5	0	1	0	2	5	6	10	5	7	3	3
6	0	0	0	2	4	5	5	2	3	3	2

a) What is the average number for children and beds per a family?

b) Find the covariance

c) Can we confirm that there is a high linear correlation between the number of children and number of beds per family?

Solution

a) Average number of children per family:

Contingency table:

y \ x	0	1	2	3	4	5	6	7	8	9	10	Total
1	0	2	7	5	2	0	0	0	0	0	0	16
2	2	2	10	8	15	1	0	0	0	0	0	38
3	1	3	5	6	8	6	1	0	0	0	0	30
4	0	2	8	2	6	12	10	8	0	0	0	48
5	0	1	0	2	5	6	10	5	7	3	3	42
6	0	0	0	2	4	5	5	2	3	3	2	26
Total	**3**	**10**	30	25	40	30	26	15	10	6	5	200

$$\bar{x} = \frac{1}{n}\sum_{i=1}^{k} f_i x_i, \quad \bar{y} = \frac{1}{n}\sum_{i=1}^{k} f_i y_i$$

Marginal series:

x_i	0	1	2	3	4	5	6	7	8	9	10	Total
f_i	3	10	30	25	40	30	26	15	10	6	5	$\sum f_i = 200$
$f_i x_i$	0	10	60	75	160	150	156	105	80	54	50	$\sum f_i x_i = 900$

y_j	1	2	3	4	5	6	Total
f_j	16	38	30	48	42	26	$\sum f_j = 200$
$f_j y_j$	16	76	90	192	210	156	$\sum f_j y_j = 740$

The means are

$$\bar{x} = \frac{1}{n}\sum_{i=1}^{10} f_i x_i = \frac{900}{200} = 4.5$$

And $\bar{y} = \frac{1}{n}\sum_{j=1}^{6} f_j y_j = \frac{740}{200} = 3.7$
There are about 5 children per family and about 4 beds per family.

b) The covariance is calculated as follow:

$\operatorname{cov}(x,y) = (\frac{1}{n}\sum_{(i,j)=(1,1)}^{(p,q)=(10,6)} f_{ij} x_i y_j) - \bar{x}\bar{y}$, where i assumes values from 1 to $p=10$, and j assumes values from 1 to $q=6$,or

$\operatorname{cov}(x,y) = \frac{1}{200}\sum_{i=1}^{66} f_i x_i y_i - \bar{x}\,\bar{y}$ where $\bar{y}=3.7$ and $\bar{x}=4.5$

$$= \frac{1}{200}\begin{pmatrix} 0+2+14+15+8+0+4+40+48+120+10+0 \\ +9+30+54+96+90+18+0+8+64+24+96 \\ +240+240+224+0+5+0+30+100+150 \\ +300+175+280+135+150+0+36+96+150 \\ +180+84+144+162+120 \end{pmatrix} - 4.5\times 3.7$$

$$= \frac{3751}{200} - 16.65$$
$$= 18.7555 - 16.65$$
$$= 2.105$$

c) Correlation coefficient is given by $\frac{\operatorname{cov}(x,y)}{\sigma_x \sigma_y}$

$$\sigma_y^2 = \frac{1}{200}\sum_{i=1}^{6} f_i y_i^2 - (\bar{y})^2$$
$$= \frac{1}{200}(16+38\times 4+30\times 9+48\times 16+42\times 25+26\times 36) - (3.7)^2$$
$$= 15.96 - 13.69$$
$$= 2.27$$

Therefore, the correlation coefficient is

$$r = \frac{2.105}{\sqrt{4.96}\sqrt{2.27}} \approx 0.63$$

There is a high linear correlation.

NOTICE:

Spearman's coefficient of rank correlation

A Spearman coefficient of rank correlation or Spearman's rho is measure of statistical dependence between two variables. It assesses how well the relationship between two variables can be described using a monotonic function.

The **Spearman's coefficient of rank correlation** is denoted and defined by

$$\rho = 1 - \frac{6\sum_{i=1}^{k} d_i^2}{n\left(n^2 - 1\right)}$$

Where, d refers to the difference of ranks between paired items in two series and n is the number of observations. It is much easier to calculate the Spearman's coefficient of rank correlation than to calculate the Pearson's coefficient of correlation as there is far less working involved. However, in general, the Pearson's coefficient of correlation is a more accurate measure of correlation.

METHOD OF RANKING

Ranking can be done in ascending order or descending order.

Examples:

1) Suppose that we have the marks, x, of seven students in this order:

12, 18, 10, 13, 15, 16, 9

We assign the rank 1, 2, 3, 4, 5, 6, 7 such that the smallest value of x will be ranked 1.

That is

x	12	18	10	13	15	16	9
$Rank(x)$	3	7	2	4	5	6	1

If we have two or more equal values we proceed as follow:

Consider the following series

x	66	65	66	67	66	64	68	68

To assign the rank to this series we do the following:

$x=64$ will take rank 1, since it is the smallest value of x

$x=65$ will be ranked 2.

$x=66$ appears 3 times, since the previous value was ranked 2 here 66 would be ranked 3, another 66 would be ranked 4 and another 5 but since there are three 66's we need to find the average of those ranks which is $\frac{3+4+5}{3}=4$ so that each 66 will be ranked 4.

$x=67$ will be ranked 6 since we are on the 6^{th} position

$x=68$ appears 2 times, since the previous value was ranked 6 here 68 would be ranked 7, and another 66 would be ranked 8 but since there are two 68's we need to find the average of those ranks which is $\frac{7+8}{2}=7.5$ so that each 68 will be ranked 7.5

Thus we have the following

x	66	65	66	67	66	64	68	68
$Rank(x)$	4	2	4	6	4	1	7.5	7.5

2) Calculate the Spearman's coefficient of rank correlation for the series

x	12	8	16	12	7	10	12	16	12	9
y	6	5	7	7	4	6	8	13	10	10

Solution

x	y	$Rank(x)$	$Rank(y)$	$Rank(x)-Rank(y)=d$	d^2
12	6	6.5	3.5	3	9
8	5	2	2	0	0
16	7	9.5	5.5	4	16
12	7	6.5	5.5	1	1
7	4	1	1	0	0
10	6	4	3.5	0.5	0.25
12	8	6.5	7	0.5	0.25
16	13	9.5	10	0.5	0.25
12	10	6.5	8.5	2	4
9	10	3	8.5	5.5	30.25
					$\sum_{i=1}^{10} d_i^2=61$

Then

$$\rho = 1 - \frac{6 \times 61}{10(100-1)}$$

$$\Leftrightarrow \rho = 1 - \frac{366}{990}$$

$$\Leftrightarrow \rho = \frac{990-366}{990}$$

Or

$$\rho = 0.63$$

APPLICATION ACTIVITY 4.3

1) The scores of 12 students in their mathematics and physics classes are

Mathematics	2	3	4	4	5	6	6	7	7	8	10	10
Physics	1	3	2	4	4	4	6	4	6	7	9	10

Find the correlation coefficient distribution and interpret it.

2) The values of the two variables X and Y are distributed according to the following table:

X \ Y	0	2	4
1	2	1	3
2	1	4	2
3	2	5	0

Calculate the correlation coefficient.

2) The marks of eight candidates in English and Mathematics are

Candidate	1	2	3	4	5	6	7	8
English (x)	50	58	35	86	76	43	40	60
Mathematics (y)	65	72	54	82	32	74	40	53

Rank the results and hence find Spearman's rank correlation coefficient between the two sets of marks. Comment on the value obtained.

4.4 Regression lines

ACTIVITY 4.4

The regression line y on x has the form $y = ax + b$. We need the distance from this line to each point of the given data to be small, so that the sum of the square

of such distances to be very small. That is $D = \sum_{i=1}^{k}\left[y_i - (ax_i + b)\right]^2$ or

$D = \sum_{i=1}^{k}(y_i - ax_i - b)^2 \quad (1)$ is minimum.

1) Differentiate relation (1) with respect to b. In this case y, x and a will be considered as constants.

2) Equate relation obtained in 1) to zero, divide each side by n and give the value of b.

3) Take the value of b obtained in 2) and put it in relation obtained in 1). Differentiate the obtained relation with respect to a, equate it to zero and divide both sides by n to find the value of a.

4) Using the relations: The variance for variable x is $\sigma_x^2 = \frac{1}{n}\sum_{i=1}^{k}\left(x_i - \overline{x}\right)^2$ and the

variance for variable y is $\sigma_y^2 = \frac{1}{n}\sum_{i=1}^{k}\left(y_i - \overline{y}\right)^2$ and the covariance of

these two variables is $\operatorname{cov}(x, y) = \frac{1}{n}\sum_{i=1}^{k}\left(x_i - \overline{x}\right)\left(y_i - \overline{y}\right)$, give the

simplified expression equal to a.

Put the value of b obtained in 2) and the value of a obtained in 4) in relation $y = ax + b$ and give the expression of regression line y on x

Content summary

We use the regression line of y on x to **predict** a value of y for any given value of x and vice versa, we use the regression line of x *on* y, to predict a value of x for a given value of y. The "best" line would make the best predictions: the observed y-values should stray as little as possible from the line. This straight line is the regression line from which we can adjust its algebraic expressions and it is written as $y = ax + b$, where a *is* the gradient and c is the y-intercept.

The regression line y on x is written as $y = \frac{\operatorname{cov}(x,y)}{\sigma_x^2}x + \left(\overline{y} - \frac{\operatorname{cov}(x,y)}{\sigma_x^2}\overline{x}\right)$

We may write

$$L_{y/x} \equiv y - \overline{y} = \frac{\operatorname{cov}(x,y)}{\sigma_x^2}\left(x - \overline{x}\right)$$

Note that the regression line x on y is $x = cy + d$, where c is the gradient of the line and d is the x-intercept, it is given by $x - \overline{x} = \frac{\operatorname{cov}(x,y)}{\sigma_y^2}\left(y - \overline{y}\right)$

This line is written as $L_{x/y} \equiv x - \overline{x} = \frac{\operatorname{cov}(x,y)}{\sigma_y^2}\left(y - \overline{y}\right)$

Shortcut method of finding regression line

To abbreviate the calculations, the two regression lines can be determined as follow:

a) The equation of the regression line of y on x is $L_{y/x} \equiv y = ax + b$ and the values of a and b are found by solving the simultaneous equations:

$$\begin{cases} \sum_{i=1}^{k} f_i y_i = a\sum_{i=1}^{k} f_i x_i + b\,n \\ \sum_{i=1}^{k} f_i x_i y_i = a\sum_{i=1}^{k} f_i x_i^2 + b\sum_{i=1}^{k} f_i x_i \end{cases}$$

These equations are called the **normal equations** for y on x.

b) The equation of the regression line of x on y is $L_{x/y} \equiv x = cy + d$ and the values of c and d are found by solving the simultaneous equations:

$$\begin{cases} \sum_{i=1}^{k} f_i x_i = c\sum_{i=1}^{k} f_i y_i + d\,n \\ \sum_{i=1}^{k} f_i x_i y_i = c\sum_{i=1}^{k} f_i y_i^2 + d\sum_{i=1}^{k} f_i y_i \end{cases}$$

These equations are called the **normal equations** for x on y.

Examples:

1) *Find the equation of the regression line of* y *on* x *, and the equation of the regression line of* x *on* y *,for the following data and estimate the value of* y *for* $x=4, x=7, x=16$ *and the value of* x *for* $y=7, y=9, y=16$.

x	3	5	6	8	9	11
y	2	3	4	6	5	8

Solution

x	y	$x-\bar{x}$	$y-\bar{y}$	$\left(x-\bar{x}\right)^2$	$\left(y-\bar{y}\right)^2$	$\left(x-\bar{x}\right)\left(y-\bar{y}\right)$
3	2	-4	-2.6	16	6.76	10.4
5	3	-2	-1.6	4	2.56	3.2
6	4	-1	-0.6	1	0.36	0.6
8	6	1	1.4	1	1.96	1.4
9	5	2	0.4	4	0.16	0.8
11	8	4	3.4	16	11.56	13.6
$\sum_{i=1}^{6} x_i = 42$	$\sum_{i=1}^{6} y_i = 28$			$\sum_{i=1}^{6}\left(x_i-\bar{x}\right)^2 = 42$	$\sum_{i=1}^{6}\left(y_i-\bar{y}\right)^2 = 23.36$	$\sum_{i=1}^{6}\left(x_i-\bar{x}\right)\left(y_i-\bar{y}\right) = 30$

$$\bar{x} = \frac{42}{6} = 7 \;,\; \bar{y} = \frac{28}{6} = 4.7$$

$$\operatorname{cov}(x,y) = \frac{1}{n}\sum\left(x-\bar{x}\right)\left(y-\bar{y}\right) = \frac{30}{6} = 5$$

$$\sigma_x^2 = \frac{42}{6} = 7 \;,\; \sigma_y^2 = \frac{23.36}{6} = 3.89$$

$$L_{y/x} \equiv y - \bar{y} = \frac{\operatorname{cov}(x,y)}{\sigma_x^2}\left(x-\bar{x}\right)$$

$$L_{y/x} \equiv y - 4.7 = \frac{5}{7}(x-7)$$

Finally, the equation of the regression line of y on x is $L_{y/x} \equiv y = \frac{5}{7}x - 0.3$

And $L_{x/y} \equiv x - \bar{x} = \frac{\operatorname{cov}(x,y)}{\sigma_y^2}(y - \bar{y})$

$$L_{x/y} \equiv x - 7 = \frac{5}{3.89}(y - 4.7)$$

Finally, the equation of the regression line of *x* on *y* is $L_{x/y} \equiv y = 1.3x + 1$.

Alternative method

x	y	x^2	y^2	xy
3	2	9	4	6
5	3	25	9	15
6	4	36	16	24
8	6	64	36	48
9	5	81	25	45
11	8	121	64	88
$\sum_{i=1}^{6} x_i = 42$	$\sum_{i=1}^{6} y_i = 28$	$\sum_{i=1}^{6} x_i^2 = 336$	$\sum_{i=1}^{6} y_i^2 = 154$	$\sum_{i=1}^{6} x_i y_i = 226$

$L_{y/x} \equiv y = ax + b$

$$\begin{cases} \sum_{i=1}^{k} f_i y_i = a\sum_{i=1}^{k} f_i x_i + b\,n \\ \sum_{i=1}^{k} f_i x_i y_i = a\sum_{i=1}^{k} f_i x_i^2 + b\sum_{i=1}^{k} f_i x_i \end{cases}$$

$$\begin{cases} 28 = 42a + 6b \\ 226 = 336a + 42b \end{cases} \Leftrightarrow \begin{cases} a = \frac{5}{7} \\ b = -0.3 \end{cases}$$

Thus, the line of y on x is $L_{y/x} \equiv y = \frac{5}{7}x - 0.3$

$x = 4 \Rightarrow y = 2.5$

$x = 7 \Rightarrow y = 4.7$

$x = 16 \Rightarrow y = 11.1$

$L_{x/y} \equiv x = cy + d$

$$\begin{cases} \sum_{i=1}^{k} f_i x_i = c\sum_{i=1}^{k} f_i y_i + d\,n \\ \sum_{i=1}^{k} f_i x_i y_i = c\sum_{i=1}^{k} f_i y_i^2 + d\sum_{i=1}^{k} f_i y_i \end{cases}$$

$$\begin{cases} 42 = 28c + 6d \\ 226 = 154c + 28d \end{cases} \Leftrightarrow \begin{cases} c = 1.3 \\ d = 1 \end{cases}$$

Thus, the line of x on y is $L_{x/y} \equiv x = 1.3y + 1$

$y = 7 \Rightarrow x = 10.1$

$y = 9 \Rightarrow x = 12.7$

$y = 16 \Rightarrow x = 21.8$

APPLICATION ACTIVITY 4.4

1. Consider the following table

x	y
60	3.1
61	3.6
62	3.8
63	4
65	4.1

Find the regression line of y on x

a) Calculate the approximate y value for the variable $x = 64$

2. The values of two variables x and y are distributed according to the following table

y \ x	100	50	25
14	1	1	0
18	2	3	0
22	0	1	2

Find the regression lines

4.5 Interpretation of statistical data (Application)

ACTIVITY 4.5

Explain in your own words how statistics, especially bivariate statistics, can be used in our daily life.

Content summary

Bivariate statistics can help in prediction of a value for one variable if we know the value of the other.

Example

One measure of personal fitness is the time taken for an individual's pulse rate to return to normal after strenuous exercise, the greater the fitness, the shorter the time. Following a short program of strenuous exercise Norman recorded his pulse rates P at time t minutes after he had stopped exercising. Norman's results are given in the table below.

t	0.5	1.0	1.5	2.0	3.0	4.0	5.0
P	125	113	102	94	81	83	71

Estimate Norman's pulse rate 2.5 minutes after stopping the exercise program.

Solution

t	P	t^2	P^2	tP
0.5	125	0.25	15625	62.5
1	113	1	12769	113
1.5	102	2.25	10404	153
2	94	4	8836	188
3	81	9	6561	243
4	83	16	6889	332
5	71	25	5041	355
$\sum_{i=1}^{7} t_i = 17$	$\sum_{i=1}^{7} P_i = 669$	$\sum_{i=1}^{7} t_i^2 = 57.5$	$\sum_{i=1}^{7} P_i^2 = 66125$	$\sum_{i=1}^{7} t_i P_i = 1446.5$

We need the line $P = at + b$

Use the formula

$$\begin{cases} \sum_{i=1}^{7} P_i = a\sum_{i=1}^{7} t_i + bn \\ \sum_{i=1}^{7} t_i P_i = a\sum_{i=1}^{7} t_i^2 + b\sum_{i=1}^{7} t_i \end{cases}$$

We have

$$\begin{cases} 669 = 17a + 7b \\ 1446.5 = 57.5a + 17b \end{cases}$$

Solving we have

$$\begin{cases} a = -11 \\ b = 122.3 \end{cases}$$

Then $P = -11t + 122.3$

So, the Norman's pulse rate 2.5 minutes after stopping the exercise program is estimated to be $P = -11(2.5) + 122.3$ or 94.8.

APPLICATION ACTIVITY 4.5

1) An old film is treated with a chemical in order to improve the contrast. Preliminary tests on nine samples drawn from a segment of the film produced the following results.

Sample	A	B	C	D	E	F	G	H	I
x	1.0	1.5	2.0	2.5	3.0	3.5	4.0	4.5	5.0
y	49	60	66	62	72	64	89	90	90

The quantity x is a measure of the amount of chemical applied, and y is the constant index, which takes values between 0 (no contrasts) and 100 (maximum contrast).

i) Plot a scatter diagram to illustrate the data.

ii) It is subsequently discovered that one of the samples of film was damaged and produced an incorrect result. State which sample you think this was.

In all subsequent calculations this incorrect sample was ignored. The remaining data can be summarized as follows: $\sum x=23.5$, $\sum y=584$, $\sum x^2=83.75$, $\sum y^2=44622$, $\sum xy=1883$, $n=8$

iii) Calculate the product moment correlation coefficient,

iv) State with a reason whether it is sensible to conclude from your answer to part(iii) that **x** and **y** are linearly related.

v) The line of regression of **y** on **x** has equation **y=a+bx**. Calculate the value of **a** and **b**, each correct to three significant figures.

vi) Use your regression line to estimate what the contrast index corresponding to the damaged piece of film would have been if the piece has been undamaged.

vii) State with a reason, whether it would be sensible to use your regression equation to estimate the contrast index when the quantity of chemical applied to the film is zero.

4.6 END UNIT ASSESSMENT

1) The following results were obtained from line-ups in Mathematics and Physics examinations:

	$Mathematics\ (x)$	$Physics\ (y)$
Mean	475	39.5
Standard deviation	16.8	10.8

$r = 0.95$

Find both equations of the regression lines. Also estimate the value of y for $x = 30$.

2) For a set of 20 pairs of observation s of the variables x and y, it is known that $\sum_{i=1}^{k} f_i x_i = 250$, $\sum_{i=1}^{k} f_i y_i = 140$, and that the regression line of y on x passes through $(15,10)$. Find the equation of that regression line and use it to estimate y when $x = 10$.

3) In a partially destroyed laboratory record of an analysis of correlation data, the following results only are legible: Variance of x is 9

Equations of regression lines: $8x-10y+66=0$ and $40x-18y-214=0$

What were

a) the mean values of x and y

b) the standard deviation of y, and

c) the coefficient of correlation between x and y.

4) The table below shows the marks awarded to six students in a competition:

Student	A	B	C	D	E	F
Judge 1	6.8	7.3	8.1	9.8	7.1	9.2
Judge 2	7.8	9.4	7.9	9.6	8.9	6.9

Calculate a coefficient of rank correlation.

5) A company is to replace its fleet of cars. Eight possible models are considered and the transport manager is asked to rank them, from 1 to 8, in order of preference. A saleswoman is asked to use each type of car for a week and grade them according to their suitability for the job (*A*-very suitable to *E*-unsuitable).

The price is also recorded:

Model	Transport manager's ranking	Saleswoman's grade	Price (£10s)
S	5	B	611
T	1	B+	811
U	7	D-	591
V	2	C	792
W	8	B+	520
X	6	D	573
Y	4	C+	683
Z	3	A-	716

a. Calculate the Spearman's coefficient of rank correlation between

i. price and transport manager's rankings,

ii. price and saleswoman's grades.

b. Based on the result of a. state, giving a reason, whether it would be necessary to use all three different methods of assessing the cars.

c. A new employee is asked to collect further data and to do some calculations. He produces the following results:

The coefficient of correlation between

i. price and boot capacity is 1.2,

ii. maximum speed and fuel consumption in miles per gallons is -0.7,

iii. price and engine capacity is -0.9

For each of his results say, giving a reason, whether you think it is reasonable.

d. Suggest two sets of circumstances where Spearman's coefficient of rank correlation would be preferred to the Pearson's coefficient of correlation as a measure of association.

UNIT 5

POLYNOMIAL, RATIONAL AND IRRATIONAL FUNCTIONS

Key Unit competence: Use concepts and definitions of functions to determine the domain of polynomial, rational and irrational functions

5.0. INTRODUCTORY ACTIVITY

1)Consider the following sentences:

i. the **function** of the heart is to pump blood

ii. Last Saturday, my sister got married; many people attended the **function.**

iii. The area of a square is **function** of the length of its side.

Explain what is meant by the word "Function" in each of the three sentences above.

2. Which of the following illustrates the idea of a function as "**a quantity whose value depends on the value of another quantity":**

i. $y = \dfrac{4x-4}{(x-1)^2}$

ii. $A = \pi r^2$

iii. $s = \sqrt{A}$

3. Any function involves at least two variables. Suggest, in each case in point (2) above, what is the "**independent variable**" and what is the "**dependent variable**"

4. Describe the similarity and the difference between the functions in part (2) above; make a proposal for the name of each type of function described in part (2) above.

5. Classify the following functions as "**polynomial**", "**rational**" or "**irrational**"

$f(x)=\sqrt{\dfrac{x^2+1}{x-2}}$; $f(x)=\dfrac{x+1}{x-5}$; $f(x)=\sqrt{x^2-1}$; $f(x)=2x-7$; $f(x)=\dfrac{x^3+2x-4}{5x}$

6. If we agree that the set of all possible values the independent variable can assume is called the " **Domain**" of the function and the set of all possible values, the dependent value can assume is called the "**Range**" of the function, determine the range and the domain of each of the functions in part (2) above.

7. For each of the following functions $f(x)=\dfrac{1}{x}$; $g(x)=\sqrt{x}$; $h(x)=x^2$, write down the set of real numbers that are not in the:

i. domain of the function

ii. the range of the function

5.1. Types of functions

ACTIVITY 5.1

Differentiate rational from irrational numbers. Guess which of the following functions is a polynomial, rational or irrational function

1. $f(x)=(x+1)^2$
2. $h(x)=\dfrac{x^3+2x+1}{x-4}$
3. $f(x)=\sqrt{x^2+x-2}$

Content summary

Definition: a function is a relationship between the members of two sets(not necessarily distinct), such that to each value of the independent variable there corresponds at most one value of the dependent variable.

a) Polynomial

A function that is expressible as the sum of finitely many monomials in x is called **polynomial in** x.

Example: x^3+4x+7; $17-\frac{2}{3}x$; y and x^5 are polynomials. Also $(x-2)^3$ is a polynomial in x because it is expressible as a sum of monomials.

In general, f is a polynomial in x if it is expressible in the form

$f(x)=a_0+a_1x+a_2x^2+......+a_nx^n$ where n is a positive integer, and $a_0, a_1, ..., a_n$ are real constants. The greatest positive integer n, *for which* $a_n \neq 0$, *is called the* ***degree*** *of the polynomial, and, in this case,* a_n *is called the* ***leading coefficient*** *of the polynomial.*

A polynomial is called **linear** if it is of degree 1,that is, if it can be expressed in the form a_0+a_1x ,where $a_1 \neq 0$

The polynomial is said to be **quadratic** if it is of degree 2, which means it can be expressed in the form $a_0+a_1x+a_2x^2,\ a_2 \neq 0$,

The polynomial is **cubic,** or of degree 3**,** if it is possible to re write it as $a_0+a_1x+a_2x^2+a_3x^3,\ a_3 \neq 0$,

An **n^th^ degree polynomial** has the form $a_0+a_1x^1+a_2x^2+a_3x^n+...+a_nx^n; a_n \neq 0$

b) Rational function

A function that is expressible as ratio of two polynomials is called **rational function.** It has the form $f(x)=\frac{a_0+a_1x+a_2x^2+......+a_nx^n}{b_0+b_1x+b_2x^2+......+b_mx^m}$.

Example:

$f(x)=\frac{x^2+4}{x-1},\ g(x)=\frac{1}{3x-5}$ are rational functions

c) Irrational function

A function that can be expressed as $\sqrt[n]{f(x)}$, where $f(x)$ contains the variable *x.*

Example:

$f(x)=\frac{\sqrt{x^2+4}}{\sqrt[3]{x-1}},\ g(x)=\sqrt{\frac{x}{x-5}}$ are irrational functions

APPLICATION ACTIVITY 5.1

Observe the given functions and categorize them into polynomial, rational or irrational functions.

$f(x)=x^3+2x^2-2$ $\qquad g(x)=\dfrac{x^3+2x^2-2}{x-5}$ $\qquad h(x)=\sqrt{x^3+2x^2-2}$

5.2. Injective, surjective and bijective functions

ACTIVITY 5.2

a. Consider the function f defined by $f:\mathbb{R}\to\mathbb{R}$

$$x\mapsto f(x)=x^2$$

i. Is there any real number missing an image?

ii. State whether f is a mapping or not

iii. Is there any real number that is image of more than one real number? If yes, give an example. If not, explain.

iv. State whether f is *one to one* or not.

v. Is there any real number that is not image under function f? If yes, give an example. If not, explain.

vi. State whether f is *onto* or not.

b. Consider now the function f defined by $f:\mathbb{R}\to\mathbb{R}^+$

$$x\mapsto f(x)=x^2$$

Repeat questions iii, iv, v and vi

c. Finally, consider the function f defined by $f:\mathbb{R}^+\to\mathbb{R}^+$

$$x\mapsto f(x)=x^2$$

Repeat questions iii, iv, v and vi

d. Consider the function f defined by $f:\mathbb{R}\to\mathbb{R}$

$$x\mapsto f(x)=-x^2+4x$$

Determine the greatest subsets A and B of $\mathbb{R}$ such that the function defined

by $f:A\to B$

$x\mapsto f(x)=-x^2+4x$ is bijective (that is, both one to one and onto)

Content summary

Given sets A and B, a **function** from A to B is a correspondence, or a rule that associates to any element of A either one image in B, or no image in B

A function such that no element of A is missing an image is called a **mapping**, thus, under a mapping any element of A has exactly one image in B (not less than one, and not more than one)

A mapping such that any element of B is image of either one element of A, or of no element of A, is called a **one- to- one mapping**, or an **injective mapping** or simply an **injection;** under a one-to-one mapping no two elements of A share the common image in B.

Mathematically, $(\forall x_1\in A)(\forall x_2\in A); f(x_2)=f(x_1)\Rightarrow x_2=x_1$

A mapping such that any element of B is image of at least one element of A (image of one element of A, or image of more than one element of A), is called an **onto mapping**, or a **surjective mapping** or simply a **surjection**

Mathematically, $(\forall y\in B)(\exists x\in A); f(x)=y$

A mapping both one-to-one and onto is said to be a bijective mapping , or simply ,a bijection

In particular, linear function $f:\mathbb{R}\to\mathbb{R}$

$x\mapsto f(x)=ax+b$, where $a\neq 0$, is bijective, there is no restrictions on the variables(independent or dependent)

Quadratic function $f:\mathbb{R}\to\mathbb{R}$

$x\mapsto f(x)=ax^2+bx+c$, where $a\neq 0$, is not bijective, since some real numbers share images, or some real numbers are not images under function *f.*

But the restrictions

$f:[\frac{-b}{2a},+\infty[\to[-\frac{\Delta}{4a},+\infty[$

$x\mapsto f(x)=a\text{x}^2+bx+c$， $a>0$；

$f:]-\infty,-\frac{b}{2a}]\to[-\frac{\Delta}{4a},+\infty[$

$x\mapsto f(x)=a\text{x}^2+bx+c$， $a>0$；

$f:[\frac{-b}{2a},+\infty[\to]-\infty,-\frac{\Delta}{4a}]$

$x\mapsto f(x)=a\text{x}^2+bx+c$，$a<0$ and

$f:]-\infty,-\frac{b}{2a}]\to]-\infty,\frac{\Delta}{4a}]$

$x\mapsto f(x)=a\text{x}^2+bx+c$， $a<0$ are bijective

The homographic function $f:\mathbb{R}\to\mathbb{R}$

$x\mapsto f(x)=\frac{ax+b}{cx+d}$ is not bijective, since it is not a mapping($x=-\frac{d}{c}$ has no image under function f, or $y=\frac{a}{c}$ is not image under function f)

But the restriction $f:\mathbb{R}-\{-\frac{d}{c}\}\to\mathbb{R}-\{\frac{a}{c}\}$

$x\mapsto f(x)=\frac{ax+b}{cx+d}$ is bijective

Example1

Consider the set of pigeons and the set of pigeonholes on the diagram below to answer the questions:

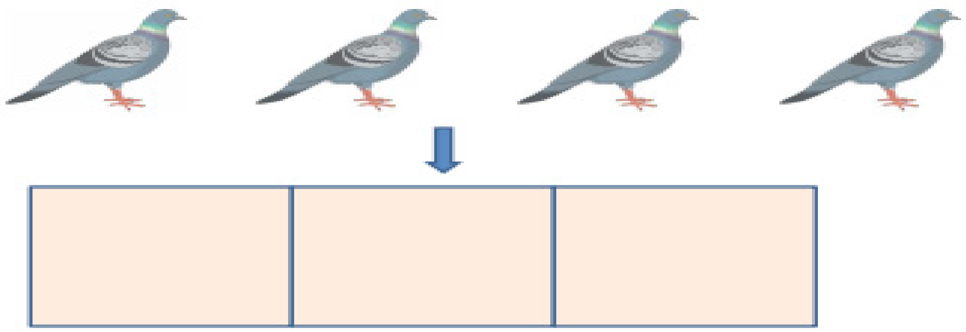
Determine whether it can be established or not between the two sets:

a. A mapping,

b. A one-to-one mapping,

c. An onto mapping,

d. A bijective mapping:



Solution:

Let the pigeons be numbered *a, b, c,* d and the pigeonholes be numbered *1,2,3.*

a. It is possible to establish a mapping between the two sets. For example, $\{(a,1);(b,2);(c,3);(d,3)\}$.This function is a mapping since each pigeon is accommodated in exactly one pigeonhole, though pigeons *c* and *d* are in the same pigeonhole.

b. It is not possible to establish a one-to-one mapping, since sharing images is not allowed.A function from one finite set to a smaller finite set cannot be one-to-one: there must be at least two elements that have the same image

c. The example given in part (a) illustrates a mapping that is onto: no pigeonhole is empty.

d. It is impossible to define a bijection, since it is already impossible to establish a one-to-one mapping

Example2

Determine whether function $f:\mathbb{R}\to\mathbb{R}$

$x\mapsto f(x)=3\text{x}-5$ is (or is not)

a. One-to-one

b. Onto

c. Bijective.

Solution:

a. Let x_1 and x_2 be real numbers such that $f(x_1)=f(x_2)$.Then $3x_1-5=3x_2-5$

This is equivalent, successively to $3x_1=3x_2$ (by adding 5 on both sides);

$x_1=x_2$ (Dividing both sides by 3)

Since the equality $f(x_1)=f(x_2)$ implies $x_1=x_2$, the function is one-to-one.

b. Suppose y a real number. Let us look for real number x, if possible, such that $f(x)=y$.

Then $3x-5=y$. It follows that $x=\frac{y+5}{3}$; such x exists for any value of y ;

$$f(\frac{y+5}{3})=3\left(\frac{y+5}{3}\right)-5=y$$

Therefore, function f is onto.

c. Since, from points (a) and (b), f is one-to-one and onto, function f is bijective.

Example3

Show that function f defined by $f:\mathbb{R}\to\mathbb{R}$

$x\mapsto f(x)=x^2+2x-3$ is neither one-to-one, nor onto

Solution

$f(-2)=(-2)^2+2(-2)-3=-3$ and $f(0)=(0)^2+2(0)-3=-3$

Since $f(-2)=f(0)$ and $-2\neq 0$, the function is not one-to-one.

On the other side, there is no x such that $f(x)=-5$;

in fact, $f(x)=-5\Leftrightarrow x^2+2x-3=-5$

$\Leftrightarrow x^2+2x+2=0$

No such x since $\Delta=2^2-4(1)(2)=-4<0$

Therefore, the function is not onto

Example4

Consider the function f defined by $f:\mathbb{R}\to\mathbb{R}$

$$x\mapsto f(x)=-x^2+4x$$

Determine the greatest subsets A and B of $\mathbb{R}$ such that function $f:A\to B$

$x \mapsto f(x) = -x^2 + 4x$ is bijective

Solution:

The maximum value of the function occurs for $x = 2$ and $f(2) = 4$ .Therefore,

$A = [2, +\infty[$ and $B =]-\infty, 4]$,or $A =]-\infty, 2]$ and $B =]-\infty, 4]$

Example5

Function $f : \mathbb{R} - \{a\} \to \mathbb{R} - \{b\}$

$x \mapsto f(x) = \dfrac{2x-5}{3-x}$ is bijective

a. Find the values of a and b

b. Show that f is one-to-one

c. Find the real number whose image is 2

Solution:

a. f is bijective if $a = 3$ and $b = -2$

b. Let $x_1 \neq 3$ and $x_2 \neq 3$ be such that $f(x_1) = f(x_2)$, that is $\dfrac{2x_1 - 5}{3 - x_1} = \dfrac{2x_2 - 5}{3 - x_2}$

Then $6x_1 - 2x_1x_2 - 15 + 5x_2 = 6x_2 - 2x_1x_2 - 15 + 5x_1$,

which is equivalent to $6(x_1 - x_2) - 5(x_1 - x_2) = 0 \Leftrightarrow x_1 - x_2 = 0 \Leftrightarrow x_1 = x_2$

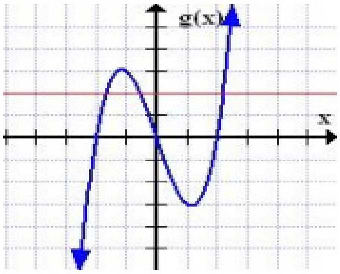
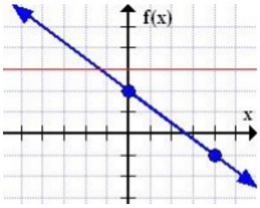
Therefore, f is one- to- one

c. Let x be the number. Then $f(x) = 2 \Leftrightarrow \dfrac{2x-5}{3-x} = 2$

Solving this equation, we get $x = 2$

Horizontal Line Test

Horizontal Line Test states that a function is a one to one(injective) function if there is no horizontal line that intersects the graph of the function at more than one point.

Graph representation	Interpretation	Conclusion
	You can see that for this graph, there are horizontallines that intersect the graph more than once.	Not injective
	You can see that for this graph any horizontal line intersects the graph only once.	Injective function

APPLICATION ACTIVITY 5.2

1. Let A and B be two none-empty sets where $A=\{1,2,3,4\}$ and $B=\{a,b,c\}$ Consider each of the following relations:

$$T=\{(1,a),(2,b),(2,c),(3,c),(4,b)\}$$

$$U=\{(1,a),(2,b),(4,b)\}$$

$$V=\{(1,a),(2,b),(3,c),(4,b)\}$$

Which of these relations $(\mathrm{T},\mathrm{U}\ and V)$ qualify as functions?

2. i) Although the relation V in Question 1 above is a function, it is not a one-to-one (or injective) function. Why?

   ii) Is the relation V above defined an onto (surjective) function? Why?

   (iii) Does the function f, defined by the relation V, have an inverse?

3. a) Find a function $f:\mathbb{N}\to\mathbb{N}$ that is injective, but not surjective.

   b) Find a function $f:\mathbb{N}\to\mathbb{N}$ that is surjective, but not injective.

4. Discuss whether the following functions are bijective: $f(x)=3^x$ and $g(x)=x^3$ if no, which condition fails?

5. Let W and Z be sets; $W \times Z$ the Cartesian product of W and Z ;

C be the set of all correspondences (relations) from W to Z ;

F be the set of all functions from W to Z ;

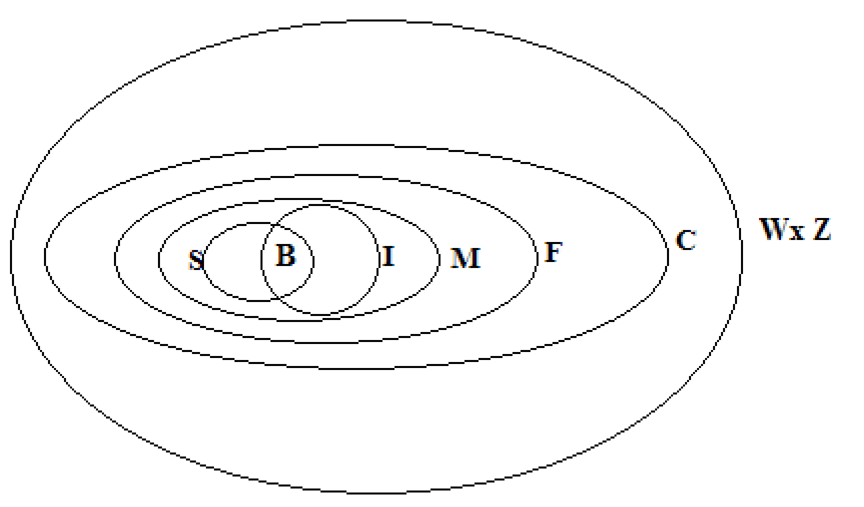
M be the set of all mappings from W to Z ;

I be the set of all one to one mappings from W to Z ;

S be the set of all onto mappings from W to Z ;

B be the set of all bijective mappings from W to Z ;

Then we have the following sequence of inclusion of sets



Using examples, explain in your own words the relationship amongst these sets, and decide on the following inclusion: $S \subset M \subset F \subset C \subset (W \times Z)$.

5.3. Domain and range of a numerical function

5.3.1. Existence condition for a given function

ACTIVITY 5.3.1

Consider the following functions and calculate their numerical value at the given value(s) of x. Discuss your findings

a) $f(x)=\frac{1}{x}$ at $x=0$

b) $f(x)=\frac{2+x}{x-3}$ at $x=1,\ x=2,\ x=3$

c) $f(x)=\frac{2(x-1)}{x-1}$ at $x=0,\ x=1$

d) $f(x)=\frac{x+1}{x^2-1}$ at $x=-1, x=1$ *and* $x=2$

e) $f(x)=\frac{2(x-1)}{x^2+1}$ at $x=-1, x=1$ *and* $x=2$

Content summary

The conditions or restrictions on the independent variable can be formulated as follow:

- Is there any denominator containing the independent variable? If yes, we set any denominator containing the independent variable to be different from zero, and we solve to find the restrictions on the independent variable.
- Is there any nth root, with even index, greater or equal to two, where the radicand is containing the independent variable? If yes, we set the radicand to be greater or equal to zero, and we solve the inequality to find the restrictions on the independent variable.

The goal to succeed in getting the restrictions on the independent variable is to know to solve equations and inequalities.

Remember:

$$ax+b\neq 0 \Leftrightarrow x\neq -\frac{b}{a}; a\neq 0$$

For ax^2+bx+c:

If $b^2-4ac>0$, then $x_1=\dfrac{-b+\sqrt{b^2-4ac}}{2a}; x_2=\dfrac{-b-\sqrt{b^2-4ac}}{2a}$

$ax^2+bx+c\neq 0 \Leftrightarrow x\neq x_1$ and $x\neq x_2$

If $b^2-4ac=0$, then $ax^2+bx+c\neq 0 \Leftrightarrow x\neq -\dfrac{b}{2a}$

If $b^2-4ac<0$, then $ax^2+bx+c\neq 0$ for all real values of *x*.

To solve an inequality, first study the sign of the corresponding expression, then choose the convenient interval(s).

In particular, if $a\neq 0$,then:

x	$-\infty$ .... $-\dfrac{b}{a}$ .... $+\infty$
$ax+b$	Sign of $-a$ 0 sign of a

If $a\neq 0$ and $b^2-4ac>0$,then:

x	$-\infty$ .... x_1 .... x_2 .... $+\infty$
ax^2+bx+c	Sign of a 0 sign of $-a$ 0 sign of a

Where x_1 and x_2 are the roots of the equation $ax^2+bx+c=0$

If $a\neq 0$ and $b^2-4ac=0$,then:

x	$-\infty$ .... $-\dfrac{b}{2a}$ .... $+\infty$
ax^2+bx+c	Sign of a 0 sign of a

Where x_1 and x_2 are the roots of the equation $ax^2+bx+c=0$

If $b^2 - 4ac < 0$,then:

x	$-\infty$ .... $+\infty$
$ax^2 + bx + c$	Sign of a

the equation $ax^2 + bx + c = 0$ has no real roots.

A radical function is a function that contains the independent variable under the square root sign. For example, $f(x) = \sqrt{7 - x}$ is a radical function

An irrational function is a function that contains the independent variable in the radicand; the index may be any positive integer ≥ 2 .Thus all radical functions are irrational functions, but the converse is not true.

The following table gives examples of restrictions for irrational functions.

Function	**Restriction**
$f(x) = \sqrt{x}$	$x \geq 0$
$f(x) = \sqrt{x+10}$	$x + 10 \geq 0$
$f(x) = \sqrt{-x}$	$-x \geq 0 ; x \leq 0$
$f(x) = \sqrt{x^2 - 1}$	$x^2 - 1 \geq 0$
$f(x) = \sqrt{x^2 + 1}$	$x^2 + 1 \geq 0$ (always true for all real values of x) , therefore , there is no restriction for this function.

Examples

For each of the following functions determine the restrictions on the independent variable $f(x) = \sqrt{x^2 - 1} + \frac{1}{x}$

1. $f(x) = \tan x$
2. Let $f(x) = x + 2$ and $g(x) = \sqrt{x+1}$, hence find the restrictions for the function $h(x) = \frac{g(x)}{f(x)}$.
3. $k(x) = \frac{3x^2}{5x - 1}$

Solution:

1) $x^2-1\geq 0$ and $x\neq 0$.This is equivalent to $x\leq -1$ or $x\geq 1$

2) $\cos x\neq 0$,that is $x\neq \pm\frac{\pi}{2}+2k\pi;k\in\mathbb{Z}$

3) For $f(x)$: no restriction

For $g(x): x+1\geq 0$; that is $x\geq -1$

For $\frac{f(x)}{g(x)}$: $x+1\geq 0$ and $x+2\neq 0$, then $x\geq -1$

4) $5x-1\neq 0; x\neq \frac{1}{5}$

APPLICATION ACTIVITY 5.3.1

Find out the condition (s) or the restriction(s) for the existence of the image under the following functions:

i) $h(x)=\frac{x^2-3}{x-4x^3}$ ii) $f(x)=\frac{9}{\sqrt{4-x^2}}$

iii) $f(x)=\frac{\sqrt[3]{x^3+x}}{5x}$ iv) $f(x)=\frac{3x}{x^2+4}$

5.3.2. Domain and range of polynomial functions

ACTIVITY 5.3.2

1) Given the function $f(x)=x-4$. Plot the graph of $f(x)$ and discuss whether it continues endlessly or not.

