MATHEMATICS FOR TTCs

TUTOR'S GUIDE

YEAR

3

OPTIONS:

SCIENCE AND MATHEMATICS EDUCATION (SME)

FOREWORD

Dear Tutor,

Rwanda Education Board is honoured to present the tutor's guide for Mathematics in the option of Science and mathematics Education (SME). This book serves as a guide to competence-based teaching and learning to ensure consistency and coherence in the learning of the mathematics content.

The Rwandan educational philosophy is to ensure that learners achieve full potential at every level of education which will prepare them to be well integrated in society and exploit employment opportunities.

Specifically, TTC curriculum was reviewed to train quality teachers who will confidently and efficiently implement the Competence Based Curriculum in pre-primary and primary education. The rationale of the changes is to ensure that TTC leavers are qualified for job opportunities and further studies in Higher Education in different programs under education career advancement.

In line with efforts to improve the quality of education, the government of Rwanda emphasizes the importance of aligning teaching and learning materials with the syllabus to facilitate their learning process. Many factors influence what they learn, how well they learn and the competences they acquire. Those factors include the relevance of the specific content, the quality of teachers' pedagogical approaches, the assessment strategies and the instructional materials.

The ambition to develop a knowledge-based society and the growth of regional and global competition in the jobs market has necessitated the shift to a competence-based curriculum. After a successful shift from knowledge to a competence based curriculum in general education, TTC textbooks also were elaborated to align them to the new curriculum.

The book provides active teaching and learning techniques that engage student teachers to develop competences. In view of this, your role is to:

- Plan your lessons and prepare appropriate teaching materials.
- Organize group discussions for students considering the importance of social constructivism suggesting that learning occurs more effectively when the students works collaboratively with more knowledgeable and experienced people.
- Engage students through active learning methods such as inquiry methods, group discussions, research, investigative activities and group and individual work activities.
- Provide supervised opportunities for students to develop different competences by giving tasks which enhance critical thinking, problem

solving, research, creativity and innovation, communication and cooperation.

- Support and facilitate the learning process by valuing students' contributions in the class activities.
- Guide students towards the harmonization of their findings.
- Encourage individual, peer and group evaluation of the work done in the classroom and use appropriate competence-based assessment approaches and methods.

To facilitate you in your teaching activities, the content of this book is self-explanatory so that you can easily use it. It is divided in 3 parts:

The part I explain the structure of this book and give you the methodological guidance;

The part II gives a sample lesson plan;

The part III details the teaching guidance for each concept given in the student book.

Even though this Teacher's guide contains the guidance on solutions for all activities given in the learner's book, you are requested to work through each question before judging student's findings.

I wish to sincerely express my appreciation to the people who contributed towards the development of this book, particularly, REB staff, UR-CE Lecturers, TTC Tutors, Teachers from general education and Educate! for their technical support. A word of gratitude goes also to the Head Teachers and TTCs principals who availed their staff for various activities.

Dr. NDAYAMBAJE Irénée

Director General, REB

ACKNOWLEDGEMENT

I wish to express my appreciation to the people who played a major role in the development of this tutor`s guide for Mathematics in the option of Science and Mathematics Education (SME). It would not have been successful without active participation of different education stakeholders.

I owe gratitude to different universities and schools in Rwanda that allowed their staff to work with REB in the in-house textbooks production initiative.

I wish to extend my sincere gratitude to lecturers and teachers whose efforts during writing exercise of this tutor`s guide were very much valuable.

Finally, my word of gratitude goes to the Rwanda Education Board staffs who were involved in the whole process of in-house textbook writing.

Joan MURUNGI

Head of CTLR Department

Table of Contents

TPART I. GENERAL INTRODUCTION

1.1 The structure of the guide

The tutor's guide of Mathematics is composed of three parts:

The Part I concerns general introduction that discusses methodological guidance on how best to teach and learn Mathematics, developing competences in teaching and learning, addressing cross-cutting issues in teaching and learning and Guidance on assessment.

Part II presents a sample lesson plan. This lesson plan serves to guide the teacher on how to prepare a lesson in Mathematics.

The Part III is about the structure of a unit and the structure of a lesson. This includes information related to the different components of the unit and these components are the same for all units. This part provides information and guidelines on how to facilitate student teachers while working on learning activities. More other, all application activities from the textbook have answers in this part.

1.2 Methodological guidance

1.2.1 Developing competences

Since 2015 Rwanda shifted from a knowledge based to a competency-based curriculum for pre-primary, primary, secondary education and recently the TTC curriculum. This called for changing the way of learning by shifting from teacher centred to a learner centred approach. Teachers are not only responsible for knowledge transfer but also for fostering learners' learning achievement and creating safe and supportive learning environment. It implies also that learners have to demonstrate what they are able to transfer the acquired knowledge, skills, values and attitude to new situations.

The competence-based curriculum employs an approach of teaching and learning based on discrete skills rather than dwelling on only knowledge or the cognitive domain of learning. It focuses on what learner can do rather than what learner knows. Learners develop competences through subject unit with specific learning objectives broken down into knowledge, skills and attitudes through learning activities.

In addition to the competences related to Mathematics, student teachers also develop generic competences which should promote the development of the higher order thinking skills and professional skills in Mathematics teaching. Generic competences are developed throughout all units of Mathematics as follows:

Generic competences	Ways of developing generic competences
Critical thinking	All activities that require learners to calculate, convert, interpret, analyze, compare and contrast, etc have a common factor of developing critical thinking into learners
Creativity and innovation	All activities that require learners to plot a graph of a given algebraic data, to organize and interpret statistical data collected and to apply skills in solving problems of economics have a common character of developing creativity into learners
Research and problem solving	All activities that require learners to make a research and apply their knowledge to solve problems from the real-life situation have a character of developing research and problem solving into learners.
Communication	During Mathematics class, all activities that require learners to discuss either in groups or in the whole class, present findings, debate …have a common character of developing communication skills into learners.
Co-operation, interpersonal relations and life skills	All activities that require learners to work in pairs or in groups have character of developing cooperation and life skills among learners.
Lifelong learning	All activities that are connected with research have a common character of developing into learners a curiosity of applying the knowledge learnt in a range of situations. The purpose of such kind of activities is for enabling learners to become life-long learners who can adapt to the fast-changing world and the uncertain future by taking initiative to update knowledge and skills with minimum external support.

Professional skills	Specific instructional activities and procedures that a teacher may use in the class room to facilitate, directly or indirectly, students to be engaged in learning activities. These include a range of teaching skills: the skill of questioning, reinforcement, probing, explaining, stimulus variation, introducing a lesson; illustrating with examples, using blackboard, silence and nonverbal cues, using audio – visual aids, recognizing attending behaviour and the skill of achieving closure.

The generic competences help learners deepen their understanding of Mathematics and apply their knowledge in a range of situations. As students develop generic competences they also acquire the set of skills that employers look for in their employees, and so the generic competences prepare students for the world of work.

1.2.2 Addressing cross cutting issues

Among the changes brought by the competence-based curriculum is the integration of cross cutting issues as an integral part of the teaching learning process-as they relate to and must be considered within all subjects to be appropriately addressed. The eight cross cutting issues identified in the national curriculum framework are: *Comprehensive Sexuality Education, Environment and Sustainability, Financial Education, Genocide studies, Gender, Inclusive Education, Peace and Values Education, and Standardization Culture*.

Some cross-cutting issues may seem specific to particular learning areas/ subjects but the teacher need to address all of them whenever an opportunity arises. In addition, learners should always be given an opportunity during the learning process to address these cross-cutting issues both within and out of the classroom.

Below are examples of how crosscutting issues can be addressed:

Cross-Cutting Issue	Ways of addressing cross-cutting issues
Comprehensive Sexuality Education: The primary goal of introducing Comprehensive Sexuality Education program in schools is to equip children, adolescents, and young people with knowledge, skills and values in an age appropriate and culturally gender sensitive manner so as to enable them to make responsible choices about their sexual and social relationships, explain and clarify feelings, values and attitudes, and promote and sustain risk reducing behaviour.	Using different charts and their interpretation, Mathematics tutor should lead students to discuss the following situations: "Alcohol abuse and unwanted pregnancies" and advise student teachers on how they can instil learners to fight those abuses. Some examples can be given when learning statistics, powers, logarithms and their properties.
Environment and Sustainability: Integration of Environment, Climate Change and Sustainability in the curriculum focuses on and advocates for the need to balance economic growth, society well-being and ecological systems. Learners need basic knowledge from the natural sciences, social sciences, and humanities to understand to interpret principles of sustainability.	Using Real life models or students' experience, Mathematics Tutor should lead student teachers to illustrate the situation of "population growth" and discuss its effects on the environment and sustainability.
Financial Education: The integration of Financial Education into the curriculum is aimed at a comprehensive Financial Education program as a precondition for achieving financial inclusion targets and improving the financial capability of Rwandans so that they can make appropriate financial decisions that best fit the circumstances of one's life.	Through different examples and calculations on interest rate problems, total revenue and total cost, Mathematics Tutor can lead student teachers to discuss how to make appropriate financial decisions.

Gender: At school, gender will be understood as family complementarities, gender roles and responsibilities, the need for gender equality and equity, gender stereotypes, gender sensitivity, etc.	Mathematics Tutor should address gender as cross-cutting issue through assigning leading roles in the management of groups to both girls and boys and providing equal opportunity in the lesson participation and avoid any gender stereotype in the whole teaching and learning process.
Inclusive Education: Inclusion is based on the right of all learners to a quality and equitable education that meets their basic learning needs and understands the diversity of backgrounds and abilities as a learning opportunity.	Firstly, Mathematics Tutors need to identify/recognize students with special needs. Then by using adapted teaching and learning resources while conducting a lesson and setting appropriate tasks to the level of students, they can cater for students with special education needs. They must create opportunity where student teachers can discuss how to cater for learners with special educational needs.
Peace and Values Education: Peace and Values Education (PVE) is defined as education that promotes social cohesion, positive values, including pluralism and personal responsibility, empathy, critical thinking and action in order to build a more peaceful society.	Through a given lesson, a tutor should: ▪ Set a learning objective which is addressing positive attitudes and values, ▪ Encourage students to develop the culture of tolerance during discussion and to be able to instil it in colleagues and cohabitants; ▪ Encourage students to respect ideas for others.

Standardization Culture: Standardization Culture in Rwanda will be promoted through formal education and plays a vital role in terms of health improvement, economic growth, industrialization, trade and general welfare of the people through the effective implementation of Standardization, Quality Assurance, Metrology and Testing.	With different word problems related to the effective implementation of Standardization, Quality Assurance, Metrology and Testing, students can be motivated to be aware of health improvement, economic growth, industrialization, trade and general welfare of the people.

1.2.3 Guidance on how to help students with special education needs in classroom

In the classroom, students learn in different way depending to their learning pace, needs or any other special problem they might have. However, the teacher has the responsibility to know how to adopt his/her methodologies and approaches in order to meet the learning need of each student in the classroom. Also teachers need to understand that student with special needs, need to be taught differently or need some accommodations to enhance the learning environment. This will be done depending to the subject and the nature of the lesson.

In order to create a well-rounded learning atmosphere, teachers need to:

- Remember that learners learn in different ways so they have to offer a variety of activities (e.g. role-play, music and singing, word games and quizzes, and outdoor activities);
- Maintain an organized classroom and limits distraction. This will help learners with special needs to stay on track during lesson and follow instruction easily;
- Vary the pace of teaching to meet the needs of each child. Some learners process information and learn more slowly than others;
- Break down instructions into smaller, manageable tasks. Learners with special needs often have difficulty understanding long-winded or several instructions at once. It is better to use simple, concrete sentences in order to facilitate them understand what you are asking.
- Use clear consistent language to explain the meaning (and demonstrate or show pictures) if you introduce new words or concepts;
- Make full use of facial expressions, gestures and body language;
- Pair a learner who has a disability with a friend. Let them do things together and learn from each other. Make sure the friend is not over protective and does not do everything for the one with disability. Both

learners will benefit from this strategy;

- Use multi-sensory strategies. As all learners learn in different ways, it is important to make every lesson as multi-sensory as possible. Learners with learning disabilities might have difficulty in one area, while they might excel in another. For example, use both visual and auditory cues.

Below are general strategies related to each main category of disabilities and how to deal with every situation that may arise in the classroom. However, the list is not exhaustive because each child is unique with different needs and that should be handled differently.

Strategy to help learners with developmental impairment:

- Use simple words and sentences when giving instructions;
- Use real objects that learners can feel and handle. Rather than just working abstractly with pen and paper;
- Break a task down into small steps or learning objectives. The learner should start with an activity that she/he can do already before moving on to something that is more difficult;
- Gradually give the learner less help;
- Let the learner with disability work in the same group with those without disability.

Strategy to help learners with visual impairment:

- Help learners to use their other senses (hearing, touch, smell and taste) and carry out activities that will promote their learning and development;
- Use simple, clear and consistent language;
- Use tactile objects to help explain a concept;
- If the learner has some sight, ask him/her what he/she can see;
- Make sure the learner has a group of friends who are helpful and who allow him/her to be as independent as possible;
- Plan activities so that learners work in pairs or groups whenever possible;

Strategy to help learners with hearing disabilities or communication difficulties

- Always get the learner's attention before you begin to speak;
- Encourage the learner to look at your face;
- Use gestures, body language and facial expressions;
- Use pictures and objects as much as possible.
- Keep background noise to a minimum.

Strategies to help learners with physical disabilities or mobility difficulties:

- Adapt activities so that learners who use wheelchairs or other mobility aids, can participate.
- Ask parents/caregivers to assist with adapting furniture e.g. the height of a table may need to be changed to make it easier for a learner to reach it or fit their legs or wheelchair under;
- Encourage peer support when needed;
- Get advice from parents or a health professional about assistive devices if the learner has one.

Adaptation of assessment strategies:

At the end of each unit, the tutor is advised to provide additional activities to help students achieve the key unit competence. These assessment activities are for remedial, consolidation and extension designed to cater for the needs of all categories of students; slow, average and gifted students respectively. Therefore, the tutor is expected to do assessment that fits individual student.

Remedial activities	After evaluation, slow students are provided with lower order thinking activities related to the concepts learnt to facilitate them in their learning. These activities can also be given to assist deepening knowledge acquired through the learning activities for slow students.
Consolidation activities	After introduction of any concept, a range number of activities can be provided to all students to enhance/ reinforce learning.
Extended activities	After evaluation, gifted and talented students can be provided with high order thinking activities related to the concepts learnt to make them think deeply and critically. These activities can be assigned to gifted and talented students to keep them working while other students are getting up to required level of knowledge through the learning activity.

1.2.4. Guidance on assessment

Assessment is an integral part of teaching and learning process. The main purpose of assessment is for improvement of learning outcomes. Assessment for learning/ Continuous/ formative assessment intends to improve students' learning and tutor's teaching whereas assessment of learning/summative assessment intends to improve the entire school's performance and education system in general.

Continuous/ formative assessment

It is an on-going process that arises during the teaching and learning process. It includes lesson evaluation and end of sub unit assessment. This formative assessment should play a big role in teaching and learning process. The teacher should encourage individual, peer and group evaluation of the work done in the classroom and uses appropriate competence-based assessment approaches and methods.

Formative assessment is used to:

- Determine the extent to which learning objectives are being achieved and competences are being acquired and to identify which students need remedial interventions, reinforcement as well as extended activities. The application activities are developed in the learner book and they are designed to be given as remedial, reinforcement, end lesson assessment, homework or assignment
- Motivate students to learn and succeed by encouraging students to read, or learn more, revise, etc.
- Check effectiveness of teaching methods in terms of variety, appropriateness, relevance, or need for new approaches and strategies. Mathematics tutors need to consider various aspects of the instructional process including appropriate language levels, meaningful examples, suitable methods and teaching aids/ materials, etc.
- Help students to take control of their own learning.

In teaching Mathematics, formative or continuous assessment should compare performance against instructional objectives. Formative assessment should measure the student's ability with respect to a criterion or standard. For this reason, it is used to determine what students can do, rather than how much they know.

Summative assessment

The assessment can serve as summative and informative depending to its purpose. The end unit assessment will be considered summative when it is done at end of unit and want to start a new one.

It will be formative assessment, when it is done in order to give information on the progress of learners and from there decide what adjustments need to be done.

The assessment done at the end of the term, end of year, is considered as summative assessment so that the teacher, school and parents are informed of the achievement of educational objective and think of improvement strategies.

There is also end of level/ cycle assessment in form of national examinations.

When carrying out assessment?

Assessment should be clearly visible in lesson, unit, term and yearly plans.

- Before learning (diagnostic): At the beginning of a new unit or a section of work; assessment can be organized to find out what students already know / can do, and to check whether the students are at the same level.
- During learning (formative/continuous): When students appear to be having difficulty with some of the work, by using on-going assessment (continuous). The assessment aims at giving students support and feedback.
- After learning (summative): At the end of a section of work or a learning unit, the Mathematics Tutor has to assess after the learning. This is also known as Assessment of Learning to establish and record overall progress of students towards full achievement. Summative assessment in Rwandan schools mainly takes the form of written tests at the end of a learning unit or end of the month, and examinations at the end of a term, school year or cycle.

Instruments used in assessment.

- **Observation:** This is where the Mathematics tutor gathers information by watching students interacting, conversing, working, playing, etc. A tutor can use observations to collect data on behaviours that are difficult to assess by other methods such as attitudes, values, and generic competences and intellectual skills. It is very important because it is used before the lesson begins and throughout the lesson since the tutor has to continue observing each and every activity.

- **Questioning**

a) Oral questioning: a process which requires a student to respond verbally to questions

b) Class activities/ exercises: tasks that are given during the learning/ teaching process

c) Short and informal questions usually asked during a lesson

d) Homework and assignments: tasks assigned to students by their tutors to be completed outside of class.

Homework assignments, portfolio, project work, interview, debate, science fair, Mathematics projects and Mathematics competitions are also the different forms/ instruments of assessment.

1.2.5. Teaching methods and techniques that promote active learning

The different learning styles for students can be catered for, if the teacher uses active learning whereby learners are really engaged in the learning process.

The main teaching methods used in mathematics are the following:

- **Dogmatic method** (the teacher tells the students what to do, What to observe, How to attempt, How to conclude)
- **Inductive-deductive method:** Inductive method is to move from specific examples to generalization and deductive method is to move from generalization to specific examples.
- **Analytic-synthetic method:** Analytic method proceeds from unknown to known, 'Analysis' means 'breaking up' of the problem in hand so that it ultimately gets connected with something obvious or already known. Synthetic method is the opposite of the analytic method. Here one proceeds from known to unknown.
- **Skills Laboratory method:** Laboratory method is based on the maxim "learning by doing." It is a procedure for stimulating the activities of the students and to encourage them to make discoveries through practical activities.
- **Problem solving method, Project method and Seminar Method.**

The following are some active techniques to be used in Mathematics:

- Group work
- Research
- Probing questions
- Practical activities (drawing, plotting, interpreting graphs)
- Modelling
- Brainstorming
- Quiz Technique
- Discussion Technique
- Scenario building Technique

What is Active learning?

Active learning is a pedagogical approach that engages learners in doing things and thinking about the things they are doing. Learners play the key role in the active learning process. They are not empty vessels to fill but people with ideas, capacity and skills to build on for effective learning. Thus, in active learning, learners are encouraged to bring their own experience and knowledge into the learning process.

The role of the teacher in active learning	The role of learners in active learning
- The teacher engages learners through active learning methods such as inquiry methods, group discussions, research, investigative activities, group and individual work activities. - He/she encourages individual, peer and group evaluation of the work done in the classroom and uses appropriate competence-based assessment approaches and methods. - He provides supervised opportunities for learners to develop different competences by giving tasks which enhance critical thinking, problem solving, research, creativity and innovation, communication and cooperation. - Teacher supports and facilitates the learning process by valuing learners' contributions in the class activities.	A learner engaged in active learning: - Communicates and shares relevant information with peers through presentations, discussions, group work and other learner-centred activities (role play, case studies, project work, research and investigation); - Actively participates and takes responsibility for his/her own learning; - Develops knowledge and skills in active ways; - Carries out research/investigation by consulting print/online documents and resourceful people, and presents their findings; - Ensures the effective contribution of each group member in assigned tasks through clear explanation and arguments, critical thinking, responsibility and confidence in public speaking - Draws conclusions based on the findings from the learning activities.

Main steps for a lesson in active learning approach

All the principles and characteristics of the active learning process highlighted above are reflected in steps of a lesson as displayed below. Generally, the lesson is divided into three main parts whereby each one is divided into smaller steps to make sure that learners are involved in the learning process. Below are those main part and their small steps:

1) Introduction

Introduction is a part where the teacher makes connection between the current and previous lesson through appropriate technique. The teacher opens short discussions to encourage learners to think about the previous learning experience and connect it with the current instructional objective. The teacher reviews the prior knowledge, skills and attitudes which have a link with the new concepts to create good foundation and logical sequencings.

2) Development of the new lesson

The development of a lesson that introduces a new concept will go through the following small steps: discovery activities, presentation of learners' findings, exploitation, synthesis/summary and exercises/application activities.

- **Discovery activity**

Step 1

- The teacher discusses convincingly with learners to take responsibility of their learning
- He/she distributes the task/activity and gives instructions related to the tasks (working in groups, pairs, or individual to instigate collaborative learning, to discover knowledge to be learned)

Step 2

- The teacher let learners work collaboratively on the task;
- During this period the teacher refrains to intervene directly on the knowledge;
- He/she then monitors how the learners are progressing towards the knowledge to be learned and boosts those who are still behind (but without communicating to them the knowledge).

- **Presentation of learners' findings/productions**
    - In this episode, the teacher invites representatives of groups to present their productions/findings.
    - After three/four or an acceptable number of presentations, the teacher decides to engage the class into exploitation of learners' productions.

- **Exploitation of learner's findings/ productions**
    - The teacher asks learners to evaluate the productions: which ones are correct, incomplete or false
    - Then the teacher judges the logic of the learners' products, corrects those which are false, completes those which are incomplete, and confirms those which are correct.

- **Institutionalization or harmonization (summary/conclusion/ and examples)**
    - The teacher summarizes the learned knowledge and gives examples which illustrate the learned content.

- **Application activities**
  - Exercises of applying processes and products/objects related to learned unit/sub-unit
  - Exercises in real life contexts
  - Teacher guides learners to make the connection of what they learnt to real life situations. At this level, the role of teacher is to monitor the fixation of process and product/object being learned.

3) Assessment

In this step the teacher asks some questions to assess achievement of instructional objective. During assessment activity, learners work individually on the task/activity. The teacher avoids intervening directly. In fact, results from this assessment inform the teacher on next steps for the whole class and individuals. In some cases, the teacher can end with a homework/ assignment. Doing this will allow learners to relay their understanding on the concepts covered that day. Teacher leads them not to wait until the last minute for doing the homework as this often results in an incomplete homework set and/or an incomplete understanding of the concept.

PART II: SAMPLE LESSON PLANS

Sample Lesson for unit 5: Integration

School Name:….. Teacher's name: …..

Term	Date	Subject	Class	Unit Nº	Lesson Nº	Duration	Class size
III	…/…/2019	Mathematics	Year three SME	5	5 of 21	40 min	…
Type of Special Educational Needs to be catered for in this lesson and number of learners in each category				3 slow student teachers and 2 low vision student teachers:			

Unit title	Integration
Key unit competency:	Determine correctly integration as the inverse of differentiation or limit of a sum and apply it to find area of plane surfaces, volumes of solid of revolution and lengths of curved lines.
Title of the lesson	Basic integration formulas or immediate integration

Instructional Objective	Given a simple function, student teachers will be able to accurately determine its integrations by considering the basic integration formulas .
Plan for this Class (location: in / outside)	The lesson is held indoors, the class is organized into groups ,3 slow student teachers are scattered in different groups ,and 2 low vision student teachers seat on the front desks near the blackboard in order to see and participate fully in all activities
Learning Materials (for ALL student teachers)	Textbooks and charts containing the table of derivatives and integrations.
References	Mathematics SME Year 3 textbook and Tutor's guide.

<table>
<tr><th rowspan="3">Timing for each step</th><th colspan="2">Description of teaching and learning activity</th><th rowspan="3">Generic competences and cross cutting issues to be addressed + a short explanation</th></tr>
<tr><td colspan="2">- student teachers work individually the introductory activity, and the correction is done on the chalk board by two student teachers, one after another under the guidance of the tutor.
- Then they discuss in groups the discovery activity, followed by the presentation by a sample group, interaction of student teachers and harmonization of the results under the facilitation of the tutor.
- Next, they discuss in pairs the solved example and compare their results with the answer proposed in the book .
- Finally, the student teachers are assigned individual tasks, and the correction is done on the chalk board, and the tutor winds up the lesson.</td></tr>
<tr><td>Tutor activities</td><td>Student teacher activities</td></tr>
</table>

Introduction: 5 minutes	The tutor asks student teachers to work individually: a) $f(x)=\sin x$ b) $f'(x)=....$; c) $h(x)=.....$ d) $h'(x)=2x+1$ - The tutor links the introduction to the lesson of the day.	- student teachers work individually. - Two student teachers, one after another, write the answers on the chalkboard: a) $f'(x)=\cos x$ b) $h(x)=x^2+x+c$ The answer for $h(x)$ is not unique; because of that constant c.	**Communication** skills developed through the presentation and sharing ideas
Development of the lesson			
Discovery activity: 10 minutes	-The tutor organizes the student teachers into groups -Tutor gives student teachers activity 5.5 to discuss in groups and gives instructions related to the task -Tutor goes round to monitor the work of each group and provide assistance where needed	- Student teachers form groups. -Each group analyses and discusses the activity 5.5 under the direction of the task manager of the group. - Student teachers present to the tutor their eventual problems.	▪ **Cooperation and communication** skills through discussions ▪ **Peace and values education;** Cooperation , mutual respect, tolerance through discussions with people with different views and respect one's views

Presentation of student teachers' findings and exploitation: 15 minutes	-Tutor invites the reporter of a sample group to present the findings of the group -The tutor encourages student teachers to follow attentively -Tutor takes notes on key points from student teachers' presentation. -The tutor asks student teachers to amend the presentation and to evaluate their work	-The reporter presents the work on the behalf of the group. **Expected answers** **(Refer** to solution of activity 5.5, in Tutor Guide) - Student teachers follow the presentation. - Student teachers evaluate the findings of other student teachers. - Student teachers evaluate their own findings.	▪ **Cooperation and communication/** attentive listening during presentations and group discussions ▪ **Critical thinking** through evaluating other's findings

Conclusion/ Summary: 5 minutes **Assessment** 5 minutes	-Tutor facilitates the student teachers to elaborate the summary of the presentation -Tutor requests student teachers to write down the main points in their note books. - Tutor asks student teachers to individually work out the application activity 5.5	-The student teachers come to the main point: **Basic integration formulas or immediate integration: they will be put on manila paper and hung on the wall.** - Student teachers take notes in their books. -Individually student teachers work out the application activity 5.5. and finally they make a correction on the chalk board. **Expected answers** (Refer to solution of application activity 5.5, in Tutor Guide).	- **Critical thinking** and problem solving skills are developed through analysing and solving real life Mathematical problem. - **Financial education** is addressed through good management of the school fees brought by the elder brother - **Standardisation culture:** good habit of paying school fees.
Observation on lesson delivery	To be completed after receiving the feed-back from the student teachers (what did the learners like, what challenged them,...)		

PART III: UNIT DEVELOPMENT

UNIT 1

COMPLEX NUMBERS

1.1 Key unit competence

Perform operations on complex numbers in different forms and use them to solve related problems in Physics, etc.

1.2 Prerequisite knowledge and skills

- Points, straight lines and circle in 2D (Year 1: unit 9)
- Addition/ subtraction of two vectors and their geometrical representation in Cartesian plane (Senior 2: unit 7 and Year 1: unit 9)
- Solving equations in the set of real numbers (Senior 1: unit 3; Senior 2: unit 1 and Year 1: unit 5).
- Definition of the principle trigonometric ratios(Year 1: Unit 8)
- Solving trigonometric equations (Year 2: unit 3)

1.3 Cross-cutting issues to be addressed

- Inclusive education(promote education for all while teaching)
- Peace and value Education (respect others views and thoughts during class discussions)
- Gender (equal opportunity of boys and girls in the lesson participation)
- Standardization culture (regulation of the current intensity)

1.4 Guidance on the introductory activity

a) Student-teachers work on the introductory activity to understand the importance of imaginary number "j" in solving equations which can't be solved in the set of real numbers.

b) Let Student-teachers read the introductory activity in the Student-teacher's book.

c) Through question-answer, facilitate Student-teachers to realise that $x=\pm 2$ are solutions of equation $x^2-4=0x^2-4=0$in the set of real numbers $\mathbb{R}$ and that the equation $x^2+4=0$ does not have solutions in $\mathbb{R}$ because the square root of a negative number is not possible in $\mathbb{R}$.

d) Through class discussions, let Student-teachers think of different ways of getting solutions of the equation $x^2+4=0$ and by introduction of imaginary number " j ", let them use " $j^2=-1$ " to find that $x=\pm 2i$ are solutions of the equation $x^2+4=0$.

e) Help Student-teachers to understand that the imaginary number " i "is element of a new set of numbers known as set of complex numbers " $\mathbb{C}$ ".

1.5 List of lessons

UNIT 1: COMPLEX NUMBERS			
SUB-UNIT 1: Algebraic form of Complex numbers and their geometric representation (40 periods)			
Introductory activity : 1 period: 40 minutes			
#	**Lesson title**	Learning objectives (from the syllabus including knowledge, skills and attitudes):	**Number of periods**
SUB-UNIT 1: Concept of complex numbers and their algebraic form (4 periods**)**			
1	Definition and properties of " the imaginary number i "	Identify the real part and the imaginary part of a complex number	2
2	Geometric representation of complex numbers	Represent a complex number on Argand diagram	2
SUB-UNIT 2: Operation on complex numbers (14 periods)			

3	Addition and subtraction in the set of complex numbers	Apply the properties of complex numbers to perform operations (addition , subtraction, conjugate, multiplication, powers, and division) on complex numbers in algebraic form	2
4	Conjugate of a complex number		1
5	Multiplication and powers of complex number		2
6	Division in the set of complex numbers		1
7	Modulus of a complex number and its interpretation	Find and give interpretation of the modulus of a complex number	2
8	Loci related to the distances on Argand diagram	Appreciate the importance of complex numbers	2
9	Square roots of a complex number	Find the square roots of a complex number	2
10	Equations in the set of complex numbers	Solve a simple linear or quadratic equation in the set of complex numbers	2
SUB-UNIT 3: Polar form of complex numbers (13 periods)			
11	Definition and properties of a complex number z in polar form	Define a complex number in a polar form and convert a complex number from algebraic form to polar form and vice versa.	2

12	Multiplication and division of complex numbers in polar form	Apply the properties of complex numbers to perform operations on complex numbers in polar form	2
13	Powers and De Moivre's formula	Apply De Moivre's theorem to calculate power of complex number	2
14	Transformation of trigonometric numbers of a multiple of an angle	Apply De Moivre's formula to transform trigonometric expressions	2
15	N^{th} roots of a complex number	Determine the n^{th} roots of a complex number and represent them on the Argand diagram	2
16	Construction of regular polygons	Construct on the Argand diagram the points representing the n^{th} roots of a complex number and deduce the names of the polygons obtained by joining the points representing the n^{th} roots of unity	3
SUB-UNIT 4: Exponential form of complex numbers (6 periods)			
17	Definition and properties of a complex number z in exponential form	Define a complex number in exponential form and convert a complex number from algebraic or polar form to exponential form and vice versa	2
18	Euler's formula of complex numbers	Apply Euler's formula to transform trigonometric expressions	2

19	Application of complex numbers in physics.	Appreciate the importance of complex numbers to solve related problems such as in Physics (problem related to voltage and alternating current),…	2
19	**End unit assessment**		2

Notice:

For application of mathematics content to other subjects, the teacher will consider the prerequisite of Student-teachers in this domain then act accordingly; the time spent and importance given to application activity will depend on the learner's level of knowledge.

Lesson 1: Definition and properties of "the imaginary number i"

a) Learning objective

Identify the real part and the imaginary part of a complex number

b) Teaching resources

- Student-teacher's book and other Reference textbooks to facilitate research
- Flash cards containing the complex numbers, real numbers and pure imaginary numbers

c) Prerequisites/Revision/Introduction

Through examples, let Student-teachers discuss how to solve quadratic equations in $\mathbb{R}$ (set of real numbers). For example use the quadratic equation $x^2 - 9 = 0$ and let Student-teachers find that the solution is $x = \pm 3$ because the square root of a positive number exists.

d) Learning activities

- Ask Student-teachers to use the formula for solving quadratic equation $x^2 + 16 = 0$ in the set of real numbers from activity 1.1, and by introducing the imaginary number $i = \sqrt{-1}$ or $i^2 = -1$, let them find the square root of a negative number which does not exist in $\mathbb{R}$.
- Lead Student-teachers to realize that the solution $x = \pm 4i$ is the solution of the equation $x^2 + 16 = 0$.

- Explain to the Student-teacher that the solution $x = \pm 4i$ containing the imaginary number i is an element of a new set called "set of complex numbers $\mathbb{C}$" and that $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$.
- Through generalization, let Student-teachers discover that given two real numbers a and b the complex number z is defined as, $z = a + ib$ where the number a is called the **real part** of z the number b is called the **imaginary part** of z and $i^2 = -1 \, i^2 = -1$.
- Let Student-teachers be aware that the definition of complex number is mathematically written as follows:

$$\mathbb{C} = \left\{ a + ib; \quad a, b \in \mathbb{R} \quad and\ i^2 = -1 \right\}$$

Answers for Activity 1.1.1

Through different examples, help Student-teachers to discover the properties of imaginary number "i" and understand the importance of complex numbers.

The solution set of the following equation $x^2 + 16 = 0 \Leftrightarrow x^2 = -16 \Leftrightarrow x^2 = 16i^2$;

Thus $x = 4i\ or\ x = -4i.$

The solution set is $\{-4i; 4i\}$; these solutions are not real numbers; they are elements of a new set called set of complex numbers.

Answer for activity 1.1.2

$i^3 = -i; i^4 = 1; i^5 = i; i^7 = -i; i^8 = 1$

Answers for Application activity 1.1.

Solution1:

$a)\ z = 4 + 2i, \quad \mathrm{Re}(z) = 4, \mathrm{Im}(z) = 2$

$b)\ z = i, \quad \mathrm{Re}(z) = 0, \mathrm{Im}(z) = 1$

$c)\ \mathrm{z} = \sqrt{2} - i, \quad \mathrm{Re}(z) = \sqrt{2}, \mathrm{Im}(z) = -1$

$d)\ z = -3.5, \quad \mathrm{Re}(z) = -3.5, \mathrm{Im}(z) = 0$

Solution2

$a)\ 25 = (6 \times 4) + 1, \quad i^{25} = i$

$b)\ 2310 = (577 \times 4) + 2, \quad i^{2310} = -1$

$c)71=(17\times 4)+3, \quad i^{71}=i^{3}=-i$

$d)\ 51=4\times 12+3, \quad i^{28}=i^{3}=-i$

$e)\ 28=4\times 7, \quad i^{28}=i^{0}=1$

Lesson 2: Geometric representation of complex numbers

a) Learning objective

Represent a complex number on Argand diagram

b) Teaching resources:

- T-square, ruler, if possible Math draw software as geogebra, Math lab, graph,….
- Student-teacher's book and other textbooks to facilitate research

c) Prerequisites/Revision/Introduction:

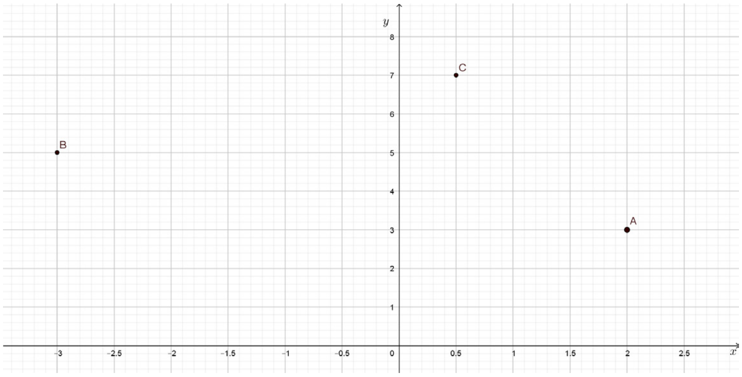
- Using the activity 1.3, guide Student-teachers in plotting points: $A(2,3)$ and $B(-3,5)$ in Cartesian plane.

d) Learning activities :

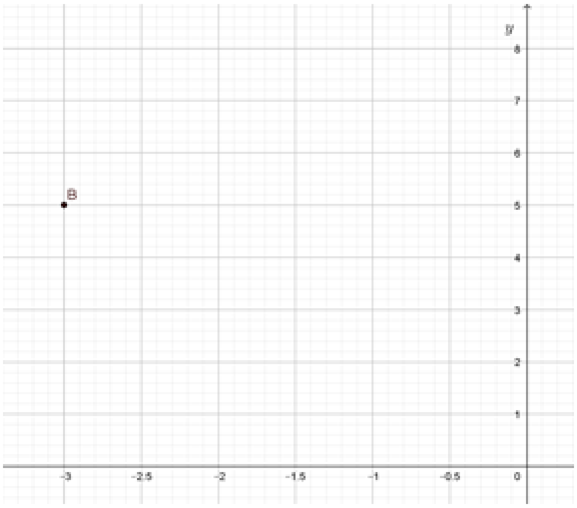
- Given the complex number $z=-3+5i$, ask Student-teachers to plot the point $z(-3,5)$ in Cartesian plane and then from activity 1.2 , facilitate them to plot the point $z(a,b)$ affix of $z=a+bi$ in a complex plane.
- Through class discussions, ask Student-teachers to establish a relationship between Cartesian plane and complex plane by considering $x-axis$ as real axis and $y-axis$ axis as imaginary axis.
- Using figure 1.2, let Student-teachers deduce that the **complex plane** comprises two number lines that intersect in a right angle at the point $(0,0)$. The horizontal number line (or $x-axis$ in Cartesian plane) is the **real axis** while the vertical number line (or $y-axis$ in Cartesian plane) is the **imaginary axis**.

Answer for Activity 1.2

1.

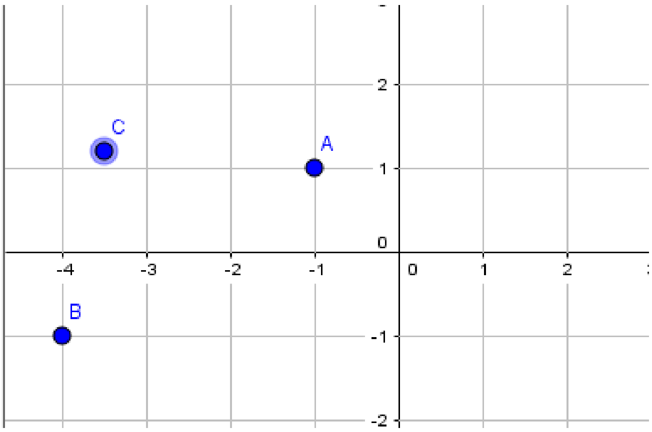


2.



Yes, all complex numbers can be represented on an Argand diagram

Answers for Application Activity 1.2.



Lesson 3: Addition and subtraction in the set of complex numbers

a) Learning objective

Apply the properties of complex numbers to perform addition and subtraction on complex numbers in algebraic form

b) Teaching resources:

- T-square, ruler, if possible Math draw software as geogebra, Mathlab,….
- Student-teacher's book/ Internet or textbooks to facilitate research

c) Prerequisites/Revision/Introduction:

- Guide Student-teachers to plot the point $A(1,2)$ and $B(-2,4)$ and deduce the coordinate of the vector $\overrightarrow{OA}+\overrightarrow{OB}=\overrightarrow{A+B}$ in the Cartesian plane.

d) Learning activities :

- From **Activity 1.3.1** and basing on the answer found in a), guide Student-teachers in plotting respectively the affix $z_1(1,2)$ and $z_2(-2,4)$ of $z_1=1+2i$ and $z_2=-2+4i$.
- Basing on the geometrical representation of $z_1(1,2)$ and $z_2(-2,4)$, facilitate Student-teachers to deduce the affix $z(-1,6)$ of the complex number z_1+z_2.

Through the interpretation of the figure 1.2 by Student-teachers using parallelogram rule, let Student-teacher to find that the sum of two complex numbers $z_1=a+bi \text{ and } z_2=c+di$ is obtained by adding the sum of real parts to the sum of imaginary part as follow: $[\text{Re}(z_1)+\text{Re}(z_2)]+i[\text{Im}(z_1)+\text{Im}(z_2)]$,therefore the sum of $z_1=a+bi \text{ and } z_2=c+di$ is obtained by $(a+bi)+(c+di)=(a+c)+(b+d)\text{i}$.

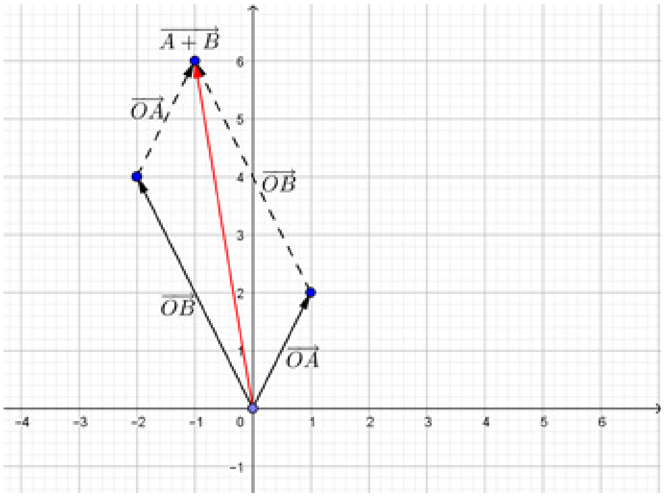
- Through a variety of examples and graphical representations of two different complex numbers in the complex plane, lead Student-teachers to calculate the difference of two complex numbers $z_1=a+bi \text{ and } z_2=c+di$ as follows: $(a+bi)-(c+di)=(a-c)+(b-d)\text{i}$
- Facilitate Student-teachers to remember how to **add vectors graphically:**

  Vector addition is found geometrically by constructing a *parallelogram*, using the two vectors $z_1=a+bi$ and $z_2=c+di$ as two of the sides.

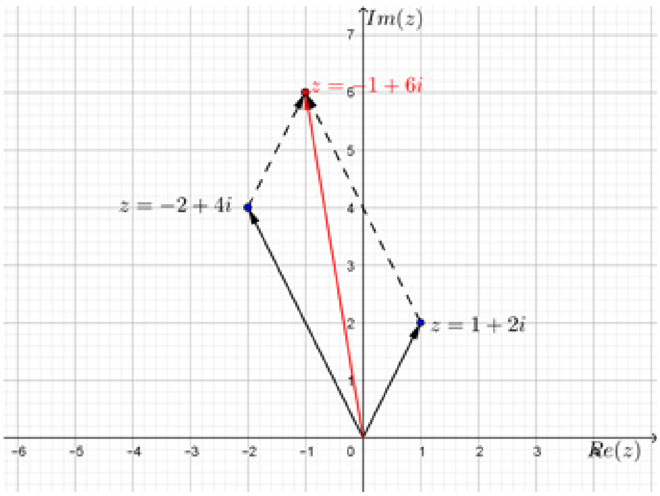
  Then, the diagonal is the resultant (or the sum vector).

Answers for Activity 1.3.1

a)



b)

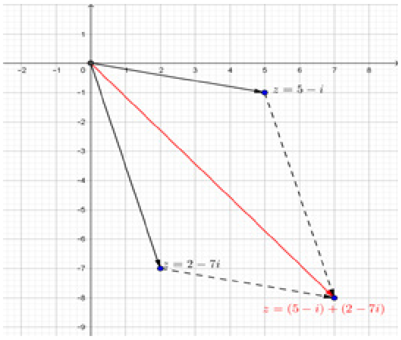


c) $z_1 + z_2 = (1 + 2i) + (-2 + 4i) = -1 + 6$

d) The affix of the sum of two complex numbers is the end point of the diagonal of the parallelogram obtained from the affixes of the terms

Answers for Application Activity 1.3.1

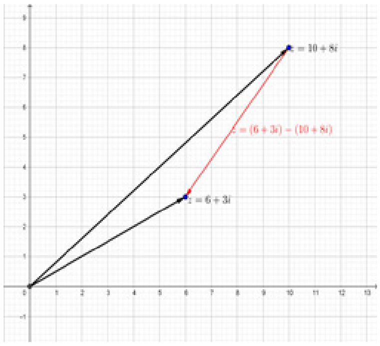
a)



From graphical representation

$$(5-i)+(2-7i)=7-8i=5+2-i(1+7)$$

b)

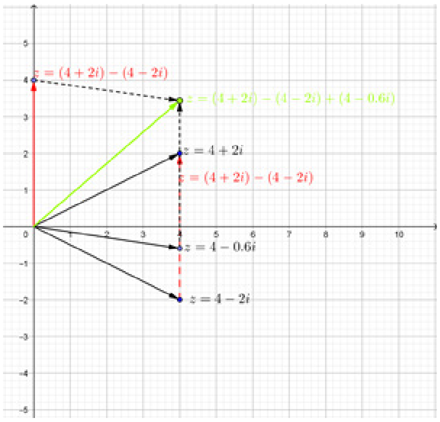


After representing graphically, $6+3i$ and $10+8i$, we deduce that

$$(6+3i)-(10+8i)=-4-5i=6-10+i(3-8)$$

Note that $(6+3i)-(10+8i)=(6+3i)+\left[-(10+8i)\right]$

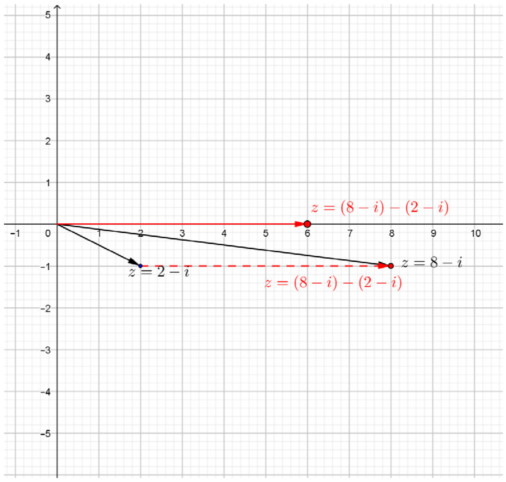
c)



From graphical representation, we get

$$(4+2i)-(4-2i)+(4-0.6i)=4i+(4-0.6i)=4+3.4i$$

d)



From geometric presentation, we deduce that

$$(8-i)-(2-i)=6=8-2-i(1-1)=6$$

Lesson 4: Conjugate of a complex number

a) Learning objective

Apply the properties of complex numbers to determine conjugate of complex numbers in algebraic form

b) Teaching resources:

- T-square, ruler, if possible Math draw software as geogebra, Mathlab,….
- Student-teacher's book and reference textbooks for developing Student-teachers' self-confidence through research activity.

c) Prerequisites/Revision/Introduction:

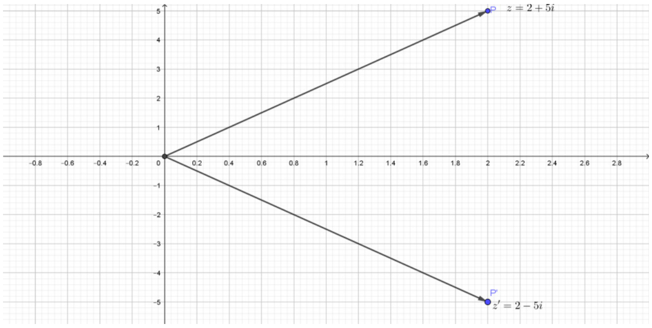
- Use the activity1.3.2 and facilitate Student-teachers to plot the number $z=2+5i$ in the complex plane and lead them to find the image $P'(2,-5)$ of the point $P(2,5)$ affix of z by the reflection of real axis.

d) Learning activities :

- Through the interpretation of the figure 1.8 and class discussion, let Student-teachers to discover that $z'=2-5i$ is the conjugate of $z=2+5i$.
- Using figure 1.8, the answer of the activity 1.5 and a variety of example 1.5, lead Student-teachers to discover that for every complex number

$z=a+bi$ there is a corresponding complex number $\bar{z}=a-bi$ called conjugate of z that is the reflection of z by real axis.

Answers for Activity 1.3.2

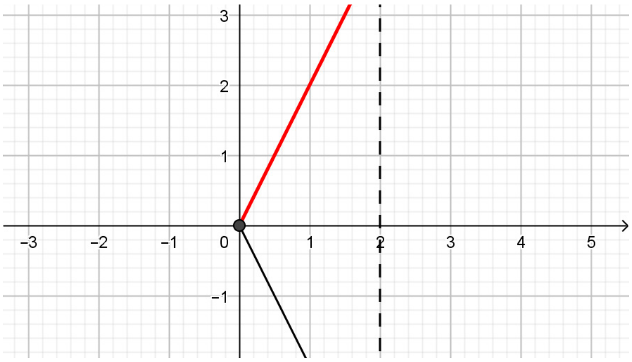


2. $(2,-5)$; $z'=2-5i$

$\mathrm{Re}(z')=\mathrm{Re}(z); \mathrm{Im}(z')=-\mathrm{Im}(z)$

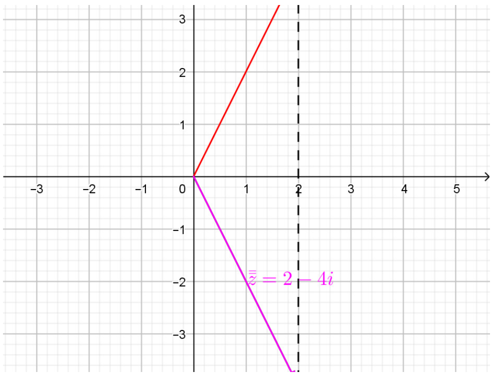
Answers for Application Activity 1.3.2

1.



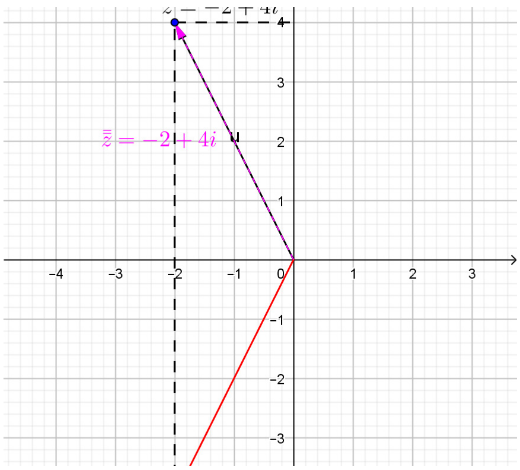
$z=2+4i$, $\bar{z}=2-4i$, $\bar{\bar{z}}=2+4i$

2.



