MATHEMATICS FOR TTC

STUDENT'S BOOK

YEAR

2

OPTION:

LANGUAGE EDUCATION (LE)

FOREWORD

Dear Student,

Rwanda Education Board (REB) is honoured to present Year 2 Mathematics book for Language Education (LE) student teachers. This book will serve as a guide to competence-based teaching and learning to ensure consistency and coherence in the learning of the Mathematics. The Rwandan educational philosophy is to ensure that you achieve full potential at every level of education which will prepare you to be well integrated in society and exploit employment opportunities.

The government of Rwanda emphasizes the importance of aligning teaching and learning materials with the syllabus to facilitate your learning process. Many factors influence what you learn, how well you learn and the competences you acquire. Those factors include the relevance of the specific content, the quality of teachers' pedagogical approaches, the assessment strategies and the instructional materials available. In this book, we paid special attention to the activities that facilitate the learning process in which you can develop your ideas and make new discoveries during concrete activities carried out individually or with peers.

In competence-based curriculum, learning is considered as a process of active building and developing knowledge and meanings by the learner where concepts are mainly introduced by an activity, situation or scenario that helps the learner to construct knowledge, develop skills and acquire positive attitudes and values.

For efficiency use of this textbook, your role is to:

- Work on given activities which lead to the development of skills;
- Share relevant information with other learners through presentations, discussions, group work and other active learning techniques such as role play, case studies, investigation and research in the library, on internet or outside;
- Participate and take responsibility for your own learning;
- Draw conclusions based on the findings from the learning activities.

To facilitate you in doing activities, the content of this book is self explanatory so that you can easily use it yourself, acquire and assess your competences. The book is made of units as presented in the syllabus.

Each unit has the following structure: the key unit competence is given and it is followed by the introductory activity before the development of mathematical concepts that are connected to real world problems or to other sciences.

The development of each concept has the following points:

- It starts by a learning activity: it is a hand on well set activity to be done by students in order to generate the concept to be learnt;
- Main elements of the content to be emphasized;
- Worked examples; and
- Application activities: those are activities to be done by the user to consolidate competences or to assess the achievement of objectives.

Even though the book has some worked examples, you will succeed on the application activities depending on your ways of reading, questioning, thinking and grappling ideas of calculus not by searching for similar-looking worked out examples.

Furthermore, to succeed in Mathematics, you are asked to keep trying; sometimes you will find concepts that need to be worked at before you completely understand. The only way to really grasp such a concept is to think about it and work-related problems found in other reference books.

I wish to sincerely express my appreciation to the people who contributed towards the development of this book, particularly, REB staff, UR-CE Lecturers and TTC Tutors for their technical support. A word of gratitude goes to Head Teachers and TTCs principals who availed their staff for various activities.

Any comment or contribution would be welcome to the improvement of this text book for the next edition.

Dr. NDAYAMBAJE Irénée

Director General, REB

ACKNOWLEDGEMENT

I wish to express my appreciation to the people who played a major role in the development of this Mathematics book for Year Two student teachers in the option of Language Education (LE). It would not have been successful without active participation of different education stakeholders.

I owe gratitude to different universities and schools in Rwanda that allowed their staff to work with REB in the in-house textbooks production initiative.

I wish to extend my sincere gratitude to lecturers and TTC tutors whose efforts during writing exercise of this book were very much valuable.

Finally, my word of gratitude goes to the Rwanda Education Board staffs who were involved in the whole process of in-house textbook Elaboration.

Joan MURUNGI

Head of CTLR Department

TABLE OF CONTENT

UNIT 1

FUNCTIONS AND GRAPHS

Key unit Competence: Apply graphical representation of function in economics models

1.0 INTRODUCTORY ACTIVITY

Suppose that average weekly household expenditure on food depends on average net household weekly income according to the relationship $C=12+0.3Y$.

a) Can you find a value of Y for which C is not a real number?

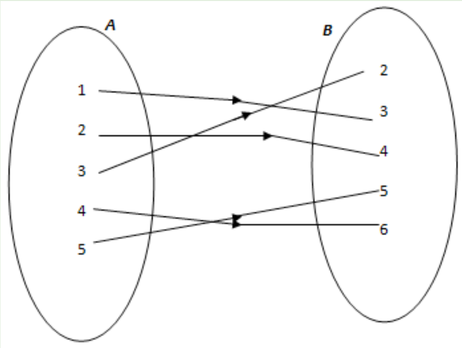
b) Complete a table of value from Y= 0 to Y= 10 and use it to draw the graph of

c) If Y= 90, what is the value of C?

1.1. Generalities on numerical functions

ACTIVITY 1.1

In the following arrow diagram, for each element of set A state which element of B is mapped to.



a) What is the set of elements of A which have images in B?

b) Determine the set of elements in B which have antecedent in A.

c) Is there any element of A which has more than one image?

Content summary

1.1.1 Function

A function is a rule that assigns to each element in a set *A* one and only one element in set *B*. We can even define a function as any relationship which takes one element of one set and assigns to it one and only one element of second set.

If *x* is an element in the domain of a function *f*, then the element that *f* associates with *x* is denoted by the symbol $f(x)$ (**read *f* of *x***) and is called the **image of *x* under *f*** or the **value of *f* at *x***.

The set of all possible values of $f(x)$ as *x* varies over the domain is called the **range** of *f* and it is denoted $R(f)$. The set of all values of A which have images in B is called **Domain of f** and denoted $Domf$.

We shall write $f(x)$ to represent the image of *x* under the function *f*. The letters commonly used for this purpose are *f, g* and *h*.

Examples

1. *Given that* $f(x)=x^2$*, find the values of* $f(0), f(2), f(3), f(4)$ *and* $f(5)$

Solution

$f(0)=0^2=0$ $\quad\quad$ $f(2)=2^2=4$

$f(3)=3^2=9$ $\quad\quad$ $f(4)=4^2=16$

$f(5)=5^2=25$

Note:

$f(x)=x^2$ can also be written as $f:x\to x^2$ which is read as

"*f* is a function which maps *x* onto x^2"

2. Draw arrow diagrams for the functions. Use the domain $\{1,2,3\}$

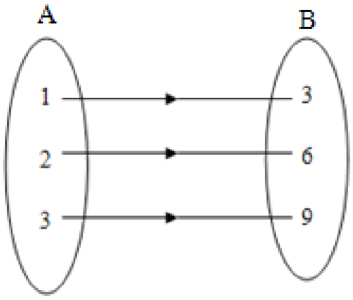
a. $f: x \to 3x$

b. $h: x \to x^2 + 1$

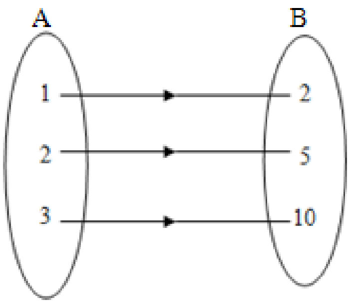
c. $g: x \to 2x + 1$

Solution

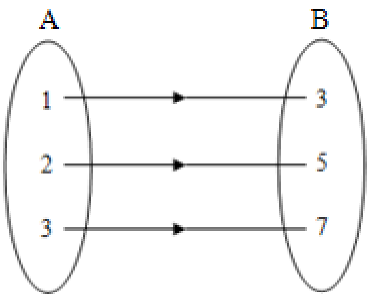
a. $f: x \to 3x$



b. $h: x \to x^2 + 1$



c. $g: x \to 2x + 1$



3. The functions f and g are given as $f(x)=x+3$ for $x\geq 0$ and $g(x)=x^2$ for $-2\leq x\leq 3$

State the range of each of these functions.

Solution

If $x\geq 0$, then $x+3\geq 3$. Thus, the range of *f* will be $f(x)\geq 3$

If $-2\leq x\leq 3$, then $(-2)^2\leq x^2\leq (3)^2$. Thus, the range of *g* will be $4\leq g(x)\leq 9$

1.1.2 Injective, surjective and bijective functions

Given sets A and B, a **function** defined from A to B is a correspondence, or a rule that associates to any element of A either one image in B, or no image in B.

A function such that every elements of A has an image in B is called a **mapping**, thus, under a mapping any element of A has exactly one image in B (not less than one, and not more than one).

A mapping such that every element of B is image of either one element of A, or of no element of A, is called a **one- to- one mapping**, or an **injective mapping** or simply an **injection;** under a one-to-one mapping no two elements of A share the common image in B.

Mathematically, $(\forall x_1\in A)(\forall x_2\in A);\mathrm{f}(\mathrm{x}_2)=f(x_1)\Rightarrow x_2=x_1$

A mapping such that every element of B is image of at least one element of A (image of one element of A, or image of more than one element of A), is called an **onto mapping**, or a **surjective mapping** or simply a **surjection**

Mathematically, $(\forall y\in B)(\exists x\in A);\mathrm{f(x)}=y$ or simply $B=f(A)$

A mapping satisfying properties of both one-to-one and onto is said to be a **bijective mapping**, or simply a bijection.

In particular, linear function $f:\mathbb{R}\to\mathbb{R}$

$x\mapsto f(x)=a\,\mathrm{x}+\mathrm{b}$, where $a\neq 0$, is bijective, there is no restrictions on the variables(independent or dependent).

Quadratic function $f:\mathbb{R}\to\mathbb{R}$

$x\mapsto f(x)=a\,\mathrm{x}^2+bx+c$, where $a\neq 0$, is not bijective, since some real numbers share images, or some real numbers are not images under function *f.*

But the following restrictions are bijective:

1) $f:[\frac{-b}{2a},+\infty[\to[-\frac{\Delta}{4a},+\infty[$

$x\mapsto f(x)=a\mathrm{x}^2+bx+c\,,\;\; a>0\,;$

2) $f:]-\infty,-\frac{b}{2a}]\to[-\frac{\Delta}{4a},+\infty[$

$x\mapsto f(x)=a\mathrm{x}^2+bx+c\,,\; a>0\,;$

3) $f:[\frac{-b}{2a},+\infty[\to]-\infty,-\frac{\Delta}{4a}]$

$x\mapsto f(x)=a\mathrm{x}^2+bx+c\,,a<0$ and

4) $f:]-\infty,-\frac{b}{2a}]\to]-\infty,\frac{\Delta}{4a}]$

$x\mapsto f(x)=a\mathrm{x}^2+bx+c\,,\; a<0\,.$

The homographic function

$f:\mathbb{R}\to\mathbb{R}$

$x\mapsto f(x)=\frac{ax+b}{cx+d}$ is not bijective, since it is not a mapping ($x=-\frac{d}{c}$ has no image under function *f*, or $y=\frac{a}{c}$ is not image under function *f*).

But the restriction

$$f:\mathbb{R}-\{-\frac{d}{c}\}\to\mathbb{R}-\{\frac{a}{c}\}$$
$$x\mapsto f(x)=\frac{ax+b}{cx+d}$$
is bijective

Examples:

1) Consider the set of pigeons and the set of pigeonholes on the diagram below to answer the questions:-

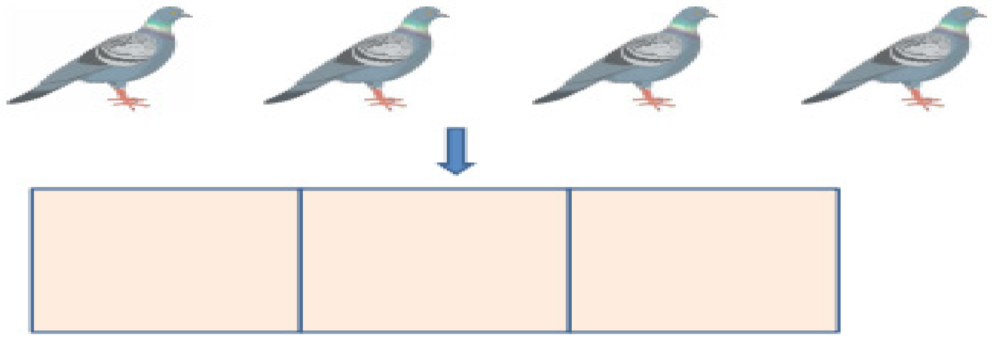
Determine whether it can be established or not between the two sets:

a. A mapping,

b. A one-to-one mapping,

c. An onto mapping,

d. A bijective mapping:



Solution:

Let the pigeons be numbered *a, b, c,* d and the pigeonholes be numbered *1,2,3.*

a. It is possible to establish a mapping between the two sets. For example, $\{(a,1);(b,2);(c,3);(d,3)\}$.This function is a mapping since each pigeon is accommodated in exactly one pigeonhole, though pigeons *c* and *d* are in the same pigeonhole.

b. It is not possible to establish a one-to-one mapping, since sharing images is not allowed. A function from one finite set to a smaller finite set cannot be one-to-one: there must be at least two elements that have the same image

c. The example given in part (a) illustrates a mapping that is onto: no pigeonhole is empty.

d. It is impossible to define a bijection, since it is already impossible to establish a one-to-one mapping

2) Determine whether function $f:\mathbb{R}\rightarrow\mathbb{R}$

$x \mapsto f(x)=3\text{x}-5$ is (or is not)

a. One-to-one
b. Onto
c. Bijective.

Solution:

a. Let x_1 and x_2 be real numbers such that $f(x_1)=f(x_2)$.Then $3x_1-5=3x_2-5$

This is equivalent, successively to $3x_1=3x_2$ (by adding 5 on both sides);

$x_1=x_2$ (Dividing both sides by 3)

Since the equality $f(x_1)=f(x_2)$ implies $x_1=x_2$, the function is one-to-one.

b. Suppose y a real number. Let us look for real number x, if possible, such that $f(x) = y$.

Then $3x - 5 = y$. It follows that $x = \frac{y+5}{3}$; such x exists for any value of y;

$f(\frac{y+5}{3}) = 3\left(\frac{y+5}{3}\right) - 5 = y$

Therefore, function f *is onto.*

c. Since, from points (a) and (b), f is one-to-one and onto, function f is bijective.

3) Show that function f defined by $f: \mathbb{R} \to \mathbb{R}$

$x \mapsto f(x) = x^2 + 2x - 3$ is neither one-to-one, nor onto

Solution

$f(-2) = (-2)^2 + 2(-2) - 3 = -3$ and $f(0) = (0)^2 + 2(0) - 3 = -3$

Since $f(-2) = f(0)$ and $-2 \neq 0$, the function is not one-to-one.

On the other side, there is no x such that $f(x) = -5$;

in fact, $f(x) = -5 \Leftrightarrow x^2 + 2x - 3 = -5$

$\Leftrightarrow x^2 + 2x + 2 = 0$

No such x since $\Delta = 2^2 - 4(1)(2) = -4 < 0$

Therefore, the function is not onto

4) Consider the function f defined by $f: \mathbb{R} \to \mathbb{R}$

$x \mapsto f(x) = -x^2 + 4x$

Determine the greatest subsets A and B of $\mathbb{R}$ such that function

$f: A \to B$

$x \mapsto f(x) = -x^2 + 4x$

is bijective

Solution:

The maximum value of the function occurs for $x = 2$ and $f(2) = 4$. Therefore,

$A = [2, +\infty[$ and $B =]-\infty, 4]$, or $A =]-\infty, 2]$ and $B =]-\infty, 4]$

5) Function $f:\mathbb{R}-\{a\}\to\mathbb{R}-\{b\}$

$$x\mapsto f(x)=\frac{2x-5}{3-x} \text{ is bijective}$$

a. Find the values of a and b

b. Show that f is one-to-one

c. Find the real number whose image is 2

Solution:

a. f is bijective if $a=3$ and $b=-2$

b. Let $x_1\neq 3$ and $x_2\neq 3$ be such that $f(x_1)=f(x_2)$, that is $\frac{2x_1-5}{3-x_1}=\frac{2x_2-5}{3-x_2}$

Then $6x_1-2x_1x_2-15+5x_2=6x_2-2x_1x_2-15+5x_1$,

which is equivalent to $6(x_1-x_2)-5(x_1-x_2)=0\Leftrightarrow x_1-x_2=0\Leftrightarrow x_1=x_2$

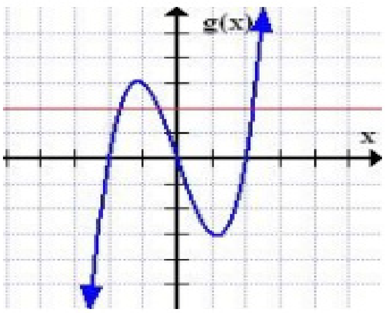
Therefore, f is one- to- one

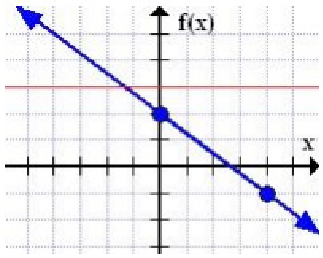
c. Let x be the number. Then $f(x)=2\Leftrightarrow\frac{2x-5}{3-x}=2$

Solving this equation, we get $x=2$

Horizontal Line Test

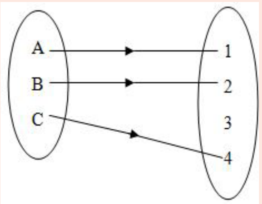
Horizontal Line Test states that a function is a one to one(injective) function if there is no horizontal line that intersects the graph of the function at more than one point.

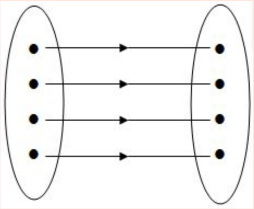
Graph representation	Interpretation	Conclusion
	You can see that for this graph, there are horizontal lines that intersect the graph more than once.	Not injective

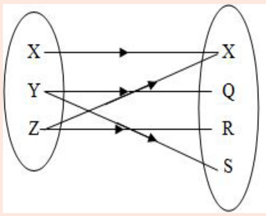
	You can see that for this graph any horizontal line intersects the graph only once.	Injective function

APPLICATION ACTIVITY 1.1

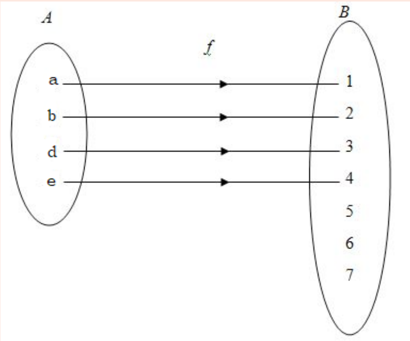
1. State which of the following relations shows a function

(a) 

(b) 

(c) 

2. In the following arrow diagram, state the domain, co-domain and range



3. If $f(x)=2x+4$, find
   a. $f(2)$
   b. $f(-2)$
   c. $f(d)$
   d. The value of a if $f(a)=a$

4. Let W and Z be sets;

$W \times Z$ the Cartesian product of W and Z ;

C be the set of all correspondences (relations) from W to Z ;

F be the set of all functions from W to Z ;

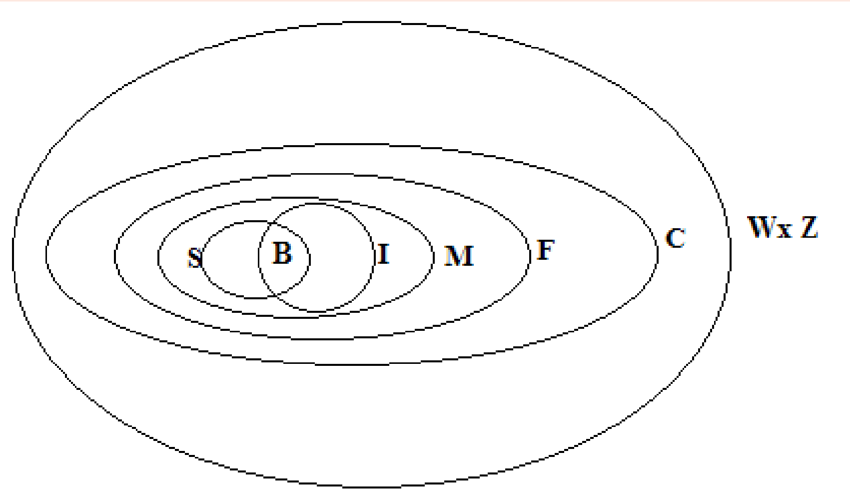
M be the set of all mappings from W to Z ;

I be the set of all one to one mappings from W to Z ;

S be the set of all onto mappings from W to Z ;

B be the set of all bijective mappings from W to Z ;

Then we have the following sequence of inclusion of sets



Using examples, explain in your own words the relationship amongst these sets and decide on the following inclusion: $S \subset M \subset F \subset C \subset (W \times Z)$

1.2 Types of numerical functions

ACTIVITY 1.2

Differentiate rational from irrational numbers. Guess which of the following functions is a polynomial, rational or irrational function

1. $f(x)=(x+1)^2$
2. $h(x)=\dfrac{x^3+2x+1}{x-4}$
3. $f(x)=\sqrt{x^2+x-2}$

a) Constant function

A function that assigns the same value to every member of its domain is called **a constant function**. This is $f(x)=c$ where c is a given real number.

Example: $f(x)=4$

The function f given by $f(x)=3$ *is constant.*

Remark

The constant function that assigns the value *c* to each real number is sometimes called **the constant function *c***.

b) Identity: The identity function is of the form $f(x)=x$

c) Monomial

A function of the form cx^n, where *c* is constant and *n* a nonnegative integer is called a **monomial in *x***.

Examples

1. $f(x)=0.3x$ is a monomial in x.
2. $2x^3$; πx^7; $4x^0$; $-6x$ *and* x^{17} are also monomials

The functions $4x^{\frac{1}{2}}$ and x^{-3} are not monomials because the powers of *x* are not nonnegative integers.

d) Polynomial

A function that is expressible as the sum of finitely many monomials in x is called

Polynomial in *x*.

Examples:

1. x^3+4x+7 and $17-\frac{2}{3}x$ are polynomials.
2. $(x-2)^3$ is a polynomial in x because it is expressible as a sum of monomials.

In general, f is a polynomial in x if it is expressible in the form

$f(x)=a_0+a_1x+a_2x^2+......+a_nx^n$ where n is a nonnegative integer and

$a_0, a_1, ..., a_n$ are real constants.

A polynomial is called

- **Linear** if it has the form $a_0+a_1x,\ a_1\neq 0$, with degree 1.
- **Quadratic** if it has the form $a_0+a_1x+a_2x^2,\ a_2\neq 0$, with degree 2
- **Cubic** if it has the form $a_0+a_1x+a_2x^2+a_3x^3,\ a_3\neq 0$, with degree 3
- **n^th^ degree polynomial** if it has the form $a_0+a_1x^1+a_2x^2+a_3x^n+...+a_nx^n; a_n\neq 0$, with degree n.

e) Rational function

A function that is expressible as ratio of two polynomials is called **rational function**. It has the form $f(x)=\dfrac{a_0+a_1x+a_2x^2+......+a_nx^n}{b_0+b_1x+b_2x^2+......+b_mx^m}$.

Example:

$f(x)=\dfrac{x^2+4}{x-1}$ *and* $g(x)=\dfrac{1}{3x-5}$ are rational functions

f) Irrational function

A function that is expressed as root extractions is called irrational function.

It has the form $\sqrt[n]{f(x)}$, where $f(x)$ is a polynomial or rational function and n is positive integer greater or equal to 2.

Example: $f(x)=\frac{\sqrt{x^2+4}}{\sqrt[3]{x-1}}$, $g(x)=\sqrt{\frac{x}{x-5}}$ are irrational functions.

APPLICATION ACTIVITY 1.2

What is the type of the following function?

1. $f(x)=x^3+2x^2-2$
2. $g(x)=-2$
3. $h(x)=\frac{x^3+2x^2-2}{x-5}$
4. $f(x)=\frac{x^2-2}{x^2-8x+15}$
5. Provide other types of functions and explain your reasons with examples

1.3 Domain of definition for a numerical function

ACTIVITY 1.3.1

For which value(s) the following functions are not defined

1. $f(x)=x^3+2x+1$
2. $f(x)=\frac{1}{x}$
3. $g(x)=\frac{x+2}{x-1}$

ACTIVITY 1.3.2

Find the domain of definition for each of the following functions

1. $f(x)=x^3+2x^2-2$
2. $g(x)=-2$
3. $h(x)=\frac{x^3+2x^2-2}{x-5}$
4. $f(x)=\frac{x^2-2}{x^2-8x+15}$

ACTIVITY 1.3.3

For each of the following functions, give a range of values of the variable x for which the function is not defined

1. $f(x)=\sqrt{2x+1}$
2. $f(x)=\sqrt[3]{x^2+x-2}$
3. $g(x)=\sqrt{\frac{x-2}{x+1}}$

Content summary

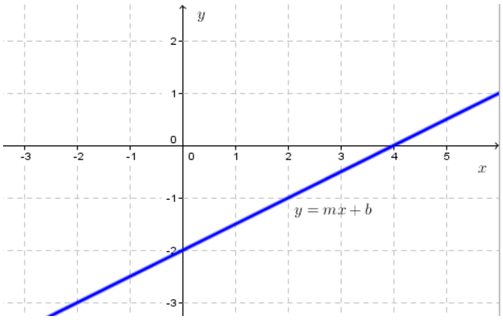
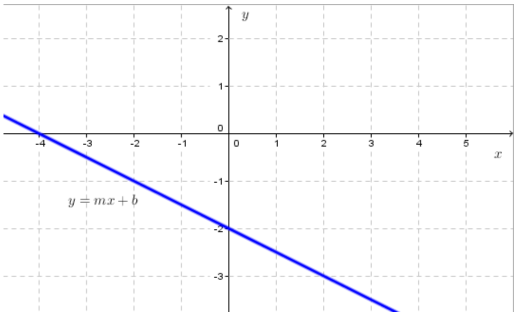
(1) Case of polynomial functions

The domain of any **linear function** f is the set of all real numbers, that is

Domf= $\mathbb{R}$ or Dom f $=]-\infty,+\infty[$.

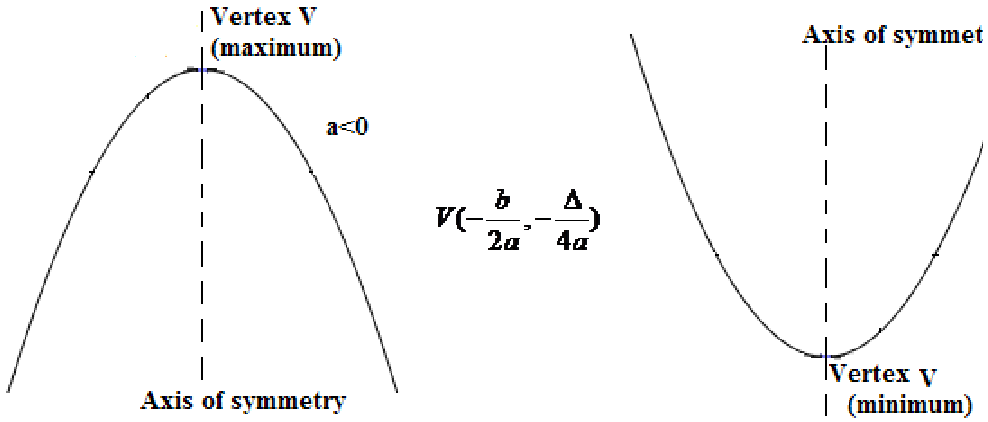
Similarly, the **range** (the set of all values $f(x)$ for all $x \in Domf$) of a linear function f, denoted $\operatorname{Im} f$ is the set of all real numbers, that is $\operatorname{Im} f=]-\infty,+\infty[$.

Depending on the sign of *m in the equation* $y = mx + b$, the trend of the graph is as follows:

$m > 0$	$m < 0$
	

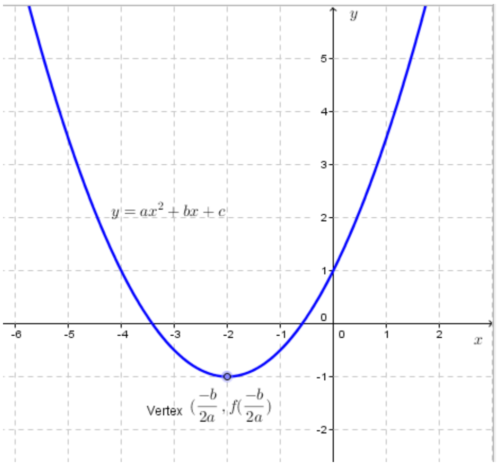
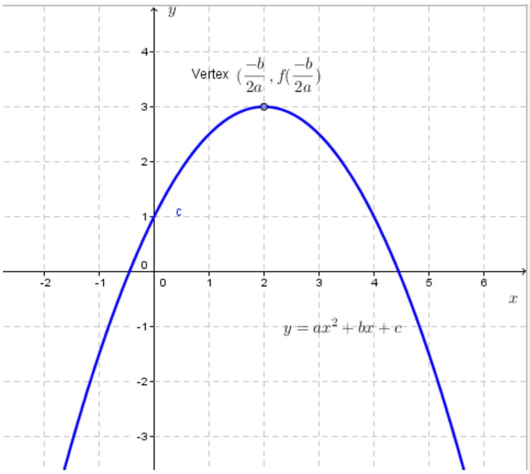
From the graphs, one can observe that each value of x has its corresponding y value.