2) Consider the function $f(x)=\frac{1}{2}(x-2)^2-4$ .complete the following table and use it to plot the graph of $f(x)$. Can you predict all values of x which can make $f(x)$ not defined.

x	-5	-4	-3	-2	-1	0	1	2	3	4
y										

Consider the function $f(x)=x^3-3x^2+3x-2$ .complete the following table and

use it to plot the graph of $f(x)$. Can you predict all values of x which can make $f(x)$ defined.

x	-8	-4	-2	0	1	2	3	4	5
y									

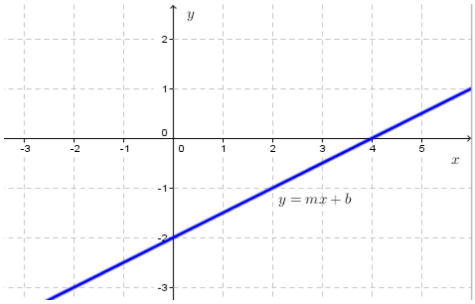
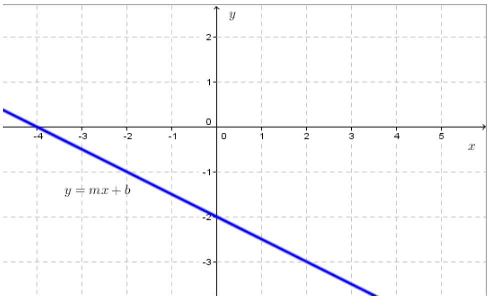
4) For which value(s) of x is the function $f(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_2 x^2 + a_1 x + a_0$ defined ? $a_n \ , a_{n-1}, \ldots, a_2, a_1, a_0$ are real coefficients and $n = 0, 1, 2, 3, 4, \ldots$

Content summary

The domain of any **linear functions** is the set of all real numbers ,

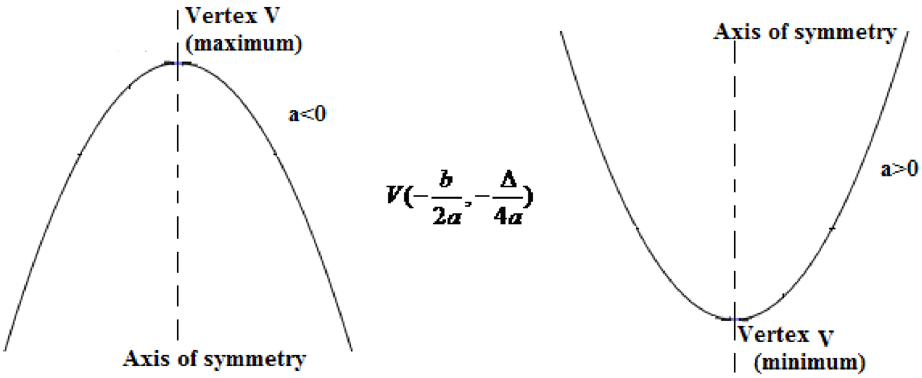
that is Domf= $\mathbb{R}$ or Domf= $]-\infty, +\infty[$ Similarly, the range of a linear function f, denoted Imf, is the set of all real numbers, that is $\operatorname{Im} f =]-\infty, +\infty[$

Depending on the sign of m *in the equation* $y = mx + b$, the trend of the graph is as follows:

$m > 0$	$m < 0$
	

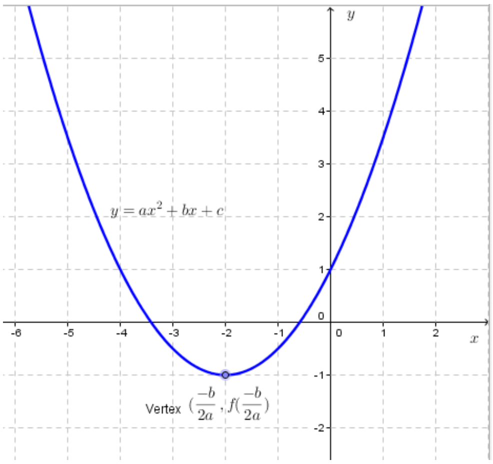
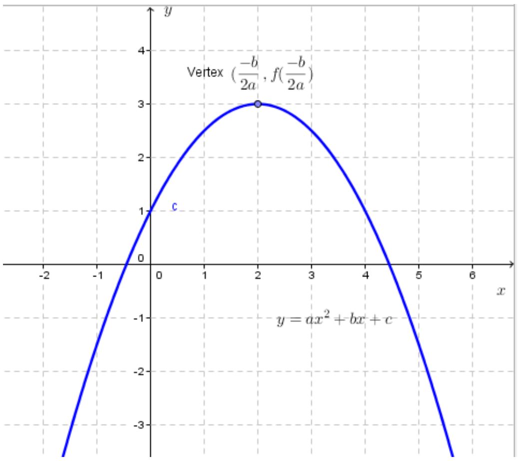
From the graphs, one can observe that each value of x has its corresponding y value.

For quadratic functions $y = ax^2 + bx + c$, the main features are summarized on the graph below:

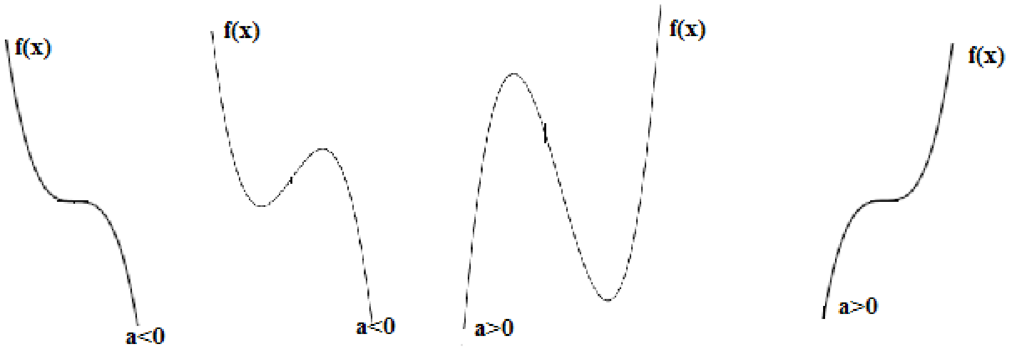


If $a>0$, then the range of function $y=ax^2+bx+c$ is $[-\frac{\Delta}{4a},+\infty[$

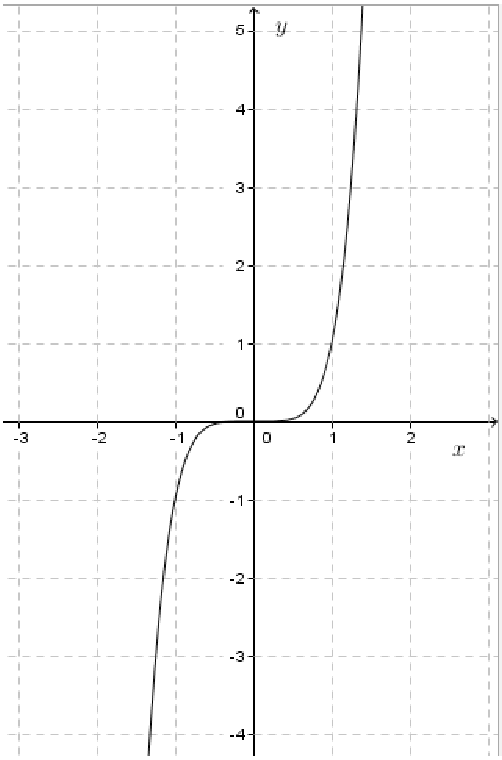
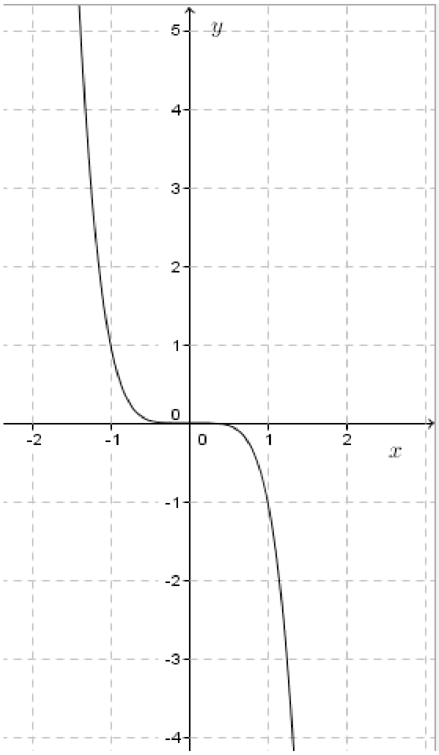
If $a<0$, then the range of function $y=ax^2+bx+c$ is $]-\infty,-\frac{\Delta}{4a}]$

$a>0$, minimum	$a<0$, maximum
	

For cubic functions $f(x)=ax^3+bx^2+cx+d; a\neq 0$,the trends of the graphs are as shown below:



In each case, the domain is $]-\infty,+\infty[$ and the range is $]-\infty,+\infty[$

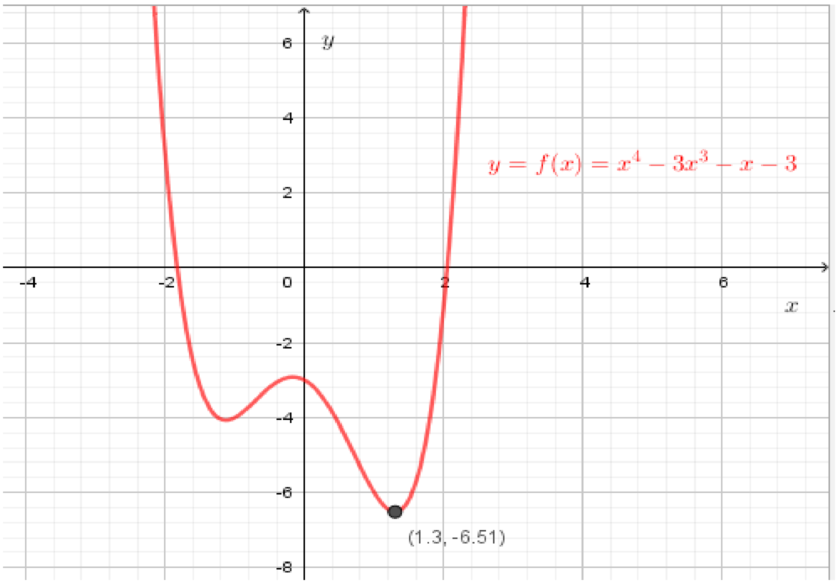
$a > 0$	$a < 0$
	

$D = \left]-\infty, \infty\right[\quad R = \left]-\infty, \infty\right[$.

If for polynomials of odd degrees the range is the set of all real numbers, it is not the case for polynomials of even degree, greater or equal to 4.The determination of the range is not easy unless the function is given by its graph; in this case, find by inspection, on the *y*-axis, the set of all points such that the horizontal lines through those points cut the graph.

Example1

Determine the domain and range of $f(x)=x^4-3x^3-x-3$ shown on the graph below:



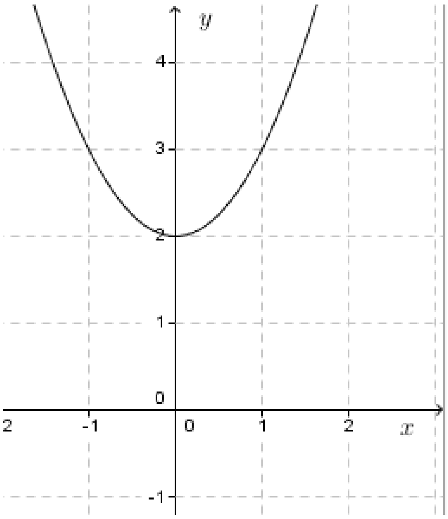
Solution:

$domf=]-\infty,+\infty[$ and $\operatorname{Im} f=[-6.51,+\infty[$

Example 2: Find the range for the function $f(x)=x^2+2$

Solution:

Let $f:\mathbb{R}\to\mathbb{R}$ be the function given by $f(x)=x^2+2$. Then the range is $[2,+\infty[$ as shown on the graph below:



APPLICATION ACTIVITY 5.3.2

Find the domain and range of the following functions.

$a)\, f(x) = -x^2 + 1$

$b)\, f(x) = 2x^3 - x + 1$

$c)\, f(x) = 3x + 1$

5.3.3. Domain and range of rational functions

ACTIVITY 5.3.3

For which value(s) the following functions are not defined

a. $f(x) = \dfrac{1}{x-1}$

b. $f(x) = \dfrac{x}{(x+1)(x-3)}$

Content summary

Given that $f(x) = \dfrac{g(x)}{h(x)}$ where $g(x)$ and $h(x)$ are polynomials, then the domain of definition is the set of real numbers excluding all values where the denominator is zero. That is $Domf = \{x \in \mathbb{R} : h(x) \neq 0\}$

From activity 5.3.3, one can determine the intervals for which the given functions are valid.

Example1

Find the domain of each of the functions:

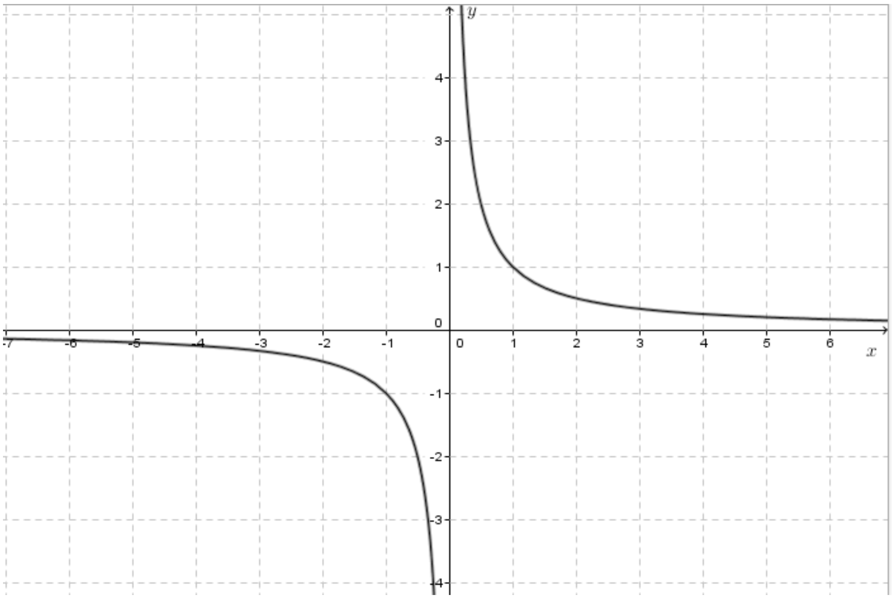
a. $f(x) = \dfrac{1}{x}$

b. $f(x) = \dfrac{x}{(x-1)(x+3)}$

c. $f(x)=\dfrac{x+1}{3x+6}$

Solution:

a) The denominator should be different from zero ($x \neq 0$), domain of definition is $\{x \in \mathbb{R} : x \neq 0\}$ or $\mathbb{R}^{*} or\ \mathbb{R}^{+}$. The domain can be written as an interval as follows: $]-\infty, 0[\cup]0, +\infty[$. Observing the graph of the function $f(x)=\dfrac{1}{x}$, one can early realize that the function has no value only if $x=0$

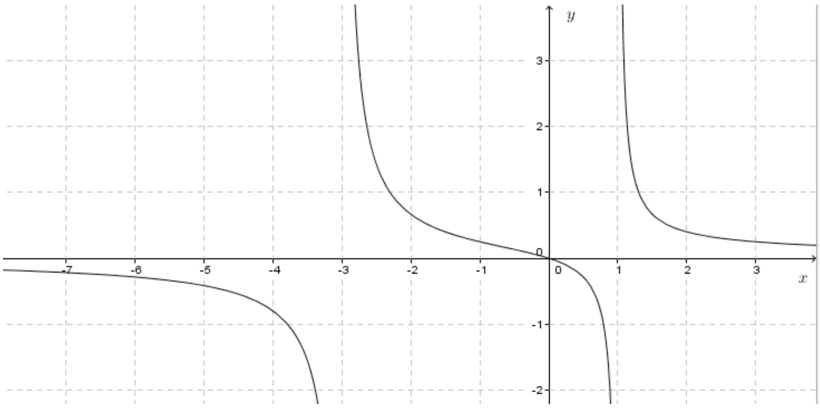


b) $f(x)=\dfrac{x}{(x-1)(x+3)}$. The denominator should be different from zero,

$(x-1)(x+3) \neq 0$. domain of definition is $\{x \in \mathbb{R} : x \neq 1 \text{ and } x \neq -3\}$ or simply $]-\infty, -3[\cup]-3, 1[\cup]1, +\infty[$. Observing the graph of the function

$f(x)=\dfrac{x}{(x-1)(x+3)}$, one can early realize that the function has no value only if

$x=1\ and\ x=-3$



c) Condition: $3x+6\neq 0$

$3x+6=0 \Rightarrow x=-2$

Then, $Domf=\mathbb{R}\setminus\{-2\}$ or $Domf=]-\infty-2[\cup]-2,+\infty[$

Example 2: Find the range for the function $f(x)=\dfrac{1}{x-2}$

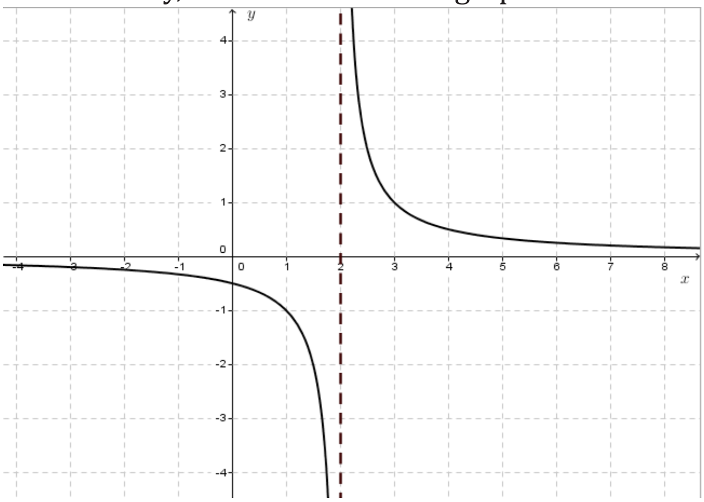
Solution

(1) Put $y=f(x)=\dfrac{1}{x-2}$

(2) Solve for x, $y=\dfrac{1}{x-2} \Leftrightarrow x=\dfrac{1}{y}+2$. Note that x can be solved if and only if $y\neq 0$.

The range of $f(x)$ is $\{y\in\mathbb{R}: y\neq 0\}=\mathbb{R}\setminus\{0\}$.

Alternatively, one can see on the graph that the range of $f(x)$ is $\mathbb{R}\setminus\{0\}$.



Example 3: Find the range for the function $f(x)=\dfrac{2x+1}{x^2+2}$

Solution

(1) Put $y=f(x)=\dfrac{2x+1}{x^2+1}$

(2) Solve for x, $y=\dfrac{2x+1}{x^2+1}\Leftrightarrow yx^2+y=2x+1$.

$yx^2+y=2x+1\Leftrightarrow yx^2-2x+(y-1)=0$

$$x=\frac{2\pm\sqrt{4-4y(y-1)}}{2y}\ \text{ if } y\neq 0,\quad x=-\frac{1}{2}\ \text{ if } y=0$$

$$=\frac{1\pm\sqrt{1-y^2+y}}{y}\ \text{ if } y\neq 0$$

Comparing the two, we see that x exists in the set of real numbers if and only if $1-y^2+y\geq 0$, that is $y^2-y-1\leq 0$

The range of $f(x)$ is $\{y\in\mathbb{R}: y^2-y-1\leq 0\}$. Solving the inequality

$y^2-y-1\leq 0$, we get $y=\dfrac{1\pm\sqrt{5}}{2}$. Then studying the sign of the quadratic expression, $y^2-y-1=\left(y-\dfrac{1-\sqrt{5}}{2}\right)\left(y-\dfrac{1+\sqrt{5}}{2}\right)$

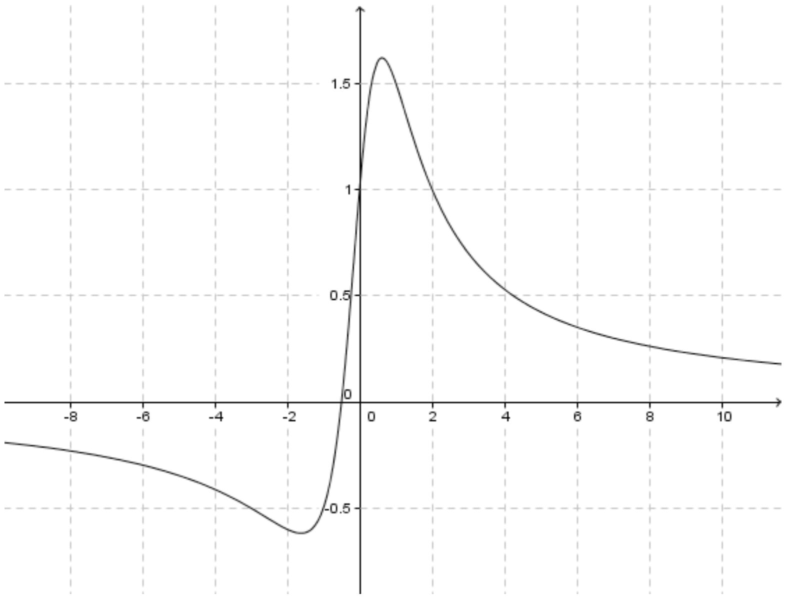
	$y<\frac{1-\sqrt{5}}{2}$	$y=\frac{1-\sqrt{5}}{2}$	$\frac{1-\sqrt{5}}{2}<y<\frac{1+\sqrt{5}}{2}$	$y=\frac{1+\sqrt{5}}{2}$	$y>\frac{1+\sqrt{5}}{2}$
$y-\frac{1-\sqrt{5}}{2}$	———	0	+ + + + + +	+ + +	+ + +
$y-\frac{1+\sqrt{5}}{2}$	———	———	—————	0	+ + +
y^2-y-1	+ + +	0	—————	0	+ + +

From the table , we see that the range of the function $f(x)=\dfrac{2x+1}{x^2+2}$ is:

$$R(f)=\left\{y\in\mathbb{R}:\frac{1-\sqrt{5}}{2}\leq y\leq\frac{1+\sqrt{5}}{2}\right\}=\left[\frac{1-\sqrt{5}}{2},\frac{1+\sqrt{5}}{2}\right]$$

One can see on the graph that the range of $f(x)$ is

$$\left[\frac{1-\sqrt{5}}{2},\frac{1+\sqrt{5}}{2}\right]\simeq[-0.618034;1.61803]$$



APPLICATION ACTIVITY 5.3.3

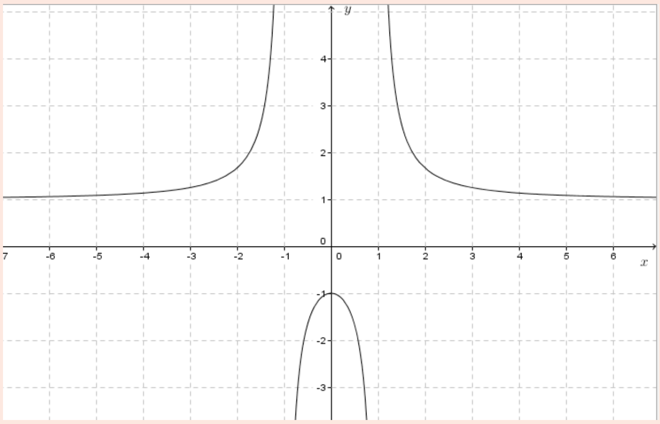
1. Find the domain and range of the following functions

a) $f(x)=\dfrac{2x+1}{6x^2-x-2}$ b) $f(x)=\dfrac{-6}{25x^2-4}$ c) $f(x)=\dfrac{2x-9}{x^3+2x^2-8x}$

d) $f(\mathrm{x})=\dfrac{5}{x-2}$

2. Given the function $f(x)=\dfrac{x^2+1}{x^2-1}$, observe its graph and write down x -values

where the function is not defined.



5.3.4. Domain and range of irrational functions

ACTIVITY 5.3.4

For each of the following functions, give the range of values of the variable x for which the function is not defined

1. $f(x)=\sqrt{2x+1}$ 2. $f(x)=\sqrt[3]{x^2+x-2}$ 3. $g(x)=\sqrt{\dfrac{x-2}{x+1}}$

Content summary

Given that $f(x)=\sqrt[n]{g(x)}$ where $g(x)$ is a polynomial, there are two cases

a) If n is odd number, then the domain is the set of real numbers. That is $domf=\mathbb{R}$

b) If n is even number, then the domain is the set of all values of x such $g(x)$ is positive or zero. That is $domf=\{x\in\mathbb{R}:g(x)\geq 0\}$

Example 1: Given the function $g(x)=\sqrt{x^2-1}$, determine the domain.

Solution

Condition of existence: $x^2\text{-}1\geq 0$ this implies that we need to determine the interval where $x^2\text{-}1$ is positive.

The corresponding equation is $x^2\text{-}1=0$, solving for the variable, we obtain:

$$x^2\text{-}1=0 \Leftrightarrow (x-1)(x+1)=0$$

$$x\text{-}1=0 \Rightarrow x=1$$

$$x+1=0 \Rightarrow x=\text{-}1$$

x	$-\infty$ -1 1 $+\infty$
$x+1$	- - - 0 + + + + + + +
$x-1$	- - - - - - - 0 + + + + + + + +
x^2-1	+ + + 0 - - - - 0 + + + + + +

Therefore $Domf=]-\infty,-1]\cup[1,+\infty[$

Example 2: Find the domain of the function $h(x)=\dfrac{\sqrt{1-x^2}}{x}$

Solution

Conditions of existence $1-x^2\geq 0$ and $x\neq 0$

- For the corresponding equation is $1-x^2=0$ and solving for the variable,

  we get: $(1-x)(1+x)=0 \Rightarrow \begin{cases} 1-x=0 \text{ or } x=1 \\ 1+x=0 \text{ or } x=-1 \end{cases}$

- For $x\neq 0$ all real numbers are accepted except from zero. Combining the two conditions we get:

x	$-\infty$ -1 0 1 $+\infty$
$1+x$	- - - - 0 + + + + + + + + + +
$1-x$	+ + + + + + + + + 0 - - - - -
$1-x^2$	- - 0 + + + + + + 0 - - - - -
x	- - - 0 + + + + + + +

$f(x)=\frac{\sqrt{1-x^2}}{x}$	undefined0 - - - $\Vert$ + + + + + 0 undefined

$-1\le x\le 1$ and $x\ne 0$.Therefore, $domf=[-1,0[\cup]0,1]$

Example 3: Find domain of definition of $f(x)=\sqrt[3]{x+1}$

Solution

Since the index in radical sign is odd number, then $Domf=\mathbb{R}$

Example 4: Find the domain of definition of $g(x)$ if $g(x)=\sqrt[4]{x^2+1}$

Solution

Condition: $x^2+1\ge 0$

Clearly x^2+1 is always positive. Thus $Domg=\mathbb{R}$

Example 5: Find domain of $f(x)=\dfrac{x}{\sqrt{x^3-4x^2+x+6}}$

Solution

Conditions: $x^3-4x^2+x+6\ge 0$ and $x^3-4x^2+x+6\ne 0$.The two conditions are combined in one: $x^3-4x^2+x+6>0$

$x^3-4x^2+x+6=(x+1)(x-2)(x-3)$

x	$-\infty$	-1		2		3		$+\infty$
$x+1$	-	0	+	+	+	+	+	
$x-2$	-	-	-	0	+	+	+	
$x-3$	-	-	-	-	-	0	+	
x^3-4x^2+x+6	-	0	+	0	-	0	+	

Then, $Domf=]-1,2[\cup]3,+\infty[$

Example 6: Find the range for the function $f(x)=\sqrt{1+5x}$

Solution

$1+5x\ge 0$ (Restrictions on x);

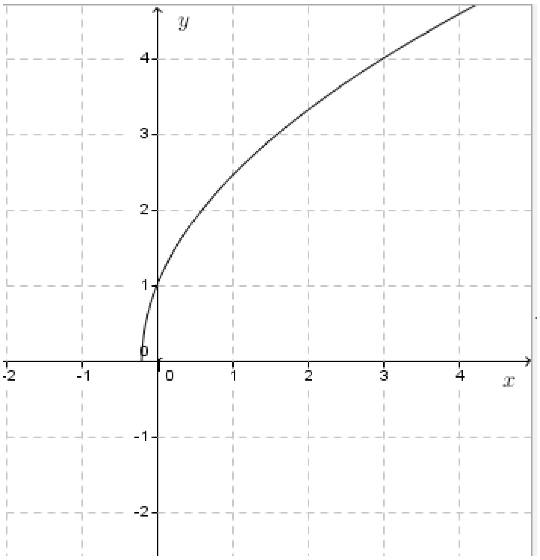
$\sqrt{1+5x}\ge 0$ (Taking the square roots);

But $f(x)=\sqrt{1+5x}$;

Therefore, $f(x)\geq 0$

The range of $f(x)$ is $\operatorname{Im} f=[0,+\infty[$.

The graph below illustrates the range:



APPLICATION ACTIVITY 5.3.4

1. Find the domain of definition for each of the following functions

$f(x)=\sqrt{4x-8}$	$g(x)=\sqrt{x^2+5x-6}$	$h(x)=\dfrac{x^3+2x^2-2}{\sqrt[3]{x+4}}$	$f(x)=\dfrac{x-2}{\sqrt[4]{x^2-25}}$
$f(x)=\sqrt{\dfrac{(x-1)^2}{x+4}}$	$h(x)=\sqrt{\dfrac{(x-1)(x+3)}{8-2x}}$	$f(x)=\dfrac{x-1}{\sqrt{2-x}}$	$f(\mathrm{x})=\sqrt{4-x^2}$

2. Find the range of each of the following functions

$a)\, f(x)=\sqrt{1-x}$

$b)\, f(x)=\sqrt{-x-3}$

c) $f(x)=\dfrac{\sqrt{x+2}}{x^2-9}$

5.3.5. Composition of functions

5.3.5. Composition of functions

ACTIVITY 5.3.5

Given two functions $f(x) = 2x^2 + 2x - 5$ and $g(x) = 3x + 2$

a)Substitute the variable in $f(x)$ by the value of $g(x)$. Is the obtained function a function of x ?

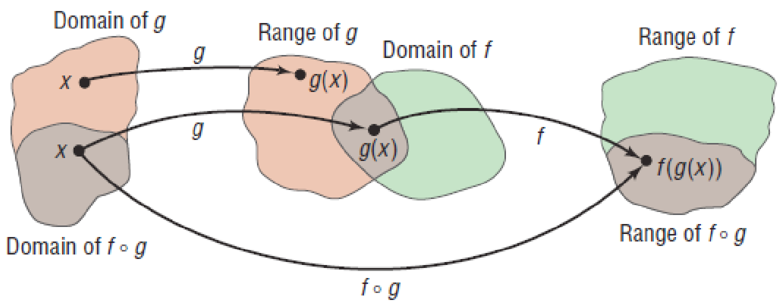
b) Substitute the variable in $g(x)$ by the value of $f(x)$. What do you notice?

c) Name the obtained functions by $h(x)$ and $j(x)$ respectively. Compare the two new functions and explain if they are the same or different.

Content summary

Given two functions $f(x)$ and $g(x)$, one can "combine" them by substituting one function into the other as follow: $f[g(x)]$ or $g[f(x)]$.

Definition: Let $f(x)$ and $g(x)$ be two functions of x such that the codomain of $f(x)$ is a subset of the domain of $g(x)$. The composition of $f(x)$ with $g(x)$, denoted by $(fog)(x)$ is given by $(fog)(x) = f(g(x))$ Indicates that $f(x)$ is a function from A to B and $g(x)$ is a function from C to D where $B \subseteq C$. The function given by $(fog)(x) = f(g(x))$ is the composite of $f(x)$ and $g(x)$. The domain of $(fog)(x)$ is the set of all x in the domain of $g(x)$ such that $g(x)$ is in the domain of $f(x)$ as shown by the figure below.



Only the $x's$ in the domain of g for which $g(x)$ is in the domain of f, can be in the domain of $f \circ g$. The reason is that if $g(x)$ is not in the domain of f then $f[g(x)]$ is not defined. Because of this, the domain of $f \circ g$ is a subset of the domain of g; the range of $f \circ g$ is a subset of the range of f.

The domain of $(f \circ g)(x)$ is the set of all numbers x in the domain of g such that $g(x)$ is in the domain of f

Here, we note that $(f \circ g)(x) \neq (g \circ f)(x)$

Examples

1. *Let* $f(x) = x^2$ *and* $g(x) = 2x+1$. *Find* $(fog)(x)$ *and* $(gof)(x)$.

Solution

By the definition of composition, we have,

$$(fog)(x) = f(g(x)) = f(2x+1)$$
$$= (2x+1)^2 = 4x^2 + 4x + 1$$
$$(gof)(x) = g(f(x)) = g(x^2)$$
$$= 2x^2 + 1.$$

Remark If the range of $f(x)$ is not contained in the domain of $g(x)$, then we have to restrict $f(x)$ to a smaller set so that for every x in that set, $f(x)$ belongs to the domain of $g(x)$. The domain of $(gof)(x)$ is taken to be the following:

$$\textbf{dom } (gof)(x) = \{x \in \textbf{dom } (f(x)) \ : f(x) \in \textbf{dom } (g(x))\}.$$

2. Let $f(x) = x+1$ and $g(x) = \sqrt{x}$ .find $(gof)(x)$

Solution

Note that the domain of $f(x)$ is $\mathbb{R}$ and the domain of $g(x)$ is $[0, +\infty[$. Thus the domain of $(gof)(x)$ is Dom $(gof) = \{x \in \mathbb{R}; x+1 \in [0, +\infty[\}$ $= \{x \in \mathbb{R}; x \geq -1\} = [-1, +\infty[$

APPLICATION ACTIVITY 5.3.5

Given the two functions $f(x)=x^2+1$ and $g(x)=x+1$. Find the following

a) $fog(x)$ b) $gof(x)$ c) $fog(1)$ d) $gof(1)$ e) $fog(x^2)$ e) $gof(\sqrt{x})$

5.3.6. Inverse Functions

ACTIVITY 5.3.6

Given the function $f(x)=4x+6$

a) Substitute $f(x)$ by y and solve for x

b) After solving for x, replace x on (a) by $f^{-1}(x)$ and y by x.

c) Find $f\left[f^{-1}(x)\right]$ and $f^{-1}\left[f(x)\right]$ and compare the results .What have you noticed?

Content summary

Definition: Let $f(x)$ and $g(x)$ be two functions such that $f[g(x)]=x$ for each x in the domain of g and $g[f(x)]=x$ for each x in the domain of f.

Under these conditions, the function g is the inverse of f. The function g is denoted by $f^{-1}(x)$, which is read as " f - inverse". So $f\left(f^{-1}(x)\right)=x$ and $f^{-1}\left(f(x)\right)=x$

Note:

- If $f(x)$ is inverse of $g(x)$, then $g(x)$ is inverse of $f(x)$.
- The domain of f must be equal to the range of f^{-1}, and the range of f must be equal to the domain of f^{-1}.
- The notation $f^{-1}(x)\neq\dfrac{1}{f(x)}$

Example

Let the function $f(x)=\sqrt{2x-3}$ find the inverse function $f^{-1}(x)$.

Solution

$f(\mathrm{x})=\sqrt{2x-3}$ Rewrite the original function

$y=\sqrt{2x-3}$ Replace $f(\mathrm{x})$ with y

$x=\sqrt{2y-3}$ Interchange x and y

$x^2=2y-3$ Square each side to find y

$$x^2+3=2y \Rightarrow y=\frac{x^2+3}{2}$$

Therefore, $f^{-1}(x)=\dfrac{x^2+3}{2}$

APPLICATION ACTIVITY 5.3.6

1. The demand function for a commodity is $p=\dfrac{14.75}{1+0.01x}, x\geq 0$. Where

p is the price per unit and x is the number of units sold.

a) Find x as a function of p

b) Find the number of units sold if the unit price is 10$.

2. For each of the following pair of functions $f(x)$ and $\mathrm{g(x)}$ show that they are inverse to each other.

a) $f(x)=5x+1$ and $g(x)=\dfrac{x-1}{5}$

b) $f(x)=9-x^2, x\geq 0$ and $g(x)=\sqrt{9-x}, x\leq 9$

c) $f(x)=1-x^3$ and $g(x)=\sqrt[3]{1-x}$

5.4. Even functions and odd functions

5.4.1. Even functions

ACTIVITY 5.4.1

Consider the function $f(x)=x^2-1$

i. Complete the table below

x	…..	….	-3	-2	-1	0	1	2	3	…..	….
$y=x^2-1$											

ii. How are the images of any two opposite values of *x?* Express this mathematically,

iii. Plot each pair of obtained points (x, y) in the Cartesian plane and deduce the graph.

iv. From the graph of $f(x)=x^2-1$, what can you say about the line $x=0$

Content summary

Let $f(x)$ be a numerical function whose domain is $Domf$.

$f(x)$ is said to be an **even function** if and only if:

i. $(\forall x \in Domf), -x \in Domf$

ii. $f(-x)=f(x)$, that is, any two opposite values of the independent variable have the same image under the function.

The graph of an even function is symmetrical about the *y*-axis.

Example1

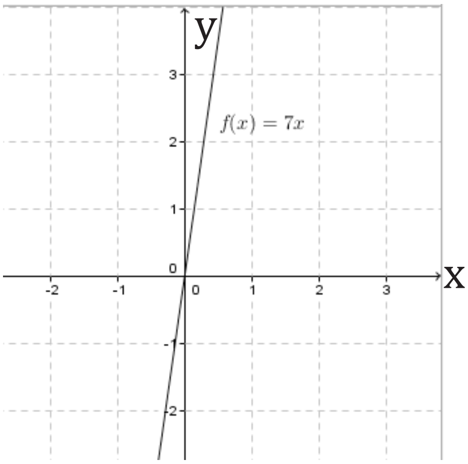
Determine whetherthe function $f(\text{x})=7\,\text{x}$ is even or not.

Solution

The domain of function *f* is the set of all real numbers. For any real number x, the opposite –x is also a real number and $f(-x)=-7x \neq 7x=f(x)$

Since $f(-x) \neq f(x)$, function f is not even.

Graphically,



The graph is not symmetrical about the y-axis

Example2

Determine whether the function $f(x) = 3x^2 - 4$ is even or not.

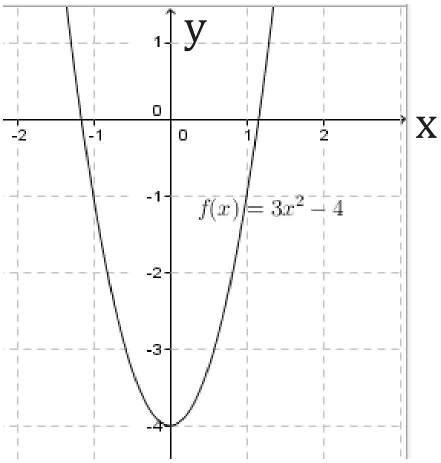
Solution

$f(-x) = 3(-x)^2 - 4 = 3x^2 - 4 = f(x)$

Since $f(-x) = f(x)$, function f is even.

Remember that $(-x)^n = \begin{cases} x^n, \text{if n is even} \\ -x^n, \text{if n is odd} \end{cases}$

The graph of the function is symmetrical about the y-axis as shown on the diagram below:



Example3

Determine whether the function $g(x)=x^6-x^4+x^2+9$ is even or not.

Solution

$$g(-x)=(-x)^6-(-x)^4+(-x)^2+9$$
$$=x^6-x^4+x^2+9$$
$$=g(x)$$

Therefore, the function is even.

Example 4

Determine whether the function $f(x)=\dfrac{3x+1}{x^2-25}$ is even or not.

Solution

$f(-x)=\dfrac{3(-x)+1}{(-x)^2-25} \Leftrightarrow f(-x)=\dfrac{-3x+1}{x^2-25}$. Therefore the function is not even since $f(-x)\neq f(x)$

Example 5

Given functions $f(x)=\sqrt{x}$ and $g(x)=\sqrt{x-4}$, find the $f(x).g(x)$ and determine if the result is an even function or not.

Solution

Provided $x\geq 0$ and $x-4\geq 0$,

$$f(x).g(x)=\sqrt{x}\,.\,\sqrt{x-4} \Leftrightarrow f(x).g(x)=\sqrt{x^2-4x}$$

$f.g(-x)=\sqrt{(-x)^2-4(-x)} \Leftrightarrow f.g(-x)=\sqrt{x^2+4x}$. Therefore the function is not even since $f(-x)\neq f(x)$

Notice that the conclusion could have been drawn from the fact that $x\geq 4$ does not imply $-x\geq 4$, thus function f is not even

APPLICATION ACTIVITY 5.4.1

Determine whether the following functions are even or not.

1. $f(x)=\dfrac{x^2+1}{x^4+3}$ 2. $f(x)=x^{\frac{2}{3}}(x-4)$ 3. $f(x)=x\sqrt{9-x}$

5.4.2. Odd function

ACTIVITY 5.4.2

Given the function $f(x)=x^3$

i. Complete the table below

x	-5	-4	-3	-2	-1	0	1	2	3	4	5
y											

How are the images of any two opposite values of *x?* Express this mathematically,

iii. Plot each pair obtained in the Cartesian plane.

iv. How is the graph, with respect to the origin O?

v. Which name can you give to the origin $(0,0)$ with respect to the graph of $f(x)$?

Content summary

Let $f(x)$ be a numerical function whose domain is $Domf$.

$f(x)$ is said to be an **odd function** if and only if:

i. $(\forall x \in Domf), -x \in Domf$

ii. $f(-x)=-f(x)$, that is, any two opposite values of the independent variable have opposite images under the function.

The graph of an even function is symmetrical about the *origin*.

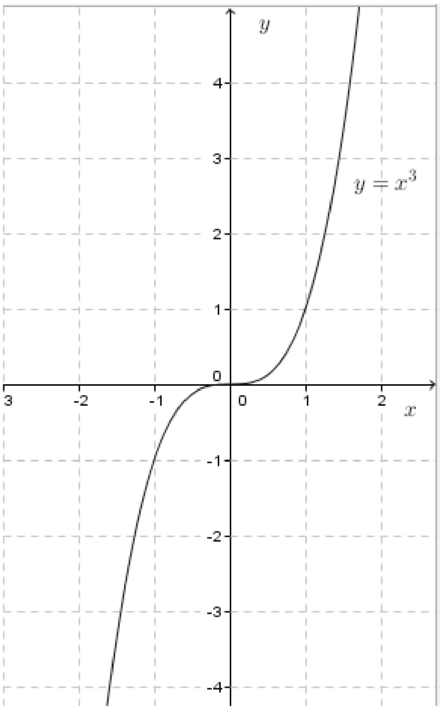
Note: Some functions are neither even nor odd

Example 1: Determine whether the function $f(x)=x^3$ is odd or not

Solution

$f(-x)=(-x)^3 \Leftrightarrow f(-x)=-x^3$ *and* $-f(x)=-x^3$. Therefore, $f(-x)=-f(x)$ and the function $f(x)=x^3$ is odd.

Graphically, the point $(0,0)$ is the center of symmetry for the graph of the function $f(x)=x^3$.



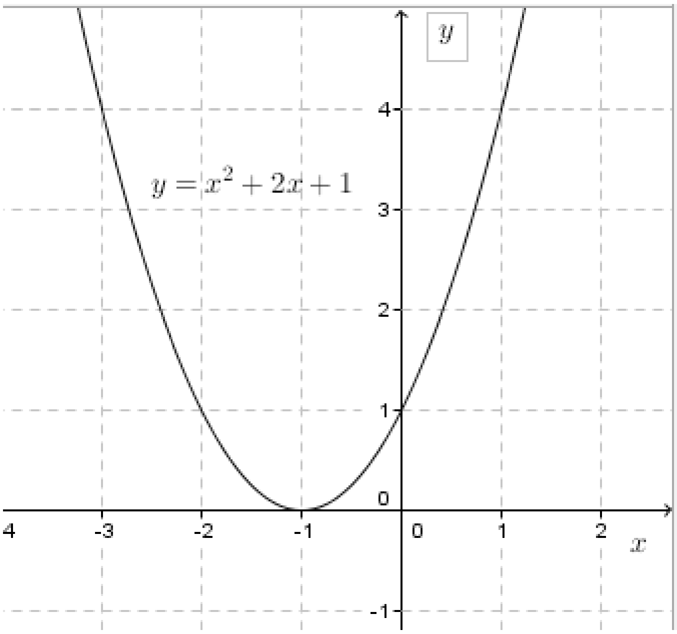
Example 2: Determine whether the function $f(x)=x^2+2x+1$ is odd ,even or neither

Solution

$f(-x)=(-x)^2+2(-x)+1 \Leftrightarrow f(-x)=x^2-2x+1$ *and* $-f(x)=-x^2-2x-1$.