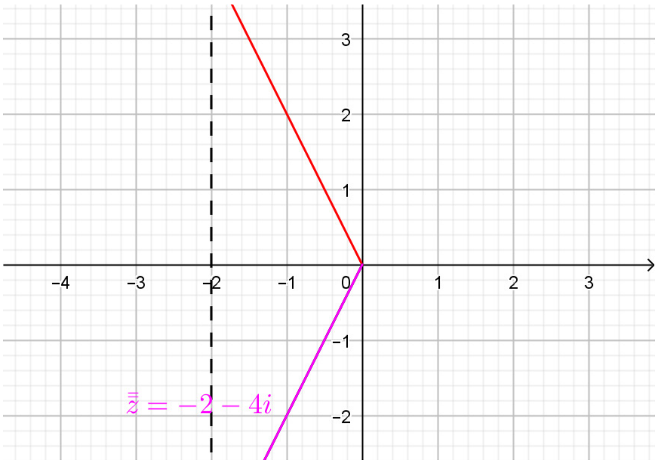
$z = 2 - 4i$, $\bar{z} = 2 + 4i$, $\bar{\bar{z}} = 2 - 4i$

3.



$z = -2 + 4i$, $\bar{z} = -2 - 4i$, $\bar{\bar{z}} = -2 + 4i$

4.



$z = -2 - 4i$, $\bar{z} = -2 + 4i$, $\bar{\bar{z}} = -2 - 4i$

From graphical representation or numerical analysis, we find that $\bar{\bar{z}} = z$.

Lesson 5: Multiplication and powers of complex numbers

a) Learning objective

Apply the properties of complex numbers to perform operations (addition, subtraction, conjugate, multiplication) on complex numbers in algebraic form

b) Teaching resources:

Student-teacher's book and other reference textbooks to facilitate research, calculator, manila paper, markers, pens, pencils...

c) Prerequisites/Revision/Introduction:

- By means of examples, help Student-teachers to calculate the product in the set of real numbers.
- Let them apply the following distributive property $(a+b).(c+d) = a(c+d)+b(c+d)$

d) Learning activities :

- Given two complex numbers $z_1 = 4-7i$ and $z_2 = 5+3i$, lead Student-teachers to determine the product

$$z_1 \cdot z_2 = (4-7i)\cdot(5+3i) = 4(5+3i) - 7i(5+3i) = 20+12i-35i+21 = 41-23i$$

by applying the distributive property and converting i^2 into -1

- Using the same procedure, facilitate Student-teachers to calculate the power $z_1^{\ 2} = (4-7i)^2 = (4-7i)(4-7i) \Leftrightarrow z_1^{\ 2} = -33-56i.$
- Let Student-teachers go through the example and work out application activity 1.3.3 to emphasize their skills in calculating the product and powers of complex numbers.

Answers for Activity 1.3.3

$$z_1 \cdot z_2 = (4-7i)\cdot(5+3i) = 4(5+3i) - 7i(5+3i) = 20+12i-35i+21 = 41-23i$$

Distributive property of multiplication over addition

Answers for application Activity 1.3.3

a) $z = i(3-7i)(2-i) = 17-i$

b) $(1+i)^2 - 3(2-i)^3 = -6+35i$

Lesson 6: Division in the set of complex numbers

a) Learning objective

Apply the properties of complex numbers to perform operations (addition , subtraction, conjugate, multiplication, powers and division) on complex numbers in algebraic form

b) Teaching resources:

- Internet and textbooks to facilitate research

c) Prerequisites/Revision/Introduction:

- Through different exercises, let Student-teachers determine the conjugate of the complex number $z = 5 + i$

d) Learning activities :

- Through the activity 1.3.4, let Student-teachers apply the rules of rationalizing the denominator in $\mathbb{R}$ and convert i^2 into -1, and assist them to transform the denominator of $z = \frac{2+3i}{5+i}$ into real part without changing the value of z as follow: $z = \frac{(2+3i)(5-i)}{(5+i)(5-i)} \Leftrightarrow z = \frac{(2+3i)(5-i)}{26}$
- Facilitate Student-teachers to determine the quotient of $2+3i$ and $5+i$ which is

$$\frac{2+3i}{5+i} = \frac{(2+3i)(5-i)}{(5+i)(5-i)} = \frac{10-2i+15i+3}{26} = \frac{1}{2} + \frac{1}{2}i$$

- From the activity 1.3.4, let Student-teachers work in small groups and make generalization on how to find $\frac{z_1}{z_2}$ given two complex numbers $z_1 = a + bi$ and $z_2 = c + di$.
- From the harmonization of groups work, help Student-teachers to realize that the quotient $\frac{z_1}{z_2}$ is obtained by $\frac{z_1}{z_2} = \frac{(a+bi)(c-di)}{(c+di)(c-di)} = \frac{(ac+bd)+(bc-ad)i}{c^2+d^2}$
- Let Student-teachers go through the example and work out application activity 1.3.4 to emphasize their skills in calculating the quotient of complex numbers .

Answers for Activity 1.3.4

$$\frac{2+3i}{5+i} = \frac{(2+3i)(5-i)}{(5+i)(5-i)} = \frac{10-2i+15i+3}{26} = \frac{1}{2} + \frac{1}{2}i$$

Answers for Application Activity 1.3.4

1. a) $z = \frac{1}{(2+i)(1-2i)}$

Or, $(2+i)(1-2i)=(2+2)+i(1-4)=4-3i$

Hence $z=\frac{1}{(2+i)(1-2i)}=\frac{1}{4-3i}=\frac{4+3i}{16+9}=\frac{1}{25}(4+3i)$.

b) $z=\left(\frac{\sqrt{3}-i}{\sqrt{3}+i}\right)^4\left(\frac{1+i}{1-i}\right)^5$

Let $z_1=\frac{\sqrt{3}-i}{\sqrt{3}+i}$ and $z_2=\frac{1+i}{1-i}$

, thus $z_1=\frac{(\sqrt{3}-i)(\sqrt{3}-i)}{3+1}=\frac{3-2\sqrt{3}i-1}{4}=\frac{1}{2}(1-i\sqrt{3})$ and

$z_2=\frac{1+i}{1-i}=\frac{(1+i)(1+i)}{1+1}=\frac{2i}{2}=i.$

Therefore, $z=z_1^4\cdot z_2^5=\left(\frac{\sqrt{3}-i}{\sqrt{3}+i}\right)^4\left(\frac{1+i}{1-i}\right)^5=\left[\frac{1}{2}(1-i\sqrt{3})\right]^4\cdot i^5=\frac{1}{16}(1-i\sqrt{3})^4\cdot i$

But, $(1-i\sqrt{3})^4=1-4(i\sqrt{3})+6(i\sqrt{3})^2-4(i\sqrt{3})^3+(i\sqrt{3})^4$.

$$=1-4i\sqrt{3}-18+12i\sqrt{3}+9=-8+8i\sqrt{3}.$$

Hence, $z=\frac{1}{16}(-8+8i\sqrt{3})\cdot i=\frac{1}{2}(-1+i\sqrt{3})\cdot i=-\frac{\sqrt{3}}{2}-\frac{1}{2}i$.

2. The solution of this type of problem is based on the fact that two complex numbers are equal if and only if their real parts and imaginary parts respectively are equal to each other.

a) $x+4y+xyi=12-16i \Leftrightarrow \begin{cases} x+4y=12 & (1) \\ xy=-16 & (2) \end{cases}$

From equation (1), we get $x=12-4y$.

Replacing x in equation (2), we obtain:

$(12-4y)y=-16 \Leftrightarrow 4y^2-12y-16=0 \Leftrightarrow y^2-3y-4=0 \Leftrightarrow (y-4)(y+1)=0$.

Hence, $y=-1$ or $y=4$.

If $y=-1$, then $x=16$; if $y=4$, then $x=-4$.

The solutions of the given problem are $(16,-1)$ and $(-4,4)$.

b) $x-7y+8xi=6y+(6y-100)i$

$$\Leftrightarrow \begin{cases} x-7y=6y \\ 8x=6y-100 \end{cases} \Leftrightarrow \begin{cases} x-13y=0 & (1') \\ 8x-6y=-100 & (2') \end{cases}$$

From (1') : $x = 13y$

Substituting the value of x in (2 ') gives

$$104y-6y=-100 \Leftrightarrow 98y=-100 \Leftrightarrow y=-\frac{100}{98}=-\frac{50}{49}.$$

Putting the value of y in (1') : $x=-\frac{650}{49}$.

Hence, $x=-\frac{650}{49}$ and $y=-\frac{50}{49}$.

c) $\frac{1}{x+iy}+\frac{1}{1+2i}=1 \Leftrightarrow \frac{x-iy}{x^2+y^2}+\frac{1-2i}{5}=1 \quad (x\neq 0\, and\ y\neq 0).$

$$\Leftrightarrow \frac{5(x-iy)+(x^2+y^2)(1-2i)}{5(x^2+y^2)}=1$$

$$\Leftrightarrow \frac{5(x-iy)+(x^2+y^2)(1-2i)}{5(x^2+y^2)}=1$$

$$\Leftrightarrow \frac{(x^2+y^2+5x)-(2x^2+2y^2+5y)i}{5(x^2+y^2)}=1$$

Hence, $\begin{cases} \frac{x^2+y^2+5x}{5(x^2+y^2)}=1 & (a) \\ \frac{2x^2+2y^2+5y}{5(x^2+y^2)}=0 & (b) \end{cases}$

$$(a)\&(b)\Leftrightarrow\begin{cases}x^2+y^2+5x=5x^2+5y^2\\2x^2+2y^2+5y=0\end{cases}\Leftrightarrow\begin{cases}4x^2+4y^2-5x=0 & (a')\\2x^2+2y^2+5y=0 & (b')\end{cases}$$

The linear combination $(a')-2(b')$ gives us

$-5x-10y=0\Leftrightarrow 5(x+2y)=0\Leftrightarrow x+2y=0$

Letting $y=\alpha$, we get $x=-2\alpha$, $\alpha\in\mathbb{R}$.

Substituting these values in (a') yields

$$16\alpha^2+4\alpha^2+10\alpha=0\Leftrightarrow 20\alpha^2+10\alpha=0$$

$$\Leftrightarrow 10\alpha(2\alpha+1)=0\Leftrightarrow\alpha=0\ or\ \alpha=-\frac{1}{2}.$$

The value $\alpha=0$ is to be eliminated since it is incompatible with the condition $x\neq 0\ and\ y\neq 0.$

Let $\alpha=-\frac{1}{2}\Rightarrow x=1$ and $y=-\frac{1}{2}$.These values satisfy the equations $(a')$and (b'), therefore the solution of the problem is $x=1$ and $y=-\frac{1}{2}$.

3)

$$\frac{1+T^2}{2T}=\frac{1+\left(\frac{x-iy}{x+iy}\right)^2}{2\frac{x-iy}{x+iy}}=\frac{\frac{(x+iy)^2+(x-iy)^2}{(x+iy)^2}}{2\frac{x-iy}{x+iy}}$$

$$=\frac{\left(x^2+2xyi-y^2+x^2-2xyi-y^2\right)(x+iy)}{2(x+iy)^2(x-iy)}$$

$$=\frac{2\left(x^2-y^2\right)}{2(x+iy)(x-iy)}=\frac{x^2-y^2}{x^2+y^2}\ as\ required.$$

Lesson 7: Definition and interpretation of Modulus of a complex number

a) Learning objective:

Find and give interpretation of the modulus of a complex number

b) Teaching resources:

- T-square, ruler, if possible Math draw software as geogebra, Mathlab, graph,….
- Internet and textbooks to facilitate research

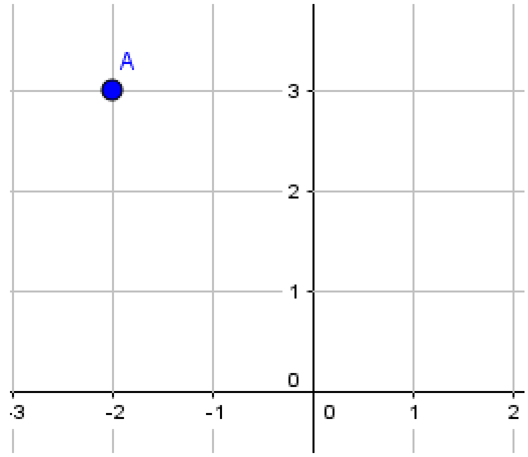
c) Prerequisites/Revision/Introduction:

- Facilitate Student-teachers to plot the point $A(0,0), B(3,0)$ and $C(4,3)$ in the Cartesian plane, and join those points to find a triangle and then find the distance of the side AC.

d) Learning activities :

- From the activity 1.3.5.1, lead Student-teachers to calculate the modulus of $z = a + bi$ and let them discover that the modulus is always a positive real number denoted by $|z|$, such that $|z| = \sqrt{z.\overline{z}} = \sqrt{a^2 + b^2}$.
- In small groups, facilitate the Student-teachers to calculate the modulus by explaining that the modulus value looks like the length of the vector with origin $(0,0)$ to the affix of the complex number z.
- By using the figure , harmonize the Student-teachers' group works and help them to get the real meaning of modulus through observation and interpretation.
- Individually, let Student-teachers go through the examples, and work out application activity 1.3.5.1 to emphasize their skills in calculating the modulus of complex numbers.

Answers for Activity 1.3.5.1



The distance from $(0,0)$ to $(-2,3)$ is $\sqrt{13}$

Answers for Application Activity 1.3.5.1

1. $z_1 = 2-3i$ then $|z_1| = \sqrt{4+9} = \sqrt{13}$,

   $z_2 = 3-4i$ then $|z_2| = \sqrt{9+16} = \sqrt{25} = 5$,

   $z_3 = 6+4i$ then $|z_3| = \sqrt{36+16} = \sqrt{52} = 2\sqrt{13}$,

   $z_4 = 15-8i$ then $|z_4| = \sqrt{225+64} = \sqrt{289} = 17$.

$$z = \frac{(2-3i)(3+4i)}{(6+4i)(15-8i)} = \frac{z_1 \cdot z_2}{z_3 \cdot z_4} \Rightarrow |z| = \frac{|z_1|\cdot|z_2|}{|z_3|\cdot|z_4|} = \frac{5\sqrt{13}}{2\sqrt{13}\times 17} = \frac{5}{34}.$$

2. If $z_1 = 1 - i$, $z_2 = -2 + 4i$, $z_3 = 3 - 2i$, then

a) $|2z_2 - 3z_1| = |-7+11i| = \sqrt{170}$

b) $|z_1.\overline{z}_2 + \overline{z}_1 . z_2| = 12$

c) $\left|\frac{z_1+z_2+1}{z_1-z_2+1}\right| = \left|\frac{(1-i)+(-2+4i)+1}{(1-i)-(-2+4i)+1}\right| = \left|\frac{3i}{4-5i}\right| = \frac{|3i|}{|4-5i|} = \frac{3}{\sqrt{41}} = \frac{3\sqrt{41}}{41}$

d) $|z_1^2 + z_2^2|^2 + |z_3^2 - z_2^2|^2 = |(1-i)^2 + (-2+4i)^2|^2 + |(3-2i)^2 - (-2+4i)^2|^2$

$$=\left|-2i+(-12-16i)\right|^2+\left|(5-12i)-(-12-16i)\right|^2=\left|-12-18i\right|^2+\left|17+4i\right|^2$$

$$=\left(\sqrt{144+324}\right)^2+\left(\sqrt{289+16}\right)^2=773$$

Lesson 8: Loci related to distances

a) Learning objectives:

Appreciate the importance of complex numbers

b) Teaching resources

Student-learner's book and other Reference books to facilitate research, calculator, Manila paper, markers, pens, pencils and other geometric instruments.

c) Prerequisites

Student-teachers will learn better this lesson if they have good knowledge and skills on how finding modulus of a complex number learnt in the previous lesson and general form of equation of a circle, a straight line,… in Cartesian plane learnt in 9th unit of year 1.

d) Learning activities:

- Invite student-teachers to work in groups and do the activity 1.3.5.2 in their Mathematics books, they have to sketch the set of points determined by the given condition.
- Move around in the class for facilitating student-teachers where necessary and give more clarification on eventual challenges they may face during their work; Verify and identify groups with different working steps;
- Invite one member from each group with different working steps to present their work where they must explain the working steps;
- As a tutor, harmonize the findings from presentation.
- After this step, through examples, guide students to do the application activity 1.3.5.2 and evaluate whether lesson objectives were achieved.

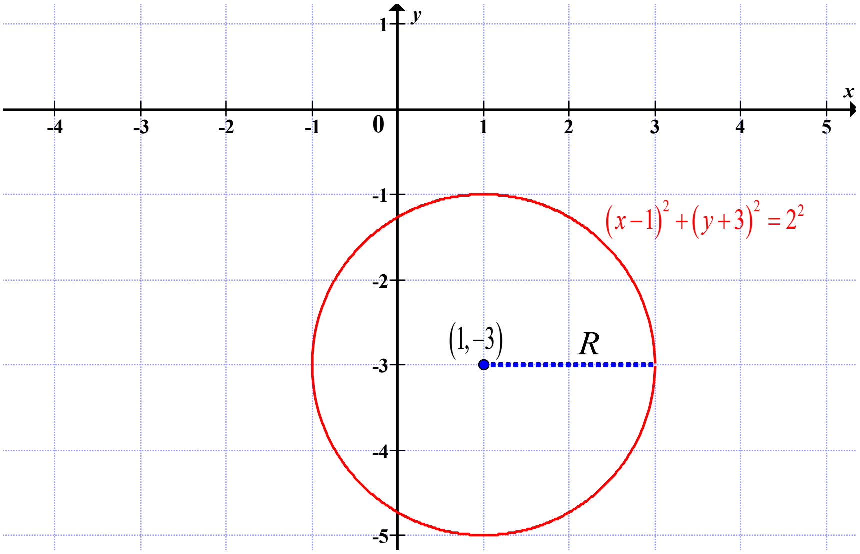
Answers for Activity 1.3.5.2

Let $z=x+yi$, we have

$$|x+yi-1+3i|=2\Leftrightarrow|x-1+i(y+3)|=2\Leftrightarrow\sqrt{(x-1)^2+(y+3)^2}=2\Leftrightarrow(x-1)^2+(y+3)^2=2^2$$

Which is the circle of centre $(1,-3)$ or $1-3i$ and radius $R=2$.

Curve



As conclusion, $|z|=R$ represents a circle with centre P and radius R, $|z-z_1|=R$ represents a circle with centre z_1 and radius R and $|z-z_1|=|z-z_2|$ represents a straight line- the perpendicular bisector (mediator) of the segment joining the points z_1 and z_2.

Answers for Application Activity 1.3.5.2

1. Circle: $3x^2+3y^2+4x+1=0$, radius is $\frac{1}{3}$ and centre is $\left(-\frac{2}{3},0\right)$
2. a) Circle: $x^2+y^2=4$, radius 2, centre at origin

   b) Interior of the circle: $x^2+y^2=4$, radius 2, centre at origin

   c) Exterior of the circle: $x^2+y^2=4$, radius 2, centre at origin

   d) Circle: $(x+1)^2+y^2=1$, radius 1, centre $(-1,0)$

   e) Vertical line: $x=0$, mediator of the line segment joining points $z_1=-1$ and $z_2=1$

   f) Circle: $(x-1)^2+(y+3)^2=4$, radius 2, centre $(1,-3)$

Lesson 9: Square roots of a complex number

a) Learning objectives:

Determine the square roots of a given complex number

b) Teaching resources:

Student-teacher's book and other Reference textbooks to facilitate research

c) Prerequisites/Revision/Introduction:

- Given $z = a + bi$, let Student-teachers discuss and calculate z^2, the square of a complex number $z = a + bi$.

d) Learning activities :

- From activity 1.3.6, let Student-teacherswork in groups andsolve $(a+ib)^2 = 6-4i$, where a and b are real numbers.
- Facilitate them to discuss the value of a and b and determine the square root of $z = 6-4i$. Help Student-teachers to solve the equation from activity 1.9 by matching similar terms of the eqation both sides or identification technique and then solve the obtained similitaneous equations $\begin{cases} a^2 - b^2 = 6 \\ 2ab = -4 \end{cases}$
- While working in groups, help Student-teachers remember how to solve bi-quadratic equation of the form: $ax^4 + bx^2 + c = 0$.
- Facilitate Student-teachers to find solutions of $\begin{cases} a^2 - b^2 = 6 \\ 2ab = -4 \end{cases}$ by solving bi-quadraticequation $b^4 + 6b^2 - 4 = 0$.

Let them suppose that $x = b^2$ and solve the quadratic equation $x^2 + 6x - 4 = 0$.

Or alternativelly, facilitate Student-teachers to find solutions of

$\begin{cases} a^2 - b^2 = 6 & (1) \\ 2ab = -4 & (2) \end{cases}$ by considering $|z|^2 = |z^2|$ and $a^2 + b^2 = 2\sqrt{13}$ (3)

- Generally , harmonize the Student-teachers' group works and let them realize that when the complex number $z = x + iy$ is the square root of a complex number $a + bi$, this means that

$$(x+yi)^2 = (a+bi) \Rightarrow |x+iy|^2 = |a+ib| \Leftrightarrow x^2 + 2xyi - y^2 = a + ib.$$

Equality of two complex numbers gives $\begin{cases} x^2 - y^2 = a \\ 2xy = b \end{cases}$.

Therefore, $\begin{cases} x = \pm\sqrt{\frac{1}{2}\left(a+\sqrt{a^2+b^2}\right)} \\ y = \pm\sqrt{\frac{1}{2}\left(-a+\sqrt{a^2+b^2}\right)} \end{cases}$.

Individually, let Student-teachers go through example and work out application activity 1.3.6 to emphasize their skills in determination of the square root of complex numbers .

Answers for Activity 1.3.6.

1. Yes, it is possible: $a=0; b=2; d=-3; f=0; g=-c$

2. $\begin{cases} a=\sqrt{29} \\ b=-\frac{2\sqrt{29}}{29} \end{cases}$ or $\begin{cases} a=-\sqrt{29} \\ b=\frac{2\sqrt{29}}{29} \end{cases}$

The square roots are $\sqrt{29}-\frac{2\sqrt{29}}{29}i$ and $-\sqrt{29}+\frac{2\sqrt{29}}{29}i$

Answers for Application Activity 1.3.6.

a) $z=4+3i$, let us take $|x+iy|^2=|4+3i|$

$$\begin{cases} x = \pm\sqrt{\frac{1}{2}\left(a+\sqrt{a^2+b^2}\right)} \\ y = \pm\sqrt{\frac{1}{2}\left(-a+\sqrt{a^2+b^2}\right)} \end{cases}$$

$$\begin{cases} x = \pm\sqrt{\frac{1}{2}\left(-3+\sqrt{9+16}\right)} \\ y = \pm\sqrt{\frac{1}{2}\left(3+\sqrt{9+16}\right)} \end{cases} \Leftrightarrow \begin{cases} x=\pm\sqrt{1}=\pm1 \\ y=\pm\sqrt{4}=\pm2 \end{cases}$$

Therefore, square roots of $-3+4i$ are $1+2i$ and $-1-2i$

b) $z=-2i$

$$\begin{cases} a=0 \\ b=-2<0 \end{cases}$$

$$\begin{cases} x=\pm\sqrt{\frac{2}{2}}=\pm 1 \\ y=\pm\sqrt{\frac{2}{2}}=\pm 1 \end{cases}$$

As b is less than zero, we take the different signs.

Square roots of $z=-2i$ are $1-i$ and $-1+i$

c) $z=2-2i\sqrt{3}$

Find x and y.

$$\begin{cases} x=\pm\sqrt{\frac{1}{2}\left(2+\sqrt{4+12}\right)}=\pm\sqrt{3} \\ y=\pm\frac{1}{5}\sqrt{\frac{1}{2}\left(-2+\sqrt{4+12}\right)}=\pm 1 \end{cases}$$

As b is less than zero, we take the different signs:

Square roots of $z=2-2i\sqrt{3}$ are $\sqrt{3}-i$ and $-\sqrt{3}+i$.

Lesson 10: Equations in the set of complex numbers

A. Simple equations of the form $Az+B=0$ where A and B are two given complex numbers, $A\neq 0$

a) Learning objectives:

Solve a simple linear equation in the set of complex numbers

b) Teaching resources:

Student-teacher's book and other Reference textbooks to facilitate research

c) Prerequisites/Revision/Introduction:

Give Student-teachers different exercises on solving equation of the form $ax+b=0$ in the set of real numbers.

d) Learning activities :

- From the activity 1.4.1, facilitate Student-teachers to work in small groups and find the solution set of the equation $4z+5i=12-i$

From the activity 1.4. 1 ,facilitate Student-teachers to have a general idea on how to find the solution set of the equation $Az+B=0$ (where A and B are two complex numbers, A different to zero) by helping them to remember that they apply the same process as solving equation of the form $ax+b=0$

in the set of real numbers. Let them find out that $Az+B=0 \Rightarrow z=-\frac{B}{A}$ and z must be written in the form of $z=x+yi$

- Individually, let Student-teachers go through example and work out application activity 1.4.1, to emphasize their skills in solving equations of the form $Az+B=0$.

Answer for Activity 1.4.1.

$$4z=12-i-5i \Leftrightarrow 4z=12-6i \Rightarrow z=3-\frac{3}{2}i$$

Answers for Application Activity 1.4.1.

1. $(1+3i)\mathrm{z}=2\mathrm{i}+4\mathrm{i};\quad \mathrm{z}=\frac{2\mathrm{i}+4\mathrm{i}}{1+3i}=\frac{9}{5}+\frac{3i}{5}$

2. $\begin{cases} 7z+(8-2i)w=4-9i \\ (1+i)+(2-i)w=2+7i \end{cases}$

Find w from the second equation, $w=\frac{2+7i-1-i}{2-i}=\frac{(1+6i)(2+i)}{(2-i)(2+i)}=\frac{-4}{5}+\frac{13}{5}i$

Replace w by its value in the first equation, $7z+(8-2i)(\frac{-4}{5}+\frac{13}{5}i)=4-9i$

Then $z=\frac{26}{35}-\frac{157}{35}i$.

B.**Quadratic equations** $Az^2+Bz+C=0$ **where A, B and C are three given complex numbers**

a) Learning objectives:

Solve a simple quadratic equations in the set of complex numbers

b) Teaching resources:

Student-teacher's book and other Reference textbooks to facilitate research

c) Prerequisites/Revision/Introduction:

Students will learn better in this lesson if they revise the content on quadratic equations learnt in year 1: unit 5 and refer to the lesson 9 of this unit.

d) Learning activities :

- From the activity 1.4.2, facilitate Student-teachers to work in small groups and find the solution set of the equation $z^2 = 1+i$, by considering the square roots of $1+i$
- From the activity 1.4.2 , facilitate Student-teachers to have a general idea on how to find the solution set of simple quadratic equations in the set of complex numbers. Let them remember that it recalls the procedure of how to solve the quadratic equations in the set of real numbers considering that $\sqrt{-1} = i$. There fore, $Az^2 + C = 0 \Rightarrow z^2 = \frac{-c}{A} \Leftrightarrow z = \sqrt{\frac{-c}{A}}$
- Through group activiy, help Student-teachers to solve the equation of the form $Az^2 + Bz + C = 0$, $(A \neq 0)$ and lead them to the following: $z = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}$. By considering the case where A, B, and C are **real numbers,** and A different to zero, facilitate sudents to find out that the equation $Az^2 + Bz + C = 0$ has either two real roots, one double real root or two conjugate complex roots depending on the value of $\Delta = B^2 - 4AC$

1. If $\Delta = 0$, there is a double real root $z_1 = z_2 = \frac{-B}{2A}$
2. If $\Delta > 0$, there are two distinct real roots $z_1 = \frac{-B + \sqrt{\Delta}}{2A}$ and $z_2 = \frac{-B - \sqrt{\Delta}}{2A}$
3. If $\Delta < 0$, there is no real root. In this case, there are two conjugate complex roots $z_1 = \frac{-B + i\sqrt{-\Delta}}{2A}$ and $z_2 = \frac{-B - i\sqrt{-\Delta}}{2A}$

- Individually, let Student-teachers go through example and work out application activity 1.4.2, to emphasize their skills in solving equations of the form $Az^2 + Bz + C = 0$

Answer for Activity 1.4.2.

Let $x+yi$ be a square root of $1+i$.Then $\begin{cases} x^2+y^2=\sqrt{2} \\ x^2-y^2=1 \\ 2xy=1 \end{cases}$

Solving simultaneously, we obtain: $\begin{cases} x=\sqrt{\dfrac{1+\sqrt{2}}{2}} \\ y=\dfrac{1}{2\sqrt{\dfrac{1+\sqrt{2}}{2}}} \end{cases}$ or $\begin{cases} x=-\sqrt{\dfrac{1+\sqrt{2}}{2}} \\ y=-\dfrac{1}{2\sqrt{\dfrac{1+\sqrt{2}}{2}}} \end{cases}$

The square roots are $\sqrt{\dfrac{1+\sqrt{2}}{2}}+\dfrac{1}{2\sqrt{\dfrac{1+\sqrt{2}}{2}}}i$ and $-\sqrt{\dfrac{1+\sqrt{2}}{2}}-\dfrac{1}{2\sqrt{\dfrac{1+\sqrt{2}}{2}}}i$

Answers for Application Activity 1.4.2.

1. a) $z^2-(3+i)z+4+3i=0$

$$z=\frac{(3+i)\pm\sqrt{(3+i)^2-4.(4+3i)}}{2}=\frac{(3+i)\pm\sqrt{9+6i-1-16-12i}}{2}=\frac{(3+i)\pm\sqrt{-8-6i}}{2}$$

Let us first find $\sqrt{-8-6i}$:

$$x=\pm\sqrt{\frac{a+\sqrt{a^2+b^2}}{2}}=\pm\sqrt{\frac{-8+\sqrt{84+36}}{2}}=\pm1$$

$$y=\pm\sqrt{\frac{-a+\sqrt{a^2+b^2}}{2}}=\pm\sqrt{\frac{8+\sqrt{84+36}}{2}}=\pm3$$

$\sqrt{-8-6i}=1-3i$ or $-1+3i$.

Then $z=\dfrac{(3+i)\pm(1-3i)}{2}=2-i$ or $1+2i$.

b) $z^2+9=0$ $z=\pm3i$

2. $(z-4)$ is a factor of $z^3-15z-4$ if $z=4$ is a zero of $p(z)=z^3-15z-4$.

Or $p(4)=4^3-15(4)-4=64-60-4=0$;

Hence, $(z-4)$ is a factor of $z^3-15z-4$.

Since $(z-4)$ is a factor of $z^3-15z-4$, $z=4$ is one of the solutions of $z^3-15z-4=0$; other solutions could be found by factorization.

To get other factors, let us use synthetic division,

	1	0	–15	–4
4		4	16	4
	1	4	1	0

Thus, $z^3-15z-4=(z-4)(z^2+4z+1)$

Solve $z^2+4z+1=0$ by discriminant method:

$\Delta=16-4=12$

Then, $z_1=\dfrac{-4+2\sqrt{3}}{2}=-2+\sqrt{3}$ and $z_2=\dfrac{-4-2\sqrt{3}}{2}=-2-\sqrt{3}$

Therefore, the solutions of $z^3-15z-4=0$ are $4;-2+\sqrt{3}$ and $-2-\sqrt{3}$.

Lesson 11: Definition and properties of a complex number z in polar form

a) Learning objectives:

Define a complex number in a polar form and convert a complex number from algebraic form to polar form and vice versa.

b) Teaching resources:

Student-teacher's book and other Reference textbooks to facilitate research, Mathematical set, calculator, Manila paper, markers.

c) Prerequisites/Revision/Introduction:

From the activity 1.5.1, help Student-teachers to remember how to write a vector $\overrightarrow{M}$ using trigonometric relations. Using figure 1.10, let Student-teachers discuss and find that $a=r\cos\theta$ and $a=r\sin\theta$. Finally, facilitate Student-teachers to deduce that $M(r,\theta)=\text{r}(\cos\theta+\sin\theta)$.

d) Learning activities :

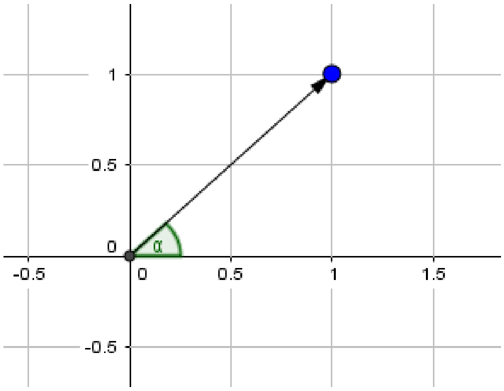
- From the activity 1.5.1 and the figure, let Student-teachers discuss in small groups to find out $\vec{r} = a\vec{e_1} + b\vec{e_2} = r\cos\theta\vec{e_1} + r\sin\theta\vec{e_2}$ and simillarly, facilitate them to discover that when given the complex number z with the affix $P(x, y)$ and the modulus $|z| = \sqrt{x^2 + y^2} = r$, the complex number z will be written as follow: $z = x + iy = r\cos\theta + ir\sin\theta = r(\cos\theta + i\sin\theta)$

- Help Student-teachers to understand that $z = r(\cos\theta + i\sin\theta)$is called the **polar form**. Let them find out that its affix is represented by the coordinates (r,θ) where $r = \sqrt{x^2 + y^2}$ is its modulus and θ the angle between the corresponding vector and $x - axis$.
- Explain to Student-teachers that the angle θ is called the **argument** of z and then given $z = x + iy$, the formula used to convert to the polar form are the following:

$$r = \sqrt{x^2 + y^2},\ x = r\cos\theta,\ y = r\sin\theta \text{ and } \theta = \arg(z) = \arctan\frac{y}{x}$$

- In small groups , let Student-teachers go through example1.14 and work out application activity 1.5.1 to emphasize their skills in converting a complex number from algebraic form to polar form and vice versa .

Answer for Activity 1.5.1.



Answers for Application Activity 1.5.1

The table of trigonometric values can help the Student-teacher to find the argument of z.

Let θ be argument of complex number z

1. a) $z = -2i$

$$|z|=2,\begin{cases}\cos\theta=0\\ \sin\theta=\dfrac{-2}{2}=-1\end{cases}$$, thus θ lies on negative $y-axis$ i.e. $\theta=-\dfrac{\pi}{2}[\pi]$.

The principal argument is $-\dfrac{\pi}{2}$.

b) $z=1-i$

$$|z|=\sqrt{2},\begin{cases}\cos\theta=\dfrac{1}{\sqrt{2}}\\ \sin\theta=\dfrac{-1}{\sqrt{2}}\end{cases}$$, thus θ lies in 4th quadrant and $\theta=-\dfrac{\pi}{4}[2\pi]$.

The principal argument is $-\dfrac{\pi}{4}$.

a) $z=1-i\sqrt{3}$

$$|z|=2,\begin{cases}\cos\theta=\dfrac{1}{2}\\ \sin\theta=-\dfrac{\sqrt{3}}{2}\end{cases}$$, thus θ lies in 4th quadrant and $\theta=-\dfrac{\pi}{3}[2\pi]$.

The principal argument is $-\dfrac{\pi}{3}$.

b) $z=-1+i\sqrt{3}$

$$|z|=2,\begin{cases}\cos\theta=-\dfrac{1}{2}\\ \sin\theta=\dfrac{\sqrt{3}}{2}\end{cases}$$, thus θ lies in 2nd quadrant and $\theta=\dfrac{2\pi}{3}[2\pi]$.

The principal argument is $\dfrac{2\pi}{3}$.

c) $z=-\sqrt{3}-i$

$$|z|=2,\begin{cases}\cos\theta=-\dfrac{\sqrt{3}}{2}\\ \sin\theta=-\dfrac{1}{2}\end{cases}$$, thus θ lies in 3rd quadrant and $\theta=-\dfrac{5\pi}{6}[2\pi]$.

The principal argument is $-\dfrac{5\pi}{6}$.

1. a) $z=-1+i$

$|z| = \sqrt{2}, \begin{cases} \cos\theta = -\frac{1}{\sqrt{2}} \\ \sin\theta = \frac{1}{\sqrt{2}} \end{cases}$, thus θ lies on 2nd quadrant and $\theta = \frac{3\pi}{4}[2\pi]$.

Therefore, $z = \sqrt{2}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)$.

b) $z = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$

$|z| = 1, \begin{cases} \cos\theta = -\frac{1}{2} \\ \sin\theta = \frac{\sqrt{3}}{2} \end{cases}$, thus θ lies in 2nd quadrant and $\theta = \frac{2\pi}{3}[2\pi]$.

Therefore, $z = \sqrt{2}\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)$.

a) $z = -2\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) = -1 - i\sqrt{3}$

$|z| = 2, \begin{cases} \cos\theta = -\frac{1}{2} \\ \sin\theta = \frac{\sqrt{3}}{2} \end{cases}$, thus θ lies in 3rd quadrant and $\theta = -\frac{2\pi}{3}[2\pi]$.

Therefore, $z = 2\left[\cos\left(-\frac{2\pi}{3}\right) + i\sin\left(-\frac{2\pi}{3}\right)\right]$.

b) $z = 2$

$|z| = 2, \begin{cases} \cos\theta = \frac{2}{2} = 1 \\ \sin\theta = 0 \end{cases}$, thus θ lies on positive $x - axis$ and $\theta = 0[2\pi]$.

Therefore, $z = 2(\cos 0 + \text{i}\sin 0)$.

c) $z = -i$

$|z|=1, \begin{cases}\cos\theta = 0\\ \sin\theta = -1\end{cases}$, thus θ lies on negative $y-axis$ and $\theta = -\frac{\pi}{2}[2\pi]$

Therefore, $z = 2\left[\cos\left(-\frac{\pi}{2}\right) + i\sin\left(-\frac{\pi}{2}\right)\right]$.

1. a) $z = 5cis270^0 = 5\left(\cos 270^0 + i\sin 270^0\right) = 0 + 5i$

   b) $z = 4cis300^0 = 4\left(\cos 300^0 + i\sin 300^0\right) = 4\left[\cos\left(-60^0\right) + i\sin\left(-60^0\right)\right]$

   $= 4\left(\frac{1}{2} - i\frac{\sqrt{3}}{2}\right) = 2 - 2i\sqrt{3}$

   c) $z = \sqrt{2}cis\left(-\frac{\pi}{4}\right) = \sqrt{2}\left[\cos\left(-\frac{\pi}{4}\right) + i\sin\left(-\frac{\pi}{4}\right)\right]$

   $= \sqrt{2}\left(\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}\right) = 1 - i$

   d) $z = 3cis\left(\frac{\pi}{2}\right) = 3\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right) = 0 + 3i$

Lesson12: Multiplication and division of complex numbers in polar form

a) Learning objectives:

Apply the properties of complex numbers to perform operations on complex numbers in polar form.

b) Teaching resources:

Student-teacher's book, Internet and reference textbooks for research

c) Prerequisites/Revision/Introduction:

- Give an exercise involving trigonometric identities:

  $\cos\phi\cos\theta \pm \sin\phi\sin\theta = \cos(\phi \mp \theta); \cos\phi\sin\theta \pm \cos\phi\sin\theta = \sin(\phi \pm \theta)$

- Given two complex numbers $z_1 = \sqrt{3} - i$ and $z_2 = 1 + i$, let Student-teachers work in groups or individually to find out the modulus and argument of each number.

d) Learning activities :

- Write on the board the two complex numbers $z_1 = \sqrt{3} - i$ and $z_2 = 1 + i$. In small groups, ask Student-teachers to express these numbers in polar form, then compute the product $z_1.z_2$ and the quotient $\frac{z_1}{z_2}$ Ensure that Student-teachers use above trigonometric identities to come up with the following final results: $z_1 = 2(\cos\frac{\pi}{6} - i\sin\frac{\pi}{6})$ and $z_2 = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$

  Therefore $z_1.z_2 = z_1.z_2 = r_1.r_2\left[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)\right]$

  and $\frac{z_1}{z_2} = \frac{r_1}{r_2}\left[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)\right]$.

Ask Student-teachers to express these results in their own words, e.g., the modulus of the product of two complex numbers is the product of the moduli and the argument is the sum of arguments.

- Let Student-teachers go through the example. Individually, ask them to work out application activity 1.5.2.1 to develop their skills and increase their self confidence in calculating the product and quotient of complex numbers in polar form.

Answers for Activity 1.5.2.1.

a) $z_1 = 2\left[\cos(-\frac{\pi}{6}) + i\sin(-\frac{\pi}{6})\right]; z_2 = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$

b) $z_1z_2 = 2\sqrt{2}\left[\cos(-\frac{\pi}{6} + \frac{\pi}{4}) + i\sin(-\frac{\pi}{6} + \frac{\pi}{4})\right] = 2\sqrt{2}\left(\cos\frac{\pi}{12} + i\sin\frac{\pi}{12}\right)$

c) $|z_1z_2| = 2\sqrt{2}; Arg(z_1z_2) = \frac{\pi}{12}$

d) $\arg(z_1z_2) = Arg(z_1) + Arg(z_2)$:the argument of a product equals the sum of the arguments of the factors

Answers for Application Activity 1.5.2.1.

1. Conversion in polar form

- $z = 1+i$

$|z| = \sqrt{2}, \begin{cases} \cos\theta = \frac{1}{\sqrt{2}} \\ \sin\theta = \frac{1}{\sqrt{2}} \end{cases}$, thus θ lies in 1st quadrant and $\theta = \frac{\pi}{4}[2\pi]$.

Therefore, $z = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$

- $w = -\sqrt{3} + i$

$|w| = 2, \begin{cases} \cos\theta = -\frac{\sqrt{3}}{2} \\ \sin\theta = \frac{1}{2} \end{cases}$, thus θ lies in the 2nd quadrant and $\theta = \frac{5\pi}{6}[2\pi]$.

Therefore, $z = 2\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right)$.

- $y = -3 + i\sqrt{3}$

$|y| = 2\sqrt{3}, \begin{cases} \cos\theta = -\frac{\sqrt{3}}{2} \\ \sin\theta = \frac{1}{2} \end{cases}$, thus θ lies in 2nd quadrant and $\theta = \frac{5\pi}{6}[2\pi]$.

Therefore, $y = 2\sqrt{3}\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right)$.

2. Computation

- $z \cdot w = 2\sqrt{2}\left(\cos\frac{13\pi}{12} + i\sin\frac{13\pi}{12}\right) = 2\sqrt{2}\left[\cos\left(-\frac{11\pi}{12}\right) + i\sin\left(-\frac{11\pi}{12}\right)\right]$

- $z \cdot y = 2\sqrt{6}\left(\cos\frac{13\pi}{12} + i\sin\frac{13\pi}{12}\right) = 2\sqrt{6}\left[\cos\left(-\frac{11\pi}{12}\right) + i\sin\left(-\frac{11\pi}{12}\right)\right]$

- $\frac{w}{y}=\frac{2}{2\sqrt{3}}\left(\cos 0+i\sin 0\right)=\frac{\sqrt{3}}{3}$.

- $\frac{y}{z}=\frac{2\sqrt{3}}{\sqrt{2}}\left(\cos\frac{7\pi}{12}+i\sin\frac{7\pi}{12}\right)=\sqrt{6}\left(\cos\frac{7\pi}{12}+i\sin\frac{7\pi}{12}\right)$.

Lesson 13: Powers and De Moivre's formula

a) Learning objectives:

Apply De Moivre's theorem to calculate power of complex number

b) Teaching resources:

- Student-teacher's textbook and any other reference textbooks for enabling research

c) Prerequisites/Revision/Introduction:

- From activity 1.5.2.2, let Student-teachers work individually and convert $z=1-i\sqrt{3}$ in polar form.

d) Learning activities :

- After checking whether all Student-teachers have converted $z=1-i\sqrt{3}$ in polar form accurately , let Student-teachers work in groups and calculate z^2 .
- In the same groups, ask Student-teachers to determine $z^3=z^2.z$ in polar form
- From different products, let Student-teachers deduce **De Moivre's theorem:** $\left(\cos\theta+i\sin\theta\right)^n=\cos\left(\mathrm{n}\,\theta\right)+i\sin\left(\mathrm{n}\,\theta\right)$.
- Let Student-teachers go through the example and work out application activity 1.5.2.2 to develop independently their skills in calculating the power of complex numbers in polar form

Answers for Activity 1.5.2.2.

1. $z^2=r^2(\cos 2\theta+i\sin 2\theta)$
2. $z^3=r^3(\cos 3\theta+i\sin 3\theta)$
3. $z^n=r^n(\cos n\theta+i\sin n\theta)$
4. Answers will vary from one student to another depending on the English words. Ensure that they are correct

Answers for Application Activity 1.5.2.2.

1. a) $(\cos 3\pi + i\sin 3\pi)^9 = \cos 27\pi + i\sin 27\pi = \cos\pi + i\sin\pi = -1$

b) $\left(\cos\frac{5\pi}{2} + i\sin\frac{5\pi}{2}\right)^{\frac{2}{5}} = \cos\pi + i\sin\pi = -1$

c) $\left(\cos\frac{\pi}{5} + i\sin\frac{\pi}{5}\right)^5 = \cos\pi + i\sin\pi = -1$

d) $\left(\cos 45^0 + i\sin 45^0\right)^2 = \cos 90^0 + i\sin 90^0 = i$

e) $\left(\cos 60^0 + i\sin 60^0\right)^3 = \cos 180^0 + i\sin 180^0 = -1$

2. $a)\left(1+i\sqrt{3}\right)^6 = 2^6\left(\cos\frac{6\pi}{3} + i\sin\frac{6\pi}{3}\right) = 2^6\left(\cos 2\pi + i\sin 2\pi\right) = 2^6$

$b)\left(-\sqrt{3}+i\right)^{10} = 2^{10}\left(\cos\frac{50\pi}{6} + i\sin\frac{50\pi}{6}\right) = 2^{10}\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right) = 2^9\left(1+i\sqrt{3}\right)$

$c)\left(1-i\right)^7 = \sqrt{2^7}\left[\cos\left(-\frac{7\pi}{4}\right) + i\sin\left(-\frac{7\pi}{4}\right)\right] = 2^3\sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) = 8\left(1+i\right)$

Lesson 14. The n^{th} root of a complex number

a) Learning objectives:

Determine the n^{th} roots of a complex number and represent them on the Argand diagram

b) Teaching Aids

Student-teacher's book and other Reference textbooks to facilitate research, Mathematical set, calculator, Manila paper, markers.

c) Prerequisites

In this lesson, Student-teachers will perform better if they have good back ground on polar form of a complex number learnt in the content and refer to the lessons 11 and 12 of this unit.

d) Learning activities :

- Invite student-teachers to work in small groups the activity 1.5.2.3 found in their Mathematics books;

- Give them time to explore and workout the activity and move around in the class for facilitating students where necessary and give more clarification on eventual challenges they may face during their work;
- Invite randomly one member from any group to present their work.
- As a tutor, harmonize the findings from presentation and guide them to mention main points on how to determine n[th] roots of a complex number.
- Through different examples, guide students to do the application activity 1.5.2.3 and evaluate whether lesson objectives are achieved or not for eventual improvement for the following lessons.

Answers for Activity 1.5.2.3.

1. $|z| = 4, \ \arg(z) = \arctan(0) = ($

   $\Rightarrow z = 4\,\text{cis}\,0$

2. $(z_k)^4 = z$

   But $(z_k)^4 = (r')^4\,\text{cis}\,4\theta'$

   Then $(r')^4\,\text{cis}\,4\theta' = 4\,\text{cis}\,0$

$$\begin{cases}(r')^4 = 4\\ 4\theta' = 2k\pi\end{cases} \Rightarrow \begin{cases} r' = \sqrt[4]{4} \Rightarrow r' = \sqrt{2}\\ \theta' = \dfrac{k\pi}{2}\end{cases}$$

Now, $z_k = \sqrt{2}\,\text{cis}\left(\dfrac{k\pi}{2}\right)$

If $k = 0, \ z_0 = \sqrt{2}\,\text{cis}\,0$

If $k = 1, \ z_1 = \sqrt{2}\,\text{cis}\left(\dfrac{\pi}{2}\right)$

If $k = 2, \ z_2 = \sqrt{2}\,\text{cis}\,\pi$

If $k = 3, \ z_3 = \sqrt{2}\,\text{cis}\left(\dfrac{3\pi}{2}\right) = \sqrt{2}\,\text{cis}\left(-\dfrac{\pi}{2}\right)$

Synthesis

To find n^{th} roots of a complex number z, you start by expressing z in polar form $z = rcis\theta$, where r is modulus of z and θ argument of z.

Then, n^{th} roots of a complex number z is given by

$$z_k = \sqrt[n]{r}\, cis\left(\frac{\theta + 2k\pi}{n}\right) \qquad k = 0,1,2,3,\ldots\ldots, n-1.$$

If the complex number for which we are computing the n^{th} roots is $z = rcis\theta$, the radius of the circle will be $R = \sqrt[n]{r}$ and the first root z_0 correspond to $k = 0$ will be at an amplitude of $\varphi = \frac{\theta}{n}$. This root will be followed by the $n-1$ remaining roots at equal distances apart.

Answers for Application Activity 1.5.2.3.

1. $= \text{cis}\left(\frac{\pi}{8}\right),\ z_1 = \text{cis}\left(\frac{5\pi}{8}\right),\ z_2 = \text{cis}\left(-\frac{7\pi}{8}\right),\ z_3 = \text{cis}\left(-\frac{3\pi}{8}\right)$

2. $= 1,\ z_1 = \text{cis}\left(\frac{2\pi}{5}\right),\ z_2 = \text{cis}\left(\frac{4\pi}{5}\right),\ z_3 = \text{cis}\left(-\frac{4\pi}{5}\right),\ z_4 = \text{cis}\left(-\frac{2\pi}{5}\right.$

3. $= 2,\ z_1 = 2\,\text{cis}\left(\frac{2\pi}{5}\right),\ z_2 = 2\,\text{cis}\left(\frac{4\pi}{5}\right),\ z_3 = 2\,\text{cis}\left(-\frac{4\pi}{5}\right),\ z_4 = 2\,\text{cis}\left(-\frac{2\pi}{5}\right.$

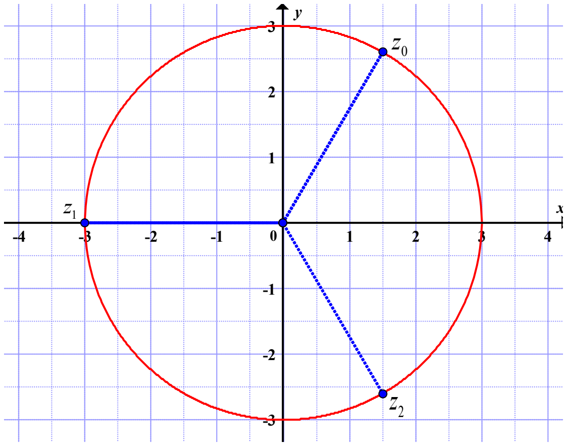
4. $= 2\,\text{cis}\left(\frac{\pi}{12}\right),\ z_1 = 2\,\text{cis}\left(\frac{7\pi}{12}\right),\ z_2 = 2\,\text{cis}\left(-\frac{11\pi}{12}\right),\ z_3 = 2\,\text{cis}\left(-\frac{5\pi}{12}\right.$

5. $\sin\frac{2\pi}{5} = \frac{\sqrt{10+2\sqrt{5}}}{4}.$

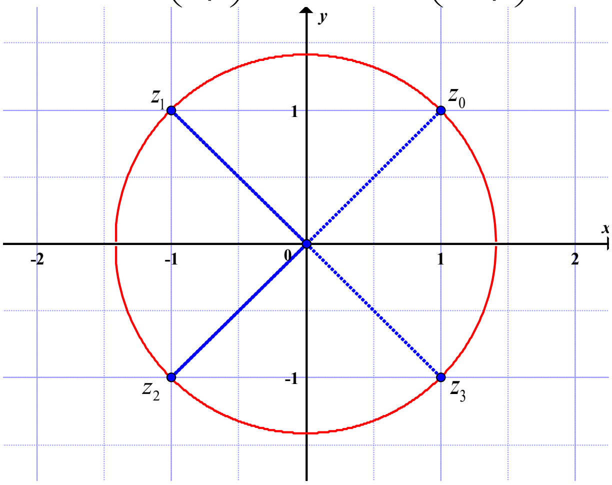
Hint: First, find $\cos\frac{2\pi}{5}$ and then use the relation $\sin\frac{2\pi}{5} = \sqrt{1-\cos^2\frac{2\pi}{5}}$

1.