For quadratic functions $y = ax^2 + bx + c$, the main features are summarized on the graph below:



If $a > 0$, then the range of function $y = ax^2 + bx + c$ is $[-\frac{\Delta}{4a}, +\infty[$

If $a < 0$, then the range of function $y = ax^2 + bx + c$ is $]-\infty, -\frac{\Delta}{4a}]$

<table>
<tr><th>$a > 0$, minimum</th><th>$a < 0$, maximum</th></tr>
<tr><td>
</td><td>
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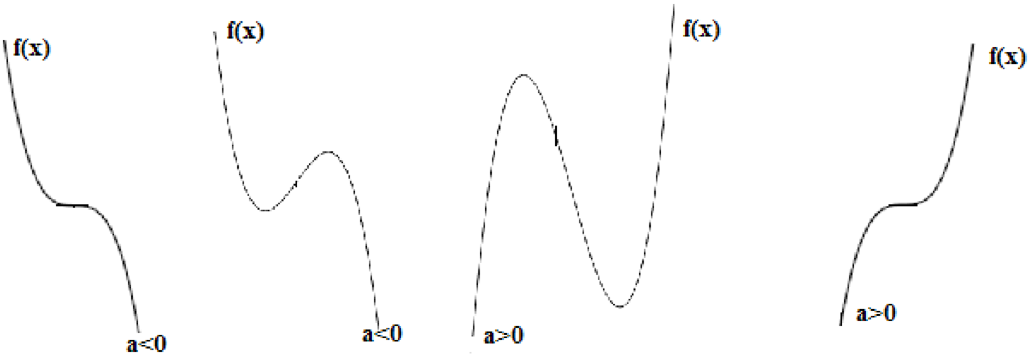
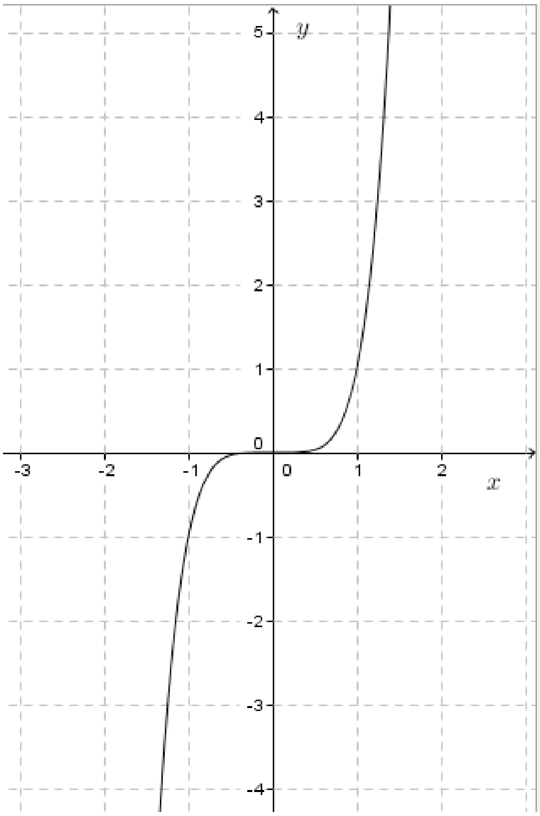
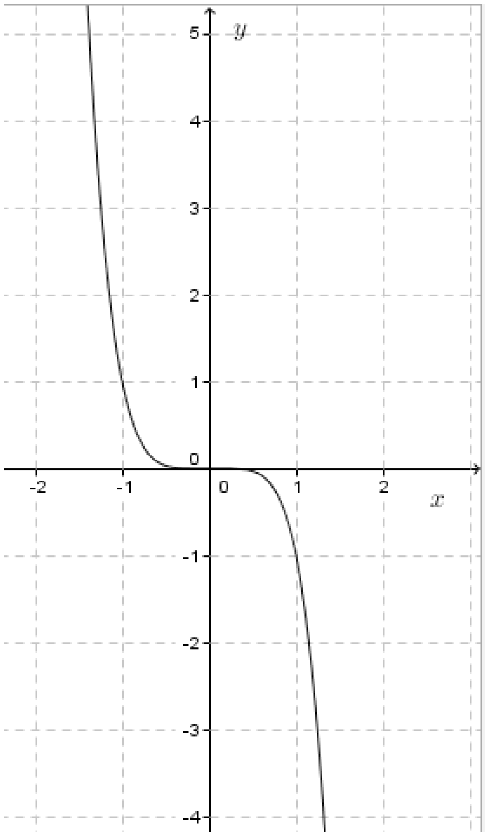
</table>

For cubic functions $f(x) = ax^3 + bx^2 + cx + d; a \neq 0$,the trends of the graphs are as shown below:



In each case, the domain is $]-\infty,+\infty[$ and the range is $]-\infty,+\infty[$

$a>0$	$a<0$
	

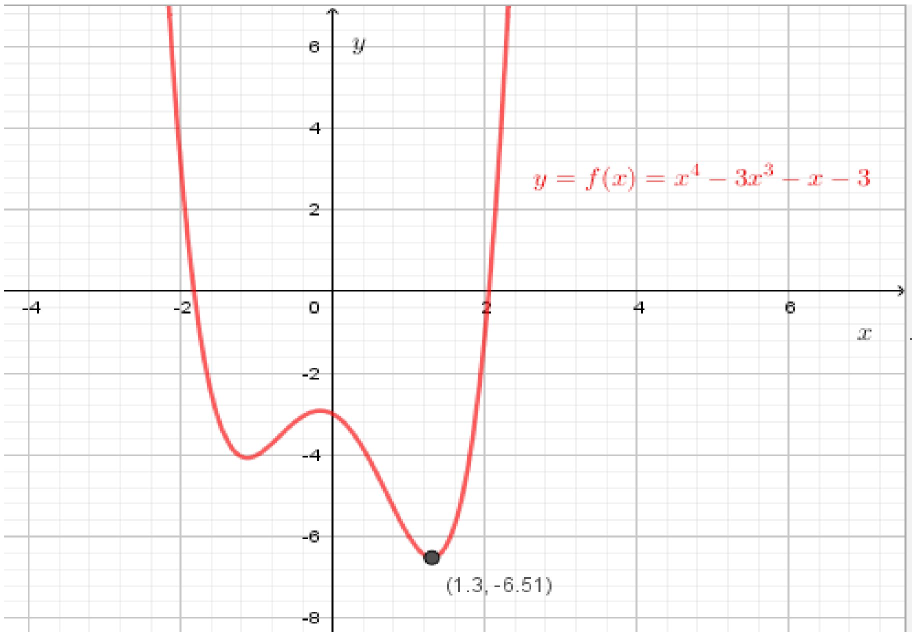
$D =]-\infty,\ +\infty[$ and $\mathbb{R} =]-\infty,\ +\infty[$.

Even though the range for polynomials of odd degrees is the set of all real numbers, it is not the case for polynomials of even degree greater or equal to 4.

The determination of the range is not easy unless the function is given by its graph; in this case, find by inspection, on the *y*-axis, the set of all points such that the horizontal lines through those points cut the graph.

Example

1) Determine the domain and range of $f(x)=x^4-3x^3-x-3$ shown on the graph below:



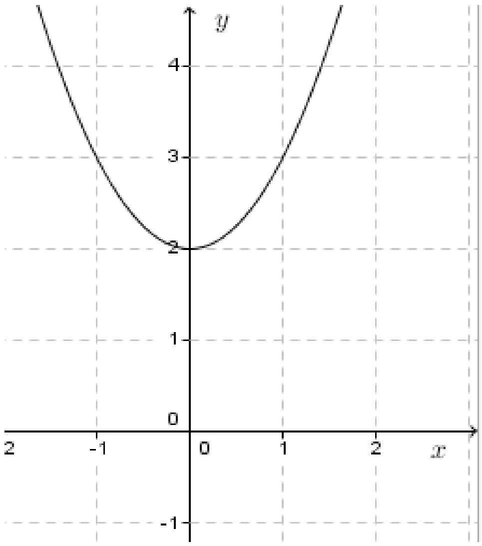
Solution:

$dom f =]-\infty, +\infty[$ and $\text{Im} f = [-6.51, +\infty[$

2) Find the range for the function $f(x) = x^2 + 2$

Solution:

Let $f : \mathbb{R} \to \mathbb{R}$ be the function given by $f(x) = x^2 + 2$. Then the range is $[2, +\infty[$ as shown on the graph below:



(2) Case for rational functions

Given that $f(x)=\dfrac{g(x)}{h(x)}$ where $g(x)$ and $h(x)$ are polynomials, then the domain of definition is the set of real numbers excluding all values where the denominator is zero. That is $Domf=\{x\in\mathbb{R}:h(x)\neq 0\}$

Example

1) Find the domain of each of the functions:

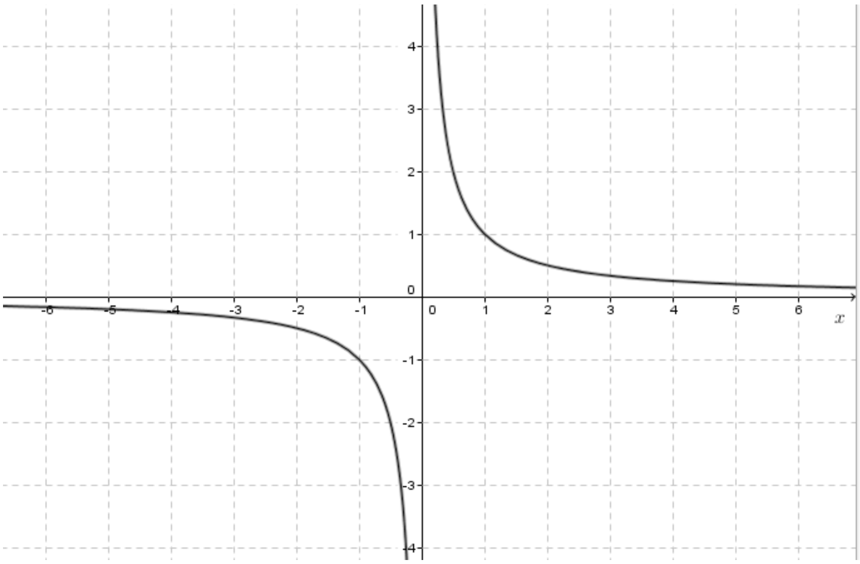
a. $f(x)=\dfrac{1}{x}$

b. $f(x)=\dfrac{x}{(x-1)(x+3)}$

c. $f(x)=\dfrac{x+1}{3x+6}$

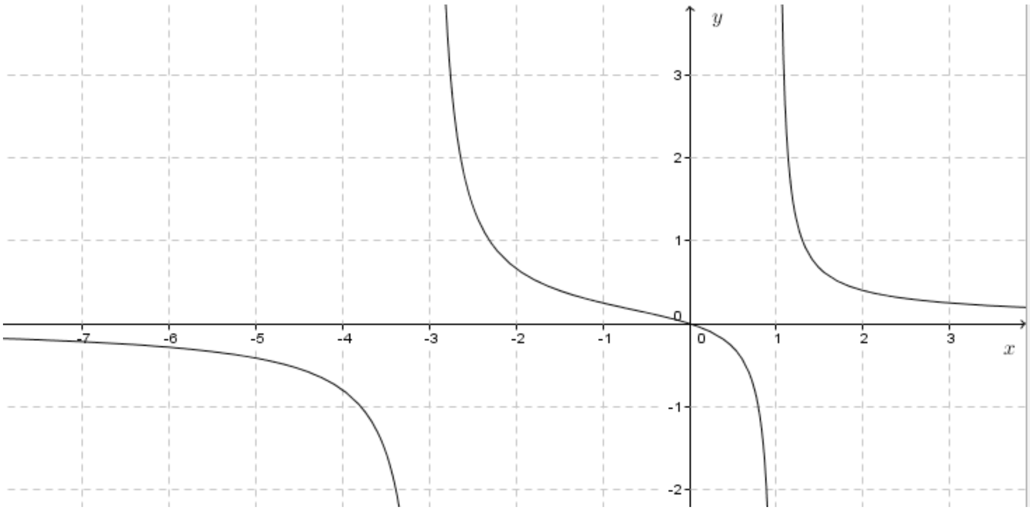
Solution

a) The denominator should be different from zero ($x\neq 0$), domain of definition is $\{x\in\mathbb{R}:x\neq 0\}$ or $\mathbb{R}^{*} or\ \mathbb{R}^{+}$. The domain can be written as an interval as follows: $]-\infty,0[\cup]0,+\infty[$. Observing the graph of the function $f(x)=\dfrac{1}{x}$, one can easily realize that the function has no value only if $x=0$



b) $f(x)=\dfrac{x}{(x-1)(x+3)}$. The denominator should be different from zero, $(x-1)(x+3)\neq 0$. domain of definition is $\{x\in\mathbb{R}: x\neq 1 \text{ and } x\neq -3\}$ or simply $]-\infty,-3[\cup]-3,1[\cup]1,+\infty[$.

Observing the graph of the function $f(x)=\dfrac{x}{(x-1)(x+3)}$, one can early realize that the function has no value only if $x=1$ *and* $x=-3$



c) Condition: $3x+6\neq 0$

$3x+6=0 \Rightarrow x=-2$

Then, $Domf=\mathbb{R}\setminus\{-2\}$ or $Domf=]-\infty-2[\cup]-2,+\infty[$

2) Find the range for the function $f(x)=\dfrac{1}{x-2}$

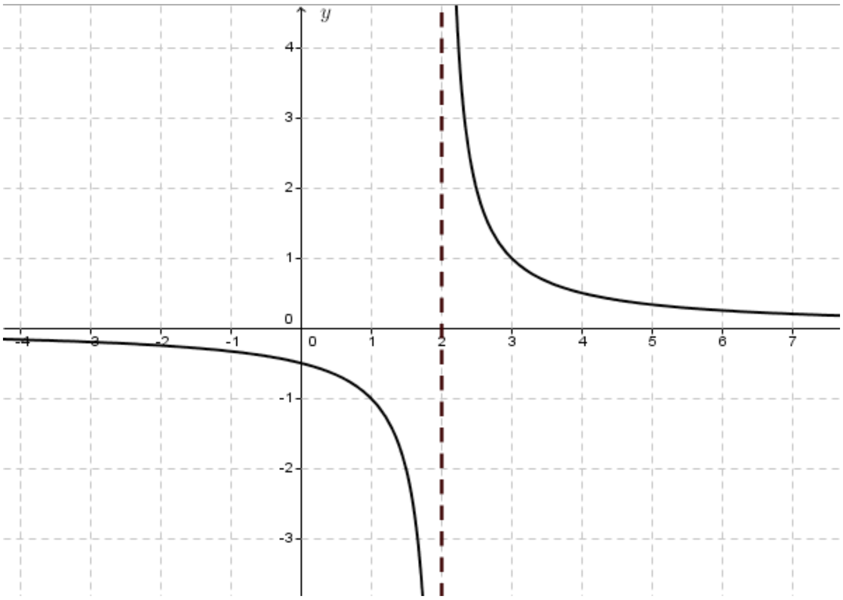
Solution

1) Put $y=f(x)=\dfrac{1}{x-2}$

2) Solve for x, $y=\dfrac{1}{x-2}\Leftrightarrow x=\dfrac{1}{y}+2$. Note that x can be solved if and only if $y\neq 0$.

The range of $f(x)$ is $\{y \in \mathbb{R} : y \neq 0\} = \mathbb{R} \setminus \{0\}$.

Alternatively, one can see on the graph that the range of $f(x)$ is $\mathbb{R} \setminus \{0\}$.



3) Find the range for the function $f(x) = \dfrac{2x+1}{x^2+2}$

Solution

(1) Put $\quad y = f(x) = \dfrac{2x+1}{x^2+1}$

(2) Solve for x, $y = \dfrac{2x+1}{x^2+1} \Leftrightarrow yx^2 + y = 2x+1$.

$yx^2 + y = 2x+1 \Leftrightarrow yx^2 - 2x + (y-1) = 0$

$$x = \frac{2 \pm \sqrt{4-4y(y-1)}}{2y} \text{ if } y \neq 0, \quad x = -\frac{1}{2} \text{ if } y = 0$$

$$= \frac{1 \pm \sqrt{1-y^2+y}}{y} \text{ if } y \neq 0$$

Comparing the two, we see that x exists in the set of real numbers if and only if

$1 - y^2 + y \geq 0$, that is $y^2 - y - 1 \leq 0$

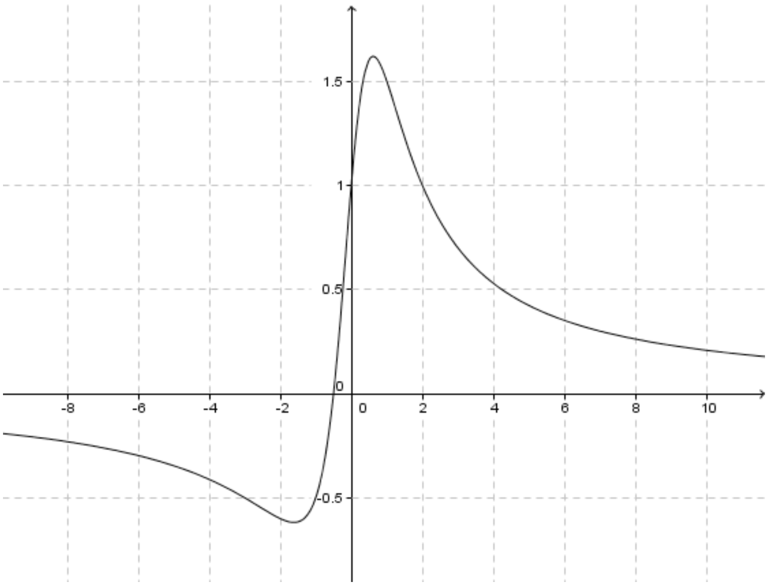
The range of $f(x)$ is $\{y \in \mathbb{R} : y^2 - y - 1 \leq 0\}$. Solving the inequality $y^2 - y - 1 \leq 0$, we get $y=\frac{1\pm\sqrt{5}}{2}$. Then studying the sign of the quadratic expression

$, y^2-y-1=\left(y-\frac{1-\sqrt{5}}{2}\right)\left(y-\frac{1+\sqrt{5}}{2}\right)$

	$y<\frac{1-\sqrt{5}}{2}$	$y=\frac{1-\sqrt{5}}{2}$	$\frac{1-\sqrt{5}}{2}<y<\frac{1+\sqrt{5}}{2}$	$y=\frac{1+\sqrt{5}}{2}$	$y>\frac{1+\sqrt{5}}{2}$
$y-\frac{1-\sqrt{5}}{2}$	- - - - - -	0	+ + ++	+ + ++ +	+ + +
$y-\frac{1+\sqrt{5}}{2}$	- - - - -	- - -	- - - - - -	0	+ + +
y^2-y-1	+ +	0	- - -	0	+ + + +

From the table, we see that the range of the function $f(x)=\frac{2x+1}{x^2+2}$ is:

$R(f)=\left\{y\in\mathbb{R}:\frac{1-\sqrt{5}}{2}\leq y\leq\frac{1+\sqrt{5}}{2}\right\}=\left[\frac{1-\sqrt{5}}{2},\frac{1+\sqrt{5}}{2}\right]$ One can see on the graph that the range of $f(x)$ is $\left[\frac{1-\sqrt{5}}{2},\frac{1+\sqrt{5}}{2}\right]\simeq[-0.618034;1.61803]$



(3) Case for irrational functions

Given that $f(x)=\sqrt[n]{g(x)}$ where $g(x)$ is a polynomial, there are two cases

a) If n is odd number, then the domain is the set of real numbers. That is $domf=\mathbb{R}$

b) If n is even number, then the domain is the set of all values of x such $g(x)$ is positive or zero. That is $domf=\{x\in\mathbb{R}:g(x)\geq 0\}$

Examples:

1) Given the function $g(x)=\sqrt{x^2-1}$, determine the domain.

Solution

Condition of existence: $x^2\text{-}1\geq 0$ this implies that we need to determine the interval where $x^2\text{-}1$ is positive.

The corresponding equation is $x^2\text{-}1=0$, solving for the variable, we obtain:

$x^2\text{-}1=0 \Leftrightarrow (x-1)(x+1)=0$

$$x-1=0 \Rightarrow x=1$$

$$x+1=0 \Rightarrow x=-1$$

x	$-\infty$ -1 1 $+\infty$
$x+1$	- - 0 + + + + + + +
$x-1$	- - - - - - 0 + + + + + + + +
x^2-1	+ + + 0 - - - - 0 + + + + + + +

Therefore $Domf=]-\infty,-1]\cup[1,+\infty[$

2) Find the domain of the function $h(x)=\dfrac{\sqrt{1-x^2}}{x}$

Solution

Conditions of existence $1-x^2\geq 0$ and $x\neq 0$

- For the corresponding equation is $1-x^2=0$ and solving for the variable, we get: $(1-x)(1+x)=0 \Rightarrow \begin{cases}1-x=0 \text{ or } x=1\\ 1+x=0 \text{ or } x=-1\end{cases}$
- For $x\neq 0$ all real numbers are accepted except from zero. Combining the two conditions we get:

x	$-\infty$ -1 0 1 $+\infty$
$1+x$	- - - 0 + + + + + + + +
$1-x$	+ + + + + + + + + + 0 - - - - -
$1-x^2$	- - - 0 + + + + + + 0 - - - - -
x	- - - - - - - 0 + + + + + + + + +
$f(x)=\dfrac{\sqrt{1-x^2}}{x}$	undefined 0 - - - || + + + + + 0 undefined

$-1\leq x\leq 1$ and $x\neq 0$. Therefore, $domf=[-1,0[\cup]0,1]$

3) Find domain of definition of $f(x)=\sqrt[3]{x+1}$

Solution

Since the index in radical sign is odd number, then $Domf = \mathbb{R}$

4) Find the domain of definition of $g(x)$ if $g(x) = \sqrt[4]{x^2 + 1}$

Solution

Condition: $x^2 + 1 \geq 0$

Clearly $x^2 + 1$ is always positive. Thus $Domg = \mathbb{R}$

5) Find domain of $f(x) = \dfrac{x}{\sqrt{x^3 - 4x^2 + x + 6}}$

Solution

Conditions: $x^3 - 4x^2 + x + 6 \geq 0$ and $x^3 - 4x^2 + x + 6 \neq 0$.

The two conditions are combined in one: $x^3 - 4x^2 + x + 6 \geq 0$ and $x^3 - 4x^2 + x + 6 \neq 0$

$x^3 - 4x^2 + x + 6 = (x+1)(x-2)(x-3)$

x	$-\infty$		-1		2		3		$+\infty$
$x+1$			0	+		+ +		+ +	
$x-2$		- -			0	+	+	+	
$x-3$		- -		-	-		0	+	
$x^3 - 4x^2 + x + 6$			0	+	0	-	0	+	

Then, $Domf =]-1, 2[\cup]3, +\infty[$

6) Find the range for the function $f(x) = \sqrt{1 + 5x}$

Solution

$1 + 5x \geq 0$ (Restrictions on x);

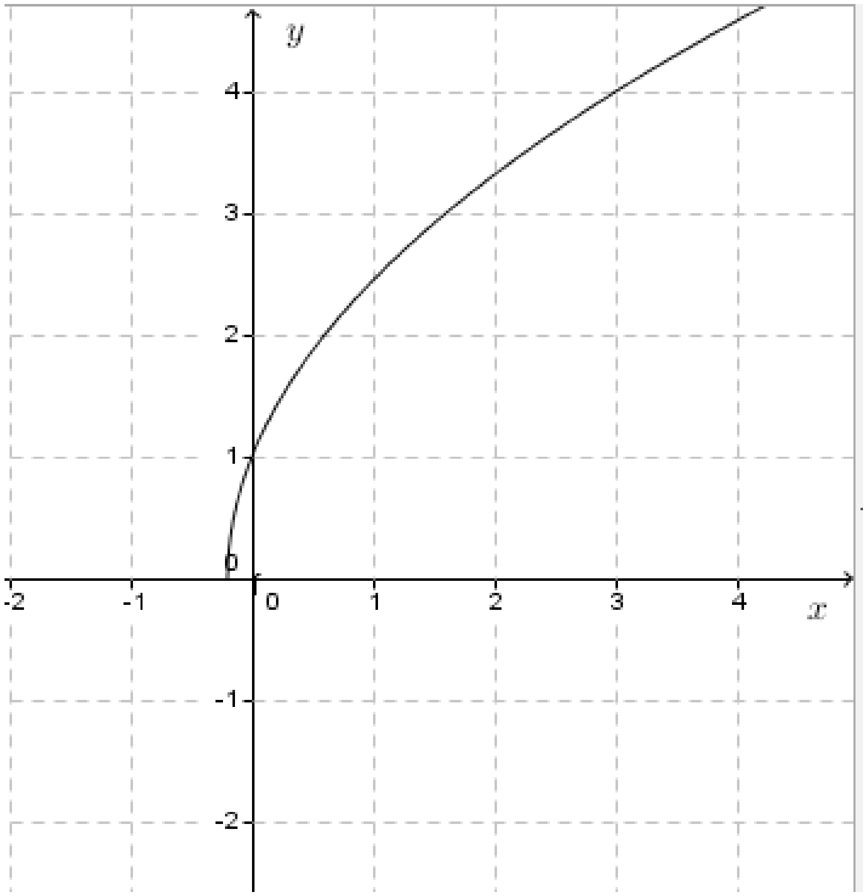
$\sqrt{1 + 5x} \geq 0$ (Taking the square roots);

But $f(x) = \sqrt{1 + 5x}$;

Therefore, $f(x) \geq 0$

The range of $f(x)$ is $\operatorname{Im} f = [0, +\infty[$.

The graph below illustrates the range:



APPLICATION ACTIVITY 1.3

Find the domain of definition for each of the following functions

1. $f(x)=\sqrt{4x-8}$
2. $g(x)=\sqrt{x^2+5x-6}$
3. $h(x)=\dfrac{x^3+2x^2-2}{\sqrt[3]{x+4}}$
4. $f(x)=\dfrac{x-2}{\sqrt[4]{x^2-25}}$
5. $f(x)=\sqrt{\dfrac{(x-1)^2}{x+4}}$

1.4 Parity of a function (odd or even)

ACTIVITY 1.4

For each of the following functions, find $f(-x)$ *and* $-f(x)$. *Compare* $f(-x)$ *and* $-f(x)$ *using* $=$ *or* $\neq$

1. $f(x)=x^2+3$ 2. $f(x)=\sqrt[3]{x^3+x}$ 3. $f(x)=\dfrac{x^2-3}{x^2+1}$

Even function

Let $f(x)$ be a numerical function whose domain is $Domf$.

$f(x)$ is said to be an **even function** if and only if:

i) $(\forall x \in Domf), -x \in Domf$

ii) $f(-x)=f(x)$, that is, any two opposite values of the independent variable have the same image under the function.

The graph of an even function is symmetrical about the y-axis.

Examples

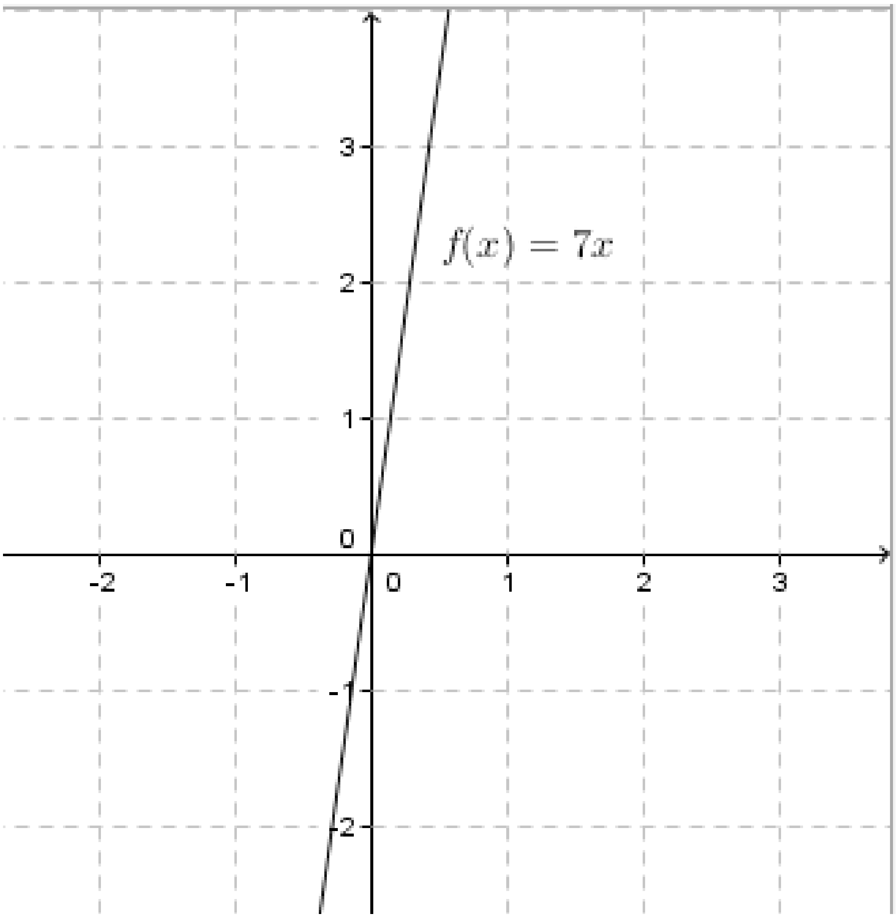
1) Determine whether the function $f(\mathrm{x})=7\mathrm{x}$ is even or not.

Solution

The domain of function f is the set of all real numbers. For any real number x, the opposite –x is also a real number and $f(-x)=-7x\neq 7x=f(x)$.

Since $f(-x)\neq f(x)$, function f is not even.

Graphically,-



The graph is not symmetrical about the *y*-axis

2) Determine whether the function $f(x) = 3x^2 - 4$ is even or not.

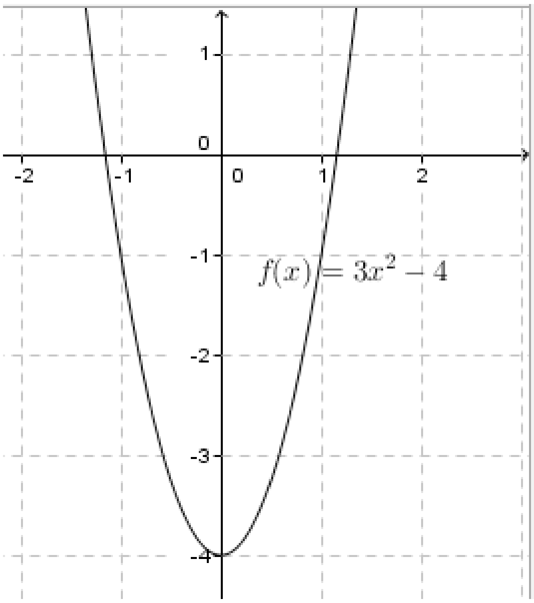
Solution

$f(-x) = 3(-x)^2 - 4 = 3x^2 - 4 = f(x)$

Since $f(-x) = f(x)$, function f is even.

Remember that $(-x)^n = \begin{cases} x^n, if \text{ n is even} \\ -x^n, if \text{ n is odd} \end{cases}$

The graph of the function is symmetrical about the y-axis as shown on the diagram below:



3) Determine whether the function $g(x) = x^6 - x^4 + x^2 + 9$ is even or not.

Solution

$$g(-x) = (-x)^6 - (-x)^4 + (-x)^2 + 9$$
$$= x^6 - x^4 + x^2 + 9$$
$$= g(x)$$

Therefore, the function is even.

4) Determine whether the function $f(x) = \dfrac{3x+1}{x^2-25}$ is even or not.

Solution

$f(-x) = \dfrac{3(-x)+1}{(-x)^2-25} \Leftrightarrow f(-x) = \dfrac{-3x+1}{x^2-25}$. Therefore the function is not even since

$f(-x) \neq f(x)$

5) Given functions $f(x) = \sqrt{x}$ and $g(x) = \sqrt{x-4}$, find the $f(x).g(x)$ and determine if the result is an even function or not.

Solution

Provided $x \geq 0$ and $x - 4 \geq 0$,

$$f(x).g(x) = \sqrt{x} \,.\, \sqrt{x-4} \Leftrightarrow f(x).g(x) = \sqrt{x^2 - 4x}$$

$f.g(-x)=\sqrt{(-x)^2-4(-x)} \Leftrightarrow f.g(-x)=\sqrt{x^2+4x}$. Therefore the function is not even since $f(\text{-}x)\neq f(x)$

Notice that the conclusion could have been drawn from the fact that $x\geq 4$ does not imply $-x\geq 4$, thus function f is not even

Odd function

Let $f(x)$ be a numerical function whose domain is $Domf$.

$f(x)$ is said to be an **odd function** if and only if:

i. $(\forall x\in Domf), -x\in Domf$

ii. $f(-x)=-f(x)$, that is, any two opposite values of the independent variable have opposite images under the function.