$f(-x)\neq -f(x)$ and $f(-x)\neq f(x)$, therefore,the function $f(x)=x^2+2x+1$ is not odd neither even.

Graphically, point $(0,0)$ is not the center of symmetry for the graph of the function $f(x)$,and the line $x=0$ is not the axis of symmetry for the graph of function $f(x)=x^2+2x+1$.



Example 3: Determine if the function $f(x)=\dfrac{x^3}{x^2-1}$ is even , odd ,or neither and deduce the symmetry of its graph.

Solution

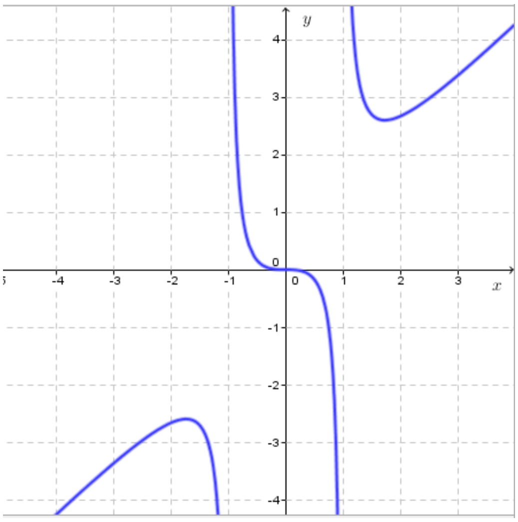
$$f(-x)=\frac{(-x)^3}{(-x)^2-1}=\frac{-x^3}{x^2-1}$$

$$\neq f(x)$$

Therefore, the function is not even

But, $f(-x)=-f(x)$;it follows that f is an odd function.

The graph of $f(x)=\dfrac{x^3}{x^2-1}$ is shown below. It can be seen that point $(0,0)$ is the center of symmetry for the graph



APPLICATION ACTIVITY 5.4.2

1. Consider the function $f(x)$ defined in the set of real numbers $\mathbb{R} \to \mathbb{R}$ by

$$f(x) = \frac{x^3}{9 - x^2}$$

   i. Find the domain of definition of $f(x)$

   ii. Determine if $f(x)$ is even, or odd function and deduce the symmetry of the graph of $f(x)$

2. For which value(s) of x the function $f(x) = \sqrt{x^2 + 5x + 6}$ is valid? Determine if $f(x)$ is an even or odd function.

5.5 END UNIT ASSESSMENT

1. Determine if the given functions are even, odd or neither and deduce the symmetry of their graphs.

   a) $f(x) = \dfrac{x^2 + 4}{x^3 - x}$ b) $f(x) = 3 - 2\sqrt{x+2}$

2. Determine if the relation h is a function which is injective, surjective or neither and explain why.

$$h = \{(-3,8), (-11,-9), (5,4), (6,-9)\}$$

3. Given the function $h(x) = \sqrt{x^2 - 9}$ find

i) $h(5)$

ii) The value(s) of x if $h(x) = 2$

iii) The domain and range of h

4. Use the functions $f(x) = -2x + 1$, $g(x) = \dfrac{x^2}{x-3}$, $h(x) = \sqrt{5 - x}$ to determine

i) $h(-4) - f(-1)$

ii) x if $g(x) = 0$

iii) The domain of $g(x)$ and $h(x)$

5. Find the inverse of $f(x) = 3x - 5$ and $g(x) = \dfrac{2}{x-4}$

6. Given the functions $f(x) = x + 2$ and $g(x) = x - 2$, show that the two functions are inverse to each other.

7. Given $f(x) = \dfrac{2}{x-1}$ and $g(x) = \dfrac{3}{x}$, find the domain of $(fog)(x)$

8. The regular price of a computer in dollars is modelled by $f(x) = x - 400$ or $g(x) = 0.75x$, with x the unit price of a computer.

a. In your own words, give the meaning of the function f or g in terms of the price of the computer.

b. Find $\left(fog\right)(x)$ and describe the meaning of the new model in terms of the price of the computer.

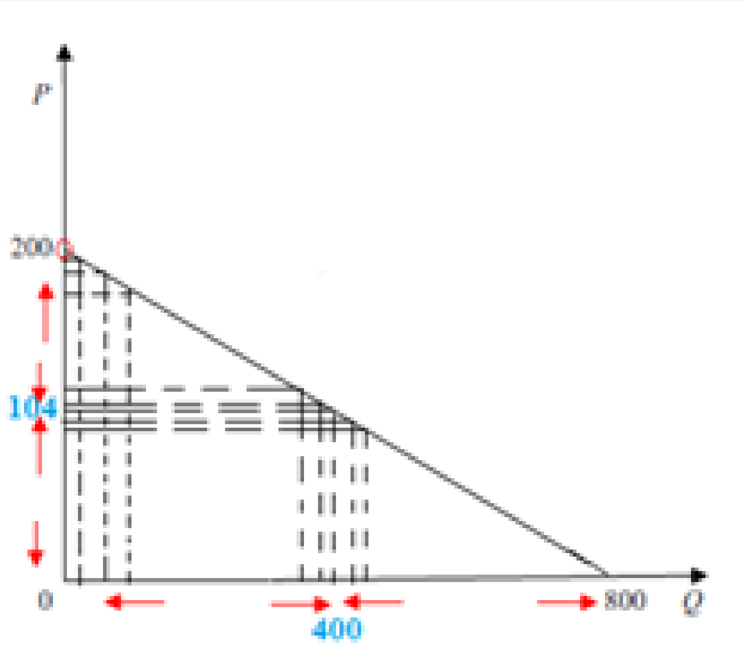
UNIT 6

LIMITS OF POLYNOMIAL, RATIONNAL AND IRRATIONAL FUNCTIONS

Key unit competence: Evaluate correctly limit of function and deduce asymptotes of a real function.

6.0. INTRODUCTORY ACTIVITY

Consider and observe the graph of the price function of quantity Q given by $P = 200 - 0.24Q$



1) Complete the table and approximate the value of P when Q approaches 20.

Q	19.5	19.9	19.999	20	20.1	20,2	20,5	21
P								

When Q approaches 20 , from left and from right, the price gets "closer and closer" to ……, this can be written mathematically as $\lim\limits_{Q \to 20} P = \ldots$

6.1 Concepts of limits

6.1.1 Meaning of limit of a function

ACTIVITY 6.1.1

To find the value of a function $f(x)$ when x approaches 2, a student used a calculator and dressed a table as follows:

x	$f(x)$	x	$f(x)$
2.5	3.4	1.5	5.0
2.1	3.857142857	1.9	4.157894737
2.01	3.985074627	1.99	4.015075377
2.001	3.998500750	1.999	4.001500750
2.0001	3.999850007	1.9999	4.000150008
2.00001	3.999985000	1.99999	4.000015000

a) Is it possible to put the values of x on a number line? Try to do it and locate the point $x = 2$

b) Write 2 possible open intervals of the number line such that their centre is $x = 2$

c) Try to approximate the value of $f(x)$ when x approaches 2.

Content summary

Neighborhood of a real number

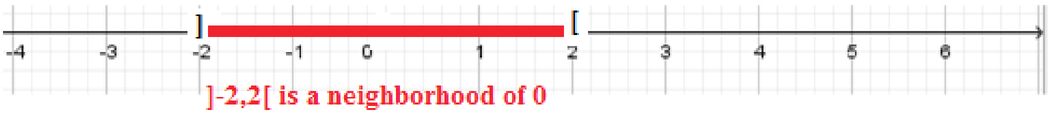
Let x_0 be a real number. A **δ-neighborhood of** x_0 is any open interval centered at x_0 and with radius δ, that is the interval $]x_0-\delta, x_0+\delta[$

A **deleted δ-neighborhood of** x_0 is the set $]x_0-\delta, x_0+\delta[-\{x_0\}$, that is a δ**-neighborhood of** x_0, excluding the center x_0.

A neighborhood can be large or small; for example the intervals $]-2,2[$;

$]-100,100[$ and $]-0.001,0,001[$ are all neighborhoods of 0, while the intervals

$]-2,0[$ and]-1.01;-0.99[are neighborhoods of -1.

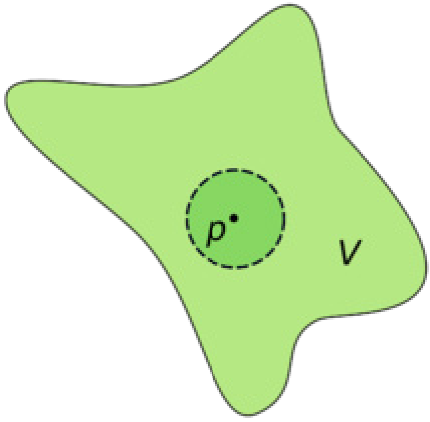


Note that the values of x are on both sides of $x=2$ and that they become closer and closer to $x=2$

Mathematically, a set N is called a neighborhood of point P if there exist an open interval I such that $x \in I$ and $\subset N$. The collection of all neighborhoods of a point is called the **neighborhood system** at the point.

The center and the radius of the interval $]a,b[$ are $\frac{a+b}{2}$ and $\frac{b-a}{2}$, respectively.

A set V in the plane is a neighborhood of a point p if there exists a small disk around p is contained in V as illustrated below.



Example 1

a. Determine whether interval I is a neighborhood of x_0 in each of the following cases"

i. $I =]2,3[$; x_0 is any element of I

ii. $I = [2,3]$; $x_0 = 3$

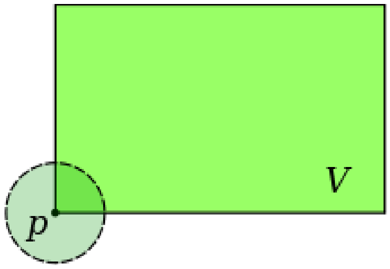
b. Determine whether a rectangle is a neighborhood of any of its corners.

Solution

a. i. I is a neighborhood of any of its elements x_0 , since it is always possible to center an open interval at x_0, the interval being contained in I

ii. I is not a neighborhood of x_0, since it is not possible to center an open interval at x_0, the interval being contained in I, that is any interval centered at x_0, no matter how small it is, will have some elements outside I

b. A rectangle is not a neighborhood of any of its corners.



In fact, it is not possible to center a disc at p so that the disc is entirely contained in the rectangle, no matter how small is the disc.

Example 2

Determine the 1-neighborhood of $x=0$ and the deleted 1-neighborhood of $x=0$

Solution:

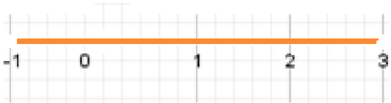
The interval $]-1,1[=\{y:-1<y<1\}$ is the 1-neighborhood of $x=0$ on the real line $]-1,0[\cup]0,1[=]-1,1[\setminus\{0\}$ is the 1-deleted neighborhood of 0.

Example 3

Find the center and the radius of the interval $]-1,3[$

Solution:

The center is $\frac{-1+3}{2}=1$ and the radius is $\frac{3-(-1)}{2}=2$, as shown on the diagram below:



In everyday language, people refer to "a speed limit", "the limit of one's endurance" or "stretching a spring to its limit". All these phrases suggest that limit is a bound. Which on some occasions may not be reached but on the other occasions may be reached or exceeded.

Consider a spring that will break if a weight of 10 newton or more is applied. To determine how far the spring will be stretched without breaking you could attach increasingly heavier weights, as long as they are less that 10 newton, and measure the spring length s for each weight w. If the length s of the spring approaches a value L, then it is said that the limit of s as w approaches 10 is L.

If $f(x)$ becomes arbitrarily close to a unique number L as x approaches c from either side, then the limit of $f(x)$ as x approaches c is L.

This is written mathematically as $\lim_{x \to c} f(x) = L$

Example

You are given 24m of wire and asked to form a rectangle.

a. Express the length l of t5he rectangle as function of the width w of the rectangle

b. Express the area A of the rectangle as function of w.

c. Complete the following table

width,w	5.0	5.5	5.9	6.0	6.1	6.5	7.0
Area,A							

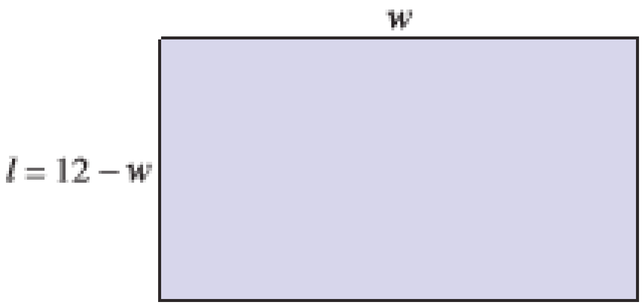
d. Predict, from the table, the maximum area of the rectangle and the value of w for which it occurs.

Solution

a. Let w be the width of the rectangle and l represent the length of the rectangle.

Because $2w + 2l = 24$ (perimeter is 24)

It follows that $l = 12 - w$ as shown below.



b. Knowing that the area of the rectangle is $A = lw$, we have $A = (12 - w)w$ (Substitute $(12 - w)$ for l). $A = 12w - w^2$ After removing brackets.

c. Using this model for area, you can obtain the following:

width,w	5.0	5.5	5.9	6.0	6.1	6.5	7.0
Area,A	35.00	37.75	35.99	36.00	35.99	37.75	35.00

d. From the table, it appears that the maximum area is 36 m^2 and it occurs when $w=6$.

e. In limit terminology, we can say that "the limit of A as w approaches 6 is 36." This is written and calculated as :

$$\lim_{w\to 6} A = \lim_{w\to 6}(12w - w^2)$$

$$= 12(6) - (6)^2$$

$$= 72 - 36$$

$$= 36$$

To estimate numerically the $\lim_{x\to a} f(x)$ we have to construct a table that shows values of $f(x)$ for two sets of x-values: one set whose elements approach a a from the left and another whose elements approach a a from the right.

APPLICATION ACTIVITY 6.1.1

1. Apart of The Kingdom of Lesotho, give two examples of countries or Cities in the world that are surrounded by a single country or city.

2. Give three examples of intervals that are neighborhoods of -5

3. Is a circle a neighborhood of each of its points? Why

4. Draw any plane and show three points on that plane for which the plane is their neighborhood.

5. Use a table to estimate limits numerically.

   a. $\lim_{x\to 2}(3x-2)$

x	1.9	1.99	1.999	2.00	2.001	2.01	2.1
$f(x)$				?			

b) $\lim_{x\to 0}\frac{x}{\sqrt{x+1}-1}$

x	-0.1	-0.01	-0.001	-0.0001	0	0.0001	0.001	0.01	0.1
$f(x)$					?				

c) $\lim_{x\to 2}\frac{x^2-4}{x-2}$

x	1.9	1.99	1.999	2.00	2.001	2.01	2.1
$f(x)$				?			

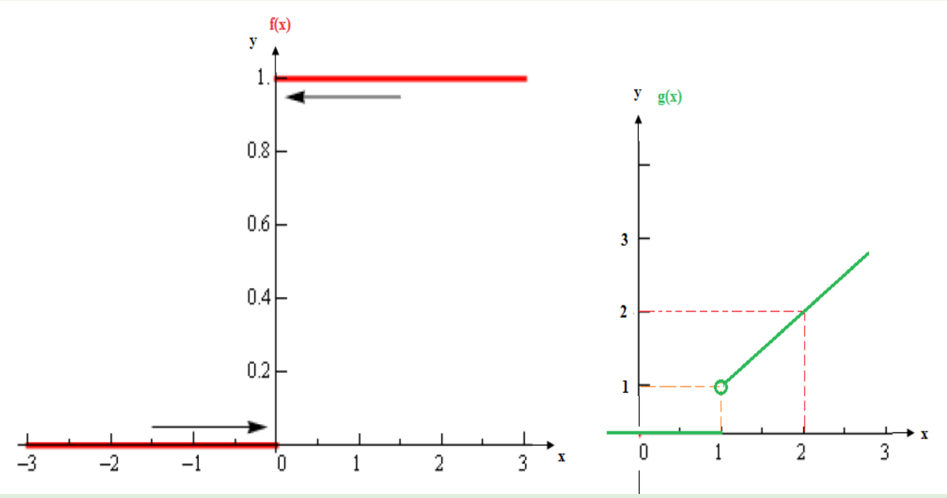
d) $\lim_{x\to 1}\left(2x^2+x-4\right)$.

e) $\lim_{x\to -1}\frac{x+1}{x^2-x-2}$.

6.2. Graphical interpretation and one sided limits of a function

ACTIVITY 6.2

Consider the graphs of two functions *f(x)* and *g(x)* represented below:



a) Complete the following:

$$f(x)=\begin{cases}\ldots.\text{ for } x<0\\ \ldots..\text{ for } x\geq 0\end{cases} \quad\text{and}\quad g(x)=\begin{cases}\ldots.\text{ for } x\leq 1\\ \ldots..\text{ for } x>1\end{cases}$$

b) If we stay to the left side, as x approaches 0, $f(x)$ gets closer to .....;

c) If we stay to the right side, as x approaches 0, $f(x)$ gets closer to ......

d) If we stay to the left side, as x approaches 1, $g(x)$ gets closer to .....;

e) If we stay to the right side, as x approaches 1, $g(x)$ gets closer to .....;.

Content summary

Right-handed limit

We say that $\lim_{x\to a^+} f(x) = L$ provided that we can make *f(x)* as close to L , as we want, for all x sufficiently close to *a* and $x > a$,without actually letting x be *a*.

Left-handed limit

We say $\lim_{x\to a^-} f(x) = L$ provided that we can make $f(x)$ as close to L ,as we want, for all x sufficiently close to a and $x < a$ without actually letting x be a.

For the right-handed limit, we now have $x \to a^+$ (note the "+") which means that we will only look at $x>a$.

Likewise for the left-handed limit, we have $x \to a^-$ (note the "-") which means that we will only be looking at $x<a$.

So when we are calculating limits, it's important to pay attention to whether we are dealing with one-sided limit or not.

Condition of existence for a limit

If the value of $f(x)$ approaches L_1 as x approaches x_0 from the right side we write $\lim_{x\to x_0^+} f(x) = L$ and we read "**the limit of $f(x)$ as x approaches x_0 from the right equals L_1** .

If the value of $f(x)$ approaches L_2 as x approaches x_0 from the left side we write $\lim_{x\to x_0^-} f(x) = L_2$ and we read "**the limit of $f(x)$ as x approaches x_0 from the left equals L_2**

If the limit from the left side is the same as the limit from the right side, that is, $\lim_{x\to x_0^-} f(x) = \lim_{x\to x_0^+} f(x) = L$**,** then we write $\lim_{x\to x_0} f(x)$ and we read "**the limit of** $f(x)$ **as** x **approaches** x_0 **equals** L.

Note that $\lim_{x\to x_0} f(x)$ exists if and only if $\lim_{x\to x_0^-} f(x) = \lim_{x\to x_0^+} f(x)$

Always recall that the value of a limit does not actually depend upon the value of the function at the point into consideration. The value of a limit depends only on the behavior of the values of the function around the point into consideration . Therefore, even if the function doesn't exist at this point the limit can still have a value. To find graphically the limit $\lim_{x\to a} f(x)$, we must consider the behavior of the values of the independent variable x in the neighborhood of a, and analyze the corresponding behavior of $f(x)$, and then draw conclusion about the limit of the function.

Example 1:

Calculate the limit of function $f(x) = \frac{|x-2|}{x^2+x-6}$ as x approaches *2*

Solution

$$\lim_{x\to 2} \frac{|x-2|}{x^2+x-6}$$

$$|x-2| = \begin{cases} x-2, if \ \ x \geq 2 \\ -(x-2), if \ \ x < 2 \end{cases}$$ Applying the properties of absolute value.

- $$\lim_{x\to 2^+} \frac{x-2}{x^2+x-6}$$

  $$= \lim_{x\to 2^+} \frac{x-2}{(x-2)(x+3)}$$ Factorizing the denominator.

  $$= \lim_{x\to 2^+} \frac{1}{x+3}$$ Simplifying like terms.

  $$= \frac{1}{(2)+3} = \frac{1}{5}$$ Replacing x by 2 and evaluating.

- $$\lim_{x\to 2^-} \frac{-(x-2)}{x^2+x-6}$$

$$= \lim_{x\to 2^-} \frac{-(x-2)}{(x-2)(x+3)}$$ Factorizing the denominator.

$$= \lim_{x\to 2^-} \frac{-1}{x+3}$$ Simplifying like terms.

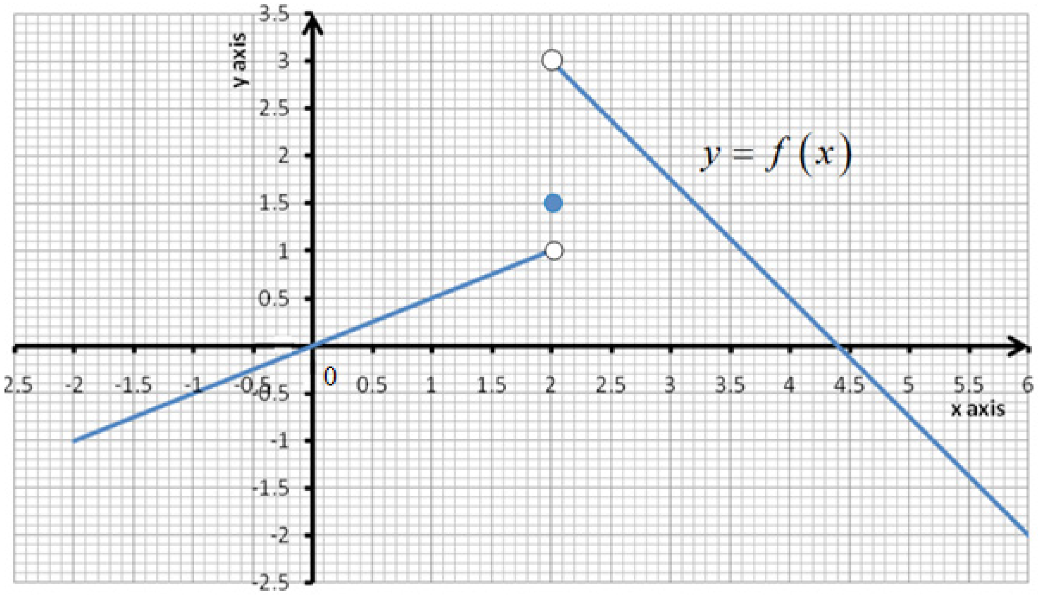
$$= \frac{-1}{(2)+3} = \frac{-1}{5}$$ Replacing x by 2 and evaluating.

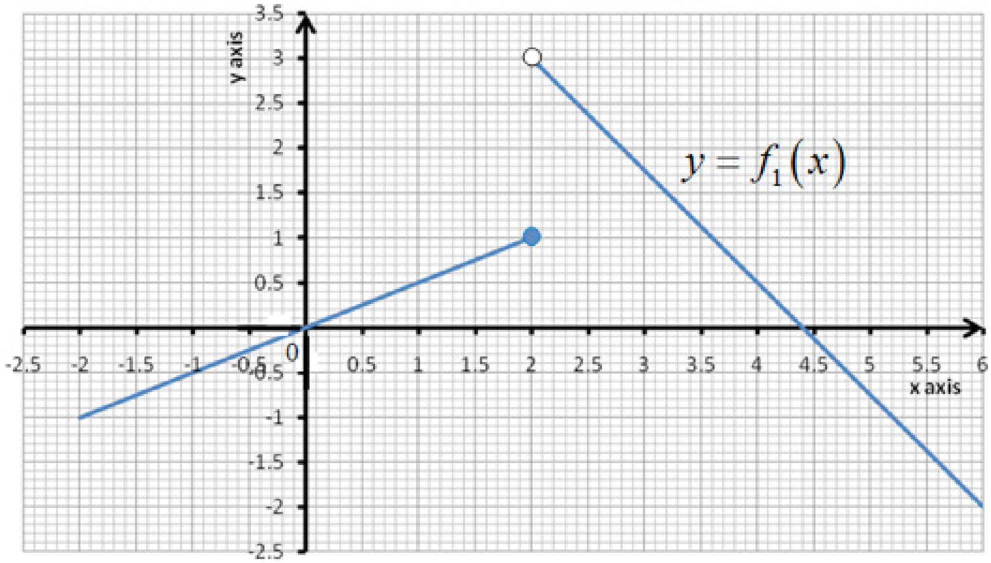
Note: $2^- \neq -2$

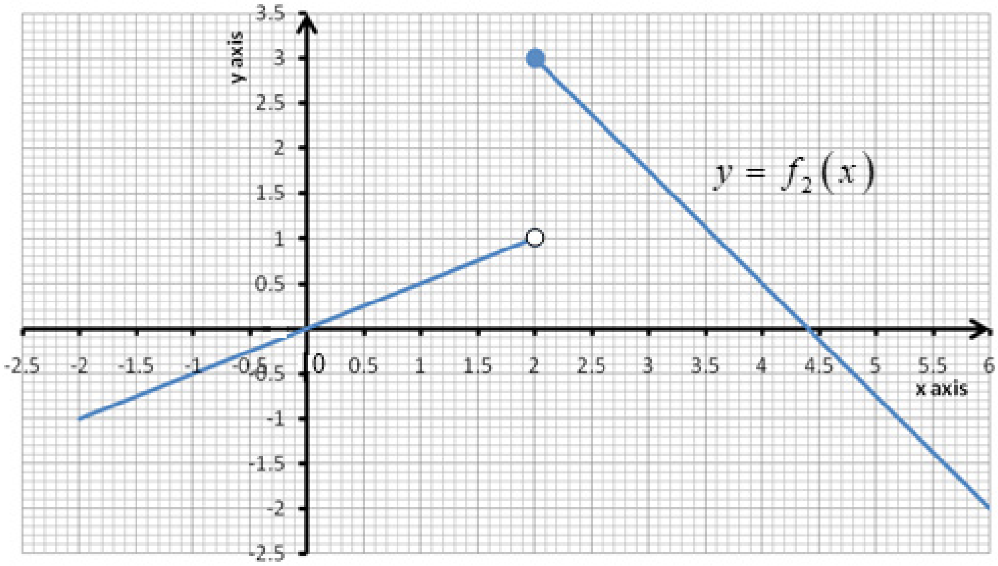
Conclusion $\lim_{x\to 2} f(x)$ does not exist since $\lim_{x\to 2^-} f(x) \neq \lim_{x\to 2^+} f(x)$

Example2

Let $y = f(x)$, $y = f_1(x)$ and $y = f_2(x)$ be the functions whose graphs are shown below







a.Determine:

i. $f(2)$

ii. $f_1(2)$

iii. $f_2(2)$

b. Determine the limit of each function as:

i. x approaches 2 from left side

ii. x approaches 2 from right side

c. Determine whether the limit of function *f, as x approaches 2* exist or not.

Solution:

a.We have:

i. $f(2)=1.5$

ii. $f_1(2)=1$

iii. $f_2(2)=3$

b. i.As x approaches 2 from the left, $f(x)$ approaches 1, so $\lim\limits_{x\to 2^-} f(x)=1$.

As x approaches 2 from the left, $f_1(x)$ approaches 1,so $\lim\limits_{x\to 2^-} f_1(x)=1$

As x approaches 2 from the left, $f_2(x)$ approaches 1,so $\lim\limits_{x\to 2^-} f_2(x)=1$

ii. As x approaches 2 from the right, $f(x)$ approaches 3, so $\lim\limits_{x\to 2^+} f(x)=3$

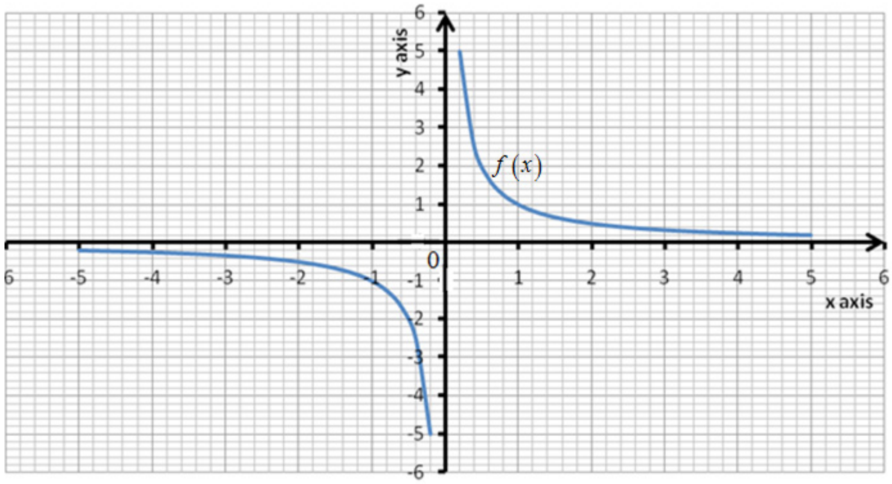
As x approaches 2 from the right, $f_1(x)$ approaches 3,so $\lim\limits_{x\to 2^+} f_1(x)=3$

As x approaches 2 from the right, $f_2(x)$ approaches 3,so $\lim\limits_{x\to 2^+} f_2(x)=3$

c. Since the left hand and right hand limits at *2* are different, the limit of function f at *x=2* does not exist, though the function *f* is defined at *2;* $f(2)=1.5$

Notice

Let f be the function whose graph is shown below:

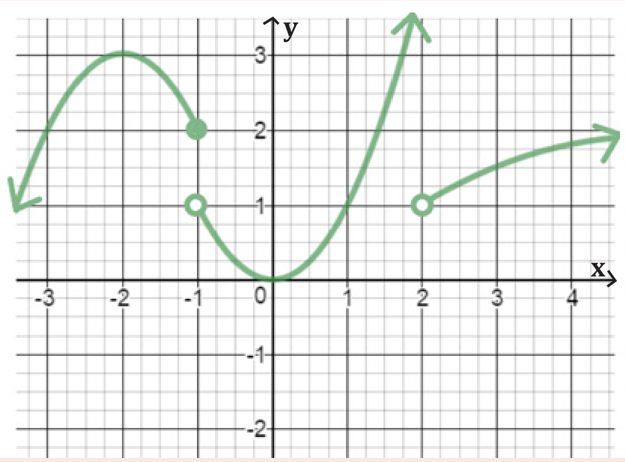


As x approaches 0 from the right side, $f(x)$ gets larger and larger without bound and consequently approaches no fixed value. In this case, we would write $\lim_{x\to 0^+} f(x) = +\infty$ to indicate that the limit fails to exist because $f(x)$ is increasing without bound.

As x approaches 0 from the left side, $f(x)$ becomes more and more negative without bound and consequently approaches no fixed value. In this case, we write $\lim_{x\to 0^-} f(x) = -\infty$ to indicate that the limit fails to exist because $f(x)$ is decreasing without bound. As $\lim_{x\to 0^+} f(x) \neq \lim_{x\to 0^-} f(x)$ then $\lim_{x\to 0}$ f(x)does not exist as x gets larger and larger, $f(x)$ gets close to zero. Also as x becomes more and more negative, $f(x)$ is close to zero. Thus, $\lim_{x\to+\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$

APPLICATION ACTIVITY 6.2

1. Given the graph of the function illustrated below,



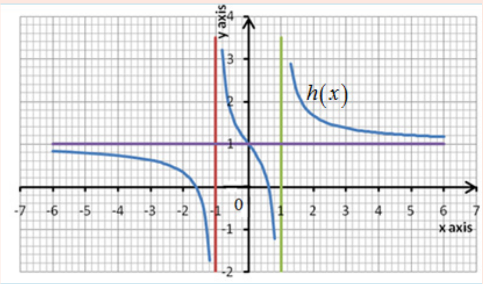
From the graph

a) $\lim_{x\to -1^+} f(x)$ and $\lim_{x\to -1^-} f(x)$ and conclude about $\lim_{x\to -1} f(x)$.

b) $\lim_{x\to 2^+} f(x)$ and $\lim_{x\to 2^-} f(x)$ and conclude about $\lim_{x\to 2} f(x)$.

2. Evaluate $\lim_{x\to -1} \frac{|x+1|}{x^2-1}$

3. Find $\lim_{x\to -1} h(x), \lim_{x\to 1} h(x), \lim_{x\to -\infty} h(x), \lim_{x\to +\infty} h(x)$ using the following graph of $h(x)$



5. In the same Cartesian plane sketch the curves of \

$f(x)=x^2+5, g(x)=-x^2+5, h(x)=5$ and $h(x)=5$. What can you say about the three curves?

6.2.1. Properties of limits

ACTIVITY 6.2.1

Evaluate the following limits and make a comparison of the answers of each sub-question.

a. $\lim\limits_{x\to 0}[3(3x-1)], 3\left[\lim\limits_{x\to 0}(3x-1)\right]$

b. $\lim\limits_{x\to 0}(x^2), \lim\limits_{x\to 0}(3x-1), \lim\limits_{x\to 0}(x^2+3x-1)$

c. $\lim\limits_{x\to 1}(x^2+3x-6), \lim\limits_{x\to 1}(x+4), \lim\limits_{x\to 1}\dfrac{x^2+3x-6}{x+4}$

d. $\lim\limits_{x\to 2}(x-1), \quad \lim\limits_{x\to 2}(x+4), \quad \lim\limits_{x\to 2}(x^2+3x-4)$

e. $\lim\limits_{x\to -4}\left[(x^2+1)^2\right], \left[\lim\limits_{x\to -4}(x^2+1)\right]^2$

Content summary

Assume that $\lim\limits_{x\to a} f(x)$ and $\lim\limits_{x\to} g(x)$ exist and that c is any constant. Then,

1. $\lim\limits_{x\to a}[cf(x)] = c\lim\limits_{x\to a} f(x)$

2. $\lim\limits_{x\to a}[f(x)\pm g(x)] = \lim\limits_{x\to a} f(x) \pm \lim\limits_{x\to a} g(x)$, the limit of a sum or difference is equal to the sum or difference of the limits of the terms . This is also true for more

than two functions.

3. $\lim_{x\to a}[f(x)g(x)] = \lim_{x\to a} f(x) \lim_{x\to a} g(x)$

The limit of a product is the same as the product of limits of the factors. This property applies for more than two factors.

4. $\lim_{x\to a}\left[\frac{f(x)}{g(x)}\right] = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$, provided that $\lim_{x\to a} g(x) \neq 0$

5. $\lim_{x\to a}[f(x)]^n = \left[\lim_{x\to a} f(x)\right]^n$

In this property n can be any real number (positive or negative, integer, fraction, irrational, zero, etc.). In the case that n is an integer this rule can be thought of as an extended case of 3.

6. $\lim_{x\to a}\left[\sqrt[n]{f(x)}\right] = \sqrt[n]{\lim_{x\to a} f(x)}$

provided $f(x) \geq 0$ in a $\delta - neighborhood$ of a for some $\delta > 0$,if n is an even positive integer.

$$\lim_{x\to a}\left[\sqrt[n]{f(x)}\right] = \lim_{x\to a}[f(x)]^{\frac{1}{n}}$$

$$= \left[\lim_{x\to a} f(x)\right]^{\frac{1}{n}} = \sqrt[n]{\lim_{x\to a} f(x)}$$

7. $\lim_{x\to a} c = c$ In other words, the limit of a constant is just the constant.

8. $\lim_{x\to a} x = a$

9. $\lim_{x\to a} x^n = a^n$

It can be observed that the calculation of the limit of a function is equivalent to the calculation of the numerical value of the function, that is $\lim_{x\to a} f(x) = f(a)$ provided $f(\text{a})$ is meaningful.

Example: Evaluate the following limits

a. $\lim_{x\to 1}(4x^3 - 3x^2 + 2x - 1)$

Solution

$$\lim_{x\to 1}(4x^3 - 3x^2 + 2x - 1)$$

$= 4(1)^3 - 3(1)^2 + 2(1) - 1$ Direct substitution of x by 1

=4-3+2-1

=2

b. $\lim_{x\to 0}\frac{x^2-2x-3}{x+6}$

Solution

$\lim_{x\to 0}\frac{x^2-2x-3}{x+6}$

$= \frac{(0)^2-2(0)-3}{(0)+6}$ Direct substitution of x by 0

$= \frac{-3}{6} = \frac{-1}{2}$

c. $\lim_{x\to 1}\frac{\sqrt{x+3}}{\sqrt{3x-2}}$

Solution

$= \frac{\sqrt{(1)+3}}{\sqrt{3(1)-2}} = \frac{2}{1} = 1$ Direct substitution and evaluating.

APPLICATION ACTIVITY 6.2.1

If $\lim_{x\to 3} f(x) = 3$ and $\lim_{x\to 3} g(x) = -3$ .find

a) $\lim_{x\to 3}\left[f(x)+g(x)\right]$

b) $\lim_{x\to 3}\left[f(x)g(x)\right]^3$

c) $\lim_{x\to 3} 5f(x)$

6.3. Finite and infinite limits

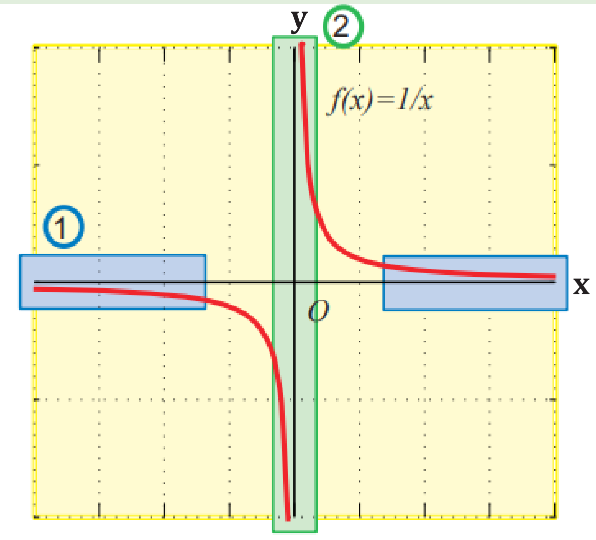
ACTIVITY 6.3

1. Consider the function $f(x)=\dfrac{x+1}{x-1}$ .Find

   a) $f(0.97)$ b) f(0.98) c) f(0.99) c) f(1.01)

   d) f(1.02) e) f(1.03)

2. Observe the following graph of $f(x)=\dfrac{1}{x}$ and hence complete the following table under it.



x	-100	-10	-1	-0.1	0	0.1	1	10	100
$f(x)$									

a) What can you say about y -values as x increases without bound or decreases without bound?

b) What can you say about y -values as x approaches to zero from the right and left?

3. Evaluate the following:

   a) $-2+\infty$ b) $3+\infty$ c) $\dfrac{+\infty}{-\infty}$ d) $\dfrac{3}{\infty}$ e) $+\infty-\infty$ f) $-\infty\times(+\infty)$ g) $\dfrac{\infty}{-2}$

Content summary

Finite and Infinity limits.

A function whose values grow arbitrarily large are said to have an infinite limit. Infinity is not a real number, infinite limits provide a way of describing the behaviour of functions that grow arbitrarily large, in absolute value, positive or negative.

Example 1

Describe the behaviour of the function $f(x)=\frac{1}{x^2}$ near $x=0$.

Solution

As x approaches 0 from either side, the values of $f(x)=\frac{1}{x^2}$ are positive and grow larger and larger, so the limit of $f(x)$ as x approaches 0 is $+\infty$. It is , nevertheless, convenient to describe the behaviour of f near 0 by saying that $f(x)$ increases without bound as x approaches zero. We write

$$\lim_{x\to 0^-} f(x)=\lim_{x\to 0}\frac{1}{x^2}=+\infty \text{ And } \lim_{x\to 0^+} f(x)=\lim_{x\to 0}\frac{1}{x^2}=+\infty$$

Example 2

Describe the behaviour of the function $f(x)=\frac{1}{x}$ near $x=0$.

Solution

Let x successively assumes x - values $=1,\frac{1}{10},\frac{1}{100},\ldots$, then $\frac{1}{x}=1,10,100,\ldots$ successively. As x approaches 0 from the right the value of $\frac{1}{x}$ gets larger and larger without bound, then $\lim_{x\to 0^+}\frac{1}{x}=+\infty$

Let x successively assumes values $x=-1,\frac{-1}{10},\frac{-1}{100},\ldots$, then $\frac{1}{x}=-1,-10,-100,\ldots$ successively. As x approaches 0 from the left the value of $\frac{1}{x}$ decreases and becomes more and more negative without bound, then $\lim_{x\to 0^-}\frac{1}{x}=-\infty$

Another way to see this is to construct the sign table:

x	$-\infty$	0	$+\infty$
1	+	+	+
x	-	0	+
$\frac{1}{x}$	-	$\|\|$ ∞	+

Considering the last row, we see that for $x=0$ the value of $\frac{1}{x}$ does not exist (∞). At the left there is a negative sign, thus $\lim_{x\to 0^-}\frac{1}{x}=-\infty$. At the right there is a positive sign, thus $\lim_{x\to 0^+}\frac{1}{x}=+\infty$.

It follows that $\lim_{x\to 0}\frac{1}{x}$ does not exist because the one sided limits as x approaches zero do not exist.

Example 3: a) Evaluate $\lim_{x\to 1^-}\frac{1}{x-1}$ b) $\lim_{x\to 1^+}\frac{1}{x-1}$

If we use direct substitution we will get $\frac{1}{(1)-1}=\frac{1}{0}=\infty$

To determine whether this infinity is positive or negative, this will depend on the signs of the denominator.

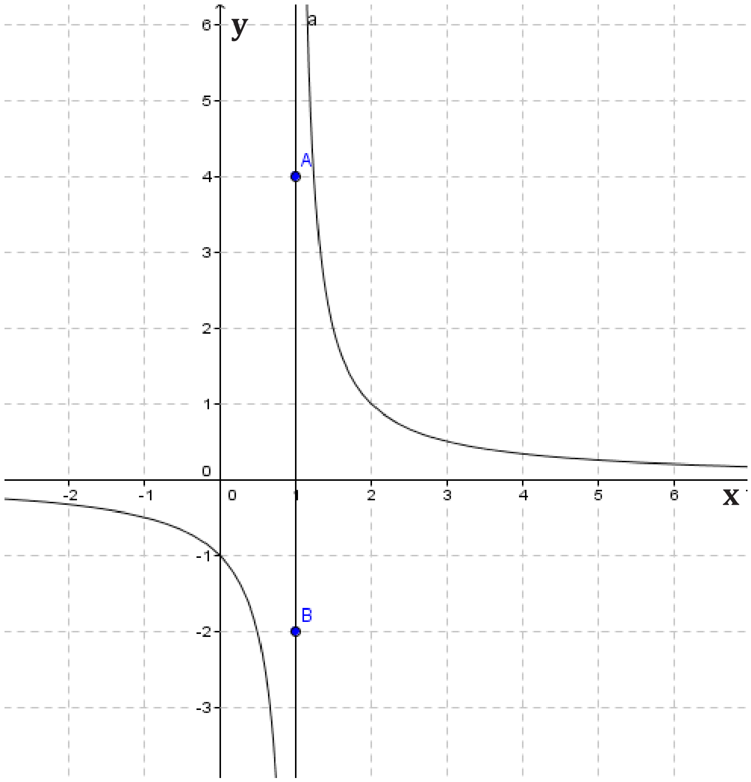
Sign table of the denominator.

x	$-\infty$	1	$+\infty$
$x-1$	-	0	+

We can see that the signs of the denominator at the left of Zero is negative this means at the left of zero we would be dividing 1 by a negative number while at the right we would be dividing 1 by a positive.