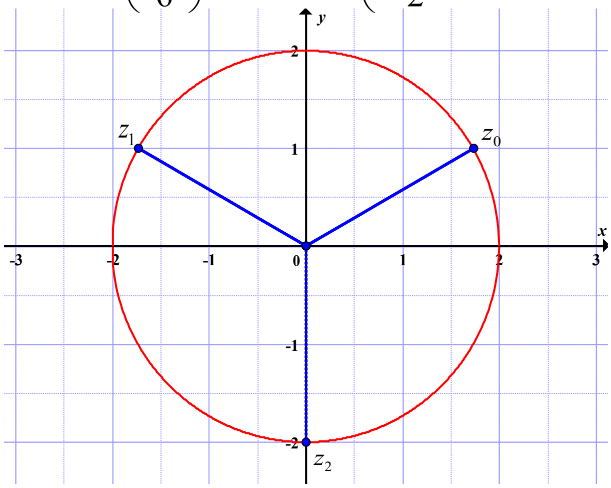
a) $z_0 = 3\,\text{cis}\left(\frac{\pi}{3}\right),\ z_1 = 3\,\text{cis}(\pi),\ z_2 = 3\,\text{cis}\left(-\frac{\pi}{3}\right)$



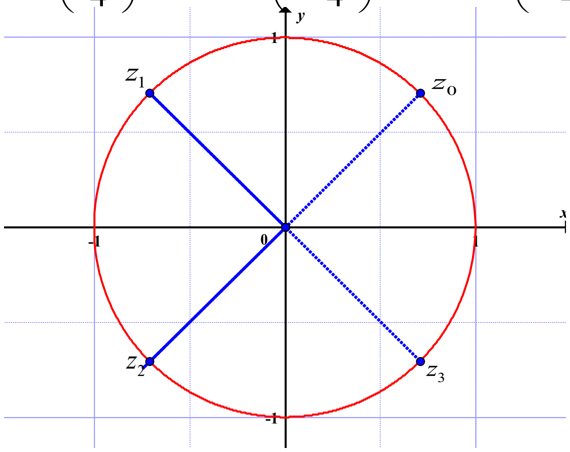
b) $= \sqrt{2}\,\text{cis}\left(\frac{\pi}{4}\right)$, $z_1 = \sqrt{2}\,\text{cis}\left(\frac{3\pi}{4}\right)$, $z_2 = \sqrt{2}\,\text{cis}\left(-\frac{3\pi}{4}\right)$, $z_3 = \sqrt{2}\,\text{cis}\Big(-$



c) , $= 2\,\text{cis}\left(\frac{\pi}{6}\right)$, $z_1 = 2\,\text{cis}\left(\frac{5\pi}{6}\right)$, $z_2 = 2\,\text{cis}\Big(-\frac{\pi}{2}$



d) $, = \text{cis}\left(\frac{\pi}{4}\right)$, $z_1 = \text{cis}\left(\frac{3\pi}{4}\right)$, $z_2 = \text{cis}\left(-\frac{3\pi}{4}\right)$, $z_3 = \text{cis}\left(-\frac{\pi}{4}\right.$



Lesson 15: Construction of regular polygon

a) Learning objectives:

Construct on the Argand diagram the points representing the n[th] roots of a complex number and deduce the names of the polygons obtained by joining the points representing the n[th] roots of unity.

b) Teaching Aids

Student-teacher's book and other Reference textbooks to facilitate research, Mathematical set, calculator, Manila paper, markers.

c) Prerequisites

In this lesson, Student-teachers will perform better if they refer to the lesson 13 of this unit.

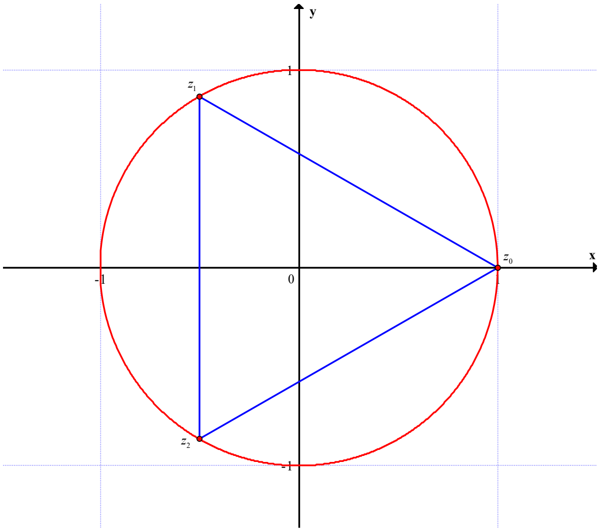
d) Learning activities :

- Invite student-teachers to work in pairs and do the activity 1.5.3 found in their Mathematics books;
- Move around in the class for facilitating students where necessary and give more clarification on eventual challenges they may face during their work;
- Request two pairs to exchange their works and to discuss on their findings;
- Verify and identify groups with different working steps;
- Invite one member to present the work of his/her group.
- As a tutor, harmonize the findings from presentation and guide them to explain how to construct a regular polygon
- Through different examples, lead them to perform well application activity 1.5.3.

Answers for Activity 1.5.3

$z_k = \text{cis}\frac{2k\pi}{3},\ k = 0,1,2$

$z_0 = \text{cis}\,0 = 1,\ z_1 = \text{cis}\frac{2\pi}{3},\ z_2 = \text{cis}\frac{4\pi}{3}$



The obtained figure is an equilateral triangle

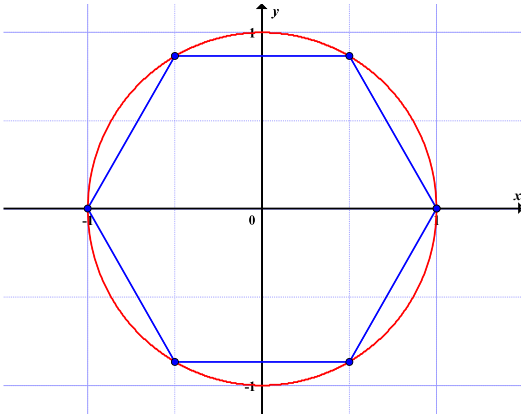
Synthesis

To draw a regular polygon with *n* sides follows the following steps:

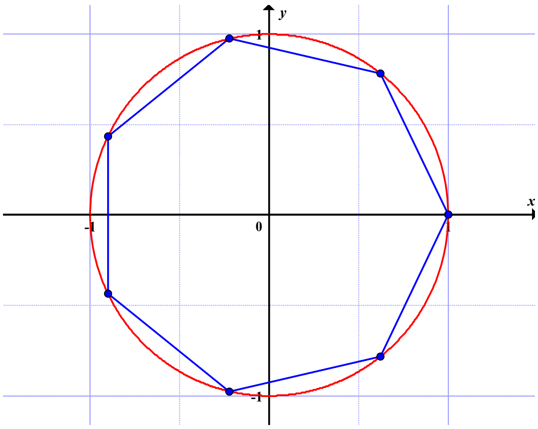
- Start by drawing a unit circle in Argand diagram. The radius and the centre of this circle will be the radius and centre of the regular polygon.
- Around the circle, place the points with affixes $z_k = \text{cis}\frac{2k\pi}{n},\ k = 0,1,2,.....,n-1$. Those points are the vertices of the polygon.
- Using a ruler join the obtained points around the circle.
- The obtained figure is the needed regular polygon.

Answers for Application Activity 1.5.3

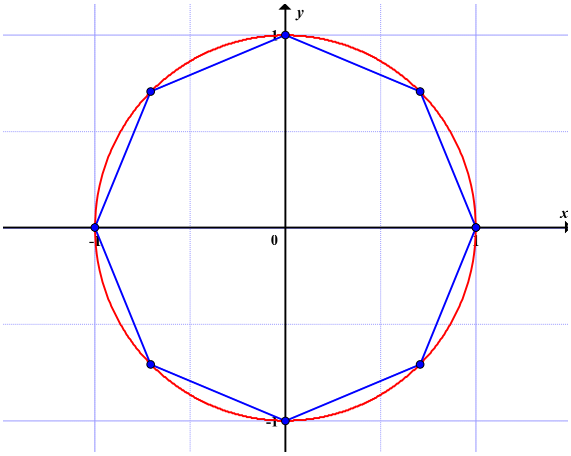
1. A regular hexagon (6 sides)



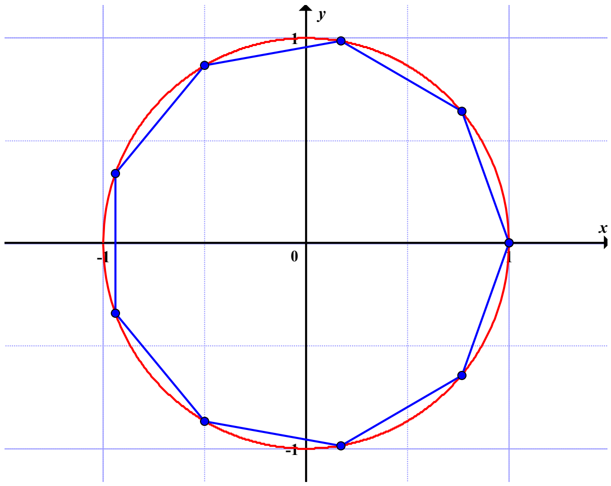
2. A regular heptagon (7 sides)



3. A regular octagon (8 sides)



4. A regular nonagon (9 sides)



Lesson 16: Definition and properties of a complex number z in exponential form

a) Learning objectives:

Define a complex number in exponential form and convert a complex number from algebraic or polar form to exponential form and vice versa

b) Prerequisites/Revision/Introduction:

Using examples and activities, help Student-teachers to review exponent concept learnt in year 1: unit 6.

c) Teaching resources:

Student-teacher's book, internet and other appropriate textbooks for research

d) Learning activities :

- Guide Student-teachers to make a research in the library or on the internet wherever possible to find out the relationship between complex number in trigonometric form with exponential function, that is, expressing $\cos\theta + i\sin\theta$ into $e^{i\theta}$. Considering that $z = \cos\theta + i\sin\theta$ is the polar form of a complex number, let Student-teachers find out the form of $z = e^{i\theta}$. Inform students that this notation is due to the mathematician Euler.
- From the findings of activity 1.6.1, harmonize the group findings by explaining that any complex number z of modulus $|z|$ and argument θ can be written as $z = |z|(\cos\theta + i\sin\theta) = |z|e^{i\theta}$.
- Explain to Student-teachers that $z = r.e^{i\theta}$ is called an **exponential**

form of the complex number z, where r and θ are the modulus and the argument of z respectively and then let them know that all power properties applicable for other forms of complex numbers are also applicable for exponential form.

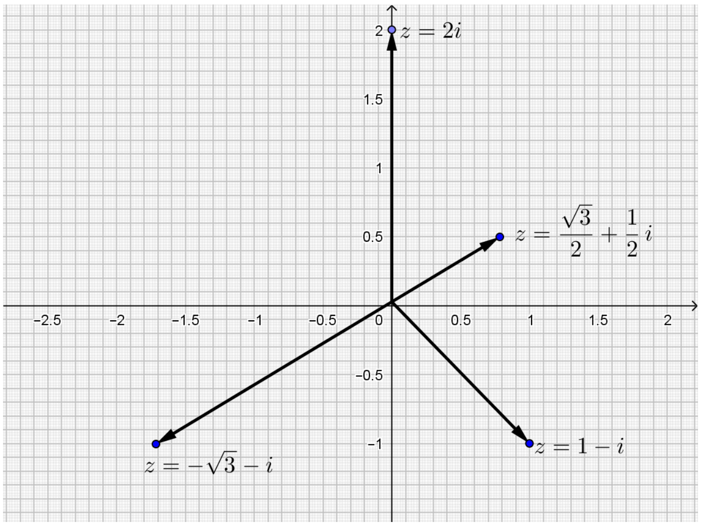
- Let Student-teachers go through the example and work out application activity 1.6.1 , to emphasize their skills in converting a complex number in exponential form and vice versa.

Answers for Activity 1.6.1

The findings will differ from one student to another, all yielding to the same result

Answers for Application Activity 1.6.1

1.



a) $1-\mathrm{i} = \sqrt{2}e^{-\frac{\pi}{4}i}$ b) $2\mathrm{i} = 2e^{\frac{\pi}{2}i}$ c) $\frac{\sqrt{3}}{2} + \frac{1}{2}i = e^{\frac{\pi}{6}i}$ d) $-\sqrt{3} - i = 2e^{-\frac{5\pi}{6}i}$

2. a) $z = e^{i\frac{\pi}{3}} = \cos\frac{\pi}{3} + i\sin\frac{\pi}{3} = \frac{1}{2} + i\frac{\sqrt{3}}{2}$

b) $z = e^{-\frac{\pi}{4}i} = \cos\left(-\frac{\pi}{4}\right) + i\sin\left(-\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}$

c) $z = 3e^{i\frac{\pi}{6}} = 3\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2} + \frac{1}{2}i$

d) $z = 2e^{i\frac{2\pi}{3}} = 3\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right) = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$

3. a) $z = -1 + i\sqrt{3} = 2.e^{i\frac{2\pi}{3}}$

b) $z = 3 + 4i = 5.e^{i\arctan\frac{4}{3}}$

c) $z = 2 - 2i = 2\sqrt{2}.e^{-i\frac{\pi}{4}}$

d) $z = -3 + i\sqrt{3} = 2\sqrt{3}.e^{i\frac{5\pi}{6}}$

Lesson 17: Euler's formulae

a) Learning objectives:

Apply Euler's formula to transform trigonometric expressions

b) Prerequisites/Revision/Introduction:

- Facilitate Student-teachers to remember De Moivre's formula or theorem and convert the polar form of a complex number into exponential form.

c) Teaching resources:

Student-teacher's textbook and any reference textbooks to enable research

d) Learning activities :

- Organise Student-teachers in groups to work out the activity 1.6.2. Basing on the group findings and using question-answer technique, facilitate Student-teachers to realize that $\cos\theta = \frac{1}{2}\left(e^{i\theta} + e^{-i\theta}\right)$ and $sin\theta = \frac{1}{2i}\left(e^{i\theta} - e^{-i\theta}\right)$.

  Inform Student-teachers that these identities are called **Euler's formulae**.
- In groups, let Student-teachers go through the example. Individually Student-teachers work out application activity 1.6.2 to reinforce their learning and develop their mathematical skills. Ensure all Student-teachers are capable to achieve high level of performance through providing adequate feedback to individual Student-teacher.

Answers for Activity 1.6.2

$e^{i\theta} = \cos\theta + i\sin\theta;$

$e^{-i\theta} = \cos\theta - i\sin\theta$

Solving simultaneously for $\cos\theta$ and $\sin\theta$, we obtain:

$\cos\theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta})$ and $\sin\theta = \frac{1}{2i}(e^{i\theta} - e^{-i\theta})$

Answers for Application Activity 1.6.2

a) $\sin^2 x\cos x = \left(\frac{e^{ix} - e^{-ix}}{2i}\right)^2\left(\frac{e^{ix} + e^{-ix}}{2}\right) = -\frac{1}{4}\cos 3x + \frac{1}{4}\cos x$

$b)$ $\cos^2 x\cos y = \frac{\cos(2x+y)}{4} + \frac{\cos y}{2} + \frac{\cos(2x-y)}{4}$

$c)$ $\cos^3 x = \left(\frac{e^{ix} + e^{-ix}}{2}\right)^3 = \frac{\cos 3x}{4} + \frac{3\cos x}{4}$

Lesson 18: Applications of complex numbers in physics.

a) Learning objectives:

Appreciate the importance of complex numbers to solve related problems such as in Physics (problem related to voltage and alternating current),…

b) Prerequisites/Revision/Introduction:

Student-teachers will perform well if he/she has a good back ground on how to convert the algebraic form of a complex number into the polar form and then into the exponential form learnt in the previous lessons (especially lesson 11 and 17) of this unit.

c) Teaching resources:

Student-teacher's book, Internet or any reference textbooks for enabling research

d) Learning activities

- In groups, depending on the school facilities, let Student-teachers explore internet or textbooks to look for different applications of complex numbers in other subjects especially in Physics. This intends to support Student-teachers in developing their skills and increasing their curiosity about complex numbers and their applications in daily situations.

Answers for Activity 1.7.1

Answers may vary from one student to another. They may mention Alternating current, fractals, etc.

Answers for Application Activity 1.7.1

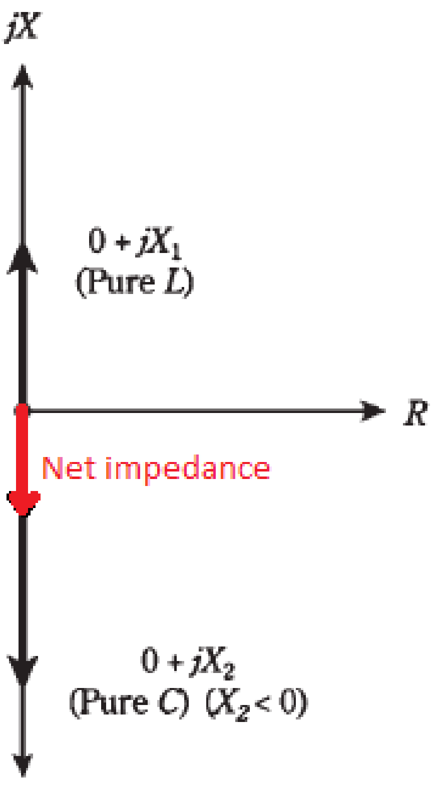
$\text{Im}(Z_R) = 0$;

$\text{Im}(Z_L) = \omega L$;

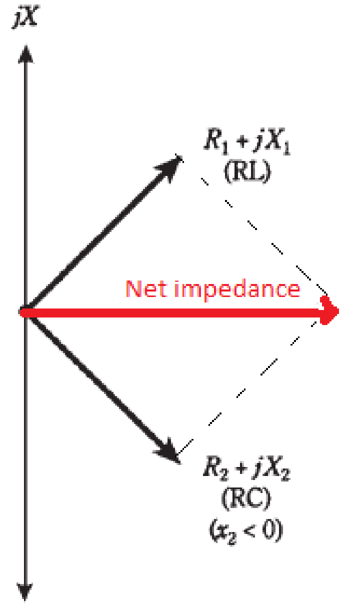
$\text{Im}(Z_C) = \frac{-1}{\omega C}$

Answers for Activity 1.7.2

1.



2.



Answers for Application Activity 1.7.2

1. Since $X = X_L + X_C$, the net reactance vector is $jX_L + jX_C = 200j - 150j = 50j$
2. Consider the resistor to be part of the coil (inductor), obtaining two complex vectors, $560 + 400j$ and $0 - 410j$. Adding these gives the resistance component of $(560+0)\Omega = 560\Omega$, and the reactive component of $(400j - 410j)\Omega = -10j\Omega$. Therefore, the net impedance is $Z = (560 - 10j)\Omega$.

Answers for Activity 1.7.3

$V = V_0(\cos\omega t + j\sin\omega t)$: **polar form**

$V = V_0 e^{j\omega t}$: **exponential form**

Answers for Application Activity 1.7.3

1. Remember that, in the Resistance and Inductance $(R-L)$ series circuit, impedance

$Z = R + jX_L$ where R is the resistance and X_L is inductive reactance (equivalent to $2\pi fL$ ohms).

For the Resistance and Capacitance $(R-C)$ series circuit impedance
$Z = R - jX_C$ where R is the resistance and X_C is the capacitive reactance (equivalent to $\frac{1}{2\pi fC}$ ohms). Hence,

a) From $Z=(3+j8)\Omega$, we find that $R=3\Omega$ and inductance

$$L=\frac{X_L}{2\pi f}=\frac{8}{2\pi\times 50}=25.5mH\ .$$

b) From $Z=(2-j3)\Omega$, we find that $R=2\Omega$ and capacitance

$$C=\frac{1}{2\pi fX_C}=\frac{1}{2\pi\times 50\times 3}=1061.6\mu F\ .$$

c) From $Z=j14\Omega$, we find that $R=0\Omega$ and inductance

$$L=\frac{X_L}{2\pi f}=\frac{14}{2\pi\times 50}=44.6mH\ .$$

d) From $Z=8\,\text{cis}\left(-60^0\right)\Omega=4-j4\sqrt{3}$, we find that $R=4\Omega$ and capacitance

$$C=\frac{1}{2\pi fX_C}=\frac{1}{2\pi\times 50\times 4\sqrt{3}}=459.5\mu F\ .$$

2. Impedance

$$Z=Z_1+Z_2=(3+j6+4-j3)\Omega=(7+j3)\Omega$$

$|Z|=\sqrt{49+9}=\sqrt{58}$ and $Arg\left(Z\right)=\tan^{-1}\left(\frac{3}{7}\right)=0.404892\simeq 23.20^0$

$$I=\frac{V}{Z}=\frac{120cis0^0}{\sqrt{58}cis23.20^0}=\frac{120}{\sqrt{58}}cis\left(-23.20^0\right).$$

The magnitude of the current $\left|I\right|=\frac{120}{\sqrt{58}}=15.76\,A$; phase angle relative to the voltage is -23.20^0.

Simply $\left[15.76A, 23.20^0\ lagging\right]$.

3. For the 2–branch parallel circuit, Impedance Z is given by

$$\frac{1}{Z}=\frac{1}{Z_1}+\frac{1}{Z_2}=\left(\frac{1}{3+j6}+\frac{1}{4-j3}\right)\Omega\Rightarrow Z=\frac{255+j15}{\sqrt{58}}\Omega$$

Which leads to $\left[27.25A, 3.37^0\ lagging\right]$

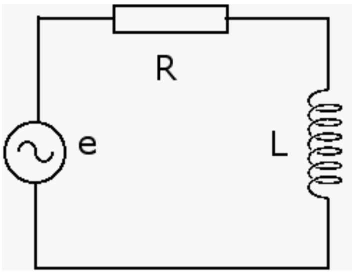
4. From inductance $L = 2.0H$, we get inductive reactance $X_L = 2\pi fL = 200\pi$,

from which $Z = Z_1 + Z_2 = (80 + j200\pi + 420)\Omega = (500 + j200\pi)\Omega$.

5. Thus, a) $|I| = \frac{|V|}{|Z|} = 0.3A$ b) V leads I by 52^0

$Z_0 = 390.2cis\left(-10.43^0\right), \gamma = 0.1029cis61.92^0$

6.



a) The e.m.f. that is supplied to the circuit is distributed between the resistor and the inductor.

Since the elements are in series the common current is taken to have the reference phase $I = I_m.e^{j\omega t}$

Adding the potentials around the circuit:

$$E = V_R + V_L = RI + j\omega LI = \left(R + j\omega L\right)I = \left(R + j\omega L\right)I_m e^{j\omega t}$$

$$= \sqrt{R^2 + \left(\omega L\right)^2} e^{j\arctan\frac{\omega L}{R}} \cdot I_m \cdot e^{j\omega t} = \sqrt{R^2 + \left(\omega L\right)^2} \cdot I_m \cdot e^{j\left(\omega t + \arctan\frac{\omega L}{R}\right)}$$

$$= \sqrt{R^2 + \left(\omega L\right)^2} \cdot I_m \cdot e^{j(\omega t + \theta)} = |Z| \cdot I_m \cdot e^{j(\omega t + \theta)}$$

b) In a) we have found that $E = \sqrt{R^2 + \left(\omega L\right)^2} \cdot I_m \cdot e^{j(\omega t + \theta)} = |Z| \cdot I_m \cdot e^{j(\omega t + \theta)}$, therefore

$|E| = \sqrt{R^2 + \left(\omega L\right)^2} \cdot I_m = |Z| \cdot I_m \cdot e^{j(\omega t + \theta)}$ and the phase is $\theta = \arctan\frac{\omega L}{R}$.

The physical current and potentials are:

$$i = \text{Im}\left\{I_m e^{j\omega t}\right\} = I_m \sin \omega t$$

$$V_R = \text{Im}\left\{RI_m e^{j\omega t}\right\} = RI_m \sin \omega t$$

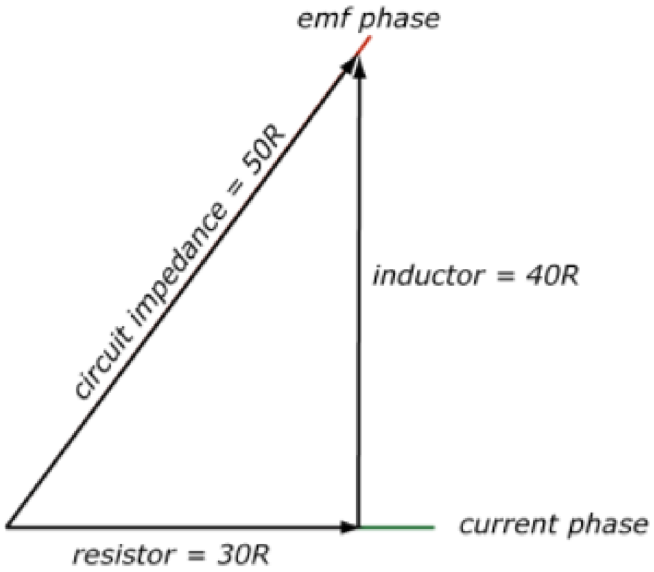
$$V_L = \mathrm{Im}\left\{ j\omega L I_m e^{j\omega t} \right\} = \omega L I_m \sin\left(\omega t + \frac{\pi}{2} \right)$$

$$e = \mathrm{Im}\left\{ Z I_m e^{j(\omega t + \theta)} \right\} = Z I_m \sin\left(\omega t + \theta \right).$$

3. $\omega = 2\pi \cdot \dfrac{1000}{\pi} = 2000 \dfrac{rad}{s}$; as $potential\ difference = 100V$ then

$$E_{\max} = 100\sqrt{2} = 141V \Rightarrow E = 141e^{j2000t}$$

The **complex impedance** $Z = R + j\omega L = 30 + j40 \Leftrightarrow Z = 50e^{0.93j}$,



4. The complex current $I = \dfrac{E}{Z} = \dfrac{141}{50} e^{j(2000t - 0.93)} = 2.82e^{j(2000t - 0.93)}$,

The physical current $i = 2.82\sin\left(2000t - 0.93\right) A$,

The rms current or equivalent dc current is $\left|I\right| = \dfrac{2.82}{\sqrt{2}} = 2A$ and has no phase.

5. Across the resistor R:

The complex potential difference is $V_R = R.I = 30 \times 2.82e^{j(2000t - 0.93)} = 84.8e^{i(2000t - 93)}$, the physical is $V_R = 84.8\sin\left(2000t - 0.93\right) Volts$, the one equivalent to dc current (rms potential difference) is $V_R = \dfrac{84.8}{\sqrt{2}} volts = 60\ volts.$

Across the inductor L:

The complex potential difference is

$$V_L = j\omega L.I = 40j \times 2.82e^{j(2000t+0.64)} = 40 \times 2.82e^{j\left(2000t-0.93+\frac{\pi}{2}\right)} = 112.8e^{j(2000t+0.64}.$$

The physical pd is $V_L = 112.8\sin(2000t + 0.64)\,Volts$, the one equivalent to dc current (rms potential difference) is $V_L = \frac{112.8}{\sqrt{2}} volts = 80\ volts.$

1.6. Unit summary

1. Definitions

- **Complex number**

Given two real numbers *a* and *b*, the complex number *z* is a number $z = a + ib$ where the number *a* is called the **real part** of *z* and the number *b* is called the **imaginary part** of *z* and $i^2 = -1$ $i^2 = -1$.

The set of all complex numbers is $\mathbb{C} = \{z = a + ib : a, b \in \mathbb{R}\ and\ i^2 = -1\}$.

- **Conjugate of a complex number**

Conjugate of complex number $z = a + bi$ is its reflection by $x - axis$ and is denoted and defined by $\bar{z} = a - bi$.

- **Modulus of a complex number**

The modulus of $z = a + bi$ is a positive real number denoted by $|z|$, such that

$|z| = \sqrt{z \cdot \bar{z}} = \sqrt{a^2 + b^2}$.

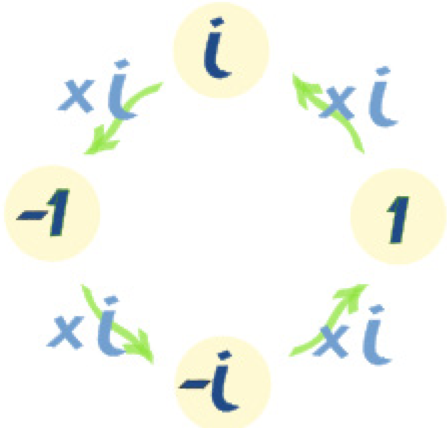
- **Argument of a complex number**

Argument of a complex number $z = a + bi$ denoted by θ is an angle between a vector of affix $z = a + bi$ with $positive\, x - axis$ and defined as follows

$$\theta = arg(z) = \begin{cases} \arctan\frac{b}{a}, if\ a>0 \\ \pi + \arctan\frac{b}{a}, if\ a<0\ and\ b>0 \\ -\pi + \arctan\frac{b}{a}, if\ a<0\ and\ b<0 \\ \frac{\pi}{2}, if\ a=0\ and\ b>0 \\ -\frac{\pi}{2}, if\ a=0\ and\ b<0 \\ Undefined, if\ a=0\ and\ b=0 \end{cases}$$

2. Properties of the imaginary number "i"

The imaginary unit, i , "cycles" through 4 different values $\{1, i, -1, -i\}$ each time we multiply .

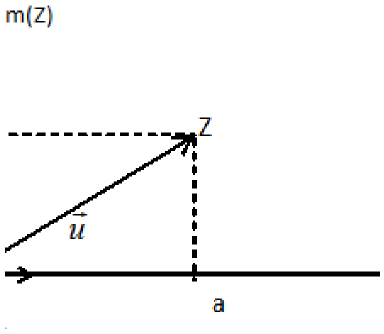


$i^{4n} = 1, \quad i^{4n+1} = i, i^{4n+2} = -1, i^{4n+3} = -i.$

3. Presentation of a complex number

There are four different ways of presenting a complex number: algebraic form, geometric form, polar form and exponential form as illustrated in the following table

Form	Formula/Graphical representation	Additional information
Algebraic	$z = a + bi$	$a, b \in \mathbb{R}$ and $i^2 = -1$

Geometric		$x-axis \equiv \mathrm{Re}(z)$ and $y-axis \equiv \mathrm{Im}(z)$
Polar	$z = r(\cos\theta + i\sin\theta)$ or $z = rcis\theta$ or $r\angle\theta$	$r = \|z\|, and\ \theta = \arg(z)$
Exponential	$z = re^{i\theta}$	$r = \|z\|, and\ \theta = \arg(z)$

4. Operations in set of complex numbers

- **Addition and subtraction in the set of complex numbers**

To perform **addition** and **subtraction** of complex numbers we combine real parts together and imaginary parts separately:

Given two complex numbers $z_1 = a + bi\ and\ z_2 = c + di$, then

$z_1 + z_2 = (a+c) + (b+d)i$ and $z_1 - z_2 = (a-c) + (\mathrm{b}-d)i.$

- **Multiplication, division and powers of complex numbers**

The **product** of two complex numbers $z_1 = a + bi\ and\ z_2 = c + di$ is given by:

$$z_1.\mathrm{z}_2 = (a+bi)(c+di) = (ac-bd) + i(ad+cb);$$

$$\frac{z_1}{z_2} = \frac{a+bi}{c+di} = \frac{(ac+bd)+(bc-ad)i}{c^2+d^2}$$

The **power** n of $z = a + bi$ is given by:

$$z^n = \underbrace{z \ldots z}_{n\ times} = \underbrace{(a+bi)\ldots(a+bi)}_{n\ times} = (a+bi)^n$$

Square roots of a complex number

When the complex number $z = x + yi$ is the square root of a complex number $a+bi$, this means that $\begin{cases} x = \pm\sqrt{\frac{1}{2}\left(a+\sqrt{a^2+b^2}\right)} \\ x = \pm\sqrt{\frac{1}{2}\left(-a+\sqrt{a^2+b^2}\right)} \end{cases}$.

Multiplication or division of complex numbers in polar form and exponential form

From two complex numbers in polar form $z_1 = r_1 cis\theta_1$ and $z_2 = r_2 cis\theta_2$,

- The product of z_1 and z_2 is

$$z_1 \cdot z_2 = \underbrace{r_1 \cdot r_2 cis(\theta_1+\theta_2)}_{polar\ form} = \underbrace{r_1 \cdot r_2 e^{i(\theta_1+\theta_2)}}_{\exp onential\ form};$$

- The quotient of z_1 and z_2 is $\frac{z_1}{z_2} = \underbrace{\frac{r_1}{r_2} cis(\theta_1-\theta_2)}_{polar\ form} = \underbrace{\frac{r_1}{r_2} e^{i(\theta_1-\theta_2)}}_{\exp onential\ form}$

De Moivre's formula and Power of a complex number using the polar/ exponential form

Given the complex number $z = r(\cos\theta + i\sin\theta)$,then,

$$z^n = r^n\left[\cos(n\theta) + i\sin(n\theta)\right] = r^n e^{in\theta}$$

For $r=1$, we have **De Moivre's formula** that is

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$$

n^{th} root of a complex number

To find n^{th} roots of a complex number z, you start by expressing z in polar form $z = rcis\theta$, where r is modulus of z and θ argument of z.

Then, n^{th} roots of a complex number z is given by

$$z_k = \sqrt[n]{r}\ cis\left(\frac{\theta + 2k\pi}{n}\right) \quad k = 0, 1, 2, 3, ..., n-1.$$

Construction of regular polygon

While drawing a regular polygon with *n* sides follows the following steps:

- Start by drawing a unit circle in Argand diagram. The radius and the centre of this circle will be the radius and centre of the regular polygon.
- Around the circle, place the points with affixes

  $z_k = \text{cis}\frac{2k\pi}{n},\ k = 0, 1, 2, ..., n-1$. Those points are the vertices of the polygon.
- Using a ruler join the obtained points around the circle and the obtained figure is the required regular polygon.

Euler's formula

From polar form and exponential form of a complex number $r\angle\theta$, you get the following formulae called Euler's formulae:

$$\begin{cases} \cos\theta = \frac{1}{2}\left(e^{i\theta} + e^{-i\theta}\right) \\ \sin\theta = \frac{1}{2i}\left(e^{i\theta} - e^{-i\theta}\right) \end{cases}.$$

Applicationof complex numbers in other subjects

Complex numbers are applied in other subjects to express certain variables or facilitate the calculation in complicated expressions. They are mostly used in Electrical Engineering, electronics engineering, Signal analysis, Quantum mechanics, Relativity, Applied mathematics, Fluid dynamics, Electromagnetism, Civil and Mechanical Engineering.

In sciences, it is better to use j as the imaginary number instead of using i to avoid the confusion of the expression of the imaginary number i and the expression of the electrical current i.

In alternating current, if the angular velocity of the wire is $\omega = 2\pi f$, respective impedances are $Z_R = R, Z_L = j\omega L$ and $\frac{1}{j\omega C}$,their modulus are the **resistance** R

, the **capacitive reactance** is $|Z_C| = \dfrac{1}{\omega C}$ and the **inductive reactance** is given by $|Z_L| = \omega L$.

The voltage in an AC circuit can be represented as

$$V = V_0 e^{j\omega t}$$

$$= V_0(\cos \omega t + j \sin \omega t)$$

which denotes **Impedance**, V_o is peak value of impedance and $\omega = 2\pi f$ where f is the frequency of supply. To obtain the measurable quantity, the real part is taken:

$\mathrm{Re}(V) = V_0 \cos \omega t$ and is called **Resistance** while imaginary part denotes **Reactance** (inductive or capacitive).

Briefly, the current, I (cosine function) leads the applied potential difference (p.d.), V (sine function) by one quarter of a cycle i.e. $\dfrac{\pi}{2}$ radians or 90^0

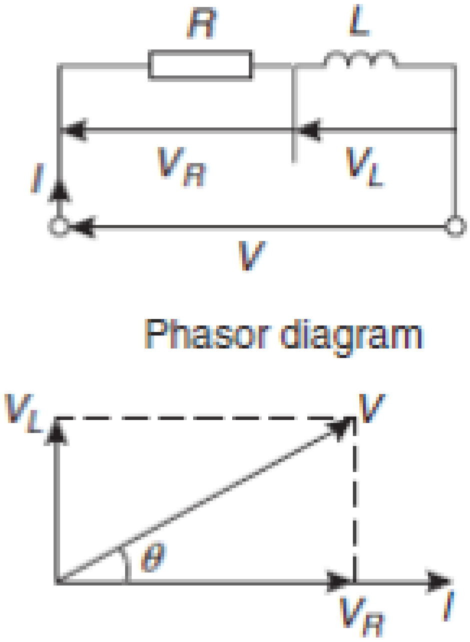


Figure 1.8: $R-L$ series circuit

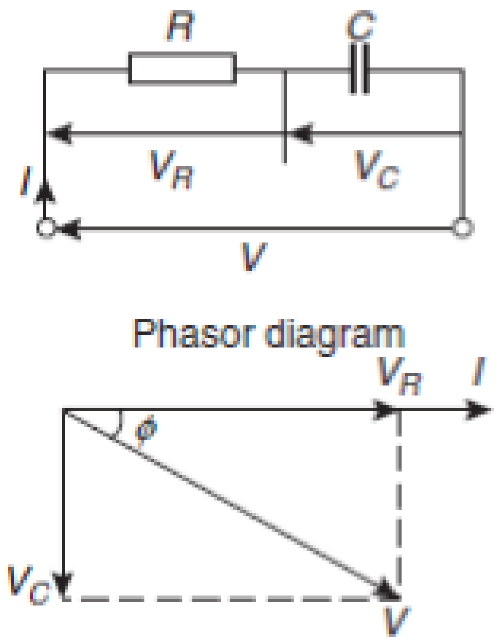


Figure1.9: $R-C$ series circuit

In the **Resistance and Inductance** ($R-L$) *series* circuit, as shown in figure 1.8

$V_R + jV_L = V$ as $V_R = IR, V_L = IX_L$ (where X_L is the inductive reactance $2\pi f L$ ohms) and $V = IZ$ (where Z is the impedance) then $R + jX_L = Z$.

In the **Resistance and Capacitance** ($R - C$) series circuit, as shown in figure 1.9

$V_R - jV_C = V$, from which $R - jX_C = Z$ (where X_C is the capacitive reactance

$X_C = \dfrac{1}{2\pi fC}\ \Omega$).

1.7. Additional information for the teacher

Application of complex numbers in other subjects

Complex numbers are applied in other subjects to express certain variables or facilitate the calculation in complicated expressions. They are mostly used in electrical engineering, electronics engineering, signal analysis, quantum mechanics, relativity, applied mathematics, fluid dynamics, electromagnetism, civil and mechanical engineering.

Application in electronics engineering

Information that expresses a single dimension, such as linear distance, is called a scalar quantity. Scalar numbers are the kind of numbers that Student-teachers use most often. In relation to science, the voltage produced by a battery, the resistance of a piece of wire (ohms), and current through a wire (amps) are scalar quantities.

When electrical engineers analyzed alternating current circuits, they found that quantities of voltage, current and resistance (called impedance in AC) were not the familiar one-dimensional scalar quantities that are used when measuring DC circuits. These quantities which now alternate in direction and amplitude possess other dimensions (frequency and phase shift) that must be taken into account.

In order to analyze AC circuits, it became necessary to represent multi-dimensional quantities. In order to accomplish this task, scalar numbers were abandoned and complex numbers were used to express the two dimensions of frequency and phase shift at one time.

In mathematics, i is used to represent imaginary unit. In the study of electricity and electronics, j is used to represent imaginary unit to avoid confusion with i which represents current in electronics. It is also customary for scientists to write the complex number in the form $a + jb$.

Complex numbers play a great role in electronics. The main reason for this is that they make the whole topic of analyzing and understanding alternating signals much easier.

We can now consider oscillating currents and voltages as being complex values that have a real part we can measure and an imaginary part which we can't. At first it seems pointless to create something we can't see or measure, but it turns out to be useful in a number of ways.

1. It helps us understand the behavior of circuits which contain reactance (produced by capacitors or inductors) when we apply A.C signals.

2. It gives us a new way to think about oscillations. This is useful when we want to apply concepts like the conservation of energy to understanding the behavior of systems which range from a simple mechanical pendulum to a quartz-crystal oscillator.

In Signal analysis

Complex numbers are used in signal analysis and other fields for a convenient description for periodically varying signals. For given real functions representing actual physical quantities, often in terms of sines and cosines, corresponding complex functions are considered of which the real parts are the original quantities. For a sine wave of a given frequency, the absolute value$|Z|$ of the corresponding z is the amplitude and the argument $\arg(z)$ the phase.

If Fourier analysis is employed to write a given real-valued signal as a sum of periodic functions, these periodic functions are often written as complex valued functions of the form $\omega f(t) = z$ where ω represents the angular frequency and the complex number Z encodes the phase and amplitude as explained above.

In Quantum mechanics

The complex number field is relevant in the mathematical formulation of quantum mechanics, where complex Hilbert spaces (infinite dimensional space over $\mathbb{C}$) provide the context for one such formulation that is convenient and perhaps most standard. The original foundation formulas of quantum mechanics - the Schrödinger equation and Heisenberg's matrix mechanics - make use of complex numbers.

The quantum theory provides a quantitative explanation for two types of phenomena that classical mechanics and classical electrodynamics cannot account for:

Some observable physical quantities, such as the total energy of a black body, take on discrete rather than continuous values. This phenomenon is called quantization, and the smallest possible intervals between the discrete values are called quanta (singular: quantum, from the Latin word for "quantity", hence the name "quantum mechanics."). The size of the quanta typically varies from system to system.

Under certain experimental conditions, microscopic objects like atoms or electrons exhibit wave-like behavior, such as interference. Under other conditions, the same species of objects exhibit particle-like behavior ("particle" meaning an object that can be localized to a particular region of space), such as scattering. This phenomenon is known as wave-particle duality.

In Relativity

In special and general relativity, some formulas for the metric on space time become simpler if one takes the time variable to be imaginary. (This is no longer standard in classical relativity, but is used in an essential way in quantum field theory.) Complex numbers are essential to spinors, which are a generalization of the tensors used in relativity.

In Applied mathematics

In differential equations, it is common to first find all complex roots r of the characteristic equation of a linear differential equation and then attempt to solve the system in terms of base functions of the form $f(t)=e^{rt}$.

1.8. End unit assessment

ANSWER FOR QUESTION ONE

I. $a)\quad z_1+z_2=16+11i$

$b)\quad \frac{z_1}{z_2}=\frac{(6+3i)(10-8i)}{164}=84-18i$

$c)\quad z_1.z_2=36-78i$

$d)\quad (z_1-\overline{z_2})(z_1+\overline{z_2})=(6+3i-10+8i)(6+3i+10-8i)=-119+196i$

II.

$$z=R+j\omega L+\frac{1}{j\omega C}\Leftrightarrow z=R+j\omega L+\frac{-j\omega C}{j\omega C(-j\omega C)}$$

$$\Leftrightarrow z=R+j\omega L-\frac{j\omega C}{\omega^2C^2}\Leftrightarrow z=R+j\omega L-\frac{j}{\omega C}$$

$$\Leftrightarrow z=R+j(\omega L-\frac{1}{\omega C})$$

When $R=10, L=5, C=0.04$ and $\omega=4$, we get

$$z=10+j(4\times5-\frac{1}{0.04\times4})$$
$$z=10+j(20-\frac{1}{0.16})$$
$$z=10+j13.75$$

III. a) $z=3+3i=3\sqrt{2}\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)=3\sqrt{2}.e^{i\frac{\pi}{4}}$

b) $(3+3i)^5=\left[3\sqrt{2}\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)\right]^5=\left(3\sqrt{2}\right)^5\left(\cos\frac{5\pi}{4}+i\sin\frac{5\pi}{4}\right)=\left(3\sqrt{2}\right)^5 e^{i\frac{5\pi}{4}}$

c) Let $z=(x+yi)$ be square root of $Z=3+3i$ then, $(x+yi)^2=3+3i$
$x^2-y^2+2xyi=3+3i$

Equate real parts and imaginary parts to get:

$$x^2-y^2=3 \ (1)$$
$$2xy=3 \quad (2)$$
$$x^2+y^2=\sqrt{3^2+3^2}=\sqrt{18}=3\sqrt{2}$$

Solving (1) and (3) to get: $\begin{cases} x^2-y^2=3 \\ x^2+y^2=3\sqrt{2} \end{cases} \Leftrightarrow \begin{cases} x=\pm\sqrt{\frac{3+3\sqrt{2}}{2}} \\ y=\sqrt{\frac{-3+3\sqrt{2}}{2}} \end{cases}$

Then in algebraic form $z=\pm\left(\sqrt{\frac{3+3\sqrt{2}}{2}}+i\sqrt{\frac{-3+3\sqrt{2}}{2}}\right)$

ANSWER FOR QUESTION TWO

Let $z=r(\cos\theta+i\sin\theta)$ be complex number in polar form , and

$i=\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}$, Then,

$$r(\cos\theta + i\sin\theta)\times\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right) = r\cos\left(\theta+\frac{\pi}{2}\right) + i\sin\left(\theta+\frac{\pi}{2}\right)\pi$$

$\therefore z$ is rotated in anticlockwise with angle $\frac{\pi}{2}$.