The graph of an even function is symmetrical about the *origin*.

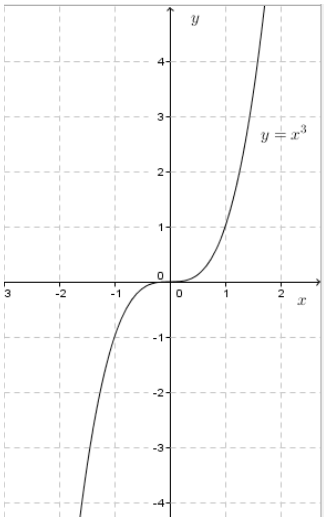
Note: Some functions are neither even nor odd

1) Determine whether the function $f\left(x\right)=x^3$ is odd or not

Solution

$f(-x)=(-\text{x})^3 \Leftrightarrow f(-x)=-x^3$ *and* $-f(x)=-x^3$. Therefore, $f(-x)=-f(x)$ and the function $f(x)=x^3$ is odd.

Graphically, the point $(0,0)$ is the center of symmetry for the graph of the function $f(x)=x^3$.



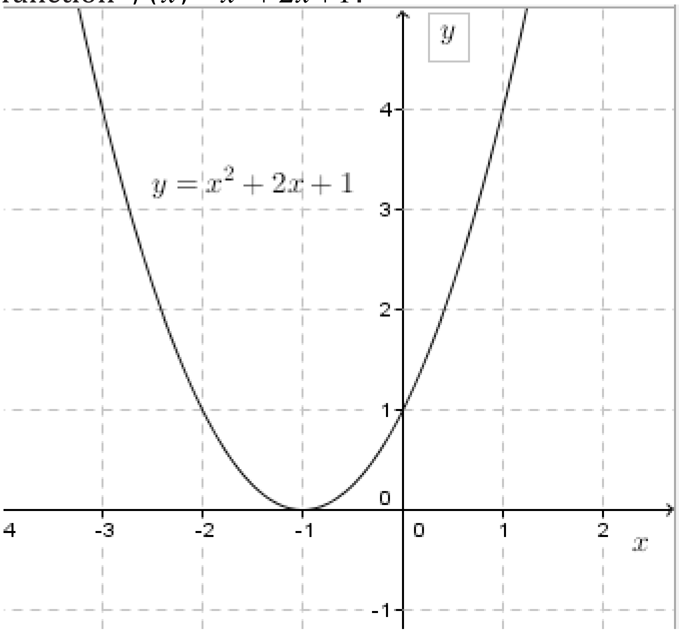
3) Determine whether the function $f(x)=x^2+2x+1$ is odd ,even or neither

Solution

$f(-x)=(-x)^2+2(-x)+1 \Leftrightarrow f(-x)=x^2-2x+1$ *and* $-f(x)=-x^2-2x-1$.

$f(-x)\neq -f(x)$ and $f(-x)\neq f(x)$. Therefore, the function $f(x)=x^2+2x+1$ is not odd neither even.

Graphically, point $(0,0)$ is not the centre of symmetry for the graph of the function $f(x)$,and the line $x=0$ is not the axis of symmetry for the graph of function $f(x)=x^2+2x+1$.



3) Determine if the function $f(x)=\dfrac{x^3}{x^2-1}$ is even , odd ,or neither and deduce the symmetry of its graph.

Solution

$$f(-x)=\frac{(-x)^3}{(-x)^2-1}=\frac{-x^3}{x^2-1}$$

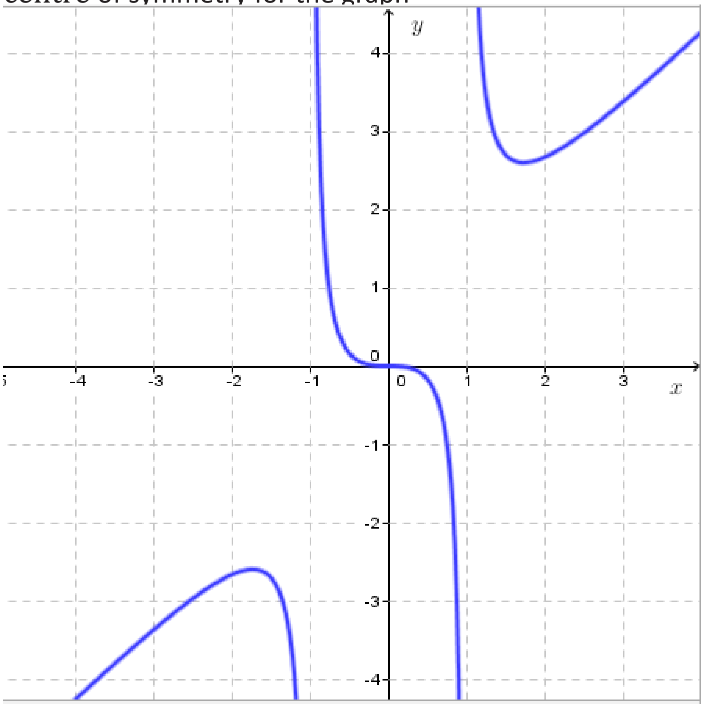
$$\neq f(x)$$

Therefore, the function is not even

But, $f(-x)=-f(x)$;it follows that f is an odd function.

The graph of $f(x)=\dfrac{x^3}{x^2-1}$ is shown below. It can be seen that point $(0,0)$ is the

centre of symmetry for the graph



APPLICATION ACTIVITY 1.4

Study the parity of the following functions

1. $f(x)=2x^2+2x-3$
2. $f(x)=\dfrac{3x^3+2x^2+8}{x-5}$
3. $g(x)=x^3-x$
4. $h(x)=\dfrac{x^2+4}{x^2-4}$
5. $g(x)=x(x^2+x)$

1.5 Operations on functions

1.5.1 Addition, subtraction, multiplication and division

LEARNING ACTIVITY 1.5.1

Given the functions $f(x)=\frac{x+1}{2x-3}$ and $g(x)=x+1$, find

1. $f(x)+g(x)$

2. $f(x)-g(x)$

3. $f(x)\cdot g(x)$

4. $\frac{f(x)}{g(x)}$

Just as numbers can be added, subtract, multiplied and divided to produce other numbers, there is a useful way of adding, subtracting, multiplying and dividing functions to produce other functions. These operations are defined as follows:

Given functions f and g, **sum** $f+g$, **difference** $f-g$, **product** $f\cdot g$ and **quotient** $\frac{f}{g}$, are defined by

- $(f+g)(x)=f(x)+g(x)$
- $(f-g)(x)=f(x)-g(x)$
- $(f\cdot g)(x)=f(x)\cdot g(x)$
- $\left(\frac{f}{g}\right)(x)=\frac{f(x)}{g(x)}$

For the functions, $f+g$, $f-g$ and $f\cdot g$, the domain is defined to be the intersection of the domains of *f* and g and for $\frac{f}{g}$, as we have seen it, the domain is this intersection with **the points where** $g(x)=0$ **excluded.**

Example:

1) Let f and g be the functions $f(x)=3x^4-5x^3+x-4$ and $g(x)=4x^3-3x^2+4x+3$. Find $(f+g)(x)$ and $(f-g)(x)$

Solution

$$\begin{array}{rrrrrr} & f(x)=3x^4 & -5x^3 & & +x & -4 \\ + & g(x)= & 4x^3 & -3x^2 & +4x & +3 \\ \hline (f+g)(x)= & 3x^4 & -x^3 & -3x^2 & +5x & -1 \end{array}$$

$$\begin{array}{rrrrrr} & f(x)=3x^4 & -5x^3 & & +x & -4 \\ - & g(x)= & 4x^3 & -3x^2 & +4x & +3 \\ \hline (f-g)(x)= & 3x^4 & -9x^3 & +3x^2 & -3x & -7 \end{array}$$

2) If $f(x)=\dfrac{9}{x+2}$ and $g(x)=x^3$. Find

a. $h(x)=f(x)+g(x)$

b. $t(x)=f(x)\times g(x)$

c. $k(x)=\dfrac{f(x)}{g(x)}$

Solution

$h(x)=f(x)+g(x)$ $=\dfrac{9}{x+2}+x^3$ $=\dfrac{9+x^3(x+2)}{x+2}$ $=\dfrac{x^4+2x^3+9}{x+2}$	$t(x)=f(x)\times g(x)$ $=\dfrac{9}{x+2}\cdot x^3$ $=\dfrac{9x^3}{x+2}$	$k(x)=\dfrac{f(x)}{g(x)}$ $=\dfrac{\frac{9}{x+2}}{x^3}$ $=\dfrac{9}{x+2}(\dfrac{1}{x^3})$ $=\dfrac{9}{x^2+2x^3}$

APPLICATION ACTIVITY 1.5.1

1. Given the functions $f(x)=2x^3+5x-1$ and $g(x)=3x-4$. Find $(f+g)(x)$
2. Given the functions $f(x)=3x^3-5x^2+7x-4$ and $g(x)=2x^2-x+3$. Find $(f\cdot g)(x)$

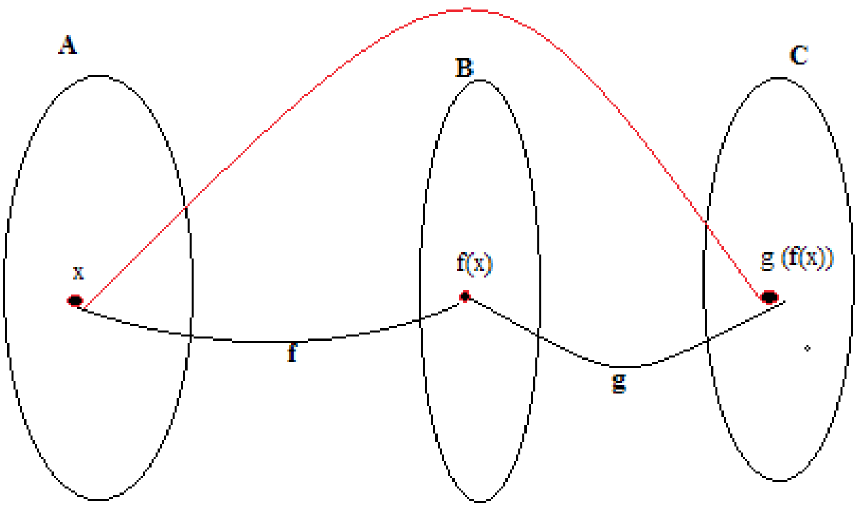
1.5.2 Composite functions

ACTIVITY 1.5.2

Consider two functions $f(x)=3x+2$ and $g(x)=x^2-1$:

1. Find the expressions $f(y)$ and $f(g(x))$
2. Find $g(f(x))$
3. Compare $f(g(x))$ and $g(f(x))$

Consider the functions $f(x)=2x-1$ and $g(x)=x^2$.



$g_o f(x) = g(f(x))$

This combined or composite function is written $(g\circ f)(x)$ or $g[f(x)]$, simply gf. The function f is performed first and so is written nearer to the variable x. The set $\{1,3,5,7\}$ is the domain for the composite function and $\{1,25,81,169\}$ is the range.

Note that $(f\circ g)(x)\neq(g\circ f)(x)$

Example

1. If $f(x)=2x$ and $g(x)=3x+1$, express $g\circ f$ as a single function $h(x)$.

Solution

$f(x)=2x$ so $(g\circ f)(x)=g(2x)=3(2x)+1=6x+1$

$\therefore\ h(x)=6x+1$

2. Let $f(x)=x-1$ and $g(x)=\sqrt{x}$, find $(f\circ g)(x)$ and $(g\circ f)(x)$

- $g(x)=\sqrt{x}, so\ fg(x)=f(\sqrt{x})=\sqrt{x}-1$ $\quad\therefore\ (f\circ g)(x)=\sqrt{x}-1$
- $f(x)=x-1, so\ gf(x)=g(x-1)=\sqrt{x-1}$ $\quad\therefore\ (g\circ f)(x)=\sqrt{x-1}$

APPLICATION ACTIVITY 1.5.2

1. Let $f(x)=x^2 and\ g(x)=2x+1. find\ (fg)(x)\ and (gf)(x)$
2. Given that
   $f(x)=x+3, g(x)=2x, find:$
   $a) fg(x)$
   $b) gf(x)$
3. The functions
   $f(x)=2x-1$ and $g(x)=x^2+2$
   Find:
   $a) fg(x)$ $\quad$ b)$gf(x)$ $\quad$ c)$gf(3)$
4. Find $(fog)(x)$ and $(gof)(x)$
   a) if $f(x)=x^3-3x^2+1$ and $g(x)=2$
   b) if $f(x)=2x^2+x-3$ and $g(x)=6x$

1.5.3 The inverse of a function

ACTIVITY 1.5.3

Find the value of x in function of y if

1. $y = x + 1$
2. $y = 3x - 2$
3. $y = \dfrac{-x+3}{2x-1}$

Consider a function f which maps each element x of the domain X onto its image y in the range Y that is $f : x \to y$ where $x \in X, y \in Y$. If this map can be reversed, i.e $f^{-1} : y \to x$ and the resulting relationship is a function, it is called the **inverse of the original function**, and is denoted by f^{-1}.

Only one-to-one functions can have an inverse function. To find the inverse of one-to-one functions, we change the subject of a formula.

Definition: Let $f(x)$ and $g(x)$ be two functions such that $f[g(x)] = x$ for each x in the domain of g and $g[f(x)] = x$ for each x in the domain of f.

Under these conditions, the function g is the inverse of f. The function g is denoted by $f^{-1}(x)$, which is read as "f - inverse". So $f\left(f^{-1}(x)\right) = x$ and $f^{-1}\left(f(x)\right) = x$

Note:

- If $f(x)$ is inverse of $g(x)$, then $g(x)$ is inverse of $f(x)$.
- The domain of f must be equal to the range of f^{-1}, and the range of f must be equal to the domain of f^{-1}.
- The notation $f^{-1}(x) \neq \dfrac{1}{f(x)}$

Steps to find inverse functions

Let $f : x \to \mathbb{R}$ be an injective function where $x \subseteq \mathbb{R}$. To find the inverse function of f means to find the domain of f^{-1} as well as a formula for $f^{-1}(y)$. If the formula for $f(x)$ is not very complicated, dom (f^{-1}) and $f^{-1}(y)$ can be found by solving the equation $y = f(x)$ for x.

($Step\ 1$) Put $y = f(x)$.

($Step\ 2$) Solve x in terms of y. The result will be in the form x = an expression in y.

($Step\ 3$) From the expression in y obtained in Step 2, the range of f can be determined. This is the domain of f^{-1}. The required formula is $f^{-1}(y)$ = the expression in y obtained in Step 2.

Examples

1) Find the inverse function of $f(x) = 2x + 3$.

Solution

Let us make x the subject of $y = 2x + 3$ as follows:

$y = 2x + 3$ (Solve for the variable x);

Then, $y - 3 = 2x$, thus, $x = \frac{y-3}{2}$

$\therefore f^{-1}(x) = \frac{x-3}{2}$

2) Find the inverse of the function $f(x) = 3x - 1$

Solution

If $f(x) = 3x - 1$, we require $f^{-1}(y) = x$. If $y = 3x - 1$ then $x = \frac{y+1}{3}$

So, given y, we can return to x using the expression $\frac{y+1}{3}$. Thus, $f^{-1}(x) = \frac{x+1}{3}$

APPLICATION ACTIVITY 1.5.3

1) Find the inverse of the following functions

a) $f(x) = 5x + 2$

b) $g(x) = -7x - 2$

c) $h(x) = \frac{-2x+1}{x-2}$

1.6 Graphical representation and interpretation of linear and quadratic functions

ACTIVITY 1.6

1.Copy and complete the tables below.

a)

x	-3	-2	-1	0	1	2	3
$y = 2x - 1$							

b)

x	-3	-2	-1	0	1	2	3
$y = x^2 - 1$							

2. Use the coordinates from each table to plot the graphs on separate Cartesian planes.

3. What is your conclusion about the shapes of the graphs?

1.6.1 Linear function

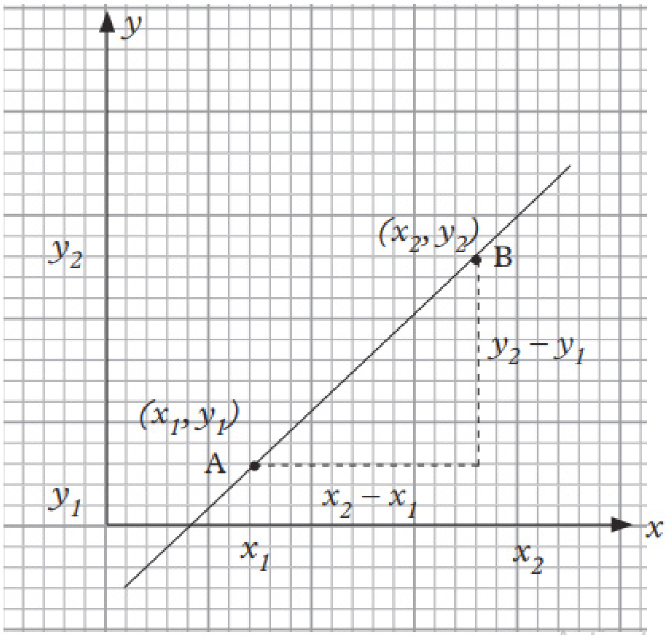
Definition: Any function of the form $f(x) = mx + b$, where m is not equal to 0 is called a linear function. The ***domain*** of this function is the set of all real numbers. The ***range*** of f is the set of all real numbers. The graph of *f* is a line with slope m and y intercept b.
Note: A function $f(x) = b$, where b is a constant real number is called a constant function. Its graph is a horizontal line at $y = b$.

Examples: a) $a)\ y = x + 1,\quad b)\ y = 2x - 3,\ c)\ y = -3x + 4, \ldots$

Graphs of linear functions

The ordered pair (x, y) represents coordinates of any point on the Cartesian plane.

Consider a line passing through the points $A(x_1, y_1)$ and $B(x_2, y_2)$.



From A to B, the change in the x-coordinate (horizontal change) is $x_2 - x_1$ and the

change in the y-coordinate (vertical change) is $y_2 - y_1$.

By definition, gradient / slope is equal to $\frac{change\ in\ y-coordinates}{change\ in\ x-coordinates} = \frac{y_2 - y_1}{x_2 - x_1}$

In the Cartesian plane, the gradient of a line is the measure of its slope or inclination to the x-axis. It is defined as the ratio of the change in y-coordinate (vertical) to the change in the x-coordinate (horizontal).

When drawing a graph of a linear function, it is sufficient to plot only two points and these points may be chosen as the x and y intercepts of the graph. In practice, however, it is wise to plot three points. If the three points lie on the same line, the working is probably correct, if not you have a chance to check whether there could be an error in your calculation.

If we assign x any value, we can easily calculate the corresponding value of y.

Determine the x intercept, set $f(x)$ is equal to zero and solve for x and then determine the y intercept, set x equals zero to find $f(0)$.

Consider the equation y = 2x + 3.

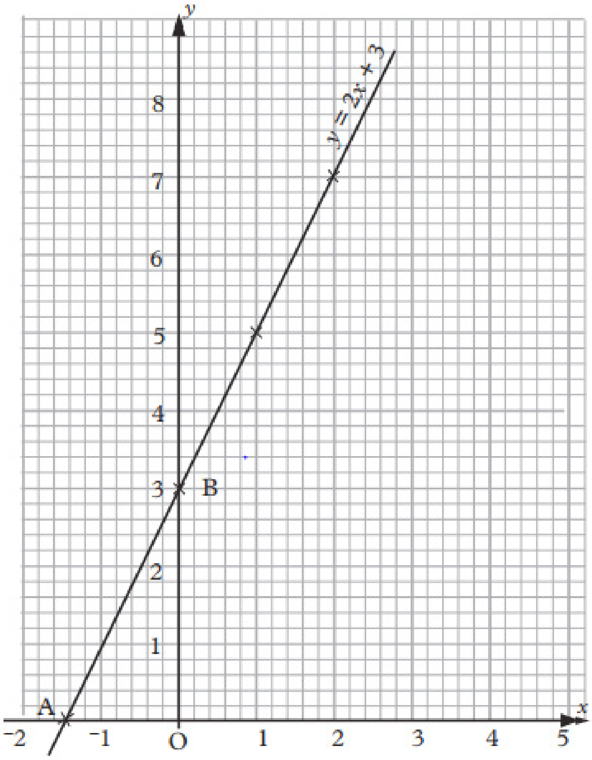
- When x = 0, y = 2 × 0 + 3 = 3
- When x = 1, y = 2 × 1 + 3 = 5
- When x = 2, y = 2 × 2 + 3 = 7 and so on.

For convenience and ease while reading, the calculations are usually tabulated as shown below in the table of values for y = 2x + 3.

x	0	1	2	3	4
$2x$	0	2	4	6	8
$+3$	3	3	3	3	3
$y=2x+3$	3	5	7	9	11

From the table the coordinates (x, y) are (0, 3), (1, 5), (2, 7), (3, 9), (4, 11)....

When drawing the graph, the dependent variable is marked on the vertical axis generally known as the y – axis. The independent variable is marked on the horizontal axis also known as the x – axis.



1.6.2 Quadratic function

A polynomial equation in which the highest power of the variable is 2 is called a quadratic function. The expression $y=ax^2+bx+c$, where a,b and c are constants and a ≠ 0, is called a quadratic function of x or a function of the second degree (highest power of x is two).

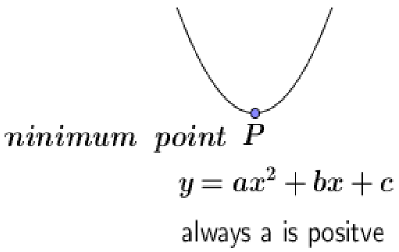
Table of values are used to determine the coordinates that are used to draw the graph of a quadratic function. To get the table of values, we need to have the domain (values of an independent variable) and then the domain is replaced in a given quadratic function to find range (values of dependent variables). The values obtained are useful for plotting the graph of a quadratic function. All quadratic function graphs are parabolic in nature.

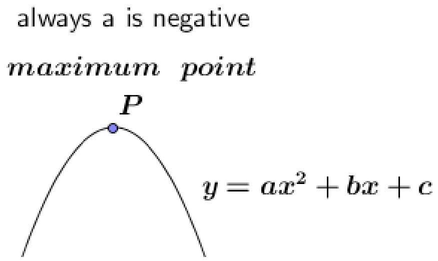
Any quadratic function has a graph which is symmetrical about a line which is parallel to the y-axis i.e. a line $x = h$ where h is constant value. This line is called **axis of symmetry.**

For any quadratic function $f(x) = ax^2 + bx + c$ whose axis of symmetry is the line $x = h$, the vertex is the point $(h, f(h))$.

The vertex of a quadratic function is the point where the function crosses its axis of symmetry.

If the coefficient of the x^2 term is positive, the vertex will be the lowest point on the graph, the point at the bottom of the U-shape. If the coefficient of the term x^2 is negative, the vertex will be the highest point on the graph, the point at the top of the ∩-shape. The shapes are as below.





Since the quadratic function written as $f(x) = ax^2 + bx + c$, then we can get the y-coordinate of the vertex by substituting the x-coordinate which is $x = -\frac{b}{2a}$.

So the vertex becomes $\left(-\frac{b}{2a}, f(-\frac{b}{2a})\right)$

The axis of symmetry of a quadratic function is the x-coordinate of the quadratic function and it is calculated from $x = -\frac{b}{2a}$.

The intercepts with axes are the points where a quadratic function cuts the axes.

There are two intercepts i.e. x-intercept and y-intercept. x-intercept for any quadratic function is calculated by letting y = 0 and y- intercept is calculated by letting x = 0

Graph of a quadratic function

The graph of a quadratic function can be sketched without table of values as long as the following are known.

- The vertex
- The x-intercepts
- The y-intercept

Example:

Find the vertex and axis of symmetry of the parabolic curve $y = 2x^2 - 8x + 6$

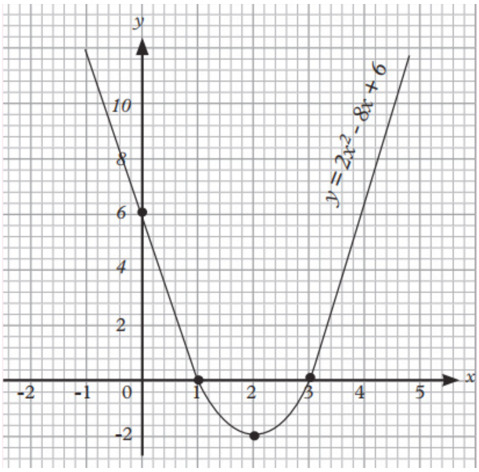
Solution

- The coefficients are a = 2, b = –8 and c = 6
- The x-coordinate of the vertex is $h = -\frac{b}{2a} = \frac{-(-8)}{2(2)} = \frac{8}{4} = 2$
- The y-coordinate of the vertex is obtained by substituting the x-coordinate of the vertex to the quadratic function. We get

  $y = 2(2)^2 - 8(2) + 6 = -2$
- The vertex is (2, –2) and the axis of symmetry is $x = 2$.
- When x = 0, $y = 2(0)^2 - 8(0) + 6 = 6$.
- The y-intercept is (0, 6)

When y = 0, $0 = 2x^2 - 8x + 6$, we therefore solve the quadratic equation for the values of x and we find the x-intercepts are (1,0) or (3,0)

The graph is as below.



APPLICATION ACTIVITY 1.6

1.Using the table of values, sketch the graph of the following functions:

a) y = – 3x + 2

b) y = x2 – 3x + 2

2. Without tables of values, state the vertex, intercept with axis, axis of symmetry, and sketch the graph of

$-3x^2+6x+1=y$

1.7 Graphical representation and interpretation of functions in economics and finance

ACTIVITY 1.7

Considering that C is the dependent variable, measured in the vertical axis, and *Y* is the independent variable, measured on the horizontal axis, Draw the graph of the function

$C = 200 + 0.6Y$ Where *C* is consumer spending and *Y* is income. Note that the income cannot be negative.

Determine the point (Y,C) at which the line cuts the vertical axis.

Content summary

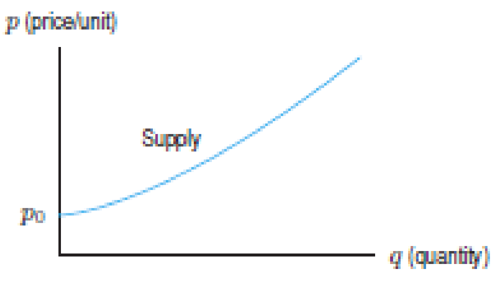
1. Price as function of quantity supplied

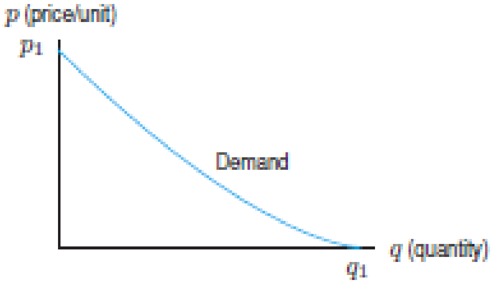
The quantity *q* of an item that is manufactured and sold depends on its price *p* As the price increases, manufacturers are usually willing to supply more of the product, whereas the quantity demanded by consumers falls.

The supply curve, for a given item, relates the quantity *q* of the item that manufacturers are willing to make per unit time to the price *p* for which the item can be sold.

The demand curve relates the quantity *q* of an item demanded by consumers per unit time to the price *p* of the item.

Economists often think of the quantities supplied and demanded Q as functions of price P. However, for historical reasons, the economists put price (the independent variable) on the vertical axis and quantity (the dependent variable) on the horizontal axis. (The reason for this state of affairs is that economists originally took price to be the dependent variable and put it on the vertical axis





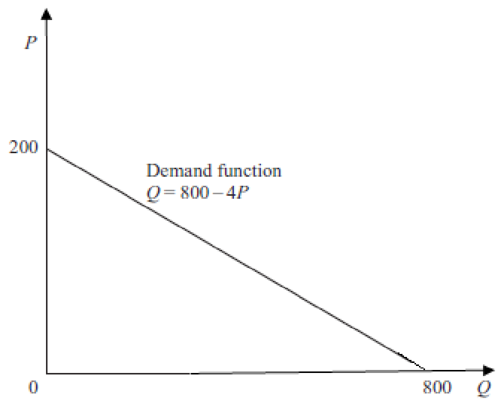
Theoretically, it does not matter which axis is used to measure which variable. However, one of the main reasons for using graphs is to make analysis clearer to understand. Therefore, if one always has to keep checking which axis measures which variable this defeats the objective of the exercise. Thus, even though it may upset some mathematical purists, the economists sometimes stick to the convention of measuring **quantity on the horizontal axis and price on the vertical axis**, even if price is the independent variable in a function.

This means that care has to be taken when performing certain operations on functions. If necessary, one can transform monotonic functions to obtain the inverse function (as already explained) if this helps the analysis.

Examples

1) The demand function $Q = 800 - 4P$ has the inverse function

$$P = \frac{800 - Q}{4} = 200 - 0.24Q$$



This figure shows that when the quantity Q is increasing, the price P reduces progressively. This can be caused by the fact that every consumer has sufficient quantity of goods and does not want to by any more.

2) Suppose that a firm faces a linear demand schedule and that 400 units of output Q are sold when price is \$40 and 500 units are sold when price is \$20. Once these two price and quantity combinations have been marked as points A and B, then the rest of the demand schedule can be drawn in. Use this data to determine the function that can help to predict quantities demanded at different prices and draw the corresponding graph.

Solution:

Accurate predictions of quantities demanded at different prices can be made if the information that is initially given is used to determine the algebraic format of the function.

A linear demand function must be in the form $P = a - bQ$, where a and b are parameters that we wish to determine the value of.

When P = 40 then Q = 400 and so 40 = a – 400b (1)

When P = 20 then Q = 500 and so 20 = a – 500b (2)

Equations (1) and (2) are what is known as simultaneous linear equations.

$$\begin{cases} 40 = a - 400b \\ 20 = 1 - 500b \end{cases}$$

We can solve these simultaneous linear equations by one of the methods we used above and find

$a = 120, b = 0.02$.

Our function can now be written as $P = 120 - 0.2Q$

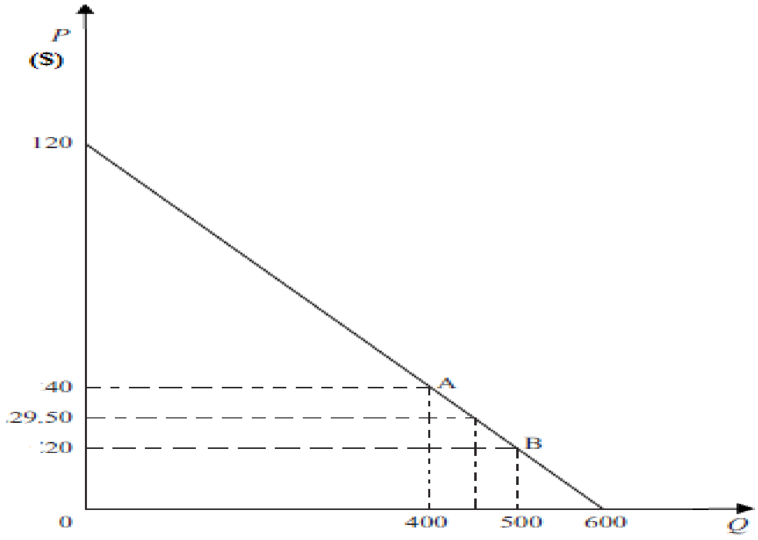
We can check that this is correct by substituting the original values of Q into the function.