Therefore $\lim_{x\to 1^-}\frac{1}{x-1}=\frac{1}{0^-}=-\infty$ and $\lim_{x\to 1^+}\frac{1}{x-1}=\frac{1}{0^+}=+\infty$

See on the graph



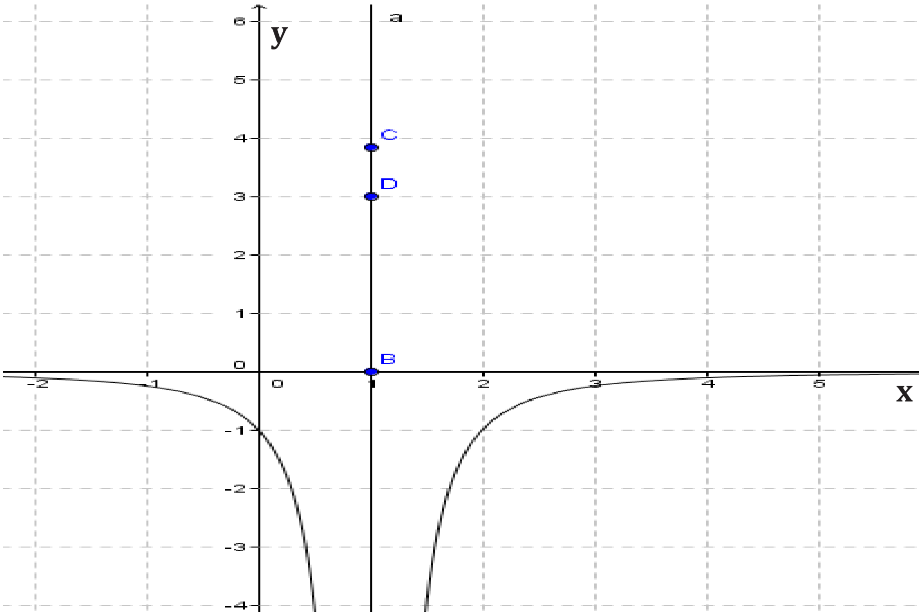
Example 4. Evaluate a) $\lim_{x\to 1^-}\frac{-1}{(x-1)^2}$ b) $\lim_{x\to 1^+}\frac{-1}{(x-1)^2}$

The foremost key is to find the dividing signs. That is simply studying the signs of the denominator.

x	-∞	1	+∞
$(x-1)^2$	+-	0	+

Because $(x-1)^2$ is always positive either at the left or right, then we will be dividing -1 by a positive number. That is

$$\lim_{x\to 1^-}\frac{-1}{(x-1)^2}=\frac{-1}{0^+}=-\infty \text{ and } \lim_{x\to 1^+}\frac{-1}{(x-1)^2}=\frac{-1}{0^+}=-\infty$$

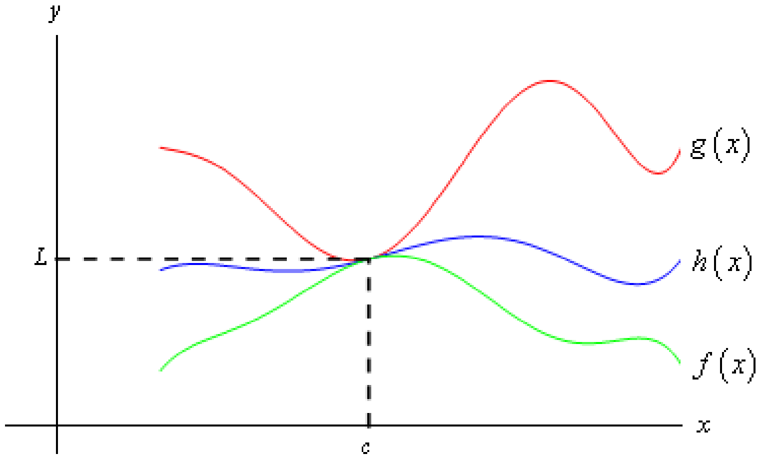


From the graph you can see that as x approaches to 1 from left or right is always $-\infty$

The squeeze theorem (or Sandwich theorem or Pinching theorem)

Suppose that $f(x) \le h(x) \le g(x)$. If $\lim_{x \to c} f(x) = \lim_{x \to c} g(x) = L$, then $\lim_{x \to c} h(x)$ =L

The following figure illustrates what is happening in this theorem



From the figure we can see that if the limits of $f(x)$ and $g(x)$ are equal at $x = c$ then the function values must also be equal at $x = c$. However, because $h(x)$ is "**squeezed**" between $f(x)$ and $g(x)$ at this point then $h(x)$ must have the same value. Therefore, the limit of $h(x)$ at this point must also be the same. Similar statements hold for left and right limits.

Examples:

1) Given that $1-\frac{x^2}{4}\le u(x)\le 1+\frac{x^2}{2}$. Find $\lim_{x\to 0} u(x)$.

Solution

Since $\lim_{x\to 0} 1-\frac{x^2}{4}=1$ and $\lim_{x\to 0} 1+\frac{x^2}{2}=1$, then according to Sandwich theorem $\lim_{x\to 0} u(x)=1$

2) Show that if $-|f(x)|\le f(x)\le |f(x)|$ then $\lim_{x\to a} f(x)$

Solution

Since $-|f(x)|\le f(x)\le |f(x)|$, and both $-|f(x)|$ and $|f(x)|$ have limit 0 as x approaches a, so does $f(x)$ by the Sandwich theorem.

APPLICATION ACTIVITY 6.3

1. Evaluate the following limits by using direct replacement.

a. $\lim_{x\to 2^+} \frac{3}{x-2}$ b. $\lim_{x\to -2^-} \frac{5}{x+2}$ c. $\lim_{x\to 3}\left(-3x^2+7x\right)$ d. $\lim_{x\to -1^+} \frac{4x-5}{3-x}$

e. $\lim_{x\to 5} \frac{\sqrt{x+4}-2}{x}$ f. $\lim_{x\to 3} \frac{\sqrt{x+1}-1}{x}$ e. $\lim_{x\to 4} \sqrt[3]{x+4}$ f. $\lim_{x\to +\infty} \frac{x}{x^2}$

2. The cost function for a new supermarket to recycle x tons of organic material is given by C=60x+1650 , where C is the cost in dollars

   a. Write a model for the average cost per ton of organic material recycled

   b. Find the average cost of recycling 100tons and 1000tons of organic material.

   c. Determine the limits of the average cost function as x approaches to infinity. Explain the meaning of this limit in the context of the problem.

6.4.Infinity limits and rational functions.

ACTIVITY 6.4

Evaluate the following limits

a. $\lim_{x\to\infty}\frac{5}{x+2}$ b. $\lim_{x\to 10}\frac{x}{2}$ c. $\lim_{x\to 0}\frac{x}{x^2}$ d. $\lim_{x\to\infty}\frac{x}{x^2}$

Content summary

We can evaluate limits by direct replacement method. But in some cases we need, to factorise and simplify the common factor in the case we have a rational function.

Examples

1. Evaluate the following limits $\lim_{x\to 2}\frac{x-2}{x^2-4}$

Solution

$$\lim_{x\to 2}\frac{x-2}{x^2-4}$$

$$=\lim_{x\to 2}\frac{(x-2)}{(x-2)(x+2)}$$ **Factorising the denominator.**

$$=\lim_{x\to 2}\frac{1}{x+2}$$ **Simplifying like terms.**

$$=\frac{1}{4}$$ **Direct replacement .**

1. $\lim_{x\to\infty}\frac{x-2}{x^2-4}$

$$\lim_{x\to\infty}\frac{x\left(1-\frac{2}{x}\right)}{x^2\left(1-\frac{4}{x^2}\right)}$$

$$\lim_{x\to\infty}\frac{\left(1-\frac{2}{x}\right)}{x\left(1-\frac{4}{x^2}\right)}$$

$$\frac{1}{\infty}=0$$

Dividing through by the term with greatest power on both numerator and denominator and simplyifying

Note: $\frac{c}{\infty}=0$ if $c\in\mathbb{R}$

APPLICATION ACTIVITY 6.4

I. Evaluate the following limits and say whether they are finite or infinite.

1. $\lim\limits_{x\to-\infty}\frac{4x^3+5x-3}{x^2+3x+1}$ 2. $\lim\limits_{x\to\infty}\frac{5x+2}{3x^2+1}$ 3. $\lim\limits_{x\to-\infty}(-6)$

II. Factorize the numerator and evaluate the limits.

a) $\lim\limits_{x\to7}\frac{49-x^2}{x-7}$ b) $\lim\limits_{x\to-2}\frac{x^3+5x^2+6x}{x+2}$ c) $\lim\limits_{x\to3}\frac{x^3-4x^2+5x-6}{x-3}$

III. Simplify the following expressions and evaluate the limits.

a) $\lim\limits_{x\to2}\left(\frac{1}{x-2}-\frac{2(2x-3)}{x^3-3x^2+2x}\right)$ b) $\lim\limits_{x\to1}\frac{\frac{1}{x+4}-\frac{1}{4}}{x}$

6.5. limits of functions with indeterminate cases

6.5.1 Indeterminate forms in irrational functions

ACTIVITY 6.5.1

Is it possible to evaluate the following limits by using direct substitution? If yes evaluate them. If no, what can be done before?

a. $\lim\limits_{x\to1}\frac{x^2-1}{x-1}$

b. $\lim\limits_{x\to2}\frac{x^3+x^2-5x-2}{x^2-4}$

Content summary.

A limit is said to be in an indeterminate form if it cannot be evaluated using properties of elementary operations(addition, subtraction, multiplication and division) .An indeterminate case, or form, hides the true value, if any, towards which the function is approaching . There are several types of indeterminate forms such as: $\frac{0}{0}, \frac{\infty}{\infty}, 0.\infty, \infty-\infty, 0^0, 1^\infty$ and ∞^0 . In this section we shall study the indeterminate forms: $0\times\infty$, $\frac{0}{0}, \frac{\infty}{\infty}, \infty-\infty$

To determine the true value towards which the function is approaching, it is necessary to remove the indetermination.

The indeterminate forms may be produced in the following ways:

- Suppose that $\lim_{x\to a} f(x)=0$ and $\lim_{x\to a} g(x)=\infty$.

The limit of the product $f(x)g(x)$ has the indeterminate form, $0\times\infty$ at $x=a$.

To evaluate this limit we change the limit into one of the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$ in this way:

$$f(x)g(x)=\frac{f(x)}{\frac{1}{g(x)}}=\frac{g(x)}{\frac{1}{f(x)}}$$

- If $\lim_{x\to a} f(x)=\lim_{x\to a} g(x)=+\infty$, then $\lim_{x\to a}\left[f(x)-g(x)\right]=\infty-\infty\left(IF\right)$. To evaluate this limit, we perform the algebraic manipulations by converting the limit into a form of $\frac{0}{0}$ or $\frac{\infty}{\infty}$.

If $f(x)$ or g(x) is expressed as a fraction, let first reduce them on the common denominator and evaluate the limit after.

6.5.2. Indeterminate forms in irrational functions

ACTIVITY 6.5.2

What is the conjugate of the irrational expression in each of the following functions?

a. $f\left(x\right)=\sqrt{x^2-2}+3$

b. $f\left(x\right)=\frac{\sqrt{x-2}-1}{x-3}$

When we are computing the limits of irrational functions, in case of indeterminate form, we need to know the conjugate of the irrational expression in that function. We may need to find the domain of the given function.

Example 1

Evaluate $\lim_{x\to 4}\frac{\sqrt{x-3}-1}{x-4}$.

Solution

Let us first substitute the independent variable for *4*:

$\lim_{x\to 4}\frac{\sqrt{x-3}-1}{x-4}=\frac{\sqrt{4-3}-1}{4-4}=\frac{0}{0}$, which is an indeterminate form.

To evaluate this limit, we multiply the numerator and denominator by the conjugate of $\sqrt{x-3}-1$ which is $\sqrt{x-3}+1$, then we get

$$\lim_{x\to 4}\frac{\sqrt{x-3}-1}{x-4}=\lim_{x\to 4}\frac{\left(\sqrt{x-3}-1\right)\left(\sqrt{x-3}+1\right)}{(x-4)\left(\sqrt{x-3}-1\right)}$$

$$=\lim_{x\to 4}\frac{x-3-1}{(x-4)\left(\sqrt{x-3}+1\right)}$$

$$=\lim_{x\to 4}\frac{x-4}{(x-4)\left(\sqrt{x-3}+1\right)}$$

$$=\lim_{x\to 4}\frac{1}{\left(\sqrt{x-3}+1\right)}$$

$$=\frac{1}{2}$$

Example 2

Evaluate $\lim_{x\to\infty}\left(\sqrt{4x^2+2}-2x\right)$

Solution

$$\lim_{x\to\infty}\left(\sqrt{4x^2+2}-2x\right)=+\infty-\infty\,(IF)$$

To evaluate this limit we multiply and divide by the conjugate of

$$\lim_{x\to\infty}\left(\sqrt{4x^2+2}-2x\right).$$

$$=\lim_{x\to\infty}\frac{\left(\sqrt{4x^2+2}-2x\right)\left(\sqrt{4x^2+2}+2x\right)}{\left(\sqrt{4x^2+2}+2x\right)}$$ Multiplying the conjugate on both numerator and denominator.

$$=\lim_{x\to\infty}\frac{4x^2+2-4x^2}{\left(\sqrt{4x^2+2}+2x\right)}=\lim_{x\to\infty}\frac{2}{\left(\sqrt{4x^2+2}+2x\right)}$$ After multiplying and simplifying.

$$=\frac{2}{\infty}=0$$

Example 3: Evaluate $\lim_{x\to\infty}\frac{\sqrt{4x^2-11x-3}}{x}$

Solution

$$\lim_{x\to\infty}\frac{\sqrt{4x^2-11x-3}}{x}=\frac{\sqrt{\infty-\infty}}{\infty}(IF)$$

To evaluate this limit, we try the algebraic manipulations such that the denominator will be cancelled.

$$=\lim_{x\to\infty}\frac{\sqrt{4x^2\left(1-\frac{11}{4x}-\frac{3}{4x^2}\right)}}{x}$$

$$=\lim_{x\to\infty}\frac{\sqrt{4x^2}}{x}\sqrt{\left(1-\frac{11}{4x}-\frac{3}{4x^2}\right)}$$

Be careful!

$$\sqrt{x}=|x|\begin{cases}-x, if\ x\geq 0\\ x,\ if\ x<0\end{cases}$$

Now, let us consider separately the limit at $-\infty$ and at $+\infty$

We have: $\lim\limits_{x\to-\infty}\frac{\sqrt{4x^2}}{x}\sqrt{1-\frac{11}{4x}-\frac{3}{4x^2}}=\lim\limits_{x\to-\infty}\frac{-2x}{x}\sqrt{1-\frac{11}{4x}-\frac{3}{4x^2}}=-2$,

And $\lim\limits_{x\to+\infty}\frac{\sqrt{4x^2}}{x}\sqrt{1-\frac{11}{4x}-\frac{3}{4x^2}}=\lim\limits_{x\to+\infty}\frac{2x}{x}\sqrt{1-\frac{11}{4x}-\frac{3}{4x^2}}=2$,

$\lim\limits_{x\to-\infty}f(\text{x})=-2$ and $\lim\limits_{x\to+\infty}f(x)=2$

APPLICATION ACTIVITY 6.5

1. Evaluate the following limits (if they exists).

a. $\lim\limits_{x\to\infty}\sqrt{x^2+2x-1}-\sqrt{x^2-x+2}$. b. $\lim\limits_{x\to1}\frac{1-\sqrt{x}}{x-1}$

c. $\lim\limits_{u\to0}\frac{\sqrt{4+u}-2}{u}$ d. $\lim\limits_{v\to1}\frac{\sqrt{2v+1}-\sqrt{3}}{v-1}$

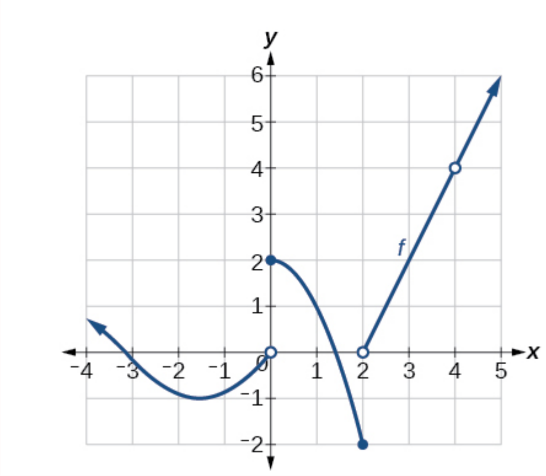
2. Simplify the following expressions and evaluate the limits.

$$\lim_{x\to2}\left(\frac{1}{x-2}-\frac{2(2x-3)}{x^3-3x^2+2x}\right)$$

6.6. Graphs and limits of function

ACTIVITY 6.6

Observe the following graph and answers about it.



a) How many parts does the graph of the function have?

b) What is the limit of the function as x approaches 0 from the left?

c) What is the limit of the function as x approaches 0 from the right?

d) What is the limit of the function as x approaches 2 from left?

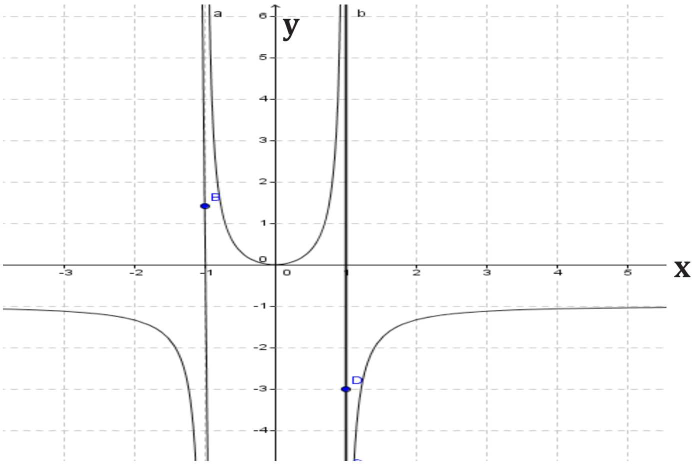
e) What is the limit of the function as x approaches 2 from right?

f) What is the limit of the function as x approaches 4?

Content summary

We can determine the limits by using the graph of a function. A gragh which has a jump at a given point is said to be **discontinuous** at that point. On the other hand if the graph has no jump is said to be **continuous** at that point.

Example: Observe the function whose graph is shown below:



Evaluate the following limits by using the graph

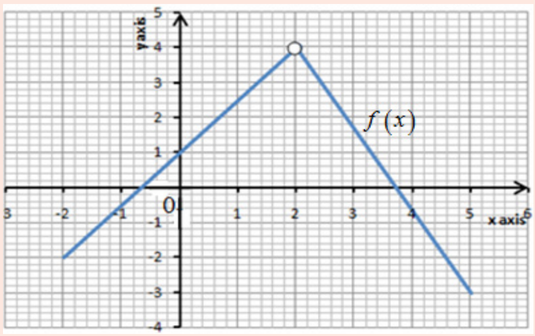
a. $\lim\limits_{x \to -1^-} f(x)$ b. $\lim\limits_{x \to -1^+} f(x)$ c. $\lim\limits_{x \to 1^-} f(x)$ d. $\lim\limits_{x \to 1^+} f(x)$ e. $\lim\limits_{x \to +\infty} f(x)$ f. $\lim\limits_{x \to -\infty} f(x)$

Solution

As x approaches to -1 at the left, the function decreases to $-\infty$ this implies that $\lim\limits_{x \to -1^-} f(x) = -\infty$. $\lim\limits_{x \to -1^+} f(x) = +\infty$, $\lim\limits_{x \to 1^-} f(x) = +\infty$, $\lim\limits_{x \to -\infty} f(x) = 0$ and $\lim\limits_{x \to +\infty} f(x) = 0$.

APPLICATION ACTIVITY 6.6

1. Find $\lim\limits_{x \to 2} f(x)$ using the following graph of $f(x)$



2. Given the functions:

a) $f(x) = x^3 - x$, find i) $\lim\limits_{x \to -\infty} f(x)$ ii) $\lim\limits_{x \to +\infty} f(x)$

b) $f(x) = \dfrac{1}{x-1}$ find i) $\lim\limits_{x \to 1^+} f(x)$ ii) $\lim\limits_{x \to 1^-} f(x)$ iii) $\lim\limits_{x \to +\infty} f(x)$ iv) $\lim\limits_{x \to -\infty} f(x)$

6.7. Applications of limits in mathematics

6.7.1. Continuity of a function

ACTIVITY 6.7.1

Given the function $f(x)=\begin{cases}\dfrac{x^2-4}{x-2}, x\neq 2\\ 4, \qquad \mathrm{x}=2\end{cases}$, find

a. $f(2)$

b. $\lim\limits_{x\to 2} f(x)$

c. What can you say about $f(2)$ and $\lim\limits_{x\to 2} f(x)$

Content summary

1. **Continuity of a function at a point or on interval *I***

A function $f(x)$ is said to be **continuous at point *c*** if the following conditions are satisfied:

a) $f(c)$ is defined

b) $\lim\limits_{x\to c} f(x)$ exists

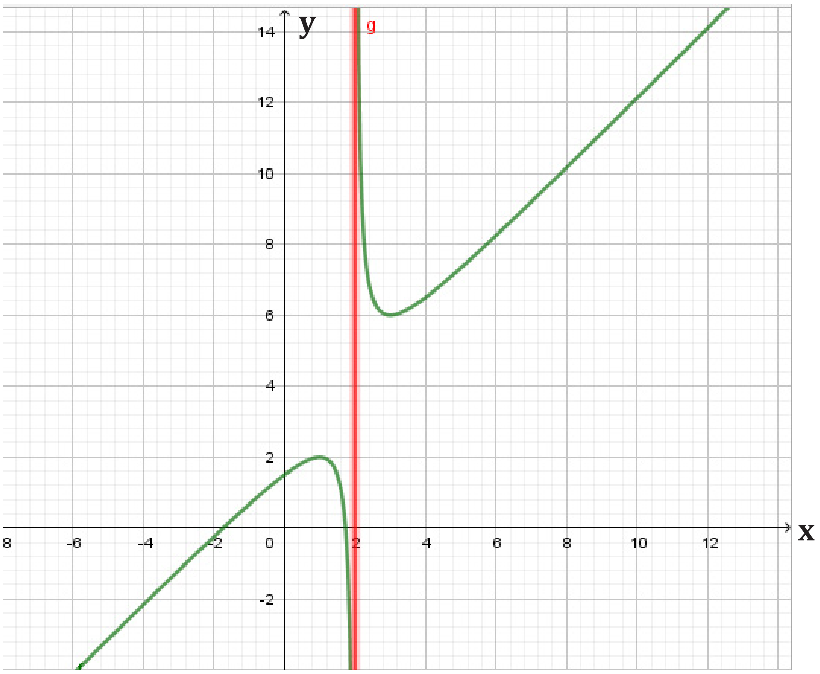
c) $\lim\limits_{x\to c} f(x)=f\left(c\right)$

If one or more conditions in this definition fails to hold, then $f(x)$ is said to be **discontinuous at point *c***, and c is called **a point of discontinuity** of $f(x)$. If f is continuous at all point of an open interval $\left]a,b\right[$, then $f(x)$ is said to be continuous on $\left]a,b\right[$.

A function that is continuous on $\left]-\infty,+\infty\right[$ is said to be **continuous everywhere** or simply **continuous.**

Examples:

1) The function $f(x)=\dfrac{x^2-3}{x-2}$ is discontinuous at 2 because $f(2)$ is undefined, see the graph below.



2) The function $g(x)=\begin{cases}\dfrac{x^2-9}{x-3}, x\neq 3\\ 6, \quad x=3\end{cases}$ is continuous at 3 because $g(3)$

and $\lim\limits_{x\to 3^+} g(x)=\lim\limits_{x\to 3^-} g(x)=\lim\limits_{x\to 3}\dfrac{(x-3)(x+3)}{x-3}=\lim\limits_{x\to 3}(x+3)=6$ so that

$\lim\limits_{x\to 3} g(x)=g(3)=6$.

2. Continuity at the left and continuity at the right of a point

A function f(x) is continuous at the left of point *c* if the following conditions are satisfied:

a) $f(c)$ is defined

b) $\lim\limits_{x\to c^-} f(x)$ exists

c) $\lim\limits_{x\to c^-} f(x)=$ f(c)

A function *f* is continuous at the right at of point *c* if the following conditions are satisfied:

a) $f(c)$ is defined

b) $\lim\limits_{x\to c^+} f(x)$ exists

c) $\lim\limits_{x\to c^+} f(x) = \text{f(c)}$

Example

Find the value of the constant a such that the function $f(x)=\begin{cases}-x+1, & \text{x} \le 3\\ 2x+a, & \text{x>3}\end{cases}$ is continuous on the entire line.

Solution

The function will be continuous if and only if $\lim\limits_{x\to 3^-} f(x) = \lim\limits_{x\to 3^+} f(x) = f(3)$

$$\lim_{x\to 3^+}\left(2x+a\right)=6+a$$

$$\lim_{x\to 3^-}\left(-x+1\right)=-3+1=-2$$

Equating the two limits we get, $6+a=-2 \Rightarrow a=-8$

Therefore, the function is continuous if $a=-8$

Continuity on an interval

We say that $f(x)$ is **continuous on the interval** I if it is continuous at each point of I. in particular, we will say that $f(x)$ is a continuous function if $f(x)$ is continuous at every point of its domain.

Examples

1) The function $f(x)=\sqrt{x}$ is a continuous function on its domain. Its domain is $[0,+\infty[$. It is continuous at the left endpoint 0 because it is right continuous there. Also, f is continuous at every number $c>0$. Since $\lim\limits_{x\to c}\sqrt{x}=\sqrt{c}$.

2) Discuss the continuity of the function $f(x)=\dfrac{|x-1|}{x^2-1}$ at $x=1$

Solution

$$|x-1|=\begin{cases}x-1, x\ge 1\\ -\left(x-1\right), x<1\end{cases}$$

$$\lim_{x\to 1^-}\frac{-\left(x-1\right)}{\left(x+1\right)\left(x-1\right)}=\frac{-1}{2} \text{ And } \lim_{x\to 1^+}\frac{\left(x-1\right)}{\left(x+1\right)\left(x-1\right)}=\frac{1}{2}$$

It can be seen that the limits at $x=1$ does not exist, since the left limit is different from right limit. Hence, the function is discontinuous at $x=1$.

On the other hand, the function is continuous at the right of 1.

Theorem 1

a) Polynomials are continuous functions

b) If the functions f and g are continuous at c, then

i) $f(x)+g(x)$ is continuous at c

ii) $f(x)-g(x)$ is continuous at c

iii) $f(x)g(x)$ is continuous at c

iv) $\dfrac{f(x)}{g(x)}$ is continuous at c if $g(\mathrm{c})\neq 0$, and is discontinuous at c if $g(\mathrm{c})=0$.

c) A rational function is continuous everywhere except at the point where the denominator is zero.

d) Piecewise functions (functions that are defined on a sequence of intervals by different formulas) are continuous if every function is in its interval of definition, and if the functions match their side limits at the boundaries points of their component intervals.

APPLICATION ACTIVITY 6.7.1

1. Discuss the continuity of the function defied by $f(x)=\begin{cases}1, \mathrm{x}\leq 1\\ x, 1\leq x\leq 3\\ -x+6, 3<x\leq 6\\ 0, x>6\end{cases}$

2. Determine the points at which the function below is not continuous.

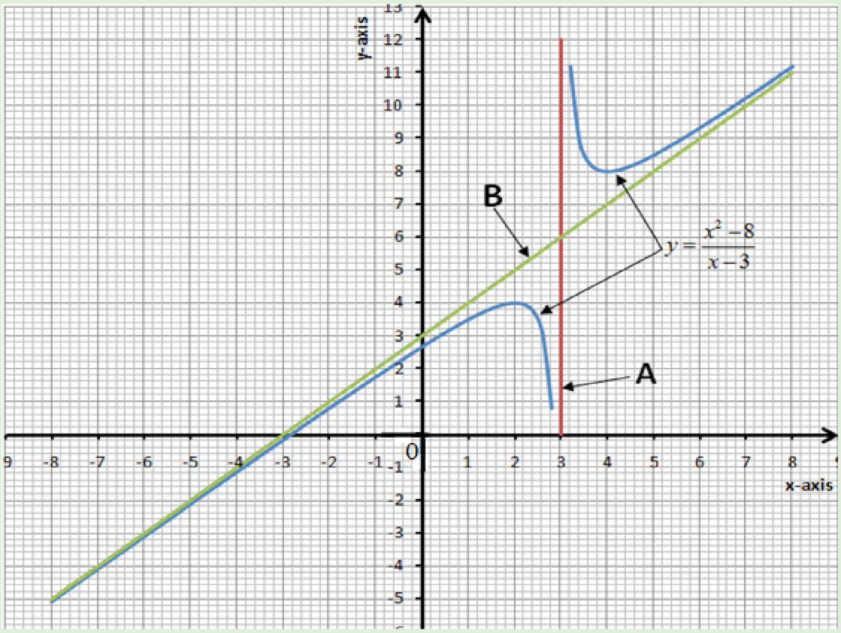
$$f(x)=\frac{4x+10}{x^2-2x-15}$$

3. For which value of k is the function: $f(x)=\begin{cases}\dfrac{x^2-9}{x-3}, x\neq 3\\ k, x=3\end{cases}$ continuous at x = 3.

6. 8. Asymptotes to curve of a function

ACTIVITY 6.8

Consider the following curve of function y



a) What is the position of the line A ?

b) What is the position of line B ?

c) What is the other type of line that you know?

d) Do the lines A and B cross the curve of the function?

Content summary

Recall that if $P(x)$ and $Q(x)$ are polynomials, then their ratio $f(x) = \dfrac{P(x)}{Q(x)}$ is called a rational function of x. The discontinuity occurs at points where $Q(x) = 0$

An asymptote on the curve is a straight line that is closely approached by that curve so that the perpendicular distance between them decreases to zero.

To find any asymptote of the function first we need to determine its domain of definition and evaluate the limits at the boundaries of the domain.

6.9. Vertical asymptote, horizontal and oblique asymptotes.

ACTIVITY 6.9

Given the function $f(x)=\begin{cases}\dfrac{x^2-4}{x-2}, x\neq 2\\ 4, \quad\quad x=2\end{cases}$, find

a. $f(2)$

b. $\lim\limits_{x\to 2} f(x)$

c. What can you say about $f(2)$ and $\lim\limits_{x\to 2} f(x)$

Content summary

A function $f(x)$ admits a vertical asymptotes at a given point $x=a, a\in\mathbb{R}$, if $f(x)$ is undefined at $x=a$ .that is $\lim\limits_{x\to a^+} f(x)=\pm\infty$ and $\lim\limits_{x\to a^-} f(x)=\pm\infty$ .the vertical asymptote is denoted by $V\equiv x=a$

A function $f(x)$ admits a horizontal asymptotes at a given point $y=b, \text{b}\in\mathbb{R}$, if $\lim\limits_{x\to\pm\infty} f(x)=b$ it is denoted by $H.A\equiv y=b$.

A function $f(x)$ admits an oblique asymptotes $O.A\equiv y=ax+b$ where a is the gradient and b is the y-axis intercept if $a=\lim\limits_{x\to\pm\infty}\dfrac{f(x)}{x}, \text{a}\in\mathbb{R}^*$ and $b=\lim\limits_{x\to\pm\infty} f(x)-\text{ax}, \text{a}\in\mathbb{R}^*$ and $\text{b}\in\mathbb{R}$

Examples

1. Given the function $f(x)=\dfrac{1}{4-x^2}$

   a. Find the domain of definition

   b. Find the limits at the endpoints of the domain and deduce all possible asymptotes.

Solution

$$dom f = \{x \in \mathbb{R} / 4 - x^2 \neq 0\}$$

$$(2-x)(2+x) \neq 0 \Rightarrow \begin{cases} 2-x \neq 0 \text{ or } x \neq 2 \\ 2+x \neq 0 \text{ or } x \neq -2 \end{cases}$$

$$dom f = \{x \in \mathbb{R} \setminus \{-2, -2\}\}$$

$$\begin{cases} \lim\limits_{x \to -2^+} f(x) = -\infty \\ \lim\limits_{x \to -2^-} f(x) = -\infty \end{cases} \Rightarrow V.A \equiv X = -2 \text{ and } \begin{cases} \lim\limits_{x \to 2^+} f(x) = -\infty \\ \lim\limits_{x \to 2^-} f(x) = -\infty \end{cases} \Rightarrow V.A \equiv X = 2$$

$$\begin{cases} \lim\limits_{x \to -\infty} f(x) = 0 \\ \lim\limits_{x \to +\infty} f(x) = 0 \end{cases} \Rightarrow H.A \equiv y = 0 \text{ There is no } O.A$$

2. Find the oblique(Slant) asymptote for the following function $f(x) = \dfrac{x^3}{x^2 - 1}$

Solution.

Slant asymptote has for equation $y = ax + b$

$$a = \lim_{x \to \infty} \frac{f(x)}{x} = \lim_{x \to \infty} \frac{x^3}{x(x^2 - 1)} = 1 \;, \quad b = \lim_{x \to \infty} f(x) - ax = \lim_{x \to \infty} \frac{x^3}{(x^2 - 1)} - (1)x$$

$b = \lim\limits_{x \to \infty} \dfrac{x^3 - x^3 + x}{x^2 - 1} = \lim\limits_{x \to \infty} \dfrac{x}{x^2 - 1} = 0$, hence the oblique asymptote has for equation $y = x$.

APPLICATION ACTIVITY 6.9

For each of the following functions find

a) The domain

b) Limits at the endpoints of the domain and deduce all possible asymptotes.

i) $f(x)=\frac{x^2+1}{x^2-x}$

ii) $f(x)=\frac{x^3}{1-x^2}$

iii) $f(x)=\frac{1}{x}$

6.10. Real life problems about limits

ACTIVITY 6.10

The total cost evolution of a small business is modeled by the function $C=ax+b$ where C is the cost and x is the units sold. If $a=0.5$ and $b=5000$

a) Find the model function for the evolution of the business

b) Find the average cost function $\overline{C}$, if x units are sold.

c) Using the function on (b) find the average if 100 units are sold, if 1000 units are sold and if 100000 units are sold.

d) Using the values obtained at (c), find the average cost as x approaches to infinity.

e) Discuss the meaning of the limit obtained at (d).

Content summary

Limits can be applied in different fields in real life. In economics the average cost per unit sold is calculated using limits.

1. **Instantaneous rate of change of a function in physics**

The **instantaneous rate of change** of $f(x)$ at a, also called the **rate of change** of $f(x)$ at a, is defined to be the limit of the average rate of change of $f(x)$ over shorter and shorter intervals around a.

Since the average rate of change is a difference quotient of the form $\frac{\Delta y}{\Delta t}$, the instantaneous rate of change is a limit of difference quotient. In practice, we often approximate a rate of change by one of these difference quotients.

Example:

1. The quantity (in mg) of a drug in the blood at time t (in minutes) is given by $Q == 25(0.8)^t$. Estimate the rate of change of the quantity at t = 3 and interpret your answer.

Solution:

We estimate the rate of change at t = 3 by computing the average rate of change over intervals near t = 3. We can make our estimate as accurate as we like by choosing our intervals small enough.

Let's look at the average rate of change over the interval 3 ≤ t ≤ 3.01:

Average rate of change = $\frac{\Delta Q}{\Delta t} = \frac{25(0.8)^{3.01} - 25(0.8)^3}{3.01 - 3.00} = -2.85$

A reasonable estimate for the rate of change of the quantity at $t = 3$ is -2.85 Since Q is in mg and t in minutes, the units of $\frac{\Delta Q}{\Delta t}$ are mg/minute. Since the rate of change is negative, the quantity of the drug is decreasing. After 3 minutes, the quantity of the drug in the body is decreasing at $2.85 mg / \text{minute}$

2. Instantaneous velocity

Instantaneous velocity of a moving body is the limit of average velocity over an infinitesimal interval of time.

$$v = \lim_{t \to 0} \frac{\Delta s}{\Delta t}.$$

3. Instantaneous acceleration

Instantaneous acceleration for a moving body is the limit of average acceleration over an infinitesimal interval of time $a = \lim_{\Delta t \to 0} \frac{\Delta v}{\Delta t}$

(Here and elsewhere, if motion is in a straight line, vector quantities can be substituted by scalars in the equations.)

Examples

1) The cost of removing $p\%$ in a given place is given by the model
$$C(x)=\frac{80000p}{100-p}, 0\le p<100$$

a)What is the vertical asymptote for this function?

b) What does this vertical asymptote mean in the context of the problem?

Solution

Vertical asymptote is found at $p=100$

To assess the behavior of the model, let us choose some values

Example: the cost of removing 85% of pollutants is

$$C(85\%)=\frac{80000(85)}{100-85}\approx 453.333$$

But the cost of removing 90% is

$$C(90\%)=\frac{80000(90)}{100-90}=720.$$

We can see that the higher and higher percentage of pollutants is removed, the cost increases dramatically.

APPLICATION ACTIVITY 6.10

1. The cost C (in millions dollars) for the federal government to seize $p\%$ of a type of illegal drug as it enters the country is modeled by the

$$C=\frac{528p}{100-p}, 0\le p<100$$

a. Find the cost of seizing (stopping) $25\%, 50\%$ and 75%.

b. Find the limit as $p\to 100^-$, interpret this limit in the context of the problem.

2. A business has a cost in dollars of $C = 0.5x + 500$ for producing x units.

a. Find the average cost function $\overline{C}$.

b. Find $\overline{C}$ when x=250 and when x=1250

c. What is the limit of $\overline{C}$ as x approaches to infinity? Interpret the results in the context of the problem.

6.11 END UNIT ASSESSMENT

1. Use limits to find the slope of the tangent line to the graph of $s = t^2$ at the point $(1,1)$.

2. Given the function $f(x) = \dfrac{x^2 + x - 6}{x^2 - 4}$

Find the limits:

a) domain of $f(x)$

b) Limits at the endpoints of the domain and deduce all possible asymptotes.

3. The cost and revenue functions for a product are $C = 25.5x + 1000$ and R=75.5x

a) Find the average profit function $\overline{P} = \dfrac{R - C}{x}$.

b) Find the average profits when x is 100, 500 and 1000 .

c) What is the limit of the average profit as x approaches to infinity? Explain your reasoning.

4. You are given 24m of wire and are asked to form a rectangle whose area is as large as possible. What dimensions should the rectangle have? (Hint: use numerical approach).

5. given a function $f(x) = \begin{cases} -2x, x \le 2 \\ x^2 - 4x + 1, x > 2 \end{cases}$ find i) $\lim\limits_{x \to 2^+} f(x)$ ii) $\lim\limits_{x \to 2^-} f(x)$ iii) $\lim\limits_{x \to 2} f(x)$

iv) Discuss the continuity of $f(x)$ at the point where $x = 2$.

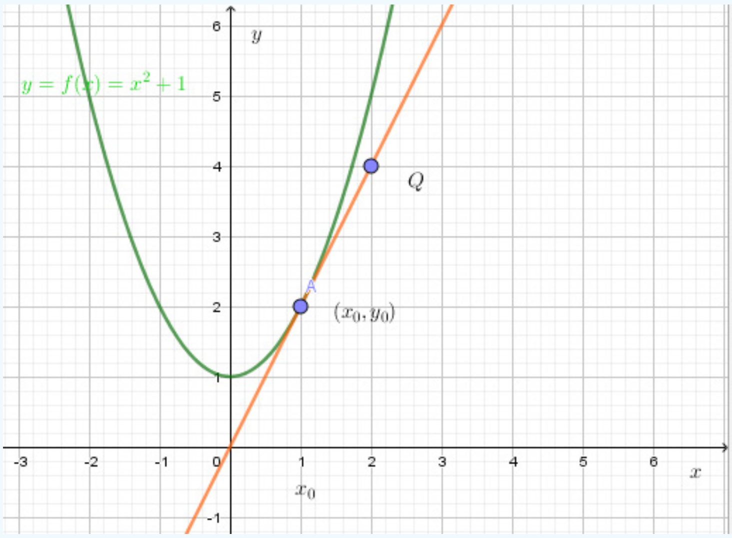
UNIT 7

DIFFERENTIATION OR DERIVATIVE OF NUMERICAL FUNCTIONS

Key unit competence: Differentiate a real function and apply derivatives to sketch the graphs and solve problems involving optimization

7.0. INTRODUCTORY ACTIVITY

1. Consider the function $f(x) = x^2 + 1$ illustrated on the following graph;



It is defined that the slope m_P of the tangent of the curve of $f(x)$ at a point $P(x_0, y_0)$ is obtained by $\lim\limits_{h \to 0} \dfrac{f(x_0 + h) - f(x_0)}{h}$,

a) Determine the slope of $f(x)$ at the point for which $x_0 = 1$.

b) Deduce the value of the function $f'(x) = \lim\limits_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$ for $f(x) = x^2 + 1$ and compare the slope m_P and $f'(x_0)$ for $x_0 = 1$.

2) Go in library or computer lab, do research and make a short presentation on the following:

a. Derivative of a function

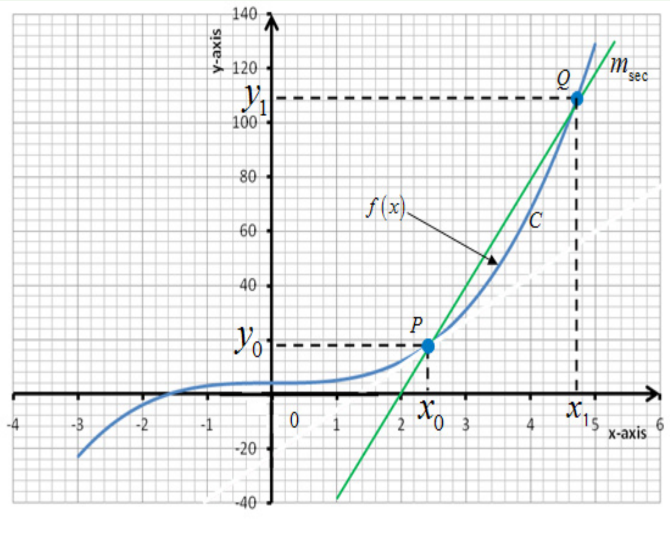
b. Find 2 examples of applications of derivatives.

7.1 Concepts of derivative of a function

7.1.1 Definition and graphical interpretation

ACTIVITY 7.1.1

1. Consider the figure below, analyse it and answer the questions that follows



a. If $P(x_0, y_0)$ and $Q(x_1, y_1)$ are two points on the graph of a function f, refer to what you learned in S3 and find the slope of secant line (m_{sec}) passing through P and Q. Since $y_0 = f(x_0)$ and $y_1 = f(x_1)$, express the slope in terms of $f(x_0)$ and $f(x_1)$.

b. If we let x_1 approach x_0, how can you conclude about position of Q to P?

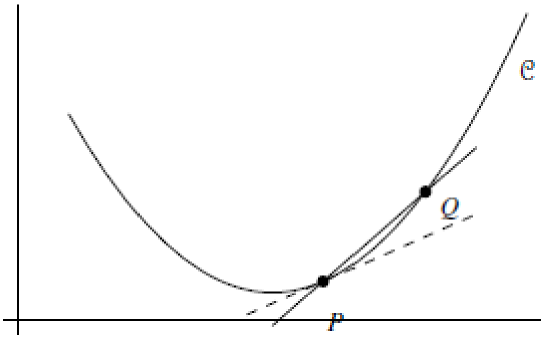
c. Let $m_{tan} = \lim\limits_{x_1 \to x_0} m_{sec}$, write down expression of m_{tan} in terms of $f(x_0)$ and $f(x_1)$.

d. After letting $h = x_1 - x_0$, rewrite m_{tan} in terms of $f(x_0)$ and $f(x_0 + h)$.

Content summary

Slope of a function at a point

To define the slope of the curve C at P, take a point Q on the curve different from P. The line PQ is called a secant line at P. Its slope, denoted by m_{PQ}, can be found using the coordinates of P and Q. If we let Q move along the curve, the slope m_{PQ} changes.



Suppose that as Q approaches P, the number m_{PQ} approaches a fixed value and the increment $h = x_Q - x_P$ of x approaches 0. This value, denoted simply by m_P, is called the slope of C at P; and the line with slope m_P and passing through P is called the tangent line to the curve C at P.

In view of the concept "limit of a function at a point", the slope m_{PQ} of the secant line PQ is $m_{pq} = \dfrac{f(x_0+h)-f(x_0)}{(x_0+h)-x_0} = \dfrac{f(x_0+h)-f(x_0)}{h}$

Note that as Q approaches P, the number h approaches 0. From these, we see that the slope of C at P (denoted by m_P) is

$$m_P = \lim_{h\to 0}\frac{f\left(x_0+h\right)-f\left(x_0\right)}{h}$$

If this limit exists, the number m_P is called slope of tangent line to the graph of f at p or at $x = x_0$

Definition of derivative of a function

The slope m_P has a special notation, we denote it by $f'(x_0)=\lim\limits_{h\to 0}\dfrac{f(x_0+h)-f(x_0)}{h}$ and $f'(x_0)$ is read f prime of x_0.

Dropping the subscript on x_0 in notation of m_P, we get one of the most important concept in mathematics, the ***derivative of function f at*** $x=x_0$

The derivative of a function $f(x)$ with respect to x is denoted by $f'(x)$ or $\dfrac{d}{dx}f(x)$ and defined as $f'(x)=\lim\limits_{h\to 0}\dfrac{f(x+h)-f(x)}{h}$ provided that the limit exists.