ANSWER FOR QUESTION THREE

$$a)\quad \sin^2 x\cos x = \left(\frac{e^{ix}-e^{-ix}}{2i}\right)^2 .\left(\frac{e^{ix}+e^{-ix}}{2}\right) = \frac{e^{2ix}-2+e^{-2ix}}{-4}\cdot\frac{e^{ix}+e^{-ix}}{2}$$

$$= \frac{e^{3ix}-2e^{ix}+e^{-ix}+e^{ix}-2e^{-ix}+e^{-3ix}}{-8} = \frac{e^{3ix}-e^{ix}-e^{-ix}+e^{-3ix}}{-8}$$

$$= \frac{1}{4}\left(\frac{e^{3ix}+e^{-3ix}}{2} - \frac{e^{ix}+e^{-ix}}{2}\right) = \frac{1}{4}\cos 3x - \frac{1}{4}\cos x$$

$$\text{b)}\quad \cos^2 x\sin x = \left(\frac{e^{ix}+e^{-ix}}{2}\right)^2 .\left(\frac{e^{ix}-e^{ix}}{2i}\right) = \frac{e^{2ix}+2+e^{-2ix}}{4}\cdot\frac{e^{ix}-e^{-ix}}{2i}$$

$$= \frac{e^{3ix}+2e^{ix}+e^{-ix}-e^{ix}-2e^{-ix}-e^{-3ix}}{8i} = \frac{\left(e^{3ix}-e^{-3ix}\right)+\left(e^{ix}-e^{-ix}\right)}{8i}$$

$$= \frac{1}{4}\left(\frac{e^{3ix}-e^{-3ix}}{2i} + \frac{e^{ix}-e^{-ix}}{2i}\right) = \frac{1}{4}(\sin 3x + \sin x) = \frac{1}{4}\sin 3x + \frac{1}{4}\sin x$$

$$\text{c)}\sin^2 x\cos^2 x = \left(\frac{e^{ix}-e^{-ix}}{2i}\right)^2\left(\frac{e^{ix}+e^{-ix}}{2}\right)^2$$

$$= \left(\frac{e^{2x}-2e^{ix}e^{-ix}+e^{-2ix}}{-4}\right)\left(\frac{e^{2ix}+2e^{ix}e^{-ix}+e^{-2ix}}{4}\right)$$

$$= \left(\frac{e^{2ix}-2+e^{-2ix}}{-4}\right)\left(\frac{e^{2ix}+2+e^{-2ix}}{4}\right)$$

$$= \left(\frac{e^{4ix}+2e^{2ix}+1-2e^{2ix}-4-2e^{-2ix}+1+2e^{-2ix}+e^{-4ix}}{-16}\right)$$

$$= \left(\frac{e^{4ix}+e^{-4ix}-2}{-16}\right) = -\frac{1}{8}\left(\frac{e^{4ix}+e^{-4ix}}{2}\right) - \frac{2}{-16} = -\frac{1}{8}\cos 4x + \frac{1}{8} = \frac{1}{8} - \frac{1}{8}\cos 4x$$

d) $\sin^3 x = \left(\dfrac{e^{ix} - e^{-ix}}{2i}\right)^3 = \left(\dfrac{e^{3ix} - 3e^{2ix}e^{-ix} + 3e^{ix}e^{-2ix} - e^{-3ix}}{-8i}\right)$

$$= \left(\frac{e^{3ix} - 3e^{ix} + 3e^{-ix} - e^{-3ix}}{-8i}\right) = \frac{\left(e^{3ix} - e^{-3ix}\right) - 3\left(e^{ix} - e^{-ix}\right)}{-8i}$$

$$= -\frac{1}{4}\left(\frac{e^{3ix} - e^{-3ix}}{2i} - 3\frac{e^{ix} - e^{-ix}}{2i}\right) = -\frac{1}{4}(\sin 3x - 3\sin x) = \frac{3}{4}\sin x - \frac{1}{4}\sin 3x$$

ANSWER FOR QUESTION FOUR

Analytically/ numerically, let O be the starting point and A, B and C be the turning points in northeast, west of north and south of west respectively. The result of all displacements is represented by the vector $\overrightarrow{OA} + \overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{OC}$.Or

$$\overrightarrow{OA} = 12\left(\cos 45^0 + i\sin 45^0\right) = 12e^{\frac{\pi}{4}i},\ \overrightarrow{AB} = 20\left[\cos\left(30^0 + 90^0\right) + i\sin\left(30^0 + 90^0\right)\right] = 20e^{\frac{2\pi}{3}i}$$

and

$$\overrightarrow{BC} = 18\left[\cos\left(60^0 + 180^0\right) + i\sin\left(60^0 + 180^0\right)\right] = 18\left[\cos\left(180^0 - 60^0\right) + i\sin\left(180^0 - 60^0\right)\right] = 18e^{-\frac{2\pi}{3}i}.$$

Therefore,

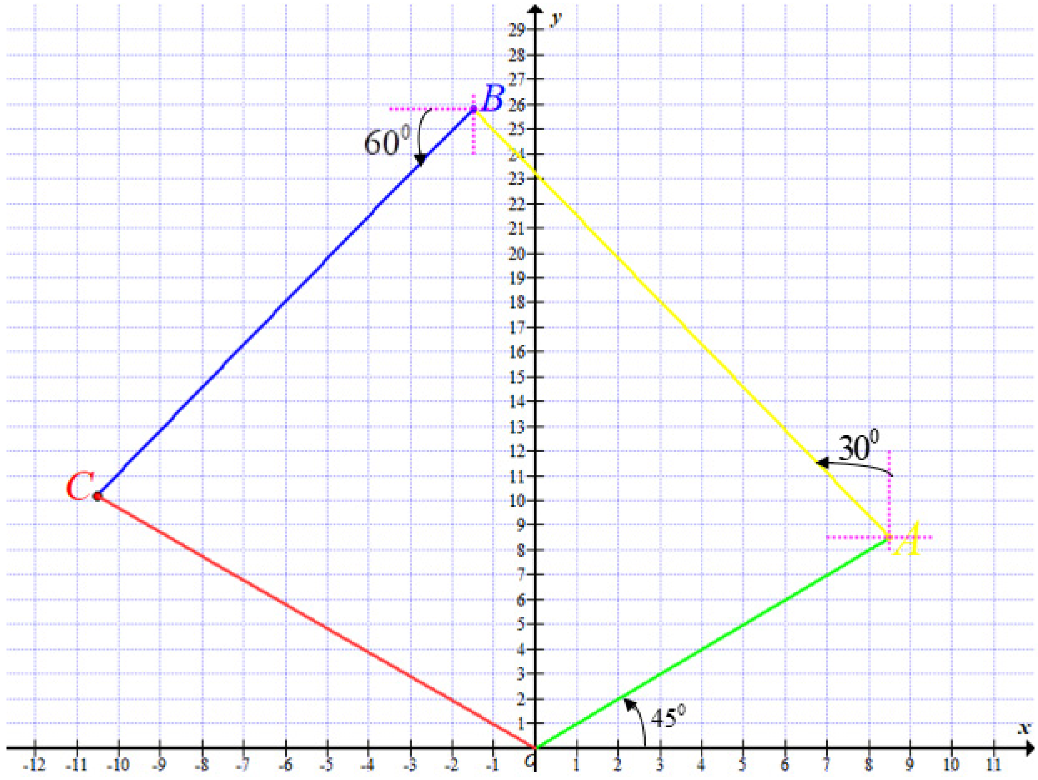
$$\overrightarrow{OC} = 12e^{\frac{\pi}{4}i} + 20e^{\frac{2\pi}{3}} + 18e^{-\frac{2\pi}{3}i} = 12\left(\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right) + 20\left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) + 18\left(-\frac{1}{2} - i\frac{\sqrt{3}}{2}\right)$$

$$= \left(6\sqrt{2} - 10 - 9\right) + i\left(6\sqrt{2} + 10\sqrt{3} - 9\sqrt{3}\right) = \left(6\sqrt{2} - 19\right) + i\left(6\sqrt{2} + \sqrt{3}\right).$$

$$\left|\overrightarrow{OC}\right| = \sqrt{\left(6\sqrt{2} - 19\right)^2 + \left(6\sqrt{2} + \sqrt{3}\right)^2} \simeq 14.7;$$

$$\arg\left(\overrightarrow{OC}\right) = \pi + \arctan\frac{6\sqrt{2} + \sqrt{3}}{6\sqrt{2} - 19} = 2.37 = 135^0 49'.$$

Thus, the man is $14.7\ km$ from his starting point in a direction $135^0 49' - 90^0 = 45^0 49'$ west of north.



Using a convenient unit of length which represent $1km$, and a protractor to measure angles, construct vectors $\overrightarrow{OA}, \overrightarrow{AB}$ and $\overrightarrow{BC}$. Then by determining the number of units in $\overrightarrow{OC}$ and the angle which $\overrightarrow{OC}$ makes with $y-axis$, we obtain the approximately $14.7\,km$ of displacement from origin (his starting point) in a direction of $45^0 49'$ west of north.

ANSWER FOR QUESTION FIVE

a) It is an open question for which the answer shows with examples that complex numbers are applicable in engineering.

b) $\omega = 2\pi.\dfrac{250}{\pi} = 500 rad/s$, RMS emf =220V, R=60 ohms, $C = 50.10^{-6} F$, $L = 180.10^{-3} H$

(i) The complex impedance is: $Z = R + j\omega L - \dfrac{j}{\omega C}$

$$= 60 + j.500.180.10^{-3} - \frac{j}{500.50.10^{-6}}$$

$$= 60 + 50j = \sqrt{60^2 - 50^2}.e^{\left(j \arctan\frac{50}{60}\right)} = 78,1.e^{0,69j}$$

The impedance makes an angle $\theta = 0.69\, radians = 39.8\, degrees$ with the applied electromotive force.

(ii) The complex current:

$I_m = \frac{E_m}{Z}$ where E_m is the complex impedance. $E = 220.\sqrt{2} = 311 volts$, which implies that

$I_m = \frac{311}{78,1} e^{j(500t - 0,69)} = 4.e^{j(500t - 0,69)}$ because E_m is a reference phase.

The physical current is the imaginary part, $i = 4.\sin(500t - 0,69)$, it is behind emf at $0.69\, radians$

The root mean square current in the circuit is $i_{eq} = \frac{220}{78,1} A \quad or \quad i_{eq} = \frac{4}{\sqrt{2}} A = 2.8A.$

1.9. Additional activities

The teacher's guide suggests additional questions and answers to assess the key unit competence.

1.9.1. Remedial Activities:

Suggestion of Questions and Answers for remedial activities for slow Student-teachers.

1. Calculate the product of the following complex numbers and write the result in the form of $a + bi$

   a) Multiply $-4(13 + 5i)$

   b) Multiply $2i(3 - 8i)$

   c) Multiply $(1 + 4i)(5 + i)$

Solution

$a)\quad 3(2+4i) = 3(2) + 3(4i) = 6 + 12i$

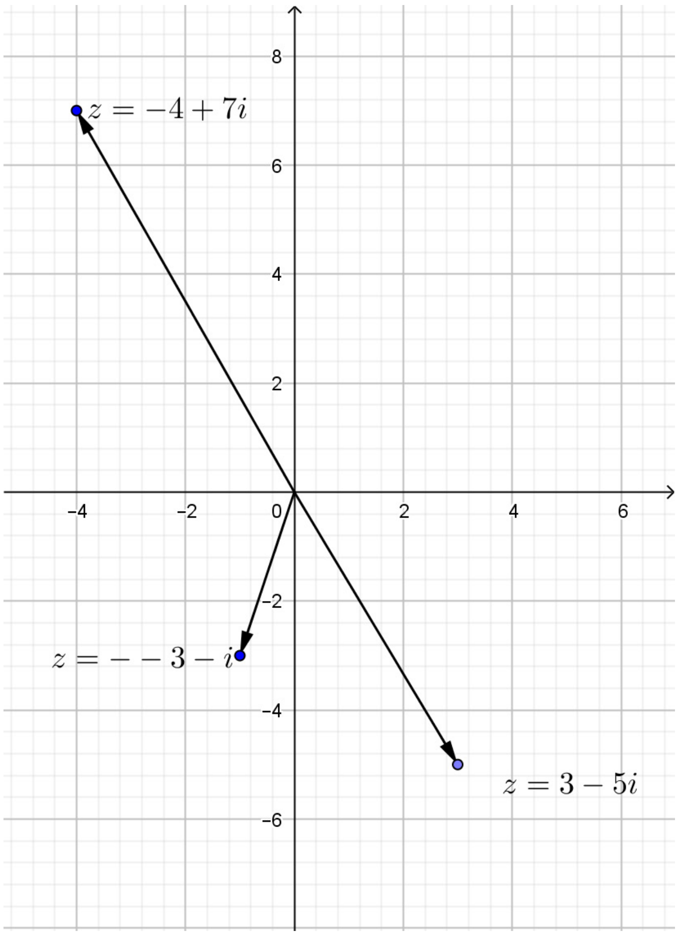
$b)\quad (5+3i)i = 5i + 3i^2 = -3 + 5i$

$c)\quad (2-7i)(3+4i) = (2)(3) - (7i)(3) + (2)(4i) - (7i)(4i) = 34 - 13i$

2. Plot the following numbers in the complex plane or on the Argand diagram

$a)\quad 3 - 5i; \qquad b)\quad -4 + 7i \qquad c)\quad -i - 3$

Solution



3. Solve each of the following equations for the complex number z

$a)\ 4+5i = z-(1-i)$ $\quad$ $b)\ (1+2i)z = 2+5i$

Solutions

$a)\ 4+5i = z-(1-i) \Leftrightarrow 4+5i+1-i = z \Rightarrow z = 5+4i$

$b)\ (1+2i)z = 2+5i \Leftrightarrow z = \dfrac{2+5i}{1+2i} \Rightarrow z = \dfrac{(2+5i)(1-2i)}{(1+2i)(1-2i)} \Leftrightarrow z = \dfrac{12}{5}+\dfrac{1}{5}i$

4. Solve the following equations for real x and y.

$3+5i+x-yi = 6-2i$

Solution

$3+5i+x-yi = 6-2i \Leftrightarrow (3+x)+(5-y)i = 6-2i$

By identification of real part and imaginary part both sides, we get the following simple equations:

$3+x=6$ and $5-y=-2$. Therefore $x=3$ and $y=7$.

1.9.2. Consolidation activities:

Suggestion of questions and answers for deep development of competences.

1. The table below shows examples of pure imaginary numbers in both unsimplified and simplified form.

Unsimplified form	**Simplified form**
$\sqrt{-9}$	$3i$
$\sqrt{-5}$	$i\sqrt{5}$
$-\sqrt{-144}$	$-12i$

a) Explain how these pure imaginary numbers are simplified

b) Explain why $\sqrt{-18}$ is an imaginary number and write it in a simplified form

Solution

a) The following property explains how the pure imaginary numbers can be simplified: For $a>0, \sqrt{-a}=i\sqrt{a}$.

Example: the square root of -9 is an imaginary number. The square root of 9 is 3, so the square root of *negative* 9 is 3 *imaginary units*, or $3i$.

b) $\sqrt{-18}$ is an imaginary number, since it is the square root of a negative number. So, we can start by rewriting $\sqrt{-18}$ as $i\sqrt{18}$.Next we can simplify $\sqrt{18}$ using what we already know about simplifying radicals: $\sqrt{-18}=i\sqrt{9\times 2}=3i\sqrt{2}$.

3. It is known that $i=\sqrt{-1}$ and that $i^2=-1$, use the properties of exponents and find the value of i^3 and i^4.

Solution

We know that $i^3=i^2\cdot i$. But since $i^2=-1$, we see that $i^3=i^2\cdot i=-i$.

Similarly $i^4=i^2\cdot i^2$. Again, using the fact that $i^2=-1$, we have the following:

$i^4=i^2\cdot i^2=(-1)(-1)=1$.

4. The table below summarize the powers of i.

i^1	i^2	i^3	i^4	i^5	i^6	i^7	i^8
i	-1	$-i$	1	i	-1	$-i$	1

From the table above, it appears that the powers of i cycle through the sequence $i, -1, -i, 1$.

a) Using this pattern, find i^{20} ?

b) Suppose that it is possible to list the sequence $i, -1, -i, 1$ up to the 138th term, but the work can take too much time. Consider that $i^4 = 1, i^8 = 1, i^{12} = 1$ etc, and 136 is a multiple of 4, calculate i^{138}.

Solution

a) $i^{20} = i^{4\times5} = (1)^5 = 1$

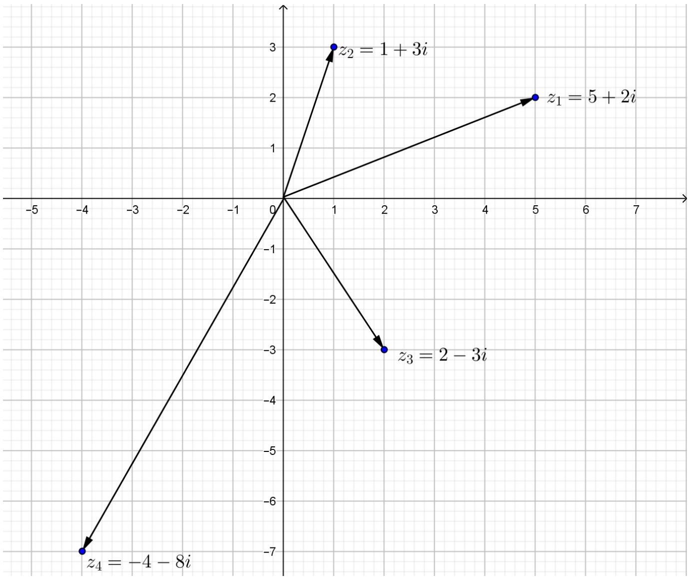
b) While 138 is not a multiple of 4, the number 136 is a multiple of 4. Let's use this to help us simplify i^{138}.

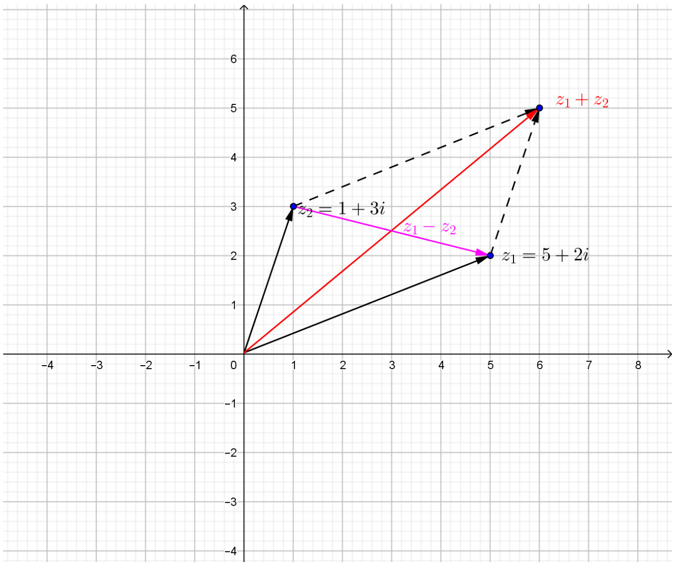
$$i^{138} = i^{136} \cdot i^2 = i^{4\times34} \cdot i^2 = (1)^{34} \cdot i^2 = 1(-1) = -1$$

5. Let $z_1 = 5+2i$, $z_2 = 1+3i$, $z_3 = 2-3i$ and $z_4 = -4-7i$.

Plot the complex numbers z_1, z_2, z_3 and z_4 on Argand diagram or complex plane and label them. Plot the complex numbers $z_1 + z_2$ and $z_1 - z_2$ on the same Argand diagram. Geometrically, explain how do the positions of the numbers $z_1 + z_2$ and $z_1 - z_2$ relate to z_1 and z_2.

Solution





$z_1 + z_2$ is at the endpoint of leading diagonal of parallelogram constructed from affixes of z_1 and z_2 while $z_1 - z_2$ is at the second diagonal of the same parallelogram from affix of z_2.

6. Convert $z = -1 - i$ in polar form

Solution

Let $z = a + bi$ where $a = -1$ and $b = -1$. In polar form the modulus of

$$z = |z| = \sqrt{1^2 + 1^2} = \sqrt{2}$$

$x = r\cos\theta \Leftrightarrow -1 = \sqrt{2}\cos\theta \Rightarrow \cos\theta = -\frac{\sqrt{2}}{2}$ and

$y = r\sin\theta \Leftrightarrow -1 = \sqrt{2}\sin\theta \Rightarrow \sin\theta = -\frac{\sqrt{2}}{2}$

$\theta = \frac{5\pi}{4}$, the coordinates of z in polar form are $z(r,\theta) = z\left(\sqrt{2}, \frac{5\pi}{4}\right)$

7. Find the quotient of two complex numbers $\frac{20-4i}{3+2i}$

Solution

$$\frac{20-4i}{3+2i} = \frac{(20-4i)(3-2i)}{(3+2i)(3-2i)} = \frac{52}{13} - \frac{52}{13}i.$$

1.9.3 Extended activities:

Suggestion of Questions and Answers for gifted and talented Student-teachers.

1. Given that $z = x + iy$, do the following activities

a) Prove that there is no complex number such that $|z| - z = i$

b) Find $z \in \mathbb{C}$ such that $z^2 \in \mathbb{R}$

c) Find $z \in \mathbb{C}$ such that $\mathrm{Re}[z(1+i)] + z\bar{z} = 0$

Solution

a) Suppose that some $z \in \mathbb{C}$ satisfies the equation $|z| - z = i \Leftrightarrow |z| = z + i$.

Then $|z| = \mathrm{Re}(z) + i(\mathrm{Im}(z) + 1)$. Since $|z| \in \mathbb{R}$, necessarily $\mathrm{Im}(z) = -1$.

Then, the given equation becomes $\sqrt{(\mathrm{Re}(z))^2 + 1} = \mathrm{Re}(z)$, and, squaring, we obtain $1 = 0$ that does not exist.

b) If $z = a + bi,\ a, b \in \mathbb{R}$, then $z^2 \in \mathbb{R} \Leftrightarrow a^2 - b^2 + 2abi \in \mathbb{R}$, that is if and only if $ab = 0 \Leftrightarrow a = 0$ or $b = 0$. Hence $z^2 \in \mathbb{R}$ if and only if $z \in \mathbb{R}$ $(b = 0)$ or if z is a pure imaginary number $(a = 0)$.

c) Let $z = a + bi,\ a, b \in \mathbb{R}$, thus

$$\mathrm{Re}(z(1+i) = \mathrm{Re}[(a+bi)(1+i)] = \mathrm{Re}[(a-b) + i(a+b)] = a - b.$$

The equation $\mathrm{Re}\left[z(1+i)\right]+z\overline{z}=0$ is then equivalent to

$a-b+a^2+b^2=0 \Leftrightarrow \left(a+\frac{1}{2}\right)^2+\left(b-\frac{1}{2}\right)=\frac{1}{2}$ whose solutions the points of the circle with center in $\left(-\frac{1}{2},\frac{1}{2}\right)$ and radius $\frac{\sqrt{2}}{2}$.

2. Given that p and q are real and that $1+2i$ is a root of the equation $z^2+(P+5i)z+q(2-i)=0$, determine

a) The values of p and q

b) The other root of the equation

3. Find the least value of the positive integer n for which $\left(\sqrt{3}+i\right)^n$ is

a) real b) pure imaginary

Solution

$$\left(\sqrt{3}+i\right)^n=2^n\left(\cos\frac{n\pi}{6}+i\sin\frac{n\pi}{6}\right)$$

a) $\left(\sqrt{3}+i\right)^n$ is real $\Leftrightarrow \sin\frac{n\pi}{6}=0 \Leftrightarrow \frac{n\pi}{6}=k\pi, k\in\mathbb{N} \Leftrightarrow n=6k$;

Therefore, $\left(\sqrt{3}+i\right)^n$ is real if n is a positive multiple of 6 and the least value of the positive integer multiple of 6 is $n=6$.

b) $\left(\sqrt{3}+i\right)^n$ is pure imaginary

$$\Leftrightarrow \cos\frac{n\pi}{6}=0 \Leftrightarrow \frac{n\pi}{6}=\frac{\pi}{2}+k\pi, k\in\mathbb{N} \Leftrightarrow n=3(2k+1);$$

Therefore, $\left(\sqrt{3}+i\right)^n$ is pure imaginary if n is a positive odd number that is multiple of 3 and the least value is $n=3$.

UNIT 2

ARRANGEMENT, PERMUTATION AND COMBINATION

2.1 Key unit competence

Apply formulae of combinatory analysis to count possible outcomes of a random experiment.

2.2 Prerequisite

Student-teachers will perform well in this unit if they make a short revision on the sets of numbers (year 1: unit 1), set theory (year 1: unit 2).

2.3 Cross-cutting issues to be addressed

- Inclusive education (promote education for all while teaching)
- Peace and value Education (respect others view and thoughts during class discussions)
- Gender (equal opportunity of boys and girls in the lesson participation and making groups)

2.4 Guidance on introductory activity.

- Form groups of students and invite student-teachers to work on questions for **Introductory activity** found in student's book unit 2;
- Guide students to read and analyse the problem related to registering motor vehicles in Rwanda.
- Guide student-teachers to find all possible roads from A to C via B
- Invite students with different working steps to present their findings to the whole class discussion;
- Basing on their experience, prior knowledge and abilities shown in answering the questions for this activity, use different questions to guide student-teachers to give their predictions and ensure that you arouse their curiosity on what is going to be leant in this unit.

Answer for introductory activity:

To find all possible roads, students can use allows to join points or a try and fail method.

$\Omega = \left\{ AB_1C_1, AB_1C_2, AB_1C_3, AB_2C_1, AB_2C_2, AB_2C_3 \right\}$ so they are 6.

2.5 List of lessons/sub-heading

No	Lesson title	Learning objectives	Number of periods
0	Introductory activity	Arouse the curiosity of student- teacher.	1
1.	Venn diagram, tree diagrams, contingency table and basic product principle of counting	Use different counting techniques (Venn diagram, tree diagram and tables) to determine the number of possibilities or occurrence outcomes for an event.	2
2	Basic product principle of counting		1
3	Basic sum principle of counting (Mutually exclusive situations)		1
4	Permutations without repetitions	Define and perform the permutations	1
5	Permutations with repetitions		1
6	Circular arrangements	Determine the number of arrangements of n unlike things in a circle.	2
7	Permutations of r unlike objects selected from n distinct objects	Determine the number of different permutations (ways) of r (like& unlike) objects selected from n different objects.	2
8	Permutations of r objects selected from the mixture of n alike and unlike objects		2
9	Combination of distinguishable objects	-Determine the number of different groups of r items that could be formed from a set of n distinct objects where the order of selections is being ignored.	2
10	Combination of r objects taken from the mixture of n alike and unlike objects.		2

11	Binomial expansion and Pascal's triangles	Apply Pascal's triangle to complete a binomial expansion in mathematics expressions.	2
12	End Unit assessment	Apply formulae of combinatory analysis to count possible outcomes of a random experiment.	1
Total number of periods			**20**

Lesson 1: Venn diagram, tree diagrams, contingency table and basic product principle of counting.

a) Learning objective

Use different counting techniques (Venn diagram, tree diagram and tables) to determine the number of possibilities or occurrence outcomes for an event.

b) Teaching resources

Graph papers, manila papers, calculators, Mathematical set.

c) Prerequisites/Revision/Introduction:

Student-teachers will perform well in this unit if they make a short revision on the sets of numbers (year 1: unit 1), set theory (year 1: unit 2).

d) Learning activities

- Form small groups of student-teachers and give them instructions on how to work on the **activity 2.1.1**;
- Walk around to each group and ask probing questions leading them to use principle of counting to answer the activity.
- Ask each group to share their answers with neighbouring group and ask them to support each other for improvement;
- Invite groups with different working steps to present their findings to the whole class for discussion;
- As a tutor, harmonize their answers highlighting the definitions of the key terms such as, ***Experiment***, ***Trial***, ***Outcom***, ***Sample space and*** *event*.
- Let them know the different techniques of counting like Venn diagram, tree diagrams and Contingency table.
- Use different probing questions and guide them to explore provided examples in the student's book and lead them to explain the main concepts

of probability accompanied with examples: events and their types, outcome, sample space, etc.

- After this step, guide students to do the **application activity 2.1.1** and evaluate whether lesson objectives were achieved.

Answer for activity 2.1.1

a) $30 = 10 + 9 + 15 - n(A_1 \cap A_2) \Rightarrow n(A_1 \cap A_2) = 4$

b) $n(A_1 \setminus A_2) = 10 - 4 = 6$

c) $n(A_2 \setminus A_1) = 9 - 4 = 5$

Application Activity 2.1.1

1)

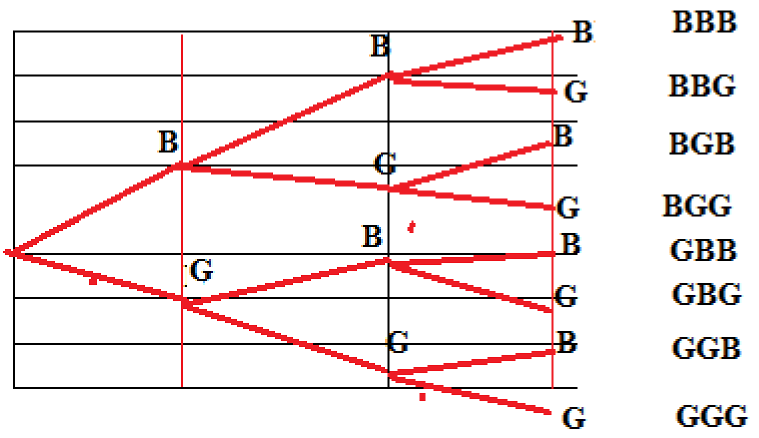
$$n(S) = n(M) + n(P) + n(C) - n(M \cap P) - n(M \cap C) - n(P \cap C) + n(M \cap P \cap C) + n(\overline{M \cup P \cup C})$$

a) $n(\overline{M \cup P \cup C}) = n(S) - n(M) - n(P) - n(C) + n(M \cap P) + n(M \cap C) + n(P \cap C) - n(M \cap P \cap C)$

$n(\overline{M \cup P \cup C}) = 100 - 77 - 47 - 44 + 43 + 37 + 12 - 12 = 12$

b) $n(M \cap P \setminus C) = n(M \cap P) - n(M \cap P \cap C) = 43 - 12 = 31$

2) Using the tree diagram, one can find:



$S = \{\text{BBB}, \text{BBG}, \text{BGB}, \text{BGG}, \text{GBB}, \text{GBG}, \text{GGB}, \text{GGG}\}$

3) The coin can land either head up or tails up.

	1	2	3	4	5	6
H	$(H,1)$	$(H,2)$	$(H,3)$	$(H,4)$	$(H,5)$	$(H,6)$
T	$(T,1)$	$(T,2)$	$(T,3)$	$(T,4)$	$(T,5)$	$(T,6)$

Lesson 2: Basic product principle of counting

a) Learning objective

Use different counting techniques (Venn diagram, tree diagram and tables) to determine the number of possibilities or occurrences outcomes for an event.

b) Teaching resources:

Graph papers, manila papers, calculators, Mathematical set, markers.

c) Prerequisites/Revision/Introduction:

Student-teachers will perform well in this unit if they make a short revision on the sets of numbers (year 1: unit 1), set theory (year 1: unit 2).

d) Learning activities

- Form small groups of student-teachers and give them instructions on how to work on the **activity 2.2**
- Walk around to each group and ask probing questions leading them to use one of the techniques of counting to determine the total number of roads from A to C via B;
- Ask each group to share their answers with neighbouring group and ask them to support each other for improvement;
- Invite groups with different working steps to present their findings to the whole class for discussion;
- As a tutor, harmonize their answers highlighting that **Basic product principle of counting** is one of the techniques of counting without necessarily listing the total number of all possible outcomes.
- Guide them to discover that if a sequence of n events in which the first one has n_1 possibilities, the second with n_2 possibilities the third with n_3 possibilities, and so forth until n_k, the total number of possibilities of the sequence will be= $n_1.n_2.n_3...n_k$.
- Use different probing questions and guide them to explore **examples 2.5 and 2.6** given in the student's book and lead them to determine total number of outcomes for a given random experiment using **basic product principle of counting**.

- After this step, guide students to do the **application activity 2.2** and evaluate whether lesson objectives were achieved.

Answer for activity 2.1.2

Since, the registration follows the following pattern RLLoooL, where "R" denotes Rwanda, L" denotes a letter from English alphabet and "o" denotes a digit, where plates containing only 0 as digit are excluded, the plate **RAB001A** follows the plate **RAA999Z** .

Following the provided pattern, $from\ 26\ letters \Rightarrow 999\times 26\ registrations$:

$$\left.\begin{array}{l} From\ the\ series\ RAA001A\ to\ RAA999A\ ,\ there\ are\ 999\ registrations \\ From\ the\ series\ RAA001B\ to\ RAA999B\ ,\ there\ are\ 999\ registrations \\ From\ the\ series\ RAA001C\ to\ RAA999C\ ,\ there\ are\ 999\ registrations \\ \vdots \\ From\ the\ series\ RAA001Y\ to\ RAA999Y\ ,\ there\ are\ 999\ registrations \\ From\ the\ series\ RAA001Z\ to\ RAA999Z\ ,\ there\ are\ 999\ registrations \end{array}\right\}$$

$$\left.\begin{array}{l} From\ the\ series\ RAB001A\ to\ RAB999A\ ,\ there\ are\ 999\ registrations \\ From\ the\ series\ RAB001B\ to\ RAB999B\ ,\ there\ are\ 999\ registrations \\ From\ the\ series\ RAB001C\ to\ RAB999C\ ,\ there\ are\ 999\ registrations \\ \vdots \\ From\ the\ series\ RAB001Y\ to\ RAB999Y\ ,\ there\ are\ 999\ registrations \\ From\ the\ series\ RAB001Z\ to\ RAB999Z\ ,\ there\ are\ 999\ registrations \end{array}\right\}$$

This means: $From\ 26\ letters \Rightarrow 2\times 999\times 26\ registrations$ = 51948 cars

Alternatively, from multiplication principle,

$(1\times 1\times 2\times 10\times 10\times 10\times 26)-(1\times 1\times 2\times 1\times 1\times 1\times 26)=51948$ cars

Application Activity 2.1.2

1. $20\times 19\times 18\times 17 = 116\,280$ ways
2. a) There are 4 ways to go from A to B and 3 ways from B to C; hence $n = 4\times 3 = 12$.

   b) There are 12 ways to go from A to C by way of B, and 12 ways to return.Thus

   $n = 12\times 12 = 144$.

   c) The man will travel from A to B to C to B to A. Enter these letters with connecting arrows as follows:

$$A \longrightarrow B \longrightarrow C \longrightarrow B \longrightarrow A$$

The man can travel four ways from A to B and three ways from B to C, but he can only travel two ways from C to B and three ways from B to A since he does not want to use a bus line more than once. Enter these numbers above the corresponding arrows as follows:

$$A \xrightarrow{4} B \xrightarrow{3} C \xrightarrow{2} B \xrightarrow{3} A$$

Thus, by the Product Rule, n $=4\cdot 3\cdot 2\cdot 3=72$

Lesson 3: Basic sum principle of counting (Mutually exclusive situations)

a) Learning objective

Use different counting techniques (Venn diagram, tree diagram and tables) to determine the number of possibilities or occurrence outcomes for an event.

b) Teaching resources:

Graph papers, manila papers, markers, calculators, and Mathematical set.

c) Prerequisites/Revision/Introduction:

Student-teachers will perform well in this unit if they make a short revision on the sets of numbers (year 1: unit 1), set theory (year 1: unit 2) and the previous lesson.

d) Learning activities

- Invite students to discuss in pairs the **activity 2.1.3**
- Walk around to each pair and ask probing questions leading them to determine the total number of outcomes given that the number of outcomes for event one and the outcomes for event 2 are known;
- Invite two neighbouring pairs to work together, exchange ideas and improve their work;
- Visit each new formed group and identify groups with different working steps;
- Invite representatives from groups with different working steps to present their work for a whole class discussion;
- As a tutor, harmonize their answers and guide students discover that , If experiment 1 has **m** possible outcomes and if experiment 2 has **n** possible outcomes, then an experiment which might be experiment 1 or experiment 2, called **experiment 1 or 2**, has $m+n$ possible outcomes;
- Guide them to explore different **examples** given in the student's book.

- . After this step, guide students to do the **application activity 2.1.3** and evaluate whether lesson objectives were achieved.

Answer for activity 2.1.3

1. Answers will vary from group to another. But help them to conclude that one has chances of picking either the soup or the juice but not all together. One is allowed two chances.
2. We need the numbers that end in 3: the last digit can be chosen in one way, as it must be a 3, the first digit can then be chosen in 3 ways, the second in 2 ways and the third in 1 way. Thus, there are $1\times3\times2\times1=6$ numbers that end in a 3.

Similarly, there are $1\times3\times2\times1=6$ numbers that end in a 4.

The number that ends in a 3 cannot also end in a 4, so these are mutually exclusive situations.

Thus, there are $6+6=12$ numbers end either in a 3 or in a 4.

Alternatively, this can be solved as follows:

The last digit can be chosen in 2 ways (3 or 4); the first digit can be chosen in 3 ways, the second in 2 ways and the third in 1 way, i.e, $2\times3\times2\times1=12$ numbers end either in a 3 or in a 4.

Application Activity 2.1.3

1. a) $5\times5\times4\times3=300$ numbers.

   b) $\underbrace{(5\times4\times3\times1)}_{\text{numbers ending by } 0}+\underbrace{(4\times4\times3\times2)}_{\text{numbers ending by } 2 \text{ or } 8}=156$ numbers.

   c) $\underbrace{(5\times4\times3\times1)}_{\text{numbers ending by } 0}+\underbrace{(4\times4\times3\times1)}_{\text{numbers ending by } 5}=108$ numbers.

2. 336.

Lesson 4: Permutations without repetitions

a) Learning objective:

Define and perform the permutations

b) Teaching resources:

Graph papers, manila papers, markers, calculators, and Mathematical set.

c) **Prerequisites/Revision/Introduction:**

Student-teachers will perform well in this unit if they make a short revision on the sets of numbers (year 1: unit 1), set theory (year 1: unit 2) and previous lessons in this unit.

d) **Learning activities**

- Form small groups of student-teachers and give them instructions on how to work on the **activity 2.2.1**, give each group the letter cards to be used and ask them to make all possible arrangements and permutations of those letters (for example: letter R, E and B);
- Walk around to each group and ask probing questions leading them to determine the total number of ways starting by the number of ways to chose the first letter, the second letter and the third letter;
- Ask each group to share their answers with neighbouring group and ask them to support each other for improvement;
- Invite groups with different working steps to present their findings to the whole class for discussion;
- As a tutor, harmonize their answers highlighting that the arrangement of letters is the same as ways of sitting different people on the same bench and that a permutation is an arrangement of n objects in a specific order.
- Use different probing questions and guide them to explore different **examples** given in the student's book and lead them to discover the formula which gives the number of different permutations of *n* different objects (unlike objects) in a row.
- Guide them to discover that this number corresponds to $n!$(read *n* factorial) and explore the related properties.
- After this step, guide students to do the **application activity 2.2.1** and evaluate whether lesson objectives were achieved.

Answer for activity 2.2.1

Possible arrangements for three letters R, E and B are
$\{REB, \text{R}BE, ERB, EBR, BER, BRE\}$;

The possible arrangement for these three letters is 6. This can be found from
$3! = 3.2.1 = 6$

Application Activity 2.2.1

1) a) $\frac{5!}{2!} = \frac{5\times4\times3\times2\times1}{2\times1} = 60$; b) $\frac{10!}{6!7!} = \frac{10\times9\times8\times7!}{6\times5\times4\times3\times2\times1\times7!} = \frac{5\times2\times3\times3\times4\times2}{3\times2\times5\times4\times3\times2\times1} = 1$

2) ${}^{n}P_2 = 72 \Leftrightarrow \dfrac{n!}{(n-2)!} = 72 \Leftrightarrow \dfrac{n(n-1)(n-2)!}{(n-2)!} = 72 \Leftrightarrow n(n-1) = 72$

Thus we get $n^2 - n = 72 \Leftrightarrow n^2 - n - 72 = 0 \Leftrightarrow (n-9)(n+8) = 0$

Since n must be positive, the only answer is $n = 9$.

3)Four different English books, five different Biology books and ten other books are to be arranged on a bookshelf:

a) (4+5+10)!= 19!= $1.216451004 \times 10^{17}$;

b) Since the 3 biology books have to be together, consider these bound together as one book, there are now $(16+1)! = 17!$ books to be arranged and these can be calculated using a calculator and find $2.134124569 \times 10^{15}$.

Lesson 5: Permutations with repetitions

a) Learning objective:

Define and perform the permutations.

b) Teaching resources:

Graph papers, manila papers, markers, calculators, and Mathematical set.

c) Prerequisites/Revision/Introduction:

Student-teachers will perform well in this lesson if they refer to the content of the previous lesson and the properties for multiplication learnt in unit 1 for this level (year 1).

d) Learning activities

- Form groups of student-teachers and give them instructions on how to work on the **activity 2.2.2,** give each group the letter cards to be used and ask them to make all possible permutations of those letters in which some letters are the same (for example: letter of the word MOON, the two O are not separable);
- Walk around to each group and ask probing questions leading them to determine the total number of permutations considering that it is not possible to distinguish the two letters "O";
- Ask each group to share their answers with neighbouring group and ask them to support each other for improvement;
- Invite groups with different working steps to present their findings to the whole class for discussion;
- As a tutor, harmonize their answers highlighting that the number of

permutations of will reduce: the total number of permutations 4! Will be divided by the number of permutations of identical letters which is 2! and find $\frac{4!}{2!}$.

- Use different probing questions and guide students to explore **examples** given in the student's book and lead them to discover the formula which gives the number of different permutations of *n* **indistinguishable** objects with n_1 alike, n_2 alike, …, which is $\frac{n!}{n_1!n_2!...}$.
- After this step, guide students to do the **application activity 2.2.2** and evaluate whether lesson objectives were achieved.

Answer for activity 2.2.2

Let us take numbers from 1, 2,3 and 4.

1) $\Omega = \begin{Bmatrix} 1234, 1243, 1324, 1342, 1423, 1432 \\ 2134, 2143, 2314, 2341, 2413, 2431 \\ 3124, 3142, 3214, 3241, 3412, 3421 \\ 4123, 4132, 4213, 4231, 4312, 4321 \end{Bmatrix}$, the possible number is 4!=24;

2. Students will try to make possible arrangements but some of them will be the same.

$$\Omega = \begin{Bmatrix} MOON, MONO, MNOO, \text{NOOM}, \text{NOMO}, \text{NMOO}, \\ OMON, OMNO, ONMO, ONOM, OOMN, OONM \end{Bmatrix}$$

The possible numbers is 12.

The number of all possible arrangements when writing once the identical arrangement is $\frac{4!}{2!}$.

Application Activity 2.2.2

1. a) Arrangements that can be made from the letters of the word ENGLISH are 7!=5,040;

   b) Arrangements that can be made from the letters of the word MATHEMATICS are $\frac{11!}{2!2!2!}$ as there are 2M, 2A and 2T which are indistinguishable.

c) Arrangements that can be made from the letters of the wordSOCIOLOGICAL are $\frac{12!}{3!2!2!2!}$ as there are three O, two C, two I and 2L which are indistinguishable.

2) Number of arrangements that can be made from the letters of English alphabets $26! = 4.032914611 \times 10^{26}$;

3) $\#\Omega = \frac{9!}{4!3!2!} = 1,260$.

Lesson 6: Circular arrangements

b) Learning objective:

Determine the number of arrangements of *n* unlike things in a circle**.**

c) Teaching resources:

Graph papers, manila papers, markers, calculators, and Mathematical set.

d) Prerequisites/Revision/Introduction:

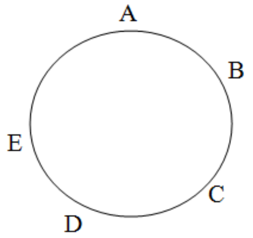
Student-teachers will perform well in this lesson if they learnt well the content of previous lessons of this unit.

e) Learning activities

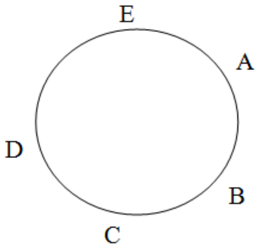
- Form groups of student-teachers and give them instructions on how to work on the **activity 2.2.3** each group may have a circular table and objects to be arranged on that table;
- Walk around to each group and ask probing questions leading them to determine the total number of arrangements when one item is fixed and the remaining items arranged around it;
- Invite groups with different working steps to present their findings to the whole class for discussion;
- As a tutor, harmonize their answers and guide them to discover that the number of arrangements of *n* unlike things in a circle will therefore be $(n-1)!$. Guide students to note that where clockwise and anticlockwise arrangements are not considered to be different, this reduces to $\frac{1}{2}(n-1)!$.
- Use different probing questions and guide students to explore **examples** given in the student's book,
- After this step, guide students to do the **application activity 2.2.3** and evaluate whether lesson objectives were achieved.

Answer for activity 2.2.3

As one notebook will be fixed, for example A, must be: $(n-1)! = (5-1)! = 24$



is not a different arrangement from



Application Activity2.2.3

1. Five men will seat on a circular table in $(5-1)!\text{ways} = 24\text{ ways}$.
2. Eleven different books will be placed on a circular table in $(11-1)!\text{ways} = 3,268,800\text{ ways}$.
3. a) This is a permutation from which the seven people will be sitting in a row; that is the number of ways that 7 people can arrange themselves in a row of chairs is $7! = 5040$.

   b) This is a circular permutation from which one person can sit at any place at the table. The other 6 people can arrange themselves in $6!$ ways around the table; that is the number of ways that 7 people can arrange themselves around a circular table is $6! = 720$.

Lesson 7: Permutations of r unlike objects selected from n distinct objects

a) Learning objective:

Determine the number of different permutations (ways) of *r* unlike objects selected from *n* different objects.

b) Teaching resources:

Graph papers, manila papers, markers, calculators, and Mathematical set.

c) Prerequisites/Revision/Introduction:

Students will perform well if they learnt well the previous lessons of this unit.

d) Learning activities

- Form groups of student-teachers and give them instructions on how to work on the **activity 2.3.1**, give each group the letter cards to be used and ask them to make all possible ways of selecting 3 letters from 7 given letters of the word *PRODUCT*.

- Walk around to each group and ask probing questions leading them to determine the total number of ways of selecting the first letter, the second letter and the third letter;
- Invite groups with different working steps to present their findings to the whole class for discussion;
- As a tutor, harmonize their answers highlighting that the number of ways for selecting 3 letters from 7 is the number of permutations of 3 objects taken from 7 written as $^{7}P_{3}$. Guide them to enhance that the first letter can be selected in 7 ways, the second can be selected in 6 ways and the third can be selected in 5 ways;
- Ask all students to guess how they can write the product $7.6.5$ using the factorial notation which lead them to guess $7.6.5=\frac{7.6..5.4.3.2.1}{4.3.2.1}=\frac{7!}{(7-3)!}$;
- Use different probing questions and guide students to explore **examples** given in the student's book and lead them to discover the formula which gives the number of different permutations (ways) of *r* unlike objects selected from *n* different objects given by $^{n}P_{r}=\frac{n!}{(n-r)!}$ which can also be written as $P(n,r)=\frac{n!}{(n-r)!}$;
- After this step, guide students to do the **application activity 2.3.1** and evaluate whether lesson objectives were achieved.

Answer for activity 2.3.1

Students will work in different ways; as a tutor, verify whether all possible ways are given:

Selection: PRO: permutations: PRO,POR, OPR, ORP, RPO, ROP

Selection: ROD: permutations: ROD, RDO, ORD, ODR, DRO, DOR

Selection: ODU: permutations: ..., ..., ..., ..., ..., ...

Selection: DUC: Permutations: ..., ..., ..., ..., ..., ...

Selection: UCT: Permutations: ..., ..., ..., ..., ..., ...

...... : .....: ....

There are 35 lines with 6 permutations which gives the number $7.6.5 = 210$ permutations.

Application Activity 2.3.1

1. a) $\frac{(n-1)!}{n!} = \frac{(n-1)!}{n(n-1)!} = \frac{1}{n}$

   b) $\frac{(n+2)!}{n!} = \frac{(n+2)(n+1)n!}{n!} = (n+2)(n+1) = n^2 + 3n + 2$

   c) $\frac{(n+2)!-n!}{(n-1)!} = \frac{(n+2)(n+1)n(n-1)!-n(n-1)!}{(n-1)!}$

   $= (n-1)!\frac{(n+2)(n+1)n-n}{(n-1)!}$

   $= \frac{(n+2)(n+1)n(n-1)!-n(n-1)!}{(n-1)!} = (n+2)(n+1)n-n$

   $= n\left[(n+2)(n+1)-1\right] = n\left(n^2+3n+2-1\right) = n^3+3n^2+n$

2. Number of permutations with 4 letters chosen from letters of the word **ANGELIUS** is ${}^8P_4 = 1680$.
3. Number of permutations with 10 letters chosen from English alphabet is ${}^{26}P_{10}$.
4. The number of arrangements of four different letters chosen from the word **PROBLEM** which
   a) begin with vowel is $2\times6\times5\times4 = 240$.
   b) end with consonant is $6\times5\times4\times5 = 600$.

Lesson 8: Permutations of r objects selected from the mixture of n alike and unlike objects

a) Learning objective:

Determine the number of different permutations (ways) of *r* objects selected from *n* like objects.

b) Teaching resources:

Graph papers, manila papers, markers, calculators, and Mathematical set.

c) Prerequisites/Revision/Introduction:

Students will perform well if they learnt well the previous lessons of this unit.

d) Learning activities

- Form groups of student-teachers and give them instructions on how to work on the **activity 2.3.2** give each group the letter cards to be used and ask them to make all possible ways of selecting 3 letters from 7 given letters of the word *PRODUCT*.
- Walk around to each group and ask probing questions leading them to determine the total number of ways of selecting the first letter, the second letter and the third letter;
- Invite groups with different working steps to present their findings to the whole class for discussion;
- As a tutor, harmonize their answers highlighting that the number of ways for selecting 3 letters from 7 is the number of permutations of 3 objects taken from 7 written as ${}^{7}P_{3}$. Guide them to enhance that the first letter can be selected in 7 ways, the second can be selected in 6 ways and the third can be selected in 5 ways;
- Ask all students to guess how they can write the product $7.6.5$ using the factorial notation which lead them to guess $7.6.5 = \frac{7.6..5.4.3.2.1}{4.3.2.1} = \frac{7!}{(7-3)!}$;
- Use different probing questions and guide students to explore **example** given in the student's book and lead them to discover the formula which gives the number of different permutations (ways) of *r* unlike objects selected from *n* different objects given by ${}^{n}P_{r} = \frac{n!}{(n-r)!}$ which can also be written as $P(n,r) = \frac{n!}{(n-r)!}$;
- After this step, guide students to do the **application activity 2.3.2** and evaluate whether lesson objectives were achieved.

Answer for activity 2.3.2

Student-teachers will work in different ways; as a tutor, verify whether all possible ways are given:

Selection: BOO; permutations: BOO, OBO, OOB

Selection: OOM; permutations: OOM, OMO, MOO

Selection: BOM; permutations: BOM, BMO, OBM, OMB, MBO, MOB

There are $3+3+6=12$ permutations.

Alternatively:

Possible selections:

- A selection containing two "O" and one other letter for example "OOB"
- A selection for which all three letters are distinct (B,O,M)

Permutations formed from three letters containing two O and another letter (M or B)		
O	**O**	**M**
O	**M**	**O**
M	**O**	**O**
O	**O**	**B**
O	**B**	**O**
B	**O**	**O**

Permutations formed from three distinct letters (B,M, O)		
B	**M**	**O**
B	**O**	**M**
M	**B**	**O**
M	**O**	**B**
O	**B**	**M**
O	**M**	**B**

There are 12 permutations of 3 letters selected from the letters of the word "**BOOM**" in which 6 contain two O's and one other letter and 6 in which all 3 letters are different.

Application Activity 2.3.2

1. There are 7 letters including two A's and two N's. To find the total number of different arrangements we consider the possible arrangements as three

mutually exclusive situations.

- The number of arrangements in which all 2 letters are different is ${}^5P_2 = 20$;
- The number of arrangements containing two A's is 1;
- The number of arrangements containing two N's is 1;

Therefore, the number of permutations with 2 letters chosen from letters of the word RWANDAN is $20+1+1=22$

2. There are 8 letters including two M's. To find the total number of different arrangements we consider the possible arrangements as two mutually exclusive situations.

- The number of arrangements in which all 3 letters are different is ${}^7P_3 = 210$
- The number of arrangements containing two M's and one other letter; the other letter can be one of six letters (E, A, N, U, E or L) and can appear in any of the three positions (before the two M'S, between the two M's, or after the two M's)i.e$3\times 6=18$.

Thus the total number of arrangements of 3 letters chosen from the word EMMANUELis$210+18=228$.

Lesson 9: Combination of distinguishable objects

a) Learning objective:

Determine the number of different groups of *r* items that could be formed from a set of *n* distinct objects where the order of selections is being ignored.

b) Teaching resources:

Graph papers, manila papers, markers, calculators, and Mathematical set.

c) Prerequisites/Revision/Introduction:

Students will perform well if they learnt well the content for the previous lessons for this level.

d) Learning activities

- Form small groups of student-teachers and give them instructions on how to work on the **activity 2.4.1.**
- Walk around to each group and ask probing questions leading them to determine the total number of arrangements of three days selected from 5 working days ;
- Invite groups with different working steps to present their findings to the whole class for discussion;
- As a tutor, harmonize their answers by highlighting that in this case,

the order in which days are placed is not important (For example the selection $(Monday, Tuesday)$ is the same as selection $(Tuesday, Monday)$; $(Monday, Tuesday, Wednesday)$, $(Monday, Wednesday, Tuesday)$, $(Tuesday, Monday, Wednesday)$, $(Tuesday, Wednesday, Monday)$, $(Wednesday, Monday, Tuesday)$ and $(Wednesday, Tuesday, Monday)$ are the 6 arrangements containing the same days but in a different order) which is contrary to the permutation of r unlike objects selected from n different objects where the order in which those objects are placed is important.

- Lead students to see that in this case, generally, we must divide by the as the order is not important; we get $\frac{^nP_r}{r!}$.
- Use different probing questions and guide students to explore **examples** given in the student-teacher's book and lead them to discover the formula which gives the number of different selections of r items that could be formed from a set of n distinct objects with the order of selections being ignored is $^nC_r = \frac{n!}{(n-r)!r!}$
- After this step, guide students to do the **application activity 2.4.1** and evaluate whether lesson objectives were achieved.

Answer for activity 2.4.1

a) Each selection of two days is made from 5 working days (Monday, Tuesday, Wednesday, Thursday, and Friday).

There are 5 weekdays from which to make a first selection and for each such selection there are 4 days left from which to make the second selection. This gives a total of $5 \times 4 = 20$ possible arrangements.