If $Q = 400$ then $P = 120 - 0.2(400) = 120 - 80 = 40$

If $Q = 500$ then $P = 120 - 0.2(500) = 120 - 100 = 20$

These are the values of P originally specified and so we can be satisfied that the line that passes through points A and B is the linear function $P = 120 - 0.2Q$.

The inverse of this function will be $Q = 600 - 5P$. Precise values of Q can now be derived for given values of P. For example, when P = £29.50 then $Q = 600 - 5(29.50) = 452.5$.



2. Consumption as function of income

It is assumed that consumption C depends on income Y and that this relationship takes the form of the linear function $C = a + bY$.

Example:

When *the income* is \$600, the consumption observed is \$660. When *the income* is \$1,000, the consumption observed is \$900. Determine the equation "consumption function of income".

Solution:

To determine the required equation, we can solve this system of equations

$$\begin{cases} 660 = a + 600b \\ 900 = a + 1000b \end{cases}$$

by using different methods to find the value of a and b.

However, let us use another way as follows: We expect b to be positive, i.e. consumption increases with income, and so our function will slope upwards. As this is a linear function then equal changes in Y will cause the same changes in C.

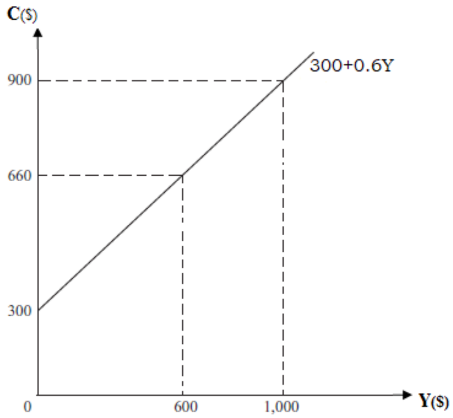
A decrease in Y of \$400, from \$1,000 to \$600, causes C to fall by \$240, from \$900 to \$660.

If Y is decreased by a further \$600 (i.e. to zero) then the corresponding fall in C will be 1.5 times the fall caused by an income decrease of \$400, since \$600 = 1.5 × \$400.

Therefore, the fall in C is 1.5 × \$240 = \$360. This means that the value of C when Y is zero is \$660 − \$360 = \$300. Thus a = 300.

A rise in Y of \$400 causes C to rise by \$240. Therefore, a rise in Y of \$1 will cause C to rise by \$240/400 = \$0.6. Thus b = 0.6.

The function can therefore be specified as $C = 300 + 0.6Y$.



The graph shows that when the income increases, the consumption increases also.

3. Price as function of quantity demanded

The linear demand function is in the form $P = a - bQ$, where a and b are parameters, P is the price and Q is the quantity demanded.

Example:

Consider the function $P = 60 - 0.2Q$ where P is price and Q is quantity demanded. Assume that P and Q cannot take negative values, determine the slop of this function and sketch its graph.

Solution:

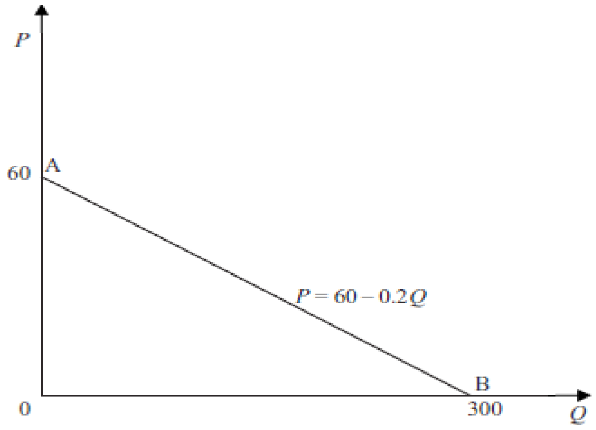
When $Q = 0$ then $P = 60$

When $P = 0$ then $0 = 60 - 0.2Q$

$0.2Q = 60$

$$Q = \frac{60}{0.2} = 300$$

Using these points: $(0,60)$ and $(300,0)$, we can find the graph as follows:



The slope of a function which slopes down from left to right is found by applying the formula

$$slope = (-1)\frac{height}{base}$$

To the relevant right-angled triangle. Thus, using the triangle 0BA, the slope of our function is

$$(-1)\frac{60}{300} = -0.2$$

This, of course, is the same as the coefficient of Q in the function $P = 60 - 0.2Q$.

Remember that in economics the usual convention is to measure P on the vertical axis of a graph. If you are given a function in the format $Q = f(\mathrm{P})$ then you would need to derive the inverse function to read off the slope.

Example:

What is the slope of the demand function $Q = 830 - 2.5P$ when P is measured on the vertical axis of a graph?

Solution:

If $Q= 830-2.5P$; then $2.5P = 830-Q$

$P = 332 - 0.4Q$

Therefore, the slope is the coefficient of Q, which is -0.4.

4. Point elasticity of demand

Elasticity can be calculated at a specific point on a linear demand schedule. This is called '*point elasticity of demand*' and is defined as

$$e = (-1)\left(\frac{P}{Q}\right)\left(\frac{1}{slope}\right)$$

where P and Q are the price and quantity at the point in question. The slope refers to the slope of the demand schedule at this point although, of course, for a linear demand schedule the slope will be the same at all points.

Example:

Calculate the point elasticity of demand for the demand schedule $P = 60 - 0.2Q$ where price is

(i) Zero, (ii) \$20, (iii) \$40, (iv) \$60.

Solution

This is the demand schedule referred to earlier and illustrated above. Its slope must be -0.2 at all points as it is a linear function and this is the coefficient of Q.

To find the values of Q corresponding to the given prices we need to derive the inverse function. Given that

$P = 60 - 0.2Q$ then $0.2Q = 60 - P$

$Q = 300 - 5P$

(i) When P is zero, at point B, then $Q = 300 - 5(0) = 300$. The point elasticity will therefore be

$$e = (-1)\frac{P}{Q}\left(\frac{1}{slope}\right) = -1\frac{0}{30}\left(\frac{1}{-0.2}\right) = 0$$

(ii) When $P = 20$ then $Q = 300 - 5(20) = 200$.

$$e = (-1)\frac{P}{Q}\left(\frac{1}{slope}\right) = -1\frac{20}{200}\left(\frac{1}{-0.2}\right) = 0.5$$

(iii) When $P = 40$ then $Q = 300 - 5(40) = 100$

$$e=(-1)\frac{P}{Q}\left(\frac{1}{slope}\right)=-1\frac{40}{100}\left(\frac{1}{-0.2}\right)=2$$

(iv) When $P = 60$ then $Q = 300 - 5(60) = 0$.

If $Q = 0$, then $\frac{P}{Q}\rightarrow\infty$.

Therefore, $e=(-1)\frac{P}{Q}\left(\frac{1}{slope}\right)=-1\frac{60}{0}\left(\frac{1}{-0.2}\right)\rightarrow\infty$

5. The Cost Function

The **cost function**, C(q), gives the total cost of producing a quantity q of some good. Costs of production can be separated into two parts: the *fixed costs,* which are incurred even if nothing is produced, and the *variable costs*, which depend on how many units are produced.

Example:

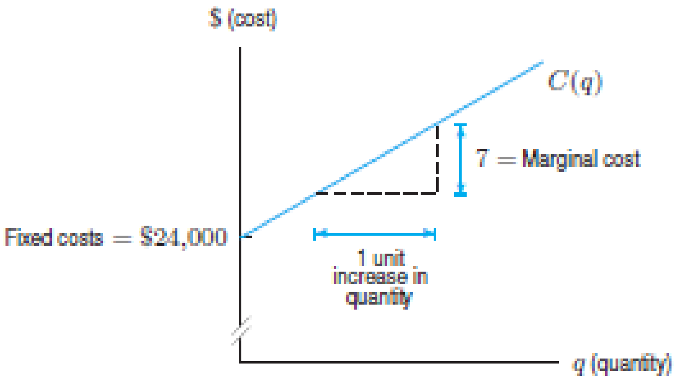
Let's consider a company that makes radios. The factory and machinery needed to begin production are fixed costs, which are incurred even if no radios are made. The costs of labor and raw materials are variable costs since these quantities depend on how many radios are made. The fixed costs for this company are $24,000 and the variable costs are $7 per radio.

Then, Total costs for the company = Fixed costs + Variable costs = 24,000 + 7.(Number of radios),

so, if q is the number of radios produced,

$C(q) = 24{,}000 + 7q$.

This is the equation of a line with slope 7 and vertical intercept 24,000.



If C(q) is a linear cost function,

- Fixed costs are represented by the vertical intercept.
- Marginal cost is represented by the slope.

6. The Revenue Function

The **revenue function**, R(q), gives the total revenue received by a firm from selling a quantity, q, of some good.

If the good sells for a price of p per unit, and the quantity sold is q, then Revenue = Price · Quantity,

so, R = pq.

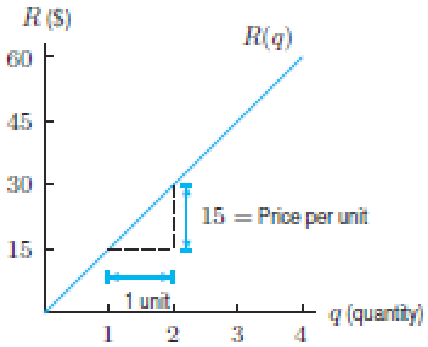
If the price does not depend on the quantity sold, so p is a constant, the graph of revenue as a function of q is a line through the origin, with slope equal to the price p.

Example:

1. If radios sell for $15 each, sketch the manufacturer's revenue function. Show the price of a radio on the graph.

Solution:

Since R(q) = pq = 15q, the revenue graph is a line through the origin with a slope of 15. See the figure. The price is the slope of the line.



2. Graph the cost function C(q) = 24,000 + 7q and the revenue function R(q) = 15q on the same axes. For what values of q does the company make money?

Solution:

The company makes money whenever revenues are greater than costs, so we find the values of q for which the graph of R(q) lies above the graph of C(q). See Figure 1.45.

We find the point at which the graphs of R(q) and C(q) cross:

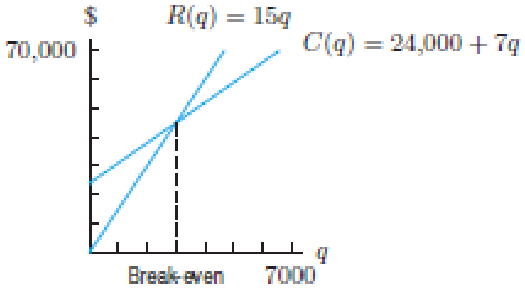
Revenue = Cost

15q = 24,000 + 7q

8q = 24,000

q = 3000.

The company makes a profit if it produces and sells more than 3000 radios. The company loses money if it produces and sells fewer than 3000 radios.



7. The Profit Function

Decisions are often made by considering the profit, usually written as π to distinguish it from the price, p.

We have: **Profit = Revenue – Cost.**

So, $\pi = R - C$

The *break-even point* for a company is the point where the profit is zero and revenue equals cost.

Example:

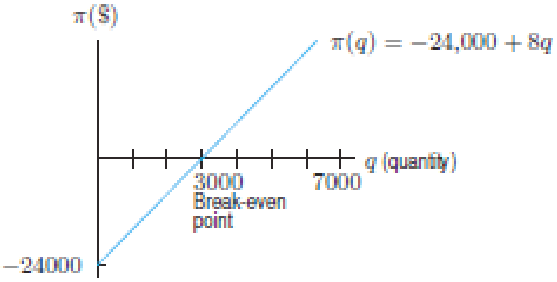
Find a formula for the profit function of the radio manufacturer. Graph it, marking the break-even point

Solution:

Since R(q) = 15q and C(q) = 24,000 + 7q, we have

$$\pi(q) = R(q) - C(q)$$
$$= 15q - (24000 + 7q) = -2400 + 8q$$

Notice that the negative of the fixed costs is the vertical intercept and the break-even point is the horizontal intercept. See the figure;



8. The Marginal Cost, Marginal Revenue, and Marginal Profit

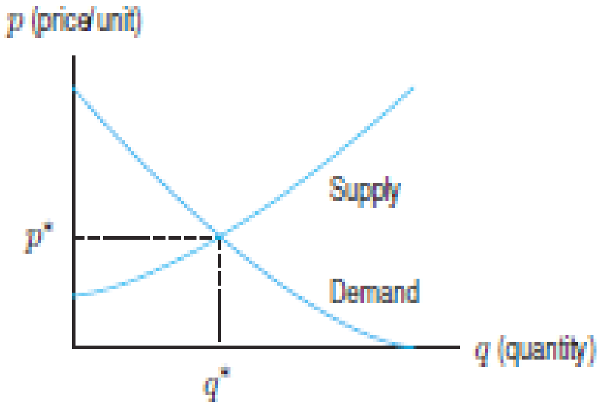
Just as we used the term marginal cost to mean the rate of change, or slope, of a linear cost function, we use the terms *marginal revenue* and *marginal profit* to mean the rate of change, or slope, of linear revenue and profit functions, respectively.

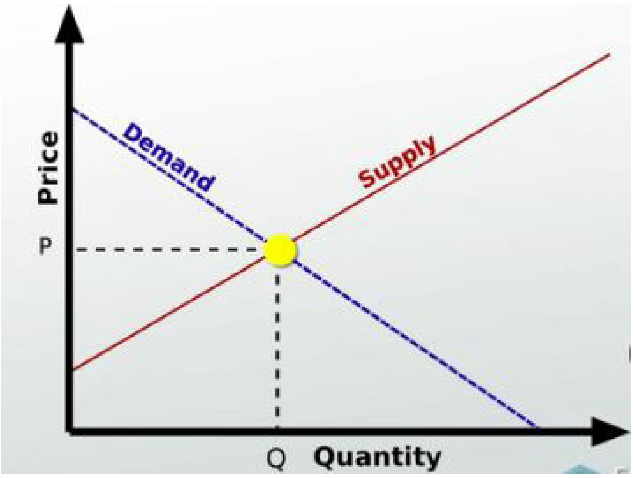
The term *marginal* is used because we are looking at how the cost, revenue, or profit change "at the margin," that is, by the addition of one more unit.

For **example**, for the radio manufacturer, the marginal cost is 7 dollars/item (the additional cost of producing one more item is $7), the marginal revenue is 15 dollars/item (the additional revenue from selling one more item is $15), and the marginal profit is 8 dollars/item (the additional profit from selling one more item is $8).

9. Equilibrium Price and Quantity

If we plot the supply and demand curves on the same axes, the graphs cross at the *equilibrium point*. The values p^* and q* at this point are called the *equilibrium price* and *equilibrium quantity*, respectively. It is assumed that the market naturally settles to this equilibrium point.





Example:

Find the equilibrium price and quantity if Quantity supplied = 3p – 50 and Quantity demanded = 100 – 2p.

Solution:

To find the equilibrium price and quantity, we find the point at which

Supply = Demand

$3p - 50 = 100 - 2p$

$5p = 150$

$p = 30.$

The equilibrium price is \$30. To find the equilibrium quantity, we use either the demand curve or the supply curve. At a price of \$30, the quantity produced is $100 - 2(30) = 40$ items.

The equilibrium quantity is 40 items.

In the figure, the demand and supply curves intersect at $p^* = 30$ and $q^* = 40$.

APPLICATION ACTIVITY 1.7

1. Assume that consumption C depends on income Y according to the function

   $C = a + bY$, where a and b are parameters. If C is \$60 when Y is \$40 and C is \$90 when Y is \$80,

   What are the values of the parameters a and b?

   Sketch the graph of C(Y) and interpret it.

2. Suppose that q = f(p) is the demand curve for a product, where p is the selling price in dollars and q is the quantity sold at that price.

   (a) What does the statement f(12) = 60 tell you about demand for this product?

   (b) Do you expect this function to be increasing or decreasing? Why?

3. A demand curve is given by 75p + 50q = 300, where p is the price of the product, in dollars, and q is the quantity demanded at that price. Find p^* and q^* intercepts and interpret them in terms of consumer demand.

1.8 END UNIT ASSESSMENT

1) The total cost C for units produced by a company is given by $C(q) = 50000 + 8q$ where q is the number of units produced.

   a) What does the number 50000 represent?

   b)What does the number 8 represent?

   d) Plot the graph of C and indicate the cost when $q = 5$.

   e) Determine the real domain and the range of $C(q)$.

   f) Is $C(q)$ an odd function?

2) Bosco was working for his boss Kamana and they agreed to start a job where the monthly salary $f(t)$ was depending on the time t representing the t^{th} month Bosco spends on service. The salary $f(t)$ was the sum of a monthly bonus of 50,000Fr and product of 10,000Frw by the inverse of the time t.

   a) Give the function $f(t)$ which models the monthly salary of Bosco;

   b) Determine the domain of $f(t)$ and explain what it means

   c) Suppose that Bosco can continue to work indefinitely, determine the Maximum Salary and the Minimum Salary can Bosco get and deduce the Range of $f(t)$.

   d) Bosco has a monthly bonus, is this bonus motivating? Explain your answer.

   e) If you were Bosco, how many months can you work for Kamana? Explain your answer.

UNIT 2

INTRODUCTION TO LOGIC

Key Unit competence: Use Mathematical logic as a tool of reasoning and decision making in daily life

2.0 INTRODUCTORY ACTIVITY 2

A) Discuss the meaning of propositional statement

B) In the following sentences which of them are propositions and which are not propositions?

1. Don't eat the daisies!
2. My dog is called Didi.
3. Do you enjoy reading novels?
4. The jokes are great.
5. Mozart composed classical music.
6. The camera is not a Kodak.
7. This statement is false.
8. Use the quadratic formula on that one.

2.1 Simple statement and compound statements

ACTIVITY 2.1

From the following expressions, give your answer by true or false

1. Every integer larger than 1 is positive
2. Kampala is in Rwanda
3. How old are you?
4. Every liquid is water.
5. Write down the names of Rwandan president.
6. $1-x^2=0$.
7. Rwanda is an African country or Rwanda is a member of Commonwealth.

Content summary

A sentence which is either true or false but not both simultaneously is named **statement or proposition**. In the context of logic, a proposition or a statement is the sentence in the grammatical sense conveying a situation which is neither imperative, interrogative nor exclamatory.

The expressions 1, 2, 4 and 7 are statements: the 2^{nd} and 4^{th} are false while 1^{st} and 7^{th} are true.

- The expression "How old are you?" is not a proposition since you cannot reply by true nor false (grammatically this sentence is interrogative).
- The equality "$1-x^2=0$" is not a proposition because for some values of x the equality is true, whereas for others it is false.
- The expression "Write down the names of Rwandan president" is not a proposition as the answer will be given by neither true nor false. It is a command.

A statement that cannot be broken into two or more sentences is called **simple statement**. Combining two or more simple statements we form a **compound statement.**

Example: In activity 7.1.1, the 1st, 2nd and 4th expressions are simple statements while the 7th is a compound statement.

In this unit, statements will be denoted by small letters such as ***p, q, r, …***

The logical statements are required to have a definite truth-value, or, to be either true or false, but never both, and to always have the same truth value.

The two truth values of proposition are **true** and **false** and are denoted by the symbols **T** and **F** respectively. Occasionally, they are also denoted by the symbols **1** and **0** respectively.

APPLICATION ACTIVITY 2.1

1) Find out which of the following sentences are statements and which are not. Justify your answer.

   a) Uganda is a member of East African Community.

   b) The sun is shining.

   c) Come to class!

   d) The sum of two prime numbers is even.

   e) It is not true that China is in Europe.

   f) May God bless you!

2) Write down the truth value (T or F) of the following statements

   a) Paris is in Italy.

   b) 13 is a prime number.

   c) Kigeri IV Rwabugiri was the king of the Kingdom of Rwanda

   d) Lesotho is a state of South Africa.

2.2 Truth values and truth tables

ACTIVITY 2.2

Are these sentences proposition? If yes, give their truth values

a) Uganda is a member of East African Community.

b) The sun shines.

c) Paris is in England.

d) Come to class!

e) The sum of two prime numbers is even.

f) It is not true that Uganda is in Europe.

Content summary

The way we will define compound statements is to list all the possible combinations of the truth-values (abbreviated **T** and **F** or **1**and **0**) of the simple statements (that are being combined into a compound statement) in a table, called a **truth table**. The name of each statement is at the top of a column of the table.

If the compound statement contains n distinct simple statements, we will consider 2^n possible combinations of truth values in order to obtain the truth table.

Examples

1) One proposition p *has two truth values* ($2^1 = 2$ possible combinations), the truth table is

p
T
F

Or

p
1
0

2) Suppose we are given two mathematical statements, named **P** and **Q**, *new* mathematical statements that incorporate **P** and Q are called **compound statements**. Their truth values will be determined *solely* by the truth-values of **P** and of **Q**.

For two propositions p and q, *we* have $2^2 = 4$ possible combinations of truth values, the truth table is:

p	q
T	T
T	F
F	T
F	F

Or

p	q
1	1
1	0
0	1
0	0

Any proposition can be represented by a truth table

- It shows truth values for all combinations of its constituent variables

Example:

Proposition involving 2 variables p and q. Let us take the true statement "p and q" where p: I am at home, and q: It is raining. One combination of r = p ∧ q is I am at home and it is raining. Others are the following: I am at home and it is not raining (F), I am not at home and it is raining (F) and I am not at home and it is not raining (F).

All possible combinations of truth values of 2 propositions p and q		**Truth values of compound proposition**
P	q	r
True	*True*	*T*
True	False	F
False	True	F
False	False	F

A proposition can involve any number of variables; each row corresponds to a possible combination of variables. With n variables the truth table has:

n+1 column (one for each of the n variables and one for the compound expression);

It has also 2^n rows plus a header.

The following lesson will develop how to make a compound statement using different connectives.

APPLICATION ACTIVITY 2.2

Write down the truth table for

1) Three propositions p, q and r

2) Four propositions p, q, r *and* s

2.3 Logical connectives

2.3.1 Negation

ACTIVITY 2.3.1

1) Given statements P: I am strong and Q: I can jump, try to make a compound statement formed by P and Q in different ways. What are the different connecting words that can be used?

2) Let p, q, ... are the given propositions, put these propositions in negative form

1. Jack is running.
2. Ronald does not smile.
3. She isn't a foot ball player.
4. -3 is a natural number.
5. Mathematics is needed in languages Education option.

Content summary

The negation of a statement P is made by introducing the word "not" denoted by prefixing the statement P. It has opposite truth value from the statement. It is denoted by $\neg P$ or $\overline{P}$ or $\sim P$

From this definition, it follows that the negation of a true statement is false while the negation of false statement is true; simply If P is true, then $\neg P$ is false and if P is false, then $\neg P$ true.

Example

1) Let P *: "Kamana is a student", then* $\neg P$: "Kamana is not a student".

2)P: The earth is round, $\neg P$: The earth is not round.

Let P be a proposition. Construct the truth table of $\neg P$

Solution

p	$\neg p$
T	F
F	T

APPLICATION ACTIVITY 2.3.1

1. Write the negation of each of the following statements:

   a) Today is raining.

   b) The sky is blue

   c) My native country is Rwanda.

   d) Bony is smart and healthy.

2. Complete the following truth table

p	q	r	$\neg p$	$\neg q$	$\neg r$
T		T		F	
	T	F	F		
T	F	T			
T	F				T
	T	T	T		
F	T	F			
F				T	F
F	F	F			

2.3.2 Conjunction

ACTIVITY 2.3.2

Given two propositions which are true;

p: I am at school, q: it is raining, Discuss the truth value of the compound propositions:

a) "I am at school and it is raining"

b) "I am not at school and it is raining",

c)"I am not at school and it is not raining".

Content summary

If two simple statements p and q are connected by the word **"and"**, then the resulting compound statement ***p*** and ***q*** is called a conjunction of ***p*** and ***q*** and is written in symbolic form $p \wedge q$. It has the truth value **true** whenever both ***p*** and ***q*** have the truth value **true**; otherwise it has the truth value **false.**

Examples

Let ***p*** be "It is raining today" and ***q*** *be* "there are fifteen chairs in this class room" assuming that p and q are true statements, construct simple sentences which describe each of the following statements and construct the truth table.

a) $p \wedge q$

Solution

a) From the given two simple statements, one of the resulting compound statements is

(i) "It is raining today and there are fifteen chairs in this class room" which is True.

(ii) "It is not raining and there are not fifteen chairs in this classroom: Which is False.

You can give the remaining two sentences and the truth table of related compound statements is as follows:

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

2) Let p and q be propositions. Construct the truth table for

a) $\neg p \wedge q$

b) $(\neg p \wedge q) \wedge \neg p$

Solution

p	q	$\neg p$	$\neg p \wedge q$	$(\neg p \wedge q) \wedge \neg p$
T	T	F	F	F
T	F	F	F	F
F	T	T	T	T
F	F	T	F	F

APPLICATION ACTIVITY 2.3.2

1) If p stands for the statement " It is cold" and q stands for the statement " It is raining", then what does $\neg q \wedge \neg q$ stands for? Construct its truth table

2) Let p and q be two propositions. Construct the truth table of

a) $p \wedge q$

b) $\neg p \wedge q$

c) $p \wedge \neg q$

d) $\neg(p \wedge q)$

e) $\neg q \wedge (\neg p \wedge q)$

3) Determine the truth value of each of the following statements

1. Paris is in France **and** it is a Capital city
2. $4+4=9$ **and** $5+8=11$
3. Paris is in England **and** $3+4=7$
4. Kigali is the Capital city of Burundi **and** $1+1=2$
5. Alphabets are the basic of any languages **and** the digits is not the basic in counting
6. m^2 is the unit of area **and** *kg* being one of the units of weight.

2.3.3 Disjunction

ACTIVITY 2.3.3

1)There were boys and girls in the classroom.

a) If they ask you to choose 2 boys and two girls. How many students will you choose?

b) If they ask you to select two girls or two boys how many students will you select?

2) Given the true proposition p: I am at home, q: it is raining. What is the truth value of

a) "I am at home or it is raining".

b) "I am not at home or it is not raining".

c) "I am at home or it is not raining".

Content summary

If two simple statements p and q are connected by the word **"or"**, then the resulting compound statement "*p* or *q*" is called a **disjunction** of *p* and *q* and it is written in symbolic form by $p \vee q$.

It has the truth value **false** only when *p* and *q* have truth value **false**, otherwise it has **true** as a the truth value.

Examples

1) Let p be "Paris is in France" and q be "London is in England". Construct simple verbal sentences which describe different related compound statements and construct the truth table. of $p \vee q$.

Solution

From the given two simple statements, the resulting compound statement is

(i) "Paris is in France or London is in England" (True)

(ii) Paris is in France or London is not in England (Tue)

(iii) Paris is not in France or London is in England (True)

(iii) Paris is not in France or London is not in England (False).

The Truth table of related compound statement is as follows:

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

2) Construct the truth tables for any $\neg(p \vee q)$ and $\neg p \wedge \neg q$ and compare them.

Solution:

First, we make a table that displays all the possible combinations of truth-values for p and q, then we add two columns and put $p \vee q$ and $\neg(p \vee q)$ into its top cell and then start calculations.

p	q	$p \vee q$	$\neg(p \vee q)$
T	T	T	F
T	F	T	F
F	T	T	F
F	F	F	T

Note: Ambiguity of "or" in English language

In natural language **"or" has two meanings:**

Inclusive or: where $p \vee q$ is true if either p or both are true.

Example: Numbers or measurements may be taken as prerequisites for geometry.

Meaning: take either one but you may also take both.

Exclusive or: denoted as $\oplus$: where $p \oplus q$ is true if either p or q but not both are true.

Example: "You will be paid money or a computer".

Meaning: do not expect to get both. Bellow is the table for $p \oplus q$

P	Q	$p \oplus q$
T	T	F
T	F	T
F	T	T
F	F	F

APPLICATION ACTIVITY 2.3.3

1.Translate each of the following compound statements into symbolic form

i) Bwenge reads News Paper or Mathematics book.

ii) Rwema is a student-teacher or not a book seller.

2. Suppose that p is a false statement, and q is a true statement.

a) What is the truth-value of the compound statement $\neg p \vee q$?

b) What is the truth-value of the compound statement $p \vee \neg q$?

3. Let p and q be two propositions. Construct the truth table of

$p \vee q$; $p \vee \neg q$; $\neg(p \vee q) \wedge (\neg p \vee \neg q)$ and $(\neg q \vee p) \wedge (\neg p \vee q)$

2.3.4 Conditional statement

ACTIVITY 2.3.4

1) Given the following conditional statement: "If Samantha's health is good, then she will go to the party". Discuss and identify the hypothesis (premise) and conclusion (consequent) of the conditional statement.

2) Complete these sentences referring to the conditional statement given in question one.

   If Samantha's health is not good, ...

   Samantha will go to the party, ...

   Samantha will not go to the party, ...

Content summary

A conditional statement is a logic statement used when the statement is in the form "if ...then ...." It can be written as "$If\ p\ then\ q$" or $p \to q$,(or $p \Rightarrow q$) read as $p\ implies\ q$.

In this case the proposition p is called antecedent, hypothesis or premise while the proposition q is the conclusion or the consequent. In language, p can be called the first clause and q is the second clause.

The conditional $p \Rightarrow q$ can also be read:

- If p, then q.
- q follows from p.
- q if p.
- p only if q.
- p is sufficient for q.
- q is necessary for p.

Let us consider the logical conditional as an obligation or contract, for example:

"If you get 100% on the final, then you will earn an A"

p: If you get 100% on the final, q: you will earn an A.

$p \Rightarrow q$

	Q: earn an A	$\neg q$: don't get A
P: get 100%	$p \Rightarrow q$ is **true**	$p \Rightarrow q$ is **false**
$\neg p$: don't get 100%	$p \Rightarrow q$ is **true**	$p \Rightarrow q$ is **true**

This shows that $F \Rightarrow T$ does not violate the obligation, the only time the obligation is broken is when $T \Rightarrow F$. Therefore, the statement $p \Rightarrow q$ has the truth value True in all cases except when p is true while q is false.

The related truth table is as follows:

p	q	$p \Rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

Examples

1.Rephrase the sentence "*If it is Sunday, you go to church*", then construct the related truth values.

Solution

Here are some various ways of rephrasing the sentence:

- "You go to church if it is Sunday."
- "It is Sunday only if you go to church."
- "It can't be Sunday unless you go to church."

Let denote the given statements as: ***p***: It is Sunday, q: you go to church.

The statement *If it is Sunday, you go to church* is symbolized as follow: $p \Rightarrow q$.