Examples:

1) Let $f(x)=x^2+1$

The derivative of $f(x)$ is

$$\begin{aligned}
f'(x)&=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}\\
&=\lim_{h\to 0}\frac{(x+h)^2+1-x^2-1}{h}\\
&=\lim_{h\to 0}\frac{x^2+2hx+h^2+1-x^2-1}{h}\\
&=\lim_{h\to 0}\frac{2hx+h^2}{h}\\
&=\lim_{h\to 0}(2x+h)\\
&=2x
\end{aligned}$$

Thus, $f'(x)=2x$

2) By using definition, calculate the first derivative for $y=x^2-3$

Solution:

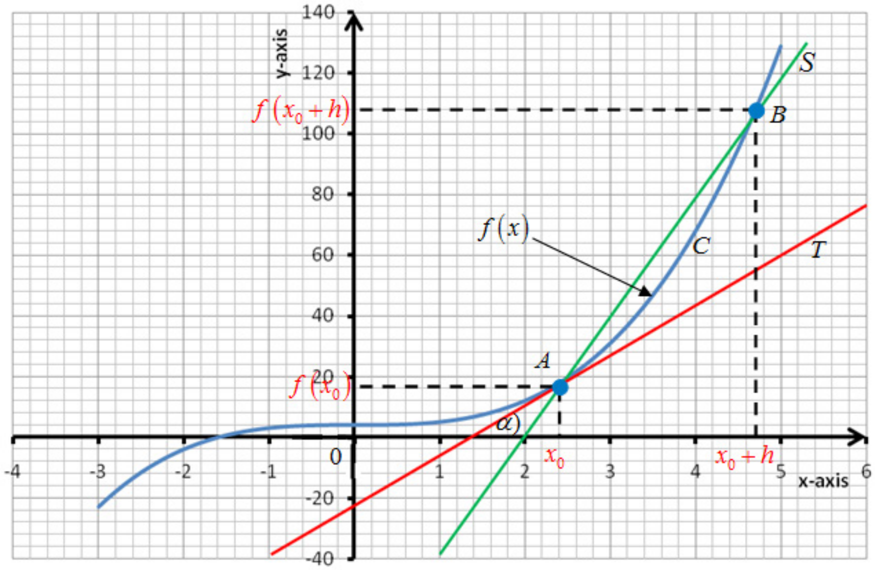
$$f'(x_0)=\lim_{h\to 0}\frac{f(x_0+h)-f(x_0)}{h}, \qquad y'=\lim_{h\to 0}\frac{(x+h)^2-3-(x^2-3)}{h}$$

$$y'=\lim_{h\to 0}\frac{x^2+2xh+h^2-x^2}{h}=\lim_{h\to 0}2x+h=2x$$

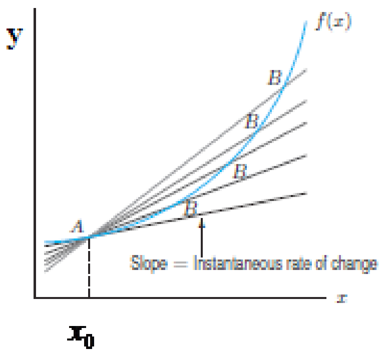
Remarks

- If $t \to f(t)$ represents the law of a moving object, then the derivative number of f represents the instantaneous speed of that moving object at instant t.
- The process of finding derivative of a function is called *differentiation* of that function.

Graphical interpretation of derivative and the slope of a function using differentiation



The graph shows the **average rate of change of a function** represented by the slope of the secant line joining points A and B.



The derivative is found by taking the average rate of change over smaller and smaller intervals. In the figure above, as point B moves toward point A, the secant line becomes the tangent line at point A. Thus, **the derivative is represented by the slope of the tangent line to the graph at the point**.

The derivative of a function at the point A is equal to:

- The slope of the graph of the function at A.
- The slope of the line tangent to the curve at A.

Then AB has slope of $\frac{f(x_0+h)-f(x_0)}{(x_0+h)-x_0}=\frac{f(x_0+h)-f(x_0)}{h}$. When h approaches to zero, the point B approaches the point A .

Then by definition $f'(x_0)=\lim_{h\to 0}\frac{f(x_o+h)-f(x_o)}{h}$ for $y=f(x)$ at point $(x_0, f(x_0))$

Alternative notations

The derivative: $f'(x)=y'=\frac{dy}{dx}=\frac{df}{dx}=\frac{d}{dx}f(x)$ where $\frac{dy}{dx}$ *and* $\frac{d}{dx}$ are called differentiation operators but $\frac{dy}{dx}$ should not be regarded as a ratio.

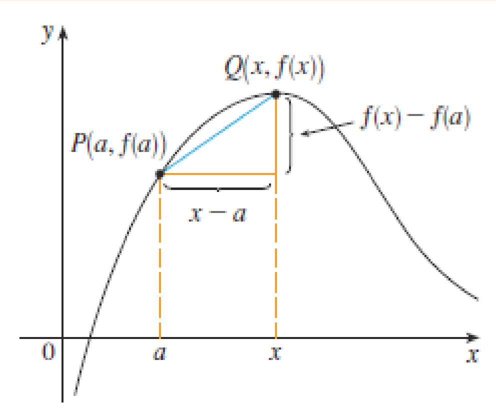
If $x = t$ is time and $y = s(t)$ is the displacement function of a moving object then $s'(t_0)$ or $\left.\frac{ds}{dt}\right|_{t=t_0}$ is the rate of change of displacement with respect to time when $t = t_0$, that is, the (instantaneous) velocity at $t = t_0$.

APPLICATION ACTIVITY 7.1

1. Find the slope of the curve given by $f(x)=x^2$ at the point $P(3, 9)$.
2. By the use of definition, find the derivative of:

    a. $f(x)=x^2+1$ b. $f(x)=x^2+2x-1$

3. Observe the graph and answer the questions that follow:



a. From your observation, interpret the graph by indicating the rate of change given that the variable x changes from a to x

b. Deduce the formula of derivative;

c. Use your interpretation and the formula deduced to find out the derivative of $f(x)=x+1$.

4. Find the derivative of the function $f(x)=x^2-8x+9$ at (the number) a.

7.2 Rules of differentiation

ACTIVITY 7.2

1) Find the derivative of

a) $\mathrm{f}(x)=x^2$ b) $\mathrm{h}(x)=3x-1$ c) $g(x)=x^2+3x-1$

Deduce the derivative of $S(x)=g(x)+\mathrm{h(x)}$

2) Calculate the derivative of $f(t)=\frac{1}{t},\ t\neq 0$

Content summary

Suppose that f is a function differentiable at every point belonging to an open interval $]a,b[$ or (a, b). Then we say that f is differentiable on $]a,b[$ or (a, b).

1) Derivative of a constant function

If f is a constant function, $f(x)=\mathrm{c}$, for all x then $\frac{df}{dx}=\frac{d}{dx}(c)=0$

Example: calculate the derivative of $f(x)=8$

Solution: $\frac{df}{dx}=\frac{d}{dx}(8)=0$

2) Derivative of identity function

The derivative of the identity function is the constant function 1, that is

$$\textit{if}\quad f(x)=x,\frac{df}{dx}=\frac{dx}{dx}=1$$

3) Multiplication by a scalar

If f is a differentiable function of x, and c is a constant, then

$$\frac{d}{dx}\left(cf\left(x\right)\right)=c\frac{d}{dx}f\left(x\right)$$

Example: find the derivative of $f(x)=3x$

Solution: $f'(x)=3(x)'=3x^{1-1}=3x^{0}=3$, knowing that $x^{0}=1$,provided $x\neq 0$

4) Derivative of a power

If n is any real number,

then $\frac{d}{dx}x^{n}=nx^{n-1}$ for all x where the powers x^{n} and x^{n-1} are defined.

This holds for any function with power. Let $D(I,\mathbb{R})$ be the set of functions differentiable on interval I. Thus, if $f\in D(I,\mathbb{R})$ for positive and negative, and fractional value of n, then $\left[f^{n}(x)\right]'=nf^{n-1}(x)f'(x)$.

Example:

Differentiate $f(x)=(2x+1)^{4}$ respecting the value of x.

Solution:

$$f'(x)=\left((2x+1)^{4}\right)'=4(2x+1)'(2x+1)^{4-1}$$

$$f'(x)=4(2)(2x+1)^{3}$$

$$f'(x)=8(8x^{3}+12x^{2}+6x+1)$$

$$f'(x)=64x^{3}+96x^{2}+48x+8$$

5) Sum rule

$$\frac{d}{dx}(u+v)=\frac{du}{dx}+\frac{dv}{dx}$$

Example: calculate the derivative of $y=x^{4}+12x$ respecting x

Solution: $y'=4x^{4-1}+12x^{1-1}$

$$y' = 4x^3 + 12$$

6) The Difference Rule

$$\frac{d}{dx}(u - v) = \frac{du}{dx} - \frac{dv}{dx}$$

Example

$y = x^4 - 2x^2 + 2$ $\qquad$ $\frac{dy}{dx} = 4x^3 - 4x$

7) Product rule

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$$

Notice that this is not just the product of two derivatives

Example

$$\frac{d}{dx}\left[(x^2+3)(2x^3+5x)\right] = (x^2+3)\,(6x^2+5) + (2x^3+5x)\,(2x)$$

$$= 6x^4 + 5x^2 + 18x^2 + 15 + 4x^4 + 10x^2$$

$$= 10x^4 + 33x^2 + 15$$

8) Quotient rule

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$ or can be written as $d\left(\frac{u}{v}\right) = \frac{v\,du - u\,dv}{v^2}$

Examples:

1. Find out the derivative for $f(x) = \frac{2x^3 + 5x}{x^2 + 3}$

Solution:

$$f'(x) = \frac{(x^2+5)(2x^3+5x)' - (2x^3+5x)(x^2+3)'}{(x^2+3)^2}$$

$$f'(x) = \frac{(x^2+5)(6x^2+5) - (2x^3+5x)(2x)}{(x^2+3)^2}$$

2. Find the derivative of $f\left(x\right)=\dfrac{2x^2+3x}{4x^3+x+1}$

$$f'(x)=\left(\frac{2x^2+3x}{4x^3+x+1}\right)'$$

$$=\frac{(2x^2+3x)'(4x^3+x+1)-(2x^2+3x)(4x^3+x+1)'}{(4x^3+x+1)^2}$$

$$f'(x)=\left(\frac{2x^2+3x}{4x^3+x+1}\right)'$$

$$=\frac{\left(2x^2+3x\right)'\left(4x^3+x+1\right)-\left(2x^2+3x\right)\left(4x^3+x+1\right)'}{\left(4x^3+x+1\right)^2}$$

$$=\frac{\left(2x^2+3x\right)'\left(4x^3+x+1\right)-\left(2x^2+3x\right)\left(4x^3+x+1\right)'}{\left(4x^3+x+1\right)^2}$$

$$=\frac{(4x+3)\left(4x^3+x+1\right)-\left(2x^2+3x\right)\left(12x^2+1\right)}{\left(4x^3+x+1\right)^2}$$

$$=\frac{16x^4+4x^2+4x+12x^3+3x+3-24x^4-2x^2-36x^3-3x}{\left(4x^3+x+1\right)^2}$$

$$=\frac{-8x^4-24x^3+2x^2+4x+3}{\left(4x^3+x+1\right)^2}$$

Derivative of polynomial

Let $y=a_nx^n+a_{n-1}x^{n-1}+\ldots+\mathrm{a}_2\,x^2+a_1x^1+a_0$ be a polynomial. Then, we have $\dfrac{dy}{dx}=na_nx^{n-1}+\left(n-1\right)a_{n-1}x^{n-2}+\ldots+2a_2x^1+a_1$

9) Derivative of the reciprocal function

Let $D\left(I,\mathbb{R}\right)$ be the set of functions differentiable on I. If $f\in D\left(I,\mathbb{R}\right)$, then $\dfrac{1}{f}\in D\left(I,\mathbb{R}\right)$ $f\left(x\right)\neq 0$. Moreover $\dfrac{d}{dx}\left(\dfrac{1}{f}\right)=-\dfrac{\frac{df}{dx}}{f^2}$ or $\dfrac{d}{dx}\left(\dfrac{1}{f}\right)=-\dfrac{f'}{f^2}$

Example

Calculate the derivative of $f(x)=\dfrac{1}{x}$ respecting the value of x

Solution: $\left(\dfrac{1}{x}\right)'=-\dfrac{(x)'}{x^2}=-\dfrac{1}{x^2}$

10) Derivative of a composite function: Chain rule

If f and g are both differentiable and F is the composite function defined by $F(x)=f(g(x))$, then F is differentiable and F' is given by the product $F'(x)=f'(g(x))\cdot g'(x)$

In Leibniz notation, if $y=f(u)$ and $u=g(x)$ are both differentiable functions, then $\frac{dy}{dx}=\frac{dy}{du}\frac{du}{dx}$

Example:

Find the derivative of $f\circ g$ if $f(x)=x^2+3x+3$ and $g(x)=\frac{2x+1}{x}$

Solution:

$$(f\circ g)'(x)=f'\left[g(x)\right]g'(x)=\left[2\left(\frac{2x+1}{x}\right)+3\right]\left(\frac{2x+1}{x}\right)'$$

$$=\left(\frac{4x+2+3x}{x}\right)\left(\frac{2x-2x-1}{x^2}\right)=\left(\frac{7x+2}{x}\right)\left(\frac{-1}{x^2}\right)$$

$$=\frac{-7x-2}{x^3}$$

11) Differentiation of radical functions

i) If we take any function in the square root function, then

$$y=f(x)\Rightarrow\frac{dy}{dx}=\frac{1}{2\sqrt{f(x)}}\frac{d}{dx}f(x)=\frac{1}{2\sqrt{f(x)}}f'(x)$$

Example 1: $y=\sqrt{x}\Rightarrow\frac{dy}{dx}=\frac{1}{2\sqrt{x}}$

Example 2: $\mathrm{y}=\sqrt{4x}\Rightarrow\frac{dy}{dx}=\frac{4}{2\sqrt{x}}=\frac{2}{\sqrt{x}}$

Example 3: Find the derivative of $y=\sqrt{2x^2+5}$

We have the given function as $y=\sqrt{2x^2+5}$ and differentiating with respect to variable x, we get $\frac{dy}{dx}=\frac{d}{dx}\sqrt{2x^2+5}$.

Now using the formula derivative of a square root, we have

$$\frac{dy}{dx}=\frac{1}{2\sqrt{2x^2+5}}\frac{d}{dx}(2x^2+5)$$

$$\frac{dy}{dx}=\frac{4x}{2\sqrt{2x^2+5}}=\frac{2x}{\sqrt{2x^2+5}}$$

ii)

$$\frac{d}{dx}\left(\sqrt[n]{f(x)}\right)=n\frac{\frac{d}{dx}(f(x))}{\sqrt[n]{f^{n-1}(x)}}$$

Example: $y=\sqrt[7]{x^5-2x}\Rightarrow y'=\dfrac{5x^4-2}{7\sqrt[7]{(x^5-2x)^6}}$

12) Implicit differentiatiation

$y=f(x)\ =\dfrac{x}{x^2+1}$ is **explicit function** when it is expressed directly in terms of x, **example**

y$=\dfrac{x}{x^2+1}$ and when this is in the form $x^2y+y-x=0$ it becomes an implicit function and y is said to be defined implicitly as a function of x . The derivative of y respect to x may be found by

considering y as a function of x and differentiating term by term:

$$\frac{d}{dx}(x^2y+y-x)=0$$

$$\frac{d}{dx}(x^2y)+\frac{d}{dx}(y)-\frac{d}{dx}(x)=0$$

$$\frac{d}{dx}(x^2)y+x^2\frac{d}{dx}(y)+\frac{dy}{dx}-1=0$$

$$2xy+x^2\frac{dy}{dx}+\frac{dy}{dx}-1=0$$

$$(x^2+1)\frac{dy}{dx}=1-2xy$$

$$\frac{dy}{dx}=\frac{1-2xy}{x^2+1}$$

13. Differentiation of function of function

Let $y = f(v)$ *with* $v = \Psi(x)$ and derivative of y respect to x is given by

$$\frac{dy}{dx} = \frac{dy}{dv}.\frac{dv}{dx}$$

Example: Let $y = 3v^2 - v$ with $v = x^2$, find $\frac{dy}{dx}$

Answer: $\frac{dy}{dv} = 6v - 1 = 6x^2 - 1$

$$\frac{dv}{dx} = 2x$$

$$\frac{dy}{dx} = \frac{dy}{dv}.\frac{dv}{dx} = (6x^2 - 1).(2x) = 12x^3 - 2x$$

$$\frac{dy}{dx} = 12x^3 - 2x$$

APPLICATION ACTIVITY 7.2

1) Given the function $f(x) = x^2 + 3x - 4$ and $g(x) = x + 1$. Find

$a. \left(f\left[g(x)\right]\right)'$ $\quad b.\ f'\left[g(x)\right]$ $\quad c.\ f'\left[g(x)\right].g'(x)$

2) Differentiate $y = \left(x^3 - 1\right)^{100}$

3) A body is moving along the $x-$axis such that its displacement is given by $x(t) = t^3 - 3t$. What will the acceleration $a(t)$ of the body be given that

$$a(t) = \frac{dv}{dt} \quad and \quad v(t) = \frac{dx}{dt}.$$

4) Find $\frac{dy}{dx}$ *given that* $x^2 - 2y^3 + 4x = 2$

7.3 Differentiation of trigonometric functions

ACTIVITY 7.3

Consider $y = f(x) = \sin x$

a) Based on definition of derivative and trigonometric formula of transformation of sum to product : $\sin a - \sin b = 2\sin\left(\frac{a-b}{2}\right)\cos\left(\frac{a+b}{2}\right)$,

Establish the formula of $\frac{dy}{dx}$

b) Knowing that $\cos x = \sin\left(\frac{\pi}{2} - x\right)$,

$\tan x = \frac{\sin x}{\cos x}, \sec x = \frac{1}{\cos x}, \operatorname{cosec} x = \frac{1}{\sin x}$, deduce the derivative of $\cos x$, $\tan x$, $\sec x$ and $\csc x$

Content summary

1. Derivative of of function $y = f(x) = \sin x$

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$f'(x) = \lim_{h\to 0}\frac{\sin(x+h)-\sin(x)}{h}$$

$$f'(x) = \lim_{h\to 0}\frac{2\sin\left(\frac{x+h-x}{2}\right)\cos\left(\frac{x+h+x}{2}\right)}{h}$$

$$f'(x) = \lim_{h\to 0}\frac{2\sin\left(\frac{h}{2}\right)\cos\left(\frac{2x+h}{2}\right)}{h}$$

$$f'(x) = \lim_{h\to 0}\frac{\sin\left(\frac{h}{2}\right)}{\frac{h}{2}}.\lim_{h\to 0}\cos\left(x+\frac{h}{2}\right)$$

$$f'(x) = 1.\lim_{h\to 0}\cos(x+h) \quad (2x^4+x)'\cos(2x^4+x)$$

$$f'(x) = \cos x$$

In general:

$[\sin u(x)]' = u'(x)\cos u(x)$ i.e if $y = \sin u$ with $u = \psi(x)$ then $\frac{dy}{dx} = \frac{du}{dx}.\cos u$

Example: given that $y = \sin(2x^4 + x)$,find $\frac{dy}{dx}$

Answer: $\frac{dy}{dx} = (2x^4 + x)'\cos(2x^4 + x) = (8x^3 + 1)\cos(2x^4 + x)$

2. Derivative of $y = \cos x$

We know that $\cos(x) = \sin\left(\frac{\pi}{2} - x\right) \Rightarrow \frac{dy}{dx} = -\sin x$

In general, $(\cos u(x))' = -u'(x)\ \sin u(x)$

$[\cos\ u(x)]' = -u'(x)\sin(u(x))$ i.e if $y = \cos(u(x))$ with $u = \psi(x)$ then $\frac{dy}{dx} = -\frac{du}{dx}.\sin(u(x))$

Example:

$y = \cos 2x \Rightarrow y' = -(2x)'\sin 2x$

$y' = -2\sin 2x$

3. Derivative of function $y = \tan x$

$y = \tan x$ can be written as $y = \frac{\sin x}{\cos x}$

then

$$\frac{dy}{dx} = \frac{d}{dx}\frac{\sin x}{\cos x} = \frac{(\sin x)'\cos x - \sin x(\cos x)'}{(\cos x)^2} = \frac{\cos x\cos x + \sin x\sin x}{(\cos x)^2} = \frac{\cos^2 x + \sin^2 x}{(\cos x)^2} = \frac{1}{(\cos x)^2}$$

$\frac{dy}{dx} = \frac{1}{(\cos x)^2}$ or $\frac{dy}{dx} = \sec^2 x$

In general, $[\tan u(x)]' = \frac{u'(x)}{\cos^2 u(x)}$ or $[\tan u(x)]' = u'(x)\sec^2 u(x)$

4. Derivative of function $y = \cot x$

In the same way with derivative of tangent, $\frac{d\cot x}{dx} = \frac{-1}{(\sin x)^2}$ **or**

$\frac{d\cot x}{dx} = -\csc^2 x$

In general, $[\cot u(x)]' = -\dfrac{u'(x)}{\sin^2 u(x)}$ or $[\cot u(x)]' = -u'(x)\operatorname{co}\sec^2 u(x)$

5. Derivative of $y = \sec x$

$$y = \sec x = \frac{1}{\cos x} \Rightarrow y' = \frac{\sin x}{\cos^2 x} \text{ or } y' = \sin x \sec^2 x$$

In general, $\left(\sec u(x)\right)' = u'(x)\dfrac{\sin u(x)}{\cos^2 u(x)}$ *or* $\left(\sec u(x)\right)' = u'(x)\sin u(x)\sec^2 u(x)$

6. Derivative of $y = \csc x$

$$y = \csc x = \frac{1}{\sin x} \Rightarrow y' = \frac{-\cos x}{\sin^2 x} \text{ or } y' = -\cos x \csc^2 x$$

In general, $\left(\csc u(x)\right)' = -u'(x)\dfrac{\cos u(x)}{\sin^2 u(x)}$ *or* $\left(\csc u(x)\right)' = -u'(x)\cos u(x)\csc^2 u(x)$

APPLICATION ACTIVITY 7.3

Find the derivative of the following functions

1. $f(x) = \sin^3\left(x^2 + 4\right)$
2. $f(x) = \cos 3x^2$

Find the derivative of the following functions

1. $f(x) = x\tan x$
2. $f(x) = \tan(3x + 2)$
3. $f(x) = \cot\left(x^2 - 5\right)$
4. $f(x) = \sin x \cot 4x$

Find the derivative of the following functions

1. $f(x) = \sec(3x + 2)$
2. $f(\theta) = \theta^3 \csc 2\theta$
3. $f(x) = \sec^4 3x$

7.4. Differentiation of inverse trigonometric function

ACTIVITY 7.4

1. We know that $f(x)=\sin^{-1}x$ and $f(x)=\cos^{-1}x$,are defined for $x\in[-1,1]$

$\sin^{-1}x=y$ means that $x=\sin y$,where $y\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$,and

$\cos^{-1}x=y$ means that $x=\cos y$, where $y\in[0,\pi]$

Use rule for derivative of composite functions to find the derivative of $\sin^{-1}x$ and $\cos^{-1}x$ the inverse of sine and cosine functions.

2. We know that $f(x)=\tan^{-1}x$ for $x\in\mathbb{R}$ and $x=\tan y$ for $y\in\left]-\frac{\pi}{2},\frac{\pi}{2}\right[$

also $f(x)=\cot^{-1}x$ for $x\in\mathbb{R}$ and $x=\cot y$ for $y\in]0,\pi[$ where $y=f(x)$.

Use rule for derivative of composite functions to find the derivative of

$\tan^{-1}x$ and $\cot^{-1}x$, the inverse of tangent and cotangent functions.

3. We know that $f(x)=\sec^{-1}x$ and $f(x)=\csc^{-1}x$ for $x\le -1$ or $x\ge 1$ and respectively $x=\sec y$ for $y\in[0,\pi], y\neq\frac{\pi}{2}$ and $x=\csc y$ for $y\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right], y\neq 0$

where $y=f(x)$. Use rule for derivative of composite functions to find the derivative of $\sec^{-1}x$ and $\csc^{-1}x$, the inverse of secant and cosecant functions.

Content summary

1) $\forall\ x\in]-1,1[,\ \left(\sin^{-1}x\right)'=\frac{1}{\sqrt{1-x^2}}$ and $\left(\cos^{-1}x\right)'=\frac{-1}{\sqrt{1-x^2}}$

If u is another function $\left(\sin^{-1}u\right)'=\frac{u'}{\sqrt{1-u^2}}$ and $\left(\cos^{-1}u\right)'=\frac{-u'}{\sqrt{1-u^2}}$

Example 1:

Find the derivative of $f(x)=\sin^{-1}x^3$

Solution

$$f'(x)=\frac{\left(x^3\right)'}{\sqrt{1-\left(x^3\right)^2}}=\frac{3x^2}{\sqrt{1-x^6}}$$

Example 2:

Find the derivative of $f(x)=\cos^{-1}(2x+1)$

Solution

$$f'(x)=\frac{-(2x+1)'}{\sqrt{1-(2x+1)^2}}$$

$$=\frac{-2}{\sqrt{1-4x^2-4x-1}}$$

$$=\frac{-1}{\sqrt{-x^2-x}}$$

Example 3:

Find the derivative of $y=\sin^{-1}\left(1-x^2\right)$

Solution

$$y'=\frac{-2x}{\sqrt{1-\left(1-x^2\right)^2}}=\frac{-2x}{\sqrt{-x^4+2x^2}}$$

Example 4:

Find the derivative of $y=3\cos^{-1}\left(x^2+0.5\right)$

Solution

$$y'=3\frac{-2x}{\sqrt{1-\left(x^2+0.5\right)^2}}=\frac{-6x}{\sqrt{0.75-x^2-x^4}}$$

Example 5:

Find the derivative of $y=\left(x^2+1\right)\sin^{-1}4x$

Solution

$$y' = (2x)\sin^{-1}4x + (x^2+1)\frac{4}{\sqrt{1-(4x^2)^2}}$$

$$= \frac{4(x^2+1)}{\sqrt{1-16x^4}} + 2x\sin^{-1}4x$$

2) The derivatives of tangent and cotangent are given by: $\left(\tan^{-1}x\right)' = \frac{1}{1+x^2}$ and

$$\left(\cot^{-1}x\right)' = \frac{-1}{1+x^2}$$

If u is another function, $\left(\tan^{-1}u\right)' = \frac{u'}{1+u^2}$ and $\left(\cot^{-1}u\right)' = \frac{-u'}{1+u^2}$

Example 1

Find the derivative of $f(x) = \left(\tan^{-1}2x\right)^4$

Solution

$$f'(x) = 4\left(\tan^{-1}2x\right)^3\left(\tan^{-1}2x\right)'$$

$$= 4\left(\tan^{-1}2x\right)^3\left(\frac{2}{1+4x^2}\right)$$

$$= \frac{8\left(\tan^{-1}2x\right)^3}{1+4x^2}$$

Example 2

Find the derivative of $f(x) = 2\cot^{-1}3x$

Solution

$$f'(x) = \frac{-2(3x)'}{1+(3x)^2} = \frac{-6}{1+9x^2}$$

3) The derivatives of secant and cosecant are given by $\left(\sec^{-1}x\right)' = \frac{1}{x\sqrt{x^2-1}}$ and

$$\left(\csc^{-1}x\right)' = \frac{-x'}{x\sqrt{x^2-1}}$$

$$=\frac{-1}{x\sqrt{x^2-1}}$$

If u is another function, $\left(\sec^{-1}u\right)'=\frac{u'}{u\sqrt{u^2-1}}$ and $\left(\csc^{-1}u\right)'=\frac{-u'}{u\sqrt{u^2-1}}$

Example 1:

Find the derivative of $f\left(x\right)=\sec^{-1}2x$

Solution

$$f'\left(x\right)=\frac{2}{2x\sqrt{4x^2-1}}$$

Example 2:

Find the derivative of $f\left(x\right)=\csc^{-1}\sqrt{x}$

Solution

$$f'(x)=\frac{-\frac{1}{2\sqrt{x}}}{\sqrt{x}\sqrt{(\sqrt{x})^2-1}}$$

$$=\frac{-1}{2x\sqrt{x-1}}$$

APPLICATION ACTIVITY 7.4

Find the derivative of the following functions

1. $f\left(x\right)=\cos^{-1}\frac{1}{x}$
2. $f\left(x\right)=\cos^{-1}x^2$
3. $f\left(x\right)=\sin^{-1}\left(1-x\right)$
4. $f\left(x\right)=\sin^{-1}\sqrt{2x}$
5. $f\left(x\right)=\cot^{-1}\sqrt{x}$

6. $f(x)=\cos^{-1}\dfrac{1}{x}-\cot^{-1}x$

7. $f(x)=\cot^{-1}\sqrt{x-1}$

8. $f(x)=\sec^{-1}(2x+1)$

9. $f(x)=\tan^{-1}\sqrt{x^2-1}+\csc^{-1}x$

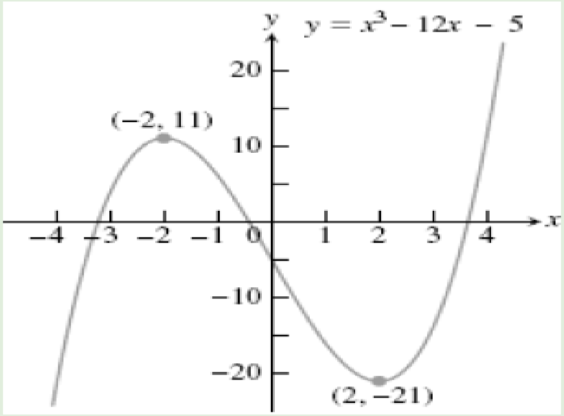
10. $f(x)=\sec^{-1}5x$

11. $f(x)=\csc^{-1}(x^2+1),\ x>0$

7.5 Derivative and the variation of a function

ACTIVITY 7.5

1. Observe the graph below and answer the questions that follow:



a. Is the given function $y=x^3-12x-5$ differentiable? Justify your answer.

b. Find y' by respecting x

c. Show the interval on which the function is increasing ,and decreasing if $y=x^3-12x-5$

2. Find where the function $f(x)=x^3-3x^2$ is concave up or down.

Content summary

Increasing and decreasing function

Theorem:

Let f be a function differentiable on an interval $]a,b[$

a) If $f'(x)>0$ on each point x of $]a,b[$, then f is increasing on $]a,b[$

b) If $f'(x)<0$ on each point x of $]a,b[$, then f is decreasing on $]a,b[$

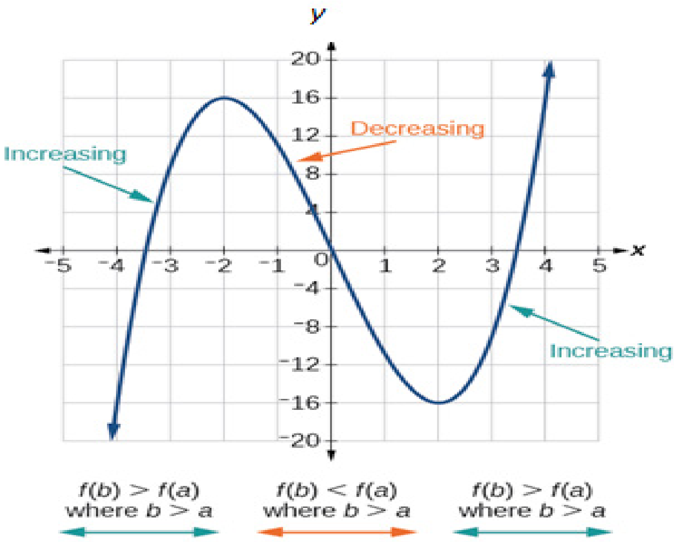
c) If $f'(x)=0$ for all $x\in]a,b[$, then f is constant on this interval, that is

$f(x_1)=f(x_2)$ for all $x_1,x_2\in]a,b[$, or equivalently, there exists a real c such that

$f(x)=c$ for all $x\in]a,b[$.

Given x_1 and x_2 from an interval I with $x_1<x_2$ if $f(x_1)<f(x_2)$ then $f(x)$ is increasing on I

Given any x_1 and x_2 from an interval I with $x_1<x_2$ if $f(x_1)>f(x_2)$ then $f(x)$ is decreasing on I.



If the derivative of a function is positive at a point then the function is increasing at that point and if the derivative is negative at a point then the function is decreasing at that point. Also, the fact that the derivative of a function is zero at a point then the function is not changing at that point. These ideas used previously to identify the intervals in which a function is increasing and decreasing. This can be summarized as follows:

- If $f'(x) > 0$ for every x on some interval I, then $f(x)$ is increasing on that interval
- If $f'(x) < 0$ for every x on some interval I, then $f(x)$ is decreasing on the interval.
- If $f'(x) = 0$ for every x on some interval I, then $f(x)$ is constant on the interval.

Example:

1) Let us determine all intervals where the following function is increasing or decreasing. $f(x) = -x^5 + \frac{5}{2}x^4 + \frac{40}{3}x^3 + 5$

To determine if the function is increasing or decreasing we will need the derivative.

$$\begin{aligned} f'(x) &= -5x^4 + 10x^3 + 40x^2 \\ &= -5x^2\left(x^2 - 2x - 8\right) \\ &= -5x^2(x-4)(x+2) \end{aligned}$$

From the factored form of the derivative we see that there are three critical points: $x = -2$, $x = 0$, and $x = 4$. We now need to determine where the derivative is positive and where it is negative by drawing a sign table of $f'(x)$, graphing the critical points and picking test points from each region to see if the derivative is positive or negative in each region.

x	$-\infty$		-2		0		4		$+\infty$
$f'(x)$		-	0	+	0	+	0	-	
$f(x)$	$+\infty$	$\searrow$	$f(-2)$	$\nearrow$	$f(0)$	$\nearrow$	$f(4)$	$\searrow$	$-\infty$

Increase is symbolized by the arrow $\nearrow$ on $-2 < x < 0$ and $0 < x < 4$

Decrease is symbolized by the arrow $\searrow$ on $-\infty < x < -2$ and $4 < x < +\infty$

First derivative test for local extrema

Knowing where a function increases and decreases also tells us how to test for the nature of **local extrema values**.

Theorem

Suppose that c is a critical point of a continuous function f, and that f is differentiable at every point in some interval containing c except possibly at c itself. Moving across c from left to right,

1. *If f' changes from negative to positive at c, then f has a* ***local minimum*** *at c*

2. *If f' changes from positive to negative at c, then f has a* ***local maximum*** *at c*

3. *If f' does not change sign at c (that is, f' is positive on both sides of c or negative on both sides), then f has* ***no*** *local extremum.*

It means that, the relative extrema of a continuous function occur at those critical points where the first derivative changes sign.

Definition: Let f be a function and let x_0 be a real number such that f is defined on an open interval containing x_0.

- If $f'(x_0)=0$, then we say that x_0 is a stationary number of f. For ex**ample**: If x is the time and $y=f(t)$ is the displacement (function) of a moving object, then $\dfrac{dy}{dt}=f'(t)$ is the velocity (function). Thus $f'(t_0)=0$ means that the velocity at time t_0 is 0, that is, the object is stationary at that moment.

Definition:

Let f be a function and let x_0 be a real number such that f is defined on an open interval containing x_0. We say that :

- f has a relative maximum at $x=x_0$ if $f(x_0)\geq f(x)$ for all x sufficiently close to x_0.
- f has a relative minimum at $x=x_0$ if $f(x_0)\leq f(x)$ for all x sufficiently close to x_0.

Example

Locate the relative extreme points of $f(x)=3x^{\frac{5}{3}}-15x^{\frac{2}{3}}$

Solution:

$$f'(x) = 5x^{\frac{2}{3}} - 10x^{-\frac{1}{3}} = 5x^{-\frac{1}{3}}(x-2)$$

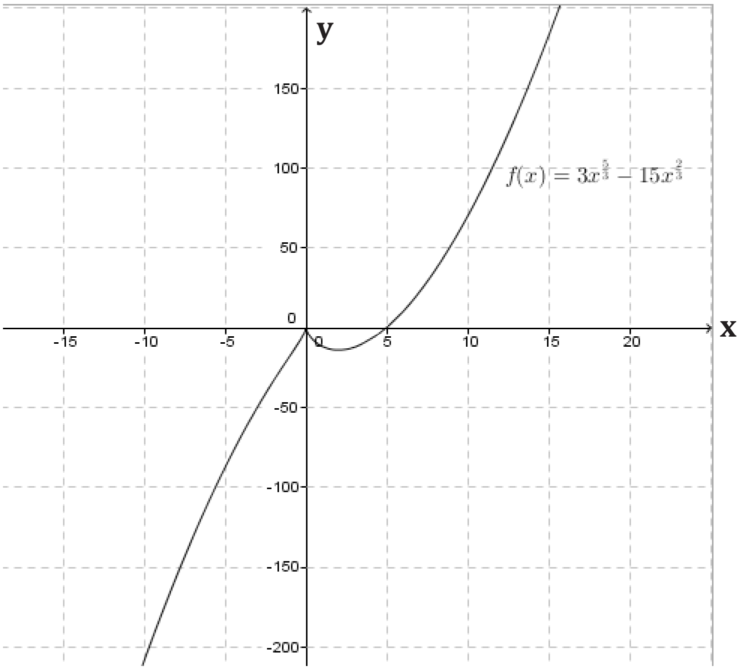
- If $x=0$ then, $f'(x)$ does not exist at $x=0$ but $f(0)$ exists and $f(0)=0$
- if $x=2$, the critical points are $(0,0)$ and $(2,-14.28)$

  Sign table of $f'(x)$

x		0		2	
$x-2$	- - -	- - -	- - -	0 + + +	+ + +
$x^{\frac{-1}{3}}$	- -	‖ + + +	+ + +	+ + +	+ + +
$f'(x)$	+ + +	‖ - - - ∞	- -	0 + + +	+ + +
$f(x)$	↗	0 ↘	↘	-14.28 ↗	↗

There is a relative maximum at 0 and a relative minimum at 2.

Graph of the function $f(x) = 3x^{\frac{5}{3}} - 15x^{\frac{2}{3}}$ is as follows:



APPLICATION ACTIVITY 7.5

Find the intervals where the following functions are increasing or decreasing, and locate the relative extreme points for each function;

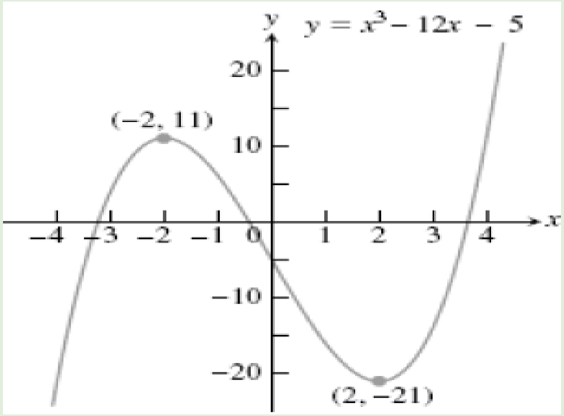
1) $f(x)=27x-x^3$

2) $f(x)=x^4-4x^3+5$.

7.6 Concavity of a function

ACTIVITY 7.6

Observe the graph below and answer the questions that follow



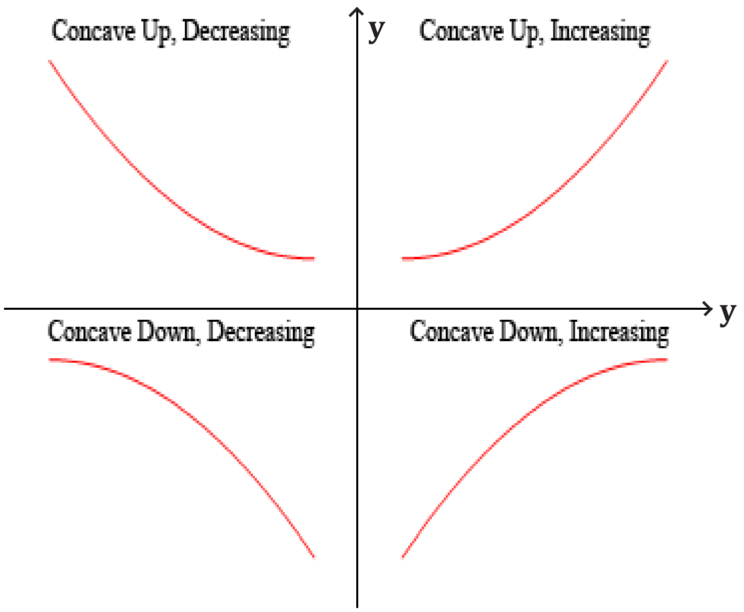
a. Give the intervals where the curve of the function opens up or down?

b. Find second derivative of $y=x^3-12x-5$ with respect to x and study the sign of $\frac{d^2y}{dx^2}$

c. What is your conclusion from the intervals obtained in a) and positive or negative second derivative in b)

Content summary

The second derivative of a function can give us information about the graph of a function such as concave up or concave down. The following figure gives us the idea of concavity



Function is **turned up (concavity up)** if it "opens" up and the function is **turned down** (**concavity down)** if it "opens" down. Notice as well that concavity has nothing to do with increasing or decreasing. A function can be concave up and either increasing or decreasing. Similarly, a function can be concave down and either increasing or decreasing.

Given the function $f(x)$ then

- $f(x)$ is **concave up** on an interval I if all of the tangents to the curve on I are below the graph of $f(x)$.
- $f(x)$ is **concave down** on an interval I if all of the tangents to the curve on I are above the graph of $f(x)$

Theorem:

Let f be a function that is defined and is twice differentiable on an open interval

$]a,b[$.

1) If $f''(x)>0$ for all $x\in]a,b[$, then f is convex or concave up on $]a,b[$

2) If $f''(x)<0$ for all $x\in]a,b[$, then f is concave down on $]a,b[$

Notice that this fact tells us that a list of possible inflection points will be those points where the second derivative is zero or doesn't exist. Be careful however to not make the assumption that just because the second derivative is zero or doesn't exist that the point will be an inflection point.

We will only know that it is an inflection point once we determine the concavity on both sides of it. It will only be an inflection point if the concavity is different on both sides of the point.

Example:

1) Let us find where the function $f(x)=x^3-3x^2$ is concave up or down.

We need the second derivative so that we will find where concave is up or down

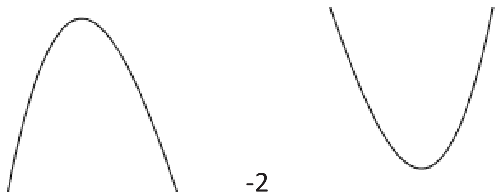
$f'(x)=3x^2-6x$,

$f''(x)=6x-6$

$f''(x)=0, x=1$

Make a table of sign for second derivative

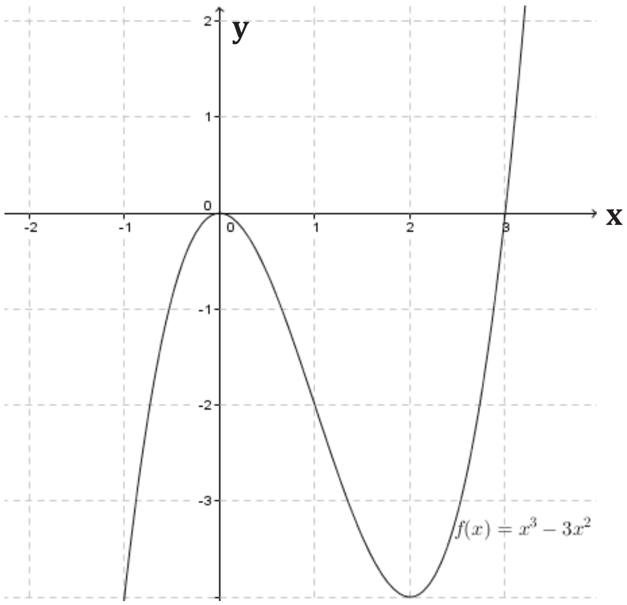
Sign of $f''(x)$

x		1	
$f''(x)$	- - - - - -	0	+ + + + + +
$f(x)$		-2	

Thus, $f(x)$ is concave up if $x>1$ or in interval $]1,+\infty[$

$f(x)$ is concave down if $x<1$ or in interval $]-\infty,1[$

The graph of the function $f(x)=x^3-3x^2$ is as follows :

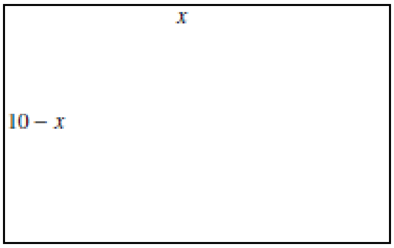


2) Find the dimensions of the rectangle that has maximum area if its perimeter is 20 cm.