List of different arrangements

$(Monday, Tuesday)$	$(Monday, Wednesday)$	$(Monday, Thursday)$	$(Monday, Friday)$
$(Tuesday, Monday)$	$(Tuesday, Wednesday)$	$(Tuesday, Thursday)$	$(Tuesday, Friday)$
$(Wednesday, Monday)$	$(Wednesday, Tuesday)$	$(Wednesday, Thursday)$	$(Wednesday, Friday)$
$(Thursday, Monday)$	$(Thursday, Tuesday)$	$(Thursday, Wednesday)$	$(Thursday, Friday)$

(Friday, $Monday$)	(Friday, $Tuesday$)	($Friday, Wednesday$)	($Friday, Thursday$)

However, not all arrangements are different. For example $(Monday, Tuesday)$ is the same as $(Tuesday, Monday)$. Every arrangement is duplicate; to avoid the fact of duplicating selections, we get that the number of possible selections is a half of permutations of 2 days selected from five days. This gives $\frac{^5P_2}{2} = \frac{5!}{3!2!} = 10$ selections.

b) There are 5 weekdays from which to make a first selection and for each such selection there are 4 days left from which to make the second selection and then 3 days from which to make the third selection. This gives a total of $5 \times 4 \times 3 = 60$ possible arrangements. However, Some arrangements contain the same days but in a different order. For example

$(Monday, Tuesday, Wednesday), (Monday, Wednesday, Tuesday), (Tuesday, Monday, Wednesday),$
$(Tuesday, Wednesday, Monday), (Wednesday, Monday, Tuesday)$ and $(Wednesday, Tuesday, Monday)$

are the 6 arrangements containing the same days but in a different order. Every arrangement appears thrice. Any one arrangement can be rearranged within itself $3 \times 2 \times 1 = 6$ times. This means that every selection of three specific days appear 6 times in the list of 60 possible arrangements. Therefore, there are 10 selections of 3 days out of the 5.

Application Activity 2.4.1

1. In fact, $^nC_r + {^nC_{r-1}} = \frac{n!}{(n-r)!r!} + \frac{n!}{(n-r+1)!(r-1)!}$

To obtain the same denominator in both fractions, multiply the first fraction by

$\frac{n-r+1}{n-r+1}$ and the second fraction by $\frac{r}{r}$.

Hence, $^nC_r + {^nC_{r-1}} = \frac{(n-r+1)\cdot n!}{(n-r+1)\cdot(n-r)!r!} + \frac{r \cdot n!}{(n-r+1)!\text{r}\cdot(r-1)!}$

$$= \frac{(n-r+1)\cdot n!}{(n-r+1)!r!} + \frac{r \cdot n!}{(n-r+1)!r!}$$

$$= \frac{(n-r+1)\cdot n! + r \cdot n!}{(n-r+1)!r!}$$

$$= \frac{\left[(n-r+1)+r\right]\cdot n!}{(n-r+1)!r!}$$

$$= \frac{(n+1)\cdot n!}{(n-r+1)!r!}$$

$$= \frac{(n+1)!}{(n-r+1)!r!}$$

$$= \frac{(n+1)!}{(n+1-r)!r!} = {}^{n+1}C_r$$

Therefore ${}^{n+1}C_r = {}^{n}C_r + {}^{n}C_{r-1}$ as required.

2. Four men can be selected from 10 men, i.e ${}^{10}C_4 = \frac{10!}{(10-4)!4!}$ ways

Two women can be selected from 12women, i.e ${}^{12}C_2 = \frac{12!}{(12-2)!2!}$ ways

By the basic product principle of counting, there are $\left({}^{10}C_4\right)\left({}^{12}C_2\right)$ ways of selecting the committee.

3. The groups containing 4 Mathematics books and 5 Physics books are formed from 9 Mathematics books and 10 Physics books in ${}^{9}C_4 \times {}^{10}C_5$ ways.

4. a) There are ${}^{14}C_2 = 91$ ways of selecting two socks from 14 socks in the box if they can be any colour.

   b) There are ${}^{8}C_2 = 28$ ways to choose 2 of the 8 blue socks, and ${}^{6}C_2 = 15$ ways to choose 2 of the 4 red socks. By the basic sum principle of counting, $n = 28 + 15 = 43$.

5. a) There are ${}^{11}C_8 = 165$ selections;

   b) There are $2 \times {}^{11}C_9 = 110$ selections;

   c) There are ${}^{5}C_3 \times {}^{8}C_7 = 80$ selections;

   d) There are ${}^{5}C_3 \times {}^{8}C_7 + {}^{5}C_4 \times {}^{8}C_6 + {}^{5}C_5 \times {}^{8}C_5 = 276$ selections.

Lesson 10: Combination of r objects taken from the mixture of n alike and unlike objects .

a) Learning objective:

Determine the number of different groups of *r* items that could be formed from a set of *n* distinct objects where the order of selections is being ignored.

b) Teaching resources:

Graph papers, manila papers, markers, calculators, and Mathematical set.

c) Prerequisites/Revision/Introduction:

Students will perform well if they learnt well the content for the previous lessons for this unit.

d) Learning activities

- Form groups of student-teachers and give them instructions on how to work on the **activity 2.4.2**;
- Walk around to each group and ask probing questions leading them to determine the possible ways of combinations of 3 letters selected from the letters of the word "**BANANA**".
- Invite groups with different working steps to present their findings to the whole class for discussion;
- As a tutor, harmonize their answers highlighting that in determining the number of all possible combinations of r objects selected from mixture of n alike and unlike objects you consider all possible mutually exclusive events corresponding to the given experiment and then apply basic sum principle of counting.
- Use different probing questions and guide students to explore **example** given in the student's book.
- Lead them to discover the formula which gives the number of different groups of *r* items that could be formed from a set of *n* distinct objects with the order of selections being ignored is ${}^{n}C_{r} = \dfrac{n!}{(n-r)!r!}$
- After this step, guide students to do the **application activity 2.4.2** and evaluate whether lesson objectives were achieved.

Answer for activity 2.4.2

Student-teachers will work in different ways; as a tutor, verify whether all possible ways are given:

	Selections of two letters from the word "**BANANA**"			
	B	A	N	
	A	A	A	
	A	A	N	
	A	A	B	
	N	N	A	
	N	N	B	
Total number of selections	**6**			

Total number of selections of three letters from the word "**BANANA**" containing no vowels is only one (NNB) .

Alternatively:

There are 6 letters including 3 A's, 2 N's and one B.

Let us consider all possible mutually exclusive selections of 3 letters:

Number of selections containing 3 A's is 1;

Number of selections containing 2 A's and one other letter (B or N) is $1\times {}^2C_1 = 2$;

Number of selections containing 2 N's and one other letter (A or B) is $1\times {}^2C_1 = 2$;

Number of selections containing 3 different letters (A, B or N) is 1;

Total number of selections of 3 letters is $1+2+2+1=6$.

Selections of three letters from the word "**BANANA**" containing no vowels is made taken from letters B, N and A that is ${}^3C_3 = 1$ selection.

Application Activity 2.4.2

1. There are 10 letters including two C's, two U's, three S's, 2 N's and three other different letters.

a) From all possible mutually exclusive selections of 3 letters we get that

Number of selections containing 3 S's is 1;

Number of selections containing 2 S's and one other letter (C,U,E,F or L) is $1\times {}^5C_1 = 5$;

Number of selections containing 2 C's and one other letter (S,U,E,F or L) is $1\times {}^{5}C_{1}=5$;

Number of selections containing 2 U's and one other letter (S,C,E,F or L) is $1\times {}^{5}C_{1}=5$;

Number of selections containing 3 different letters (S, C,U,E,F or L) is ${}^{6}C_{3}=20$;

Total number of selections of 3 letters from the word **SUCCESSFUL** is $1+5+5+5+20=36$.

b) Without vowels , there are 7 letters including two C's, three S's, 2 N's and two other different letters.

number of selections containing 3 S's is 1;

number of selections containing 2 S's and one other letter (C, F or L) is $1\times {}^{3}C_{1}=3$;

number of selections containing 2 C's and one other letter (S, F or L) is $1\times {}^{3}C_{1}=3$;

number of selections containing 3 different letters (S, C,F or L) is ${}^{4}C_{3}=4$;

Number of selections of 3 letters without any vowel is $1+3+3+4=11$.

c) The number of selections with at least one vowel is $36-11=25$.

2. There are 10 letters including three S's, three T's, two I's and two other distinct letters.

a) The number of different selections of 3 letters

$${}^{3}C_{3}+{}^{3}C_{3}+{}^{4}C_{1}+{}^{4}C_{1}+{}^{4}C_{1}+{}^{5}C_{3}=24$$

b) The number of selections with at least one T is $24-{}^{3}C_{3}-{}^{3}C_{1}-{}^{3}C_{1}-{}^{4}C_{3}=13$.

Lesson 11: Binomial expansion and Pascal's triangles

a) Learning objective:

Apply Pascal's triangle to complete a binomial expansion in mathematics expressions.

b) Teaching resources:

Manila papers, calculators, notebooks and pens.

c) Prerequisites/Revision/Introduction:

Students will perform well if they learnt well the binomial expansion and properties of powers learnt in S2 (Unit 1& Unit 2).

d) Learning activities

- In small group discussions, invite student-teachers to answer the questions in **activity 2.4.3.**
- Ask student-teachers to share their answers with another group and ask them to support each other where they became more challenged in solving that activity.
- Request student-teachers to present their findings in a whole class discussion.
- As a tutor, harmonize answers presented by students and guide theme to determine the coefficients of powers in a binomial expansion.
- Use different probing questions and guide students to explore **examples** given in the student's book and lead them to discover the coefficient of

$a^{n-r}b^r$ in the expansion of $(a+b)^n$ is given by $^nC_r = \frac{n!}{(n-r)!r!}$.

Answer for activity 2.4.3

Use these expansions and complete the table

$(a+b)^2 = a^2 + 2ab + b^2$

$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$

$(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$

$(a+b)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$

Power	Coefficient of powers of a and b					Binomial expression
0	1					$(a+b)^0$
1	1	1				$(a+b)^1$
2	1	2	1			$(a+b)^2$
3	1	3	3	1		$(a+b)^3$
4	1	4	6	4	1	$(a+b)^4$

It is clear that the coefficients of $a^{n-r}b^r$ in the expansion of $(a+b)^n$ are given by $^nC_r = \frac{n!}{(n-r)!r!}$.

Application Activity 2.4.3

1. a) The 9th row is as follows

1	9	36	84	126	126	84	36	9	1

b) The 10th row is as follows

1	10	45	120	210	252	210	120	45	10	1

2. a) The coefficient of x^2 in the expansion of $(4x+1)^6$ is 240

b) The coefficient of x^3 in the expansion of $\left(x+\frac{1}{x}\right)^4$ is 0

c) The coefficient of x^6 in the expansion of $(9x-3)^{10}$ is 9039811410.

2.6. Summary of the unit

Sample space

The totality of all possible outcome (or sample points) of a random experiment constitutes the sample space which is denoted by Ω.

When $A \cap B = 0$, the two events A and B are said to be **mutually exclusive**. This means that they cannot occur at the same time, they do not have outcomes in common.

Counting techniques

- **Use of Venn diagram,**
- **Use of tree diagrams,**
- **Use of a table,**
- **Basic product principle**

If a sequence of n events in which the first one has n_1 possibilities, the second with n_2 possibilities the third with n_3 possibilities, and so forth until n_k, the total number of possibilities of the sequence will be given by *the product* $n_1 \cdot n_2 \cdot n_3 \cdots n_k$ ".

- **Basic sum principle of counting (Mutually exclusive situations)**

"*If Experiment* 1 *has* **m** *possible outcomes*
and if experiment 2 *has* **n** *possible outcomes*,
then an experiment which might be experiment 1 *or experiment* 2,
called **experiment 1 or 2**, *has* $(m+2)$ *possible outcomes*."

- **Permutations**

- The number of different **permutations of *n* different objects** (unlike objects) in a row is

$$\times(n-1)\times(n-2)\times(n-3)\times\ldots\times 2\times 1$$

- The number of different **permutations of n indistinguishable objects** with n_1 alike, n_2 alike, …, is $\dfrac{n!}{n_1!n_2!\ldots}$.

- **Circular arrangements**

The number of arrangements of n unlike things on a circle will therefore be $(n-1)!$. In those cases where clockwise and anticlockwise arrangements are not considered to be different, this reduces to $\frac{1}{2}(n-1)!$.

- The number of different **permutations (ways) of r unlike objects selected from n different** objects is $^{n}P_{r}=\dfrac{n!}{(n-r)!}$ or we can use the denotation $P_r^n=\dfrac{n!}{(n-r)!}$ or $P(n,r)=\dfrac{n!}{(n-r)!}$

- **Combination**

The number of different groups of **r items that could be formed from a set of n distinct objects** with the order of selections being ignored is

$$^{n}C_{r}=\frac{n!}{(n-r)!r!}.$$

- **Binomial expansion**

For every integer $n\geq 1$, $(a+b)^n=\sum_{r=0}^{n} {}^{n}C_{r}a^{n-r}b^{r}$, this expression is known as Binomial theorem.

The general term U_{r+1} in the expansion $(a+b)^n$ is given by $^{n}C_{r}a^{n-r}b^{r}$

2.7 Additional Information for teachers

2.7.1 Notations

Be careful while using the following notations:

For $n \geq r$, we sometimes use ${}^{n}C_{r}$, ${}_{n}C_{r}$, C_{n}^{r}, or $\begin{pmatrix} n \\ r \end{pmatrix}$.

2.7.2 Inclusive events

For **inclusive events** the addition law of counting applies:

$$n\left(E_1 \text{ or } E_2 \text{ or } E_3 \cdots \text{ or } E_n\right) = n\left(E_1 \cup E_2 \cup E_3 \cdots \cup E_n\right) = n\left(\cup_i^n E_i\right)$$

$$= \sum_{i=1}^{n} n\left(E_i\right) - \sum_{j>i=1}^{n} n\left(E_i \cap E_j\right) + \sum_{k>j>i=1}^{n} n\left(E_i \cap E_j \cap E_k\right) + \cdots + (-1)^n n\left(E_1 \cap E_2 \cap \cdots \cap E_n\right)$$

2.8 End unit assessment

1. a) $3\times4\times4\times3=144$ b) $3\times4\times3\times2=72$
2. The number of them who own:
   a) both a foreign made car and a made in Rwanda car is

   $n(R\cap F)=n(R)+n(F)-n(S)=60+24-80=4$

   b) only a foreign made car is $n(F-R)=24-4=20$

   c) only a made in Rwanda car is $n(R-F)=60-4=56$
3. the number of
   a) codes is $26\times26\times10\times10\times10=676000$;

   b) codes with distinct letter is $26\times25\times10\times10\times10=650000$;

   c) codes with the same letters $26\times10\times10\times10=26000$.
4. The number of ways they can sit in a row where:
   a) there are no restrictions is $6!=720$;

   b) the boys and girls are each to sit together $3!3!\times2=72$;

   c) just the girls are to sit together is $4!3!3!=144$.
5. a) Here the Sum Rule is used; hence, $n=8+6=14$.

   b) Here the Product Rule is used; hence, $n=8\times6=48$.

   c) There are 14 ways to elect the president, and then 13 ways to elect the vice president. Thus $n=14\times13=182$.

6. The number of ways a customer can choose:

a) 1 dessert is ${}^6C_1 = 6$;

b) 2 of the desserts is ${}^6C_2 = 15$;

c) 3 of the desserts is ${}^6C_3 = 20$.

7. The term U_{r+1} in x^r is will be given by ${}^6C_r\left(3x^{-2}\right)^{6-r}(-2x)^r$ or ${}^6C_r3^{6-r}(-2)^r x^{-12+3r}$.

This term is independent of x if $-12+3r=0$ that is $r=4$. Hence, the required term is $U_5 = {}^6C_4(3)^2(-2)^4 = U_7 = {}^{18}C_6 3^{12}(-2)^6 = {}^{18}C_6 3^{12}2^6 = 2160$

8. ${}^{12}C_8 \times 4^4 \times x^8$

2.9 Additional activities

2.9.1 Remedial activity

1. Compute

a) $4!\times 0!$ b) $8-3!$ c) $\frac{6!}{3!}$ d) $\frac{8!}{10!}$

Solution

a) $4!\times 0! = 4! = 4\times 3\times 2\times 1 = 24$ b) $8-3! = 8-3\times 2\times 1 = 2$

c) $\frac{6!}{3!} = \frac{6\times 5\times 4\times 3!}{3!} = 120$ d) $\frac{8!}{10!} = \frac{8!}{10\times 9\times 8!} = \frac{1}{90}$

2. A farmer buys 3 cows, 2 pigs, and 4 hens from a man who has 6 cows, 5 pigs, and 8 hens. Find the number m of choices that the farmer has.

Solution

The farmer can choose the cows in ${}^6C_3$ ways, the pigs in ${}^5C_2$ ways, and the hens in ${}^8C_4$ ways. Thus the number m of choices follows: $m = {}^6C_3 \cdot {}^5C_2 \cdot {}^8C_4 = 14000$.

3. Find the term independent of x in the binomial expansion of $\left(3x - \frac{2}{x^2}\right)^{18}$.

Solution

The general term U_{r+1} in the expansion $(a+b)^n$ is given by ${}^nC_r a^{n-r}b^r$.

Substituting $3x$ for a, $\left(\frac{-2}{x^2}\right)$ for b and $n=18$, we get

$U_{r+1} = {}^{18}C_r (3x)^{18-r} \left(\frac{-2}{x^2}\right)^r = {}^{18}C_r 3^{18-r} (-2)^r (x)^{18-3r}$.

This term is independent of x if $18-3r=0$ that is $r=6$.

Hence, the required term is $U_7 = {}^{18}C_6 3^{12} (-2)^6 = {}^{18}C_6 3^{12} 2^6$.

2.9.2. Consolidation activity

1. A survey on a sample of 25 new cars being sold at a local auto dealer was conducted to see which of three popular options, air-conditioning (A), radio (R), and power windows (W), were already installed. The survey found: 15 had air-conditioning (A), 5 had A and W, 12 had radio (R), 9 had A and R, 3 had all three options. 11 had power windows (W), 4 had R and W. Using a formula, determine the number of cars that had:

a) Only W; b) only A; c) only R; d) R and W but not A; e)A and R but not W;

f) only one of the options; g) at least one option; h) none of the options.

Solution

a) $n(only\, W) = n(W) - n(W\cap A) - n(W\cap R) + n(W\cap A\cap R) = 11-5-4+3=5$

$b)\, n(only\, A) = n(A) - n(A\cap W) - n(A\cap R) + n(W\cap A\cap R) = 15-5-9+3=4$

$c)\, n(only\, R) = n(R) - n(R\cap W) - n(R\cap A) + n(W\cap A\cap R) = 12-4-9+3=2$

$d)\, n(R\cap W \setminus A) \equiv n(R\cap W\, only) = n(R\cap W) - n(W\cap A\cap R) = 4-3=1$

$e)\, n(A\cap R \setminus W) \equiv n(A\cap R\, only) = n(A\cap R) - n(W\cap A\cap R) = 9-3=6$

$f)\, n(only\, one\, of\, the\, options) = n(only\, A) + n(only\, R) + n(only\, W) = 5+4+2=11$

$g)\, n(at\, least\,\, one\, of\, the\, options) = n(A\, or\, R\, or\, W) = n(A\cup R\cup W)$

$= n(A)+n(R)+n(W)-n(A\cap R)-n(A\cap W)-n(R\cap W)+n(A\cap R\cap W)$

$=15+12+11-5-9-4+3=23$

$h)\, n(none\, of\, the\, options) = 25 - n(A\cup R\cup W) = 25-23=2$

2. Find the term independent of x in the expansion of each of the following:

a) $\left(2x^2 + \frac{4}{x}\right)^{12}$ b) $(1+x^2)\left(2x+\frac{1}{x}\right)^{10}$

Solution

a) ${}^{12}C_8 \times 4^{10}$ or ${}^{12}C_8 2^{20}$ b) $2^4\left(2\times {}^{10}C_5 + {}^{10}C_6\right)$

2.9.3 Extended activity

1. In a dormitory of a certain TTC, there are N student-teachers. Let E, F, M, K denote, respectively, English, French, Mathematics and Kinyarwanda courses. Find the number N of students in a dormitory given the data:

   12 take E, 5 take E and F, 4 take F and K, 2 take F, M, K, 20 take F, 7 take E and M, 3 take M and K, 3 take E, M, K, 20 take K, 4 take E and K, 3 take E, F, M, 2 take all four, 8 take K, 16 take F and M, 2 take E, F, K, 71 take none.

Solution

$$N = n(E)+n(F)+n(M)+n(K)-n(E\cap F)-n(E\cap M)-n(E\cap K)-n(F\cap M)$$
$$-n(F\cap K)-n(M\cap K)+n(E\cap F\cap M)+n(E\cap F\cap K)+n(F\cap M\cap K)$$
$$-n(E\cap F\cap M\cap K)+n\left(\overline{E\cup F\cup M\cup K}\right)$$

$$N=(12+20+20+8)-(5+7+4+16+4+3)+(3+2+2+3)-2+71=100$$

2. Prove the following identities

a) ${}^nC_r = {}^nC_{n-r}$ b) ${}^nC_0 + {}^nC_1 + {}^nC_2 + \cdots + {}^nC_n = 2^n$

c) ${}^nC_0 - {}^nC_1 + {}^nC_2 - {}^nC_3 + \cdots + (-1)^n\,{}^nC_n = 2^n$

Solution

a) In fact,

$${}^nC_r = \frac{n!}{(n-r)!r!}$$

$$= \frac{n!}{(n-r)!(n-n+r)!}$$

$$= \frac{n!}{(n-r)!\left[n-(n-r)\right]!}$$

$$= {}^nC_{n-r}$$

b) Hint: Expand $(1+1)^n$ b) Hint: Expand $(1-1)^n$.

UNIT 3

PROBABILITY

3. 1 Key unit competence

Determine probability of occurrence of an event from random experiment and Apply Bayes' theorem.

3.2 Prerequisite

Student-teachers will perform well in this unit if they make a short revision on the elementary probability learnt (in S1 unit 9), Tree and Venn diagram (in S2 unit 11) and Arrangements, Permutations and Combinations (in year 3 unit 2).

3.3 Cross-cutting issues to be addressed

- Inclusive education (promote education for all while teaching)
- Peace and value Education (respect others view and thoughts during class discussions)
- Gender (equal opportunity of boys and girls in the lesson participation and making groups)

3.4 Guidance on introductory activity

- Form groups of students and invite student-teachers to work on questions for **introductory activity** found in student's book unit 3;
- Guide student-teachers to read and analyse the problem related different cases of the gender that 3children can have: they have to write all those cases on a sheet of paper;
- Guide student-teachers to find out many possibilities as they can;
- Invite students with different working steps to present their findings to the whole class discussion;
- Basing on their experience, prior knowledge and abilities shown in answering the questions for this activity , use different questions to guide student-teachers to give their predictions and ensure that you arouse their curiosity on what is going to be leant in this unit.

Answer for introductory activity 3:

Three children who born can have the following gender:If F=female or Girl and M= male or Boy,

$\Omega = \{GGG, GGB, GBG, GBB, BGG, BGB, BBG, BBB\}$; There are 8 possibilities.

Therefore, there is one case under which the woman can have a girl at the first and the second delivery with a boy at the last delivery. This is $\Omega' = \{GGB\}$ which means that she has one chance among 8 possible cases.

3.5. List of lessons/sub-heading

No	Lesson title	Learning objectives	Number of periods
0	Introductory activity	Arouse the curiosity of stent teacher.	1
1.	Definitions	Define probability and explain probability as a measure of chance.	2
2.	Properties and formulas	Use and apply properties of probability to calculate the number of possible outcomes of occurring events under equally likely assumptions.	2
3.	Additional law of probability	Use different counting techniques to determine the number of possibilities or occurrence outcomes for an event.	2
4.	1.1 Independent events	2.1 Find probability for Independent events	2
5.	Dependent events and conditional probability	-Find probability for Dependent events. - Determine the probability of possible outcomes of occurring events by using basic formulae of conditional probability	3

6.	Successive trials and Tree diagram	Use successive trials and Tree diagram as techniques to determine probability.	3
7	Bayes theorem and its applications	Apply Bayes theorem to determine probability.	3
8	End assessment	Determine probability of occurrence of an event from random experiment and Apply Bayes' theorem.	2
Total number of periods			**20**

Lesson 1: Definitions

a) Learning objective

Define probability and explain probability as a measure of chance.

b) Teaching resources

Playing cards, graph papers, manila papers, calculators, coin and dice.

c) Prerequisites/Revision/Introduction:

Student-teachers will perform well in this lesson if they make a short revision on the content learnt as introduction to probability in S1 and S2.

d) Learning activities

- Form small groups of student-teachers and give them instructions on how to work on the **activity 3.1.1**;
- Walk around to each group and ask probing questions leading them to consider the total number of cards and the number of specified cards;
- Ask each group to share their answers with neighbouring group and ask them to support each other for improvement;
- Invite groups with different working steps to present their findings to the whole class for discussion;
- As a tutor, harmonize their answers highlighting that there are a number of chance of choosing a card.
- Use different probing questions and guide them to explore different

examples given in the student's book and lead them to explain the main **concepts of probability and their definitions** of events and their types, outcome, sample space, etc.

- After this step, guide students to do the **application activity 3.1.1** and evaluate whether lesson objectives were achieved.

Answer for activity 3.1.1

1. a. 52

   b. 4

   c. 1

2. Answers may vary; for example Selecting a red card, Selecting a queen of heart

Application Activity 3.1.1

1. c)
2. d)
3. d)
4. $\Omega = \{2,3,4,5,6,7,8,9,10,11,12\}$

Lesson 2: Properties and formulae

a) Learning objective:

Use and apply properties of probability to calculate the number of possible outcomes of occurring events under equally likely assumptions.

b) Teaching resources:

Manila papers, calculators, coins, dice, etc.

c) Prerequisites/Revision/Introduction:

Student-teachers will perform well if they learnt well all previous lessons for this unit and they were enough skilled in probability learnt in S1 and S2.

d) Learning activities

- Let students work in groups and do the **activity 3.1.2**;
- Go around to each group and ask probing questions to guide students to work towards the correct answer;
- Ask neighbouring groups to share their answers and ask them to support each other where they become more challenged in solving that activity.

- Request student-teachers to present their findings in a whole class discussion;
- As a tutor, harmonize answers for students and highlight how to determine the probability of an event using **properties and formulae** of probability.
- Use different probing questions and guide students to explore **examples** given in the student's book and lead them to establish and use properties of probability, determine probability of different events: **certain event, impossible event, probability of complementary event, mutually exclusive or incompatible or events.**
- After this step, guide students to do the **application activity 3.1.2** and evaluate whether lesson objectives were achieved.

Answer for activity 3.1.2

a) 11

b) 4, $\frac{4}{11}$

c) 7, $\frac{7}{11}$

d) i. Empty set

ii. $\{O, A, I, P, R, B, T, Y\}$ $\{O, A, I, I, P, R, B, B, L, T, Y\}$

iii. $\{P, R, B, B, L, T, Y\}$

iv. $\{O, A, I, I\}$

Application Activity 3.1.2

1. a) $\frac{2}{11}$ b) $\frac{2}{11}$

2. a) $\frac{13}{19}$ b) $\frac{3}{19}$ c) $\frac{3}{19}$

Lesson 3: Additional law of probability

a) Learning objective:

Use different counting techniques to determine the number of possibilities or occurrence outcomes for an event.

b) Teaching resources:

Manila papers, calculators, coins, dice, etc.

c) Prerequisites/Revision/Introduction:

Students will perform well in this lessons if they try to relate the content for all previous lessons for this units to real life problems involving events of chance and also if there are enough skilled on arrangement, permutation and combinations in unit2 (year 3).

d) Learning activities

- Form small groups and give instructions to student-teachers related to **activity 3.1.3**
- Move around to check either every student-teacher is contributing in the provided task.
- Ask neighbouring groups to share their answers and support each other by exchanging arguments on that activity;
- Invite group representatives to resent the findings in a whole class discussions;
- Tutor harmonizes answers from student-teachers of **activity 3.1.3**.
- Guide student-teachers to discuss that for any event A and *B* from a sample space *E*, then $P(A \cup B) = P(A) + P(B) - P(A \cap B)$;This is known as the **addition law** of probability from which we deduce that if *A* and *B* are mutually exclusive events, then $P(A \cup B) = P(A) + P(B)$.
- Lead them to apply the addition law
  $P(E_1 \cup E_2 \cup E_3 \cdots \cup E_n) = P(E_1) + P(E_1) + P(E_3) + \cdots + P(E_n)$

for **mutually exclusive events** and know that for **inclusive events** the addition law of probability is

$$P(E_1\, or\, E_2 or\, E_3 \cdots or\, E_n) = P(E_1 \cup E_2 \cup E_3 \cdots \cup E_n) = P(\cup_i^n E_i)$$

$$= \sum_{i=1}^{n} P(E_i) - \sum_{j>i=1}^{n} P(E_i \cap E_j) + \sum_{k>j>i=1}^{n} P(E_i \cap E_j \cap E_k) + \cdots + (-1)^n P(E_1 \cap E_2 \cap \cdots \cap E_n)$$

- Let them know that, if two events *A* and *B* are **exhaustive** $(A \cup B = \Omega)$ then $P(A \cup B) = 1$.
- Use different probing questions and guide students to explore different **examples** provided in the student-teacher's book and lead them to realize that probability is applicable in the real life.
- After this step, guide student-teachers to do the **application activity 3.1.3** and evaluate whether lesson objectives were achieved.

Answer for activity 3.1.3

The probability that a component selected at random is **either** standard or top quality is $0.65+0.18=0.83$

Application Activity 3.1.3

1. $\frac{5}{6}$ 2. $\frac{1}{2}$	3. a) $\frac{3}{8}$ b) $\frac{5}{8}$ c) $\frac{1}{32}$

Lesson 4: Independent events

a) Learning objective

Find probability for Independent events.

b) Teaching resources

Playing cards, graph papers, manila papers, calculators, coin and dice.

c) Prerequisites/Revision/Introduction:

Students will perform well in this lessons if they try to relate the content for all previous lessons for this units to real life problems involving events of chance and also if there are enough skilled on arrangement, permutation and combinations in unit2 (year 3).

d) d) Learning activities

- Form small groups of student-teachers and give them instructions on how to work on the **activity 3.2**;
- Walk around to each group and ask probing questions leading them to consider the total number of cards and the number of specified cards;
- Ask each group to share their answers with neighbouring group and ask them to support each other for improvement;
- Invite groups with different working steps to present their findings to the whole class for discussion;
- As a tutor, harmonize their answers highlighting that there are a number of chance of choosing a card.

- Lead them to know that If probability of event B is not affected by the occurrence of event A, events A and B are said to be **independent** and $P(A \cap B) = P(A) \times P(B)$.This rule is the simplest form of the **multiplication law** of probability.
- Guide them to explore different provided **examples**.
- After this step, guide students to do the **application activity 3.2** and evaluate whether lesson objectives were achieved.

Answer for activity 3.2

The occurrence of event B is not affected by occurrence of event A because after the first trial the pen is replaced in the box. It means that the sample space does not change.

Application Activity 3.2

1. $P(\text{red and red}) = \frac{1}{5} \times \frac{1}{5} = \frac{1}{25}$

2. $P(\text{head and } 3) = \frac{1}{2} \times \frac{1}{6} = \frac{1}{12}$

3. a) $\frac{1}{7} \times \frac{1}{5} = \frac{1}{35}$ b) $\frac{1}{7} \times \frac{4}{5} + \frac{6}{7} \times \frac{1}{5} = \frac{2}{7}$ c) $\frac{6}{7} \times \frac{4}{5} = \frac{24}{35}$

Lesson 5: Dependent events and conditional probability

a) Learning objective:

- Find probability for Dependent events.
- Determine the probability of possible outcomes of occurring events by using basic formulae of conditional probability.

b) Teaching resources:

Manila papers, calculators, coins, dice, etc.

c) Prerequisites/Revision/Introduction:

Students will perform well in this lessons if they try to relate the content for all previous lessons for this units to real life problems involving events of chance and also if there are enough skilled on arrangement, permutation and combinations in unit2 (year 3).

d) Learning activities

- Let students work in groups and do the **activity 3.3;**
- Go around to each group and ask probing questions to guide students to

work towards the correct answer;

- Ask neighbouring groups to share their answers and ask them to support each other where they become more challenged in solving that activity.
- Request student-teachers to present their findings in a whole class discussion;
- As a tutor, harmonize answers for students and highlight how to determine the probability of an event using the classical probability.
- Use different probing questions and guide students to explore **examples** given in the student's book.
- Lead them know, establish and use of formula for **conditional probability of** B **given** A $P(B/A)=\dfrac{P(A\cap B)}{P(A)}$ When the outcome or occurrence of the first event affects the outcome or occurrence of the second event (the two events are said to be **dependent**).
- Emphasize on this result, that we have general statement of the **multiplication law** write $P(A\cap B)=P(B)\times P(A|B)$.
- After this step, guide students to do the **application activity 3.3** and evaluate whether lesson objectives were achieved.

Answer for activity 3.3

a) Sample space for the first drawing is a set of $52\,cards$, But for the second drawing the sample space is a set of $51\,cards$.

b) The outcomes of the first draw has affected the outcome of the second i.e. the outcome of the second dependent on the first drawing.

Application Activity 3.3

1. $p(H\ and\ A)=0.53\times0.27=0.1431$

2. $P(6|even)=\dfrac{\frac{1}{6}}{\frac{3}{6}}=\dfrac{1}{3}$

3. $P(White|Black)=\dfrac{P(Black\ and\ White)}{P(Black)}=\dfrac{0.34}{0.47}=0.72$

4. $\dfrac{3}{13}$

Lesson 6: Successive trials and Tree diagram

a) Learning objective:

Use successive trials and Tree diagram as techniques to determine probability.

b) Teaching resources:

Manila papers, calculators, coins, dice, etc.

c) Prerequisites/Revision/Introduction:

Students will perform well in this lessons if they try to relate the content for all previous lessons for this units to real life problems involving events of chance and also if there are enough skilled on arrangement, permutation and combinations in unit2 (year 3).

d) Learning activities

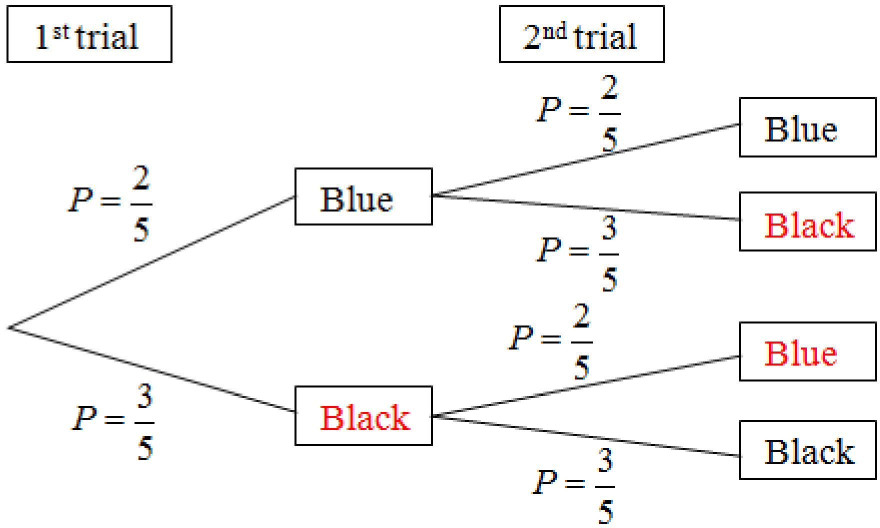
- Invite students to work in small groups, attempt the activity 3.4 and answer to related questions;
- Ask neighbouring groups to share their answers and support each other by exchanging arguments on that activity;
- Invite group representatives to resent the findings in a whole class discussions;
- Tutor harmonizes answers for students on **activity 3.4.**
- Guide students to discuss and apply a **tree diagram technique** to show the probabilities of certain **outcomes** occurring when two or more **trials** take place in succession.
- Let them know that for each **trial,** the number of branches is equal to the number of possible outcomes of that trial.
- Use different probing questions and guide students to explore different **examples** given in the student's book and lead them to realize that probability is applicable in the real life.
- After this step, guide students to do the **application activity 3.4** and evaluate whether lesson objectives were achieved.

Answer for activity 3.4

1. Probability of choosing a blue pen is $\frac{4}{10}=\frac{2}{5}$ and probability of choosing a black pen is $\frac{4}{10}=\frac{2}{5}$.
2. Probabilities on the second trial are equal to the probabilities on the first trial

since after the 1st trial the pen is replaced in the box.

3. Complete figure



Application Activity 3.4

1. $\frac{1}{2}\times\frac{1}{2}\times\frac{1}{2}=\frac{1}{8}$

2. a) $P(3\,\text{boys})=\frac{10}{16}\times\frac{9}{15}\times\frac{8}{14}=0.214$

   b) $P(2\text{ boys and 1 girl})=\frac{10}{16}\times\frac{9}{15}\times\frac{6}{14}+\frac{10}{16}\times\frac{6}{15}\times\frac{9}{14}+\frac{6}{16}\times\frac{10}{15}\times\frac{9}{14}=0.482$

   c) $P(2\text{ girls and 1 boy})=\frac{10}{16}\times\frac{6}{15}\times\frac{5}{14}+\frac{6}{16}\times\frac{10}{15}\times\frac{5}{14}+\frac{6}{16}\times\frac{5}{15}\times\frac{10}{14}=0.268$

   d) $P(3\,\text{girls})=\frac{6}{16}\times\frac{5}{15}\times\frac{4}{14}=0.0357$

3. a) $\frac{1}{21}$ b) $\frac{10}{21}$ c) $\frac{11}{21}$

4. a) $\frac{1}{816}$ b) $\frac{7}{102}$ c) $\frac{7}{34}$

Lesson 7: Bayes theorem and its applications

a) Learning objective:

Apply Bayes theorem to determine probability.

b) Teaching resources:

Manila papers, calculators, coins, dice, etc.

c) Prerequisites/Revision/Introduction:

Students will perform well in this lessons if they try to relate the content for all previous lessons for this units to real life problems involving events of chance and also if there are enough skilled on arrangement, permutation and combinations in unit2 (year 3).

d) Learning activities

- Invite student-teachers to work in small groups, discuss the betting explained in the **activity 3.5** and answer to related questions;
- Ask neighbouring groups to share their answers and support each other by exchanging arguments on that activity;
- Invite group representatives to resent the findings in a whole class discussions;
- Tutor harmonizes answers for student-teachers on **activity 3.5** and guides them to brainstorm good prediction of probability for winning.
- Guide student-teachers to discuss and apply **Bayes' formula/ Bayes' rule** to determine probability.

$$P(B_i \mid A) = \frac{P(B_i \cap A)}{P(A)} = \frac{P(A \mid B_i)P(B_i)}{\sum_{i=1}^{n} P(A \mid B_i)P(B_i)}$$

- Use different probing questions and guide them to explore **examples** given in the student-teacher's book and lead them to realize that probability is applicable in the real life.
- After this step, guide student-teachers to go through the **application activity 3.5** and evaluate whether lesson objectives were achieved.

Answer for activity 3.5

1. $P(A) = P(A \mid B_1)P(B_1) + P(A \mid B_2)P(B_2) + P(A \mid B_3)P(B_3)$

2. $P(B_1 \mid A) = \frac{P(B_1 \cap A)}{P(A)} = \frac{P(A \mid B_1)P(B_1)}{P(A \mid B_1)P(B_1) + P(A \mid B_2)P(B_2) + P(A \mid B_3)P(B_3)}$

$$P(B_2 \mid A)=\frac{P(B_2 \cap A)}{P(A)}=\frac{P(A \mid B_2)P(B_2)}{P(A \mid B_1)P(B_1)+P(A \mid B_2)P(B_2)+P(A \mid B_3)P(B_3)}$$

$$P(B_3 \mid A)=\frac{P(B_3 \cap A)}{P(A)}=\frac{P(A \mid B_3)P(B_3)}{P(A \mid B_1)P(B_1)+P(A \mid B_2)P(B_2)+P(A \mid B_3)P(B_3)}$$

Generally,

$$P(B_i \mid A)=\frac{P(B_i \cap A)}{P(A)}=\frac{P(A \mid B_i)P(B_i)}{\sum_{i=1}^{3} P(A \mid B_i)P(B_i)}$$

Application Activity 3.5

1. $P(engineer \mid managerial)=\frac{0.2\times 0.75}{0.2\times 0.75+0.2\times 0.5+0.6\times 0.2}=0.405$

2. $P(No\ accident \mid Triggered\ alarm)=\frac{0.9\times 0.02}{0.1\times 0.97+0.9\times 0.02}=0.157$

3.6. Unit summary

Probability of an event

- The probability of an event $A\subset\Omega$, is a real number obtained by applying to *A* the function *P* defined by

$$P(A)=\frac{Number\ of\ outcomes\ \text{in}\ A}{Total\ \text{nu}mber\ of\ outcomes\ \text{in}\ the\ sample\ space}=\frac{n(A)}{n(\Omega)}$$

- When *E* and E' are complementary events, $P(E)=1-P(E')$.
- When two events A and B are not mutually exclusive, $A\cap B=\phi$ the probability that A or B occurs is given by:

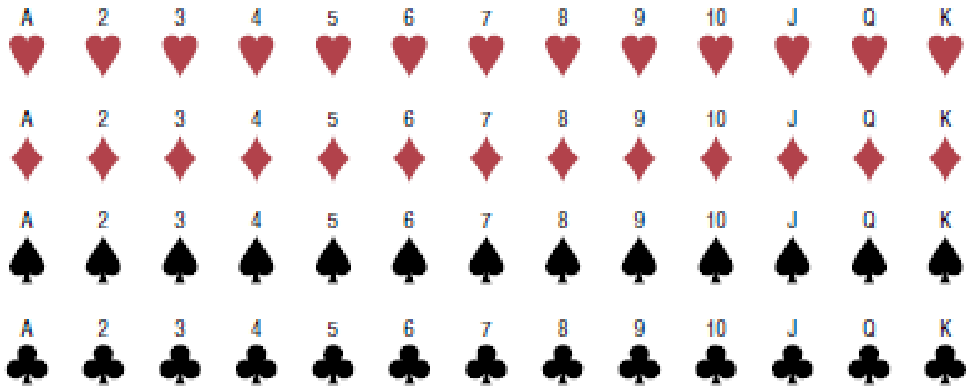
$$P(A\cup B)=\text{P(A)}+\text{P(B)}-\text{P}(A\cap B)$$

Note:

We have to think big before taking decision regarding our engagement in the games of chance. Such games are for example: betting on card games, slot machine (ikiryabarezi), lotteries, and weather forecasting. In such games, predictions are based on probability and hypotheses are tested by using probability.

3.7 Additional Information for teachers

3.7.1 Components of an ordinary deck of cards:



3.7.2 Law of Large Numbers

When a coin is tossed one time, it is common knowledge that the probability of getting a head is $\frac{1}{2}$. But what happens when the coin is tossed 50 times? Will it come up heads 25 times? Not all the time. You should expect about 25 heads if the coin is fair. But due to chance variation, 25 heads will not occur most of the time.

If the empirical probability of getting a head is computed by using a small number of trials, it is usually not exactly $\frac{1}{2}$. However, as the number of trials increases, the empirical probability of getting a head will approach the theoretical probability of $\frac{1}{2}$, if in fact the coin is fair (i.e., balanced). This phenomenon is an example of the law of large numbers.

3.7.3 Independent events

Two events *A* and *B* are **independent events** if the fact that *A* occurs does not affect the probability of *B* occurring.

When two events are independent, the probability of both occurring is

$$\text{P(FA and B)} = P(A).P(B)$$

Example:

A coin is flipped and a die is rolled. Find the probability of getting a head on the coin and a 4 on the die.

Solution:

P(Head and 4)= P(head).P(4)= $\frac{1}{2}\cdot\frac{1}{4}=\frac{1}{8}$

3.8 End unit assessment

Answers for End unit assessment

1. Choose a letter at random from the word SCHOOL 2. $\frac{9}{20}$ 3. $\frac{21}{46}$ 4. $\frac{3}{13}$ 5. a. $\frac{1}{6}$ b. $\frac{5}{126}$	6. 0.13 7. 0.56 8. a) 0.34 b) 0.714 c) 0.0833 9. a) 0.384 b) 0.512 10. $\frac{3}{13}$

3.9 Additional activities

3.9.1 Remedial activity

1. A card is drawn at random from an ordinary pack of 52 playing cards. Find the probability that the card drawn is a club or a diamond. A die is thrown once.

Solution

There are 13 clubs, then $P(\text{club})=\frac{13}{52}$

There are 13 diamonds, then $P(\text{diamond})=\frac{13}{52}$

Since a card cannot be both a club and a diamond, $P(\text{club}\cap\text{diamond})=0$

Therefore, P(a club or a diamond) = P(club) + P(diamond)

$$P(\text{a club or a diamond})=P(\text{club})+P(\text{diamond})$$
$$=\frac{13}{52}+\frac{13}{52}$$
$$=\frac{26}{52}$$
$$=\frac{1}{2}$$

2. A pen is drawn from a basket containing 10 pens of which 5 are red and 3 are black. If A is the event: "a pen is red" and B is the event: "a pen is black", find $P(A), P(B), P(A \cup B)$.

Solution

There are 5 red pens, then $P(A) = \frac{5}{10} = \frac{1}{2}$

There are 3 black pens, then $P(B) = \frac{3}{10}$

Since the pen cannot be red and black at the same time, then $A \cap B = \varnothing$ and two events are mutually exclusive so $P(A \cup B) = \frac{1}{2} + \frac{3}{10} = \frac{8}{10}$

3.9.2 Consolidation activity

1. In a group of 20 adults, 4 out of the 7 women and 2 out of the 13 men wear glasses. What is the probability that a person chosen at random from the group is a woman or someone who wears glasses?

Solution

Let *A* be the event: "the person chosen is a woman".

B be the event: "the person chosen wears glasses".

Now, there are 7 women, then $P(A) = \frac{7}{20}$

There are 6 persons who wear glasses, then $P(B) = \frac{6}{20}$

There are 4 women who wear glasses, then $P(A \cap B) = \frac{4}{20}$

The probability that a person chosen at random from the group is a woman or someone who wears glasses is given by $P(A \text{ or } B)$ which is

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$
$$= \frac{7}{20} + \frac{6}{20} - \frac{4}{20}$$
$$= \frac{9}{20}$$

On the other hand:

There are 7 women and 6 persons who wear glasses but also 4 women who wear glasses. Then

$A \cup B = 9$ and $P(A \cup B) = \frac{9}{20}$.

2. A coin is weighted so that heads is three times as likely to appear as tails. Find $P(H)$ and $P(T)$.

Solution

Let $P(T) = p_1$, then $P(H) = 3p_1$.

But $P(H) + P(T) = 1$

Therefore $p_1 + 3p_1 = 1 \Leftrightarrow 4p_1 = 1 \Rightarrow p_1 = \frac{1}{4}$

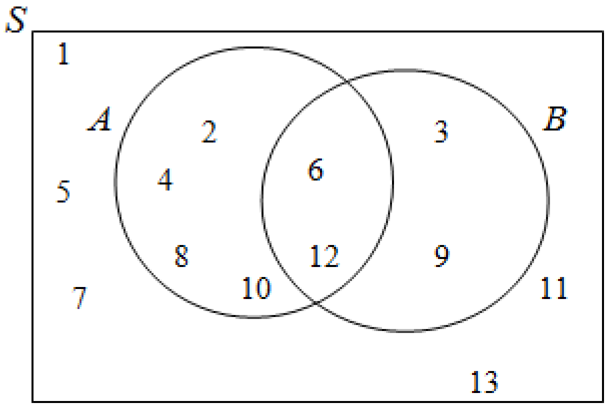
Thus, $P(H) = \frac{3}{4}$ and $P(T) = \frac{1}{4}$.

3.9.3 Extended activity

1. An integer is chosen at random from the set $S = \{x : x \in \mathbb{Z}^+, x < 14\}$. Let A be the event of choosing a multiple of 2 and let B be the event of choosing a multiple of 3.

Find $P(A \cup B)$, $P(A \cap B)$ and $P(A - B)$.

Solution



From the diagram, $\#S = 13$

$A \cup B = \{2, 3, 4, 6, 8, 9, 10, 12\} \Rightarrow \#(A \cup B) = 8$, thus $P(A \cup B) = \frac{8}{13}$

$A \cap B = \{6, 12\} \Rightarrow \#(A \cap B) = 2$, thus $P(A \cap B) = \frac{2}{13}$

$A - B = \{2, 4, 8, 10\} \Rightarrow \#(A - B) = 4$, thus $P(A - B) = \frac{4}{13}$

Solution

2. In a sample of 50 people, 21 had type O blood, 22 had type A blood, 5 had type B blood, and 2 had type AB blood. Set up a frequency distribution and find the following probabilities.

a) A person has type O blood.

b) A person has type A or type B blood.

c) A person has neither type A nor type O blood.

d) A person does not have type AB blood.

Solution

Type	**Frequency**
A	22
B	5
AB	2
O	21
Total	50

They are mutually exclusive.

a) $P(O) = \frac{f}{n} = \frac{21}{50}$

b) $P(\text{A or B}) = \frac{22}{50} + \frac{5}{50} = \frac{27}{50}$

c) Neither A nor O means that a person has either type B or type AB blood.)

$$P(\text{neither A nor O}) = \frac{5}{50} + \frac{2}{50} = \frac{7}{50}.$$

d) Find the probability of not AB by subtracting the probability of type AB from 1.

$$P(\text{not AB}) = 1 - P(AB) = 1 - \frac{2}{50} = \frac{48}{50} = \frac{24}{25}$$

UNIT 4

LOGARITHMIC AND EXPONENTIAL FUNCTIONS

4.1 Key Unit competence

Extend the concepts of functions to investigate logarithmic and exponential functions and use them to model and solve problems about interest rates, population growth or decay, magnitude of earthquake, etc.

4.2 Prerequisite knowledge and skills

Student-teachers will perform well in this unit if they have a good background on: indices and radicals and common logarithms (year 1: unit6), solving equations and inequalities in the set of real numbers (year 1: unit 5), domain and range of polynomial, rational and irrational functions (year2: unit5), limits and derivatives of polynomial, rational and irrational functions (year2: Unit 6&7).

4.3 Cross-cutting issues to be addressed

- **Inclusive education**: Promote the participation of all Student-teachers while teaching.
- **Peace and value Education**: During group activities, the Tutors will encourage student-teachers to help each other and to respect opinions of colleagues. In addition, Student-teachers will be sensitized to fight alcohol abuse in the lesson on alcohol and risk of car accident).
- **Gender**: During group activities try to form heterogeneous groups (with boys and girls) or when Student-teachers start to present their findings, encourage both (boys and girls)to present.
- **Environment and Sustainability:** During the lesson on population growth, guide Student-teachers to discuss the effect of the high rate of population growth.
- **Financial education**: Guide Student-teachers to discuss how to manage the mortgage loans taken from the bank.