1. Construct the truth value of the following statement:

"If either John takes Calculus or Betty takes Sociology then Peter will take English."

2. Let denote the statements as:

p: John takes Calculus, q: Betty takes Sociology and *r*: Peter takes English.

The given statement can be symbolized as follow: $(p \vee q) \Rightarrow r$

Use different related sentences and con*struct the truth table for* $(p \vee q) \Rightarrow r$.

Solution

p	q	r	$p \vee q$	$(p \vee q) \Rightarrow r$
T	T	T	T	T
T	T	F	T	F
T	F	T	T	T
T	F	F	T	F
F	T	T	T	T
F	T	F	T	F
F	F	T	F	T
F	F	F	F	T

Converse, Contrapositive and Inverse of a conditional propositional

a) Converse

The implication obtained by interchanging the antecedent and the consequent of an implication is called the **converse.**

Thus, the converse of $P \Rightarrow Q$ is $Q \Rightarrow P$.

Example: If you are a student, then you should study. The converse is "if you study, then you are a student" this is not true.

b) Contra-positive

Given the conditional $p \Rightarrow q$, the implication $\neg q \Rightarrow \neg p$ is called the **contra-positive** of $p \Rightarrow q$.

Example:

If it is a PC, then it is a computer, the contrapositive is "if it is not a computer, then it is not a PC". This is true.

c) Inverse

Given the conditional $p \Rightarrow q$, the implication $\neg p \Rightarrow \neg q$ is called its inverse. It is obtained by negating the antecedent and negating the consequence.

Example:

If it is a PC, then it is a computer; the inverse is "If it is not a PC, then it is not a computer". This is false.

Note:

i. A conditional statement is logically equivalent to its contrapositive. A conditional statement can be replaced with its contrapositive and keeps its truth value.

ii. Converse: $p \Rightarrow q$ does not necessary imply $q \Rightarrow p$.

iii. Inverse: $p \Rightarrow q$ does not imply $\sim p \Rightarrow \sim q$.

APPLICATION ACTIVITY 2.3.4

1) Using the statements p: Mico is fat and p: Mico is happy

Assuming that "not fat" is thin, write the following statements in symbolic form

a) If Mico is fat then she is happy.

b) Mico is unhappy implies that Mico is thin

2) Write the following statements in symbolic form and their truth values

a) If n is prime, then n is odd or n is 2.

b) If x is nonnegative, then x is positive or x is 0.

c) If Tom is Ann's father, then Jim is her uncle and Sue is her aunt.

3) Consider the conditional proposition: "If a number is a multiple of 10, then the number is a multiple of 5".

(a) What is the truth value of that proposition?

(b) Deduce the converse of the proposition and the related truth value.

(d) Deduce the contrapositive of the proposition and the related truth value.

(f) Write its inverse and the related truth value.

4) Write the converse, inverse and contrapositive of the false conditional statement below and determine whether each of the statements found is true or false.

"If x is an even number, then the last digit of x is 2.

2.3.5 Bi-conditional statements

ACTIVITY 2.3.5

1. Let p be the statement: "Anne Maria is intelligent" and q be the statement: "Anne Maria is hard working"

   a. Express $p \Rightarrow q$ and its truth value.

   b. Express $q \Rightarrow p$ and its truth value

   c. Give the truth value of $(p \Rightarrow q) \wedge (q \Rightarrow p)$.

2. Let r be the statement: "$7 = 7$" and s be the statement: "$5 = 3$"

   a. Give the truth value of $r \Rightarrow s$

   b. Give the truth value of $s \Rightarrow r$

   c. Give the truth value of $(r \Rightarrow s) \wedge (s \Rightarrow r)$.

Content summary

Let us consider the proposition p: Two lines are perpendicular, q: Two lines form a right angle.

We have: $p \Rightarrow q$: If two lines are perpendicular, then the two lines form a right angle**.(True)**

The converse is $q \Rightarrow p$ means: if two lines form a right angle, then the two lines are perpendicular. **(True).**

We see that a statement and its converse are both true statements; that is $p \Rightarrow q$ and $q \Rightarrow p$ are both true.

Generally $p \Rightarrow q$ is not the same as $q \Rightarrow p$. It may happen, however, that both $p \Rightarrow q$ and $q \Rightarrow p$ are true. The statement $p \Leftrightarrow q$ is defined to be the statement $(p \Rightarrow q) \wedge (q \Rightarrow p)$.

For this reason, the double headed arrow $p \Leftrightarrow q$ is called the **bi-conditional**.

The **bi-conditional** $p \Leftrightarrow q$, which we read "***p* if and only if *q***" or "***p* is equivalent to *q***" is **true** if both *p* and *q* have the same truth values and **false** if *p* and *q* have opposite truth values.

p	q	$p \Leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

Examples

Let denote the statements as:

p: The number is divisible by 3

q: The sum of the digits forming the number is divisible by 3.

The compound statement "*The number is divisible by 3 if and only if the sum of the digits forming the number is divisible by three*";

This means that If the sum of the digits forming the number is divisible by 3, then the number is divisible by 3 and if the number is divisible by 3, then the sum of the digits forming the number is divisible by 3.

Symbolically $p \Leftrightarrow q$ means $(p \Rightarrow q) \wedge (q \Rightarrow p)$.

3) Given the statements: p: Two lines are perpendicular and q: Two lines form a right angle. Formulate 4 different compound statements and their truth values and deduce the truth table of $p \Leftrightarrow q$.

Solution

One of these compound statements is: "If two lines are perpendicular, then the two lines form a right angle". Make others and you will see the following table.

p	q	$p \Leftrightarrow q$
T	*T*	*T*
T	F	F
F	T	F
F	F	T

Equivalent statements

Let us compare the truth values of $\neg(p \vee q)$ and $\neg p \wedge \neg q$.

a) Considering the truth values of $\neg p \wedge \neg q$, we can determine the case in which this statement $\neg p \wedge \neg q$ is true.

p	q	$\neg p$	$\neg q$	$\neg p \wedge \neg q$
T	T	F	F	F
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

The first two columns of the truth table give all possible combinations of the truth values of p and q, and the second two columns of the truth table are merely negations of the first two columns. The statement $\neg p \wedge \neg q$ is the conjunction of two statements $\neg p$ and $\neg q$. Therefore, the only case in which $\neg p \wedge \neg q$ is true is when both A and b are false.

b) The truth values of $\neg(p \vee q)$ can be summarized in the truth table:

Like the previous question, the truth table consists of four rows. The third column of the truth table gives the truth values for the disjunction $p \vee q$. The fourth column gives the truth values for the negation of disjunction. Therefore, $\neg(p \vee q)$ is true only when both p and q are false.

p	q	$p \vee q$	$\sim(p \vee q)$
T	T	T	F
T	F	T	F
F	T	T	F
F	F	F	T

Comparing the truth values of $\neg(p \vee q)$ and $\neg p \wedge \neg q$ in the last columns of the two tables, we find that their truth values are identical.

Whenever two statements have the same truth values, the statements are said to be logically equivalent. The symbol of equivalent statements is ≡. Thus we can write

$$\neg(p \vee q) \equiv \neg p \wedge \neg q .$$

In spoken language, the equivalence of two propositions means they are similar, when you express one you have the same idea with he who expressed the other.

Example:

Use a truth table to show that the conditional $p \Rightarrow q$ and its contrapositive $\neg q \Rightarrow \neg p$ are logically equivalent.

Solution

Let us construct the truth table containing both $p \Rightarrow q$ and its contrapositive $\neg q \Rightarrow \neg p$.

p	q	$p \Rightarrow q$	$\neg q$	$\neg p$	$\neg q \Rightarrow \neg p$
T	T	T	F	F	T
T	F	F	T	F	F
F	T	T	F	T	T
F	F	T	T	T	T

The order of $\neg q$ and $\neg p$ when organizing the columns of the table must be respected.

It is clear that $p \Rightarrow q$ and $\neg q \Rightarrow \neg p$ have the same truth values in the third and the sixth columns.

Note: Two propositions differ if you can find at least one row of the truth table where values differ.

APPLICATION ACTIVITY 2.3.5

1) Suppose that r is a false statement, and s is a true statement.

   a) What is the truth-value of the compound statement $(\neg r) \Leftrightarrow s$?

   b) What is the truth-value of the compound statement $r \Leftrightarrow (\neg s)$?

   c) What is the truth-value of the compound statement $r \Leftrightarrow s$?

   d) What is the truth-value of the compound statement $\neg(r \Leftrightarrow (\neg s))$?

2) Construct the truth table for

a) $p \leftrightarrow q$ and $(p \to q) \wedge (q \to p)$ b) $p \leftrightarrow q$ and $(\neg p \vee q) \wedge (\neg q \vee p)$

c) $\neg(p \leftrightarrow q)$ and $(p \vee q) \wedge \neg(p \wedge q)$

d) $\neg(p \leftrightarrow q)$ and $(p \wedge \neg q) \vee (\neg p \wedge q)$

3) Show that neither the converse nor the inverse of an implication are equivalent to the implication.

4) Formulate a conditional statement made by two compound statements p and q. Use that statement to express different forms of $\neg p \vee q$ and $p \Rightarrow q$ and show that $\neg p \vee q$ and $p \Rightarrow q$ are equivalent. Relate to the daily life situation

2.4 Tautology and contradiction

ACTIVITY 2.4

Let p and q be two propositions. Construct the truth table of

1) $p \vee \neg p$　　2) $p \wedge (\neg p)$

3) $\neg p \wedge (p \wedge q)$　　4) $\neg(p \wedge q) \vee (p \vee q)$

Do you find a column in which the truth value is always true or always false?

Content summary

A tautology is a compound statement that is always **true** regardless of the truth values of the individual statements substituted for its statement variables.

Example

- The statement "**The main idea behind data compression is to compress data**" is a tautology since it repeats the same and it is always true.
- The statement **"I will either get paid or not get paid"** is a tautology since it is always true

If you are given a statement and want to determine if it is a tautology, then all you need to do is construct a truth table for the statement and look at the truth values in the final column.

If all of the values are **T (for true)**, then the statement is a tautology.

The statement **"I will either get paid or not get paid"** is a tautology since it is always true. We can use p to represent the statement "**I will get paid**" and not p (written $\neg p$) to represent "**I will not get paid**."

p : I will get paid

$\neg p$: I will not get paid

So, $p \vee (\neg p)$: I will either get paid or not get paid

A truth table for the statement would look like:

p	$\neg p$	$p \vee (\neg p)$
T	F	T
F	T	T

Looking at the final column in the truth table, one can see that all the truth values are T (for true). Whenever all of the truth values in the final column are true, the statement is a tautology. So, our statement 'I will either get paid or not get paid' is always a true statement, a tautology.

A contradiction is a compound statement that is always **false** regardless of the truth values of the individual statements substituted for its statement variables.

Example

1) The statement "**I don't believe in reincarnation, but I did in my past life**", is always false.

2) The statement $p \wedge (\neg p)$ is always false, because p and $\neg p$ cannot both be true.

P	$\neg p$	$p \wedge \neg p$
T	F	F
F	T	F

APPLICATION ACTIVITY 2.4

1) From the following compound statements, indicate which is tautology, contradiction or neither.

a) $p \wedge \neg(p \wedge q)$ b) $\neg q \wedge (q \wedge r)$

c) $(p \wedge q) \wedge \neg(p \vee q)$ d) $(p \vee r) \vee \neg r$

2) In your own words, formulate 2 propositional statements expressing the tautology and 2 propositional statements expressing the contradiction. and show where these tautology or contradiction can be avoid in every day practice of argumentation.

2.5 Quantifiers

2.5.1 Universal quantifier

ACTIVITY 2.5.1

1) Let $A = \{1, 2, 3, 4, 5\}$. Determine the truth value of each of the following statements:

a) For all $x \in A, x + 3 < 9$

b) For all $x \in A, x + 3 \leq 7$

2) Determine the truth value of the following statement where $U = \{1, 2, 3\}$ is the universal set:

There exist $x \in U$ such that $x + 3 \leq 5$

3) What is the truth value of p, q if p: "All men eat banana", q: All men are mortal.

Content summary

Let *p(x)* be a propositional function defined on a set *A*. Consider the expression

$(\forall x \in A)\ p(x)$ or $\forall x\ p(x)$ which is read by **"For every x in A, $p(x)$ is a true statement"** or, simply, **"For all x, $p(x)$."**

The symbol universal quantifier "For all" symbolized by $\forall$ is read by **"for all"** or **"for every"** is called the ***universal quantifier***.

The statement $(\forall x \in A)\ p(x)$ means that the truth set of $p(x)$ is the entire set A.

Example:

All men are mortal. This is true if for every man, dying is applicable.

The expression $p(x)$ by itself is an open sentence or condition and therefore has no truth value. However, $(\forall x \in p(x)$, that is $p(x)$ proceeded by the quantifier $\forall$, does have a truth value which follows from the equivalence.

Specifically:

Q_1 : if $\{x / x \in A, p(x)\} = A$ then $\forall x\ p(x)$ is True; otherwise $\forall x\ p(x)$ is False.

Example

(a) The proposition $(\forall n \in \mathbb{N})(n+4>3)$ is true since $(n / n+4>3\} = \{1,2,3,..\} = \mathbb{N}$

(b) The proposition $(\forall n \in \mathbb{N})(n+2>8)$ is false since $(n / n+2>8\} = \{7,8,...\} = \mathbb{N}$

APPLICATION ACTIVITY 2.5.1

1. Given the statement p: "for every real number x, $x+y>10$." discuss the values of y which make p true.

2. Give the truth value for the following statements:

a) Some rectangles are squares

b) All squares are rectangles

c) Every language student-teacher must take a logic mathematics course.

d) $\forall x \in \mathbb{N}, x+4>4$."

2.5.2 Existence quantifier

ACTIVITY 2.5.2

1) Let $A = \{1,2,3,4,5\}$. Determine the truth value of each of the following statements:

   a) There exist $x \in A,\ x + 3 = 10$

   b) *For some* $x \in A, x + 3 < 5$

2) Let $A = \{1,2,3,...,8,9,10\}$. Consider each of the following sentences; determine the set of y for which the following statement is true. Such set is called a truth set.

   (a) $(\forall x \in A)(\exists y \in A)(x + y < 14)$

   (b) $(\exists y \in A)(x + y < 14)$.

Content summary

Let $p(x)$ be a propositional function defined on a set A. Consider the expression: *there exist some* $x \in A$ that satisfy p(x). This can be written as $(\exists x \in A)p(x)$ or $\exists x, p(x)$ or "There exists an x in A such that $p(x)$ is a true statement" or, simply, "For some x, $p(x)$." The quantifier "there exist" or "for some" or "for at least one" symbolized by $\exists$ is an existential quantifier. The statement $(\exists x)p(x)$ is read as "there exists an x such that $p(x)$ is true".

Specifically:

Q_2 : if $\{x / p(x)\} \neq 0$ then $\exists x\ p(x)$ is True; otherwise $\exists x\ p(x)$ is False.

Example

(a) The proposition $(\exists n \in \mathbb{N})(n + 4 < 7)$ is true since $\{n / n + 4 < 7\} = \{1, 2,\} \neq \varnothing$

(b) The proposition $(\exists n \in \mathbb{N})(n + 6 < 4)$ is false since $\{n / n + 6 < 4\} = \phi$.

c) Some men do not have wives, is true since some men such as Priests remain single.

d) There exist natural number between 3 and 6. This is statement is true because $\{4,5\}$ exists.

e) Some student teachers teach at university. This is false because the set of such students is empty.

APPLICATION ACTIVITY 2.5.2

1. Translate the following into symbolic form:

   a) Somebody cried out for help and called the police.

   b) Nobody can ignore her.

2. Consider the predicates: $r(x): x-7=2$ *and* $s(x): x>9$. If the universe of discourse is the real numbers, give the truth value of propositions: $(\exists x)s(x)\wedge\neg(\forall x)r(x)$

3. Formulate different statements with existential quantifiers and give their truth values.

2.5.3 Negation of quantifiers

ACTIVITY 2.5.3

Negate the following statements

1. All grapefruits have red colour.
2. Some celebrities are beautiful.
3. No one weighs more than five hundred Kilograms.
4. Some people are more than 2m tall.
5. All snakes are poisonous.
6. Some mammals can stay under water for two days without surfacing for air.
7. All birds can fly.

Consider the statement: "All dogs have tails". Its negation is "Not all dogs have tails" which means that: "There is at least one dog that does not have a tail."

Let us express these ideas symbolically:

Let $p(x)$ mean "dog x has a tail". Then all dogs have tails is represented by $\forall x, p(x)$ and dog x doesn't have a tail is $\neg p(x)$. So not all dogs have tails is $\neg[\forall x, p(x)]$. Abbreviating there is at least one dog that doesn't have a tail or there exists a tailed dog, we can say that there exists a tailless dog is $\exists x, p(x)$.

Since the last two statements have, in words, the same meaning; we can declare the predicate formulae representing them to be logically equivalent: $\neg[\forall x, p(x)] \equiv \exists x[\neg p(x)]$.

By generalizing the argument that we have used to produce two results, it can be shown that the logical equivalence holds true no matter what predicate is represented by $p(x)$. In fact, this is one of the generalized of the De Morgan's laws. For predicate logic, the other is similar: $\neg[\exists x, p(x)] \equiv \forall x[\neg p(x)]$ that can be interpreted as "there are no dogs with tails" has the same meaning as "every dog is tailless."

Examples

1. In statements that involve the words "all, every, some, none or no", forming the negation is not as easy. The following table shows examples

Statement	Negation
All men are mortal	Some men are not mortal
Some of the numbers are not positive	All of the numbers are positive
No birds are fish	Some birds are fish
Some women can jump	No women can jump
None of the small children worked	Some of the small children worked

This shows that:

"Negating a universally quantified statement changes it into an existentially quantified statement and vice-versa with the part of the statement after the quantifier becoming negated.

2. Let $p(x)$ mean "country x has a president." Interpret the following statements. Establish the equivalence of their negation.

a) $\forall x, p(x)$ b) $\exists x, p(x)$

Solution

a) Since $p(x)$ mean "country x has a president, thus $\forall x, p(x)$ means "all countries have president" The negation is $\neg[\forall x, p(x)]$ that is interpreted as "it is not true that all counties have presidents."

This sentence can be express as "there exist a country without a president" and symbolically it is written as $\exists x\left[\neg p(x)\right]$.

Thus, $\neg\left[\forall x, p(x)\right] \equiv \exists x\left[\neg p(x)\right]$

b) $\exists x, p(x)$ means "there is a country with a president."

Negating the given statements, we get

$\neg\left[\exists x, p(x)\right]$ which means "it is not true that there is a country with a president."

This sentence is equivalent to" all countries have not a president" symbolically $\forall x\left[\neg p(x)\right]$. Hence $\neg\left[\exists x, p(x)\right] \equiv \forall x\left[\neg p(x)\right]$.

These **laws can be summarized** as follow:

"Negating a universally quantified formula changes it into an existentially quantified formula and vice-versa with the part of the formula after the quantifier becoming negated."

The basic forms for negative statements that involve "all, every, some, none or no" can be summarized as follows:

Statement	**Negation**
All/every	Some .... not.
Some ... not	All/ every
None/no	Some
Some	None/No

APPLICATION ACTIVITY 2.5.3

Negate each of the following statements and write the answer in symbolic form:

1. Some students are mathematics majors.
2. Every real number is positive, negative or zero.
3. Every good boy does fine.
4. There is a broken desk in our classroom.
5. Lockers must be turned in by the last day of class.
6. Haste makes waste.

2.6 Applications of logic in real life

ACTIVITY 2.6

It is known that you must have the correct form and true premises to reason deductively toward a true conclusion.

The following are examples of arguments that need true conclusions. Try to conclude:

a) If you live in Nyarugenge, then you live in Kigali.

If you live in Kigali, then you live in Rwanda.

Therefore, ...

b) If a triangle is isosceles, then it has two equal sides.

If a triangle has two equal sides, then it has two equal angles.

Therefore, ...

c) If you study the whole student book, then you will pass the exam.

You study the whole student book.

Therefore, ...

Content summary

"A syllogism is form of argumentation in the deductive way that consists of two statements or premises, and a logical conclusion drawn from them. These premises are usually articles of faith, laws, rules, definitions, assumptions, or commonly accepted facts." In the following paragraphs, we discuss three types of syllogisms: **hypothetical syllogism**, **affirming the antecedent**, and **denying the consequent**.

2.6.1 Hypothetical Syllogism

If A, B, and C represent statements, a hypothetical syllogism is constructed from the statements, the first two lines being the premises and the third being the conclusion. The hypothetical syllogism can be written in three different ways:

If A, then B If B, then C $\therefore$ If A, then C	A implies B B implies C Therefore, A implies C	$A \Rightarrow B$ $B \Rightarrow C$ $\therefore A \Rightarrow C$

$\therefore$ **Means therefore**

In the hypothetical syllogism, the argument is correct even when one or both of the premises are false. The truth or falsehood of the premises does not affect the logic of the argument. Logic deals with the relationship between premises and conclusion, not the truth of the premises.

To say that a deductive argument is correct means that the premises are related to the conclusion in such a way that, if the premises are true, the conclusion must be true. A conclusion cannot be false if the logical form is correct and the premises are true.

Example

a) If you live in Rwamagana, then you live in Eastern province.

If you live in Eastern province, you live in Rwanda.

Therefore, if you live in Rwamagana, then you live in Rwanda.

Solution: It is the hypothetical syllogism.

b) Is the following argument a hypothetical syllogism? Why or why not?

If you have a party, you should invite your friends.

If you are graduating from college, you should invite your friends.

Therefore, if you are having a party, you are graduating from college.

Solution:

The argument is not a hypothetical syllogism. The premises do not link properly. The conclusion of the first premise should be the hypothesis of the second premise, and no logical rearrangement can accomplish the proper linking of the statements.

a) Even though the conclusion of this argument is true, explain why the following argument is a poor one.

If you are over 18 years old, then you can read.

If you can read, you can vote.

Therefore, if you are over 18 years old, then you can vote.

Solution

The argument has the form of hypothetical syllogism, so it is a correct argument. However, it is a poor argument, since neither of the premises is true, the argument does not actually prove its conclusion.

2.6.2 Affirming the Antecedent

If A and B represent statements, an argument that affirms the antecedent has the following form.

Major premise: $A \Rightarrow B$

Minor premise: A

Conclusion: $\therefore B$

The major premise is a conditional statement $A \Rightarrow B$. The minor premise states that the hypothesis of the major premise is true or has occurred.

This is called **affirming the antecedent**.

Example

If I study for 6 hours, I will pass the exam.

I studied for 6 hours.

Therefore, I will pass the exam.

Solution:

This classical argument is another example of affirming the antecedent.

All men are mortal.

Makuza is a man.

Therefore, Makuza is mortal.

This can be written so that the correct form is apparent.

If one is a man, then one is mortal.

Makuza is a man.

Therefore, Makuza is mortal.

If an argument has the correct form, it is a logical argument. However, if it is to be a convincing argument with a true conclusion, its premises must also be true. You can affirm the antecedent to reason deductively if the argument has the correct form and true premises.

Example

Is the following argument a good one? Explain.

If you want to run a marathon, then you should train for the race.

Mukamurenzi wants to run a marathon.

Therefore, Mukamurenzi should train for the race.

Solution:

The argument has the correct form for affirming the antecedent. If we take its first premise as true because of commonly accepted notions about the physical stamina needed to run a marathon, the argument is a good one.

Even though the argument has the correct form of an argument using the technique of affirming the antecedent, the major premise is not true. Thus, it is a correct argument but it does not arrive at a true conclusion. You need both the correct form and true premises to ensure true conclusions.

2.6.3 Denying the Consequent

If ***A*** and ***B*** represent statements, an argument that denies the consequent has the following form:

Major premise: $A \Rightarrow B$

Minor premise: $\sim B$

Conclusion: $\therefore \sim A$

Examples

1. If Edna is at the school, then she has notebooks.

Edna does not have notebooks.

Therefore, Edna is not at the school.

2. If you pay the bill on time, then you are not charged a penalty.

You are charged a penalty.

Therefore, you did not pay the bill on time.

The major premise is a conditional statement. The minor premise is a denial (negation) of the consequent (conclusion) of the conditional statement. For this reason, this argument is called denying the consequent.

This form of argument is based on the contra -positive principle in which the statement $A \Rightarrow B$ is logically equivalent to $\sim B \Rightarrow \sim A$. We can see that this form of argument is correct by observing that it is really an application of affirming the antecedent.

Major premise: $A \Rightarrow B$ is equivalent to $\sim B \Rightarrow \sim A$ Minor premise: $\sim B$ is equivalent to $\sim B$

Conclusion: $\sim A$ is equivalent to $\sim A$

Example

Is the following argument a good one? Explain.

If a number is not positive, then the number is negative.

Zero is not negative.

Therefore, zero is positive.

Solution:

The argument has the form of an argument using denying the con-sequent, so it is a correct argument. However, its first premise is not true, since if a number is not positive, it could be either negative or zero. Thus, the argument is faulty.

We can also make correct arguments from premises that do not at first glance seem to be one of our standard logical forms, as in the next example.

Example

Construct a logically correct argument from the following premises:

If *P*, then $\sim Q$.

If $\sim R$, then Q.

If R, then $\sim S$.

Solution:

To have the correct form of the hypothetical syllogism, the conclusion of one statement must be the hypothesis of the next statement. Since we know that if a statement is true, its contrapositive is true, we can use that principle on the second premise, that is, "If $\sim R$, then Q" implies "If $\sim Q$, then R."

If *P*, then $\sim Q$.

If $\sim R$, then Q.

If *R*, then $\sim S$.

This can be reformulated in the following way:

If *P*, then $\sim Q$.

If $\sim Q$, then *R*.

If *R*, then $\sim S$

$\therefore$ if P then $\sim S$

Thus, the conclusion of this argument is: if P then $\sim S$

Even though you may use the contrapositive statement in a logical argument, do not use the inverse or converse.

APPLICATION ACTIVITY 2.6

1. What is wrong with the following argument?

All good chess players wear glasses.

Sylvia is a good chess player.

Therefore, Sylvia wears glasses.

2. Determine whether or not the following arguments are correct. For those are not correct,

(a) Explain what is wrong with the argument;

(b) Change the minor premise and make a correct argument.

i. When it is midnight, I am asleep.

I was asleep.

Therefore, it was midnight.

ii. All Rhode Island Red hens lay brown eggs.

My hen, Motopa, is a Rhode Island Red.

Therefore, Motopa lays brown eggs

iii. If *ABCD* is a square, it has four sides.

If it has four sides, then it is a quadrilateral.

Therefore, if *ABCD* is a square, it is a quadrilateral.

iv) If a triangle is equilateral, then it has three equal sides.

ABC does not have three equal sides.

Therefore, ABC is not equilateral.

2.7 END UNIT ASSESSMENT

1. Which of the following sentences are propositions?

   a) Pretoria is the capital of South Africa

   b) Is this concept important?

   c) Wow, what a day!

2. Find the negation of the proposition "Today is Monday"

3. Find the conjunction of the propositions p and q, where p is the proposition "Today is Sunday" and q is the proposition "The moon is made of cheese".

4. Find the disjunction of the propositions p and q, where p is the proposition "Today is Sunday" and q: "The moon is made of cheese".

5. How many rows, not counting the top one, are needed to construct the truth table for a compound statement made from n statements?

6. Show that $[p \wedge (p \rightarrow q)] \rightarrow q$ is a tautology.

7. Find the truth value of the bi-conditional "The moon is made of cheese if and only if 1=2".

8. Construct the truth table for the proposition $(\neg p \rightarrow q) \wedge r$

9. In the question below are given three statements, followed by conclusions: I, II, III, IV. You have to take the given statements to be true even if they seem to be at variance from commonly known facts. Read the conclusions and then decide which of the given conclusions logically follows from the given statements disregarding commonly known facts.

Statements: Some Cats are Rats. All bats are tables. All Rats are Bats.

Conclusion:

I. Some Cats are bats   II. All bats are rats

III. All tables are cats   IV. All bats are cats

1. Only I & II follow   2. Only II follows

3. Only I & IV follow   4. None of these

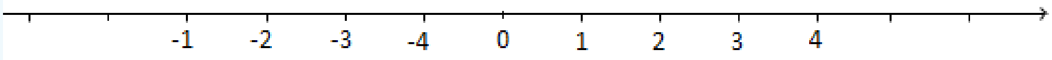
UNIT 3

POINTS, LINES AND GEOMETRIC SHAPES IN 2D

Key Unit competence: To be able to determine algebraic representations of lines and calculate the area of geometric shapes in 2D.

3.0 INTRODUCTORY ACTIVITY 3

RUKUNDO uses a number line to graph the points -4, -2, 3, and 4. Her classmate ISIMBI, notices that -4 is closer to zero than -2 as shown in the figure below. She told him that he is mistaken. Rukundo replied that his is not sure about his diagram because he did not have time to repeat his course yesterday evening.



a) Use what you know about a vertical number line to determine if RUKUNDO made a mistake or not. Support your explanation with a number line diagram

b) What is number line

c) What is Cartesian plane?

3.1. Cartesian coordinates of a point

ACTIVITY 3.1

Consider the points $A(1,2)$, $B(5,4)$

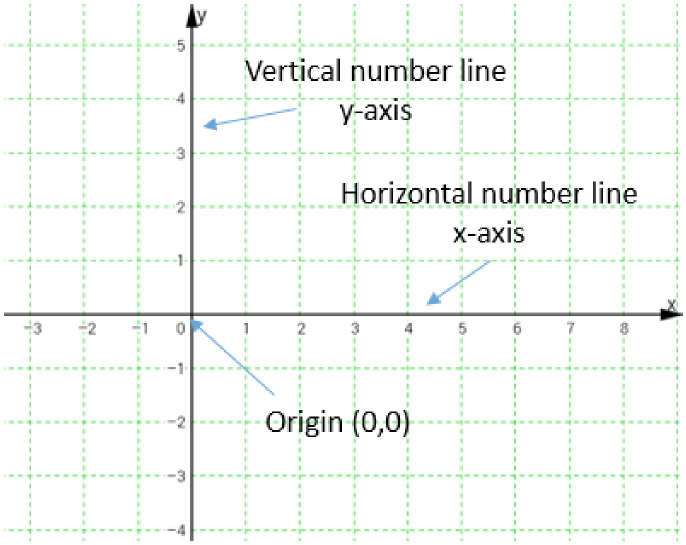
1. Represents these points in xy plane
2. Draw a line segment from A to B

Content summary

A point is represented by Cartesian coordinates (also called rectangular coordinates). In two dimensions, Cartesian coordinates are a pair of numbers that specify signed distances from the coordinate axes. They are specified in terms of the *x* coordinates and the *y* coordinates. The origin is the intersection of the two axes.

The position of a point on the Cartesian plane is represented by a pair of numbers. The pair is called an ordered pair or coordinates (x, y). The first number, x, called the x-coordinate and the second number, y, is called the y-coordinate.

Coordinate Plane or Cartesian Plane



The origin is indicated by the ordered pair or coordinates (0, 0).