Solution:

Let x be the length, the width w is such that $2(x + \text{w}) = 20$

i.e, w = 10-x



we will consider where the area A is increasing or decreasing where $A=x(10-x)$, 0<x<10.

We want to find the value of x at which A attains its maximum. Differentiating

$A(x)$, we get $A'(x)=\dfrac{d}{dx}\left(10x-x^2\right)=10-2x \quad (0<x<10)$

Solving $A^{'}(x)=0$, we obtain the critical number of $A: x_1=5$.

Since A is increasing on $(0,5)$ and decreasing on $(5,10)$, it follows that A attains its absolute maximum at $x_1=5$.The dimensions of the largest rectangle is $5cm \times 5cm$.

	$(0,5)$	$(5,10)$
A'	+ + + + + +	- - - - - - - - -
A	$\nearrow$	$\searrow$

APPLICATION ACTIVITY 7.6

1. Let f be the function $f(x)=x^3-12x-5$

   a. Calculate extreme points

   b. Show the interval of increasing and decreasing

   c. Show where the function is concave up or down

7.7 Derivative and the table of variation for a function

ACTIVITY 7.7

Let f be the function $f(x)=x^3-12x-5$

a) Calculate the first and second derivative of $f(x)$

b) In one table, present the sign table of $f'(x)$ *and* $f''(x)$ showing also the intervals where $f(x)$ is increasing or decreasing and the sense of concavity.

c) Try to analyse the function according to the above table

Content summary

Variation table or synthetic table summarize all information about the function: domain of definition, limits at the boundaries of domain, asymptotes to the curve, extrema points, variation of function (increasing or decreasing) , inflection points and the sense of concavity(opens up or down).

Example: **Establish the variation table and sketch of the function**

$$f(x)=\frac{x^2+1}{x}$$

Condition of existence: $x\neq 0$

1) $Domf=\mathbb{R}\setminus\{0\}=]-\infty,0[\cup]0,+\infty[$

2) Limits at the boundaries of Domf

a)

$$\lim_{x\to-\infty}\frac{x^2+1}{x}=\frac{\infty}{\infty}\quad (IF)$$

$$\text{T.V}:\ \lim_{x\to-\infty}\frac{x^2+1}{x}=\lim_{x\to-\infty}\frac{x^2(1+\frac{1}{x^2})}{x}=\lim_{x\to-\infty}x=-\infty\ \Rightarrow \text{There is no Horizontal asymptote}$$

b)

$$\lim_{x\to0}\frac{x^2+1}{x}=\frac{1}{0}$$

$$\lim_{x\to0^-}\frac{x^2+1}{x}=\frac{1}{0^-}=-\infty$$

$$\lim_{x\to0^+}\frac{x^2+1}{x}=\frac{1}{0^+}=+\infty\qquad \Rightarrow \textit{There is a vertical asymptote}: x=0$$

c)

$$\lim_{x\to+\infty}\frac{x^2+1}{x}=\frac{\infty}{\infty}\quad (IF)$$

$$\text{T.V}:\ \lim_{x\to+\infty}\frac{x^2+1}{x}=\lim_{x\to+\infty}\frac{x^2(1+\frac{1}{x^2})}{x}=\lim_{x\to+\infty}x=+\infty$$

$\Rightarrow$ There is no Horizontal asymptote

Let find the oblique asymptote (OA)

$$OA\equiv y=ax+b \text{ with } a=\lim_{x\to\pm\infty}\frac{f(x)}{x} \text{ and } b=\lim_{x\to\pm\infty}\left[f(x)-ax\right]$$

$$a=\lim_{x\to\pm\infty}\frac{x^2+1}{x^2}=\lim_{x\to\pm\infty}\frac{x^2}{x^2}=1$$
$$b=\lim_{x\to\pm\infty}\left(\frac{x^2+1}{x}-x\right)=\lim_{x\to\pm\infty}\left(\frac{x^2+1-x^2}{x}\right)=\lim_{x\to\pm\infty}\left(\frac{1}{x}\right)=0$$
$a=1, b=0 \Rightarrow$ *there is no OA*

3) Parity of $f(x)$

d) $f(x)$ is even function if $f(x)=\mathrm{f}(-x)$

$$f(-x)=\frac{(-x)^2+1}{-x}=-\frac{x^2+1}{x}\neq f(x) \text{ ,so } f(x)=\frac{x^2+1}{x}$$ is not even

e) $f(x)$ is odd function if $-f(x)=\mathrm{f}(-x)$

$$f(-x)=\frac{(-x)^2+1}{-x}=-\frac{x^2+1}{x}$$

f) $-f(x)=-\frac{x^2+1}{x}$,so $-f(x)=\mathrm{f}(-x)$ and $f(x)$ is odd function

4) Parity of $f(x)$

The given function is not periodic

5) First derivative of $f(x)$

$$f'(x)=\left(\frac{x^2+1}{x}\right)'=\frac{2x.x-1.(x^2+1)}{x^2}=\frac{2x^2-x^2-1}{x^2}$$
$$f'(x)=\frac{x^2-1}{x^2}$$

Critical points:

g) Condition of existence of $f'(x)$

$x^2\neq 0 \Leftrightarrow x\neq 0$

$$f'(x)=0\Leftrightarrow \frac{x^2-1}{x^2}=0\Leftrightarrow x^2-1=0$$
$\Rightarrow x=-1$ *or* $x=1$

Extrema points: if $\begin{array}{l} x=-1 \Rightarrow y=-2 \text{ ,function has a maximum point (-1,-2)} \\ x=1 \ \Rightarrow y=2, \text{function has a minimum point (1,2)} \end{array}$

Sign table of $f'(x)$

x	$-\infty$		-1		0		1		∞
$f'(x)$		+ + +	0	- - -	||	- - - -	0	+ + + +	
$f(x)$	$-\infty$	$\nearrow$	-2	$\searrow$	$-\infty$ || $+\infty$	$\searrow$	2	$\nearrow$	$+\infty$

6) Second derivative of $f(x)$

$$f''(x)=\left(\frac{x^2-1}{x^2}\right)'=\frac{2x.x^2-2x.(x^2-1)}{x^4}=\frac{2x^3-2x^3+2x}{x^4}=\frac{2x}{x^4}$$

$$f''(x)=\frac{2}{x^3}$$

Critical points:

- Existence *condition* of $f''(x): x^3 \neq 0 \Rightarrow x \neq 0$
- $f''(x)=0 \Leftrightarrow \frac{1}{x^3}=0$ It doesn't exist in $\mathbb{R}$

Sign table of $f''(x)$

x	$-\infty$		0		$+\infty$
$f''(x)$		- - - - -	||	+ + + +	
$f(x)$			||		

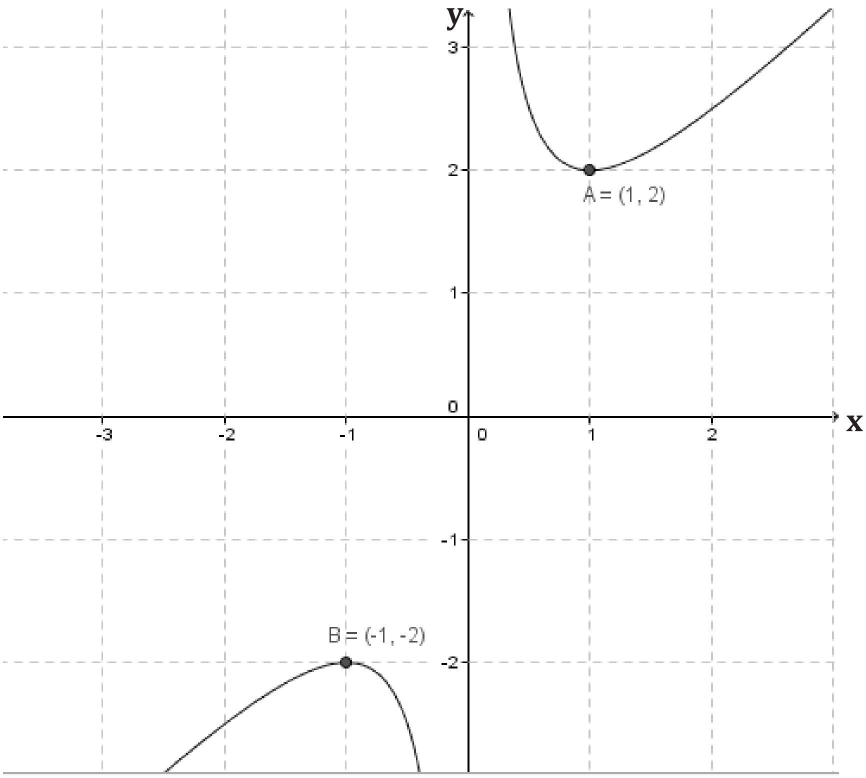
7) Variation table or synthetic table

x	$-\infty$		-1		0		1		$+\infty$
$f'(x)$		+ +	+ 0	- - -	||	- - - -	0	+ + +	
$f''(x)$		- - - -	-	-	||	+			
$f(x)$	$-\infty$	$\nearrow$	-2	$\searrow$	$-\infty$ || $+\infty$	$\searrow$	2	$\nearrow$	$+\infty$

8) Supplementary points

x	2	-2	-3	3	-4	4
$f(x)$	2.5	-2.5	$-\frac{10}{3}$	$\frac{10}{3}$	$-\frac{17}{4}$	$\frac{17}{4}$

Graph of function $f(x)=\dfrac{x^2+1}{x}$



APPLICATION ACTIVITY 7.7

1. Study completely $y = x + \frac{1}{4x}$
2. Study completely $y = \frac{1}{x-2}$

7.8 Derivative and limit with indeterminate cases: Hospital's rule

ACTIVITY 7.8

Evaluate the following limits:

a) $\lim_{t\to 1} \frac{5t^4 - 4t^2 - 1}{10 - t - 9t^2}$

b) $\lim_{x\to 4} \frac{x^2 - 16}{x - 4}$

c) $\lim_{x\to 4} \frac{4x^2 - 5x}{1 - 3x^2}$

What happen if you calculate the limits of derivative of numerator over derivative of denominator for above limits?

Content summary

Hospital rule states that, if $\lim_{x\to x_0} \frac{f(x)}{g(x)}$ *with* $g(x) \neq 0$ and x_0 a finite number or infinity is indeterminate form of $\frac{0}{0}$ or $\frac{\infty}{\infty}$ then, it can be calculated by $\lim_{x\to x_0} \frac{f'(x)}{g'(x)}$ *with* $g'(x) \neq 0$. If this result is indeterminate form, the procedure can be repeated.

Examples

By using Hospital rule , evaluate the following limits

a) $\lim_{x\to 2}\frac{\sqrt{7+x}-3}{x-2}=\frac{0}{0}$ (I.F)

$$\lim_{x\to 2}\frac{\sqrt{7+x}-3}{x-2}=\lim_{x\to 2}\frac{\left(\sqrt{7+x}-3\right)'}{(x-2)'}=\lim_{x\to 2}\frac{\frac{1}{2\sqrt{7+x}}}{1}=\frac{1}{2}\lim_{x\to 2}\frac{1}{\sqrt{7+x}}=\frac{1}{2}\cdot\frac{1}{3}=\frac{1}{6}$$

b)

$$\lim_{x\to 2}\left(\frac{4}{x^2-4}-\frac{1}{x-2}\right)=\lim_{x\to 2}\left(\frac{4-(x+2)}{x^2-4}\right)=\lim_{x\to 2}\left(\frac{2-x}{x^2-4}\right)=\frac{0}{0}\quad \text{I.F}$$

$$\lim_{x\to 2}\left(\frac{4}{x^2-4}-\frac{1}{x-2}\right)=\lim_{x\to 2}\frac{(2-x)'}{(x^2-4)'}=\lim_{x\to 2}-\frac{1}{2x}=-\frac{1}{4}$$

APPLICATION ACTIVITY 7.8

By using Hospital rule , evaluate the following limits

1) $\lim_{x\to 0}\frac{2\sin x-\sin 2x}{x-\sin x}$

2) $\lim_{x\to\infty}\frac{2x+7}{3x^2-5}$

3) $\lim_{x\to\infty}\frac{x^2-1}{x^2+3x-4}$

4) $\lim_{x\to 4}\frac{x-4}{\sqrt{x}-2}$

5) $\lim_{x\to 0}\frac{\arcsin 4x}{\arctan 5x}$

7.9 Derivative, tangent line equation and Normal line equation

ACTIVITY 7.9

Consider the function $f(x) = -x^3 + 3x$ and the line $y = 3x$ passing through point $(0,0)$.

1. Show that $(0,0)$ is the intersection of $f(x) = -x^3 + 3x$ and $y = 3x$
2. Find $f'(0)$
3. Compare the result in 2. and the gradient of the given line.

Content summary

Tangent line

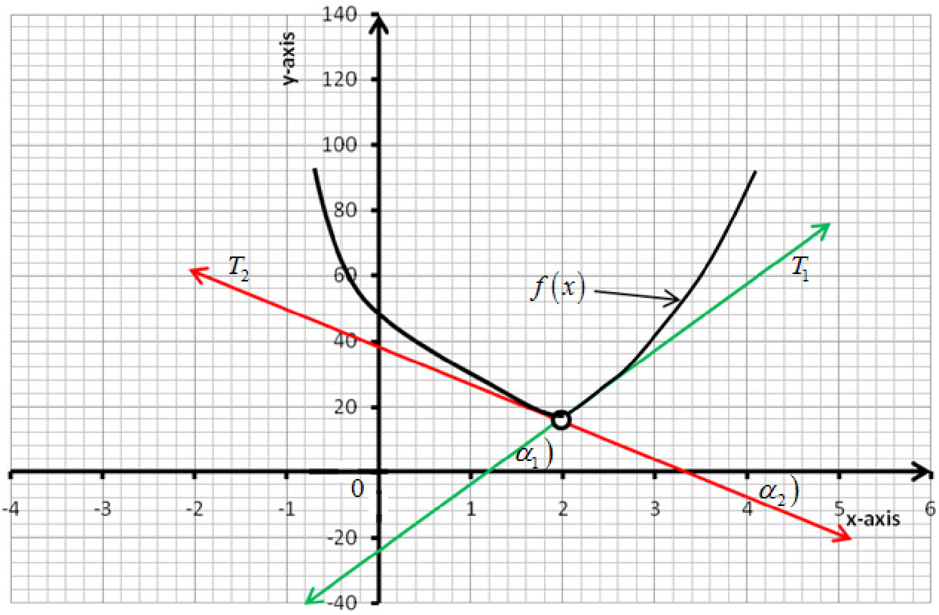
The slope of the tangent line of $y = f(x)$ at $(x_0, f(x_0) = y_0)$ is given by

$f'(x_0) = \dfrac{y - y_0}{x - x_0}$.

Then, the equation of the tangent line is $T \equiv y - y_0 = f'(x_0)(x - x_0)$

Remember that the function $f(x)$ can have distinct right-hand and left- hand derivatives at point x_0; that is, $f'(x_0^-) \neq f'(x_0^+)$.

In this case we say that the point x_0 is a **sharp.** The curve has no tangent line at x_0. Centrally, it has a half tangent at the left and another at the right with different slopes (see the following figure).



$$\tan\alpha_1 = \lim_{x\to x_0^+}\frac{f(x)-f(x_0)}{x-x_0} \text{ and } \tan\alpha_2 = \lim_{x\to x_0^-}\frac{f(x)-f(x_0)}{x-x_0}$$

Normal line

We call normal line to the curve at point (x_0, y_0), the perpendicular line to the tangent line of the curve at point (x_0, y_0). Its equation is of the form

$$N \equiv y - y_0 = -\frac{1}{f'(x_0)}(x-x_0)$$

Example

Given the parabola $f(x) = x^2$

a) Find the point where the tangent line is parallel to the bisector of the first quadrant.

b) Find the tangent line to the curve of this function at point $(2,4)$

Solution

a) The bisector of the first quadrant has the equation $y = x$ so its slope is $m = 1$. Since the two lines are parallel they have the same slope. So $f'(x_0) = 1$. Since the slope of the tangent line to the curve is equal to the derivative at $x = x_0$

$$\begin{aligned} f'(x_0) &= \lim_{h\to 0}\frac{(x_0+h)^2 - x_0^2}{h} \\ &= \lim_{h\to 0}\frac{x_0^2 + 2x_0h + h^2 - x_0^2}{h} \\ &= \lim_{h\to 0}\frac{2x_0h + h^2}{h} \\ &= \lim_{h\to 0}(2x_0 + h) \\ &= 2x_0 \end{aligned}$$

But $f'(x_0) = 1 \Rightarrow 2x_0 = 1 \Rightarrow x_0 = \frac{1}{2}$ and $y_0 = f\left(\frac{1}{2}\right) = \left(\frac{1}{2}\right)^2 = \frac{1}{4}$.

Thus, the needed point is $\left(\frac{1}{2}, \frac{1}{4}\right)$.

b) The given point is $(2,4)$, then $x_0 = 2$, $y_0 = 4$. $f'(x_0) = 2x_0 \Rightarrow f'(2) = 4$

The tangent line is

$$T \equiv y-4=4(x-2)$$
$$T \equiv y-4=4x-8$$
$$T \equiv y=4x-4$$
$$T \equiv y=4(x-1)$$

Example

Let us determine the equation of the normal line to the curve with equation $y=x^2-5x+6$ at point with abscissa $x=0$.

$f'(x)=2x-5$, $f'(0)=-5$, $f(0)=6$

The equation of normal line is $N \equiv y-6=-\frac{1}{-5}(x-0)$ or $N \equiv y=\frac{1}{5}x+6$

APPLICATION ACTIVITY 7.9

1.Determine the equation of the tangent and normal line to the curve of function

a) $f(x)=\sqrt{x^2+3}$ at the point $(-1,2)$

b) $f(x)=x^3-x+5$ at the point $(1,5)$

c) $f(x)=x^3$ at the point where $x=2$

2. Let $f(x)=x^2-x$. Find the equation of tangent line with slope $m=-3$

7.10 Applications of differentiation in Economics and finance

ACTIVITY 7.10

1) Go in library or computer lab, do research on application of differentiation in economics and finance.

2) The marginal cost (MC) is the rate of change of the total cost (TC) function. In fact, in situations where one is dealing with the concept of a marginal increase, the marginal function is equal to the rate of change of the original function. This

means that $MC = \frac{dTC}{dq}$. Determine the function MC when $TC = 6 + 4q^2$.

Content summary

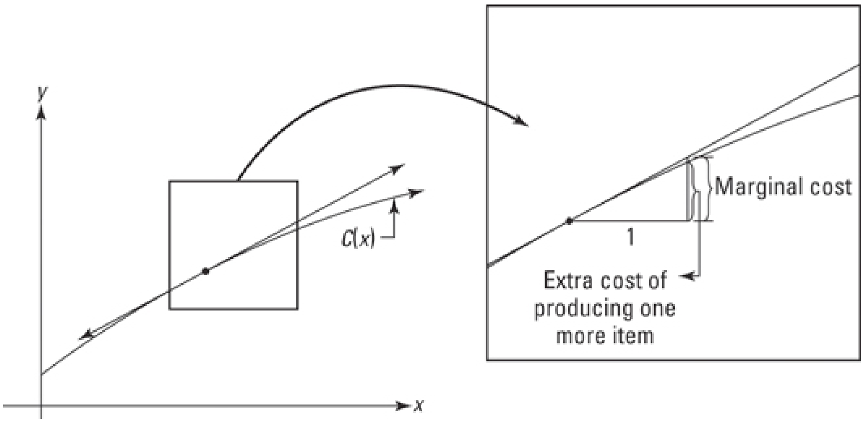
Marginal cost

Suppose a manufacturer produces and sells a product. Denote $C(q)$ to be the total cost for producing and marketing q units of the product. Thus C is a function of q and it is called the *(total) cost function*. The rate of change of C with respect to q is called the *marginal cost*, that is,

$$\text{Marginal Cost } = \frac{dC}{dq}$$

Marginal cost, marginal revenue, and marginal profit all involve how much a function goes up (or down) go through unit to the right — this is very similar to the way linear approximation works.

Say that you have a cost function that gives you the total cost, $C(x)$, of producing x items (shown in the figure below).



Example

A widget manufacturer determines that the demand function for her widgets is $p = \frac{1000}{\sqrt{x}}$ where x is the demand for widgets at a given price, p. The cost of producing x widgets is given by the following cost function:

$$C(x) = 10x + 100\sqrt{x} + 10,000$$

Determine the marginal cost at $x = 100$ widgets

Solution

Marginal cost is the derivative of the cost function, so take the derivative and evaluate it at $x = 100$.

$$C(x) = 10x + 100\sqrt{x} + 10,000$$

$$C(x) = 10 + \frac{50}{\sqrt{x}} \text{(power rule)}$$

$$C(100) = 10 + \frac{50}{\sqrt{100}} = 10 + \frac{50}{10}$$

$$C(100) = 15$$

Thus, the marginal cost at $x = 100$ is \$15 — this is the approximate cost of producing the 101st widget.

Marginal revenue

Denote $R(q)$ to be the total amount received for selling q units of the product. Thus R is a function of q and it is called the *revenue function*. The rate of change of R with respect to q is called the *marginal revenue*, that is,

$$\text{Marginal Revenue} = \frac{dR}{dq}$$

Denote $P(q)$ to be the **profit** of producing and selling q units of the product, that is,

$$P(q) = R(q) - C(q)$$

Thus P is a function of q and it is called the profit function.

In introductory economics texts, marginal revenue (MR) is sometimes defined as the increase in total revenue (TR) received from sales caused by an increase in output by 1 unit.

This only gives an approximate value for marginal revenue and it will vary if the units that output is measured in are changed. A more precise definition of *marginal revenue is that it is the rate of change of total revenue relative to increases in output*.

Denote q_{max} to be the largest number of units of the product that the manufacturer can produce. Assuming that q can take any value between 0 and q_{max}. Then for each of the functions C, R and P, the domain is $[0;\ q_{max}]$. Suppose that the cost function and the revenue function are differentiable on $]0, q_{max}[$ and suppose that producing 0 or q_{max} units of the product will not give maximum profit. Then in order to have maximum profit, we need

$$\frac{dP}{dq} = 0$$

Or eventually,

$$\frac{dC}{dq} = \frac{dR}{dq},$$

that is, marginal cost = marginal revenue.

To have a maximum profit, marginal cost = marginal revenue

Example 1:

1) Given that $\mathrm{TR(q)} = 80q - 2q^2$,

a) Find the function of the marginal revenue MR, and $P(q)$, given that $\mathrm{TR(q)} = P.q$

b) Find the exact value of the output at which TR is a maximum.

Solution:

a) $MR = \frac{dTR}{dq} = 80 - 4q$

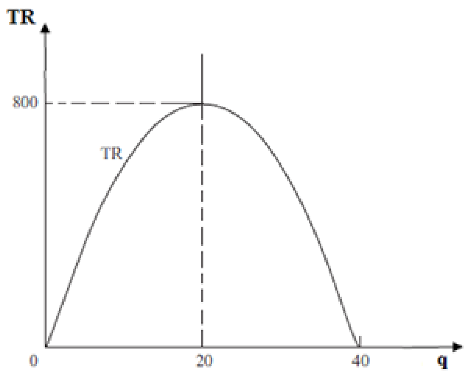
We have $TR(q) = \mathrm{Pq} \Leftrightarrow 80\mathrm{q} - 2\mathrm{q}^2 = Pq$

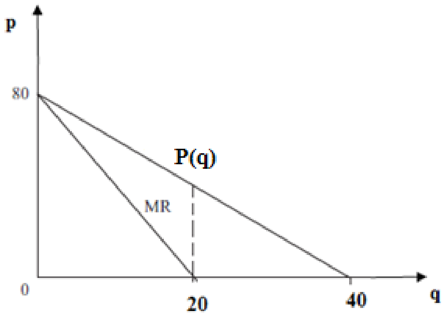
Therefore, the price function is $\mathrm{P}(q) = 80 - 2q$

b) When TR is at its maximum, $\frac{dTR(q)}{dq} = MR = 0$

Thus,

$$MR = 80 - 4q = 0$$
$$80 = 4q$$
$$20 = q$$





Example 2: For the total revenue function $TR = 500q - 2q^2$, find the value of MR when $q = 80$.

Solution

$MR = \frac{dTR}{dq} = 500 - 4q$. Thus, when $q = 80$, $MR = 500 - 4(80) = 180$

Example 3:

A widget manufacturer determines that the demand function for her widgets is $p = \frac{1000}{\sqrt{x}}$ where x is the demand for widgets at a given price, p.

The cost of producing x widgets is given by the following cost function:

$$C(x) = 10x + 100\sqrt{x} + 10,000$$

Determine the marginal revenue at x = 100 widgets

Solution

Revenue, R(x), equals the number of items sold, x, times the price, p:

$$R(x) = x.p = x.\frac{1000}{\sqrt{x}} \text{(using the above demand function)}$$

$$R(x) = \frac{1000x}{\sqrt{x}}.\frac{\sqrt{x}}{\sqrt{x}} = \frac{1000x\sqrt{x}}{x} = 1000\sqrt{x}$$

Marginal revenue is the derivative of the revenue function, so take the derivative of $R(x)$ and evaluate it at $x = 100$:

$$R(x) = 1000\sqrt{x}$$

$$R'(x) = \frac{500}{\sqrt{x}} \text{(power rule)}$$

$$R'(100) = \frac{500}{\sqrt{100}} = 50$$

Thus, the approximate revenue from selling the 101st widget is \$50.

Example 4: Profit maximization

A monopoly faces the demand schedule $p = 460 - 2q$ and the cost schedule TC $= 20 + 0.5q^2$

How much should it sell to maximize profit and what will this maximum profit be? (All costs and prices are in \$.)

Solution

To maximize profits, the rule MC = MR must be realized.

To find the output where $MC = MR$ we first need to differentiate the MC and MR functions.

Given $TC = 20 + 0.5q^2$, then $MC = \frac{dTC}{dq} = q$

As $TR = pq = (460 - 2\text{q}).\text{q} = 460\text{q} - 2\text{q}^2$

Then $MR = \frac{dTR}{dq} = 460 - 4q$

To maximize profit $MR = MC$. Therefore, equating the two quantities, we find:

$460 - 4q = q$ which implies that $q = 92$

The actual maximum profit when the output is 92 will be:

$$TR - TC = (460q - 4q^2) - (20 + 0.5q^2)$$

$$= 460q - 2.5q^2 - 20 = 460.(92) - 2.5(8,464) - 20 = 21,140$$

The actual maximum profit when the output is 92 will be \$21,140

Example 5: A firm faces the demand schedule $p = 184 - 4q$ and the TC function $TC = q^3 - 21q^2 + 160q + 40$.

What output will maximize profit?

Solution

Given that $TR = pq = (184 - 4q)q = 184q - 4q^2$, we have $MR = \frac{dTR}{dq} = 184 - 8q$

And $MC = \frac{dTC}{dq} = 3q^2 - 42q + 160$.

To maximize the profit, MC=MR. Therefore,

$$3q^2 - 42q + 160 = 184 - 8q$$
$$(q - 12)(3q + 2) = 0$$
$$q - 12 = 0 \; or \; 3q + 2 = 0$$

$$q = 12 \; or \; q = -\frac{2}{3}$$

As one cannot produce a negative quantity, the firm must produce 12 units of output in order to maximize profits.

Example 6: Suppose that a company has estimated that the cost (in Rwandan francs) of producing x items is $C(x) = 10000 + 5x + 0.01x^2$. What is the marginal cost at the production level of 500 items?

Solution: Then the marginal cost function is $c'(x) = 5 + 0.02x$

The marginal cost at the production level of 500 items is

$c'(500) = 5 + 0.02(500) = 15$ Rwandan Francs per item.

Example 7. The demand equation for a certain product is

$q - 90 + 2p = 0, \quad 0 \le q \le 90$ where q is the number of units and p is the price per unit, and the average cost function is $C_{av} = q^2 - 8q + 57 + \frac{2}{q}, \quad 0 \le q \le 90$. At what value of q will there be maximum profit? What is the maximum profit?

Solution:

Although the average cost function is undefined at $q = 0$, we may include 0 in the domain of the cost function. The cost function and the revenue function are differentiable on $]0;90[$. However, we do not know whether maximum profit would be attained in $]0;90[$ or at an endpoint. So we use the method for finding absolute extrema for functions on closed and bounded intervals.

The cost function C is given by

$C(q) = q.C_{av} = q^3 - 8q^2 + 57q + 2, \quad 0 \le q \le 90)$ and the total revenue function R is given by

$$R(q) = p.q = \frac{90-q}{2}.q; \quad 0 \le q \le 90)$$

Therefore the profit function P is given by

$$p(q) = R(q) - C(q) = (45\text{q} - \frac{q^2}{2}) - (q^3 - 8q^2 + 57q + 2)$$

$$= -q^3 + \frac{15}{2}q^2 - 12q - 2, \quad (0 \le q \le 90)$$

Differentiating $P(q)$, we get

$$p'(q) = \frac{d\left(-q^3 + \frac{15}{2}q^2 - 12q - 2\right)}{dq} = -3q^2 + 15\text{q} - 12, \quad (0 \le q \le 90)$$

Solving $P'(q)=0$, that is,

$$-3q^2+15q-12=0$$
$$-3(q-1)(q-4)=0; \quad (0<q<90)$$

We get the critical number of P where $q_1=1$ and $q_2=4$.

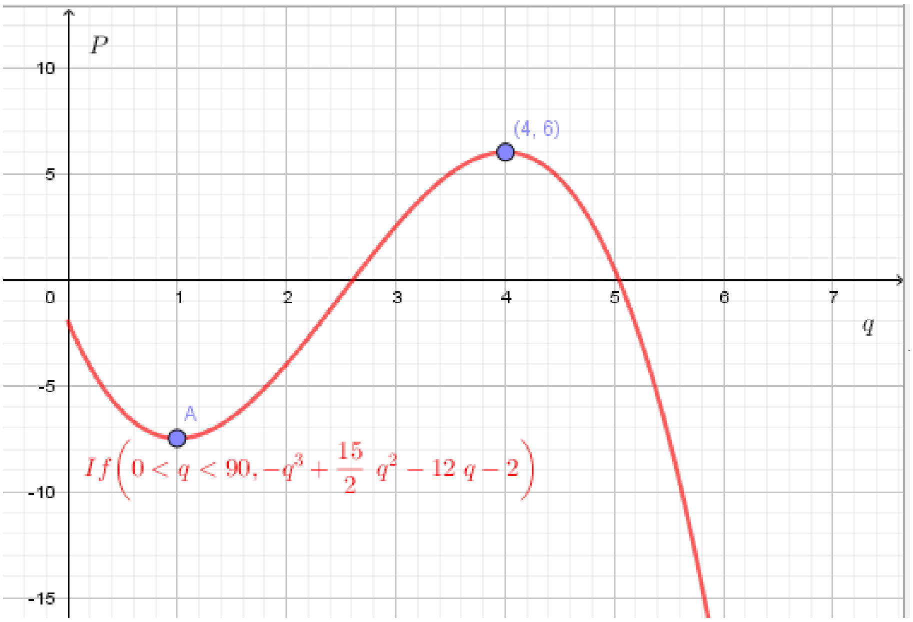
Comparing the values of P at the critical numbers as well as that at the endpoints:

q	0	1	4	90
P(q)	-2	$-\frac{15}{2}$	6	-669332

We see that maximum profit is attained at $q_2=4$ and the maximum profit is 6 (units of money).

Remarks:

If we know that maximum profit is not attained at the end points, we can simply compare the values of P at $q_1=1$ and $q_2=4$.



Example 8: A store has been selling 200 DVD burners a week at 350 dollars each. A market survey indicates that for each 10 dollars rebate offered to buyers, the number of units sold will increase by 20 a week. Find the demand function and the revenue function. How large a rebate should the store offer to maximize its revenue?

Solution

If x is the number of DVD burners sold per week, then the weekly increase in sales is $x-200$. For each increase of 20 units sold, the price is decreased by 10 dollars. So for each additional unit sold, the decrease in price will be $\frac{1}{20}\times 10$

The demand function is $p(x)=350-\frac{10}{20}(x-200)=450-\frac{2}{2}x$

The revenue function is $R(x)=xp(x)=450x-\frac{1}{2}x^2$

Since $R'(x)=450-x$, we see that $R'(x)=0$ when $x=450$

This value of x gives an absolute maximum by the First Derivative Test

$$p(450)=450-\frac{1}{2}(450)=225$$

The rebate is 350-225=125, Therefore, to maximize revenue, the store should offer a rebate of 125 Rwandan francs.

Tax yield

Elementary supply and demand analysis tells us that the effect of a per-unit tax *t* on a good sold in a competitive market will effectively shift up the supply schedule vertically by the amount of the tax. This will cause the price paid by consumers to rise and the quantity bought to fall. The change in total revenue spent by consumers will depend on the price elasticity of demand.

Example 9:

A market has the demand schedule p = 92−2q and the supply schedule p = 12+3q. What per-unit tax will raise the maximum tax revenue for the government? (All prices are in $.).

Solution:

Let the per-unit tax be t. This changes the supply schedule to $p=12+t+3q$

i.e the intercept on the price axis shifts vertically upwards by the amount t.

we now need to derive a function for q in terms of the tax t. In equilibrium, supply price equals demand price. Therefore,

$$12+3q+t=92-2q$$
$$q=16-0.2t$$

The tax yield is (amount sold) x(per unit tax). Therefore,

$$TY = q.t = (16 - 0.2t).t = 16t - 0.2t^2$$

And so the rate of change of TY with respect to t is

$$\frac{dTY}{dt} = 16 - 0.4t$$

If $\frac{dTY}{dt} > 0$, an increase in t will increase TY. However, from the formula for $\frac{dTY}{dt}$ derived above, one can see that as the amount of tax t is increased the value of $\frac{dTY}{dt}$ falls. Therefore in order to maximize TY, t should be increased until $\frac{dTY}{dt} = 0$.Any further increases in t would cause $\frac{dTY}{dt}$ to become negative and cause TY to start to fall.

Thus, $$\frac{dTY}{dt} = 16 - 0.4t = 0$$

$$t = 40$$

Therefore a per-unit tax of \$40 will maximize the tax yield.

Example 10:

A widget manufacturer determines that the demand function for her widgets is $p = \frac{1000}{\sqrt{x}}$ where x is the demand for widgets at a given price, p.
The cost of producing x widgets is given by the following cost function:

$$C(x) = 10x + 100\sqrt{x} + 10,000$$

Determine the marginal profit at x = 100 widgets

Solution

Profit, $P(x)$, equals revenue minus costs. So,

$$\begin{aligned} P(x) &= R(x) - C(x) \\ &= 1000\sqrt{x} - \left(10x + 100\sqrt{x} + 10,000\right) \\ &= -10x + 900\sqrt{x} - 10,000 \end{aligned}$$

Marginal profit is the derivative of the profit function, so take the derivative of $P(x)$ and evaluate it at $x = 100$.

$$P(x) = -10x + 900\sqrt{x} - 10,000$$

$$P'(x) = -10 + \frac{450}{\sqrt{x}} \text{(power rule)}$$

$$P'(100) = -10 + \frac{450}{\sqrt{100}}$$

$$= -10 + 45$$

$$= 35$$

So, selling the 101st widget brings in an approximate profit of \$35.

Once you know the marginal cost and the marginal revenue, you can get marginal profit with the following simple formula: Marginal Profit = Marginal Revenue – Marginal Cost.

APPLICATION ACTIVITY 7.10

A market faces the demand schedule $p = 58 - \frac{q}{2}$ and the cost schedule

$$TC = 97q - 17\frac{q^2}{2} + \frac{q^3}{3}$$

How much should it sell to maximize profit and what will be this maximum profit? (All costs and prices are in Rwandan Francs)

7.11 Applications of differentiation: rates of change problems, optimization problems

ACTIVITY 7.11

Given the function $y = f(x) = x^2$,find the ratio of the variation of y over the variation of x in each of the following intervals:

a. $[2;4]$

b. $[-3;-1]$

Content summary

Rates of Change

The purpose here is to remind us of one of the most important applications of derivatives. That is the fact that $f'(x)$ represents the rate of change of $f(x)$.

If (x_0, y_0) is a point on the graph of $y = f(x)$, then we define $m = \dfrac{y_1 - y_0}{x_1 - x_0}$ to be the **average rate** at which y changes with respect t to x over the interval $[x_0, x_1]$.

If $y = f(x)$ and $f(x)$ is differentiable at x_0, then we define $n = \dfrac{dy}{dx}\Big|_{x=x_0}$ to be the **instantaneous rate** at which y changes.

Example 1

For the curve $y = x^2 + 1$. Let us find the average rate of change of y with x over the interval $[3,5]$ and the instantaneous rate of change of y with x at point $x = 3$.

Here $x_0 = 3$ and $x_1 = 5$

$y_0 = (3)^2 + 1 = 10, y_1 = (5)^2 + 1 = 26$

So average rate of y over $[3,5]$ is $\dfrac{26-10}{5-3} = \dfrac{16}{2} = 8$. Thus, on the average y increases 8 units for each unit increase in x over the interval $[3,5]$.

$\dfrac{dy}{dx} = f'(x) = 2x$, so instantaneous rate of change of y at $x = 3$ is $\dfrac{dy}{dx}\Big|_{x=3} = 2x\Big|_{x=3} = 6$

Thus, at point $x = 3$, y is increasing 6 times as fast as x.

Example 2

Let us find all points where the function $f(x)=\sin x$ is not changing.

This function will not be changing if the rate of change is zero. Then we need to determine where the derivative is zero.

$\frac{dy}{dx}=f'(x)=\cos x$ and $\cos x=0$ for $x=\pm\frac{\pi}{2}+2k\pi$ or simply $x=\frac{\pi}{2}+k\pi$. Thus, $f(x)=\sin x$ is not changing if $x=\frac{\pi}{2}+k\pi$, $k\in\mathbb{Z}$.

Rolle's Theorem

Suppose that $f(x)$ is a function that satisfies all of the following.

- $f(x)$ is continuous on the closed interval $[a,b]$.
- $f(x)$ is differentiable on the open interval (a,b).
- $f(a)=f(b)$

Then, there is a number c such that $a<c<b$ and $f'(c)=0$.

Or, in other words $f(x)$ has a critical point on the open interval $]a,b[$.

Example

Consider the function $f(x)=x^2-1$ on $[-1,1]$

This function is continuous on $[-1,1]$ and differentiable on $]-1,1[$

Moreover $f(-1)=f(1)=0$.

Then from Rolle's theorem we must get a number c such that $-1<c<1$ and $f'(c)=0$.

The first derivative is $f'(x)=2x$ and $f'(x)=0$ for $x=0$ and we see that $-1<0<1$.

Mean Value Theorem

Suppose that $f(x)$ is a function that satisfies both of the following.

- $f(x)$ is continuous on the closed interval $[a,b]$.
- $f(x)$ is differentiable on the open interval $]a,b[$.

Then, there is a number c such that $a < c < b$ and $f'(c) = \dfrac{f(b) - f(a)}{b - a}$

Or, $f(b) - f(a) = f'(c)(b - a)$

Note that the Mean Value Theorem doesn't tell us what c is. It only tells us that there is at least one number c that will satisfy the conclusion of the theorem.

Also note that if $f(a) = f(b)$ we can think of Rolle's Theorem as a particular case of the Mean Value Theorem.

Geometrical interpretation of the Mean Value Theorem.

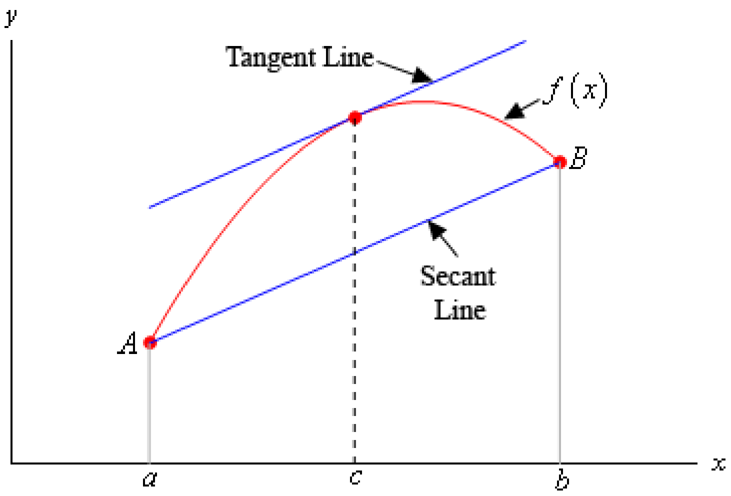
First define $A = (a, f(a))$ and $B = (b, f(b))$ and then we know from the Mean Value theorem that there is a c such that $a < c < b$ and that $f'(c) = \dfrac{f(b) - f(a)}{b - a}$

Now, if we draw in the secant line connecting A and B then we can know that the slope of the secant line is, $\dfrac{f(b) - f(a)}{b - a}$.

Likewise, if we draw the secant line to $f(x)$ at $x = c$ we know that its slope is $f'(c)$.

What the Mean Value Theorem tells us is that these two slopes must be equal or in other words the secant line connecting A and B and the tangent line at $x = c$ must be parallel.

We can see this in the following sketch.



Example

Let us determine all the numbers c which satisfy the conclusions of the Mean Value Theorem for the function $f(x)=x^3+2x^2-x$ *on* $[-1,2]$

There isn't really a lot with this problem other than to notice that since $f'(x)$ is a polynomial; it is both continuous and differentiable (i.e, the derivative exists) on the given interval.

First derivative, $f'(x)=3x^2+4x-1$

Now, to find the numbers that satisfy the conclusions of the Mean Value Theorem all we need to do is plug this into the formula given by the Mean Value Theorem.

$$f'(x)=\frac{f(2)-f(-1)}{2-(-1)}$$

$$\Leftrightarrow 3c^2+4c-1=\frac{14-2}{3}=4 \Leftrightarrow 3c^2+4c-1=4 \Leftrightarrow 3c^2+4c-5=0$$

$$\Delta=16+60=76$$

$$c=\frac{-4\pm2\sqrt{19}}{6}=\frac{-2\pm\sqrt{19}}{3}$$

Thus, the values of c which satisfy the conclusions of the Mean Value Theorem for the function $f(x)=x^3+2x^2-x$ on $[-1,2]$ is $\frac{-2+\sqrt{19}}{3}$. The value $\frac{-2-\sqrt{19}}{3}$ is excluded since it is not an element of the given interval.

Let us see a couple of facts.

Fact 1

If $f'(x)=0$ for all x in an interval $]a,b[$ then $f(x)$ is **constant** on $]a,b[$.

Fact 2

If $f'(x)=g'(x)$ for all x in an interval $]a,b[$ then in this interval we have

$f(x)=g(x)+c$ where c is some constant.

Note that in both of these facts we are assuming the functions are continuous and differentiable on the interval $[a,b]$

APPLICATION ACTIVITY 7.11

1) Find the general rate of change of the function $f(x)=2x^2+1$ then find the specific rate of change for $x_1=2$ *and* $x_2=5$

2) Find the general rate of change of the function $f(x)=x^2$, then find the specific rate of change for $x_1=0$ *and* $x_2=2$

7.12 END UNIT ASSESSMENT 7

1. By definition calculate the derivative of $f(x)=x-2$

2. A baseball team plays in a stadium that holds 55,000 spectators. With ticket prices at 10 dollars, the average attendance had been 27,000. When ticket prices were lowered to 8 dollars, the average attendance rose to 33,000.

   (a) Find the demand function, assuming that it is linear.

   (b) How should ticket prices be set to maximize revenue?

3. The given $f(x)=2x^3-5x^2+4x+2$

   a) Calculate extreme points

   b) Calculate first and second derivative

   c) Show the interval of increasing and decreasing of the given function

   d) Show where the concavity is up or down

4. Given that $g(x)=f(4-x^2)$, and $f'(1)=5; f''(1)=-1$, find **a)** $g''(\sqrt{3})$

b) Find the turning point of the curve $y=-x^2+6x-5$ and precise its nature

c) Find the equation of the tangent and the equation of the normal to curve

$y=\frac{x^3}{5}$ at point $(-1,-\frac{1}{5})$

5. Find the coordinates of the point of inflection of the graph of function

$f(x)=x^3-3x^2+2$

6.Curve (C) is defined by $\begin{cases} x=t-\dfrac{1}{t} \\ y=\dfrac{2(t+2)}{t^2} \end{cases}$

a. Find the x-intercept

b. Find the equation of the tangent to the curve at the point where t = -2

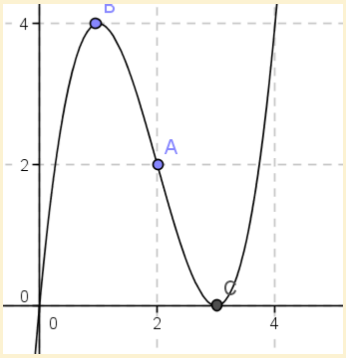
7. The distance, in km, of a train from its starting point when it is traveling along a straight track is given by the equation $s=16t^2+2t$.