4.4 Guidance on the introductory activity

- Form groups of student-teachers that are as heterogeneous as possible and guide them to work on the introductory activity.

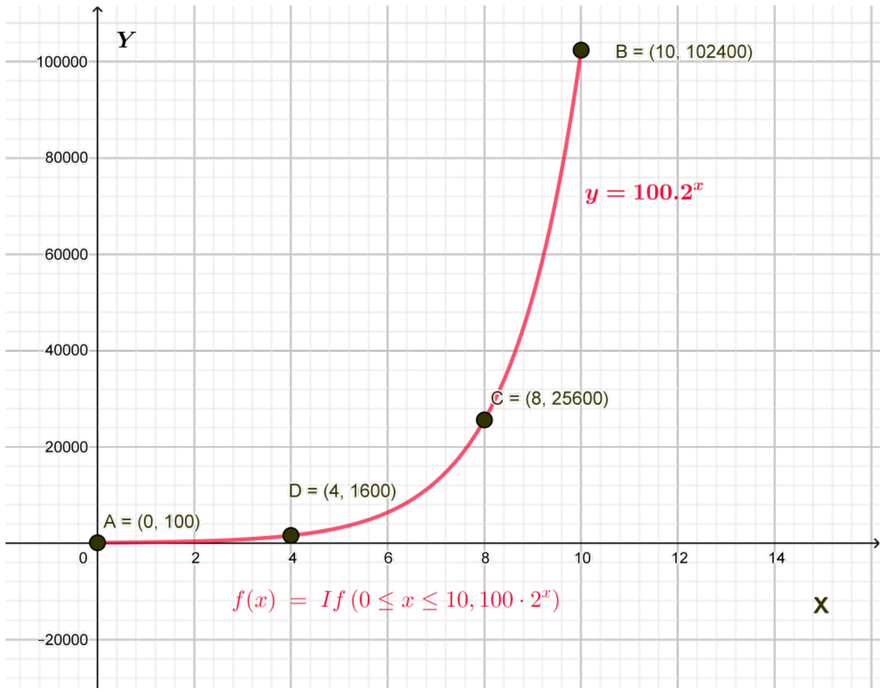
- Walk around all groups to provide pieces of advice where necessary.
- After a given time invite student-teachers to present their findings and harmonize them.

Solution:

a) Student-teachers complete the table showing the money of the businessman from the 1st day up to 10th day.

Days	Amount s	USD
1st day	$200\times1=100\times2^1$	200
2nd day	$200\times2=100\times2\times2=100\times2^2$	400
3rd day	$100\times2\times2\times2=100\times2^3$	800
4th day	$100\times2\times2\times2\times2=100\times2^4$	1600
....		
10th day	$100\times2\times2\times2\times2\times2\times2\times2\times2\times2\times2=100\times2^{10}$	102,400
nthday	$100\times\underbrace{2\times2\times...\times2\times2}_{n\ factors}=100\times2^n$	100×2^n

b) The graph plotted in a rectangular coordinate.



c) $f(n)=100\times 2^n$ USD

- During the presentation let student-teachers discover the concept of exponential function $F(t)$ starting with the property of a function with powers. $F(t)=100\times 2^t$
- Student-teachers establish the function $Y(F)$ inverse of $F(t)$

$$Y(F)=F^{-1}(t)=\ln\left(\frac{t}{100}\right)=-\ln(100)+\ln t$$

$$Y(t)=-4.6+\ln(t)$$

d) The economist wants to possess the money F under the same conditions, discuss how he/she can know the number of days necessary to get such money from the beginning of the business.

The economist wants to possess the money F, using the inverse function $Y(F)=-4.6+\ln(F)$, she/he will use the equation $t=-4.6+\ln(\text{F})$ to calculate the number t of days required.

Conclude that $F(t)$ and $Y(t)$ are respectively exponential function and logarithmic functions that are needed to be well studied so that they may be used without problems. This unit deals with the behaviour and properties of such essential functions and their application in real life situation.

4.5. List of lessons

#	Lesson title	Learning objectives	Number of periods
0.	Introductory activity	To arouse the curiosity of student-teachers on the content of unit 1.	1
1	Definition, Domain and Range of logarithmic functions (base e and for any base)	- Define logarithmic functions in base e and for any base. -Find domain and range of logarithmic functions in base e and for any base.	2

2	Properties and operations on logarithmic functions (for base e and for any base)	-State and demonstrate properties of logarithms for any base. - Carry out operations using the change of base of logarithms.	1
3	Logarithmic equations (including base e and any base)	-Use the properties of logarithms to solve logarithmic equations.	2
4	Limit of logarithmic functions with base e	Calculate limit of logarithmic function with base e	1
5	Limit of logarithmic functions with any base	Calculate limit of logarithmic function with any base	2
6	Asymptotes of the graph of logarithmic functions (including base e and any base)	Determine and interpret possible asymptotes of the graph of logarithmic functions (including base e and any base)	2
7	Derivative for logarithmic functions(for base e and for any base)	Determine the derivative of logarithmic functions	2
8	Variation and graphical representation of logarithmic functions(for base e and for any base)	Apply the 1st and the 2nd derivative to investigate the minimum and maximum (EXTREMA) of logarithmic functions and to find concavity of the graph.	2
9	Definition , Domain and range of exponential functions (base e and for any base)	- Define logarithm or exponential functions in base e and for any base. -Find domain and range of exponential functions base e and for any base.	2
10	Properties and operations on exponential functions (for base e and for any base)	-State and demonstrate properties of exponents for base e and for any base.	1

11	Exponential equations (including base e and any base)	-Use the properties of exponents to solve exponential equations.	2
12	Limit of exponential functions with base e	Calculate limit of exponential Function with base e.	1
13	Limit of exponential functions with any base	Calculate limit of exponential Function with any base	2
14	Asymptotes of the graph of exponential functions	Determine and interpret possible asymptotes of the graph of exponential functions	2
15	Derivative for exponential functions (for base e and for any base)	Determine the derivative of exponential functions (for base e and for any base)	2
16	Variation and graphical representation of exponential functions (for base e and for any base)	Apply the 1st and the 2nd derivative to investigate the minimum and maximum (EXTREMA) of exponential functions (for base e and for any base) and to find concavity of the graph.	2
17	Modelling and solving problems involving logarithmic or exponential functions	Use of logarithmic expressions to model and solve problem involving logarithms such rates problems, mortgage problems, population growth problems. Radioactive-decay problems, carbon dating problems, problems about alcohol and risk of car accident.	2
18	End unit assessment	Extend the concepts of functions to investigate logarithmic and exponential functions and use them to model and solve problems about interest rates, population growth or decay, magnitude of earthquake,	1

Lesson 1: Definition, Domain and Range of logarithmic functions

a) Learning objective

- Define logarithmic functions in base e and for any base.
- Find domain and range of logarithmic functions in base e and for any base.

b) Teaching resources:

Mathematical set, scientific calculators, textbooks, graph papers. If possible, the use of Mathematical software such as Geogebra to plot graph of logarithmic functions and/or Microsoft Excel to compute values of a function and Internet to facilitate research would be plausible.

c) Prerequisites/Revision/Introduction:

Student-teachers will perform well in this unit if they have a good background on common logarithms (year 1: unit6), solving equations and inequalities in the set of real numbers (year 1: unit 5), domain and range of polynomial, rational and irrational functions (year2: unit5).

d) Learning activities:

This lesson evaluates the set of elements for which a logarithmic function is defined. To this end,

- Form groups and ask student-teachers to work on **activity 4.1.1**
- Facilitate student-teachers to use calculator or Microsoft Excel to find images of given real numbers, discuss their existence in the set of real numbers, dress a table of values for given numbers and to plot the graph of $f(x)=\log_{10}(x)$ and $f(x)=\ln x$.
- Ask randomly some groups to present their findings to the whole class;
- Lead student-teachers to give observations about images found step by step for $x>1$, $x=1$, $0<x<1$, and values $x<0$,
- Facilitate student-teachers to deduce **the domain and the range** for $\text{f(x)}=\log_{10}(x)$ and $f(x)=\ln x$ then generalize for **the logarithmic function** of type $y=\log_a(u(x))$ or $f(x)=\ln u(x)$, with $u(x)\geq 0$, $a\neq 0, a>1$and then from their answers, write a short summary.
- Guide student-teachers to work the provided **examples**
- Call them to do individually the **application activity 4.1.1** to assess their competences.

Answer for activity 4.1.1

a) Complete the table of values for $\log_{10}(x)$

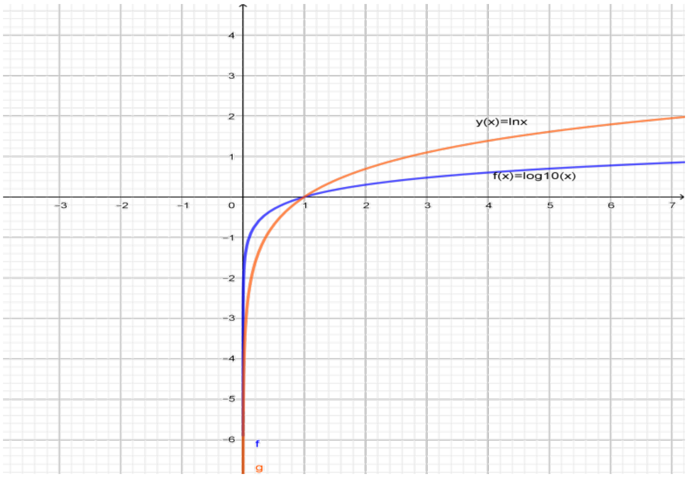
x	100	50	40	20	10	½	0.8	0.7	-5	-20	-30
$y=\log_{10}(x)$	2	1.69	1.6	1.30	10	-0.30	-0.09	0.15	Does not exist.	Does not exist.	Does not exist.

b) The values of $\log_{10}(x)$ for $x<0$ do not exist in the set of real numbers.

c) The values of $\log_{10}(x)$ for $0<x<1$, $x=1$ and $x>1$.

$\log_{10}(1)=0$, $\log_{10}(x)<0$ for $0<x<1$ and $\log_{10}(x)>0$ for $x>1$

d) The graph of $\log_{10}(x)$ for $x>0$



e) For $f:\mathbb{R}\to\mathbb{R}:x\mapsto\log_a x$,

$dom\ f=\{x\in\mathbb{R}:x>0\}=]0,+\infty[=\mathbb{R}_0^+$ and range f $=\mathbb{R}=]-\infty,+\infty[$

Application Activity 4.1.1

1. a) $y=\log_3(x-2)+4$ is defined for $x>2$.

$Domf=]2,+\infty[$

To find the range we proceed as follows:

$y=\log_3(x-2)+4 \Leftrightarrow y-4=\log_3(x-2)$ (for x in the domain)

$$\Leftrightarrow x-2=3^{y-4} \Leftrightarrow x=3^{y-4}+2$$

Since $3^{y-4}>0$ $\forall y\in\mathbb{R}$, we have $x=3^{y-4}+2>2$.

Thus the range is $\mathbb{R}$

b) $y = \log_5(8-2x)$ is defined only if $8-2x>0 \Leftrightarrow -2x>-8 \Leftrightarrow x<4$

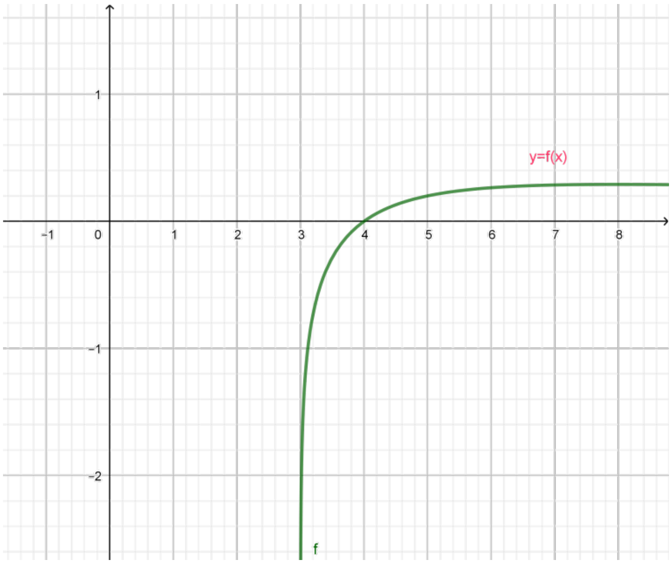
$$Domf =]-\infty, 4[$$

For the Range: $y=\log_5(8-2x) \Leftrightarrow 8-2x=5^y \Leftrightarrow x=-\frac{1}{2}(5^y)+4$

$5^y>0$ for all values of y implies $x=4-\frac{1}{2}(5^y)<4.$

Thus, the range is $\mathbb{R}$

c) From the graph



The domain of the function f is $Domf =]3,+\infty[$. The range is $\mathbb{R} =]-\infty,+\infty[$

3. a) $]0,+\infty[$ b) $]-\infty,-1[\cup]1,+\infty[$ c) $]-\infty,-1[\cup]4,+\infty[$ d) $]-5,-2[\cup]-2,0[$

4. a) Given $\log_2(2x-3)$;condition for existence is

$$2x-3>0 \Leftrightarrow x>\frac{3}{2}$$

Therefore, the expression is defined for all real number greater than $\frac{3}{2}$

b)Given $\log_{\frac{1}{2}}\frac{1-x}{x}$;condition of existence $\frac{1-x}{x}>0; x\neq 0$

x	$-\infty$ 0 1 $+\infty$
$1-x$	+ + + + + + + + + + + + 0 − − − − − − − − −
x	− − − − − − − − − 0 + + + + + + + + + + + +
$\frac{1-x}{x}$	− − − − || + + + + + 0 − − − − −

Therefore, the expression is defined in the interval $]0,1[$

c)The expression is defined in the interval $]0,+\infty[$

d)The expression is defined in the interval $]0,+\infty[$

e)The expression is defined in the interval $]-6,-2[\cup]2,+\infty[$

Lesson 2: Properties and operations on logarithmic functions (for base e and for any base)

a) Learning objective

- State and demonstrate properties of logarithms **(for base e and for any base).**
- Carry out operations using the change of base of logarithms.

b) Teaching resources:

Scientific calculators to evaluate logarithms, textbooks, graph papers. If possible, the use of Mathematical software such as Geogebra to plot graph of logarithmic functions and/or Microsoft Excel to compute values of a function and Internet to facilitate research would be plausible.

c) Prerequisites/Revision/Introduction:

Student-teachers will perform well in this unit if they have a good background on **properties and operations** common logarithms (year 1: unit6)

d) Learning activities:

This lesson deals with properties of logarithmic functions and simplification of them. To this end,

- Form groups and ask student-teachers to work on **activity 4.1.2**
- Ask randomly some groups to present their findings to the whole class;
- Lead student-teachers to give comments on previous presentation before the next one.

- Facilitate student-teachers to states all **logarithmic properties** through given real numbers discuss on their use.
- Facilitate student-teachers to deduce the $\log_b x = \frac{\log_a x}{\log_a b}$ as **Changing logarithms from one base to another** and then after, help them to make a short summary.
- Guide student-teachers to work on **examples.**
- Call them to do individually the **application activities 4.1.2** to assess their competences.

Answer for activity 4.1.2

a)

x	-2	0	1	2	4	$\frac{2}{4}$	2×4	$\frac{1}{4}$	2^4
$\ln x$	No result	No result	0	0.693	1.386	-0.693	2.079	-1.386	2.772

b)

i) $\ln 2+\ln 4=0.693+1.386=2.079$ ii) $\ln 2-\ln 4=0.693-1.386=-0.693$

ii) iii) $\ln 1-\ln 4=0-1.386=-1.386$ iv) $4\ln 2=4\times 0.693=2.772$

c) from (a) and (b); the equal values are:

$$\ln 2\times 4=\ln 2+\ln 4;\quad \ln\frac{2}{4}=\ln 2-\ln 4;\quad \ln\frac{1}{4}=\ln 1-\ln 4;\ \ln 2^4=4\ln 2$$

d) We can conclude that:

$a)\ \ln(xy)=\ln x+\ln y$ $\quad c)\ \ln\left(\frac{1}{y}\right)=-\ln y$

$b)\ \ln\left(\frac{x}{y}\right)=\ln x-\ln y$ $\quad d)\ \ln(x)^n=n\ln x$

Application Activity 4.1.2

1.

$a)\ \ln 9=\ln 3+\ln 3=1.098+1.098=2.196$

$b)\ \ln 0.3=\ln 3-\ln 2-\ln 5=1.098-0.693-1.609=-1.204$

$c)\ \ln 225=2\ln 3+2\ln 5=2\times 1.098+2\times 1.609=5.414$ $\quad d)\ \ln 20=2\ln 2+\ln 5=2.995$

$e)\ \ln 3^5=5\times\ln 3=5.49$ $\ f)\ \ln 15=\ln 3+\ln 5=2.707$

$g)\ln 0.15=\ln 3-\ln 5-2\ln 2=-1.897\quad h)\ln 75=\ln 3+2\ln 5=4.316$

2.

$$a)\log_2\frac{\left(5+\sqrt{21}\right)\left(5-\sqrt{21}\right)}{16}=\log_2\frac{25-21}{16}=\log_2\frac{4}{16}=\log_2\frac{1}{4}=-2$$

$$b)\log_3\frac{\left(17-\sqrt{19}\right)\left(17+\sqrt{19}\right)}{270}=\log_3\frac{289-19}{270}=\log_3\frac{270}{270}=\log_3 1=0$$

3. $i)\ln\frac{a^3c}{b}=\ln a^3+\ln c-\ln b=3\ln a-\ln b+\ln c$

$$ii)\ln\sqrt[5]{x^2y^{-5}z^{10}}=\ln\left(x^2y^{-5}z^{10}\right)^{\frac{1}{5}}=\ln\left(x^2\right)^{\frac{1}{5}}\left(y^{-5}\right)^{\frac{1}{5}}\left(z^{10}\right)^{\frac{1}{5}}$$

$$=\frac{2}{5}\ln x-\frac{5}{5}\ln y+\frac{10}{5}\ln z\quad=\frac{2}{5}\ln x-\ln y+2\ln z$$

4. $a)\ln\frac{2}{3}=\ln 2-\ln 3\quad b)\ln\frac{xy}{z}=\ln x+\ln y-\ln z\qquad c)\ln\sqrt{x^2+1}=\frac{1}{2}\ln\left(x^2+1\right)$

$$d)\ln\frac{3x\left(x+1\right)}{\left(2x+1\right)^2}=\ln 3+\ln x+\ln\left(x+1\right)-2\ln\left(2x+1\right)$$

$$e)\ln\frac{2x}{\sqrt{x^2-1}}=\ln 2+\ln x-\frac{1}{2}\ln\left(x^2-1\right)\quad f)\ln\left(x\sqrt[3]{x^2+1}\right)=\ln x-\frac{1}{3}\ln\left(x^2+1\right)$$

5. $a)\ln\frac{(x-2)}{(x+2)}\quad b)\ln\left(2x+1\right)\left(2x-1\right)\quad c)\ln\left[\frac{x(x+3)}{(x+4)}\right]^3\quad d)\ln\sqrt[3]{\frac{x\left(x+3\right)^2}{\left(x^2-1\right)}}$

6. $a)\log_3\frac{xy}{y^3}=\log_3\frac{x}{y^2}\qquad b)\log\frac{(x+1)}{\left(x^2-1\right)}=\log\frac{1}{(x-1)}\qquad c)\log_2 8x^{14}$

$d)\log\frac{\left(a^3+b^3\right)}{(a+b)}=\log\left(a^2-ab+b^2\right)\qquad e)\log_2 90$

Lesson 3: Logarithmic equations (including base e and any base)

a) Learning objective

Use the properties of logarithms to solve logarithmic equations.

b) Teaching resources:

Scientific calculators to evaluate logarithms, textbooks, graph papers. If possible, the use of Mathematical software such as Geogebra to plot graph of logarithmic functions and/or Microsoft Excel to compute values of a function and Internet to facilitate research would be plausible.

c) Prerequisites/Revision/Introduction:

Student-teachers will perform well in this unit if they have a good background on **properties and operations** common logarithms (year 1: unit6), solving equations and inequalities in the set of real numbers (year 1: unit 5), domain and range of polynomial, rational and irrational functions (year2: unit5).

d) Learning activities:

This lesson deals with solving logarithmic equations.

- Form groups and ask student-teachers to work on **activity 4.1.3**
- Ask randomly some groups to present their findings to the whole class;
- Lead student-teachers to give comments on previous presentation before the next one.
- Facilitate student-teachers to discuss on the **domain of validity**/ condition of existence.
- Facilitate student-teachers to apply logarithmic properties and use Changing base to another to solve the **logarithmic equations** and then after their answers, write a short summary.
- Guide student-teachers to work on **examples**.
- Call them to do individually the **application activities 4.1.3** to assess their competences.

Answer for activity 4.1.3

1. $f: x>0$; $h: x+2>0 \Leftrightarrow x>-2$; $g: x^2-5x+6>0 \Leftrightarrow x \in \mathbb{R}\setminus\{-2,-3\}$

2. Use the properties for logarithm to determine the value of x in the following expressions:

a) $\log x = 2; x>0$

$\log x = \log 10^2$

$x = 100$

b) $\ln x = \ln 10; \quad x > 0$

$x = 10$

c) $\ln x = \ln 1000; \quad x > 0$

$x = 1000$

d) $\log(100x) = 2\log 10 + \log 4; \quad x > 0$

$\log 100x = \log 400$

$x = 4$

Application Activity 4.1.3

1.

$a)\ln x = 0; \quad x > 0$

$\ln x = \ln 1$

$x = 1$

$b)\ln x + \ln 4 = 0; \quad x > 0$

$\ln x = \ln 4^{-1}$

$x = \frac{1}{4}$

$c) 2\ln x = \ln 36; \quad x > 0$

$\ln x^2 = \ln 36$

$x^2 = 36$

$x = \pm 6$

$d)\ln\left(x^2 - 1\right) = \ln(4x - 1) - 2\ln 2$

Domain of validity : $x \in \left]1, +\infty\right[$

$\ln\left(x^2 - 1\right) = \ln\frac{(4x-1)}{4}$

$x^2 - 1 = \frac{(4x-1)}{4}$

$4x^2 - 4 = 4x - 1$

$4x^2 - 4x - 3 = 0$

$4x^2 - 6x + 2x - 3 = 0$

$2x(2x-3) + 1(2x-3) = 0$

$(2x-3)(2x+1)=0$

$x=\frac{3}{2}$ and $x=-\frac{1}{2}$

$e)\ \ln 2x=\ln 2.4$

$2x=2.4$

$x=1.2$

2.

$a)\ \ln^2 x=3-2\ln x; \quad x>0$

$let \quad t=\ln x$

$t^2+2t-3=0$

$t=-3 \quad t=1$

$for \quad t=1$

$\ln x=1$

$\ln x=\ln e$

$x=e$

$for \quad t=-3$

$\ln x=\ln e^{-3}$

$x=e^{-3}$

$$b)\begin{cases}2\ln x+3\ln y=-2\\3\ln x+5\ln y=-4\end{cases}$$

By using one of the methods of solving simultaneous equations, we find that

$\ln y=-2\Leftrightarrow y=e^{-2}$ and $\ln x=2\Leftrightarrow x=e^{2}$; therefore $S=\{e^{-2},e^{2}\}$

$$c)\ \begin{cases}\ln(xy)=7\\ \ln\frac{x}{y}=1\end{cases}\Leftrightarrow\begin{cases}\ln x+\ln y=7\\ \ln x-\ln y=1\end{cases}$$

Similarly, solving the system above, we find that

$\ln x = 4 \Leftrightarrow x = e^4$ and $\ln y = 3 \Leftrightarrow y = e^3$ therefore $S = \{e^3, e^4\}$

3. Solve each equation

$a) \log(x+2) = 2; \quad x+2 > 0; \quad x > -2$

$\log(x+2) = \log 10^2$ then $x = 98$; therefore $S = \{98\}$

$b) \log x + \log(x^2 + 2x - 1) - \log 2 = 0; \quad x > 0, \quad x^2 + 2x - 1 > 0$

$S = \{1\}$

$c) \log(35 - x^3) = 3\log(5 - x)$

$\log(35 - x^3) = \log(5 - x)^3$

$35 - x^3 = 125 - 75x + 15x^2 - x^3$

$15x^2 - 75x + 90 = 0$

$x^2 - 5x + 6 = 0$

$x = 3 \text{ or } x = 2 \quad \therefore S = \{2, 3\}$

$d) \log(1 - x) = -1; \quad 1 - x > 0 \Leftrightarrow x < 1$

$\log(1 - x) = \log 10^{-1}$

$x = \frac{9}{10} \qquad \therefore S = \left\{\frac{9}{10}\right\}$

$e) \log(3x-2) + \log(3x-1) = \log(4x-3)^2; \quad x > \frac{2}{3}$

$(3x-2)(3x-1) = (4x-3)^2$

Then solving the equation and respecting the condition, we have

$S = \left\{\frac{15 + \sqrt{29}}{14}\right\}$

$f) 2\log(2x-1) - \log(2x + 3x^2) = \log(3x - 7) - \log x$

Domain and validity

$$x \in \left]\frac{7}{3}, +\infty\right[$$

$$x(2x-1)^2 = (2x+3x^2)(3x-7)$$

Then solving the equation and respecting the condition, we have

$$S = \left\{\frac{11+\sqrt{421}}{10}\right\}$$

4. In $\mathbb{R}^2$

$$a)\begin{cases} x+y=9 \\ xy=14 \end{cases} \Rightarrow \begin{cases} x=7; y=2 \\ x=2; y=7 \end{cases}$$

$$b)\begin{cases} x^2+y^2=221 \\ xy=110 \end{cases} \qquad c)\begin{cases} x-y=-8 \\ x-\frac{3}{7}y=0 \end{cases} \qquad d)\begin{cases} x^3+y^3=35 \\ x-6y=0 \end{cases}$$

for b); c) and d), we can use one of the method of simultaneous equation to get value of $x \text{ and } y$.

Lesson 4: Limit of logarithmic function with base e

a) Learning objective

Calculate limit of logarithmic function with base e.

b) Teaching resources:

Textbooks, Ruler, T-square, Scientific calculators, graph papers

If possible, students may use mathematical software such as Geogebra or Microsoft Excel to plot the graph of logarithmic functions.

c) Prerequisites/Revision/Introduction:

Student-teachers will perform well in this unit if they have a good background on **properties and operations** common logarithms (year 1: unit6), solving equations and inequalities in the set of real numbers (year 1: unit 5), domain and range of polynomial, rational and irrational functions (year2: unit5), limits of polynomial, rational and irrational functions (year2: Unit 6&7).

d) Learning activities :

- Form groups and provide each group with **activity 4.1.4.1**
- Ask them to complete the given table and discuss how to determine the required **limits logarithmic function with base e**.
- Move around to ensure all student-teachers in groups participate actively.
- Call upon groups to present their findings.
- Harmonize their findings by leading students to calculate $\lim_{x\to 0^+} \ln x$ and $\lim_{x\to +\infty} \ln x$.
- Lead students to work on **example** and let them work individually **Application activity 4.1.4.1** for assessment.

Answer for activity 4.1.4.1

x	0.5	0.001	0.0001	2	100	1001	10000
$y = \ln x$	-0.69	-6.90	-9.21	0.69	4.60	6.908	6.907

When the independent variable x takes values approaching 0 from the right, $y = \ln x$ takes the big negative values. We write $\lim_{x\to 0^+} \ln x = -\infty$.

1. When x takes greater values, $y = \ln x$ takes also greater values. Therefore, $\lim_{x\to +\infty} \ln x = +\infty$.

2. It is senseless to discuss $\lim_{x\to 0^-} \ln x$ because $y = \ln x$ is not defined for negative values of x.

Application Activity 4.1.4.1

I.

1) $-\infty$ 2) 0 3) $+\infty$ 4) $+\infty$

II.

1) $\lim_{x\to +\infty} \ln(7x^3 - x^2 + 1) = +\infty$

2) $\lim_{x\to 1^+} \left(\ln \frac{1}{x-1} \right) = \lim_{x\to 1^+} \left[\ln 1 - \ln(x-1) \right] = \lim_{x\to 1^+} \ln 1 - \lim_{x\to 1^+} \ln\left(x-1\right) = 0 - (-\infty) = +\infty$

3) $\lim_{a\to 4^+} \ln \frac{a}{\sqrt{a-4}} = \ln(\lim_{a\to 4^+} \frac{a}{\sqrt{a-4}}) = +\infty$

III. $\lim_{x\to+\infty}\frac{\ln x}{x}=0$, $\lim_{x\to0^+}\frac{\ln x}{x}=-\infty$, $\lim_{x\to1}\frac{\ln x}{x}=0$, $\lim_{x\to\frac{1}{5}}\left(\frac{\ln x}{x}\right)=\frac{\ln\frac{1}{5}}{\frac{1}{5}}=5\left(\ln 5^{-1}\right)=-5\ln 5$

Lesson 5: Limit of logarithmic functions with any base

a) Learning objective

Calculate limit of logarithmic function with any base.

b) Teaching resources:

T-square, ruler, text book, if possible mathematical software such as Geogebra, Microsoft Excel, Mathlab.

c) Prerequisites/Revision/Introduction:

Student-teachers will perform well in this unit if they have a good background on **properties and operations** common logarithms (year 1: unit6), solving equations and inequalities in the set of real numbers (year 1: unit 5), domain and range of polynomial, rational and irrational functions (year2: unit5), limits of polynomial, rational and irrational functions (year2: Unit 6&7).

d) Learning activities :

- Through group discussions invite student-teachers to do all questions of **activity 4.1.4.2** and motivate them to complete table and deduce the continuity of a logarithmic function in a given point.
- Ask them to complete the given table and discuss how to determine the required **limits logarithmic function with any base**.
- Move around to ensure all student-teachers in groups participate actively.
- Call upon groups to present their findings.
- Harmonize their findings by leading students to calculate $\lim_{x\to+\infty}\log_a u(x)$.
- Invite group representatives to present their findings, and then help all student-teachers to conclude on the continuity of a logarithmic function and how to plot its graph.
- Let student-teachers work on **example**; then work individually **application activities 4.1.4.2** to assess the competences.

Answer for activity 4.1.4.2

1. Complete the table

$x=x_0$	$y=\log_2 x$	$\lim_{x\to x_0}\log_2 x$

$\frac{1}{4}$	-2	-2
$\frac{1}{2}$	-1	-1
1	0	0
2	1	1
4	2	2

Application Activity 4.1.4.2

1.

a) $\lim_{x\to 2^-} \log_5\left(x^2-5x+6\right) = -\infty$

b) $\lim_{x\to +\infty} \frac{2+4\log x}{x} = 0$

2. a) $+\infty$ b) $+\infty$ c) $-\infty$ d) $+\infty$

Lesson 6: Asymptotes of the graph of logarithmic functions (including base e and any base)

a) Learning objective

Determine and interpret possible asymptotes of the graph of logarithmic functions (including base e and any base).

b) Teaching resources

T-square, ruler, text book, if possible Mathematical software such as Geogebra, Microsoft Excel, Mathlab.

c) Prerequisites/Revision/Introduction:

Student-teachers will perform well in this unit if they have a good background on **properties and operations** common logarithms (year 1: unit6), solving equations and inequalities in the set of real numbers (year 1: unit 5), domain and range of polynomial, rational and irrational functions (year2: unit5), limits of polynomial, rational and irrational functions (year2: Unit 6&7).

d) Learning activities :

- Through group discussions invite student-teachers to do all questions of **activity 4.1.4.3** and motivate them to complete table and deduce **the**

continuity of a logarithmic function in a given point.

- Help learners to find **the asymptotes** of the graph of logarithmic functions.
- Invite group representatives to present their findings, and then help all student-teachers to conclude on the continuity of a logarithmic function and how to plot its graph.
- Let student-teachers work on **example 4.1.4.3** and work individually **Application activities 4.1.4.3** to assess the competences.

Answer for activity 4.1.4.3

1. $Domf =]0,+\infty[$. For $a>1,\ \ln a>0$,

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+}\frac{\ln x}{\ln a} = \frac{\lim\limits_{x\to 0^+}\ln x}{\ln a} = \frac{-\infty}{\ln a} = -\infty$$

There is a vertical asymptote $VA \equiv x = 0$

$$\lim_{x\to +\infty} f(x) = \lim_{x\to +\infty}\frac{\ln x}{\ln a} = \frac{\lim\limits_{x\to +\infty}\ln x}{\ln a} = \frac{+\infty}{\ln a} = +\infty$$

There is no horizontal asymptote

$$\lim_{x\to +\infty}\frac{f(x)}{x} = \lim_{x\to +\infty}\frac{\ln x}{x\ln a} = \frac{\lim\limits_{x\to +\infty}\frac{\ln x}{x}}{\ln a} = \frac{0}{\ln a} = 0$$

There is no oblique asymptote

2. For $0<a<1,\ \ln a<0$,

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+}\frac{\ln x}{\ln a} = \frac{\lim\limits_{x\to 0^+}\ln x}{\ln a} = \frac{-\infty}{\ln a} = +\infty$$

There is a vertical asymptote $VA \equiv x = 0$

$$\lim_{x\to +\infty} f(x) = \lim_{x\to +\infty}\frac{\ln x}{\ln a} = \frac{\lim\limits_{x\to +\infty}\ln x}{\ln a} = \frac{+\infty}{\ln a} = -\infty$$

There is no horizontal asymptote

$$\lim_{x\to +\infty}\frac{f(x)}{x} = \lim_{x\to +\infty}\frac{\ln x}{x\ln a} = \frac{\lim\limits_{x\to +\infty}\frac{\ln x}{x}}{\ln a} = \frac{0}{\ln a} = 0$$

There is no oblique asymptote.

Hence,

$$\lim_{x\to 0^+} f(x) = \begin{cases} -\infty & if\ \ a>1 \\ +\infty & if\ \ 0<a<1 \end{cases}$$

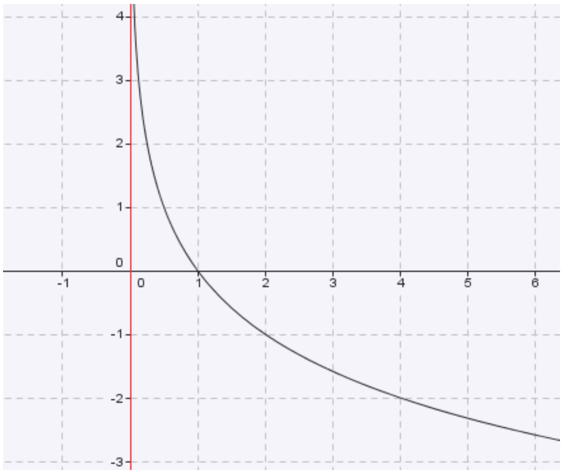
There is a vertical asymptote $VA \equiv x=0$

$$\lim_{x\to +\infty} f(x) = \begin{cases} +\infty & if\ \ a>1 \\ -\infty & if\ \ 0<a<1 \end{cases}$$

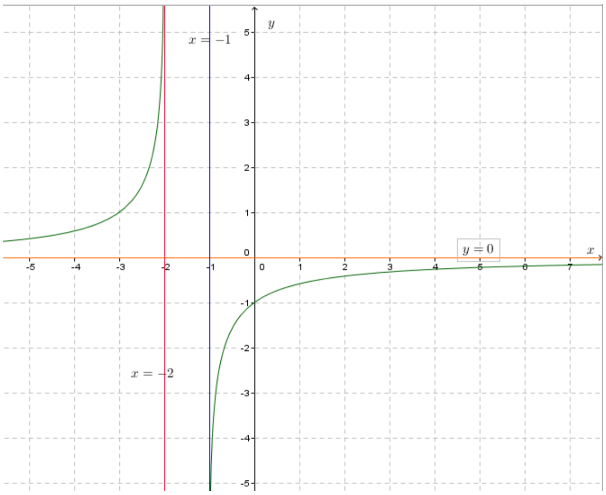
There is no horizontal asymptote. In addition no oblique asymptote

Application activity 4.1.4.3

1. $+\infty$, then the asymptotes are : $V.A \equiv x=0$ as shown in the graph.

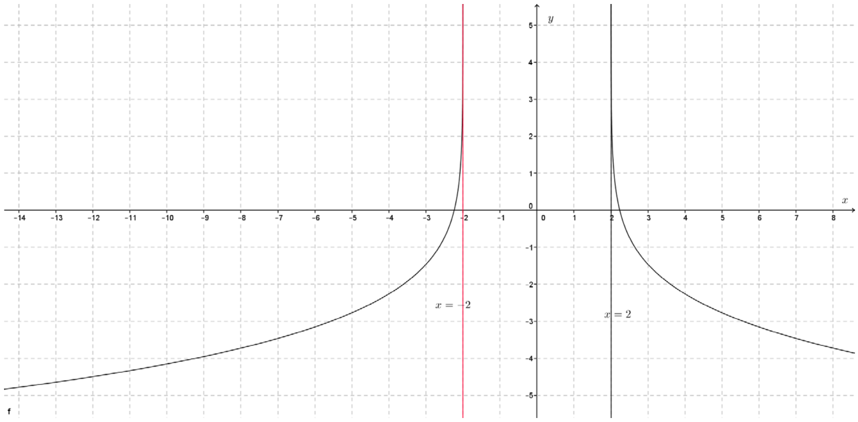


2. 2) $+\infty$, then the asymptotes are : $V.A \equiv x=-2; V.A \equiv x=-1\ and\ H.A = y=0$



3. $-\infty$, then the asymptotes are : $V.A \equiv x=-2; V.A \equiv x=-1$ and $H.A = y = 0$

4. $+\infty$, then the asymptotes are : $V.A \equiv x=-2; V.A \equiv x=2$ as shown in the graph below.



Lesson 7: Derivative for logarithmic functions

a) Learning objective

Determine the derivative of logarithmic functions.

b) Teaching resources:

- T-square, ruler, Learner's books, if possible Mathematical software such as geogebra, Microsoft Excel, Mathlab,….

c) Prerequisites/Revision/Introduction:

Student-teachers will perform well in this unit if they have a good background on **properties and operations** common logarithms (year 1: unit6), solving equations and inequalities in the set of real numbers (year 1: unit 5), domain and range of polynomial, rational and irrational functions (year2: unit5), limits and derivatives of polynomial, rational and irrational functions (year2: Unit 6&7).

d) Learning activities :

- Form groups and invite student-teachers to do tasks of **activity 4.1.5.1**.
- Walk around to different group and guide student-teachers to determine **the derivative** of the function $f(x)=\ln(x)$ in a point for which $x_0=2$ by the use of the **definition of derivative of a function.**

$$y' = \lim_{h\to 0}\frac{\ln(x+h)-\ln x}{h} = \lim_{h\to 0}\ln\left(\frac{x+h}{x}\right)^{\frac{1}{h}}$$

$$= \lim_{h\to 0}\ln\left(1+\frac{h}{x}\right)^{\frac{1}{h}} = \ln\lim_{h\to 0}\left(1+\frac{h}{x}\right)^{\frac{1}{h}} = \ln e^{\frac{1}{x}} = \frac{1}{x}$$

- Invite some group members to present their findings.

Harmonize the results by highlighting the derivative of the function $f(x)=\ln(x)$, $\frac{d}{dx}(\ln(u(x))$, $\frac{d}{dx}(\log_a x)$ and $\frac{d}{dx}[\log_a u(x)]$.

- Guide student-teachers to work through different **examples**; then work individually **application activities 4.1.5.1** to assess the competences.

Answeer for activity 4.1.5.1

h	$\frac{\ln(2+h)-\ln 2}{h}$
-0.1	0.5129329
-0.001 -0.00001	0.5001250 ≃ ½ 0.5000013 ≃ ½
-0.0000001	0.5000000 ≃ ½
0.1	0.4879016
0.001	0.4998750 ≃ ½
0.00001	0.4999988 ≃ ½
0.0000001	0.50000002 ≃ ½

$f'(2)=\lim_{h\to 0}\frac{\ln(2+h)-\ln 2}{h}\simeq\frac{1}{2}$, these results reflect that $f'(x)=\lim_{h\to 0}\frac{\ln(x+h)-\ln x}{h}=\frac{1}{x}$

The number $f'(2)$ is the slope of the tangent line to the curve $y=f(x)=\ln x$ at the point P$(2,\ln 2)$.

2.

a)

$$f(x)=\frac{\ln x}{\ln 2}\Rightarrow f'(x)=\frac{\frac{1}{x}}{\ln 2}=\frac{1}{x\ln 2}$$

b)

$$\left(\frac{\ln x^2}{\ln 2}\right)' == \frac{2x}{x^2 \ln 2} = \frac{2}{x \ln 2}$$

Application Activity 4.1.5.1

I. $y = \ln\sqrt{\frac{1+x}{1-x}}$

Here $y = \ln\sqrt{\frac{1+x}{1-x}} = \ln\sqrt{1+x} - \ln\sqrt{1-x} = \frac{1}{2}\left[\ln(1+x) - \ln(1-x)\right]$

Therefore $\frac{dy}{dx} = \frac{1}{2}\left(\frac{1}{1+x} + \frac{1}{1-x}\right) = \frac{1}{1-x^2}$

II.1) $f'(x) = \frac{2\ln x}{x}$ 2) $g'(x) = \frac{\tan^2 x + 1}{\tan x}$ 3) $h'(x) = \frac{x}{x^2-1}$ 4) $k'(x) = \frac{2}{x^2-1}$

III. 1) $f'(x) = \frac{2x+2}{\left(x^2+2x+1\right)\ln 10}$ 2) $g'(x) = -\frac{6}{\left(x^2-4x-5\right)\ln 2}$

IV. 1) $\frac{dy}{dx} = \frac{(x-2)(x+1)}{(x-1)(x+3)}\left(\frac{1}{x-2} + \frac{1}{x+1} - \frac{1}{x-1} - \frac{1}{x+3}\right)$

2) $\frac{dy}{dx} = \frac{(2x-1)\sqrt{x+2}}{(x-3)\sqrt{(x+1)^3}}\left(\frac{2}{2x-1} + \frac{1}{2(x+1)} - \frac{1}{x-3} - \frac{3}{2(x+1)}\right)$

3) $\frac{dy}{d\theta} = 3\theta \sin\theta \cos\theta\left(\frac{1}{\theta} + \tan\theta - \cot\theta\right)$

4) $\frac{dy}{dx} = \frac{x^3 \ln 2x}{e^x \sin x}\left(\frac{3}{x} + \frac{1}{x\ln 2x} - 1 - \cot x\right)$

5) $\frac{dy}{dx} = \frac{2x^4 \tan x}{e^{2x} \ln 2x}\left(\frac{4}{x} + \frac{1}{\sin x \cos x} - 2 - \frac{1}{x \ln 2x}\right)$

Lesson 8: Variation and graphical representation of logarithmic functions (for base e and for any base)

a) Learning objective

Apply the 1st and the 2nd derivative to investigate the minimum and maximum (EXTREMA) of logarithmic functions and to find concavity of the graph.

b) Teaching resources:

Learner's books, ruler, T-square, scientific calculator.

b) Prerequisites/Revision/Introduction:

Student-teachers will perform well in this unit if they have a good background on **properties and operations** common logarithms (year 1: unit6), solving equations and inequalities in the set of real numbers (year 1: unit 5), domain and range of polynomial, rational and irrational functions (year2: unit5), limits and derivatives of polynomial, rational and irrational functions (year2: Unit 6&7).

c) Learning activities :

- Through group discussions invite student-teachers to do all questions of **activity 4.1.5.2**
- During group work, motivate student-teachers to verify whether the function $f(x)=\ln u(x)$ and $g(x)=\log_a u(x)$ are **increasing** or **decreasing** on a given interval, to deduce the tables of signs for $f'(x)$ and $g'(x)$ so as to establish the variation of those functions on their domain and find **minimum point** and **maximum point.**
- Call them to deduce the tables of signs for $f''(x)$ and $g''(x)$ so as to establish the variation table of those functions and check **the concavity.**
- Assign Student-teachers to verify whether the functions $f(x)=\ln u(x)$ and $g(x)=\log_a u(x)$ **concave up** or **concave down** and state the **inflection point.**
- Assign Student-teachers to find other points in the function $f(x)=\ln u(x)$ and $g(x)=\log_a u(x)$ and use those points to sketch **the graph** of the functions.
- Invite some group members to present their findings.
- Harmonize the results emphasizing that the function $f(x)=\log_a x$ is

strictly increasing on $\mathbb{R}_0^+$ for $a > 1$ and that $f(x) = \log_a x$ is strictly decreasing on $\mathbb{R}_0^+$ for $0 < a < 1$.

- Guide student-teachers to work through **examples**, then work individually **application activities 4.1.5.2** to assess the competences.

Answer for activity 4.1.5.2

1. $f(2) = 0.693$ and $f(10) = 2.303$, $g(2) = 0.301$ and $g(10) = 1$

Therefore, both functions $f(x)$ *and* $g(x)$ are increasing on the closed interval $[2,10]$ because $f(10) - f(2) > 0$ *and* $g(10) - g(2) > 0$

2. The variation table of $f(x)$ and $f'(x) = \frac{1}{x}$ on the domain $]0, +\infty[$

x	0		e		$+\infty$
y'	+ + + +	+ +	1/e +	+ + + +	0
y	$-\infty$	↗	1	↗	$+\infty$

The variation table of $g(x)$ and $g'(x) = \frac{1}{x\ln 10}$ on the domain $]0, +\infty[$

x	0		10		$+\infty$
y'	+ + + +		1/10(ln10)	+ + + + +	0
y	$-\infty$	↗	1	↗	$+\infty$

3. The difference $f(10) - f(2) = 1.61 > 0$ *and* $g(10) - g(2) = 0.699 > 0$, prove that the function f is increasing faster than g on the interval $[2,10]$.

Application Activity 4.1.5.2

1. Variation of the function $f(x)=\dfrac{\ln(x-2)}{x-2}$

- $f(x)$ is defined $\Leftrightarrow x-2>0$. That if $x>2$.
- $Domf=\left]2,+\infty\right[$
- $\lim\limits_{x\to 2^+}\dfrac{\ln\left(x-2\right)}{x-2}=-\infty$, we have a vertical asymptote $x=2$
- $\lim\limits_{x\to +\infty}\dfrac{\ln\left(x-2\right)}{x-2}=0$, we have an horizontal asymptote $y=0$
- $f'\left(x\right)=\dfrac{d}{dx}\left[\dfrac{\ln\left(x-2\right)}{x-2}\right]=\dfrac{\dfrac{1}{x-2}\times(x-2)-1\times\ln\left(x-2\right)}{(x-2)^2}=\dfrac{1-\ln(x-2)}{(x-2)^2}$
- $f'\left(x\right)=0\Leftrightarrow 1-\ln\left(x-2\right)=0$

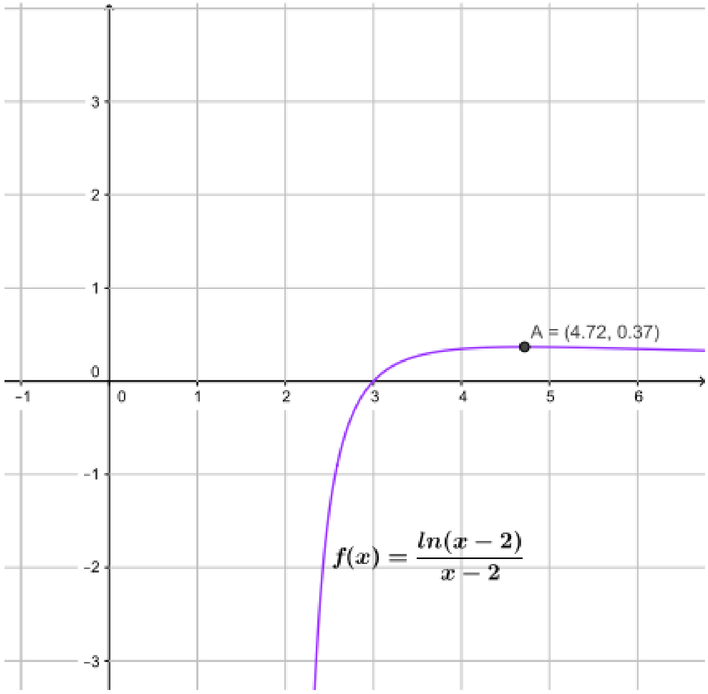
$$\ln\left(x-2\right)=1$$
$$\ln\left(x-2\right)=\ln e$$
$$x-2=e$$
$$x=e+2$$

- $f\left(e+2\right)=\dfrac{\ln(\mathrm{e}+2-2)}{e+2-2}=\dfrac{\ln e}{e}=\dfrac{1}{e}$

- Variation table of $f\left(x\right)$

x	2 $\qquad$ $(e+2)$ $\qquad$ $+\infty$
y'	+ + + + 0 -- - - - -
y	$-\infty$ ↗ $1/e$ Max[(e+2) ,1/e] ↘ 0

Graph of f(x)



2. $h(t) = 100\ln(t \;+1)$

a) $v(t) = \frac{d}{dx}\left[100\ln(t+1)\right] = \frac{100}{t+1}$

b) When $t = 2$ sec,

$v(\text{t})\ \Big|_{t=2} = \frac{100}{2+1} m/\sec = \frac{100}{3} m/\sec$ m/s

c) The velocity is increasing.