To get to the point (x, y) in cartesian plane, we start from the origin. If x is positive then we move x units right from the origin otherwise if x is negative then we move x units left from the origin. Then, if y is positive, we move y units up otherwise if y is negative, we move y units down

Example:

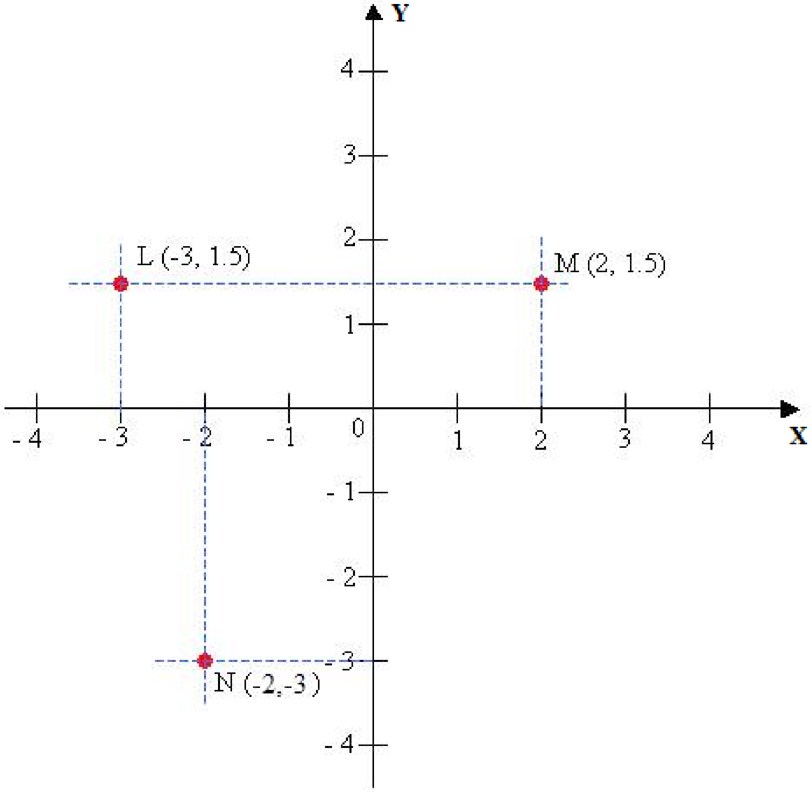
Represent the points $M(2,1.5)$, $L(-3,1.5)$ and $N(-2,-3)$ in Cartesian plane.

Solution

- Point M has coordinates $M(2,1.5)$. To get to point M, we move 2 units to the right of x-axis from the origin (positive side) and 1.5 units up on y-axis from the origin (positive side).

- Point L is represented by the coordinates $L(-3,1.5)$. To get to point L, we move 3 units to the left of x-axis from the origin (negative side) and 1.5 units up on y-axis from the origin (positive side)

- Point N has coordinates $N(-2,-3)$. To get to point N, we move 2 units to the left of x-axis from the origin of(negative) and 3 units down on y-axis from the origin (negative side).



Points on Coordinate Plane

APPLICATION ACTIVITY 3.1

Represent the points $A(-3,2)$, $B(4,2)$, $C(-3,-2)$, $D(3,-1)$ in xy-plane

3.2. Distances between two points

ACTIVITY 3.2

Kalisa and Mugisha live in the same Sector but at

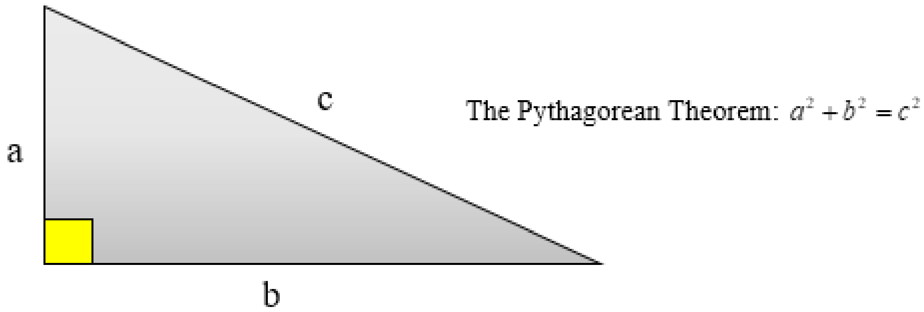
two different hills. If Kalisa's house is located at $A(1,2)$ from the Sector and Mugisha's house is at $A(4,6)$.

a) If the office of the sector is considered as the origin, present this situation on Cartesian plan.

b) Use the ruler to calculate the distance between Kalisa's house and Mugisha's.

Content summary

Recall from the Pythagorean Theorem that, in a right triangle, the hypotenuse ***c*** and sides ***a*** and ***b*** are related by $a^2+b^2=c^2$. Conversely, if $a^2+b^2=c^2$ the triangle is a right (see figure below).

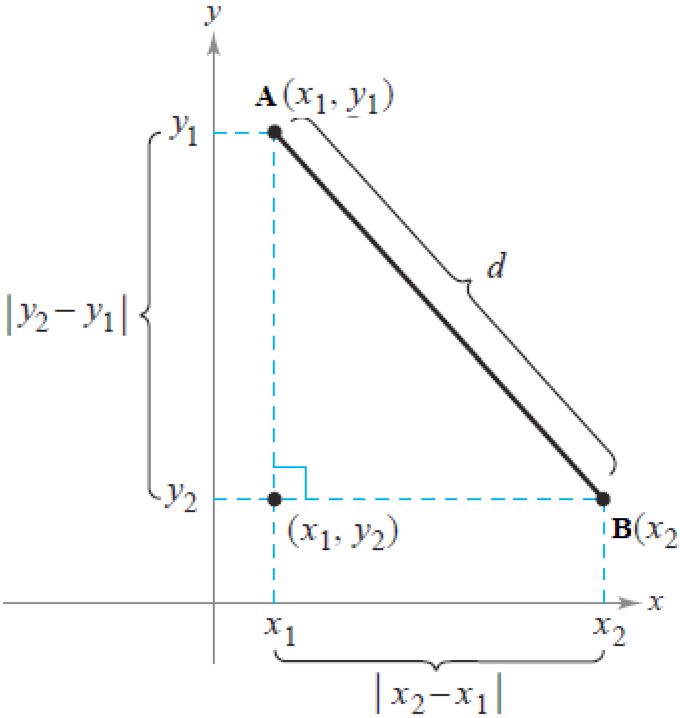


If A and B are two points on xy - coordinates, we can form a vector $\overrightarrow{AB}$ and the distance between these two points denoted $d(A,B)$ is given by $\left\|\overrightarrow{AB}\right\|$. Suppose you want to determine the distance "d" between the two points (x_1,y_1) and (x_2,y_2) in the plane.

If the points lie on a horizontal line, then $y_1=y_2$ and the distance between the points is $d=|x_2-x_1|$.

If the points lie on a vertical line, then $x_1=x_2$ and the distance between the point is $d=|y_2-y_1|$.

If the two points do not lie on a horizontal or on a vertical line, they can be used to form a right triangle, as shown in ~~in~~ the figure below.



The length of the vertical side of the triangle is $|y_2 - y_1|$, and the length of the horizontal side is $|x_2 - x_1|$. By Pythagorean Theorem, it follows that:

$d^2(A,B) = (x_2 - x_1)^2 + (y_2 - y_1)^2$ and $d(A,B) = \|\overrightarrow{AB}\| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.

Thus, if $A(x_1, y_1)$ and $B(x_2, y_2)$ are points of plane, then

The distance "*d*" between the points $A(x_1, y_1)$ and $B(x_2, y_2)$ in the Cartesian plane, is given by: $d(A,B) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

Examples:

1) Consider the points $A(1,4), B(-2,-3)$ in **Cartesian** plane. Find the distance between the point A and B.

Solution

The distance is

$$d(A,B)=\left\|\overrightarrow{AB}\right\|=\sqrt{(-2-1)^2+(-3-4)^2}$$
$$=\sqrt{9+49}$$
$$=\sqrt{58}\quad units$$

2) Consider the points $C(k,-2)$ and $D(0,1)$ in cartesian plane. Find the distance between the point A and B.

Solution

$$d(C,D)=\sqrt{k^2+9}$$

$$\sqrt{k^2+9}=5$$
$$\Leftrightarrow k^2+9=25$$
$$\Leftrightarrow k^2=16 \Rightarrow k=\pm 4$$

Thus the values of *k* are -4 and 4

APPLICATION ACTIVITY 3.2

1. Calculate the distance between the points given below:
   a) $S(-2;-5)$ and $Q(7;-2)$
   b) $A(2;7)$ and $B(-3;5)$
   c) $A(x;y)$ and $B(x+4;y-1)$
2. The length of $CD = 5$. Find the missing coordinate if:
   a) $C(6;-2)$ and $D(x;2)$.
   b) $C(4;y)$ and $D(1;-1)$.

3.3. Midpoint of a line segment

ACTIVITY 3.3

Three friends: Pascal, Steve, and Benjamin live on the same side of a street from Nyabugogo to Ruyenzi. Steve's house is halfway between Pascal's and Benjamin's houses. If the locations of Pascal can be given by the coordinates (2,5) , Benjamin's house at (4,7);

a) Draw a Cartesian plan and locate Pascal's and Benjamin's houses

b) Show the line segment joining the locations of Pascal's and Benjamin's houses;

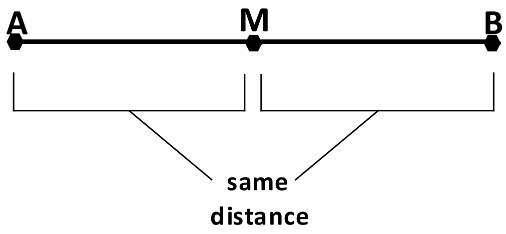
c) Given that Steve's house is in the half way, locate his house, estimate the coordinate of that location and explain how to find it.

Content summary

A line segment from point *A* to point *B*, denoted $[AB]$ is the set of all points on the part of the line that joins *A* and *B*, including *A* and *B*. The midpoint of this segment is point *M such that the distance* $[AM]=[MB]$ and it is given by given by $M=\frac{1}{2}(A+B)$.

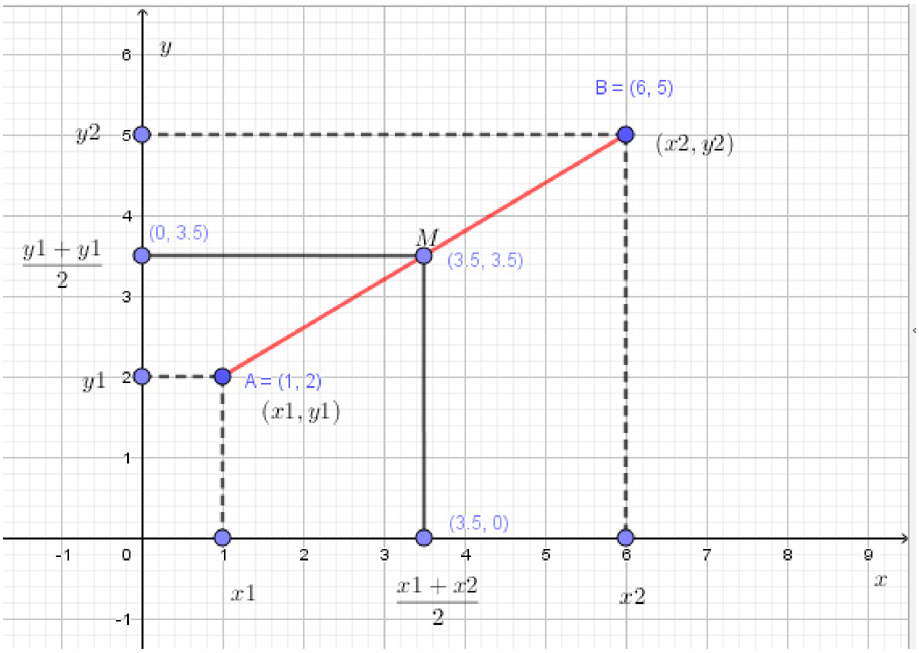
The midpoint of a segment is a point on that line segment which maintains the same distance from both of the endpoints of that line segment.

For example, consider segment to the left.



It has endpoints named A and B. The midpoint of the segment is labeled M. It is at the same distance from each of the endpoints.

Suppose that the coordinates of the line segment $\overline{AB}$, are $A(x_1,y_1)$ and $B(x_2,y_2)$
.



The coordinates of the midpoint, are obtained using the midpoint formula, such that:

The midpoint formula, is given by:

Coordinates of midpoint = $\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$

Example:

1) Find the midpoint of the segment joining points $A(3,0)$ *and* $B(1,8)$.

Solution

The midpoint is $M = \frac{1}{2}(A+B) = \frac{1}{2}(4,8) = (2,4)$.

2) If $(-3,5)$ is the midpoint of $(2,6)$ and (a,b), find the value of a and b.

Solution

$$(-3,5) = \frac{1}{2}(2+a, 6+\mathrm{b})$$

$$\Leftrightarrow (-3,5) = \left(\frac{2+a}{2}, \frac{6+b}{2}\right) \Rightarrow \begin{cases} \frac{2+a}{2} = -3 \\ \frac{6+b}{2} = 5 \end{cases} \Leftrightarrow \begin{cases} 2+a = -6 \\ 6+b = 10 \end{cases} \Rightarrow \begin{cases} a = -8 \\ b = 4 \end{cases}$$

APPLICATION ACTIVITY 3.3

Micheal and Sarah live in different cities and one day they decided to meet up for lunch. Because they both wanted to travel as little as possible they decided to meet at a point halfway between their homes. If their positions are given by $(3100, 500)$ and $(5120, 125)$.

Which of the following coordinates represents the place where they should meet?

a) $(4110, 312.5)$

b) $(4110, 375)$

c) $(2020, 375)$

d) $(8220, 625)$

3.4. Vector in 2D and dot product

3.4.1. Vectors in 2D

ACTIVITY 3.4.1

In xy plane:

1. Represent the points $A(1,2)$ and $B(-3,1)$
2. Draw arrow from point A to point B
3. Refer to the content of S2 and represent the Vector $\vec{AB}$

Content summary

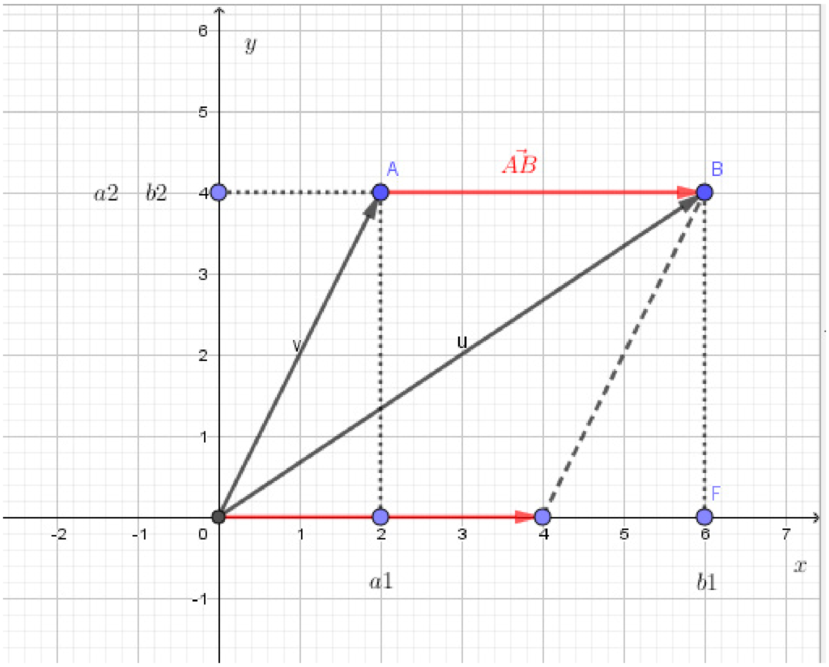
Definitions and operations on vectors

A vector is a directed line segment; it is a quantity that has both magnitude and direction. A vector is represented by an arrow.

The length of the arrow represents the **magnitude** of the vector, and the arrowhead indicates the **direction** of the vector. That is to say, a vector has a given length and a given direction.

The vector joining point *A* and point *B* is denoted by $\overrightarrow{AB}$ and to find it we subtract the coordinates of point *A* from the coordinates of point *B*.

For example the vector $\overrightarrow{AB}$ defined by two points $A(a_1, a_2)$ and $B(b_1, b_2)$



On the figure, it is clear that the vector

$$\vec{AB}=\vec{OB}-\vec{OA}=(b_1,b_2)-(a_1,a_2)=(b_1-a_1,b_2-a_2)=\begin{pmatrix}b_1-a_1\\ b_2-a_2\end{pmatrix}$$

Therefore, $\overrightarrow{AB}=(b_1-a_1,b_2-a_2)$ which can also be written in the form of a column vector $\vec{AB}=\begin{pmatrix}b_1-a_1\\ b_2-a_2\end{pmatrix}$.

The point A is called the **initial point** or tail of $\overrightarrow{AB}$ and B is called the **terminal point** or **tip**. The **zero vector** is $(0,0)$ denoted by $\vec{0}$.

Examples

1) The vector defined by point $A(1,2)$ and $B(4,3)$ is $\overrightarrow{AB}=(3,1)$

2) Vector $\overrightarrow{CD}=(-4,2)$ is defined by point $C(-3,4)$ and D. Find the coordinates of point D.

Solution: Let the point D be $D(x,y)$, then $\overrightarrow{CD}=(x+3,y-4)=(-4,2)$

$x+3=-4\Rightarrow x=-7$

$y-4=2\Rightarrow y=6$

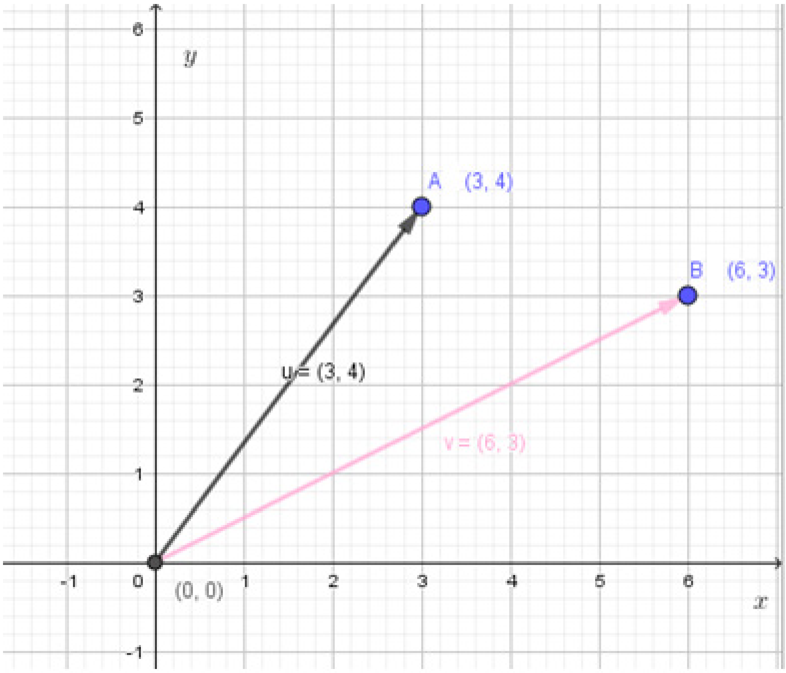
Thus, point *D has the coordinates (-7, 6).*

Position Vector

In a Cartesian plane, an **algebraic vector** $\vec{v}$ is represented as $\vec{v} = (a, \mathrm{b})$ where a and b are real numbers (scalars) called the **components** of the vector $\vec{v}$.

We use a rectangular coordinate system to represent algebraic vectors in the plane.

If $\vec{v} = (a, \mathrm{b})$ is an algebraic vector whose initial point is at the origin, then $\vec{v}$ is called a position vector. See the figure below.



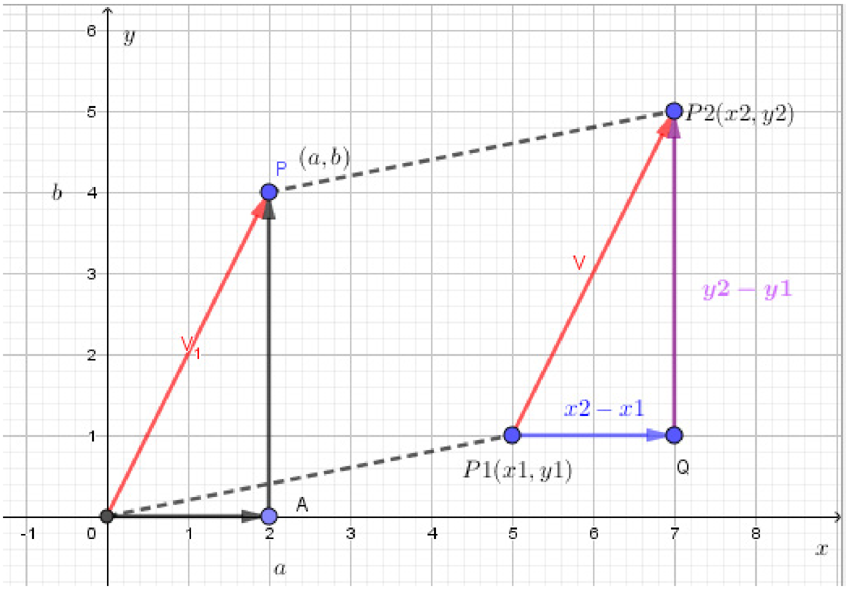
Notice that the terminal point of the position vector $\vec{v} = (a, \mathrm{b})$ is $\mathrm{A}(a, \mathrm{b})$. On the figure we have $\vec{u} = (3,4)$ and $\vec{v} = (6,3)$.

The next result states that any vector whose initial point is not at the origin is equal to a unique position vector.

Suppose that $\vec{v}$ is a vector with **initial point** $P_1(x_1, y_1)$ not necessarily the origin, and **terminal point** $P_2(x_2, y_2)$.

If $\vec{v} = \overrightarrow{P_1P_2}$, then $\vec{v}$ is equal to the vector $\vec{v} = \overrightarrow{P_1P_2} = (x_2 - x_1, y_2 - y_1)$.

See the figure below.



The triangle OPA and triangle P_1P_2Q are congruent. This is because the line segments have the same magnitude. So, $d(0,P)=d(P_1,P_2)$; and they have the same direction, so the angle $\angle POA=\angle P_2P_1Q$.
Since the triangles are right triangles, we have angle–side–angle congruence. It follows that corresponding sides are equal. As a result, $x_2-x_1=a$ and $y_2-y_1=b$, and so $\vec{v}$ may be written as

$\vec{v}=\overrightarrow{P_1P_2}=(a,b)=(x_2-x_1,y_2-y_1)$.
Because of this result, we can replace any algebraic vector by a unique position vector with initial point the origin $O(0,0)$ and vice versa.

On the figure above we have: $\vec{V_1}=(2,4)$ as a position vector.

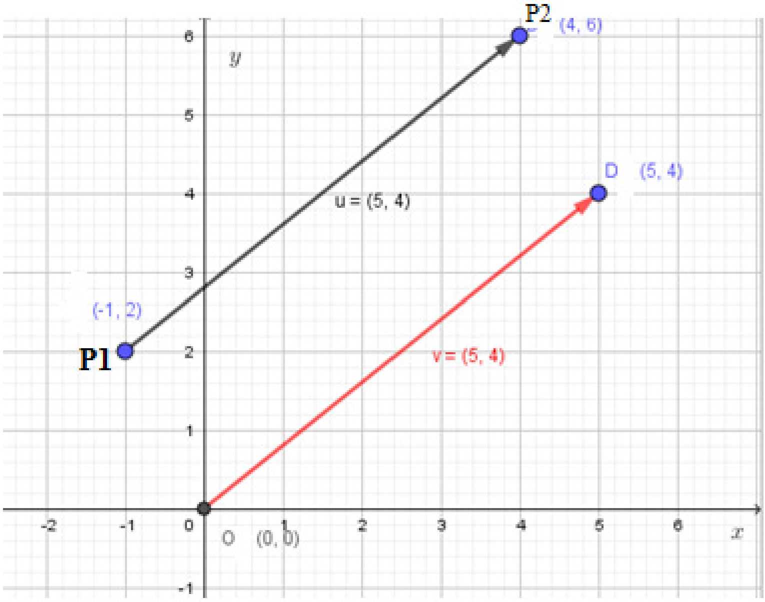
$P_1(5,1);\ P_2(7,5)$ *and* $\vec{V}=\overrightarrow{P_1P_2}=(7-5,5-1)=(2,4)$. This shows that the vector $\vec{V}_1=\vec{V}=(2,4)$ which means that $\vec{V}_1=(2,4)$ is the position vector of the vector $\vec{V}$.

Example:

Find the position vector of the vector $\vec{v}=\overrightarrow{P_1P_2}$ if $P_1(-1,2)$ and $P_2=(4,6)$.

Solution

$\vec{v} = \overrightarrow{P_1P_2} = P_2 - P_1 = (4-(-1), 6-2) = (5,4)$.



Components of a vector

We now present an alternative representation of a vector in the plane that is common in the physical sciences. Let $\vec{i}$ denote the unit vector whose direction is along the positive x-axis;

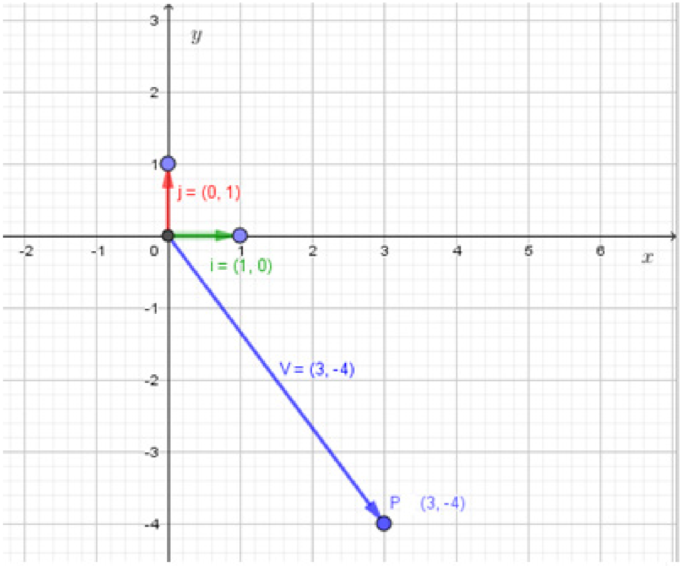
let $\vec{j}$ denote the unit vector whose direction is along the positive y-axis. Then $\vec{i} = (1,0)$ and $\vec{j} = (0,1)$.

Any vector $\vec{v} = (a,b)$ can be written using the unit vectors $\vec{i}$ and $\vec{j}$ as follows:

$\vec{v} = (a,b) = a(1,0) + b(0,1) = a\,\vec{i} + b\,\vec{j}$

We call a and b the **horizontal** and **vertical components** of $\vec{v} = (a,b)$, respectively.

For example, if $\vec{v} = (3,-4) = 3\,\vec{i} - 4\,\vec{j}$, then 3 is the **horizontal component** and -4 is the **vertical component**.



Note: It is now easy to show that if we have two points $A(a_1, a_2)$ and $B(b_1, b_2)$,

the vector $\vec{AB} = \vec{OB} - \vec{OA} = (b_1, b_2) - (a_1, a_2) = (b_1 - a_1, b_2 - a_2)$

Because

$$\vec{OA} = a_1 \vec{i} + a_2 \vec{j}, \vec{OB} = b_1 \vec{i} + b_2 \vec{j}$$

$$\vec{AB} = \vec{OB} - \vec{OA} = \left(b_1 \vec{i} + b_2 \vec{j} \right) - \left(a_1 \vec{i} + a_2 \vec{j} \right)$$

$$= b_1 \vec{i} + b_2 \vec{j} - a_1 \vec{i} - a_2 \vec{j}$$

$$= b_1 \vec{i} - a_1 \vec{i} + b_2 \vec{j} - a_2 \vec{j}$$

$$= (b_1 - a_1) \vec{i} + (b_2 - a_2) \vec{j}$$

$$\overline{AB} = (b_1 - a_1) \vec{i} + (b_2 - a_2) \vec{j} = (b_1 - a_1, b_2 - a_2)$$

Therefore, $\overline{AB} = (b_1 - a_1, b_2 - a_2)$ $\overline{AB} = \begin{pmatrix} b_1 - a_1 \\ b_2 - a_2 \end{pmatrix}$

APPLICATION ACTIVITY 3.4.1

1) In xy plane, present the following vectors:

   a. $\vec{u}=(5,6),\ \vec{v}=(-1,4)$

   b. $\vec{u}=(3,5),\ \vec{v}=(-1,6)$

2) Plot a vector with initial point P(1,1) and terminal point Q(8,5) in the Cartesian plane and illustrate its position vector.

3.4.2 Dot product

ACTIVITY 3.4.2

1. In xy plane

   a. represent the points $A(1,2)$ and $B(-3,1)$

   b. find the distance between point A to point B

2. Given that the product of two unit vectors is such that $\vec{i}\,.\,\vec{i}=1$, $\vec{i}\,.\,\vec{j}=0$ and $\vec{j}\,.\,\vec{j}=1$, apply the distributivity of multiplication under addition to evaluate the following products:

a) $(1,2)(-3,1)=\left(1\,\vec{i}+2\,\vec{j}\right)\left(-3\,\vec{i}+1\,\vec{j}\right)$

b) $(-4,2).(1,2)$

c) From your results, deduce how to calculate $(a_1,a_2).(b_1,b_2)$.

Content summary

Scalar product and properties

The scalar product or dot product (or sometimes inner product) is an algebraic operation that takes two coordinate vectors and returns a single number.

Algebraically, it is the sum of the products of the corresponding coordinates of the two vectors.

That is, the ***scalar product*** of vectors $\vec{u}=(a_1,a_2)$ and $\vec{v}=(b_1,b_2)$ of plane is defined by $\vec{u}\cdot\vec{v}=a_1b_1+a_2b_2$

Example:

$$(2,4).(10,4) = 2.10 + 4.4 = 20 + 16 = 36 \text{ or}$$

$$(2,4).(10,4) = \left(2\vec{i} + 4\vec{j}\right)\left(10\vec{i} + 4\vec{j}\right)$$

$$= 2.10\,\vec{i}\,\vec{i} + 2.4\,\vec{i}\,\vec{j} + 4.10\,\vec{i}\,\vec{j} + 4.4\,\vec{j}\,\vec{j}$$

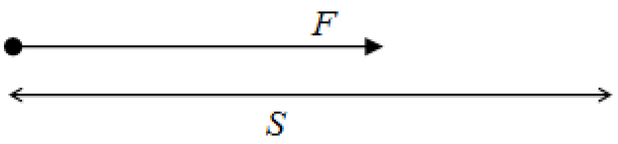
$$= 20 + 0 + 0 + 16 = 36$$

They give the same result.

We can illustrate this scalar product in terms of work done by a force on the body:

Suppose that a person is holding a heavy weight at rest. This person may say and feel he is doing hard work but in fact none is being done on the weight in the scientific sense. Work is done when a force moves its point of application along the direction of its line of action. A force F can act on a body and move it to a displacement $\boldsymbol{S}$.