Find the distance traveled after 2 hours and the velocity after 2 hours.

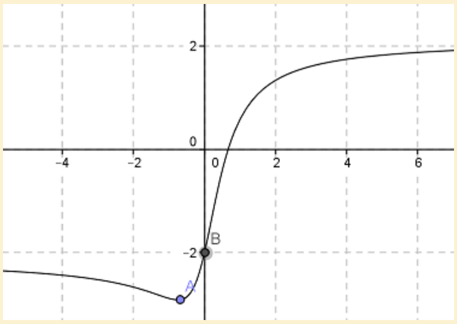
8. Given function $f(x)$, find the coordinates of the stationary points and the nature of each, the coordinates of the inflection point, the interval on which *f* is increasing, decreasing, the interval on which the graph is concave up, concave down:

a. $f(x)=\sqrt[3]{x}$ b. $f(x)=\dfrac{x^3}{1-x^2}$ c. $f(x)=4+3x-x^3$

9. a. Write down the sign of the first derivative *f'(x)* and the sign of the second derivative *f''(x)* on each of the following intervals:[0,1], [1,2], [2,3] and [3,4]



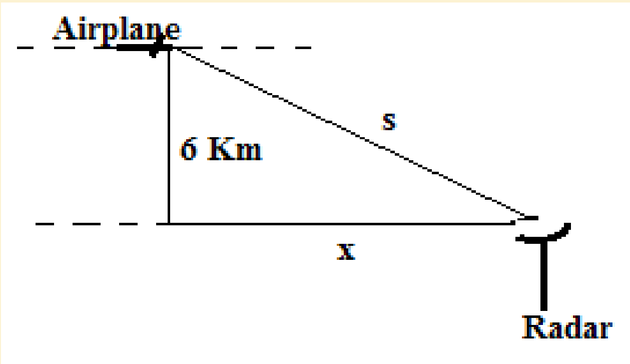
b. Precise the sign of $f'(x)$ and $f''(x)$ on the interval[1,4] and find the range of f(x)



10. a. A manufacturer wants to design an open box having a square base and a surface area of 108 cm^2.Express the volume V of the box in terms of the side *x* of the base and the height *h*, express *h* as function of variable *x*, and find the value of *x* and the value of *h* that will produce maximum volume of the box.

b. A company wishes to produce a cylindrical container with a constant volume of 1000 cm^3.The top and the bottom of the container will be made of a material that costs 5 FRW per cm^2,while the side will be made of a material that costs 3 FRW per cm^2.Find the dimensions of the container that will minimize the total cost of the container.

11. a. An airplane is flying on a path directly over a radar tracking station.



If s decreases at a rate of 400 Km/h when s is 10 Km, what is the speed of the plane?

12. a. A manufacturer has determined that the total cost C of operating a factory is $c(x) = 2x^3 - 3x^2 - 12x$, where x is the number of units produced. At what level of production will the average cost per unit be minimum? (The average cost per unit is given

by $\dfrac{C(x)}{x}$)

b. The formula for the power output P of a battery is $P = V \quad - R \quad ^2$ where V is the electromotive force in volts, R is the resistance in ohms and I is the current in amperes. Find the current that corresponds to a maximum value of P in the battery for which V=12 volts and R=0.5 ohms

c. In a metabolic experiment, the mass M of glucose decreases according to the formula $M(t) = 4.5 - 0.03t^2$, where M is measured in grams and t is the time in hours. Find the reaction rate at 1 hour.

13. a. For any demand function y=f(x), where x is the number of units demanded,

the total revenue is R=xy, and the marginal revenue is $\frac{dR}{dx}$.If the demand function is $y = 8 - x$, find the revenue and the marginal revenue

b. Given the function $f(x) = \sqrt{1+6x}$, find the coordinates of the point on the graph of f where the tangent is parallel to line $y = x + 12$

c. A second degree polynomial $P(x) = ax^2 + bx + c$ is such that $P(2) = 5, P'(2) = 3$ and $P''(2) = 2$.Find the values of a, b and c.

14. a. Given the function $f(x) = x^2 + x$.Calculate $\lim\limits_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$

b. The position of an object at time t is given by $s(t) = \dfrac{t}{t^2 - 1}$.Find its velocity and acceleration

c. A ball is thrown vertically upward. The distance s (in meters) of the ball from the ground after t seconds is $s=6+80t-16t^2$.Find the velocity of the ball at time and the distance s above the ground if the velocity is zero.

UNIT 8

MATRICES OF ORDER 2 AND ORDER 3

Key unit competence: Solve problem involving the system of linear equations using matrices

8.0. INTRODUCTORY ACTIVITY

A Farmer Kalisa bought in Ruhango Market 5 Cocks and 4 Rabbits and he paid 35,000 Frw, and the following day, he bought in the same Market 3 Cocks and 6 Rabbits and he paid 30,000 Frw.

a) Arrange what Kalisa bought according to their types in a simple table as follows

Cocks	Rabits	Prics

b) Discuss and explain in your own words how can you determine the cost of 1 Cocks and 1 Rabbit.

8.1. Definition and order of matrix

ACTIVITY 8.1

1) A shop sold 20 cell phones and 31 computers in a particular month. Another shop sold 45 cell phones and 23 computers in the same month. Present this information as an array of rows and columns.
2) a) Observe and complete the number of students in the year two classes on one Monday.

	Boys	girls	Total
SME			
SSE			

b) If every class gets new students on Tuesday such that in SME they have 2 boys and 1 girls, in SSE they receive 1 girl and 1 boy, Complete the table for new students.

c) Complete the table for all students in an array of rows and columns.

Content summary

1. A matrix is a rectangular arrangement of numbers, expressions, algebraic symbols which are arranged in rows and columns. A matrix is denoted with a capital letter: A,B,C,...and the elements are enclosed by parenthesis

$(\ )$ *or square brackets* $[\ ]$.

Example

1 $A=\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ B= $\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$ C= $\begin{pmatrix} 1 & 3 & 4 \\ 5 & 12 & 13 \\ 7 & 6 & 0 \end{pmatrix}$ M= $\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}$

From matrix B, a_{32} is element of third row and second column

From matrix C, 4 is element of first row and third column

2. The dimension or size or order of matrix

Matrix B is of order 3×3 ("read three by three") because there are 3 rows and 3 columns

Matrix M is of order $m\times n$, ("read three by three") because there are m, m rows and n, columns

A square matrix is a matrix formed by the same number of rows and columns.

The elements of the form (a_{ij}), where the two subscripts i and j are equal, constitute the **principal diagonal** (or **leading diagonal** or **main diagonal** or **major diagonal** or **primary diagonal**).

The **secondary diagonal** (or **minor diagonal** or **antidiagonal** or **counterdiagonal**) is formed by the elements with $i+j = n+1$.

Square matrix of **order 2** has the form $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$

Example of Matrix of order two : $\begin{pmatrix} 1 & 4 \\ 3 & 11 \end{pmatrix}$

Secondary diagonal

leading diagonal

Square matrix of order three has the form

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

Example of Matrix of order three:

$$\begin{pmatrix} 1 & -2 & 4 \\ 2 & 3 & 2 \\ 5 & 7 & 0 \end{pmatrix}$$

Secondary diagonal

Leading diagonal

3. Types of matrices

There are several types matrices, but the most commonly used are

1) Row matrix: matrix formed by one row

Example: $\begin{pmatrix} 2 & 4 & 7 \end{pmatrix}$

2) Column matrix or Vector matrix: matrix formed by one column

Example: $\begin{pmatrix} 1 \\ 3 \\ 6 \end{pmatrix}$

3) Zero matrix or null matrix : all elements are zero

Example: $\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$

4) Triangular Matrix : matrix whose elements located below or above the leading diagonal are zeros

a) Upper Triangular Matrix

In an upper triangular matrix, the elements located below the leading diagonal are zeros.

Examples : 1) $\begin{pmatrix} 1 & 2 \\ 0 & -5 \end{pmatrix}$ 2) $\begin{pmatrix} 1 & -2 & 4 \\ 0 & 3 & 2 \\ 0 & 0 & 4 \end{pmatrix}$

b) Lower Triangular Matrix

In a lower triangular matrix, the elements above the leading diagonal are zeros.

Examples: 1) $\begin{pmatrix} 1 & 0 \\ 18 & -5 \end{pmatrix}$ 2) $\begin{pmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \\ 5 & 7 & 9 \end{pmatrix}$

5) Diagonal Matrix

In a diagonal matrix, all the elements above and below the leading diagonal are zeros.

Examples: 1) $\begin{pmatrix} 10 & 0 \\ 0 & -5 \end{pmatrix}$ 2) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{pmatrix}$

6) Scalar Matrix

A scalar matrix is a diagonal matrix in which the leading diagonal elements are equal.

Examples: 1) $\begin{pmatrix} -5 & 0 \\ 0 & -5 \end{pmatrix}$ 2) $\begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}$

7) Identity Matrix or Unity matrix

An identity matrix by multiplication of matrices (noted by **I**) is a diagonal matrix in which the leading diagonal elements are equal to 1.

1) Identity matrix of order two $I_{2}= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

2) Identity matrix of order three I3= $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

8) Rectangular matrix: A matrix is said to be rectangular if the number of rows is not equal to the number of columns

Example: A 5x2 matrix

$$\begin{bmatrix} 1 & 2 & -3 & 4 & 5 \\ 2 & 0 & 6 & 7 & -\frac{1}{3} \end{bmatrix}$$

9) Singular matrix: a square matrix that is not invertible

10) Nilpotent matrix: A square matrix A satisfying $A^k = 0$ for some positive integer k

11) Invertible matrix: A square matrix having a multiplicative inverse, that is, a matrix B such that $AB = BA = I$

12) Idempotent matrix or projection matrix: A matrix that is equal to its square i.e $A^2 = A \times A = A$

13) Involutory matrix: A square matrix which is its own inverse**, i.e** $AA = I$

14) Transpose matrix of A: Transpose of matrix is a matrix obtained by changing rows to columns and columns to rows. (transpose of A is denoted by A^t or A^T).

15) Orthogonal matrix: a matrix whose inverse is equal to its transpose $A^{-1} = A^t$

16) Equality of matrices

Two matrices are equal if the elements of the two matrices that occupy the same position are equal i.e

If $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$, then $\begin{cases} a_{11} = b_{11} \\ a_{21} = b_{21} \\ a_{12} = b_{12} \\ a_{22} = b_{22} \end{cases}$

If $\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & a_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix}$, then $\begin{matrix} a_{11} = b_{11}, \, a_{12} = b_{12}, \, a_{13} = b_{13} \\ a_{21} = b_{21}, \, a_{22} = b_{22}, \, a_{23} = b_{23} \\ a_{31} = b_{31}, \, a_{32} = b_{32}, \, a_{33} = b_{33} \end{matrix}$

APPLICATION ACTIVITY 8.1

1) Find the dimension of each matrix a)

$a) \begin{pmatrix} 5 & 5 & 5 \\ 1 & 1 & 1 \\ 3 & 3 & 3 \end{pmatrix}$ b) $\begin{pmatrix} 6 & -9 \end{pmatrix}$ c) $\begin{pmatrix} 2 \end{pmatrix}$ d) $\begin{pmatrix} 0 & 0 & 0 & 2 & 6 \\ 1 & 2 & 3 & 4 & 5 \end{pmatrix}$ e) $\begin{pmatrix} 1 & 1 \\ 6 & 1 \end{pmatrix}$

2) Name the following matrices

a) $\begin{pmatrix} a & b & c \end{pmatrix}$ b) $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ c) $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ c) $\begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}$ d) $\begin{pmatrix} a & b & c \\ 0 & b & d \\ 0 & 0 & e \end{pmatrix}$ e) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

f) $\begin{pmatrix} a & 0 & 0 \\ c & b & 0 \\ d & 0 & c \end{pmatrix}$ g) $\begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}$

3) *If* $A = \begin{pmatrix} 3y+2 & 2 \\ 2x+1 & 6 \end{pmatrix}$ *and* $B = \begin{pmatrix} y-3 & 2 \\ 5 & 6 \end{pmatrix}$ *are equal. Find the value of* x *and* y

8.2. Operations on matrices

8.2.1. Addition and subtraction of matrices

ACTIVITY 8.2.1

1) In a survey of 900 people, the following information was obtained:

200 males thought federal defense spending was too high

150 males thought federal defense spending was too low

45 males had no opinion

315 females thought federal defense spending was too high

125 females thought federal defense spending was too low

65 females had no opinion

Discuss and arrange these data in a rectangular array as follows:

	Too high	Too Low	No opinion
Male			
Female			

Then, form a matrix from the data of this table.

2) Consider the matrix A formed by present students in two classes where students of one class make one row such that $A=\begin{bmatrix}23 & 2\\ 20 & 4\end{bmatrix}$ and B the matrix formed by students who got absent for they went to participate in a competition $B=\begin{pmatrix}2 & 1\\ 3 & 2\end{pmatrix}$. Form one matrix C representing the total number of students.

3) Consider the matrices $A=\begin{pmatrix}2 & -4 & 12\\ 1 & 0 & -4\\ 5 & 2 & 3\end{pmatrix}$, $B=\begin{pmatrix}1 & 6 & 4\\ 1 & 7 & 8\\ 3 & 21 & 3\end{pmatrix}$ and $C=\begin{pmatrix}1 & 3 & -1\\ 0 & 2 & 4\\ 3 & -2 & 0\end{pmatrix}$

- Find $A+3B$; $2A-B$; $A+(-A)$ comment on the result; $A+B$ and $B+A$. From the results give your comment; $A+(B+C)$ and $(A+B)+C$. Then comment on your result;
- Interchange the rows and column of matrix A, B and C

Content summary

Given two matrices of the same dimension, $A=(a_{ij})$ and $B=(b_{ij})$, the matrix sum is defined as: $A+B=(a_{ij}+b_{ij})$. That is, the resultant matrix's elements are obtained by adding the elements of the two matrices that occupy the same position.

i) If $A=\begin{pmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{pmatrix}$ and $B=\begin{pmatrix}b_{11} & b_{12}\\ b_{21} & b_{22}\end{pmatrix}$, then

$$A+B=\begin{pmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{pmatrix}+\begin{pmatrix}b_{11} & b_{12}\\ b_{21} & b_{22}\end{pmatrix}=\begin{pmatrix}a_{11}+b_{11} & a_{12}+b_{12}\\ a_{21}+b_{21} & a_{22}+b_{22}\end{pmatrix}$$ And

$$A-B=\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}-\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}=\begin{pmatrix} a_{11}-b_{11} & a_{12}-b_{12} \\ a_{21}-b_{21} & a_{22}-b_{22} \end{pmatrix}$$

ii) If $A=\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$ and $B=\begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix}$, then

$$A+B=\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}+\begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix}=\begin{pmatrix} a_{11}+b_{11} & a_{12}+b_{12} & a_{13}+b_{13} \\ a_{21}+b_{21} & a_{22}+b_{22} & a_{23}+b_{23} \\ a_{31}+b_{31} & a_{32}+b_{32} & a_{33}+b_{33} \end{pmatrix}$$

$$A-B=\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}-\begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix}=\begin{pmatrix} a_{11}-b_{11} & a_{12}-b_{12} & a_{13}-b_{13} \\ a_{21}-b_{21} & a_{22}-b_{22} & a_{23}-b_{23} \\ a_{31}-b_{31} & a_{32}-b_{32} & a_{33}-b_{33} \end{pmatrix}$$

Example: 1) If $A=\begin{pmatrix} 3 & 2 \\ -1 & 1 \end{pmatrix}$ and $B=\begin{pmatrix} 1 & 2 \\ 4 & 6 \end{pmatrix}$, find the sum $A+B$ and the difference $A-B$

Solution

$$A+B=\begin{pmatrix} 3 & 2 \\ -1 & 1 \end{pmatrix}+\begin{pmatrix} 1 & 2 \\ 4 & 6 \end{pmatrix}=\begin{pmatrix} 4 & 4 \\ 3 & 7 \end{pmatrix}$$

$$A-B=\begin{pmatrix} 3 & 2 \\ -1 & 1 \end{pmatrix}-\begin{pmatrix} 1 & 2 \\ 4 & 6 \end{pmatrix}=\begin{pmatrix} 2 & 0 \\ -5 & -5 \end{pmatrix}$$

2) Consider the matrices $A=\begin{pmatrix} 2 & 0 & 1 \\ 3 & 0 & 0 \\ 5 & 1 & 1 \end{pmatrix}$ and $B=\begin{pmatrix} 1 & 0 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 0 \end{pmatrix}$, find $A+B$ and $A-B$

Solution

$$A+B=\begin{pmatrix} 2+1 & 0+0 & 1+1 \\ 3+1 & 0+2 & 0+1 \\ 5+1 & 1+1 & 1+0 \end{pmatrix}=\begin{pmatrix} 3 & 0 & 2 \\ 4 & 2 & 1 \\ 6 & 2 & 1 \end{pmatrix}$$

$$A-B=\begin{pmatrix} 2-1 & 0-0 & 1-1 \\ 3-1 & 0-2 & 0-1 \\ 5-1 & 1-1 & 1-0 \end{pmatrix}=\begin{pmatrix} 1 & 0 & 0 \\ 2 & -2 & -1 \\ 4 & 0 & 1 \end{pmatrix}$$

Properties

Note: The properties are the same for matrices of order two or three

1) Closure

The sum of two matrices of order two or three is another matrix of order two or three

2) Associative

$A + (B + C) = (A + B) + C$

3) Additive identity

$A + 0 = A$, where 0 is the zero-matrix of the same dimension.

4) Additive inverse

$A + (-A) = O$

The opposite matrix of A is $-A$.

5) Commutative

$A + B = B + A$

b) Scalar multiplication

Given a matrix, $A = (a_{ij})$, and a real number, $k \in IR$, the product of a real number by a matrix is a matrix of the same dimension as **A**, and each element is multiplied by **k.**

$k \cdot A = (k\ a_{ij})$

If $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$ **, then** $kA = \begin{pmatrix} ka_{11} & ka_{12} & ka_{13} \\ ka_{21} & ka_{22} & ka_{23} \\ ka_{31} & ka_{32} & ka_{33} \end{pmatrix}$

Example 1 : If $A = \begin{pmatrix} -3 & 6 \\ 5 & 2 \end{pmatrix}$, find $2A$

Solution

$$2A = 2\begin{pmatrix} -3 & 6 \\ 5 & 2 \end{pmatrix} = \begin{pmatrix} -6 & 12 \\ 10 & 4 \end{pmatrix}$$

Example 2: Consider the matrix $A = \begin{pmatrix} 2 & 0 & 1 \\ 3 & 0 & 0 \\ 5 & 1 & 1 \end{pmatrix}$, find $2A$

Solution:

$$2A = 2\begin{pmatrix} 2 & 0 & 1 \\ 3 & 0 & 0 \\ 5 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 4 & 0 & 2 \\ 6 & 0 & 0 \\ 10 & 2 & 2 \end{pmatrix}$$

Properties:

1) $\alpha(\beta A) = (\alpha\beta)A, \quad A \in M_{m\times n}, \alpha, \beta \in IR$

2) $\alpha(A+B) = \alpha A + \alpha B, \quad A, B \in M_{m\times n}, \alpha \in IR$

3) $(\alpha+\beta)A = \alpha A + \beta A, \quad A \in M_{m\times n}, \alpha, \beta \in IR$

4) $1A = A, \quad A \in M_{m\times n}$

APPLICATION ACTIVITY 8.2.1

1. Consider the matrices $A = \begin{pmatrix} 13 & 4 \\ 6 & 10 \end{pmatrix}$ and $B = \begin{pmatrix} 7 & 10 \\ 3 & 4 \end{pmatrix}$, find

   a) $A + 3B$ b) $2A - B$

2. If $A = \begin{pmatrix} 1 & -2 & 6 \\ 2 & 2 & -5 \\ 0 & 4 & 8 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 8 & 9 \\ 14 & 4 & 6 \\ 0 & 6 & 7 \end{pmatrix}$ and $C = \begin{pmatrix} 13 & -2 & 0 \\ 10 & 0 & 3 \\ 9 & 1 & -5 \end{pmatrix}$. Evaluate

   a) $A - B$

   b) $A + B - 2C$

   c) $2A - B + C$

8.2.2. Multiplying matrices

ACTIVITY 8.2.2

1) A clothing store sells men's shirt for \$40, silk tie for \$20, and wool suit for \$400. Firstly, the store had sales consisting of 150 shirts, 120 ties, and 25 suits.

Secondary, the store had sales consisting of 100 shirts, 200 ties, and 50 suits.

Using matrix, discuss and explain in your own words how to determine the total revenue due to these sales.

2) Considering that

$$A \times B = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \times \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix}$$

$$= \begin{pmatrix} a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31} & a_{11}b_{12} + a_{12}b_{22} + a_{13}b_{32} & a_{11}b_{13} + a_{12}b_{23} + a_{13}b_{33} \\ a_{21}b_{11} + a_{22}b_{21} + a_{23}b_{31} & a_{21}b_{12} + a_{22}b_{22} + a_{23}b_{32} & a_{21}b_{13} + a_{22}b_{23} + a_{23}b_{33} \\ a_{31}b_{11} + a_{32}b_{21} + a_{33}b_{31} & a_{31}b_{12} + a_{32}b_{22} + a_{33}b_{32} & a_{31}b_{13} + a_{32}b_{23} + a_{33}b_{33} \end{pmatrix}$$

Evaluate $A \times B$, given that $A = \begin{pmatrix} -1 & 1 & 1 \\ 2 & 2 & 5 \\ 0 & 3 & 4 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & 1 & -1 \\ 3 & 2 & 3 \\ -1 & 6 & 4 \end{pmatrix}$ find

Content summary

Two matrices A and B can be multiplied together if and only if the number of columns of A is equal to the number of rows of B.

$$A_{m\times n} \times B_{m\times p} = M_{m\times p}$$

The element, c_{ij}, of the product matrix is obtained by multiplying every element in row ***i*** of matrix *A* by each element of column **j** of matrix *B* and then adding them together. This multiplication is called **ROCO** (row, column).

If $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$ and $B = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix}$, then

$$A \times B = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \times \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix}$$

$$= \begin{pmatrix} a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31} & a_{11}b_{12} + a_{12}b_{22} + a_{13}b_{32} & a_{11}b_{13} + a_{12}b_{23} + a_{13}b_{33} \\ a_{21}b_{11} + a_{22}b_{21} + a_{23}b_{31} & a_{21}b_{12} + a_{22}b_{22} + a_{23}b_{32} & a_{21}b_{13} + a_{22}b_{23} + a_{23}b_{33} \\ a_{31}b_{11} + a_{32}b_{21} + a_{33}b_{31} & a_{31}b_{12} + a_{32}b_{22} + a_{33}b_{32} & a_{31}b_{13} + a_{32}b_{23} + a_{33}b_{33} \end{pmatrix}$$

Example 1

If $A = \begin{pmatrix} 1 & 3 \\ 2 & 5 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix}$, find the product $A \cdot B$

Solution

$$A \cdot B = \begin{pmatrix} 1 & 3 \\ 2 & 5 \end{pmatrix} \cdot \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 \cdot 2 + 3 \cdot 1 & 1 \cdot 0 + 3 \cdot 1 \\ 2 \cdot 2 + 5 \cdot 1 & 2 \cdot 0 + 5 \cdot 1 \end{pmatrix}$$

$$= \begin{pmatrix} 5 & 3 \\ 9 & 5 \end{pmatrix}$$

Example 2

Consider matrices $A = \begin{pmatrix} 2 & 0 & 1 \\ 3 & 0 & 0 \\ 5 & 1 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 0 \end{pmatrix}$, find $A \times B$

Solution

$$A \times B = \begin{pmatrix} 2 & 0 & 1 \\ 3 & 0 & 0 \\ 5 & 1 & 1 \end{pmatrix} \times \begin{pmatrix} 1 & 0 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 2\times1+0\times1+1\times1 & 2\times0+0\times2+1\times1 & 2\times1+0\times1+1\times0 \\ 3\times1+0\times1+0\times1 & 3\times0+0\times2+0\times1 & 3\times1+0\times1+0\times0 \\ 5\times1+1\times1+1\times1 & 5\times0+1\times2+1\times1 & 5\times1+1\times1+1\times0 \end{pmatrix}$$

$$= \begin{pmatrix} 3 & 1 & 2 \\ 3 & 0 & 3 \\ 7 & 3 & 6 \end{pmatrix}$$

APPLICATION ACTIVITY 8.2.2

If $A=\begin{pmatrix}1 & -2 & 6\\ 2 & 2 & -5\\ 0 & 4 & 8\end{pmatrix}$, $B=\begin{pmatrix}0 & 8 & 9\\ 14 & 4 & 6\\ 0 & 6 & 7\end{pmatrix}$ and $C=\begin{pmatrix}13 & -2 & 0\\ 10 & 0 & 3\\ 9 & 1 & -5\end{pmatrix}$. Evaluate

1) $A\times B$

2) $A\times C$

3) $B\times C$

8.2.3. Properties of Multiplication of Matrices

ACTIVITY 8.2.3

Consider the matrices $A=\begin{pmatrix}3 & 0 & 1\\ 0 & -1 & 1\\ -2 & 3 & 2\end{pmatrix}$ $B=\begin{pmatrix}0 & 1 & -1\\ 1 & 1 & -2\\ -1 & 0 & 1\end{pmatrix}$ and

$C=\begin{pmatrix}1 & 1 & -1\\ 1 & -2 & 1\\ -1 & 1 & 0\end{pmatrix}$ Find

1. $A\times B$ and $B\times A$
2. $(A\times B)^t$ and $B^t\times A^t$
3. $A\times(B\times C)$ and $(A\times B)\times C$
4. $A\times(B+C)$ and $A\times B+A\times C$
5. Comment on your results

Content summary

Let $A,\ B, C$ be matrices of order two or three

1) Associative

$A\times(B\times C)=(A\times B)\times C$

2) Multiplicative Identity

$A \times I = A$, where I is the identity matrix with the same order as matrix A.

3) Not Commutative

$$A \times B \neq B \times A$$

4) **Distributive**

$$A \times (B + C) = (A \times B) + (A \times C)$$

5) $(A \times B)^t = B^t \times A^t$

Example 1: Given the matrices

$$A = \begin{pmatrix} 2 & 0 & 1 \\ 3 & 0 & 0 \\ 5 & 1 & 1 \end{pmatrix} \text{ And } B = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

Find

a) The product $A \times B$

b) The product $B \times A$

c) Conclude about the commutatively of matrices

Solution

a)

$$A \times B = \begin{pmatrix} 2 & 0 & 1 \\ 3 & 0 & 0 \\ 5 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 2\times1+0\times1+1\times1 & 2\times0+0\times2+1\times1 & 2\times1+0\times1+1\times0 \\ 3\times1+0\times1+0\times1 & 3\times0+0\times2+0\times1 & 3\times1+0\times1+0\times0 \\ 5\times1+1\times1+1\times1 & 5\times0+1\times2+1\times1 & 5\times1+1\times1+1\times0 \end{pmatrix}$$

$$= \begin{pmatrix} 3 & 1 & 2 \\ 3 & 0 & 3 \\ 7 & 3 & 6 \end{pmatrix}$$

b.
$$B\times A=\begin{pmatrix}1&0&1\\1&2&1\\1&1&0\end{pmatrix}\begin{pmatrix}2&0&1\\3&0&0\\5&1&1\end{pmatrix}$$

$$=\begin{pmatrix}1\times2+0\times3+1\times5 & 1\times0+0\times0+1\times1 & 1\times1+0\times0+1\times1\\1\times2+2\times3+1\times5 & 1\times0+2\times0+1\times1 & 1\times1+2\times0+1\times1\\1\times2+1\times3+0\times5 & 1\times0+1\times0+0\times1 & 1\times1+1\times0+0\times1\end{pmatrix}$$
$$=\begin{pmatrix}7&1&2\\13&1&2\\5&0&1\end{pmatrix}$$

c) Since $B\times A\neq A\times B$, then commutativity of multiplication of matrices is not verified.

Example 2: Given matrices $A=\begin{pmatrix}1&-1&1\\-3&2&-1\\-2&1&0\end{pmatrix}$ and $B=\begin{pmatrix}1&2&3\\2&4&6\\1&2&3\end{pmatrix}$. Find the product AB. What is your observation?

Solution

$$AB=\begin{pmatrix}1&-1&1\\-3&2&-1\\-2&1&0\end{pmatrix}\begin{pmatrix}1&2&3\\2&4&6\\1&2&3\end{pmatrix}$$
$$=\begin{pmatrix}1-2+1 & 2-4+2 & 3-6+3\\-3+4-1 & -6+8-2 & -9+12-3\\-2+2+0 & -4+4+0 & -6+6+0\end{pmatrix}$$
$$=\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}$$

Observation: If $AB=0$, it does not necessarily follow that $A=0$ or $B=0$.

Example 3: Given matrices $A=\begin{pmatrix}1&0&0\\0&1&0\\1&0&2\end{pmatrix}$ and $B=\begin{pmatrix}2&4&0\\3&1&0\\-1&-4&1\end{pmatrix}$. Find the product AB and BA. What is your observation?

Solution

$$AB=\begin{pmatrix}1&0&0\\0&1&0\\1&0&2\end{pmatrix}\begin{pmatrix}2&4&0\\3&1&0\\-1&-4&1\end{pmatrix}=\begin{pmatrix}2&4&0\\3&1&0\\0&-4&2\end{pmatrix}$$

$$BA=\begin{pmatrix}2&4&0\\3&1&0\\-1&-4&1\end{pmatrix}\begin{pmatrix}1&0&0\\0&1&0\\1&0&2\end{pmatrix}=\begin{pmatrix}2&4&0\\3&1&0\\0&-4&2\end{pmatrix}$$

$$\Rightarrow AB=BA$$

Observation: The given matrices commute in multiplication.

Notice

- If $AB=0$, it does not necessarily follow that $A=0$ or $B=0$.
- **Commuting matrices in multiplication:** In general the multiplication of matrices is not commutative, i.e, $AB\neq BA$, but we can have the case where two matrices A and B satisfy $AB=BA$. In this case A and/ B are said to be **commuting**.

Trace of matrix

The sum of the entries on the leading diagonal of a square matrix, *A*, is known as the **trace** of that matrix, noted $tr(A)$.

Example

1. Trace of $\begin{pmatrix}1&-2&4\\2&3&2\\5&7&2\end{pmatrix}=1+3+2=6$

2. Trace of $\begin{pmatrix}1&2\\3&1\end{pmatrix}=1+1=2$

Properties of trace of matrix

1) $tr(A+B)=tr(A)+tr(B)$

2) $tr(\alpha A)=\alpha\, tr(A)$

3) $tr(A)=tr(A)^t$

4) $tr(AB)=tr(BA)$

5) $tr(ABC)=tr(BCA)=tr(CAB)$, cyclic property.

6) $tr(ABC)\neq tr(ACB)$, arbitrary permutations are not allowed.

APPLICATION ACTIVITY 8.2.3

Consider the matrices $A=\begin{pmatrix}1 & 0 & 1\\ 1 & -1 & 1\\ 0 & 0 & 0\end{pmatrix}$ $B=\begin{pmatrix}-2 & 0 & 1\\ 1 & 0 & 2\\ -1 & 0 & 1\end{pmatrix}$ and

$C=\begin{pmatrix}1 & 0 & -1\\ -2 & 2 & 0\\ -1 & 1 & 0\end{pmatrix}$

find

a) $A\times B$ and $B\times A$

b) $A\times(B\times C)$ and $(A\times B)\times C$

c) $A\times(B+C)$ and $A\times B+A\times C$

d) $tr(AB)$

8.3. Transpose of Matrix and Matrix of Linear transformation in 3D

8.3.1 Transpose of Matrix

ACTIVITY 8.3.1

Consider the matrices $A = \begin{pmatrix} 1 & 3 & 1 \\ 1 & -2 & 2 \\ 3 & -2 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 12 & 3 & -1 \\ 3 & -2 & 0 \\ -4 & -1 & 0 \end{pmatrix}$

1. Interchange the rows and columns of matrix A and B
2. Add two matrices obtained in 1
3. Add A and B
4. Interchange the rows and columns of matrix obtained in 3
5. What can you say about result in 2 and 4?
6. Interchange the rows and columns of matrix A twice. What can you conclude?

Content summary

Given matrix A, **the transpose of matrix A**, noted A^t, is another matrix where the elements in the columns and rows have interchanged. In other words, the rows become the columns and the columns become the rows.

If $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$, **then** $A^t = \begin{pmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{pmatrix}$

Example $A = \begin{pmatrix} 1 & 3 & 6 \\ 0 & 2 & 0 \\ 3 & 5 & 8 \end{pmatrix}$ $A^t = \begin{pmatrix} 1 & 0 & 3 \\ 3 & 2 & 5 \\ 6 & 0 & 8 \end{pmatrix}$

Properties of transpose of matrices

Let A, B be matrices of order two or three

1) $\left(A^t\right)^t = A$

2) $\left(A + B\right)^t = A^t + B^t$

3) $\left(\alpha \times A\right)^t = \alpha \times A^t$, $\alpha \in \mathbb{R}$

APPLICATION ACTIVITY 8.3

1. If $A = \begin{pmatrix} x-1 & -3 \\ 4z+x & 3y+4 \end{pmatrix}$ and $B = \begin{pmatrix} 6x+5 & -3 \\ 1 & 1 \end{pmatrix}$. If $A = B$, find the value of x, y

and z and hence find

   a. A

   b. A^t

2. Consider matrices $A = \begin{pmatrix} 0 & 4 & 2 \\ 1 & 3 & 6 \\ 3 & -2 & 8 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 2 & 1 \\ -4 & 0 & 3 \\ 6 & 2 & 5 \end{pmatrix}$. Evaluate

   a) $\left(A+B\right)^t$

   b) $3A^t + B$

   c) $\left(-3B+4A\right)^t$

   d) Find the value of x in $M = \begin{pmatrix} 1 & 2 & x^2 \\ 4 & 1 & 0 \\ 1 & x+3 & 8 \end{pmatrix}$ if $M^t = \begin{pmatrix} 1 & 4 & 1 \\ 2 & 1 & 1 \\ 4 & 0 & 8 \end{pmatrix}$

3. Consider the matrix $A = \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix}$, find $A \times A^t$

8.3.2 Matrix of Linear transformation in 3D

ACTIVITY 8.3.2

Let $f(x,y,z)=(x+z,y-z,2x)$ and the standard basis of IR^3 is $\{\vec{e_1}=(1,0,0),\vec{e_2}=(0,1,0),\vec{e_3}=(0,0,1)\}$. Find

1) $f(\vec{e_1})$

2) $f(\vec{e_2})$

3) $f(\vec{e_3})$

4) Form matrix whose jth column is $f(\vec{e_j}),\ j=1,2,3$

Content summary:

Every linear transformation $f:\mathbb{R}^3\to\mathbb{R}^3$ can be identified with a matrix of order three, $[f]_e=(a_{ij})$, whose j$^{\text{th}}$ column is $f(\vec{e_j})$ where $\{\vec{e_j}\}$, $j=1,2,3$ is the standard basis of $\mathbb{R}^3$. The matrix $[f]_e$ is called matrix representation of f relative to the standard basis $\{\vec{e_j}\}$.

Example:

Find the matrix of f relative to the standard basis if

$f:\mathbb{R}^3\to\mathbb{R}^3: f(x,y,z)=(4x-2z,2x+y,z+y)$

Solution

The standard basis of $\mathbb{R}^3$ is $\{\vec{e_1}=(1,0,0),\vec{e_2}=(0,1,0),\vec{e_3}=(0,0,1)\}$

$$f(\vec{e_1})=(4,2,0)$$

$$f(\vec{e_2})=(0,1,1)$$

$$f(\vec{e_3})=(-2,0,1)$$

Then the matrix of f relative to the standard basis is $[f]_e=\begin{pmatrix}4 & 0 & -2\\ 2 & 1 & 0\\ 0 & 1 & 1\end{pmatrix}$

What is the procedure if the given basis is not standard? The following is the general method.

To find the matrix of a linear mapping f relative to any basis $\{\vec{e_1}, \vec{e_2}, \vec{e_3}\}$, we follow the following steps:

1. Find $f\left(\vec{e_j}\right),\ j=1,2,3$.
2. Equate $f\left(\vec{e_j}\right)$ to $\vec{e_i}a_{ij}$ to find the values of a_{ij}.
3. The matrix of f is $[f]_e=\left(a_{ij}\right)$ where $\begin{cases} i = \text{number of row} \\ j = \text{number of column} \end{cases}$

Example:

Consider the following linear mapping defined on $\mathbb{R}^3$ by

$f(x,y,z)=(4x-2z, 2x+y, z+y)$. Calculate its matrix relative to the basis

$\left\{\vec{e_1}=(1,1,1), \vec{e_2}=(-1,0,1), \vec{e_3}=(0,1,1)\right\}$

Solution

$$f\left(\vec{e_1}\right)=(4-2, 2+1, 1+1)=(2,3,2)$$
$$f\left(\vec{e_2}\right)=(-4-2, -2+0, 1+0)=(-6,-2,1)$$
$$f\left(\vec{e_3}\right)=(0-2, 0+1, 1+1)=(-2,1,2)$$

$$f\left(\vec{e_j}\right)=\vec{e_i}a_{ij}$$

- $f\left(\vec{e_1}\right)=\vec{e_i}a_{i1}$

$$\begin{pmatrix}2\\3\\2\end{pmatrix}=\begin{pmatrix}1\\1\\1\end{pmatrix}a_{11}+\begin{pmatrix}-1\\0\\1\end{pmatrix}a_{21}+\begin{pmatrix}0\\1\\1\end{pmatrix}a_{31}$$

$$\begin{pmatrix}2\\3\\2\end{pmatrix}=\begin{pmatrix}a_{11}\\a_{11}\\a_{11}\end{pmatrix}+\begin{pmatrix}-a_{21}\\0\\a_{21}\end{pmatrix}+\begin{pmatrix}0\\a_{31}\\a_{31}\end{pmatrix}$$

$$\begin{cases} a_{11} - a_{21} = 2 \\ a_{11} + a_{31} = 3 \\ a_{11} + a_{21} + a_{31} = 2 \end{cases}$$

$$\begin{cases} a_{11} = 1 \\ a_{21} = -1 \\ a_{31} = 2 \end{cases}$$

- $f\left(\overrightarrow{e_2}\right) = \overrightarrow{e_i} a_{i2}$

$$\begin{pmatrix} -6 \\ -2 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} a_{12} + \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} a_{22} + \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} a_{32}$$

$$\begin{pmatrix} -6 \\ -2 \\ 1 \end{pmatrix} = \begin{pmatrix} a_{12} \\ a_{12} \\ a_{12} \end{pmatrix} + \begin{pmatrix} -a_{22} \\ 0 \\ a_{22} \end{pmatrix} + \begin{pmatrix} 0 \\ a_{32} \\ a_{32} \end{pmatrix}$$

$$\begin{cases} a_{12} - a_{22} = -6 \\ a_{12} + a_{32} = -2 \\ a_{12} + a_{22} + a_{32} = 1 \end{cases}$$

$$\begin{cases} a_{12} = -3 \\ a_{22} = 3 \\ a_{32} = 1 \end{cases}$$

- $f\left(\overrightarrow{e_3}\right) = \overrightarrow{e_i} a_{i3}$

$$\begin{pmatrix} -2 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} a_{13} + \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} a_{23} + \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} a_{33}$$

$$\begin{pmatrix} -2 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} a_{13} \\ a_{13} \\ a_{13} \end{pmatrix} + \begin{pmatrix} -a_{23} \\ 0 \\ a_{23} \end{pmatrix} + \begin{pmatrix} 0 \\ a_{33} \\ a_{33} \end{pmatrix}$$

$$\begin{cases} a_{13} - a_{23} = -2 \\ a_{13} + a_{33} = 1 \\ a_{13} + a_{23} + a_{33} = 2 \end{cases}$$

$$\begin{cases} a_{13} = -1 \\ a_{23} = 1 \\ a_{33} = 2 \end{cases}$$

The matrix of *f* is given by $[f]_e = (a_{ij}) = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$, therefore,

$$[f]_e = \begin{pmatrix} 1 & -3 & -1 \\ -1 & 3 & 1 \\ 2 & 1 & 2 \end{pmatrix}$$

Theorems

- Let $\{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$ be a basis of E and let f be any operator on E. Then, for any vector $\vec{v} \in E$, $[f]_e \cdot [\vec{v}]_e = [f(\vec{v})]_e$. That is, if we multiply the coordinate vector of $\vec{v}$ by matrix representation of *f*, we obtain the coordinate vector of $f(\vec{v})$.

- Let $\{\vec{e}_i\}, \{\vec{f}_i\}$ and $\{\vec{g}_i\}$ be bases of E, U and V respectively. Let $f: E \to U$ and $g: U \to V$ be linear mappings. Then $[g \circ f]_{e_i}^{g_i} = [g]_{f_i}^{g_i} [f]_{e_i}^{f_i}$. That is, relative to the appropriate bases, the matrix representation of the composition of two linear mappings is equal to the product of the matrix representations of the individual mappings.

- For any $f, g \in L(E)$ and any scalar $\alpha \in K$,

  i. $[g+f]_e = [g]_e + [f]_e$ and

  ii. $[\alpha g]_e = \alpha [g]_e$.