1. a) $f(x) = \ln\left(x^2\right)$

i) Asymptote(s)

$x^2 > 0 \Rightarrow x \in \left]-\infty, 0\right[\cup \left]0, +\infty\right[\quad \therefore VA \equiv x = 0$

ii) Limits

$\lim_{x\to-\infty} \ln x^2 = +\infty \quad \lim_{x\to0^-} \ln x^2 = -\infty \quad \lim_{x\to0^+} \ln x^2 = -\infty \quad \lim_{x\to+\infty} \ln x^2 = +\infty$

iii) Increasing and decreasing.

$$f'(x) = \frac{2}{x} \quad i.e\ x \neq 0$$

Variation table

x	$-\infty$	0	$+\infty$
2	+ + + + + + + + + + + +	+ + + + + + + + + + + +	
x	- - - - - - - - - - - - -	0 + + + + + + + + + + + +	
$\frac{2}{x}$	- - - - - - - - - - - - -	‖ + + + + + + + + + + + +	

Thus, for $]-\infty, 0[$; $f(x)$ decreases, and for $]0, +\infty[$; $f(x)$ increases

There is no minimum and no maximum point.

iv) Concavity

$$f''(x) = -\frac{2}{x^2} \quad i.e\ x \neq 0$$

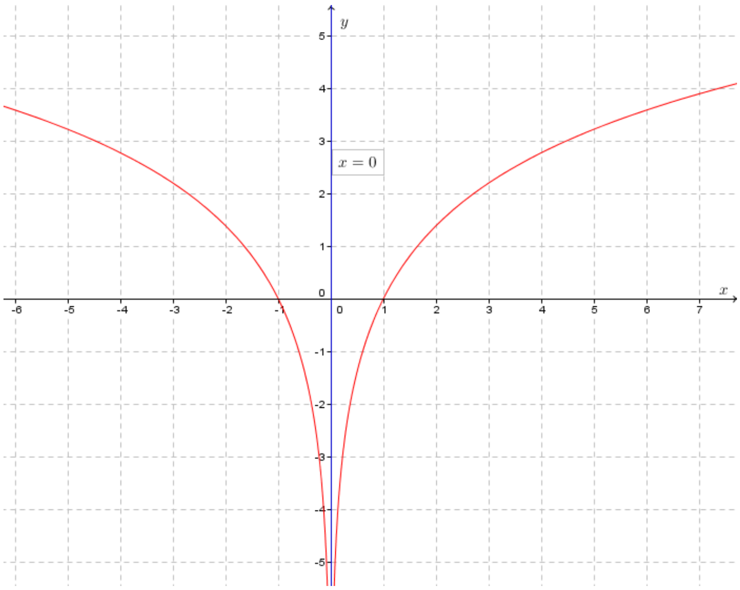
Variation table

x	$-\infty$	0	$+\infty$
-2	− − − − − − −	− − − − − − −	
x^2	+ + + + + + +	0 + + + + + + +	
$\frac{2}{x}$	− − − − − −	‖ − − − − −	

Thus, for $]-\infty, 0[$; $f(x)$ concaves down and for $]0, +\infty[$; $f(x)$ concaves down.

There is no inflection point.

v) **Sketch**



Noted: *refer to the steps completed in the question 3. a) ; then guide student-teachers to get the correct answers of 3.b),...,3.f)*

Lesson 9: Definition, Domain and range of exponential functions (base e and for any base)

a) Learning objective

- Define logarithm or exponential functions in base e and for any base.
- Find domain and range of exponential functions base e and for any base.

b) Teaching resources:

Learner's book and other reference textbooks, ruler, T-square, scientific calculator; if possible, mathematical software and internet.

c) Prerequisites/Revision/Introduction:

Student-teachers will perform well in this unit if they have a good background on: indices and radicals and common logarithms (year 1: unit6), solving equations and inequalities in the set of real numbers (year 1: unit 5), domain and range of polynomial, rational and irrational functions (year2: unit5).

d) Learning activities:

- Ask student-teachers to do all questions in **activity 4.2.1** with the aim of establishing the domain of the function $g(x)=e^x$ inverse of $f(x)=\ln x$, the domain of $h(x)=3^x$ and their ranges.

- During group discussions, move around to each group and prompt them to discuss the domain and range of the exponential functions $p(x)=a^x$ in case $a>0, a\neq 1$ and $a=1$.
- Lead student-teachers to harmonize the results by generalizing how to find **the domain and the range** of the exponential function $f(x)=a^{u(x)}$ where $u(x)$ is a function of x.
- Guide student-teachers to work through **examples**, then work individually **application activities 4.2.1** to assess the competences.

Answer for activity 4.2.1

1. If $f(x)=\ln x$, let us complete the following table:

x	0	1	e	e^2	$\ln 3$	$\ln 4$
$g(x)=f^{-1}(x)$	1	e	e^e	e^{e^2}	3	4

The set of all values of $g(x)$ is composed of all positive real numbers; that is range

$g=\mathbb{R}^+=]0,+\infty[$

2. Consider the function $h(x)=3^x$ and complete the following table

x	-10	-1	0	1	10
$h(x)=3^x$	$\frac{1}{3^{10}}$	$\frac{1}{3}$	1	3	3^{10}

a) $\forall x\in\mathbb{R}, h(x)\in\mathbb{R}^+$, the domain of $h(x)$ is $\mathbb{R}=]-\infty,+\infty[$

b) All values $h(x)$ are positive, therefore, the range of $h(x)$ is $\mathbb{R}^+=]0,+\infty[$.

Application Activity 4.3.1

I. a) $f(x)=5e^{2x}$,

$\forall x\in\mathbb{R}, f(x)x\in\mathbb{R}^+$, we realize that $domf=]-\infty,+\infty[$ and the range is the interval $]0,+\infty[$

e) $\mathbb{R}\setminus\{2,5\}$ f) $\mathbb{R}$ g) $]0,+\infty[$ h) $[4,+\infty[$

II. a) $h(x)=2^{\ln x}$

$h(x)\in\mathbb{R}$ if $x>0$, therefore, $domh=]0,+\infty[$.

The range is the set of all $h(x)=2^{\ln x}, x\in\mathbb{R}^+$. That is range h= $\mathbb{R}^+=]0,+\infty[$

b) $g(x)=3^{\left(\frac{x+1}{x-2}\right)}$

Condition for the existence of $\frac{x+1}{x-2}$ in $\mathbb{R}$: $x\neq 2$.

Therefore, $Dom\ g=\mathbb{R}\setminus\{2\}=]-\infty,2[\cup]2,+\infty[$. Its range is $]0,3[\cup]3,+\infty[$.

III.1) $\mathbb{R}\setminus\{-5,-2\}$ 2) $\mathbb{R}$ 3) $]-\infty,-1]\cup]3,+\infty[$ 4) $]-\infty,-3[\cup]-2,+\infty[$

Lesson 10: Properties and operations on exponential functions (for base e and for any base)

a) Learning objective

- State and demonstrate properties of exponents for base e and for any base.

b) Teaching resources:

Scientific calculators to evaluate exponents, textbooks, graph papers. If possible, the use of Mathematical software such as Geogebra to plot graph of exponential functions and/or Microsoft Excel to compute values of a function and Internet to facilitate research would be plausible.

c) Prerequisites/Revision/Introduction:

Student-teachers will perform well in this unit if they have a good background on properties of indices and radicals and common logarithms (year 1: unit6),.

d) Learning activities:

This lesson deals with properties of exponential functions to simplify them. To this end,

- Form groups and ask student-teachers to work on **activity 4.2.2**
- Ask randomly some groups to present their findings to the whole class;
- Lead student-teachers to give comments on previous presentation before the next one.

- Facilitate student-teachers to states **all properties of exponents for base e and for any base**. Through given real numbers, discuss their use.
- Facilitate student-teachers to simplify mathematical arguments involving exponents and then from their answers, write a short summary.
- Guide student-teachers to work on **examples**;
- Call them to do individually the **application activities 4.2.2** to assess their competences.

Answer for activity 4.2.2

1.

x	-3	-2	-1	0	1	2
$a)e^{3x}$	e^{-9}	e^{-6}	e^{-3}	1	e^{3}	e^{6}
$b)e^{x+3}$	1	e^{1}	e^{2}	e^{3}	e^{4}	e^{5}
$c)e^{-x}$	e^{3}	e^{2}	e^{1}	1	e^{-1}	e^{-2}

2. $a)2^{-0.6}=1.4$ $\quad b)\pi^{0.75}=2.359$ $\quad c)\left(1.56\right)^{\sqrt{2}}=1.875$

Application Activity 4.3.2

1. $a)\left(e^{\frac{3}{2}}\right)\left(\frac{1}{e}\right)^{\frac{3}{2}}=e^{\frac{3}{2}-\frac{3}{2}}=1$ $\quad b)\left[\left(e^{-1}\right)\left(e^{\frac{2}{3}}\right)\right]^{3}=e^{-3+2}=e^{-1}$

2. $a)e^{\frac{1}{2}}=\sqrt{e}$ $\quad b)e^{\frac{3}{5}}\times e^{-5}=e^{\frac{3}{5}-5}=e^{\frac{-22}{5}}$ $\quad c)\left(e^{-5}e^{3}\right)^{\frac{1}{2}}=e^{-1}$ $\quad d)\left(e^{2}e^{\frac{1}{2}}\right)^{3}=e^{\frac{15}{2}}$

3. $a)\left(32^{\frac{3}{2}}\right)\left(\frac{1}{2}\right)^{\frac{3}{2}}=2^{\frac{15}{2}}2^{-\frac{3}{2}}=2^{6}$ $\quad b)\left[\left(8^{-1}\right)\left(8^{\frac{2}{3}}\right)\right]^{3}=8^{-3+2}=\frac{1}{8}$

4. For $-3\le x<3$;

$a)\left(0.005\right)^{x}$	$\frac{1}{125}\times10^{9}$	$\frac{1}{25}\times10^{6}$	$\frac{1}{5}\times10^{3}$	1	5×10^{-3}	25×10^{-6}

$b) 2^x 3^x$	$\frac{1}{8\times 27}=\frac{1}{216}$	$\frac{1}{36}$	$\frac{1}{6}$	1	6	36
$c) \left(\frac{1}{5}\right)^x$	125	25	5	1	$\frac{1}{5}$	$\frac{1}{25}$

5. If $x=-5, y=3, z=\frac{1}{2}$, simplify each expression

$a) \sqrt{140625}=375 \qquad b) 225\times 15^{-5}=15^{2-5}=\frac{1}{3375}$

$$c) \left(\frac{5^3}{4^5}\right)^{\frac{1}{2}}=\frac{\sqrt{5^3}}{\sqrt{4^5}}=\frac{5\sqrt{5}}{16\sqrt{5}}=\frac{5}{16}$$

Lesson 11: Exponential equations (including base e and any base)

a) Learning objective

- Use the properties of exponents to solve exponential equations.

b) Teaching resources:

Scientific calculators to evaluate logarithms, textbooks, graph papers. If possible, the use of Mathematical software such as Geogebra to plot graph of exponential equations and/or Microsoft Excel to compute values of a function and Internet to facilitate research would be plausible.

c) Prerequisites/Revision/Introduction:

Student-teachers will perform well in this unit if they have a good background on: indices and radicals and common logarithms (year 1: unit6), solving equations and inequalities in the set of real numbers (year 1: unit 5), domain and range of polynomial, rational and irrational functions (year2: unit5).

d) Learning activities:

This lesson deals with solving exponential equations.

- Form groups and ask student-teachers to work on **activity 4.2.3**
- Ask randomly some groups to present their findings to the whole class;
- Lead student-teachers to give comments on previous presentation before the next one.
- Facilitate student-teachers to discuss on the **domain of validity/**

condition of existence.

- Facilitate student-teachers to apply exponential properties and use **them to solve the** exponential **equations** and then after their answers, write a short summary.
- Guide student-teachers to work on **examples.**
- Call them to do individually the **application activities 4.2.3** to assess their competences.

Answer for activity 4.2.3

1. For which value(s), each function $f(x)$ below can be defined. Explain.

$a)\, f(x)$ is defined for real numbers.

$b)\, f(x)=e^{x^2-5x+6}$ is defined in $\mathbb{R}$.

2. $\ln 2^{1-x}=\ln 6$

$(1-x)\ln 2=\ln 6$

$\ln 2-x\ln 2=\ln 6$

$x=\dfrac{\ln 2-\ln 6}{\ln 2}$

Application Activity 4.3.3

1. Solve each equation for x or t.

$a) e^x=6\Leftrightarrow x=\ln 6 \qquad b) 5+e^{0.2t}=10\Leftrightarrow t=5\ln 5$

$c) e^{2x}=3e^x\Leftrightarrow \left(e^x\right)^2-3e^x=0$

$let\ e^x=t\Rightarrow (t)^2-3t=0\Leftrightarrow t=0\ or\ t=3$

$\bullet For\ t=0\Rightarrow e^x=0\Leftrightarrow no\ solution$

$\bullet For\ t=0=3\Rightarrow e^x=3\Leftrightarrow x=\ln 3$

$d) e^{2x}=e^x+12\Leftrightarrow \left(e^x\right)^2-e^x-12=0$

By letting $e^x=y\Rightarrow (y)^2-y-12=0$

$then\ (y)^2-y-12=0\Leftrightarrow y=4\ or\ y=-3$

$\bullet For\ y=4 \Rightarrow x=\ln 4$

$\bullet For\ y=-3 \Rightarrow no\ solution$

2. $a)2e^{-x+1}-5=9 \Rightarrow (-x+1)=\ln 7 \Rightarrow x=1-\ln 7$

$b)\dfrac{50}{1+12e^{-0.02x}}=10.5 \Leftrightarrow 50-10.5=126e^{-0.02x}$

$e^{0.02x}=\dfrac{126}{39.5} \Rightarrow x=50\ln\dfrac{126}{39.5}$

$c)e^{\ln x^2}-9=0 \Leftrightarrow \ln x^2=\ln 9$

$\therefore x=\pm 3$

$d)e^x-12=\dfrac{-5}{e^{-x}}$

3.

$\dfrac{e^{2x}+1}{2}=1e^x \Leftrightarrow e^{2x}-2e^x+1=0$

$let\ e^x=h \Rightarrow h^2-2h+1=0 \Leftrightarrow h=1$

$then\ x=0$

4. Find the value of marked letter in each equation.

$a)9^t+3^t=12\ by\ letting\ 3^t=b$

$b^2+b-12=0 \Rightarrow b=-4\ or\ b=3$

$\bullet For\ b=-4\ ; no\ solution.$

$\bullet For\ b=3; t=\ln 3$

$c)\dfrac{2^x}{4}-\dfrac{3^x}{9}=0 \Leftrightarrow 2^{x-2}=3^{x-2}$

$\Leftrightarrow (x-2)\ln 2=(x-2)\ln 3 \quad \Leftrightarrow x=\dfrac{2(\ln 2-\ln 3)}{\ln 2-\ln 3}=2$

5. In $\mathbb{R}^2$

$c)\begin{cases}5^{3x}=25^{2y-2}\\ 9^y=3^{x+1}\end{cases} \Leftrightarrow \begin{cases}5^{3x}=5^{2(2y-2)}\\ 3^{2y}=3^{x+1}\end{cases} \Leftrightarrow \begin{cases}3x=4y-2\\ 2y=x+1\end{cases} \Leftrightarrow S=\left\{\left(0,\frac{1}{2}\right)\right\}$

$d)\begin{cases}3^{x+1}=243\\ 2^y=64\end{cases} \Leftrightarrow \begin{cases}3^{x+1}=3^5\\ 2^y=2^6\end{cases} \Leftrightarrow \begin{cases}x+1=5\\ y=6\end{cases} \Leftrightarrow S=\{(4,6)\}$

Lesson 12: Limit of exponential Function with base e

a) Learning objective

Calculate limit of exponential functions with base e

b) Teaching resources

Student-teachers' book, calculator, ruler and T-square. If possible, mathematical software such us Geogebra, Microsoft Excel, Math lab and graphical can be used.

c) Prerequisites/Revision/Introduction:

Student-teachers will perform well in this unit if they have a good background on: indices and radicals and common logarithms (year 1: unit6), solving equations and inequalities in the set of real numbers (year 1: unit 5), domain and range of polynomial, rational and irrational functions (year2: unit5), limits of polynomial, rational and irrational functions (year2: Unit 6&7).

d) Learning activities:

- Form groups of students and explain instructions related to the task to be done in the **activity 4.2.4.1.**
- Monitor how students are performing the task and provide support where necessary to guide them on how to find the graph for the inverse of a given function and how to interpret the graph of $y = e^x$ to deduce $\lim_{x\to-\infty} e^x$ and $\lim_{x\to+\infty} e^x$.
- Invite representatives of groups to present their findings.
- Decide to engage the class into exploitation of student- teachers' findings.
- Give the summary of expected feedback based on student's answers.
- Ask student-teachers to work through **examples** in the student-teachers' book and work individually **application activities 4.2.4.1** to assess the competences.

Answer for activity 4.2.4.1

1. Completed table

x	e^x
-1	0.367879441171442
-2	0.135335283236613
-5	0.00673794699909

x	e^x
1	2.7182818
2	7.3890561
5	148.4131591

-15	0.00000030590232		15	3269017.3724721
-30	0.0000000000009		30	10686474581524.5

2. From table in 1), when x takes values approaches to $-\infty$, e^x takes values closed to zero. Hence $\lim_{x\to-\infty} e^x = 0$. There exists a horizontal asymptote $y=0$, no oblique asymptote.

Also, when x takes values approaches to $+\infty$, e^x increases without bound. Hence $\lim_{x\to+\infty} e^x = +\infty$. There is no horizontal asymptote.

Application Activity 4.2.4.1

Evaluate limit of the function $f(x)$ at $+\infty$ and $-\infty$ in each of the following case.

1. $f(x) = e^{8+2x-x^3}$

$\lim_{x\to+\infty} e^{8+2x-x^3} = 0$ and $\lim_{x\to-\infty} e^{8+2x-x^3} = +\infty$

2. $f(x) = e^{\frac{6x^2+x}{5+3x}}$

$\lim_{x\to+\infty} e^{\frac{6x^2+x}{5+3x}} = e^{\lim_{x\to+\infty}\frac{6x^2+x}{5+3x}} = +\infty$ and $\lim_{x\to-\infty} e^{\frac{6x^2+x}{5+3x}} = e^{\lim_{x\to-\infty}\frac{6x^2+x}{5+3x}} = 0$

3. $f(x) = 2e^{6x} - e^{-7x} - 10e^{4x}$

$\lim_{x\to+\infty}\left(2e^{6x} - e^{-7x} - 10e^{4x}\right) = \lim_{x\to+\infty} e^{6x}\left(2 - e^{-13x} - 10e^{-2x}\right) = +\infty$

$\lim_{x\to-\infty}\left(2e^{6x} - e^{-7x} - 10e^{4x}\right) = \lim_{x\to-\infty} e^{-7x}\left(2e^{13x} - 1 - 10e^{11x}\right) = -\infty$

4. $f(x) = 3e^{-x} - 8e^{-5x} - e^{10x}$

$\lim_{x\to+\infty}\left(3e^{-x} - 8e^{-5x} - e^{10x}\right) = \lim_{x\to+\infty} e^{10x}\left(3e^{-11x} - 8e^{-15x} - 1\right) = -\infty$

$\lim_{x\to-\infty}\left(3e^{-x} - 8e^{-5x} - e^{10x}\right) = \lim_{x\to-\infty} e^{-5x}\left(3e^{4x} - 8 - e^{15x}\right) = -\infty$

5. $f(x) = \dfrac{e^{-3x} - 2e^{8x}}{9e^{8x} - 7e^{-3x}}$

$$\lim_{x\to+\infty}\frac{e^{-3x}-2e^{8x}}{9e^{8x}-7e^{-3x}}=\frac{-2}{9}$$ and

$$\lim_{x\to-\infty}\left(\frac{e^{-3x}-2e^{8x}}{9e^{8x}-7e^{-3x}}\right)=\lim_{x\to-\infty}\frac{e^{-3x}-2e^{8x}}{9e^{8x}-7e^{-3x}}$$

$$=\lim_{x\to-\infty}\frac{e^{-3x}\left(1-2e^{11x}\right)}{e^{-3x}\left(-7+9e^{11x}\right)}=-\frac{1}{7}$$

Lesson 13: Limit of exponential functions with any base

a) Learning objective

Calculate limit of exponential functions with any base.

b) Teaching resources:

Student-teachers' book, calculator, ruler and T-square. If possible, mathematical software such as Geogebra, Microsoft Excel, Math lab and graph calc can be used.

c) Prerequisites/Revision/Introduction:

Student-teachers will perform well in this unit if they have a good background on: indices and radicals and common logarithms (year 1: unit6), solving equations and inequalities in the set of real numbers (year 1: unit 5), domain and range of polynomial, rational and irrational functions (year2: unit5), limits of polynomial, rational and irrational functions (year2: Unit 6&7).

d) Learning activities:

- Form groups of students and explain instructions related to the task to be done in the **activity 4.2.4.2**
- Monitor how students are performing the task and provide support where necessary to guide them on how to find the graph for the inverse of a given function and how to interpret the graph of $\lim_{x\to-\infty}a^x$ and $\lim_{x\to+\infty}a^x$ for any values of a.
- Judge the logic of their findings, correct those which are false, complete those which are incomplete, and confirm those which are correct and guide the students to conclude about $\lim_{x\to-\infty}a^x$ and $\lim_{x\to+\infty}a^x$ for any values of a.
- Invite representatives of groups to present their findings.
- Decide to engage the class into exploitation of students' findings.

- Give the summary of expected feedback based on students' answers.
- Ask student-teachers to work through **examples** in the student-teachers' book and work individually **application activities 4.2.4.2** to assess the competences.

Answer for activity 4.2.4.2

i) Completed table

x	$\left(\frac{1}{2}\right)^x$
-1	2
-2	4
-5	32
-15	32768
-30	1073741824

x	$\left(\frac{1}{2}\right)^x$
1	0.5
2	0.25
5	0.03125
15	0.0000305176
30	0.0000000009

From table in a), when x takes values approaches to $-\infty$, $\left(\frac{1}{2}\right)^x$ increases without bound.

Hence $\lim_{x\to-\infty}\left(\frac{1}{2}\right)^x = +\infty$.

Also, when x takes values approaches to $+\infty$, $\left(\frac{1}{2}\right)^x$ takes values closed to zero.

Hence $\lim_{x\to+\infty}\left(\frac{1}{2}\right)^x = 0$.

ii) In general

If $a > 1$, $\lim_{x\to-\infty} a^x = 0$ and $\lim_{x\to+\infty} a^x = +\infty$

If $0 < a < 1$, $\lim_{x\to-\infty} a^x = +\infty$ and $\lim_{x\to+\infty} a^x = 0$

Application Activity 4.2.4.2

1) 1 2) e^4 3) e^2 4) e 5) e^k 6) e^k

Lesson 14: Asymptotes of the graph of exponential functions.

a) Learning objective

Determine and interpret possible asymptotes of the graph of exponential functions

b) Teaching resources:

Student-teachers' book, T-square, ruler, papers, if possible computers and Math draw software such as Geogebra, Microsoft Excel, Mathlab and Internet to facilitate research.

c) Prerequisites/Revision/Introduction:

Student-teachers will perform better in this lesson if they have an understanding of the following concepts: calculations of limits of polynomial, rational and irrational functions (year2, unit5), powers and radicals and common logarithmic and exponential function (year1, Unit6), Limits of logarithmic and exponential functions (year3, unit4), solving equations and inequalities in the set of real numbers (year 1: unit 5), domain and range of polynomial, rational and irrational functions (year2: unit5), limits of polynomial, rational and irrational functions (year2: Unit 6&7).

d) Learning activities

- Using small groups, invite student-teachers to understand the **activity 4.2.4.3**, in the student-teachers' book, and ask them to find the domain and range of $f(x)=2^{x-2}$. Help student-teachers to find out and realize that domain of $f(x)$ is the set of all real number $Domf =]-\infty,+\infty[$ .and the range of $f(x)$ is $R =]0,+\infty[$
- In the same groups, let student-teachers find out that $\lim\limits_{x\to-\infty} 2^{(x-2)} = 0$, $\lim\limits_{x\to+\infty} f(x) = +\infty$ and $\lim\limits_{x\to-\infty} \frac{f(x)}{x}$ =0. Deduce that $y=0$ is an horizontal asymptote to the graph of $f(x)$.
- Finally, invite student-teachers to find that

  $\lim\limits_{x\to0^+} f(x) = \frac{1}{4}, \lim\limits_{x\to0^-} f(x) = \frac{1}{4}, and\ f(0) = \frac{1}{4}$ and plot

  the graph of $f(x)=2^{x-2}$ using the following points

$$x=4; f(4)=4; \text{and } x=-1; f(-1)=\frac{1}{8}$$

- Ask groups to present their findings to the whole class and then harmonize their works to provide the lesson summary.
- Let student-teachers work on activities in the **examples** and invite them to individually work out the **application activity 4.2.4.3** to assess their competences on continuity and asymptotes of exponential functions.

Answer for activity 4.2.4.3

Let $f(x)=2^{x-2}$ and $f(x) = e^{(x-2)}$

a) $Domf =]-\infty,+\infty[$ and $\operatorname{Im} f =]0,+\infty[$

b) $\lim\limits_{x\to-\infty} 2^{x-2} = 2^{-\infty-2} = \frac{1}{2^{+\infty}} = 0$. The equation of Horizontal asymptote is $y=0$

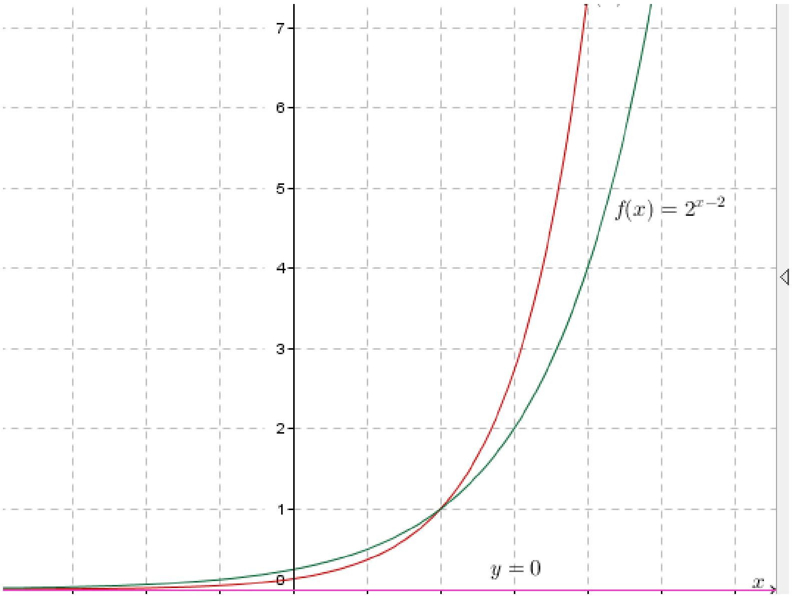
c) For $x=0, f(x)=2^{-2}=\frac{1}{4}$, therefore y-intercept of the graph is $(0,\frac{1}{4})$

d) $\lim\limits_{x\to+\infty} 2^{x-2} = 2^{+\infty-2} = 2^{+\infty} = +\infty$ and $\lim\limits_{x\to-\infty} \frac{2^{x-2}}{x} = \frac{2^{-\infty-2}}{-\infty} = \frac{1}{2^{\infty}(-\infty)} = \frac{1}{-\infty} = 0$

e) $\lim\limits_{x\to0^+} 2^{x-2} = 2^{0^+-2} = \frac{1}{2^2} = \frac{1}{4}$ and $\lim\limits_{x\to0^-} 2^{x-2} = 2^{0^--2} = \frac{1}{2^2} = \frac{1}{4}$. At $x=0, f(0)=\frac{1}{4}$.

Since the $\lim\limits_{x\to0^+} f(x) = \lim\limits_{x\to0^-} f(x) = \frac{1}{4}$, the function has the continuity at $x=0$

f) The graph of $f(x)=2^{x-2}$ and $f(x) = e^{(x-2)}$



Application Activity 4.2.4.3

Given that f is a function given by $f(x)=2^x+1$

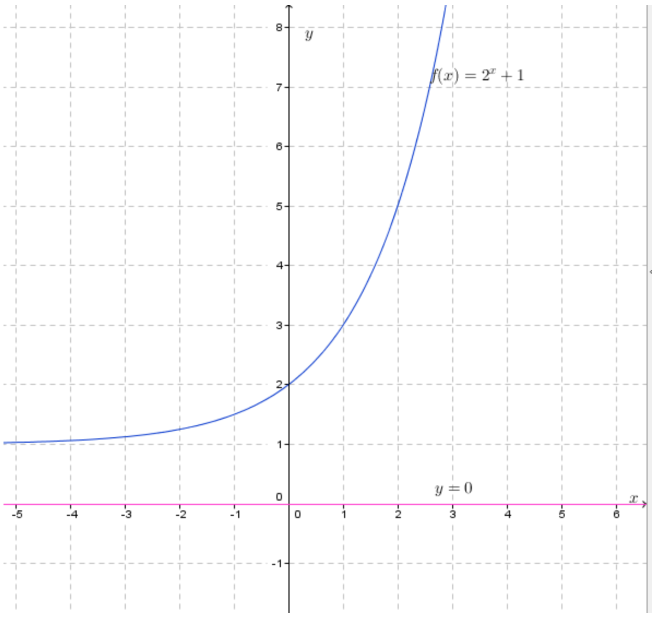
a) Dom f = $domf =]-\infty,+\infty[$ or $domf = \mathbb{R}$

b) The horizontal asymptote for the graph of $f(x)$ is the equation $y=0$,

because $\lim_{x\to 0^+} f(x)=2, \lim_{x\to 0^-} f(x)=2, and\ f(0)=2$

c) The $y-$intercept is $(0,2)$

d) d) The graph of $f(x)=2^x+1$.



Lesson 15: Derivative for exponential functions (for base e and for any base)

a) Learning objective

Determine the derivative of exponential functions (for base e and for any base)

b) Teaching resources:

Student-teachers' book, T-square, ruler, papers , if possible computers, Math draw software such as Geogebra, Microsoft Excel, Mathlab for graph sketching

c) Prerequisites/Revision/Introduction:

Student-teachers will perform well in this unit if they have a good background on: indices and radicals and common logarithms (year 1: unit6), solving equations and inequalities in the set of real numbers (year 1: unit 5), domain and range of polynomial, rational and irrational functions (year2: unit5), limits and derivatives of polynomial, rational and irrational functions (year2: Unit 6&7).

d) Learning activities :

- Form small groups and invite student-teachers to perform the **activity 4.2.5.1** in the student-teachers' book and determine the inverse of $f(x)$ and $g(x)$.
- by using the following hint: given $f(x)=e^{u(x)}$ and $g(x)=a^{u(x)}$., we find f' *and* g' as follow: $f'=u'e^{u(x)}$ and $g'=u'(x)a^{u(x)}\ln a$
- In the same groups, ask student-teachers to use the derivative $p'(x)=\frac{1}{x}$ of the function $p(x)=\ln x$, $k'(x)=\frac{1}{x.\ln 2}$ of the function $k(x)=\log_2 x$ and apply the following rule $\frac{1}{f'\left[f^{-1}(x)\right]}$ of differentiating inverse of exponential functions to determine that the derivative of $f'(x)=e^x$ and $g'(x)=2^x\ln 2$
- Finally , ask student-teachers to discuss and compare the used technique of determining the derivative of $f(x)$ and $g(x)$ to the following techniques $f'=u'e^{u(x)}$ and $g'=u'(x)a^{u(x)}\ln a$
- Ask groups to present their findings to the whole class and then lead to harmonize their works to provide the lesson summary.

- Let student-teachers work on activities of the **examples.**
- Invite student-teachers to individually work out the **application activity 4.2.5.1.**

Answer for activity 4.2.5.1

a) Given the functions $f(x)=e^x$ and $g(x)=2^x$, their inverse are : $f^{-1}(x)=\ln x$ and $g^{-1}(x)=\log_2 x$ respectively.

b) Given that $p(x)=\ln x$ and $k(x)=\log_2 x$, it is known that $p'(x)=\frac{1}{x}$ and $k'(x)=\frac{1}{x\ln 2}$. Then applying the rule for differentiating inverse of logarithmic functions we find $\frac{1}{p'\left[p^{-1}(x)\right]}=\frac{1}{p'\left(e^x\right)}$. We already know that that $p'(x)=\frac{1}{x}$, then $p'(e^x)=\frac{1}{e^x}$

Thus $\frac{1}{p'\left[p^{-1}(x)\right]}=\frac{1}{p'\left(e^x\right)}=\frac{1}{\frac{1}{e^x}}=e^x$ and $\frac{1}{k'\left[k^{-1}(x)\right]}=\frac{1}{k'\left(2^x\right)}$. Since

$k'(x)=\frac{1}{x\ln 2}$, it follows that $k'(2^x)=\frac{1}{2^x\ln 2}$ and

$$\frac{1}{k'\left[k^{-1}(x)\right]}=\frac{1}{\frac{1}{2^x\ln 2}}=2^x\ln 2.$$

Application activity 4.2.5.1

1. $f(x)=xe^{x^2+1} \Rightarrow f'(x)=e^{x^2+1}(1+2x^2)$

2. a) $2e^{2x-1}$ b) $2\left(e^{2x}+e^{-2x}\right)$ c) $\left(1+\tan^2 x\right)e^{\tan x}$

   d) $\frac{(x-2)e^x}{(x-1)|x-1|}$

3. Given the function $f(x)=4^x$.

i) $f'(x)=4^x \ln 4$

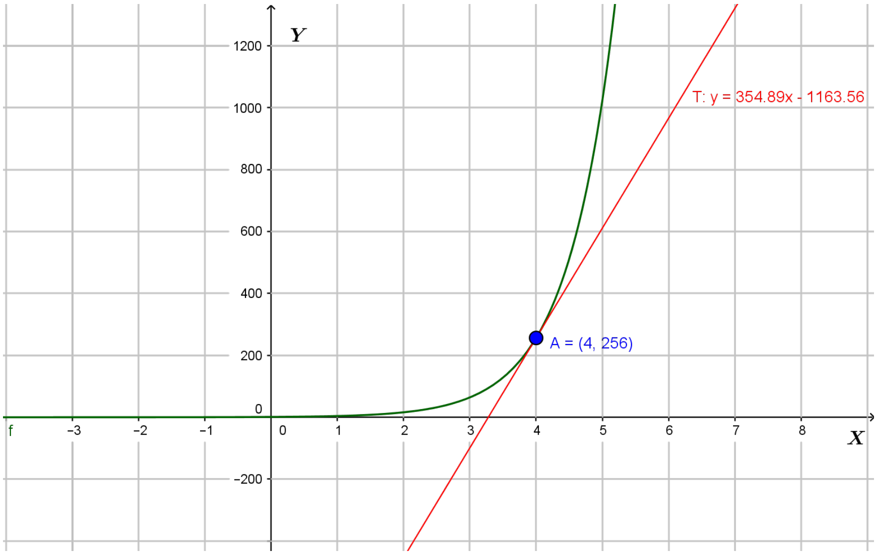
ii) $f(5)=4^5=1024$

iii) the slope of the tangent line at $x=5$ is

$f'(5)=4^5 \ln 4=1419.56$, the equation of tangent at x=4 is

$y=354.89x-1163.56$

iv) The function $f(x)=4^x$ and its tangent at $x=4$



4.

a) $f(x)=10^{3x} \Rightarrow f'(x)=3.10^{3x}\ln 10$

c) $f(x)=\dfrac{3^{4x+2}}{x} \Rightarrow f'(x)=\dfrac{\left(3^{4x+2}\right)'x-3^{4x+2}(x)'}{x^2}=\dfrac{4x.\left(3^{4x+2}\right)\ln 3-3^{4x+2}}{x^2}=\dfrac{3^{4x+2}(4x\ln 3-1)}{x^2}$

Lesson 16: Variation and graphical representation of exponential functions (for base e and for any base)

a) Learning objective

Apply the 1st and the 2nd derivative to investigate the minimum and maximum (EXTREMA) of exponential functions (for base e and for any base) and to find concavity of the graph.

b) Teaching resources:

Learner's book and other reference books to facilitate research

c) Prerequisites/Revision/Introduction:

Student-teachers will perform well in this unit if they have a good background on: indices and radicals and common logarithms (year 1: unit6), solving equations and inequalities in the set of real numbers (year 1: unit 5), domain and range of polynomial, rational and irrational functions (year2: unit5), limits and derivatives of polynomial, rational and irrational functions (year2: Unit 6&7), limits and derivatives logarithmic and exponential equations (year 3, unit4), Variation of logarithmic functions (year3,Unit4)

d) Learning activities:

- Invite student-teachers to work out the **activity 4.2.5.2** in groups and let them find out that the functions $f(x)$ is increasing in the interval $[1,10]$, the function $g = 0.5^x$ is decreasing in the interval $[1,10]$ because $a = 5$ or $5 > 1$.
- From activity 4.2.5.2, lead student-teachers to realize and conclude that the function of the form $f(x) = a^x$, with $a > 1$, is always increasing. is always and that the function

  $g(x) = a^x$ with $0 < a < 1$, is always decreasing.
- In the same groups, let student-teachers calculate and realize that the 1st derivative of $f(x)$ and $g(x)$ are $f'(x) = e^x$ and $g'(x) = 5^x \ln 5$ respectively.
- Ask them to draw table of signs for $f'(x)$ and $g'(x)$ and note that the interval of variation of those function is $]-\infty, +\infty[$
- In the same groups, let student-teachers calculate and realize that the 2nd derivative of $f(x)$ and $g(x)$ as $f''(x)$ and $g''(x)$ respectively.
- Ask them to draw table of signs for $f''(x)$ and $g''(x)$ and discuss the concavity and inflection point.
- finally, let them plot the graphs of the functions: $f(x)$ and $g(x)$
- Ask groups to present their findings to the whole class and then harmonize their works to provide the lesson summary.
- Let student-teachers read through the **examples** and invite them to

individually work out the **application activity 4.2.5.2** to increase their knowledge and skills on variation of exponential functions

Answer for activity 4.2.5.2

Given two functions $f(x)=2^x$ and $g(x)=0.5^x$,

1. $f(1)=2$ and $f(10)=2^{10}$, $f(1)<f(10)$ and the function $f(x)$ is increasing on the interval $[1,10]$.

2. $g(1)=\frac{1}{2}$ and $g(10)=\frac{1}{2^{10}}$, $g(1)>g(10)$ and the function $g(x)$ is decreasing on the interval $[1,10]$.

3. $f'(x)=2^x\ln 2$ and $g'(x)=(0.5)^x\ln 0.5$ or $g'(x)=\frac{1}{2^x}\ln\frac{1}{2}$

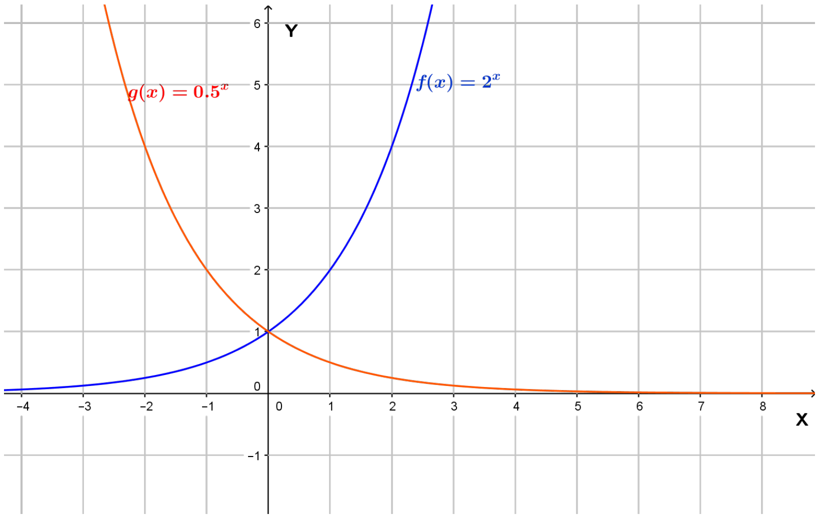
a) Table of variation of $f(x)$

x	$-\infty$		0		$+\infty$
$f'(x)$	+ + + +		$\ln 2$	+ + + + + + +	
$f(x)$	0	$\nearrow$	1	$\nearrow$	$+\infty$

b) Table of variation of $g(x)$

x	$-\infty$		0		$+\infty$
$g'(x)$	- - - - -		$\ln\frac{1}{2}$	- - - - - - - - -	
$g(x)$	$+\infty$	$\searrow$	1	$\searrow$	0

4. Plot the graphs of $f(x)$ and $g(x)$



5. The exponential *function* of the form $f(x)=a^x$, with $a>1$, is always increasing and the exponential function of the form $g(x)=a^x$ with $0<a<1$, is always decreasing.

Application Activity 4.2.5.2

1. Given the function $f(x)=xe^{x^2}$

a) The derivative of $f(x)$ is $f'(x)=e^{x^2}(1+2x^2)$

b) The derivative of $f(x)$ has no zero in $\mathbb{R}$ and is always positive,

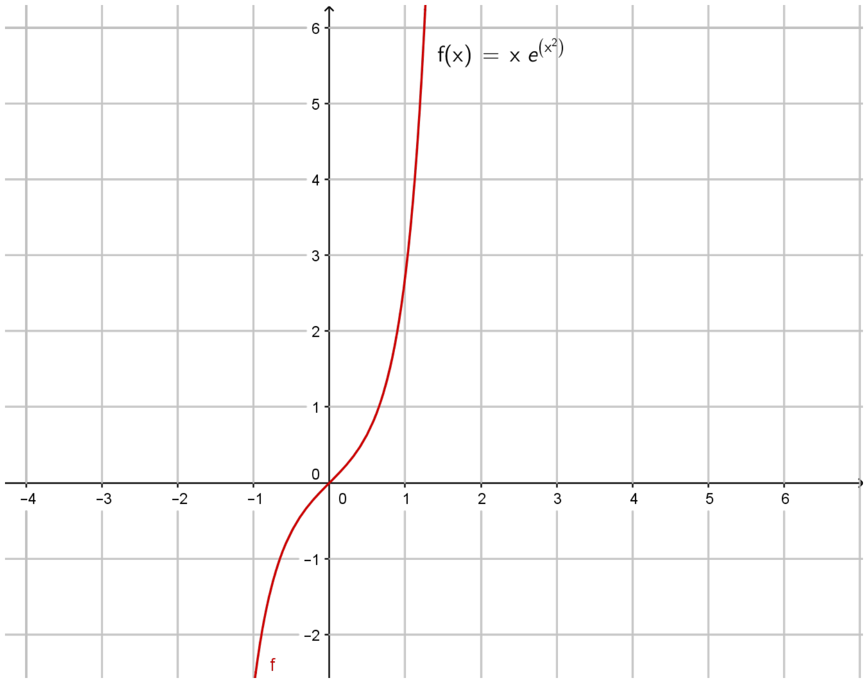
Table of variation of $f(x)$ is presented as follow:

	$-\infty$	$+\infty$
$f'(x)$	+ + + + + + + + + +	
$f(x)$	$-\infty$ ↗	$+\infty$

From the table above , the function $f(x)=xe^{x^2}$ is always increasing on the interval $]-\infty,+\infty[$

c) the function f has neither minimum nor maximum

d) Graph of the function $f(x)=xe^{x^2}$



2. Given the function $f(x)=\dfrac{e^x}{x-2}$

a) $Domf=\mathbb{R}-\{2\}$ and $Range\,f=\mathbb{R}$

b) $f'(x)=\dfrac{e^x(x-3)}{(x-2)^2}$, $f'(x)=0 \Leftrightarrow x=3$

$\lim\limits_{x\to 2} f(x)$ is not defined because $\lim\limits_{x\to 2^+} f(x)=+\infty$ and $\lim\limits_{x\to 2^-} f(x)=-\infty$, the function $f(x)=\dfrac{e^x}{x-2}$ has a vertical asymptote of equation $x=2$

c) variation table of $f(x)$

x	$-\infty$	0	2	3	$+\infty$
$f'(x)$	- - - - - - -	0 +	+ + +	+ + +	+ +
$f(x)$	$0 \searrow -\frac{1}{2}$	$\searrow -\infty$	$+\infty \searrow$	$e^3 \nearrow$	$+\infty$

3. The consumption of natural mineral resource M has risen from 4million tonnes per year. Assuming that growth of the consumption has been continuous following the function $M = M_0 e^{rt}$ where M is the final value, M_0 the initial consumption value, r the annual rate of growth and t the time in years.

a) the consumption after 6 years if $r = 20\%, t = 6$ is

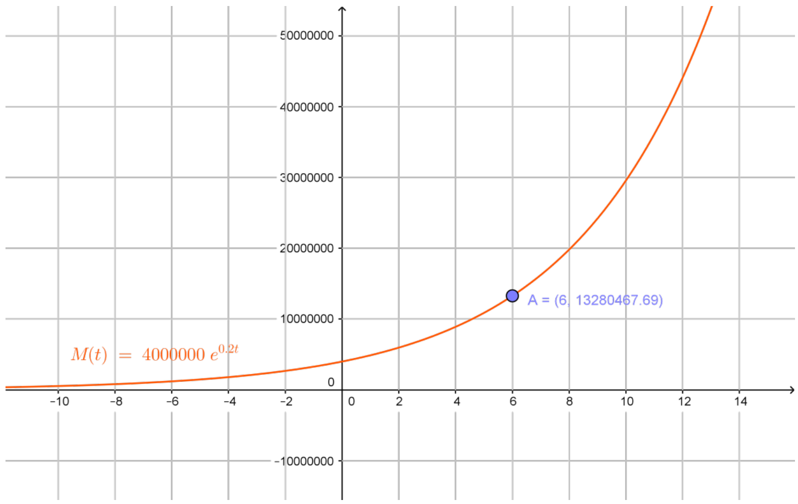
$$M = 4,000,000e^{0.2x6}$$

Using calculator we get, $M \approx 13,280,467$T

b) Draw the graph illustrating the consumption in function of time.

Graph showing the consumption in function of time or $M(t) = 4000000e^{0.2t}$

Graph of $M(t) = 4000000e^{0.2t}$



Lesson 17: Modeling and solving problems involving logarithmic or exponential functions

a) Learning objective

Use of logarithmic expressions to model and solve problem involving logarithms such rates problems, mortgage problems, population growth problems.

Radioactive-decay problems, carbon dating problems, problems about alcohol and risk of car accident.

b) Teaching resources:

Learner's book, charts containing graphical representation of exponential functions, scientific

calculators, eventual other books where the content about population growth can be found, and, if possible, a computer with mathematical software such as Geogebra and Microsoft Excel.

c) Prerequisites/Revision/Introduction:

Student-teachers will learn better the population growth problems if they have a clear understanding of:

Student-teachers will perform well in this unit if they have a good background on: indices and radicals and common logarithms (year 1: unit6), solving equations and inequalities in the set of real numbers (year 1: unit 5), domain and range of polynomial, rational and irrational functions (year2: unit5), limits and derivatives of polynomial, rational and irrational functions (year2: Unit 6&7), limits and derivatives logarithmic and exponential equations (year 3, unit4), Variation of logarithmic and exponential functions (year3,Unit4).

d) Learning activities:

- Form small groups and let student-teachers discuss the **Activity 4.3.1, Activity 4.3.2, Activity 4.3.3, Activity 4.3.4, Activity 4.3.5, Activity 4.3.6, and Activity 4.3.7** separately.
- Distribute the tasks and give clear instructions on the duration and the internal organisation of each group.
- When the student-teachers are on task, provide facilitation to the groups in need.
- Once the tasks are over, let the groups present their works one by one, and lead student-teachers to give constructive remarks in order to obtain improved information to be written by all members.
- Under your guidance, let student-teachers proceed to different

examples about solving problem involving logarithms such that rates problems, mortgage problems, population growth problems, Radioactive-decay problems, carbon dating problems, problems about alcohol and risk of car accident. Hence, check their working against the solution proposed in the learner's book,

- Ask them to work individually **application activities about** solving problem involving logarithms such rates problems, mortgage problems, population growth problems, Radioactive-decay problems, carbon dating problems, problems about alcohol and risk of car accident to check the skills they have acquired.

Answer for activity 4.3.1

To determine the total amount at the end of t years

At the end of	The total amount
The 1st year	$2000+0.1(2000)=2000(1+0.1)$
The 2nd year	$2000(1+0.1)+0.1[2000(1+0.1)]=2000(1+0.1)^2$
The 3rd year	$2000(1+0.1)^{\cdots}+\ldots=2000(1+0.1)^3$
The 4th year	$2000(1+0.1)^4$
The 5th year	$2000(1+0.1)^5$
...	...
The tth year	$2000(1+0.1)^t$

Application Activity 4.3.1

From bank I, the total amount after 10 years will be

$$A=P(1+r)^t=300000(1.1)^{10}=778122FRW$$

Then , the interest will be: $I=(778122-300000)FRW=478122FRW$

From bank II, the total amount after 10 years will be:

$$A=Pe^{rt}=300000e^{0.098\times 10}=794068.089FRW$$

Then , the interest will be: $I=(794068-300000)FRW=494068FRW$

Comparing the interest in two different banks bank II has the high interest than bank I: $494068FRW>478122FRW$

Therefore, I can advise my Aunt to go in bank II.

Answer for activity 4.3.2

1. Collect information for the meaning of the following concepts: the periodic payment(P), annual interest rate (r), mortgage amount (M), number t of years to cover the mortgage and the number n of payments per year.

$$P=\frac{\frac{rM}{n}}{1-\left(1+\frac{r}{n}\right)^{-nt}}=\frac{\frac{0.06(20000000)}{12}}{1-\left(1+\frac{0.06}{12}\right)^{-240}}=143283.2$$

The amount to pay per month should be 143286.2 Frw but in practice the bank will convert such amount into 143 287 Frw . Generally bank offices round figure to the nearest greater integer. The last payment will be less amount than 143 287 Frw, as there will be an adjustment by considering the difference between the real amount and the amount to be paid per month.

The amount to pay per month is 143 287 Frw, the balance the brother will withdraw each month is 500000Frw-143287 Frw=356713 Frw

At the end of 20 years, your brother would have paid
$143287\times 12\times 20Frw=34388880Frw$.

The interest the bank will realize is 34 388 880 Frw-20 000 000 Frw=14 338 880Frw

Application Activity 4.3.2

The periodic payment is P=200 000, the annual rate r=10%=0.01, the number of payments per year n=12(since the payment is monthly), the number of years to cover the mortgage is t=20.,

Solving in M $P=\frac{\frac{rM}{n}}{1-(1+\frac{r}{n})^{-nt}}$ yields to $M=\frac{P\left[1-(1+\frac{r}{n})^{-nt}\right]}{r}n$. Replacing each quantity by its value in the formula, we have $M=\frac{200000\times 12\times\left[1-(1+\frac{0.1}{12})^{-12(20)}\right]}{0.1}$ Calculations give M=20

808 156.36. So, the mortgage is 20 808 156 Frw

Answer for activity 4.3.3

a)

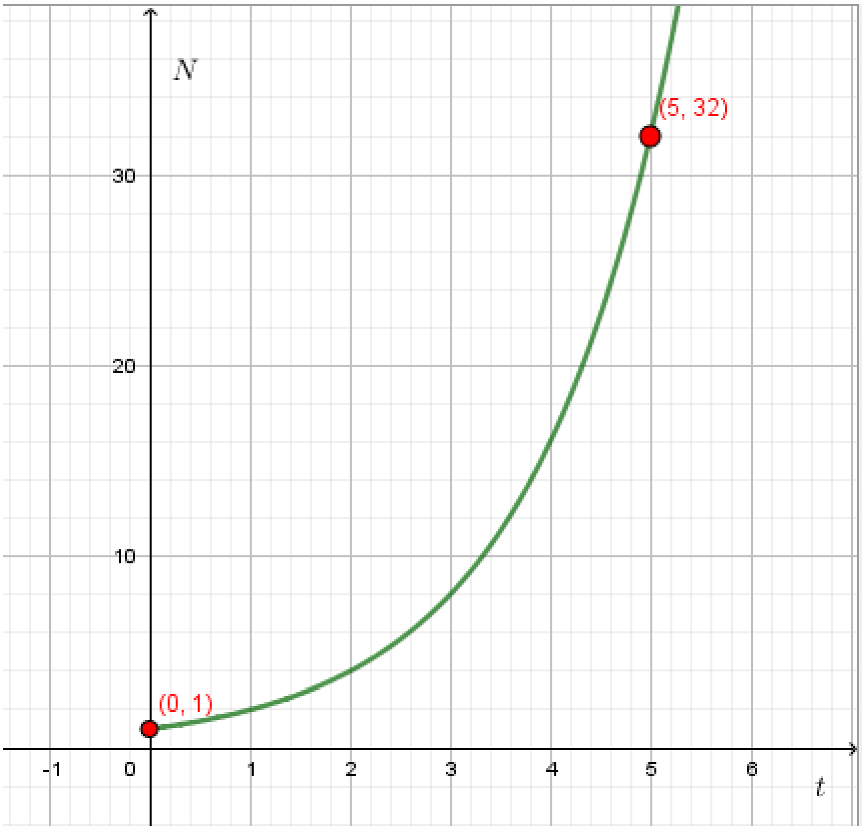
Time t(minutes)	0	1	2	3	4
Number of cells	1	2	4	8	16

b) The number of cells will be $N(\mathrm{t}) = 2^t$

If $N(\mathrm{t}) = N_0 e^{kt}$, then $N_0 = 1$,since it is independent of the base , and then $e^{kt} = 2^t$

Making k the subject of the formula and by applying natural logarithm on both sides of the equation $e^{kt} = 2^t$ and then $k = \ln 2$, we find that $N(\mathrm{t}) = e^{(\ln 2)t}$

After 5 minutes, the number of cells is $N(5) = e^{5(\ln 2)} = e^{\ln 32} = 32$.