If the constant force $\vec{F}$ and the displacement $\vec{\boldsymbol{S}}$ are in the same direction ,we define the work W done by the force on the body by $W = \boldsymbol{F} \cdot \boldsymbol{S}$



Properties of scalar product

a) If $\vec{u} = \vec{0}$ or $\vec{v} = \vec{0}$, then $\vec{u} \cdot \vec{v} = \vec{0}$

b) If $\vec{u} \parallel \vec{v}$ and $\vec{u}, \vec{v}$ have same direction, then $\vec{u} \cdot \vec{v} > 0$

c) If $\vec{u} \parallel \vec{v}$ and $\vec{u}, \vec{v}$ have opposite direction, then $\vec{u} \cdot \vec{v} < 0$

d) If $\vec{u} \perp \vec{v}$, then $\vec{u} \cdot \vec{v} = 0$

e) $\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$

f) $\vec{u} \cdot \left(a\vec{v} + b\vec{w}\right) = a\vec{v} \cdot \vec{u} + b\vec{w} \cdot \vec{u}$ $\qquad$ $\left(a\vec{u} + b\vec{v}\right).\vec{w} = a\vec{u}.\vec{w} + b\vec{v}.\vec{w}$

g) $\vec{u} \cdot \vec{u} > 0$,

We define the square of $\vec{u}$ to be $\vec{u} \cdot \vec{u} = \left(\vec{u}\right)^2$

Example

The scalar product of the vector $\vec{u}=(2,4)$ and vector $\vec{v}=(-5,0)$ is $\vec{u}\cdot\vec{v}=2(-5)+0=-10$

The square of the vector $\vec{u}=(10,4)$ is $\left(\vec{u}\right)^2=10(10)+4(4)=100+16=116$

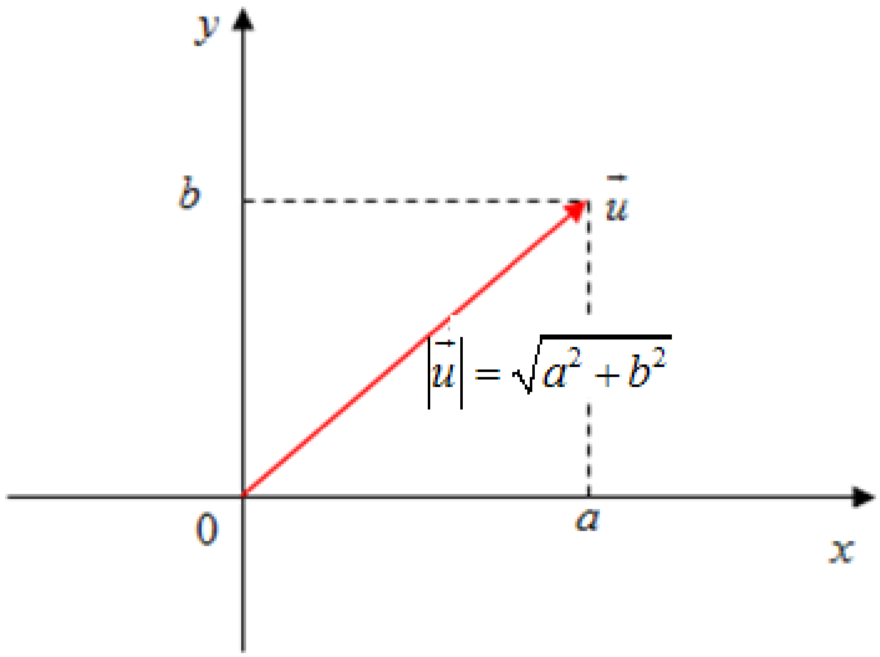
Notice

Two vectors are perpendicular if their scalar product is zero.

Magnitude or Modulus or norm of a vector

The magnitude of the vector $\vec{u}$ noted by $\left\|\vec{u}\right\|$ is defined as its length and is given by the square root of the sum of the squares of its components.

The magnitude of a vector $\vec{u}$ is also noted by $\left|\vec{u}\right|$.



That is $\left\|\vec{u}\right\|=\sqrt{\left(\vec{u}\right)^2}$ or $\left\|\vec{u}\right\|^2=\left(\vec{u}\right)^2$. Thus if $\vec{u}=(a,b)$

$\vec{u}\cdot\vec{u}=\left(\vec{u}\right)^2$

$\vec{u}=(a,b), \vec{u}.\vec{u}=(a,b)(a,b)=a^2+b^2$

$$\left(\vec{u}\right)^2=a^2+b^2$$

$$\|u\|^2=a^2+b^2$$

$$\|u\|=\sqrt{a^2+b^2}$$

Consequences

a. If $\vec{u}=\vec{0}$ then $\left\|\vec{u}\right\|=0$

b. $\left\|k\vec{u}\right\|=|k|\left\|\vec{u}\right\|$, k is a real number.

c. Geometrically, the scalar product of two vectors $\vec{u}=(a,b)$ and $\vec{v}=(c,d)$ of plane is given by $\vec{u}\cdot\vec{v}=\left\|\vec{u}\right\|\left\|\vec{v}\right\|\cos\theta$. Where θ is the angle between vectors $\vec{u}$ and $\vec{v}$. From this relation we have $\cos\theta=\frac{\vec{u}\cdot\vec{v}}{\left\|\vec{u}\right\|\left\|\vec{v}\right\|}$ or $\cos\theta=\frac{a\cdot c+b\cdot d}{\sqrt{a^2+b^2}\sqrt{c^2+d^2}}$.

Examples

1) Find the norm of the vector $\vec{v}=(3,4)$.

The norm is $\left\|\vec{v}\right\|=\sqrt{9+16}=5$

2) Find the norm of the vector $\vec{u}=(-1,4)$

The norm is $\left\|\vec{u}\right\|=\sqrt{1+16}=\sqrt{17}$

APPLICATION ACTIVITY 3.4.2

1. Find the norm(magnitude) of the following vectors:

   a) $\vec{u} = (-3,4)$

   b) $\vec{v} = (3,1)$

2. Consider the following points $A(3,4)$, $B(-2,3)$ and the vectors $\vec{u} = (4,5)$, $\vec{v} = (-3,1)$ in Cartesian plane. Find

   a) vector $\overrightarrow{AB}$

   b) vector $\vec{w} = 2\overrightarrow{AB} - 3\vec{u} + \vec{v}$

   c) the norm of vector $\overrightarrow{AB}$ and norm of vector $\vec{w}$

   d) the scalar product of vector $\vec{u}$, vector $\vec{v}$

   e) the scalar product of vector $\vec{v}$ and vector $\vec{w}$

3.5. Equation of a straight line

3.5.1. Equation of a straight line passing through a point and parallel to a direction vector

ACTIVITY 3.5.1

Given the vector $\vec{v} = (2,-3)$.

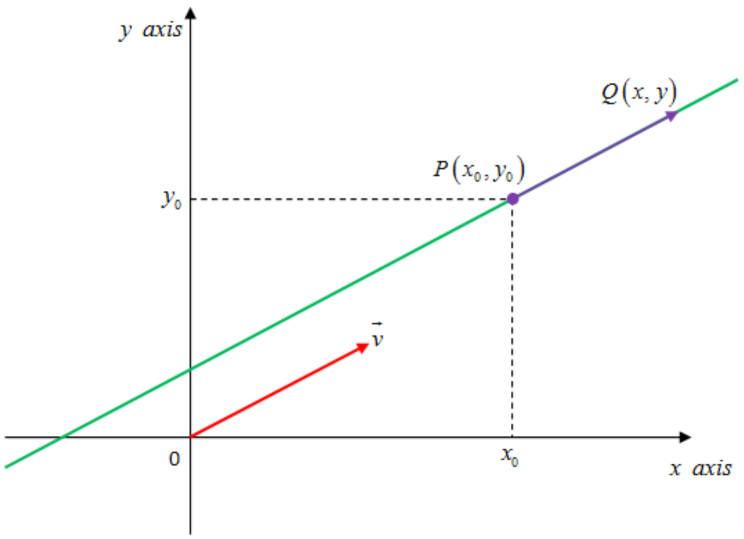
a) Determine the form of a vector $\vec{w} = (x, y)$ parallel to $\vec{v}$ and passing to the point $P(1,6)$.

b) Give 2 examples of such vectors $\vec{w}$.

c) If D is a line with the direction vector $\vec{w}$, what should be the equation of D?

To determine the equation of the line passing through the point $P(x_0, y_0)$ and parallel to the direction vector, $\vec{v} = (a,b)$, we will use our knowledge that parallel vectors are scalar multiples.

The vector $\overrightarrow{0P}$ is called the **position vector** of *P*.



Thus, the vector through $P(x_0, y_0)$ and any other point $Q(x, y)$ on the line is the product of a scalar and the direction vector $\vec{v} = (a, b)$.

$\overrightarrow{PQ} = r\vec{v}$ with r a parameter and $\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP}$

Hence, the **vector equation** of the line that is parallel to the vector $\vec{v} = (a, b)$ and which passes through the point P with position vector $\overrightarrow{0P} = (x_0, y_0)$ is given as

$\overrightarrow{OQ} = \overrightarrow{OP} + r\vec{v}$ where $Q(x, y)$ is any point of the line.

$(x, y) = (x_0, y_0) + r(a, b)$ or $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} + r\begin{pmatrix} a \\ b \end{pmatrix}$

The **parametric equations** of the line that is parallel to the vector $\vec{v} = (a, b)$ and which passes through the point P with position vector $\overrightarrow{0P} = (x_0, y_0)$ are given by:

$$\begin{cases} x = x_0 + ra \\ y = y_0 + rb \end{cases}$$

The **symmetric equation** or **Cartesian equation is** found after eliminating parameter ***r***.

$$\frac{x - x_0}{a} = \frac{y - y_0}{b}$$

This can be expanded to the standard form $bx - ay = bx_0 - ay_0$.

Example

Find the vector equation of the straight line that is parallel to the vector $\vec{u} = (2,-1)$ and which passes through the point with position vector $\vec{v} = (3,2)$.

Solution

The vector equation is $(x, y) = (3,2) + r(2,-1)$, r is a parameter

The parametric equations:

$$\begin{cases} x = 3 + 2r \\ y = 2 - r \end{cases}$$

The Cartesian equation:

$$\frac{x-3}{2} = \frac{y-2}{-1} \text{ or } 2y - 4 = -x + 3 \text{ or } x + 2y = 7$$

In general

The **vector equation** of the line can be rewritten as

$$(x, y) = (x_0, y_0) + r(a,b) \text{ or } x\vec{i} + y\vec{j} = x_0\vec{i} + y_0\vec{j} + r\left(a\vec{i} + b\vec{j}\right) \text{ where}$$

$\left\{\vec{i} = (1,0), \vec{j} = (0,1)\right\}$.

APPLICATION ACTIVITY 3.5.1.

1. Find the vector, parametric and Cartesian equation of the line passing through the point with position vector $(2,-3)$ and parallel to the line $(x, y) = (3,5) + r(1,6)$

2. Find the Cartesian equations of the lines whose vector equations are given below. Give your answers in the form $y = mx + c$

   a. $(x, y) = (-1,2) + r(-2,3)$

   b. $x\vec{i} + y\vec{j} = 3\vec{i} + 2\vec{j} + r\left(3\vec{i} - \vec{j}\right)$

3.5.2 Equation of a straight line given 2 points

ACTIVITY 3.5.2

Consider the line passing through points $A(1,4)$ and $B(3,-2)$.

a) Determine the vector $\overrightarrow{AB}$

b) Determine the form of a vector $\vec{w}=(x,y)$ parallel to $\overrightarrow{AB}$ and passing to the point $B(3,-2)$

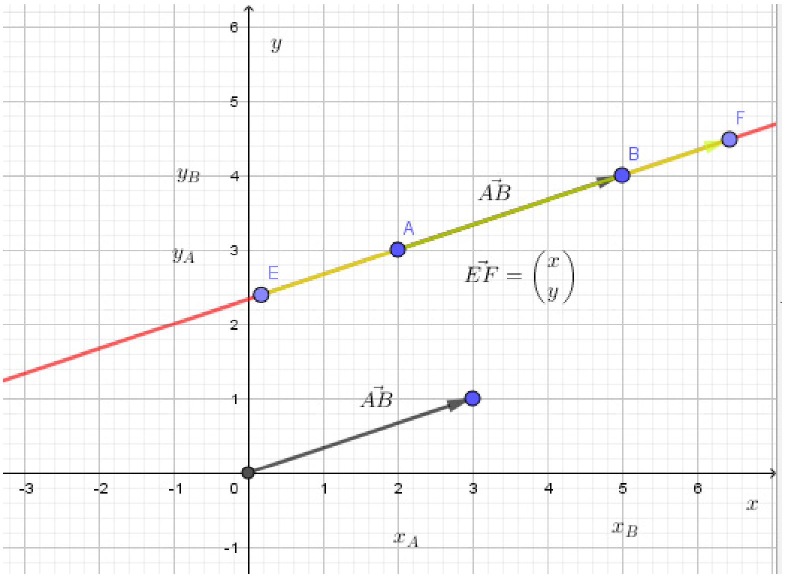
c) Give 2 examples of vectors represented by $\vec{w}$.

c) If D is a line with the direction vector $\vec{w}$, what should be the equation of D?

Content summary

Two points $A(x_A,y_A)$ *and* $B(x_B,y_B)$ form a vector $\overrightarrow{AB}=\begin{pmatrix} x_B-x_A \\ y_B-y_A \end{pmatrix}$. A vector parallel to $\overrightarrow{AB}=\begin{pmatrix} x_B-x_A \\ y_B-y_A \end{pmatrix}$ and passing to the point $B(x_B,y_B)$ is of the form

$\vec{w}=(x,y)=r\overrightarrow{AB}+A$



Therefore,

The vector equation of the line is:

$(x,y)=\overrightarrow{0A}+r\overrightarrow{AB}$ or $\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} x_A \\ y_A \end{pmatrix}+r\begin{pmatrix} x_B-x_A \\ y_B-y_A \end{pmatrix}$, where $\overrightarrow{AB}$ is the direction vector.

The parametric equations of the line are

$$\begin{cases} x=x_A+r(x_B-x_A) \\ y=y_A+r(y_B-y_A) \end{cases}$$

The Cartesian equation is obtained in this way:

$$x=x_A+r(x_B-x_A), \quad r=\frac{x-x_A}{x_B-x_A}$$

$$y=y_A+r(y_B-y_A), \quad r=\frac{y-y_A}{y_B-y_A}$$

Equalizing the value of r we find the Cartesian equation:

$$\frac{x-x_A}{x_B-x_A}=\frac{y-y_A}{y_B-y_A}$$

This can be expanded to the standard form

$$x(y_B-y_A)-y(x_B-x_A)=x_A(y_B-y_A)-y_A(x_B-x_A)$$

Example

Find the vector equation of the straight line passing through $A(3,2)$ and $B(4,1)$.

Solution

The direction vector is $\overrightarrow{AB}=(1,-1)$, r is a parameter

The vector equation is

$$(x,y)=\overrightarrow{0A}+r\overrightarrow{AB} \text{ or } (x,y)=(3,2)+r(1,-1)$$

The parametric equations is

$$\begin{cases} x=3+r \\ y=2-r \end{cases}$$

The Cartesian equation, is

$$\frac{x-3}{1}=\frac{y-2}{-1} \text{ or } y-2=-x+3 \text{ or } x+y=5$$

APPLICATION ACTIVITY 3.5.2

1) Determine the equation of a straight line passing through $P(2,4)$ and $B(3,-2)$.

2) Find the vector equation of the straight line that passes through the point with position vector $2\vec{i}+3\vec{j}$ and which is perpendicular to the line $x\vec{i}+y\vec{j}=3\vec{i}+2\vec{j}+r\left(\vec{i}-2\vec{j}\right)$.

3.5.3 Equation of a straight line given its gradient

LEARNING ACTIVITY 3.5.3

Determine the slope and the -intercept in the equations below. Hence draw the lines in the same graph.

a) $y=\frac{3}{2}x-2$

b) $y=-3x+\frac{5}{2}$

Consider a line having gradient and passing through the point $P_1(x_1,y_1)$. Suppose that the point $P(x,y)$ is an other point on the line. Then the gradient of the line is the rate at which the line rises (or falls) vertically for every unit across to the right. It is defined by the change in to the change in .

$$m=\frac{y-y_1}{x-x_1}$$

Thus, the equation of the line in point-slope form is defined by

$$y-y_1=m\left(x-x_1\right)$$

From the equation above, if we take any other point $P_2\left(x_2,y_2\right)$ that lies on this line, then:

Vector equation

$\overrightarrow{OP}=\overrightarrow{OP_1}+r\overrightarrow{P_1P_2}$ or $\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}x_1\\y_1\end{pmatrix}+r\begin{pmatrix}x_2-x_1\\y_2-y_1\end{pmatrix}$,

is the direction vector.

Parametric equations

$$\begin{cases} x = x_1 + r\left(x_2 - x_1\right) \\ y = y_2 + r\left(y_2 - y_1\right) \end{cases}$$

Cartesian equation

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_2}$$

Example :

Write the equation of the line that has slope that passes through the

Solution

$y - y_1 = m(x - x_1)$ ***use point-slope form***

$y - 7 = -3\left(x - (-1)\right)$ ***substitute*** $(-1,7)$ ***for*** (x_1, y_1) ***and -3 for m.***

$y - 7 = -3(x + 1)$

Note:

The genaral equation of the line is $Ax + By + C = 0$, where A, B, C are constantes,

And, $A \neq 0, B \neq 0$.

- If $A = 0$ the line is horizontal.
- If $B = 0$ the line is vertical.
- If $C = 0$ the line passes through the origin.

On the other hand the line has slope $m = \frac{-A}{B}$ and the intercept of , $b = \frac{-c}{B}$

- If the two lines are parallel, then their slopes/gradients are equal.

  Therefore $m_1 = m_2$

  Thus, the equations $Ax + By + C = 0$ and $Ax + By + D = 0$ are parallel.
- If two lines are peripendicular, then the product of their slopes/gradients is equal to -1.
- Therefore $m_1 \times m_2 = -1$.
- Thus, the equations $Ax + By + C = 0$ and $Bx - Ay + D = 0$ are peripendicular.

Example :

1) Find the equation for the line that contains the point $(5,1)$ and is parallel to $y=\frac{3}{5}x+3$.

Step1: Identify the slope of the given line

$y=\frac{3}{5}x+3 \Rightarrow m=\frac{3}{5}$

Step2: Write the equation of the line trougth point $(5,1)$ and $m=\frac{3}{5}$

$y-y_1=m(x-x_1)$

$y-1=\frac{3}{5}(x-5)$

$y=\frac{3}{5}x-3+1$

$y=\frac{3}{5}x-2$

2) Find the equation for the line that contains the point $(0,-2)$ and is perpendicular to $y=5x+3$?

Step1: Identify the slope of the given line and write its negative reciprocal.

$y=5x+3 \quad \Rightarrow m=5$ and it has negative reciprocal $\frac{-1}{5}$

Step2: Write the equation of the line trougth point $(0,-2)$ and $m=\frac{-1}{5}$

$y-y_1=m(x-x_1)$

$y-(-2)=\frac{-1}{5}(x-0)$

$y=\frac{-1}{5}x-2$

APPLICATION ACTIVITY 3.5.3

1. Write the equation in point-slope form for the line through the given point that has the given slope

a) $(3,-4); m=6$

b) $(-2,-7); m=\frac{-3}{2}$

c) $(-5,2); m=0$

d) $(4,2); m=\frac{-5}{3}$

e) $(4,0); m=1$

f) $(1,-8); m=\frac{-1}{5}$

2. Is $y-5=2(x-1)$ an equation of a line passing through $(4,11)$? Explain.

3. Write an equation of the line that contains the point $(-3,-5)$ and the same slope as $y+2=7(x+3)$

3.6. Problems on points and straight lines in 2D

3.6.1 Intersection, perpendicularity or parallelism of two lines

ACTIVITY 3.6.1

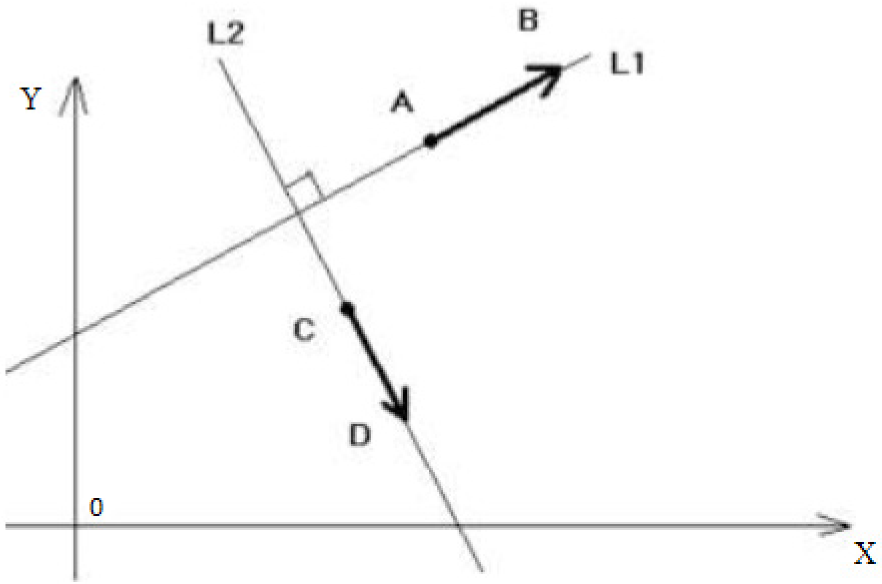
1. In the same Cartesian plane plot the straight lines containing the following points

   $(0,4)$; $(\frac{1}{2},2)$ and $(\frac{1}{2},2)$; $(4,1)$

   What is your opinion about the two lines in the Cartesian plane?

2. In the same Cartesian plane plot the straight lines containing the following points $(-3,-1)$; $(0,3)$ and $(-2,-4)$; $(1,0)$

a) Perpendicularity and intersection of two lines



Here L_1 and L_2 **are perpendicular**

To write a straight line perpendicular to a given straight line we proceed as follows:

Step I: Interchange the coefficients of x and y in equation $ax + by + c = 0$.

Step II: Interchange the sign between the terms in x and y of equation i.e., If the coefficient of x and y in the given equation are of the same signs make them of opposite signs and if the coefficient of x and y in the given equation are of the opposite signs make them of the same sign.

Step III: Replace the given constant of equation $ax + by + c = 0$ by an arbitrary constant.

For example, the equation of a line perpendicular to the line $7x + 2y + 5 = 0$ is $2x - 7y + c = 0$ again, the equation of a line, perpendicular to the line $9x - 3y = 1$ is $3x + 9y + k = 0$

Note:

Assigning different values to k in $bx - ay + k = 0$, we shall get different parallel straight lines each of which is perpendicular to the line $ax + by + c = 0$. Thus we can have a family of straight lines perpendicular to a given straight line.

b) Intersection point of two lines

Let the equations of two intersecting straight lines be

$a_1x+b_1y+c_1=0$ ....(i)
and
$a_2x+b_2y+c_2=0$.......(ii)

Suppose the above equations of two intersecting lines intersect at $P(x_1,y_1)$. Then (x_1,y_1) will satisfy both the equations (i) and (ii).

Therefore, $a_1x_1+b_1y_1+c_1=0$ and $a_2x_1+b_2y_1+c_2=0$

Solving the above two equations by using the method of cross-multiplication, we get,

$$\frac{x_1}{b_1c_2-b_2c_1}=\frac{y_1}{c_1a_2-c_2a_1}=\frac{1}{a_1b_2-a_2b_1}$$

Therefore, $x_1=\dfrac{b_1c_2-b_2c_1}{a_1b_2-a_2b_1}$ and $y_1=\dfrac{c_1a_2-c_2a_1}{a_1b_2-a_2b_1}$, $a_1b_2-a_2b_1\neq 0$

$$\left(\frac{b_1c_1-b_2c_1}{a_1b_2-a_2b_1},\frac{c_1a_2-c_2a_1}{a_1b_2-a_2b_1}\right),\ a_1b_2-a_2b_1\neq 0$$

Notes:

To find the coordinates of the point of intersection of two non-parallel lines, we solve the given equations simultaneously and the values of x and y so obtained determine the coordinates of the point of intersection.

If $a_1b_2-a_2b_1=0$ then $a_1b_2=a_2b_1$

$$\Rightarrow \frac{a_1}{b_1}=\frac{a_2}{b_2}$$

$\Rightarrow -\dfrac{a_1}{b_1}=-\dfrac{a_2}{b_2}$ **i.e., the slope of line (i) = the slope of line (ii)**

Therefore, in this case the straight lines (i) and (ii) are parallel and hence they do not intersect at any real point.

Example

Find the coordinates of the point of intersection of the lines $2x-y+3=0$ and $x+2y-4=0$

Solution:

We know that the co-ordinates of the point of intersection of the lines $a_1x+b_1y+c_1=0$ and

$a_2x+b_2y+c_2=0$ are $\left(\frac{b_1c_1-b_2c_1}{a_1b_2-a_2b_1},\frac{c_1a_2-c_2a_1}{a_1b_2-a_2b_1}\right)$, $a_1b_2-a_2b_1\neq 0$

Given equations are

$2x-y+3=0$....(i)

$x+2y-4=0$.....(ii)

Here $a_1=2$, $b_1=-1$, $c_1=3$, $a_2=1$, $b_2=2$ and $c_2=-4$

$$\left(\frac{(-1).(-4)-(2)(3)}{(2).(2)-(1).(-1)},\frac{(3)(1)-(-4)(2)}{(2)(2)-(1)(-1)}\right)$$

$$\Rightarrow\left(\frac{4-6}{4+1},\frac{3+8}{4+1}\right)$$

$$\Rightarrow\left(\frac{-2}{5},\frac{11}{5}\right)$$

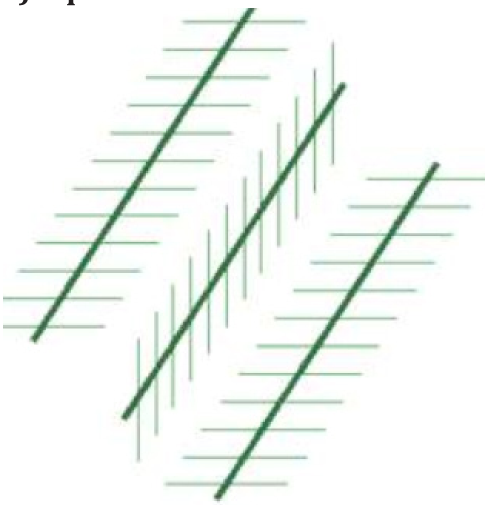
Therefore, the co-ordinates of the point of intersection of the lines

$2x-y+3=0$ and $x+2y-4=0$ are $\left(\frac{-2}{5},\frac{11}{5}\right)$

Alternatively, by solving simultaneous equations $2x-y+3=0$ (i)and

$x+2y-4=0$ (ii)using different methods you get the same answer. $\left(\frac{-2}{5},\frac{11}{5}\right)$.

c) Equation of a Line Parallel to a given line



These are three parallel antennas

Let, $ax+by+c=0 \quad (b \neq 0)$ be the equation of the given straight line.

Now, convert the equation $ax+by+c=0$ to its slope-intercept form.

$ax+by+c=0$

$\Rightarrow by=-ax-c$

Dividing both sides by b , $[b \neq 0]$ we get,

$y=-\frac{a}{b}x-\frac{c}{b}$ **which is the slope- intercept form**

Now comparing the above equation to slope-intercept form $(y=mx+b)$ we

get, The slope of the line $ax+by+c=0$ is $(-\frac{a}{b})$ .Since the required line is

parallel to the given line, the slope of the required line is also $(-\frac{a}{b})$.

Let ***k*** (an arbitrary constant) be the intercept of the required straight line. Then the equation of the straight line is $y=-\frac{a}{b}x+k$

$\Rightarrow by=-ax+bk$

$\Rightarrow ax+by=\lambda$, Where $\lambda=bk$ λ = another arbitrary constant.

Note:

(i) Assigning different values to λ in $ax+by=\lambda$, we shall get different straight lines each of which is parallel to the line $ax+by+c=0$. Thus, we can have a family of straight lines parallel to a given line.

(ii) To write a line parallel to a given line we keep the expression containing x and y same and simply replace the given constant by a new constant λ. The value of λ can be determined by some given condition.

To get it more clear let us compare the equation $ax+by=\lambda$ with equation $ax+by+c=0$. It follows that to write the equation of a line parallel to a given straight line we simply need to replace the given constant by an arbitrary constant, the terms with x and y remain unaltered. For example, the equation of a straight line parallel to the straight line;

$7x-5y+9=0$ is $7x-5y+\lambda=0$ where λ is an arbitrary constant.

Example

Find the equation of the straight line which is parallel to $5x-7y=0$ and passing through the point $(2,-3)$.

Solution:

The equation of any straight line parallel to the line $5x-7y=0$ is $5x-7y+\lambda=0$ ………… (i) [Where λ is an arbitrary constant].If the line (i) passes through the point $(2,-3)$ then we shall have:

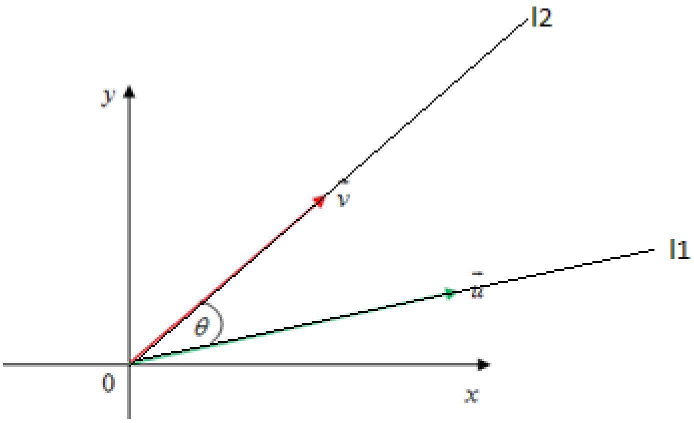
$$5.2-7(-3)+\lambda=0$$
$$\Rightarrow 10+21+\lambda=0$$
$$\Rightarrow 31+\lambda=0$$
$$\Rightarrow \lambda=-31$$

Therefore, the equation of the required straight line is $5x-7y+31=0$

d) Angles between two lines

Geometrically, the scalar product of two vectors $\vec{u}=(a,b)$ and $\vec{v}=(c,d)$ of plane is given by $\vec{u}\cdot\vec{v}=\|\vec{u}\|\|\vec{v}\|\cos\theta$. Where θ is the angle between vectors $\vec{u}$ and $\vec{v}$. From this relation we have $\cos\theta=\frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\|\vec{v}\|}$ or $\cos\theta=\frac{a\cdot c+b\cdot d}{\sqrt{a^2+b^2}\sqrt{c^2+d^2}}$.