Example

Matrices representation of linear transformation f and g are $A=\begin{pmatrix}0 & -4 & 3\\ -1 & 1 & 0\\ -1 & 4 & -2\end{pmatrix}$ and $B=\begin{pmatrix}3 & 0 & 4\\ 1 & 5 & -1\\ 2 & 1 & 2\end{pmatrix}$ respectively. Find matrix representation of

a) $4f$

b) $2f+3g$

c) $f\circ g$

Solution

a)
$$[4f]=4\begin{pmatrix}0 & -4 & 3\\ -1 & 1 & 0\\ -1 & 4 & -2\end{pmatrix}=\begin{pmatrix}0 & -16 & 12\\ -4 & 4 & 0\\ -4 & 16 & -8\end{pmatrix}$$

b)
$$[2f+3g]=2\begin{pmatrix}0 & -4 & 3\\ -1 & 1 & 0\\ -1 & 4 & -2\end{pmatrix}+3\begin{pmatrix}3 & 0 & 4\\ 1 & 5 & -1\\ 2 & 1 & 2\end{pmatrix}=\begin{pmatrix}9 & -8 & 18\\ 1 & 17 & -3\\ 4 & 11 & 2\end{pmatrix}$$

c)
$$[f\circ g]=\begin{pmatrix}0 & -4 & 3\\ -1 & 1 & 0\\ -1 & 4 & -2\end{pmatrix}\begin{pmatrix}3 & 0 & 4\\ 1 & 5 & -1\\ 2 & 1 & 2\end{pmatrix}=\begin{pmatrix}2 & -17 & 10\\ -2 & 5 & -5\\ -3 & 18 & -12\end{pmatrix}$$

APPLICATION ACTIVITY 8.3.2

1. Find matrix representation of the transformation

$f:\mathbb{R}^3\to\mathbb{R}^3$

$f(x,y,z)=(3x+2y,2z-y,z-x)$

a) Relative to the standard basis of $\mathbb{R}^3$

b) Relative to the basis $\left\{\vec{e_1}=(1,1,1),\vec{e_2}=(-1,0,1),\vec{e_3}=(0,1,1)\right\}$

2. Matrices representation of linear transformation f and g are $A=\begin{pmatrix}3 & 4 & 1\\ -1 & 2 & 0\\ 4 & -5 & -3\end{pmatrix}$ and $B=\begin{pmatrix}2 & 3 & 2\\ -1 & 0 & 3\\ 1 & 0 & 2\end{pmatrix}$ respectively. Find matrix representation of

a) $4f-5g$

b) $f\circ g$

c) $g\circ f$

8.4 Determinants and inverse of matrices of order two and three

8.4.1. Determinant of order two or three

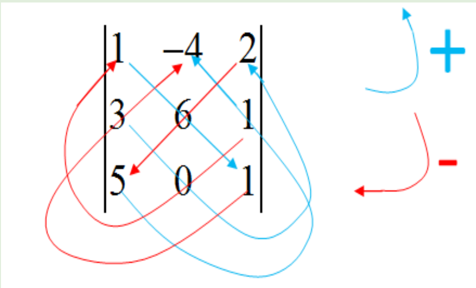
ACTIVITY 8.4.1

1) Given that $\begin{vmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{vmatrix}=a_{11}a_{22}-a_{21}a_{12}$

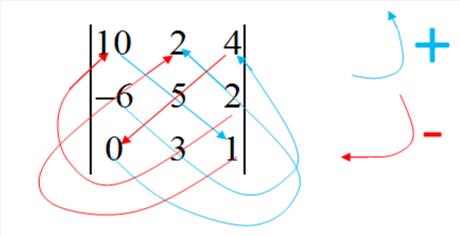
Determine: a) $\begin{vmatrix}1 & 0\\ 2 & 1\end{vmatrix}$ b) $\begin{vmatrix}-2 & -4\\ 3 & 6\end{vmatrix}$ c) $\begin{vmatrix}3 & 1\\ 6 & 8\end{vmatrix}$ d) $\begin{vmatrix}12 & 3\\ -2 & 9\end{vmatrix}$

2) Evaluate the following operations by considering the direction of arrows(sum of the blue products minus sum of the red products)

a)



b)



Content summary

Consider two matrices, one of order two and another one of order three:

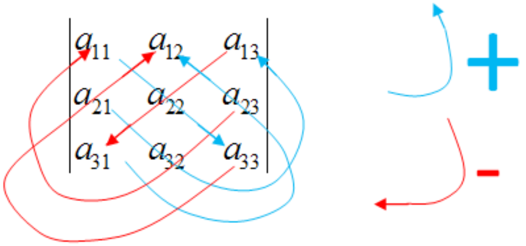
$M = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ and $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$. The determinant of A is calculated by SARRUS rule:

The terms with a **positive sign** are formed by the elements of the **principal diagonal** and those of the **parallel diagonals** with its corresponding **opposite vertex.**

The terms with a **negative sign** are formed by the elements of the **secondary diagonal** and those of the **parallel diagonals** with its corresponding **opposite vertex.**

$$\det M = |M| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \text{ or } |M| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{21}a_{12}$$

$$\det A = |A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{vmatrix} =$$

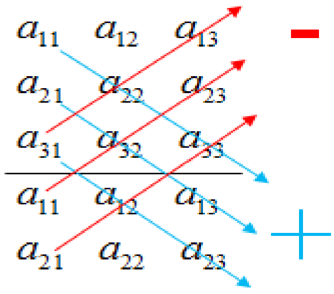


$$\begin{cases} a - 3b = -5 \\ 3a - b + 2c = 7 \\ 5a - 2b + 4c = 10 \end{cases}$$

$$\det = a_{11}a_{22}a_{33} + a_{21}a_{32}a_{13} + a_{31}a_{23}a_{12} - a_{13}a_{22}a_{31} - a_{23}a_{32}a_{11} - a_{33}a_{21}a_{12}$$

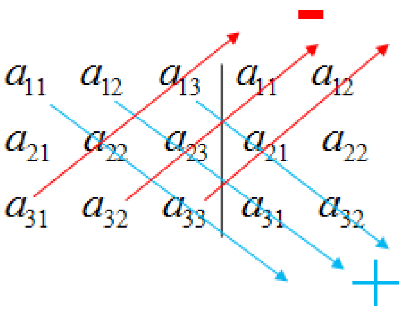
Or we can work as follow:

To calculate the 3x3 determinant we rewrite the first two rows below the determinant (or first two columns to the right of the determinant).



$$\det = a_{11}a_{22}a_{33} + a_{21}a_{32}a_{13} + a_{31}a_{12}a_{23} - a_{31}a_{22}a_{13} - a_{11}a_{32}a_{23} - a_{21}a_{12}a_{33}$$

Or



$$\det = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{31}a_{22}a_{13} - a_{32}a_{23}a_{11} - a_{33}a_{21}a_{12}$$

As multiplication of real numbers is commutative, the three are the same.

Example 1

Determine det N given that

$$N = \begin{pmatrix} 12 & 6 \\ 5 & 4 \end{pmatrix} \Rightarrow |N| = \begin{vmatrix} 12 & 6 \\ 5 & 4 \end{vmatrix} = 12 \times 4 - 5 \times 6 = 48 - 30 = 18$$

Example 2

Determine detQ if

$$Q = \begin{pmatrix} 3 & 2 & 1 \\ 0 & 2 & -5 \\ -2 & 1 & 4 \end{pmatrix} \Rightarrow |Q| =$$

$$\begin{vmatrix} 3 & 2 & 1 \\ 0 & 2 & -5 \\ -2 & 1 & 4 \end{vmatrix} = 3\times2\times4+0\times1\times1+(-2)\times(-5)\times2-1\times2\times(-2)-(-5)\times1\times3-4\times0\times2$$

$$= 24+0+20+4+15-0$$

$$= 63$$

Determinant of $n\times n$ matrices by method of minors and cofactors

General method of finding the determinant of matrix with $n\times n$ dimension $(2\times2,\ 3\times3,\ 4\times4,\ 5\times5,\ldots)$ is the use of cofactors.

Minor

An element, a_{ij}, to the value of the determinant of order $n-1$, obtained by deleting the row i and the column j in the matrix is called a **minor.**

$$\begin{vmatrix} 1 & 2 & 1 \\ 2 & [5] & 4 \\ 3 & 6 & 2 \end{vmatrix} \rightarrow \begin{vmatrix} 1 & 1 \\ 3 & 2 \end{vmatrix}$$

Cofactor

For an n by n determinant A, the cofactor of entry a_{ij}, denoted by A_{ij}, is given

$$A_{ij} = (-1)^{i+j} M_{ij}$$

where M_{ij} is the minor of entry a_{ij}.

The exponent of $(-1)^{i+j}$ is the sum of the row and column of the entry a_{ij} so if $i+j$

is even, $(-1)^{i+j}$ will equal 1, and if $i+j$ is odd, will equal -1

To find the value of a determinant, multiply each entry in any row or column by its cofactor and sum the results.

This process is referred to as **expanding across a row or column.**

For example, the value of the 3 by 3 determinant; if we choose to expand down column 2, we obtain

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = (-1)^{1+2}a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + (-1)^{2+2}a_{22}\begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} + (-1)^{3+2}a_{32}\begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix}$$

Expand down column 2.

If we choose to expand across row 3, we obtain

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = (-1)^{3+1}a_{31}\begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} + (-1)^{3+2}a_{32}\begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} + (-1)^{3+3}a_{33}\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

Expand across row 3.

It can be shown that the value of a determinant does not depend on the choice of the row or column used in the expansion. However, expanding across a row or column that has an entry equal to 0 reduces the amount of work needed to compute the value of the determinant.

Example

$$\begin{vmatrix} 3 & 2 & 1 \\ 0 & 2 & -5 \\ -2 & 1 & 4 \end{vmatrix} = 3\begin{vmatrix} 2 & -5 \\ 1 & 4 \end{vmatrix} - 2\begin{vmatrix} 0 & -5 \\ -2 & 4 \end{vmatrix} + 1\begin{vmatrix} 0 & 2 \\ -2 & 1 \end{vmatrix}$$

$$= 3(8+5) - 2(0-10) + 1(0+4)$$

$$= 39 + 20 + 4$$

$$= 63$$

Note that we choose only one line (row or column).

APPLICATION ACTIVITY 8.4.1

Find the determinants of the following matrices

1. $A = \begin{pmatrix} 1 & 3 & 1 \\ -4 & 5 & -2 \\ -3 & 1 & 3 \end{pmatrix}$

2. $B = \begin{pmatrix} 1 & 4 & -2 \\ 0 & 1 & 2 \\ 1 & 2 & 4 \end{pmatrix}$

$$C=\begin{pmatrix}1 & 4 & 2\\ -2 & 0 & 1\\ -1 & 3 & 0\end{pmatrix}$$

8.4.2. Properties of determinant

ACTIVITY 8.4.2

Consider the matrices $A=\begin{pmatrix}1 & 0 & 1\\ 1 & -1 & 1\\ 0 & 0 & 0\end{pmatrix}$, $B=\begin{pmatrix}-2 & 1 & -1\\ 1 & 3 & 4\\ -1 & 4 & 3\end{pmatrix}$, $C=\begin{pmatrix}1 & 0 & -1\\ 0 & 2 & 1\\ 0 & 0 & 3\end{pmatrix}$

and $D=\begin{pmatrix}1 & 3 & -1\\ -1 & 2 & 2\\ 0 & 1 & -1\end{pmatrix}$ find

1. $|A|$ and $|B|$
2. $|C\cdot D|$ and $|C|\cdot|D|$. What can you conclude?
3. $|C|$ and the product of elements for the leading diagonal of C. What can you conclude?

Content summary

1) Matrix A and its transpose A^t have the same determinant.

$|A^t| = |A|$

Example

$$A=\begin{pmatrix}2 & 3 & 0\\ 3 & 2 & 7\\ 2 & 1 & 6\end{pmatrix},\quad A^t=\begin{pmatrix}2 & 3 & 2\\ 3 & 2 & 1\\ 0 & 7 & 6\end{pmatrix},\quad |A|=|A^t|=-2$$

2) $|A|=0$ in the following cases:

- if matrix A has two equal rows or lines

Example

$$|A| = \begin{vmatrix} 2 & 3 & 2 \\ 3 & 2 & 3 \\ 2 & 3 & 2 \end{vmatrix} = 0$$

- All elements of a row or column are zero.

Example $$|A| = \begin{vmatrix} 2 & 3 & 2 \\ 3 & 2 & 3 \\ 0 & 0 & 0 \end{vmatrix} = 0$$

- The elements of a line are a linear combination of the others. (say line means row or column)

Example $$|A| = \begin{vmatrix} 2 & 3 & 2 \\ 1 & 2 & 4 \\ 3 & 5 & 6 \end{vmatrix} = 0$$

r3 = r1 + r2

3) A triangular matrix determinant is the product of the leading diagonal elements.

Example $$|A| = \begin{vmatrix} 2 & 0 & 0 \\ 1 & 2 & 0 \\ 3 & 5 & 6 \end{vmatrix} = 2 \times 2 \times 6 = 24$$

4) If a determinant switches two parallel lines its determinant changes sign.

Example $$|A| = \begin{vmatrix} 2 & 1 & 2 \\ 1 & 2 & 0 \\ 3 & 5 & 6 \end{vmatrix} = -\begin{vmatrix} 1 & 2 & 0 \\ 2 & 1 & 2 \\ 3 & 5 & 6 \end{vmatrix}$$

5) If the elements of a line are added to the elements of another parallel line previously multiplied by a real number, the value of the determinant is unchanged.

Example $\begin{vmatrix} 2 & 1 & 2 \\ 1 & 2 & 0 \\ 3 & 5 & 6 \end{vmatrix} = 16 \quad c_3 = 2c_1 + c_2 + c_3 \quad \begin{vmatrix} 2 & 1 & 7 \\ 1 & 2 & 4 \\ 3 & 5 & 17 \end{vmatrix} = 16$

6) If a determinant is multiplied by a real number, any line can be multiplied by the above mentioned number, but only one.

Example

$$2 \times \begin{vmatrix} 2 & 1 & 2 \\ 1 & 2 & 0 \\ 3 & 5 & 6 \end{vmatrix} = \begin{vmatrix} 2\times 2 & 1 & 2 \\ 2\times 1 & 2 & 0 \\ 2\times 3 & 5 & 6 \end{vmatrix} = \begin{vmatrix} 4 & 1 & 2 \\ 2 & 2 & 0 \\ 6 & 5 & 6 \end{vmatrix} = 32$$

$$2 \times \begin{vmatrix} 2 & 1 & 2 \\ 1 & 2 & 0 \\ 3 & 5 & 6 \end{vmatrix} = 2 \times 16 = 32$$

7) If all the elements of a line are formed by two addends, the above mentioned determinant decomposes in the sum of two determinants.

Example

$$\begin{vmatrix} 2 & 1 & 2 \\ a+b & a+c & a+d \\ 3 & 5 & 6 \end{vmatrix} = \begin{vmatrix} 2 & 1 & 2 \\ a & a & a \\ 3 & 5 & 6 \end{vmatrix} + \begin{vmatrix} 2 & 1 & 2 \\ b & c & d \\ 3 & 5 & 6 \end{vmatrix}$$

8) The determinant of a product equals the product of the determinants.

$$|A \times B| = |A| \times |B|$$

Example

Let $A = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 4 & 8 \\ 3 & 2 & 3 \end{pmatrix}$, $B = \begin{pmatrix} 3 & -1 & 1 \\ 0 & 4 & 1 \\ 3 & 2 & 2 \end{pmatrix}$ then $A \times B = \begin{pmatrix} 6 & 9 & 5 \\ 30 & 30 & 22 \\ 18 & 11 & 11 \end{pmatrix}$

$$|A \times B| = \begin{vmatrix} 6 & 9 & 5 \\ 30 & 30 & 22 \\ 18 & 11 & 11 \end{vmatrix} = 72$$

$$|A|=\begin{vmatrix}1 & 2 & 1\\ 2 & 4 & 8\\ 3 & 2 & 3\end{vmatrix}=24,\ |B|=\begin{vmatrix}3 & -1 & 1\\ 0 & 4 & 1\\ 3 & 2 & 2\end{vmatrix}=3$$

$$|A|\times|B|=24\times 3=72$$

APPLICATION ACTIVITY 8.4.2

Consider the following matrices $A=\begin{pmatrix}12 & 0 & 1\\ 34 & 0 & 2\\ -3 & 0 & 3\end{pmatrix}$, $B=\begin{pmatrix}1 & 4 & 5\\ 2 & 6 & 8\\ 3 & 2 & 5\end{pmatrix}$,

$C=\begin{pmatrix}6 & 7 & 6\\ 2 & 4 & 8\\ 1 & 3 & 9\end{pmatrix}$, $D=\begin{pmatrix}-3 & 5 & 1\\ 2 & 10 & 1\\ 1 & 8 & 1\end{pmatrix}$

Find

1) $|A|$, $|B|$, $|C|$ and $|D|$

2) $|BC|$

3) $|CD|$

8.4.3 Inverse of matrices of order two or three

ACTIVITY 8.4.3

Consider the matrix $A=\begin{pmatrix}1 & 1 & 1\\ 2 & 1 & -1\\ 3 & 2 & 1\end{pmatrix}$

1) Calculate the determinant of A, i.e $|A|$.
2) Replace every element in matrix A by its cofactor to find a new matrix called cofactor matrix. (Be careful on the signs of the cofactors).
3) Find the transpose of the cofactor matrix.
4) Multiply the inverse value of determinant obtained in 1 by the matrix obtained in 3
5) Multiply matrix A by matrix obtained in 4. Discuss your result.

Content summary

Calculating matrix inverse of matrix A, is to find matrix A^{-1} such that,

$$A \cdot A^{-1} = A^{-1} \cdot A = I$$

Where I is identity matrix.

The matrix inverse of matrix A is equal to the inverse value of its determinant multiplied by the **adjoint or adjugate matrix.**

$$A^{-1} = \frac{1}{|A|} . adj(A)$$

Where $adj(A)$ is the **adjoint** (also called **adjugate**) matrix which is the transpose of the cofactor matrix. The cofactor matrix is found by replacing every element in matrix A by its cofactor.

Notice: If $\det A = 0$ (i.e the determinant is zero), the matrix has no inverse and is said to be a **singular** matrix.

Example

Find the inverse of the following matrix

$$A = \begin{pmatrix} 2 & 0 & 1 \\ 3 & 0 & 0 \\ 5 & 1 & 1 \end{pmatrix}$$

Solution

We find its inverse as follow: $|A| = 3$

Cofactor of each element:

$$c(2) = \begin{vmatrix} 0 & 0 \\ 1 & 1 \end{vmatrix} = 0 \qquad c(0) = -\begin{vmatrix} 3 & 0 \\ 5 & 1 \end{vmatrix} = -3 \qquad c(1) = \begin{vmatrix} 3 & 0 \\ 5 & 1 \end{vmatrix} = 3$$

$$c(3) = -\begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix} = 1 \qquad c(0) = \begin{vmatrix} 2 & 1 \\ 5 & 1 \end{vmatrix} = -3 \qquad c(0) = -\begin{vmatrix} 2 & 0 \\ 5 & 1 \end{vmatrix} = -2$$

$$c(5) = \begin{vmatrix} 0 & 0 \\ 0 & 1 \end{vmatrix} = 0 \qquad c(1) = -\begin{vmatrix} 2 & 1 \\ 3 & 0 \end{vmatrix} = 3 \qquad c(1) = \begin{vmatrix} 2 & 0 \\ 3 & 0 \end{vmatrix} = 0$$

The cofactor matrix is

$$\begin{pmatrix} 0 & -3 & 3 \\ 1 & -3 & -2 \\ 0 & 3 & 0 \end{pmatrix}, \text{ and then } \quad adj(A) = \begin{pmatrix} 0 & 1 & 0 \\ -3 & -3 & 3 \\ 3 & -2 & 0 \end{pmatrix}$$

Therefore, the matrix inverse of A is $A^{-1} = \frac{1}{|A|} adj(A) = \frac{1}{3}\begin{pmatrix} 0 & 1 & 0 \\ -3 & -3 & 3 \\ 3 & -2 & 0 \end{pmatrix}$

$$A^{-1} = \begin{pmatrix} 0 & \frac{1}{3} & 0 \\ -1 & -1 & 1 \\ 1 & \frac{-2}{3} & 0 \end{pmatrix}$$

NB: Let A be a matrix of order two, $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$, the inverse of A is given by

$$A^{-1} = \frac{1}{\det A}\begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix}$$

Find the inverse of $A = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$

Solution

$\det A = 1 - 0 = 1$

$$A^{-1} = \frac{1}{1}\begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}$$

Example 2

Find the inverse of $A = \begin{pmatrix} 1 & 3 \\ 2 & 6 \end{pmatrix}$

Solution

$\det A = 6 - 6 = 0$

Since the determinant is zero, the given matrix has no inverse.

APPLICATION ACTIVITY 8.4.3

Find the inverse of the following matrices

1) $A=\begin{pmatrix}1 & 4 & 5\\ 2 & 6 & 8\\ 3 & 2 & 5\end{pmatrix}$

2) $B=\begin{pmatrix}11 & -8 & 1\\ 0 & -6 & 2\\ 3 & 2 & 7\end{pmatrix}$

3) $C=\begin{pmatrix}6 & 7 & 6\\ 2 & 4 & 8\\ 1 & 3 & 9\end{pmatrix}$

4) $D=\begin{pmatrix}-3 & 5 & 1\\ 2 & 10 & 1\\ 1 & 8 & 1\end{pmatrix}$

8.4.4. Properties of the Inverse Matrix

ACTIVITY 8.4.4

Consider the matrices $A=\begin{pmatrix}1 & 0 & 1\\ 1 & -1 & 1\\ 0 & 0 & 1\end{pmatrix}$ and $B=\begin{pmatrix}2 & 1 & 1\\ 1 & 1 & 1\\ -1 & 1 & 0\end{pmatrix}$ find

1. $(AB)^{-1}$ and $B^{-1}A^{-1}$
2. $(A^{-1})^{-1}$
3. $(4A)^{-1}$ and $\frac{1}{4}A^{-1}$
4. $(A^{t})^{-1}$ and $(A^{-1})^{t}$

What can you conclude for each result?

Content summary

For two invertible matrices A and B

1) $(A \cdot B)^{-1} = B^{-1} \cdot A^{-1}$

2) $\left(A^{-1}\right)^{-1} = A$

3) $(\alpha \cdot A)^{-1} = \alpha^{-1} \cdot A^{-1}$

4) $\left(A^{t}\right)^{-1} = \left(A^{-1}\right)^{t}$

Example 1

Consider matrix $A = \begin{pmatrix} 1 & 1 & -2 \\ 0 & 1 & -1 \\ 3 & 0 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 1 & 0 \\ 2 & -2 & 0 \\ 1 & 1 & 1 \end{pmatrix}$, find

a) A^{-1} and B^{-1}

b) $(AB)^{-1}$

c) $(3B)^{-1}$

d) $\left(B^{t}\right)^{-1}$

Solution

a)

$$|A| = 3, \quad Adj(A) = \begin{pmatrix} 0 & 0 & 1 \\ -3 & 6 & 1 \\ -3 & 3 & 1 \end{pmatrix}, \quad A^{-1} = \begin{pmatrix} 0 & 0 & \frac{1}{3} \\ -1 & 2 & \frac{1}{3} \\ -1 & 1 & \frac{1}{3} \end{pmatrix}$$

$$|B| = -2, \quad Adj(B) = \begin{pmatrix} -2 & -1 & 0 \\ -2 & 0 & 0 \\ 4 & 1 & -2 \end{pmatrix}, \quad B^{-1} = \begin{pmatrix} 1 & \frac{1}{2} & 0 \\ 1 & 0 & 0 \\ -2 & -\frac{1}{2} & 1 \end{pmatrix}$$

b)

$$(AB)^{-1}=B^{-1}A^{-1}=\begin{pmatrix}1 & \frac{1}{2} & 0\\ 1 & 0 & 0\\ -2 & -\frac{1}{2} & 1\end{pmatrix}\begin{pmatrix}0 & 0 & \frac{1}{3}\\ -1 & 2 & \frac{1}{3}\\ -1 & 1 & \frac{1}{3}\end{pmatrix}=\begin{pmatrix}-\frac{1}{2} & 1 & \frac{1}{2}\\ 0 & 0 & \frac{1}{3}\\ -\frac{1}{2} & 0 & -\frac{1}{2}\end{pmatrix}$$

c)

$$(3B)^{-1}=\frac{1}{3}B^{-1}=\frac{1}{3}\begin{pmatrix}1 & \frac{1}{2} & 0\\ 1 & 0 & 0\\ -2 & -\frac{1}{2} & 1\end{pmatrix}=\begin{pmatrix}\frac{1}{3} & \frac{1}{6} & 0\\ \frac{1}{3} & 0 & 0\\ -\frac{2}{3} & -\frac{1}{6} & \frac{1}{3}\end{pmatrix}$$

d)

$$\left(B^t\right)^{-1}=\left(B^{-1}\right)^t=\begin{pmatrix}1 & \frac{1}{2} & 0\\ 1 & 0 & 0\\ -2 & -\frac{1}{2} & 1\end{pmatrix}^t=\begin{pmatrix}1 & 1 & 2\\ \frac{1}{2} & 0 & -\frac{1}{2}\\ 0 & 0 & 1\end{pmatrix}$$

APPLICATION ACTIVITY 8.4.4

1) If $A=\begin{pmatrix}-1 & -3\\ 1 & 1\end{pmatrix}$ and $B=\begin{pmatrix}6 & 3\\ 2 & 1\end{pmatrix}$, find

a. A^{-1}

b. B^{-1}

c. $(AB)^{-1}$

d. $\left(A^t\right)^{-1}$

e. $(4B)^{-1}$

2) Consider of the following matrices $A=\begin{pmatrix}3&7&2\\1&0&2\\0&1&1\end{pmatrix}$ and

$$B=\begin{pmatrix}2&-2&1\\3&1&6\\-1&1&1\end{pmatrix}$$

Find:

a. A^{-1} and B^{-1}

b. $\left(A^{-1}\right)^{-1}$

c. $(10A)^{-1}$

d. $\left(A^{t}\right)^{-1}$

8.5 Applications of matrices and determinants

8.5.1. Solving System of linear equations using inverse matrix

ACTIVITY 8.5.1

1) A Farmer Kalisa bought in Ruhango Market 5 Cocks and 4 Rabbits and he paid 35,000Frw, on the following day, he bought in the same Market 3 Cocks and 6 Rabbits and he paid 30,000Frw.

a) Considering x as the cost for one cock and y the cost of one Rabbit, formulate equations that illustrate the activity of Kalisa;

b) Make a matrix A indicating the number of cocks and rabbits

c) If C is a matrix column made by the money paid by Kalisa,

ie $C=\begin{pmatrix}35,000\\30,000\end{pmatrix}$,

write the equation $A\begin{pmatrix}x\\y\end{pmatrix}=C$

d) Discuss and explain in your own words how you can determine $\begin{pmatrix}x\\y\end{pmatrix}$ the cost of 1 Cocks and 1 Rabbit.

a. Rewrite this system in matrix form (the matrix of coefficients times the column matrix of the unknown equate it to the column matrix of the independent terms)

b. If we pre-multiply (multiply to the left) both sides of the equality obtained in 1) by $\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}^{-1}$, what will be the new equality?

Content summary

The solution of the following system of 3 linear equations in 3 unknowns.

$$\begin{cases} a_{11}x + a_{12}y + a_{13}z = b_1 \\ a_{21}x + a_{22}y + a_{23}z = b_2 \\ a_{31}x + a_{32}y + a_{33}z = b_3 \end{cases}$$

$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$ or $AX = B$, where $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$ and

$B = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$ It is clear that $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = A^{-1}B$, provided that A^{-1}.

Notice

- If at least, one of b_i is different of zero the system is said to be **non-homogeneous** and if all b_i are zero the system is said to be **homogeneous**.
- The set of values of x, y, z that satisfy all the equations of system (1) is called **solution of the system**.
- For the homogeneous system, the solution $x = y = z = 0$ is **called trivial solution**. Other solutions are **non-trivial solutions**.

Non- homogeneous system cannot have a trivial solution as at least one of x, y, z is not zero. **Example**

Solve the system of equations

$$\begin{cases} x+y+z=6 \\ 2x+y-z=1 \\ 3x+2y+z=10 \end{cases}$$

Solution:

$$\begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ 3 & 2 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}=\begin{pmatrix} 6 \\ 1 \\ 10 \end{pmatrix}$$

$$A=\begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ 3 & 2 & 1 \end{pmatrix}, \quad X=\begin{pmatrix} x \\ y \\ z \end{pmatrix}, \quad B=\begin{pmatrix} 6 \\ 1 \\ 10 \end{pmatrix}$$

We find the inverse of A.

A is invertible if its determinant is not zero.

$\det(A)=\begin{vmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ 3 & 2 & 1 \end{vmatrix}=1+4-3-3+2-2=-1\neq 0$, then A has inverse.

We have seen that the adjugate matrix and determinant of a matrix are used to find its inverse.

Let use another useful method.

We have $A=\begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ 3 & 2 & 1 \end{pmatrix}$, to find its inverse, suppose that its inverse is given by

$$A^{-1}=\begin{pmatrix} a & d & g \\ b & e & h \\ c & f & i \end{pmatrix}$$

We know that $AA^{-1}=I$, then

$$\begin{pmatrix}1&1&1\\2&1&-1\\3&2&1\end{pmatrix}\begin{pmatrix}a&d&g\\b&e&h\\c&f&i\end{pmatrix}=\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}\Rightarrow\begin{cases}\begin{cases}a+b+c=1\\2a+b-c=0\\3a+2b+c=0\end{cases} & (1)\\ \begin{cases}d+e+f=0\\2d+e-f=1\\3d+2e+f=0\end{cases} & (2)\\ \begin{cases}g+h+i=0\\2g+h-i=0\\3g+2h+i=1\end{cases} & (3)\end{cases}$$

We solve these three systems to find value of $a,\ b,\ c,\ d,\ e,\ f,\ g,\ h,$ and i .

$$\begin{cases}a+b+c=1\\2a+b-c=0\\3a+2b+c=0\end{cases}\quad(1)\Rightarrow\begin{cases}a=-3\\b=5\\c=-1\end{cases}$$

$$\begin{cases}d+e+f=0\\2d+e-f=1\\3d+2e+f=0\end{cases}\quad(2)\Rightarrow\begin{cases}d=-1\\e=2\\f=-1\end{cases}$$

$$\begin{cases}g+h+i=0\\2g+h-i=0\\3g+2h+i=1\end{cases}\quad(3)\Rightarrow\begin{cases}g=2\\h=-3\\i=1\end{cases}$$

Then,

$$A^{-1}=\begin{pmatrix}-3&-1&2\\5&2&-3\\-1&-1&1\end{pmatrix}$$

$$X=\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}-3&-1&2\\5&2&-3\\-1&-1&1\end{pmatrix}\begin{pmatrix}6\\1\\10\end{pmatrix}=\begin{pmatrix}1\\2\\3\end{pmatrix}$$

Therefore, $S=\{(1,2,3)\}$

APPLICATION ACTIVITY 8.5.1

Solve the following system using the inverse of a matrix

$$\begin{cases} 5x+15y+56z=35 \\ -4x-11y-41z=-26 \\ -x-3y-11z=-7 \end{cases}$$

8.5.2. Solving System of linear equations using Cramer method

ACTIVITY 8.5.2

Consider the following system of 3 linear equations with 3 unknowns.

$$\begin{cases} x+2y-3z=0 \\ 3x+3y-z=5 \\ x-2y+2z=1 \end{cases}$$

a) Rewrite this system in matrix form $A\begin{pmatrix} x \\ y \\ z \end{pmatrix}=\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$

b) Calculate Δ , determinant of coefficients of unknowns

c) Calculate Δ_x , determinant of coefficients of unknowns with column in x replaced by column of independent terms $\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$

d) Calculate Δ_y , determinant of coefficients of unknowns with column in y replaced by column of independent terms

e) Calculate Δ_z , determinant of coefficients of unknowns with column in z replaced by column of independent terms

f) Find $x=\dfrac{\Delta_x}{\Delta}, y=\dfrac{\Delta_y}{\Delta}, z=\dfrac{\Delta_z}{\Delta}$,

g) Compare the obtained values to solutions of the given system ,solved by using other seen methods

Content summary

Consider the system

$$\begin{cases} a_{11}x & + & a_{12}y & + & a_{13}z = b_1 \\ a_{21}x & + & a_{22}y & + & a_{23}z = b_2 \\ a_{31}x & + & a_{32}y & + & a_{33}z = b_3 \end{cases}$$

We use Cramer's rule as follows

$$\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$\Delta_x = \begin{vmatrix} c_1 & a_{12} & a_{13} \\ c_2 & a_{22} & a_{23} \\ c_3 & a_{32} & a_{33} \end{vmatrix}$$

$$\Delta_y = \begin{vmatrix} a_{11} & c_1 & a_{13} \\ a_{21} & c_2 & a_{23} \\ a_{31} & c_3 & a_{33} \end{vmatrix}$$

$$\Delta_z = \begin{vmatrix} a_{11} & a_{12} & c_1 \\ a_{21} & a_{22} & c_2 \\ a_{31} & a_{32} & c_3 \end{vmatrix}$$

$x = \frac{\Delta_x}{\Delta}$, $y = \frac{\Delta_y}{\Delta}$ and $z = \frac{\Delta_z}{\Delta}$

Notice:

- The solution $\frac{b}{0}$, $b \neq 0$ means impossible
- The solution $\frac{0}{0}$ means indeterminate

Example: Use Cramer's method to solve following the system

$$\begin{cases} x+y+z=6 \\ 2x+y-z=1 \\ 3x+2y+z=10 \end{cases}$$

Solution:

$$\Delta = \begin{vmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ 3 & 2 & 1 \end{vmatrix} = -1$$

$$\Delta_x = \begin{vmatrix} 6 & 1 & 1 \\ 1 & 1 & -1 \\ 10 & 2 & 1 \end{vmatrix} = -1$$

$$\Delta_y = \begin{vmatrix} 1 & 6 & 1 \\ 2 & 1 & -1 \\ 3 & 10 & 1 \end{vmatrix} = -2$$

$$\Delta_z = \begin{vmatrix} 1 & 1 & 6 \\ 2 & 1 & 1 \\ 3 & 2 & 10 \end{vmatrix} = -3$$

$$x = \frac{\Delta_x}{\Delta} = \frac{-1}{-1} = 1, \ y = \frac{\Delta_y}{\Delta} = \frac{-2}{-1} = 2 \ , \ z = \frac{\Delta_z}{\Delta} = \frac{-3}{-1} = 3$$

Therefore, $S = \{(1,2,3)\}$

APPLICATION ACTIVITY 8.5.2

Use matrix inverse method to solve the following systems

1. $\begin{cases} 3x+y+z=0 \\ 2x-y+2x=0 \\ 7x+y-3z=0 \end{cases}$

2. $\begin{cases} 4x+y-z=1 \\ x-3y+z=2 \\ 5x-2y=4 \end{cases}$

3. $\begin{cases} x+y-z=3 \\ 3x-y+z=1 \\ -2x+y+z=0 \end{cases}$

8.5.3. Solving system of linear equations using Gaussian method (elimination of Gauss)

ACTIVITY 8.5.3

Consider the following system of 3 linear equations with 3 unknowns.

$$\begin{cases} x+2y-2z=1 \\ 2x+y-4z=-1 \\ 4x-3y+z=11 \end{cases}$$

equivalent to

$$\begin{bmatrix} 1 & 2 & -2 \\ 2 & 1 & -4 \\ 4 & -3 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 1 \\ -1 \\ 11 \end{bmatrix}$$

From the given matrix, complete the augmented martix of the form:

$$\left[\begin{array}{ccc|c} 1 & 2 & -2 & 1 \\ 2 & 1 & -4 & -1 \\ 4 & -3 & 1 & 11 \end{array}\right] \begin{array}{c} r_1 \\ r_2 = 2r_1 - r_2 \\ r_3 = 4r_1 - r_3 \end{array} \sim \left[\begin{array}{ccc|c} \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \end{array}\right]$$

Multiply the equivalent matrix obtained by the matrix of unkowns and deterimine the values of x, y *and* z (using the following form).

$$\left(\begin{array}{ccc|c} . & . & . & . \\ . & . & . & . \\ . & . & . & . \end{array}\right)\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

Content summary

Resolution of systems of linear equation by **Gauss's method**

Let system: $(1)\begin{cases} a_{11}x_1 + a_{11}x_2 \dots\dots\dots\dots\dots\dots a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 \dots\dots\dots\dots\dots\dots a_{2n}x_n = b_2 \\ \dots \\ a_{n1}x_1 + a_{n2}x_2 \dots\dots\dots\dots\dots\dots a_{nn}x_n = b_n \end{cases}$

The system is equivalent to the system:

$$\begin{pmatrix} a_{11}a_{12} \dots\dots a_{1n} \\ \dots \\ a_{n1}a_{n2} \dots\dots a_{nn} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ . \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ . \\ b_n \end{pmatrix}$$

A simply $Ax = B$ where $A = \left(a_{ij}\right)$ and called matrix of coefficients $B = \left(b_{ij}\right)$ and called matrix of constants $X = \left(x_{ij}\right)$ and called matrix of unknowns

$$\left(\begin{array}{ccc|c} a_{11} & \cdots & a_{1n} & b_1 \\ \vdots & \ddots & \vdots & . \\ a_{m1} & \cdots & a_{mn} & b_n \end{array}\right)$$

From the matrix called **augmented matrix**

We manipulate the **elimination of Gauss**, for getting the equivalent matrix if we perform one of the following operations:

1) Exchange the rows L_i and L_j

2) Replace the row L_i by L_j

3) Replace the row L_i by $L_i + kL_j$

4) Replace the row L_i by $L_i + \sum_{j \neq 1} kL_j$

Example: Solve by elimination method of Gauss $\begin{cases} x+y-z=0 \\ x+2y+3z=14 \\ 2x+y+4z=16 \end{cases}$

$\rightarrow$ Augmented matrix: $\left(\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 1 & 2 & 3 & 14 \\ 2 & 1 & 4 & 6 \end{array}\right) L_2 \sim L_2 - L_1$

$\rightarrow \left(\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 0 & 1 & 4 & 14 \\ 2 & 1 & 4 & 6 \end{array}\right) L_3 \sim L_3 - 2L_1$

$\rightarrow \left(\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 0 & 1 & 4 & 14 \\ 0 & 0 & 10 & 30 \end{array}\right)$;

from the last matrix we have $10z = 30 \Rightarrow z = 3$

$\Rightarrow y + 4z = 14 \Rightarrow y + 12 = 14 \Rightarrow y = 2$

$\Rightarrow x + y - z = 0 \Rightarrow x + 2 - 3 = 0 \Rightarrow x = 1$

$S = \{(1,2,3)\}$

2) Solve by using Gauss's method

1) $\begin{cases} 3x+2y+4z=-1 \\ 2x-y+2z=-2 \\ -x+y+2z=2 \end{cases}$

2) $\begin{cases} -x+2y=5 \\ 2x+3y=4 \\ 3x-6y=-15 \end{cases}$

Solutions:

1) $\left(\begin{array}{ccc|c} 3 & 2 & 4 & -1 \\ 2 & -1 & 2 & -1 \\ -1 & 1 & 1 & 2 \end{array}\right) L_2 \sim 3L_2 - 2L_1$

$$\Rightarrow \left(\begin{array}{ccc|c} 3 & 2 & 4 & -1 \\ 0 & -7 & -2 & -2 \\ -1 & 1 & 1 & 2 \end{array}\right) L_3 \sim 3L_3 - L_1$$

$$\Rightarrow \left(\begin{array}{ccc|c} 3 & 2 & 4 & -1 \\ 0 & -7 & -2 & -4 \\ 0 & 5 & 10 & 5 \end{array}\right) L_3 \sim 5L_2 + 7L_3$$

$$\Rightarrow \begin{pmatrix} 3 & 2 & 4 & -1 \\ 0 & -7 & -2 & -4 \\ 0 & 0 & 60 & 15 \end{pmatrix}$$

$$\Rightarrow 60z = 15 \Rightarrow \boldsymbol{z} = \frac{1}{4}$$

$$3x + 2y + 4z = -1 \Rightarrow 3x + 1 + 1 = -1 \Rightarrow 3x = -3 \Rightarrow \boldsymbol{x} = -1$$

$$\Rightarrow -7y - 2z = -4 \Rightarrow -7y - \frac{2}{4} = -4 \Rightarrow \boldsymbol{y} = \frac{1}{2}$$

$$S = \left\{-1; \frac{1}{2}; \frac{1}{4}\right\}$$

2) $\begin{cases} -x + 2y = 5 \\ 2x + 3y = 4 \\ 3x - 6y = -15 \end{cases}$

$$\Rightarrow [A/B] = \left[\begin{array}{cc|c} -1 & 2 & 5 \\ 2 & 3 & 4 \\ 3 & -6 & -15 \end{array}\right] L_2 \sim L_2 + 2L_1$$

$$\Rightarrow [A/B]=\left[\begin{array}{cc|c}-1 & 2 & 5\\ 0 & 7 & 14\\ 3 & -6 & -15\end{array}\right] L_3 \sim L_3+3L_1$$

$$\Rightarrow \left[\frac{A}{B}\right]=\left[\begin{array}{cc|c}-1 & 2 & 5\\ 0 & 7 & 14\\ 0 & 0 & 0\end{array}\right] L_2 \sim \frac{1}{7}L_2 \quad \Rightarrow [A/B]=\left[\begin{array}{cc|c}-1 & 2 & 5\\ 0 & 1 & 2\\ 0 & 0 & 0\end{array}\right]$$

Therefore

$y=2 \Rightarrow -x+4=5$ and $x=-1$

This gives the set of solution $S=\{(-1,2)\}$.

Note: The method of **GAUSS** helps us to solve the above system where **CRAMER'S** method cannot.

APPLICATION ACTIVITY 8.5.3

Solve by using elimination of Gauss

1. $\begin{cases}2x-2y=5\\ 4x+3y=1\end{cases}$

2. $\begin{cases}a-3b=-5\\ 3a-b+2c=7\\ 5a-2b+4c=10\end{cases}$

8.6. END UNIT ASSESSMENT 8

1. Find the values of x, y and z for $3\begin{pmatrix} x & y-1 \\ 4 & 3z \end{pmatrix} = \begin{pmatrix} 15 & 6 \\ 6z & 3x+y \end{pmatrix}$

2. If $A = \begin{pmatrix} 3 & -1 & 3 \\ 1 & 0 & -6 \\ 0 & -4 & 2 \end{pmatrix}$, $B = \begin{pmatrix} 10 & 2 & 3 \\ 1 & -4 & 6 \\ 0 & 6 & 4 \end{pmatrix}$ and $C = \begin{pmatrix} 11 & 12 & 0 \\ 1 & 0 & 8 \\ 2 & 0 & 7 \end{pmatrix}$. Evaluate

a) $A-B$

b) $A+B-2C$

c) $2A-B+C$

d) $A \times B$

e) $A \times C$

f) $B \times C$

3. Find the matrix of the following map relative to the canonical basis $f: IR^3 \to IR^3$

$f(x,y,z) = (2x+y, y-z, 2x+4y)$

4. Let f be a linear operator on IR^3 defined by $f(x,y,z) = (2y+z, x-4y, 3x)$

a) Find the matrix of f in the basis $\{e_1 = (1,1,1), e_2 = (1,1,0), e_3 = (1,0,0)\}$

b) Verify that $[f]_e [v]_e = [f(v)]_e$ for any vector $v \in IR^3$

5. Find the inverse of

a) $A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$

b) $B = \begin{pmatrix} 2 & -2 & 0 \\ 1 & 3 & 4 \\ 3 & 1 & 4 \end{pmatrix}$

c) $C = \begin{pmatrix} 5 & 0 & 1 \\ 2 & 3 & 7 \\ 1 & 8 & 4 \end{pmatrix}$

6. Using matrix inverse method , solve

$A \cdot X + 2 \cdot B = 3 \cdot C$ if

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} \quad B = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

7. Use matrix inverse method to solve

a) $\begin{cases} x+3y+3z=0 \\ 3x+4y-z=0 \\ -3x-9y+z=0 \end{cases}$

b) Use Cramer's method to solve

$$\begin{cases} x+y+z=3 \\ 2x-y=1 \\ 4x+y-z=4 \end{cases}$$

c) Use Gaussian method to solve

$$\begin{cases} -x+y-z=-4 \\ 3x+10y+z=10 \\ x-y-z=2 \end{cases}$$

8. If $f(x)=x^3-20x+8$, find $f(A)$ if $A = \begin{pmatrix} 1 & 1 & 3 \\ 1 & 3 & -3 \\ -2 & -4 & -4 \end{pmatrix}$

9. Find the condition of k such that $A = \begin{pmatrix} 1 & 3 & 4 \\ 3 & k & 6 \\ -1 & 5 & 1 \end{pmatrix}$ be no singular matrix. Obtain A^{-1} for $k=1$.

10. If $A = \begin{pmatrix} 1 & 2 & -1 \\ 0 & 1 & 2 \\ -1 & 0 & -2 \end{pmatrix}$

a) Show that $A^3-4A+7I=O$ where I, O the unit and the null matrix of are order 3 respectively. Use this result to find A^{-1}.

b) Find the matrix X such that $AX = \begin{pmatrix} 2 \\ 1 \\ -7 \end{pmatrix}$

11. Given $A = \begin{pmatrix} 2 & -2 & -1 \\ 1 & 1 & -2 \\ 3 & 1 & -3 \end{pmatrix}$, find A^3 and hence solve the equations

a) $\begin{cases} 2x-2y-z=-18 \\ x+y-2z=-2 \\ 3x+y-3z=-10 \end{cases}$ b) $\begin{cases} x+7y-5z=9 \\ x+y-z=3 \\ x+4y-2z=6 \end{cases}$

12. Find , in terms of t , the determinant of the matrix
$A = \begin{pmatrix} 2-t & 1 & 3 \\ 1 & 1-t & 1 \\ -1 & -1 & -2-t \end{pmatrix}$

13. If A is a square matrix of order 3 such that $\det A = x$, find the value of

a) $\det\left(A^2\right)$ b) $\det\left(A^n\right)$ $, n \in \mathbb{Z}$ c) $\det(2A)$ d) $\det(mA), m \in \mathbb{R}$

14. Find the value of k for which the following system of equations

$\begin{cases} 3x-2y+2z=3 \\ x+ky-3z=0 \\ 4x+y+2z=5 \end{cases}$ are consistent

15. For what value of λ and μ the following system of equations

$\begin{cases} 2x+3y+5z=9 \\ 7x+3y-z=1 \\ 4x+3y+\lambda z=\mu \end{cases}$

will have

a) No solution

b) Unique solution

c) More than one solution

16. For what values of k is the matrix $A = \begin{bmatrix} k-1 & -2 & 1 \\ -2 & k-1 & -1 \\ -1 & 6 & k+2 \end{bmatrix}$ singular?

17. Discuss the consistency of the following system of equations for different cases of λ $\begin{cases} x + y + \lambda z = 1 \\ x + \lambda y + z = \lambda \\ \lambda x + y + z = \lambda^2 \end{cases}$

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