We deduce that the angle θ between two lines l_1 and l_2 with direction vectors $\vec{u}=(a,b)$ and $\vec{v}=(c,d)$ respectively is given by $\theta=\cos^{-1}\left(\frac{a\cdot c+b\cdot d}{\sqrt{a^2+b^2}\sqrt{c^2+d^2}}\right)$.
Where $\cos^{-1}$ denotes the inverse function of cosine.

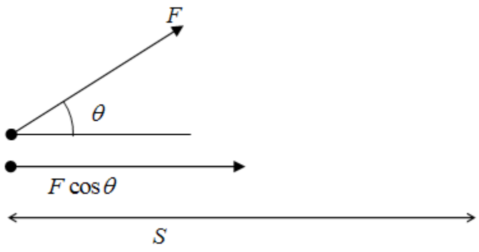


We can illustrate the scalar product in terms of work done by the force on the body:

We saw that if the constant force F and the displacement S are in the same direction, the work W done by the force on the body is $W=\vec{F}.\vec{S}$

If the force does not act in the direction in which motion occurs but an angle θ to it , then the work done is defined as the product of the component of the force in the direction of motion and the displacement in that direction.

$W = \|\vec{F}\|.\|\vec{S}\|cos\theta$



Notice

- Two lines are perpendicular if the angle between them is a multiple of a right angle
- Two lines are parallel and with the same direction if the angle between them is a multiple of a zero angle
- Two lines are parallel and with the opposite direction if the angle between them is a multiple of a straight angle

Example:

Find the angle between vectors $\vec{u} = (3,0)$ *and* $\vec{v} = (5,5)$

Solution

Let α be the angle between these two vectors

$$\alpha = \cos^{-1}\left(\frac{3\cdot5+0\cdot5}{\sqrt{3^3+0^2}\sqrt{5^2+5^2}}\right) = \cos^{-1}\left(\frac{\sqrt{2}}{2}\right)$$

$$\alpha = 45^\circ$$

APPLICATION ACTIVITY 3.6.1

1. Find the equation of a straight line that passes through the point $(-2,3)$ and perpendicular to the straight line $2x+4y+7=0$

2. Find the equation of the straight line which passes through the point of intersection of the straight lines $x+y+9=0$ and $3x-2y+2=0$ and is perpendicular to the line

   $4x+5y+1=0$

3. Find the equation of the straight line passing through the point $(5,-6)$ and parallel to the straight line $3x-2y+10=0$.

4. Calculate the dot product and the angle formed by the following vectors: $\vec{u}=(3,4)$ and $\vec{v}=(-8,6)$

3.6.2. Distance between two lines and the distance between a point and line

ACTIVITY 3.6.2

1. In one Cartesian plane plot the straight lines containing the following points

   $(0,4);\ (\frac{1}{2},2)$ and $(\frac{1}{2},2);\ (4,1)$

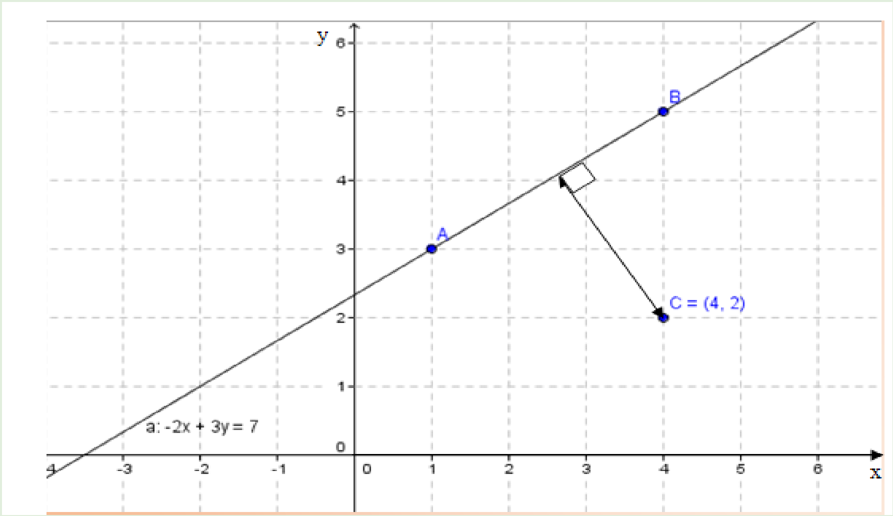
   What is the distance between these two lines?

2. Also in one Cartesian plane plot the straight lines containing the following points

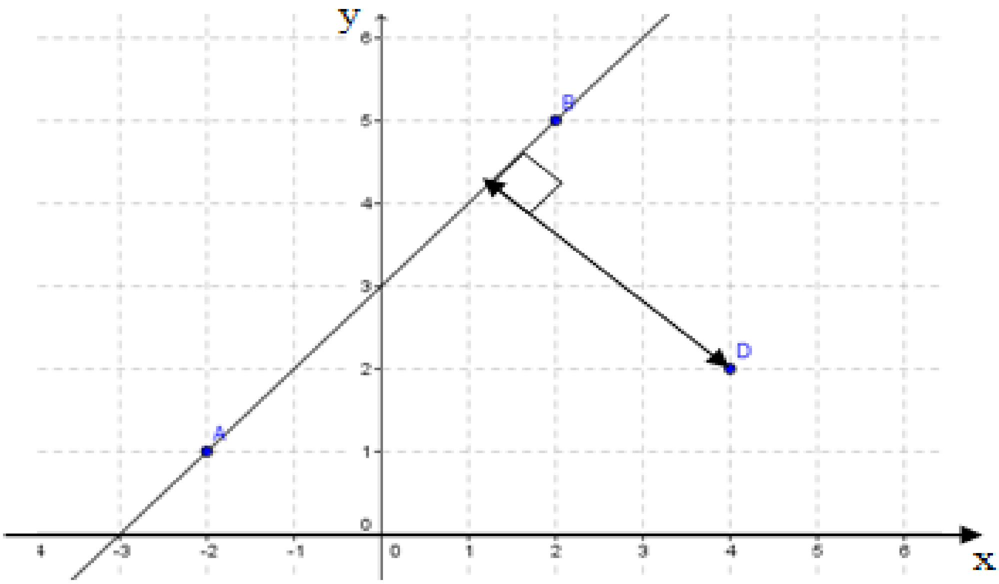
   $(-3,-1);\ (0,3)$ and $(-2,-4);\ (1,0)$

   What is the distance between these two lines?

3. Find the shortest distance between the point $C(4,2)$ and the line $a:-2x+3y=7$



Distance between a point and a line



The perpendicular distance from a point $D(x_1, y_1)$ to the line $ax + by + c = 0$ is given by

$$\frac{ax_1 + by_1 + c}{\sqrt{a^2 + b^2}}$$

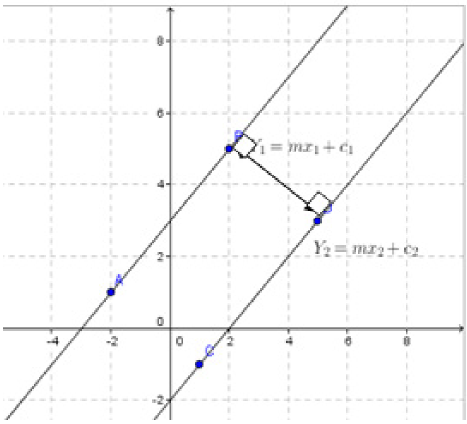
Example

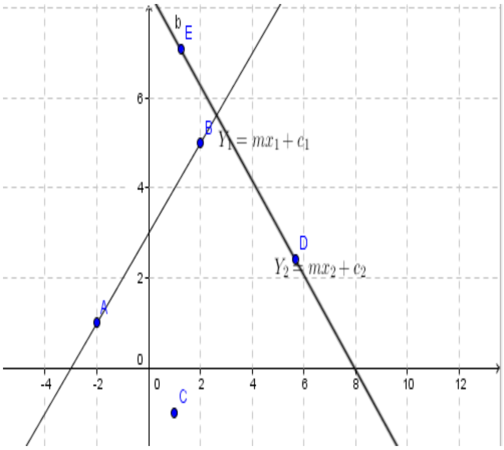
Find the shortest distance from the line $2x + y - 1 = 0$ to the point $(3, 2)$

Solution :

The distance is $\dfrac{2(3)+1(2)-1}{\sqrt{(2)^2+(1)^2}} = \dfrac{7}{\sqrt{5}} = \dfrac{7\sqrt{5}}{5}$.

Distance between two parallel lines and two intersecting lines





$y_1 = mx_1 + c_1$ **and** $y_2 = mx_2 + c_2$ **are Parallel** $y_1 = mx_1 + c_1$ **and** $y_2 = mx_2 + c_2$ **are intersecting**

If they intersect, the distance is **0** at the point of intersection.

If they're parallel: the distance between them will be given by the following formula

$$d = \frac{|c_1 - c_2|}{\sqrt{1+m^2}}$$

Example

What is the distance between the lines $3x + 4y = 9$ and $6x + 8y = 15$?

Solution:

First of all we find the slopes of the two lines. We convert the equations into slope intercept form

$$3x + 4y = 9$$

$$\Rightarrow 4y = -3x + 9$$

$$\Rightarrow y = \frac{-3}{4}x + \frac{9}{4}$$

$$6x+8y=15$$
$$\Rightarrow 8y=-6x+15$$
$$\Rightarrow y=\frac{-6}{8}x+\frac{15}{8}$$
$$\Rightarrow y=\frac{-3}{4}x+\frac{15}{8}$$

The slope m of the lines is $\frac{-3}{4}$

Hence both lines are parallel

y Intercept of the first line $=\frac{9}{4}$

y Intercept of the second line $=\frac{15}{8}$

Difference between the y- intercepts $=\left|\frac{9}{4}-\frac{15}{8}\right|$

$$=\left|\frac{18}{8}-\frac{15}{8}\right|$$

$$=\frac{3}{8}$$

$$\sqrt{1+m^2}=\sqrt{1+\frac{9}{16}}=\sqrt{\frac{25}{16}}=\frac{5}{4}$$

Distance between the lines $=\frac{3}{8}\div\frac{5}{4}=\frac{3}{8}\times\frac{4}{5}=\frac{3}{10}$.

Alternatively, the distance between two parallel lines

$L_1\equiv ax+by+c_1=0$ *and* $L_2\equiv ax+by+c_2$ is given by $\frac{|c_1-c_2|}{\sqrt{a^2+b^2}}$.

Using the previous example, $3x+4y=9$ and $6x+8y=15$ the distance between the two lines is calculated taking $3x+4y=9$ and $3x+4y=\frac{15}{2}$.

Therefore, the distance is

$$\frac{|c_1-c_2|}{\sqrt{a^2+b^2}}=\frac{\left|9-\frac{15}{2}\right|}{\sqrt{3^2+4^2}}=\frac{\frac{3}{2}}{5}=\frac{3}{10}.$$

APPLICATION ACTIVITY 3.6.2

1. What is the distance between the lines $3x+4y=9$ and $6x+8y=15$
2. What will be the distance between the lines $3x+4y=5$ *and* $6x-8y=45$
3. What is the distance between the lines $3x+4y=9$ *and* $6x-8y=18$
4. How far is the line $3x-4y+15=0$ from the origin?
5. What is the distance of the point $(-2,3)$ from the line from the line $x-y=5$?
6. What is distance from the point $(-3,-4)$ to the line $3x-4y-1=0$?
7. What is distance from the point $(2,-1)$ to the line $3x+4y=6$?

3.7 Geometric shapes in 2D

3.7.1 Identification of the Geometric shapes in two dimensions

LEARNING ACTIVITY 3.7.1

1) i) Discuss the difference between regular polygons and irregular polygons and give examples.

ii) Name the shapes with three sides and give at least 4 examples of materials in real life with three sides.

iii) Name the shapes with four sides and give at least 4 examples of materials in real life with four sides.

2) i) Plot the following points on the 2D coordinate plane, then use a ruler to connect the points in the order they are listed to form a polygon.

Polygon 1: $(8,6);(11,1);(8,6)$

Polygon 2: $(2,5);(6,5);(6,3);(2,3)$

Polygon3: $(3,5);(7,2);(7,-5);(3,-2)$

Polygon4: $(-2,5);(-2,-2),(3,-2),(5,5)$

Polygon5: $(-2,-2);(2,-2);(2,2);(6,2),(6,-6);(-4,-6);(-4,-4);(-2,4)$

ii) What is the name of each geometric figure formed?

iii) Describe the similarities and differences of those figures.

Content summary

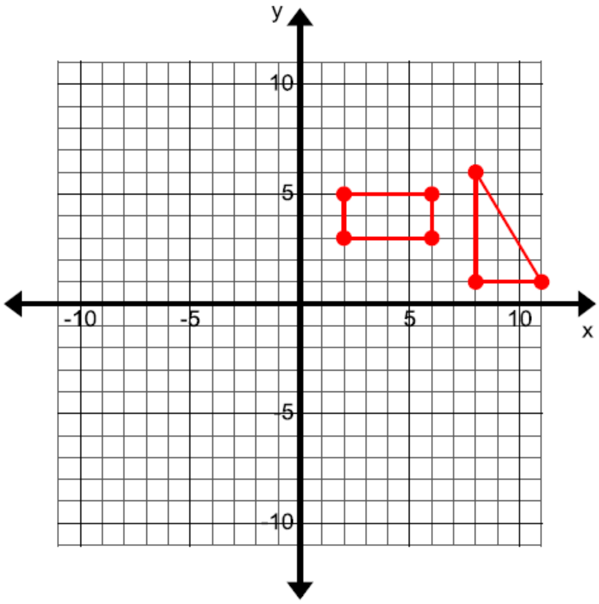
In geometry, a two-dimensional shape can be defined as a flat plane figure or a shape that has two dimensions – length and width. Length on x- axis and width on y-axis or vice versa

Two-dimensional or 2-D shapes do not have any thickness and can be measured in only two faces. In the two –dimensional coordinate, geometric figures can be plotted:

Example : The following points;

a) $(8,6);(11,1);(8,6)$ b) $(2,5);(6,5);(6,3);(2,3)$

Can make the following polygons in coordinate's graphs



Two dimensional shapes are for example: Triangle, square, rectangle, parallelogram, trapezium, rhombus, circle, pentagon, hexagon, etc.

It is also important to take note of the hierarchy, e.g. all rectangles are parallelograms but all parallelograms are not rectangles.

Example:

Explain why square is rectangle but a rectangle is not a square?

Solution:

Depending on their properties, a square is a quadrilateral with four right angles and equal sides. The two parallel sides are obviously equal, which allows a square to be a rectangle. Even though the two parallel sides of a rectangle are equal, the four sides are not equal; which means that a rectangle is not a square but a square is rectangle.

APPLICATION ACTIVITY 3.7.1

1) Given two points C(1,2) and P(5,-1).

a) Determine the distance $r = \bar{C}P$

b) Plot these points in the Cartesian plan, and draw a circle with centre C and radius $r = CP$.

c) Give other two points of the circle and their coordinates. Explain how you found those points.

2) Draw a polygon on the coordinate plane by plotting each set of vertices and connecting them in the order they are listed. Is it a regular or irregular polygon? Can you find its area?

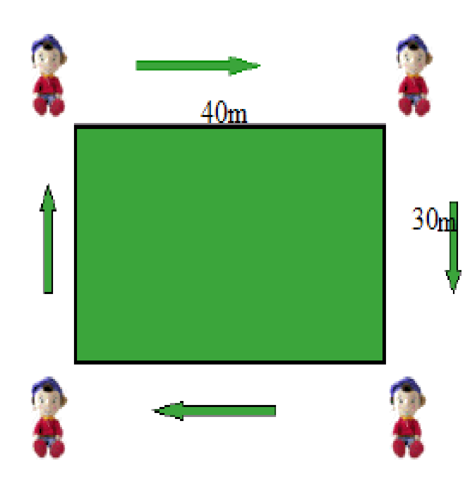
a) $(0,2),(0,-1),(5,-1),(5,-4),(8,-4),(8,5),(5,5),(5,2)$

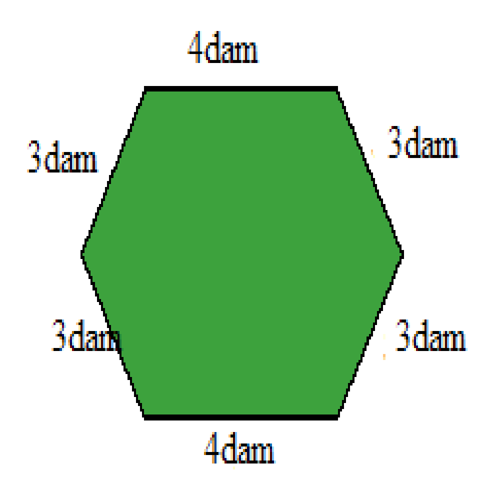
b) $(-5,7),(1,7),(1,2)(5,2),(5,-4),(-1,-4),(-1,1),-5,1)$

3.7.2 Perimeter and area of geometric shapes in 2D.

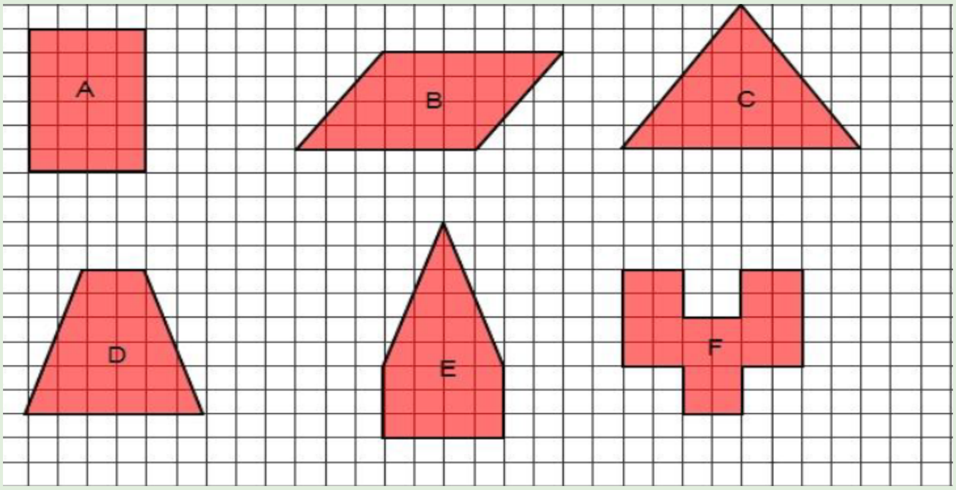
LEARNING ACTIVITY 3.7.2

1) The following fields have two dimensional shapes, find the length of one wire that can be used on one row of the fence in order to protect the plants.



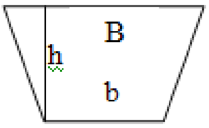
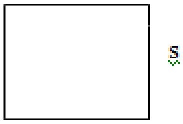
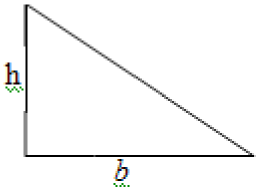
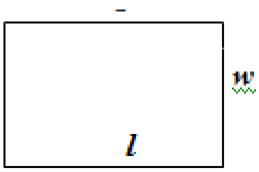
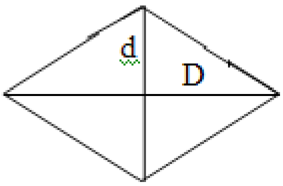


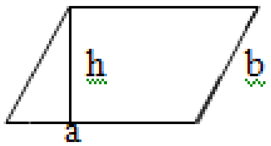
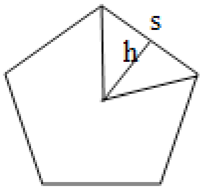
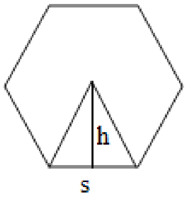
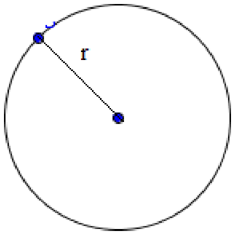
2) On the graph below, one square of the paper represents one square unit. Calculate the area of given shapes in the graph

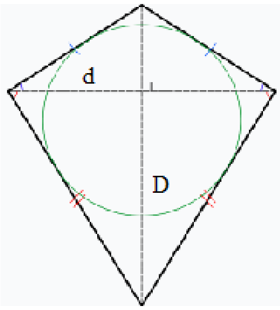


Content summary

As studied in previous levels (Primary and ordinary levels), the perimeter and area of two dimensional shapes can be summarised in the following table:

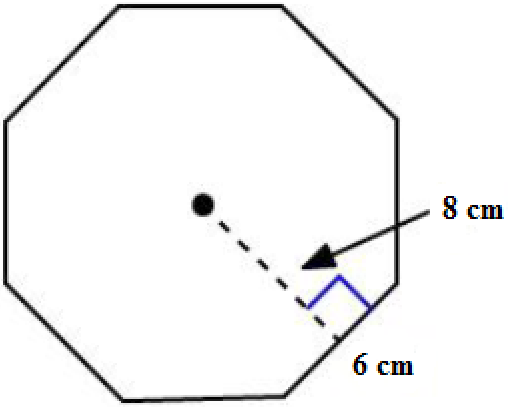
Shape	**Description**	**Perimeter formulas**	**Area Formulas**
Quadrilateral 	4-sided polygon	Sum of its sides	$\frac{(b+B)\text{h}}{2}$
Square 	A quadrilateral having all sides equal in length and forming right angles.	$4s$	s^2
Triangle 	A 3-sided polygon (sum of internal angles = 180°)	Sum of its side	$\frac{b \times h}{2}$
Rectangle 	A 4-sided polygon with all right angles.	$2(w+l)$	$l \times w$
Rhombus 	A 4-sided polygon with two equal opposites angles and 4 equal sides	$4s$	$\frac{D \times d}{2}$

Shape	Description	Perimeter formulas	Area Formulas
Parallelogram 	4-sided polygon with two pairs of parallel sides.	$2(a+b)$	$b \times h$
Pentagon 	5-sided polygon (the graphic shows a regular hexagon with "regular" meaning each of the sides are equal in length)	$5s$	$5(\frac{s \times h}{2})$
Hexagon 	6-sided polygon	$6s$	$3(s \times h)$
Circle 	Round Collection of infinite points Polygon with an infinite number of sides Any point on the circle is the same distance from the centre Made up of a closed curved line	$2\pi r$	πr^2

Shape	Description	Perimeter formulas	Area Formulas
KITE 	**(1)** The diagonals of a kite meet at a right angle. **(2)** Kites have exactly one pair of opposite angles that are congruent.	$2(a+b)$	$\frac{D\times d}{2}$

Example

Find the perimeter and the area of the regular octagon below. A regular octagon has all sides that are equal.



Solution:

Given: $s = 6cm; \quad h = 8cm$

Asked: Perimeter =? Area = ?

a) Perimeter $= number\ \ of\ \ sides \times sides\ \ lenght = 6\times 6in = 36in$

b) Area

$= Number\ of\ sides \times (area\ of\ triangle)$

$= 8(\frac{6cm \times 8cm}{2})$

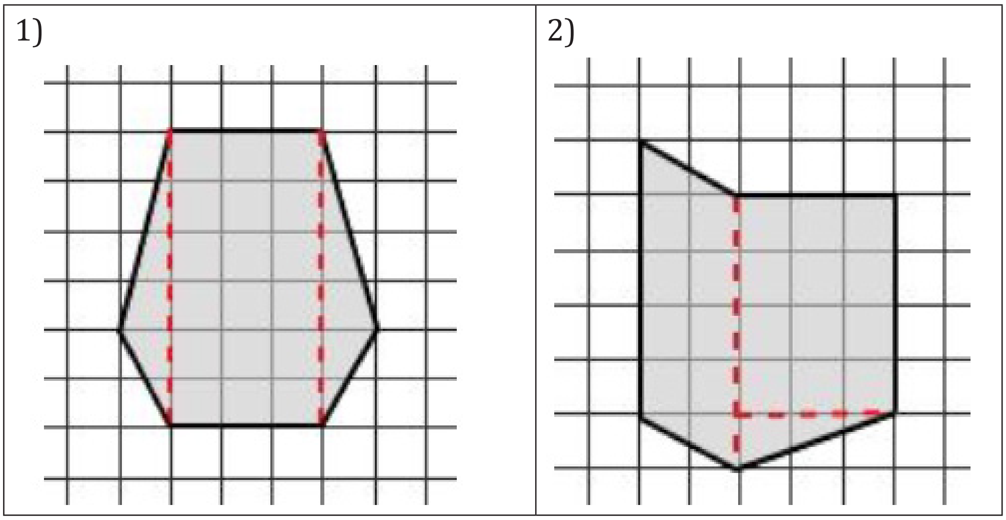
$= 4 \times 48cm^2$

$= 192cm^2$

Irregular polygons

If the shape is irregular, it can be divided into regular or other simple figures to find its area. while the perimeter is found by adding the sum of sides for the original figure.

Examples: Given that one square of the paper represents one square unit, find the area of shaded figures.



Solution:

1. Area of the rectangle is $3(6) = 18$

Area of the two triangles is

$2 \times \frac{1}{2}(6 \times 1) = 2 \times 3 = 6$

Total area is

$18 + 6 = 24$ Units

2. Area of the rectangle is $3(4) = 12$

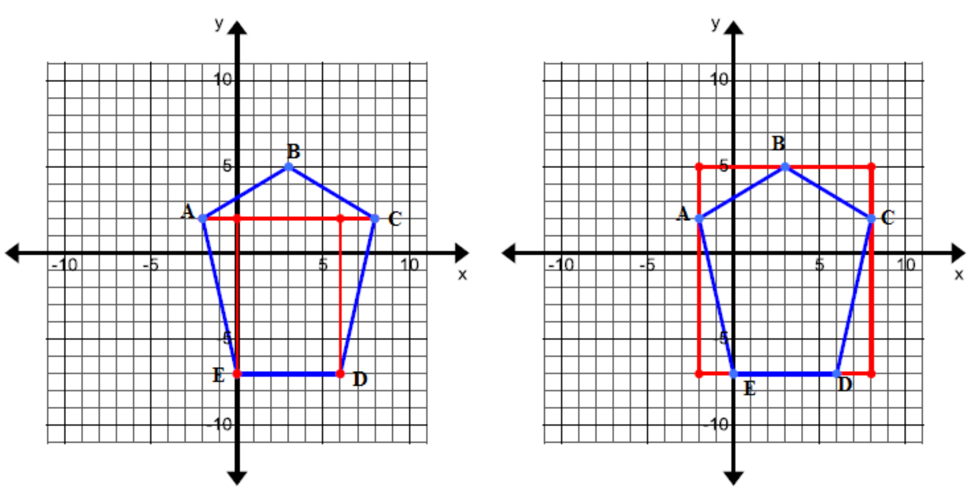
Area of the parallelogram is 5(2) =10

Area of the triangle is $\frac{1}{2}(1)(3)=1.5$

Total area is $12+10+1.5=23.5$ Units.

APPLICATION ACTIVITY 3.7.2

Two different copies of a geometric figure were provided as follow:



i) Given that one square of the paper represents one square unit on each figure, find the area of the geometric figure ABCDE in blue.

ii) Show and explain that different methods can be used when finding the area of an irregular polygon.

3.8 END UNIT ASSESSMENT

1) A new park is being designed for your city. The plans for the design are being drawn on the coordinate plane below. The vertices given form a polygon that represents the location for 5 different features in the park.

a) Plot each set of points in the order they are given to form a polygon. Label the feature that the polygon represents on the graph.

Playground:
$(-5,-2),(-8,-2),(-8,-5),(-11,-5),(-11,-8),(-8,-8),(-8,-11),(-5,-11)$

Splash Pad: $(10,4),(-4,8),(6,8)$

Restrooms: $(-4,5),(-4,8),(-7,10),(-10,7),(-7,5)$

Picnic Pavilion: $(8,-1),(6,-3),(4,-3),(2,-1),(2,1),(4,3),(6,3),(8,1)$.

If each square on the graph represents 1 square meter, answer the questions that follow.

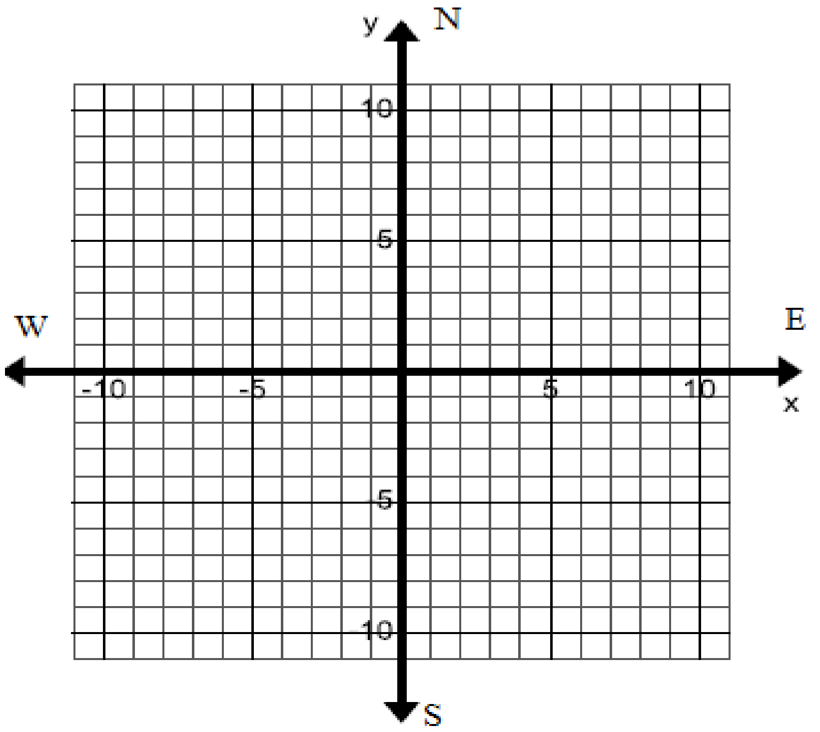
b) Cement needs to be poured to lay the foundation for the restroom and the splash pad. How many square meters of cement will the city need for the foundation of these two features?

c) The playground is going to be covered with wood chips, how many square meters of wood chips will the city need for the playground?

d) The citizens of the city have asked that the playground have a fence around its perimeter. How many meters of fencing will they need for the playground?

e) The foundation of the pavilion picnic area is going to be covered with special pavers. How many square meters of pavers will they need for the pavilion?

2) A coordinate grid represents the map of a city. Each square on the grid represents one city block.



a) Helena's apartment is at the point . She walks 4 blocks south, then 8 blocks west, then 4 blocks north, and then finally 8 blocks east back to her apartment. How many blocks did she walk total? Describe the shape of her path. Mark and label her apartment and highlight her walk.

b) Draw and describe in words at least two different ways you could walk exactly 20 blocks and end up back where you came from.

c) Carl lives at the point , Find the distance between Helena's house and Carl's house, and then mark and label Carl's house.

d) Establish the equation of the line passing through the points H(5,7) and C(5,-5). Given that one block has 10 m of length's side, calculate the distance between them, and the area of the square centered in C with radius .

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