From the graph, it is clear that at $t = 0$, $N = 1$and at $t = 5$, $N = 32$ and as the time becomes larger and larger, the number of cells grows exponentially following $N(\mathrm{t}) = e^{(\ln 2)t}$. The number of cell is growing as $k = \ln 2 > 0$.

Application Activity 4.3.3

1. $N(t) = N_0 e^{tk}$. Substituting for $N_0 = 1000000$and $N(5) = 2N_0$

,we obtain $k = \frac{1}{5}\ln 2$. The population ,in 10 years would be

$N(10) = 1000000e^{10(\frac{1}{5}\ln 2)} = 4000000$

2. $\text{A}(t) = A_0 e^{tk} = 56 \times 10^9 e^{(0.025)(1.75)} = 58.504,384$

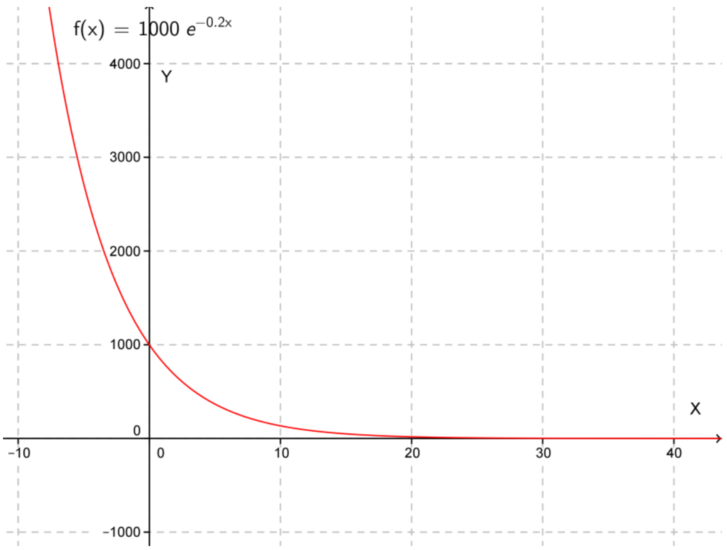
3. From $13000 == 11000e^{6k}, k = \frac{1}{6}\ln\frac{13}{11}$

Answer for activity 4.3.4

Function $N(t) = N_0 e^{-0.2t}$ passes through points (0,1000) and (5,800) ,

Then $800 = 1000e^{-5k}$ Solving for k, we have: $k = -\frac{1}{5}\ln\frac{4}{5} = 0.0446$

Using GEOGEBRA to graph the function $N(t) = 1000e^{-0.2t}$, we obtain:



Application Activity 4.3.4

1. a.The fixed price is 5

   b. As the quantity demanded becomes larger and larger, the price decreases

   c. From the formula $N(\text{x}) = N_0 e^{kx}$,using $N(0) = 5$ and $N(1) = 3$,we find $N(\text{x}) = 5e^{-0.51x}$

2. a. 50 cm² b. 22.46 cm²

3. $u(t) = 20 + (80 - 20)e^{kt} = 20 + 60e^{kt}$

$u(4) = 20 + 60e^{k(4)} = 60; k = \frac{1}{4}\ln\frac{2}{3} \approx -0.10136$

Such that $u(t) = 20 + 60e^{-0.10136t}$; $u(t) = 25 \Leftrightarrow 20 + 60e^{-0.10136t} = 25 \Leftrightarrow t = 23.971$;

It will take about 24 minutes

Answer for activity 4.3.5

The quantities involved in the measurement of an earthquake: the epicentre, the

seismographic reading x_0 at a distance of 100 kilometres when there is no earthquake and

the seismographic reading x at a distance of 100 kilometres from the epicentre when there is an earthquake.

The formula $M(x) = \log\frac{x}{x_0}$ is used to evaluate the magnitude of an earthquake.

The ratio of the seismographic readings is used to compare the intensities of two earthquakes.

Application Activity 4.3.5

a) Let x and y be the seismographic readings of the earthquakes at Ecuador and at Mexico, respectively.

Then, $\log\frac{x}{0.001} = 7.8$ and $\log\frac{y}{0.001} = 8.1$

This is equivalent to: $\frac{x}{0.001} = 10^{7.8}$ and $\frac{x}{0.001} = 10^{8.1}$

Dividing side by side, $\frac{\frac{x}{0.001}}{\frac{y}{0.001}} = \frac{10^{7.8}}{10^{8.1}} \Leftrightarrow \frac{x}{y} = 10^{7.8-8.1} = 10^{-0.0.3} = 0.998$

This means that the earthquake at Ecuador was almost as heavy as the one that happened at Mexico.

b) Let z be the seismographic reading of the earthquake at San Francisco.

Then $\log\frac{x}{0.001} = 7.8$ and $\log\frac{y}{0.001} = 6.9$

This is equivalent to: $\frac{x}{0.001}=10^{7.8}$ and $\frac{x}{0.001}=10^{6.9}$

Dividing side by side, $\frac{\frac{x}{0.001}}{\frac{y}{0.001}}=\frac{10^{7.8}}{10^{6.9}} \Leftrightarrow \frac{x}{y}=10^{7.8-6.9}=10^{0.9}=7.943$

This means that the earthquake at Ecuador was is about 8 times heavier than the one that happened at San Francisco.

Answer for activity 4.3.6

$N(t)=N_0e^{-0.693t/H}$,where H represents the half-life of Carbon-14, N_0 represents the amount of the radioactive material at time t=0 (time of the death), solving for t, in the equation $N(t)=N_0e^{-0.693t/H}$, we obtain $t=\frac{H}{-0.693}.\ln\frac{N(t)}{N_0}$

Application Activity 4.3.6

From the formula $N(t)=N_0e^{-0.693t/H}$,solving for t, $t=\frac{H}{-0.693}.\ln\frac{N(t)}{N_0}$, and substituting for H=5700, $\frac{N(t)}{N_0}=0.79$,we have: $t=\frac{5700}{-0.693}.\ln(0.79)=1938.84$

The age of the animal given that the half-life of carbon-14 is 5700 years would be about 1939 years

Answer for activity 4.3.7

a) Excess of alcohol taken by the driver can yield to car accident

b)

i) The risk when there is no alcohol in the driver's blood is 1; it is not zero because the car accident is not due only to excess of alcohol in the driver's blood.

ii) More the concentration of alcohol in the driver's blood, more the risk of accident.

c) Since the risk grows exponentially, the equation is of the type $R(x)=R_0e^{kx}$

$R(0)=1$ and $R(4)=5$ give $R(x)=e^{(\frac{1}{4}\ln 5)x}$

4.6 Unit summary

4.6.1 Logarithmic functions

- Definition: $\log_a x = y \Leftrightarrow a^y = x$, where $a > 0, a \neq 1, x > 0$;this definition is used to determine the domain and the range.

- Formula for changing the base, from base a to base e: $\log_a x = \frac{\ln x}{\ln a}$
- Limits of a logarithmic function:

For $f(x) = \log_a x$ the domain is $]0,+\infty[$, If $x_0 \in]0,+\infty[$, then $\lim_{x\to x_0} \log_a x = \log_a x_0$

$\lim_{x\to 0^+} \log_a x = \begin{cases} -\infty; & ;a>1 \\ +\infty; & 0<a<1 \end{cases}$; in particular, $\lim_{x\to 0^+} \ln x = -\infty$.

$\lim_{x\to +\infty} \log_a x = \begin{cases} +\infty; & ;a>1 \\ -\infty; & 0<a<1 \end{cases}$; in particular, $\lim_{x\to +\infty} \ln x = +\infty$

Indeterminate cases $\frac{0}{0};\frac{\infty}{\infty}$: the indeterminate can be removed by applying Hospital's rule

Indeterminate cases $\infty - \infty; 0\times\infty; 1^\infty; 0^0; \infty^0$:Re write the limit to obtain $\frac{0}{0}$ or $\frac{\infty}{\infty}$ and then apply Hospital's rule

- Derivative of a logarithmic function:

$(\log_a u)' = \frac{u'}{u \ln a}$; $(\ln u)' = \frac{u'}{u}$, where u is function of variable x;

For more elaborated functions, such as product, power, quotient, etc, containing logarithms, the rules for differentiation still apply.

- Variations and graphs of logarithmic functions: either graph the function, using software, such as Geogebra, and then analyse the graph to draw the conclusion about maximum, minimum, increasing, decreasing, concavity, inflection point or, study the sign of the first derivative (eventually the second derivative) and draw conclusion about the variations, then graph the function.

4.6.2. Exponential functions

- Definition: $f(x) = a^x$, where $a > 0, a \neq 1$, ;this definition is used to determine the domain and the range of an exponential function

- Limits of an exponential function

For $f(x)=a^x$ the domain is $]-\infty,+\infty[$

- If $x_0\in]-\infty,+\infty[$, then $\lim\limits_{x\to x_0} a^x = a^{x_0}$
- $\lim\limits_{x\to-\infty} a^x = \begin{cases} 0 \;; a>1 \\ +\infty;\; 0<a<1 \end{cases}$; in particular, $\lim\limits_{x\to-\infty} e^x = 0$
- $\lim\limits_{x\to+\infty} a^x = \begin{cases} +\infty;\; ; a>1 \\ 0;\; 0<a<1 \end{cases}$; in particular, $\lim\limits_{x\to+\infty} e^x = +\infty$

Indeterminate cases $\frac{0}{0};\frac{\infty}{\infty}$: the indeterminate form can be removed by applying Hospital's rule

Indeterminate cases $\infty-\infty; 0\times\infty; 1^\infty; 0^0; \infty^0$:Re write the limit to obtain $\frac{0}{0}$ or $\frac{\infty}{\infty}$ and then apply Hospital's rule

- **Derivative of an exponential function:**

$(a^u)' = u'a^u \ln a$;

$(e^u)' = u'e^u$, where u is function of variable x

For more elaborated functions, such as product, power, quotient, etc, containing

logarithms ,the rules for differentiation still apply

- Variations and graphs of exponential functions: either graph the function, using software, such as Geogebra, and then analyse the graph to draw the conclusion about maximum, minimum, increasing, decreasing, concavity, inflection point or, study the sign of the first derivative (eventually the second derivative) and draw conclusion about the variations, then graph the function.

4.6.3. Applications of logarithmic and exponential functions

- Interest compounded n times per year, r: rate of annual interest, P: Principal

$$A = P\left(1+\frac{r}{n}\right)^{nt}$$

- Interest compounded continuously: $A = Pe^{rt}$
- Formula connecting the quantities involved in a mortgage problem:

$$P = \frac{\frac{rM}{n}}{1-\left(1+\frac{r}{n}\right)^{-nt}}$$

where P: Amount after t years, M: Mortgage, r: Annual rate interest,

n: Number of payment per year, t: Number of years to cover the Mortgage

- Law of exponential growth or decay

$P(t) = P_0 e^{kt}$ where P: Population at time t, P_0 :Initial population, $k > 0$ or $k < 0$, t: time

- Magnitude of earthquake with seismographic reading x : $M(x) = \log \frac{x}{0.001}$:
- Risk of car accident corresponding to concentration x of alcohol in the driver's blood

  $R(x) = R_0 e^{kx}$:

4.7 Additional information for the teacher

Given that the knowledge of the teacher must be wider than the one of the student-teachers, the following information is useful for the teacher, though not stated in the learner's book:

- The study of logarithmic and exponential functions can follow the study of integrals. In this case, the natural logarithm of x is defined as $\ln x = \int_1^x \frac{dt}{t}$,where $x > 0$
- The concepts of logarithmic functions and exponential functions can be taught interchangeably. Exponential functions can be taught first: $f(x) = a^x$, where $a > 0$ and $a \neq 1$
- The roots of some equations involving logarithms or exponentials can be approximated using Taylor's expansion. Similarly, in the calculation of limits, some indeterminate cases can be removed by approximating the function involved in the limit, using Taylor's expansion.

The Taylor's expansion of f(x) at x=0,or the Maclaurin's expansion of $f(x) = e^x$ and for $f(x) = \ln(1+x)$ are given below:

$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + ...$ for any value of x

$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + ...$ for values of x in the neighbourhood of 0

Examples:

a) Approximate e^x by a quadratic function and approximate the roots of the equation $e^x - x^2 = 0$

Approximate e^x and $\ln(1+x)$ by quadratic functions then calculate

$$\lim_{x \to 0} \frac{x^3 e^x}{xe^{-x} - \ln(x+1)}$$

Solution:

a)

$$e^x - x^2 = 0 \Leftrightarrow (1 + x + \frac{x^2}{2}) - x^2 = 0 \Leftrightarrow -\frac{1}{2}x^2 + x + 1 = 0 \Leftrightarrow x_1 = -1 + \sqrt{3}; x_2 = -1 - \sqrt{3}$$

b) Substituting x in the limit, we obtain the indeterminate case $\frac{0}{0}$

Then $\lim_{x \to 0} \frac{x^2 e^x}{xe^{-x} - \ln(x+1)} = \lim_{x \to 0} \frac{x^2(1 + x + \frac{x^2}{2})}{x(1 - x + \frac{x^2}{2}) - (x - \frac{x^2}{2})} = -2$

4.8 End unit assessment

a) Learning objective

Extend the concepts of functions to investigate logarithmic and exponential functions and use them to model and solve problems about interest rates, population growth or decay, magnitude of earthquake.

b) Learning activities

This part provides the answers of end unit assessment activities designed in integrative approach to assess the key unit competence with cross reference to the textbook.

The following are steps which have been based on setting end unit assessment questions:

i) Thoroughly explore and investigate logarithmic and exponential functions; reflecting analytically and logically about their learning and setting their own goals.

ii) Correctly apply logarithmic and exponential functions in solving problems, selecting the appropriate mathematical operations and calculations.

ANSWERS FOR QUESTION ONE

a) $f(x)=\log_2(3x-2)$

$f(x)=\log_2(3x-2)$ is defined if and only if $(3x-2)>0 \Leftrightarrow 3x>2 \Leftrightarrow x>\frac{2}{3}$

Thus, $Domf = \left]\frac{2}{3},+\infty\right[$

From the function, $0<3x-2<+\infty$.

Then $-\infty<\log_2(3x-2)<+\infty$

Therefore, the range is $\mathbb{R}=]-\infty,+\infty[$

b) $f(x)=\ln(x^2-1)$

$f(x)=\ln(x^2-1)$ is defined if and only if $x^2-1>0 \Leftrightarrow (x-1)(x+1)>0$

x	$-\infty$		-1		1		$+\infty$
$x-1$	$-$	$-$	$-$	$-$	0	$+$	$+$
$x+1$	$-$	$-$	0	$+$	$+$	$+$	$+$
$(x-1)(x+1)$	$+$	$+$		$-$		$+$	$+$

$Domf =]-\infty,-1[\cup]1,+\infty[$

From the function, $0<x^2-1<+\infty$.

Then $-\infty<\ln(x^2-1)<+\infty$

Therefore, the range is $\mathbb{R}=]-\infty,+\infty[$

c) $f(x)=2e^{3x+1}$

$Domf=\mathbb{R}=]-\infty,+\infty[$

From the function, $-\infty<3x+1<+\infty$;

Then $0<e^{3x+1}<+\infty$;

$0<2e^{3x+1}<+\infty$;

Therefore, the range is $]0,+\infty[$

d) $f(t)=4^{\sqrt{3t+1}}$

$f(t)=4^{\sqrt{3t+1}}$ is defined if and only if $3t+1\geq 0 \Leftrightarrow t\geq \frac{-1}{3}$

$$Domf=\left[\frac{-1}{3},+\infty\right[$$

From the function, $0\leq 3t+1<+\infty$;

Then $0\leq \sqrt{3t+1}<+\infty$;

$1\leq 4^{\sqrt{3t+1}}<+\infty$;

Therefore, the range is $[1,+\infty[$

ANSWERS FOR QUESTION TWO

a) $\lim_{x\to 0^+}\frac{\ln x}{x}=-\infty$

We have a vertical asymptote which has equation $x=0$

b) $\lim_{x\to +\infty}\left(3+x^2\ln x\right)=+\infty$

No horizontal asymptote

ANSWERS FOR QUESTION THREE

1. a) $f(x)=\log_2\sqrt{\frac{x^2-4}{x+2}}=\frac{1}{2}\left[\log_2(x-2)+\log_2(x+2)-\log_2(x+2)\right]=\frac{1}{2}\log_2(x-2)$

$$\frac{d}{dx}f(x)=\frac{1}{(2\ln 2)(\mathrm{x}-2)}$$

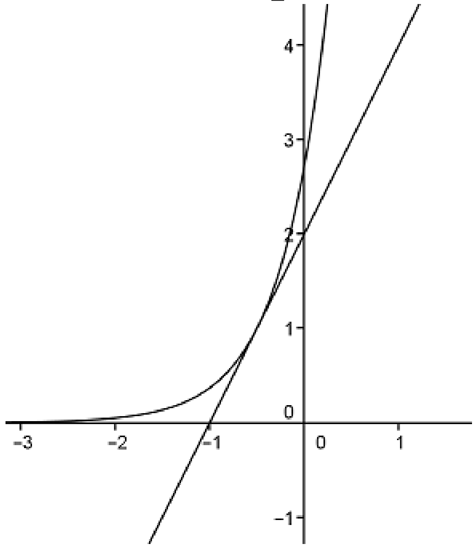
b) $\frac{d}{dx}h(x)=\frac{d}{dx}[\frac{1}{3}\left(4^{2x+5}\right)]=\frac{2}{3}(4^{2x+5})\ln 4$

2. i)The value of the function $y=e^{2x+1}$ at $x=-\frac{1}{2}$ is $y=e^{2(-\frac{1}{2})+1}=1$

The intersection point is $A(-\frac{1}{2},1)$

ii) $\frac{dy}{dx}=2e^{2x+1}$ .At $x=-\frac{1}{2}, \frac{dy}{dx}=2e^{2(-\frac{1}{2})+1}=2$

The equation of the tangent is $y-1=2(x+\frac{1}{2})$, which is equivalent to $y=2x+2$



ANSWERS FOR QUESTION FOUR

$y=xe^{-x}$;

$\frac{dy}{dx}=(e^{-x}-xe^{-x})=(1-x)e^{-x}$

$\frac{dy}{dx}=0 \Leftrightarrow x=1$, the extrema is $(1,\frac{1}{e})$ .From the study of the sign of $\frac{dy}{dx}$,

$\frac{dy}{dx}>0$ for $x<1$,and $\frac{dy}{dx}<0$ for $x>1$.Therefore, the stationary point, the extrema is a maximum

ANSWERS FOR QUESTION FIVE

1. Five applications of logarithmic or exponential functions:
   - In Geography: the magnitude of an earthquake is found using logarithms

- In Entrepreneurship: the interest rate problems can be solved using logarithm and exponential functions
- In Chemistry: the radioactive decay problems are solved using logarithmic and exponential functions
- In History: archaeology involves carbon dating whose principles are based on the use of logarithms
- In Social studies: logarithms and exponentials are used in the determination of risk corresponding to a given concentration of alcohol

2. Assuming exponential growth, $N(t)=5.7e^{0.02t}$

   The population will reach 114 billion after t years such that $114=5.7e^{0.02t}$ .solving for t,

   $t=\frac{1}{0.02}\ln\frac{114}{5.7}=26.278$ years

   The population will reach 114 billion in the year 2021.

4.9 Additional activities activities

The teacher's guide suggests additional questions and answers to assess the key unitcompetence.

4.9.1 Remedial activities

Suggestion of Questions and Answers for remedial activities for slow student-teachers.

1. Find the domain and the range of the function;

a) $f(x)=\log(x-1)$ b) $f(x)=4^{\sqrt{6x}}$

Solution

a) Domain: $]1,+\infty[$ Range: $\mathbb{R}=]-\infty,+\infty[$	b) Domain: $[0,+\infty[$ Range: $\mathbb{R}=[1,+\infty[$

2. Calculate: $\lim\limits_{x\to+\infty}\left(2-\frac{3}{\ln x}+e^{-x}\right)$

Solution

$$\lim_{x\to+\infty}\left(2-\frac{3}{\ln x}+e^{-x}\right)=2-\frac{3}{\lim\limits_{x\to+\infty}\ln x}+\lim_{x\to+\infty}e^{-x}=2-\frac{3}{+\infty}+e^{-\infty}=2-0+0=2$$

3. Given the logarithmic function $f(x)=\log_2(x-5)$

   a) What is the equation of the asymptote line? b) If $x=7$ find y

Solution

a) Vertical asymptote: $x=5$, since $\lim\limits_{x\to5^+}\log_2(x-5)=-\infty$

b) $y=f(7)=\log_2(7-5)=\log_2 2=1$

4.9.2 Consolidation activities

Suggestion of questions and answers for deep development of competences.

1. Consider the function $f(x)=6^{x-2}$

a) Determine $f'(x)$

b) Find the equation of the tangent to the graph of the function at the point where $x=3$

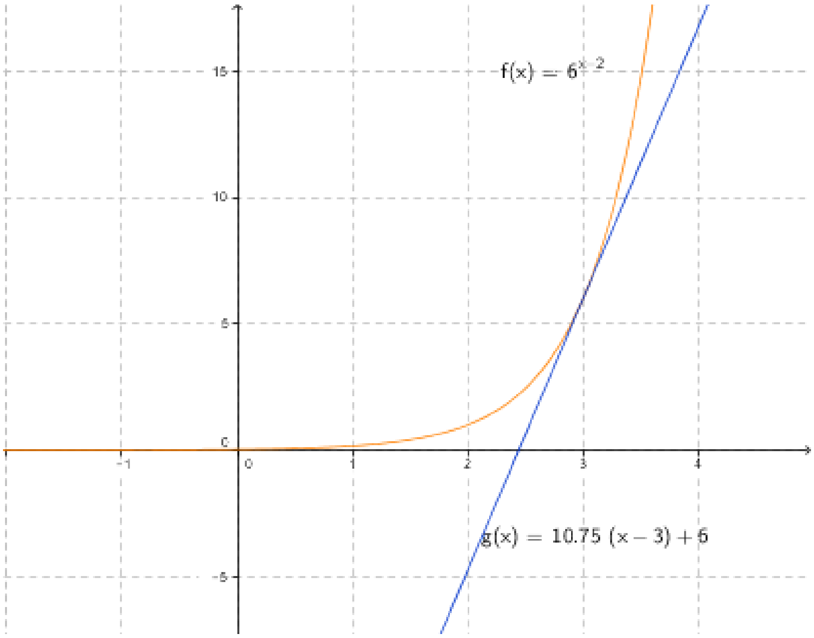
c) Graph the function and its tangent

Solution

a) $f'(x)=(6^{x-2})'=6^{x-2}\ln 6$

b) $f'(3)==6^{3-2}\ln 6=6\ln 6$ and $f(3)=6^{3-2}=6$

c) The equation of the tangent is then $y-6=(6\ln 6)(x-3)$



2. Suppose the function $f(x) = 2x - \ln x$

   a) State the domain and range

   b) Find the 1st derivative.

   c) Solve for $f^{'}(\text{x}) = 0$

   d) Determine a point through which the graph passes

   e) Draw the variation table of $f(x)$, and find the stationary point and its nature.

   f) Sketch the graph

Solution

Function $f(x) = 2x - \ln x$

a) Domain: $]0, +\infty[$

The range is $\mathbb{R} =]-\infty, +\infty[$

b) The first derivative: $f^{'}(x) = 2 - \frac{1}{x}$

c) Calculate $f^{'}(\text{x}) = 0 \quad \Leftrightarrow 2 - \frac{1}{x} = 0$

$\Leftrightarrow \frac{1}{x} = 2 \Leftrightarrow 2x = 1$

$\Leftrightarrow x = \frac{1}{2}$

d) An example of a point through which the graph of $f(x)$ passes is $A(1,2)$

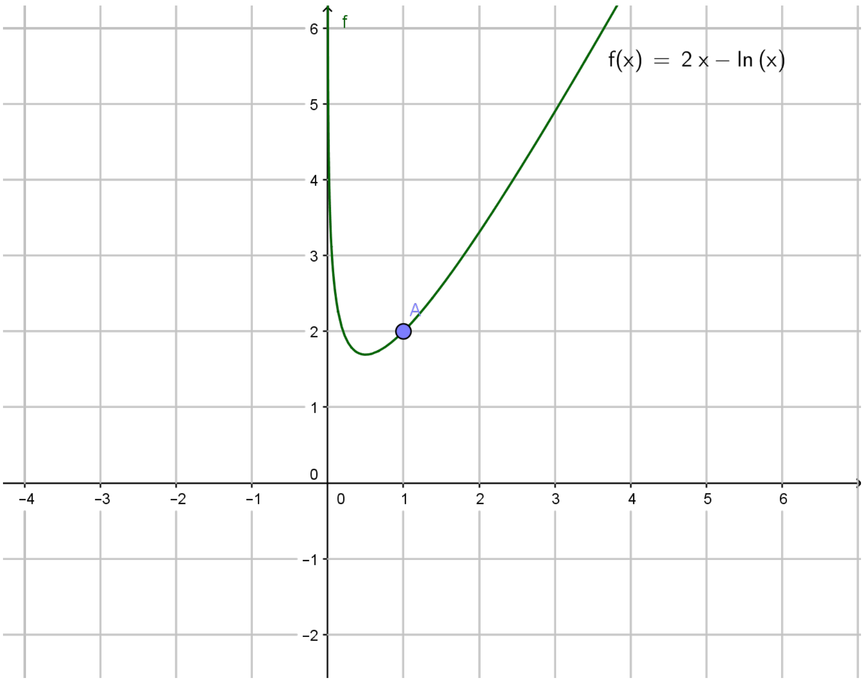
e) Variation table of $f(x)$

x	0	$\frac{1}{2}$	$+\infty$
$f'(x)$	$------$	0 $+++$	
$f(x)$	$+\infty$ ↘	$1+\ln 2$	↗ $+\infty$

Function f is decreasing on interval $]0,\frac{1}{2}[$ and increasing on $]\frac{1}{2},+\infty[$.

The graph has a minimum at point $(\frac{1}{2},1+\ln 2)$

f) The graph of $f(x)$



3. Two earthquakes took place at town B, and at town K. Their magnitudes, on Richter's scale, were 5.6 and 5.2, respectively. Compare the two earthquakes

by finding the ratio of their seismographic readings.

Solution

Let x and y be the seismographic readings of the earthquakes at B and at K, respectively.

Then, $\log\frac{x}{0.001}=5.6$ and $\log\frac{y}{0.001}=5.2$

This is equivalent to: $\frac{x}{0.001}=10^{5.6}$ and $\frac{y}{0.001}=10^{5.2}$

Dividing side by side, $\frac{\frac{x}{0.001}}{\frac{y}{0.001}}=\frac{10^{5.6}}{10^{5.2}} \Leftrightarrow \frac{x}{y}=10^{5.6-5.2}=10^{0.4}=2.5118$

This means that the earthquake at town B is about 2.5 times heavier than the earthquake at town K .

4. An amount of 1, 000 000 FRW is invested at a bank that pays an interest rate of 10% compounded annually.

a) How much will the owner have at the end of 15 years, in each of the following alternatives?

The interest rate is compounded:

i) Once a year.

ii) Twice a year

b) Compare the two types of compounding, and explain which one is the best

Solution

a) i. For once a year, at the end of 15 years the owner will have

$$A=P(1+r)^t=1{,}000{,}000(1+0.10)^{15}$$
$$=1{,}000{,}000(1.10)^{15}=4{,}177{,}248.16\,frw$$

ii) For twice a year, at the end of 15 years , the owner will have

$$A=P(1+\frac{r}{2})^{2t}=1{,}000{,}000(1+\frac{0.10}{2})^{2(15)}$$
$$=1{,}000{,}000(1.05)^{30}=4{,}321{,}942.37\,Frw$$

b) Conclusion: since $4{,}321{,}942.3>4{,}177{,}248.1$, compounding many times per year is better.

5. How long will it take money to double at 5% interest when compounded quarterly?

Solution

$$A = P(1+\frac{0.05}{4})^{4t} = 2P \Leftrightarrow (1.0125)^{4t} = 2$$

$t = 13.95$ years

4.9.3 Extended activities

Suggestion of Questions and Answers for gifted and talented Student-teachers.

The revenue R obtained by selling x units of a certain item at price p per unit is $R = xp$.

If x and p are related by $p(x) = 8.25e^{-0.02x}$. Find the price and the number of units to sell for the revenue to be maximized.

Solution

The revenue is $\mathrm{R}(x) = 8.25xe^{-0.02x}$

The revenue is maximum if $\mathrm{R}'(x) = (8.25xe^{-0.02x})' = 0$

$$\Leftrightarrow (8.25e^{-0.02x} - 0.165\,\mathrm{x}\mathrm{e}^{-0.02x})' = 0$$

$$\Leftrightarrow 1 - 0.02x = 0 \Leftrightarrow x = 50 \;;$$

$$p(50) = 8.25e^{-0.02(50)} = \frac{8.25}{e} = 3.035$$

$$\lim_{t\to+\infty}\frac{t-e^{-t}}{t} = 1 - \lim_{t\to+\infty}\frac{e^{-t}}{t} = 1 - 0 = 1$$

As the time gets larger and larger, the level of oxygen approaches 1, that is 100%

UNIT 5

INTEGRATION

5.1 Key unit competence*:*

Determine correctly integration as the inverse of differentiation or limit of a sum and apply it to find area of plane surfaces, volumes of solid of revolution and lengths of curved lines.

5.2 Prerequisites

Student-teachers will easily learn this unit, if they have a good background on : differentiation (Year 2: Unit 7), quadratic equations (year 1:unit 5), Trigonometry (year 1: unit 8), Logarithm(year 3:unit 4) .

5.3 Cross-cutting issues to be addressed

Inclusive education (promote education for all while teaching)

Peace and value Education (respect others' views and thoughts during class discussions)

Gender (provide equal opportunity to boys and girls in the lesson participation)

5.4 Guidance on introductory activity

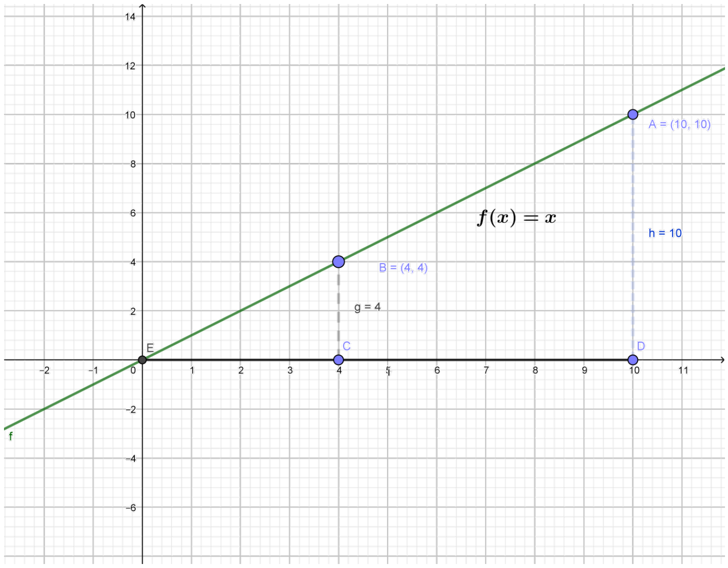
- Invite student-teachers to work in groups and lead them to work on introductory activity 5.0 found in unit of student teacher's book to understand the concept of the anti- derivative; using integration as the inverse of differentiation and to calculate the area of a plane shape
- Guide student teachers to read and analyse the questions insisting on the analysis of how to calculate the area of of a quadrilateral field BCDA shown in the given figure
- Invite some group members to present groups' findings, then try to harmonize their answers.
- Basing on student-teachers' experience, prior knowledge and abilities shown in answering the questions for this activity, use different questions to facilitate them to give their predictions and ensure that you arouse their curiosity on what is going to be leant in this unit after you harmonize their

works and ensure that they got exact solution.

Answer for introductory activity:

a) One unit stands for one meter:

Figure: The quadrilateral field



1. The area A_1 found by the first group is $A_1 = area(\Delta EDA) - area(\Delta ECB)$

Calculate the area of $area(\Delta EDA) = \frac{10 \times 10}{2} = 50m^2$

Calculate the area of $area(\Delta ECB) = \frac{4 \times 4}{2} = 8m^2$

Therefore $A_1 = (50-8)m^2 = 42m^2$

2. The second group with high critical thinking skills used a function $F(x)$ that was differentiated to find $f(x) = x$ (which means $F'(x) = f(x)$) and the $x-$ -coordinate d of D and the $x-$ coordinate c of C in the following way: $A_2 = F(d) - F(c)$.

Therefore $F(x) = \frac{x^2}{2} + c$, where c is a given constant.

$F(x)$ is said to be an integral of $f(x)$ or anti-derivative of $f(x)$, because $F'(x) = f(x) = x$ in this case.

3. The area A_2 found by the second group using $F(x)$ is

$A_2 = F(d) - F(c) = F(10) - F(4)$

Then $F(10) = \frac{(10)^2}{2} = 50$ *and* $F(4) = \frac{(4)^2}{2} = 8$

Therefore $A_2 = F(10) - F(4) = 50 - 8 = 42m^2$

4. We realize that A_1 and A_2 are equal (see results found in (1) and in (2).

Therefore, referring to the graph of the function $f(x)$ on the figure 5.1, you can find the area bounded by a function $f(x)$ and x-axis and the lines of equations are $x = x_1$ and $x = x_2$.

If $F'(x) = f(x)$, the area is calculated using $area = F(x_2) - F(x_1)$

5.5. List of lessons/sub-heading

No	Lesson title	Learning objectives	Number of periods
0	Introductory activity	To arouse the curiosity of student teachers on the content of unit 6.	1
1	Increment and differential of a function.	Determine the increment of a function and define the differential of a function.	1
2	Operation on increments	Evaluate the sum and the difference of the increments	1
3.	Properties of differentials and their applications	Determine the properties of differentials and use them to solve real life problems.	1
4.	Definition of indefinite integral	Clarify the relationship between derivative and anti-derivative of a function.	2
5.	Properties of integrals.	Use properties of integrals to simplify the calculation of integrals.	2
6.	Basic integration formulas (or immediate integration)	Calculate integrals using basic integration formulas.	2

7.	Integration by substitution or by change of variable	Calculate integrals by substitution method or by change of variable	2
8.	Integration by parts	Calculate integrals by parts	2
9.	Integration of rational function where the numerator is expressed in terms of derivative of the denominator	Calculate integrals of rational function where the numerator is expressed in terms of derivative of the denominator	2
10.	Integration of rational function where the degree of a numerator is greater or equal to the degree of the denominator	Calculate integrals of rational function where the degree of a numerator is greater or equal to the degree of the denominator.	2
11.	The denominator is factorized into linear factors	Calculate integrals for which the denominator is factorized into linear factors	2
12.	The denominator is a quadratic factor	Calculate integrals for which the denominator is a quadratic factor	2
13.	Integral of the form : $\int \sin mx \cos nxdx$, $\int \cos mx \cos nxdx$, $\int \sin mx \sin nxdx$	Calculate integrals of the form : $\int \sin mx \cos nxdx$, $\int \cos mx \cos nxdx$, $\int \sin mx \sin nxdx$	2
14.	Integral of the form: $\int \sin^m x \cos^n xdx, \left(m, n \in \mathbb{Z}^+\right)$	Calculate integrals of the form: $\int \sin^m x \cos^n xdx, \left(m, n \in \mathbb{Z}^+\right)$	2
15.	Riemann sum approximation and definition of define integrals	Define definite integrals using Riemann sum approximation.	1
16.	Properties of definite integrals	Extend the concepts of indefinite integrals and their properties to definite integrals	1
17.	Techniques of integration of definite integrals.	Evaluate definite integrals	2

18.	Calculation of area of a plane surface.	Use integrals to solve problems related to area of a plane surface.	2
19.	Calculation of volume of a solid of revolution	Use integrals to solve problems related to the volume of a solid of revolution	2
20.	Calculation of the arc length of curved surfaces.	Use integrals to solve problems related to the arc length of curved surfaces.	2
21.	Application of integrals in real life or other sciences.	Use integral calculus in solving problems from other sciences or daily life.	2
22.	**End assessment**		2
Total periods			40

Lesson 1: Increment and differential of a function

a) Learning objectives

Define the differential of a function in order to precise the increment of a function.

b) Teaching resources

Student teacher's book and other Reference books to facilitate research, scientific calculator, Manila paper, markers, pens, pencils, if possible computers with mathematical software such as Geogebra, Microsoft Excel, Math-lab or graph-application and internet can be used...

c) Prerequisites/Revision/Introduction

Students teacher will learn better in this lesson if they have a good background on: differentiation (Year 2: Unit 7), quadratic equations (year 1: unit 5), Trigonometry (year 1: unit 8), Logarithm (year 3: unit 4) .

d) Learning activities:

- Invite student-teachers to work in groups and do the activity 5.1.1 in their Mathematics books;
- Move around in the class for facilitating student teachers where necessary.
- Verify and identify groups with different working steps;
- Invite one member from each group with different working steps to present

their work where they must explain the working steps;

- As a tutor, harmonize their findings from the presentation.;
- Use different probing questions and guide student teachers to explore the content and examples given in the student teachers' book and lead them to discover how to define increment of a function.
- After this step, guide students to do the application activity 5.1.1 and evaluate whether lesson objectives were achieved.

Answer for activity 5.1.1

a) $y = f(x) = 4 + 0.5x + 0.1\sqrt{x}$;

The consumption at $x = 2$ is $f(2) = 4 + 0.5(2) + 0.1\sqrt{2} = 5 + 0.1\sqrt{2}$

The consumption at $x = 10$ is $f(10) = 4 + 0.5(10) + 0.1\sqrt{10} = 9 + 0.1\sqrt{10}$

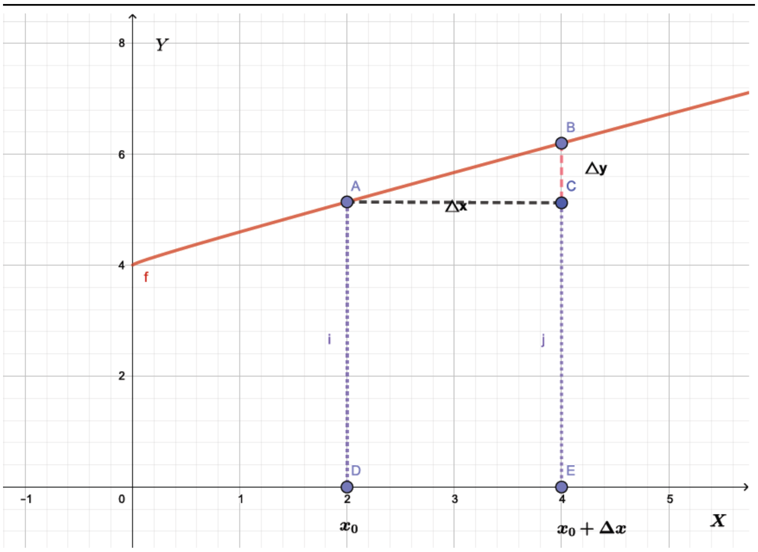
b) The corresponding increment of y is

$f(10) - f(2) = \left(9 + 0.1\sqrt{10}\right) - \left(5 + 0.1\sqrt{2}\right) \approx 4.175$

c) If x changes from x_0 *to* x_1 where $(x_1 > x_0)$,then $f(x)$ changes from $f(x_0)$ *to* $f(x_1)$ the increment of x is $\Delta x = x_1 - x_0$ and the change in y is $\Delta y = f(x_1) - f(x_0) = ...$

d) If Δx is very small, then we have $\lim\limits_{\Delta x \to 0} \Delta y = \lim\limits_{\Delta x \to 0} f'(x_0)\Delta x = dy = f'(x_0)dx$

e) Graphical interpretation:



In this regard, $dy = 0.5 + \frac{0.1}{2\sqrt{x}}dx$

Application Activity 5.1.1

L a. $d(x^2e^x) = (2x + x^2)e^x dx$ b. $d(\frac{\ln x}{x}) = (\frac{1-\ln x}{x^2})dx$

Lesson 2: Operation on increments

a) Learning objectives

Define the differential of a function in order to precise the increment of a function and its operation.

b) Teaching resources

student teacher's book and other Reference books to facilitate research, scientific calculator, Manila paper, markers, pens, pencils, if possible computers with mathematical software such as Geogebra, Microsoft Excel, Math-lab or graph-application and internet can be used...

c) Prerequisites/Revision/Introduction

Students teacher will learn better in this lesson if they have a good background on: differentiation (Year 2: Unit 7), quadratic equations (year 1: unit 5), Trigonometry (year 1: unit 8), Logarithm (year 3: unit 4) .

d) Learning activities:

- Invite student-teachers to work in groups and do the activity 5.1.2 in their Mathematics books;
- Move around in the class for facilitating student teachers where necessary.
- Verify and identify groups with different working steps;
- Invite one member from each group with different working steps to present their work where they must explain the working steps;
- As a tutor, harmonize their findings from the presentation.;
- Use different probing questions and guide student teachers to explore the content and examples given in the student teachers' book and lead them to discover how to define increment of a function.
- After this step, guide students to do the application activity 5.1.2 and evaluate whether lesson objectives were achieved.

Answer for activity 5.1.2

a) $h(x) = f(x) + g(x) = x^2 + 2x$

b) i) $\Delta f = f(1.5) - f(1) = 1$

ii) $\Delta g = g(1.5) - g(1) = 1.25$

iii) $\Delta h = h(1.5) - h(1) = 2.25$

c) $\Delta h = \Delta f + \Delta g = 1 + 1.25 = 2.25$

Application Activity 5.1.2

a) $f(x) = x^2 e^x \Rightarrow \Delta f = f(5.08) - f(5) = 438.6706195$

b) $f(x) = \frac{\ln x}{x} \Rightarrow \Delta f = f(5.08) - f(5) = -0.001944421$

Lesson 3: Properties of differentials and their applications

a) Learning objectives

State the properties of differentials and their applications.

b) Teaching resources

student teacher's book and other Reference books to facilitate research, scientific calculator, Manila paper, markers, pens, pencils, if possible computers with mathematical software such as Geogebra, Microsoft Excel, Math-lab or graph-application and internet can be used...

c) Prerequisites/Revision/Introduction

Students teacher will learn better in this lesson if they have a good background on: differentiation (Year 2: Unit 7), quadratic equations (year 1: unit 5), Trigonometry (year 1: unit 8), Logarithm (year 3: unit 4) .

d) Learning activities:

- Invite student-teachers to work in groups and do the activity 5.1.3 in their Mathematics books;
- Move around in the class for facilitating student teachers where necessary.
- Verify and identify groups with different working steps;
- Invite one member from each group with different working steps to present their work where they must explain the working steps;
- As a tutor, harmonize their findings from the presentation.;

- Use different probing questions and guide student teachers to explore the content and examples given in the student teachers' book and lead them to discover how to define increment of a function.
- After this step, guide students to do the application activity 5.1.3 and evaluate whether lesson objectives were achieved.

Answer for activity 5.1.3

a) $h(x) = x^2 + 2x$

b) i) $f(x) = 2x + 5 \Rightarrow df = 2dx$

ii) $g(x) = x^2 - 5 \Rightarrow df = 2xdx$

iii) $h(x) = x^2 + 2x \Rightarrow dh = (2x+2)dx$

c) $dh = df + dg = 2dx + 2xdx = (2+2x)dx$

Application Activity 5.1.3

1. a) $f(x) = 2x^2e^x \Rightarrow df = (2x + x^2)e^x dx$

b) $f(x) = 2x^3 - x + 5 \Rightarrow df = (6x^2 - 1)dx$

2.) Let x be the edge of the tank in the form of the shape of cube and V its volume. We have $x = 4m$ and $dx = 0.02m$. However $V = x^3$ and $dv = 3x^2dx$

, $dv = 3x^2dx \Leftrightarrow dv = 3(4)^2 \times 0.02 = 0.96m^3$,

The capacity of the container in liters is $V = x^3 = (4m)^3 = 64m^3 = 64oool$

Thus, the Therefore approximation error on the measurement of the volume is $0.96m^3$

3. The area of a rectangle is given by $A = wl$. Where w is width and l is the length of the rectangle. Then, $dA = d(wl) = ldw + wdl$. We divide this approximation by $A = wl$ to get an approximation that links the relative changes in A, w and l. $\frac{dA}{A} = \frac{ldw}{lw} + \frac{wdl}{lw} = \frac{dw}{w} + \frac{dl}{l} = 0.01 + 0.01 = 0.02 = 2\%$

4. As the period T of a simple pendulum is $T = 2\pi\sqrt{\frac{l}{g}}$, Then

$$dT = d\left(2\pi\sqrt{\frac{l}{g}}\right) = 2\pi d\left(\sqrt{\frac{l}{g}}\right) = 2\pi \times \frac{1}{2\sqrt{\frac{l}{g}}}\left(\frac{gdl - ldg}{g^2}\right)$$. We divide this

approximation by $T = 2\pi\sqrt{\frac{l}{g}}$ to get an approximation that links

the relative changes to T, l and g. Thus,

$$\frac{dT}{T} = \frac{\cancel{2\pi}\frac{1}{2\sqrt{\frac{l}{g}}} \times \frac{gdl - ldg}{g^2}}{\cancel{2\pi}\sqrt{\frac{l}{g}}} = \frac{1}{2\frac{l}{g}} \times \frac{gdl - ldg}{g^2} = \frac{1}{2}\left(\frac{dl}{l} - \frac{dg}{g}\right) = 0.75\%$$

.

Lesson 4: Definition of indefinite integral

a) Learning objectives

Clarify the relationship between derivative and anti-derivative of a function

b) Teaching resources

student teacher's book and other Reference books to facilitate research, scientific calculator, Manila paper, markers, pens, pencils, if possible computers with mathematical software such as Geogebra, Microsoft Excel, Math-lab or graph-application and internet can be used…

c) Prerequisites/Revision/Introduction

Students teacher will learn better in this lesson if they have a good background on: differentiation (Year 2: Unit 7), quadratic equations (year 1: unit 5), Trigonometry (year 1: unit 8), Logarithm (year 3: unit 4) .

d) Learning activities:

- Invite student-teachers to work in groups and do the activity 5.2.1 in their Mathematics books;
- Move around in the class for facilitating student teachers where necessary.

- Verify and identify groups with different working steps;
- Invite one member from each group with different working steps to present their work where they must explain the working steps;
- As a tutor, harmonize their findings from the presentation.;
- Use different probing questions and guide student teachers to explore the content and examples given in the student teachers' book and lead them to discover how to define increment of a function.
- After this step, guide students to do the application activity 5.2.1 and evaluate whether lesson objectives were achieved.

Answer for activity 5.2.1

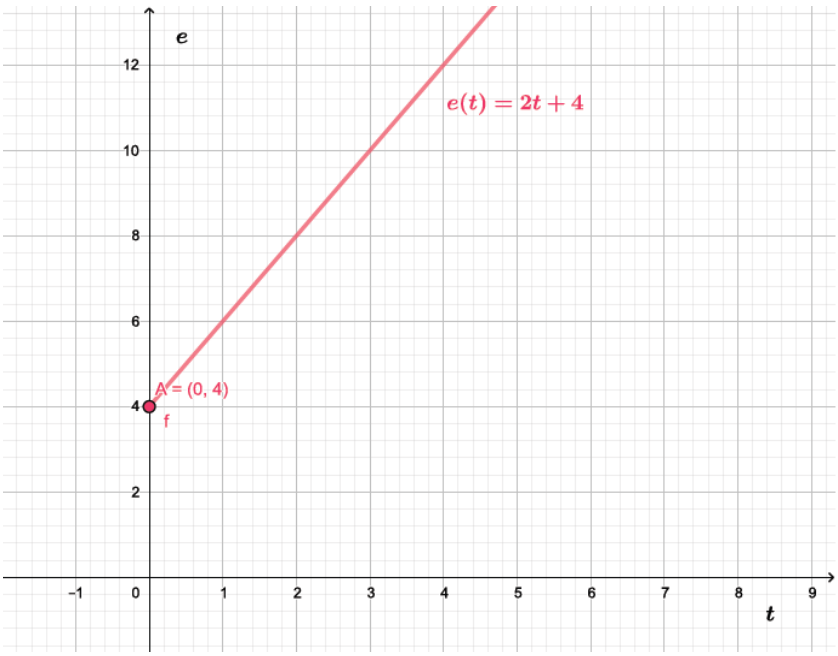
1. If v is the constant velocity, the position of a moving body at time v is given by $e(t) = vt + e_0$ where e_0 is the initial position.

   i) $e(t) = 2t + 1$

   ii) $e(t) = 2t + 2$

   iii) $e(t) = 2t + 4$

2. For the third caterpillar, $e(t) = 2t + 4$. Therefore, $e'(t) = 2m\min^{-1}$ which is the velocity.



$e'(t) = 4 = v(t)$

3. i) $F(x)=x^2+c$ where c is a constant, since $F'(x)=(x^2+c)'=2x$

ii) There are infinitely many possibilities for $F(x)$ because c can take different values in the set of real numbers.

iii) They all differ by a constant c.

Application Activity 5.2.1

1. $I=\int \cos x dx=\sin x+c$

2. $I=\int 3x dx=\frac{3}{2}x^2+c$

$I=\int x^2 dx=\frac{1}{3}x^3+c$

Lesson 5: Properties of indefinite integral

a) Learning objectives

Use properties of integrals to simplify the calculation of integrals.

b) Teaching resources

student teacher's book and other Reference books to facilitate research, scientific calculator, Manila paper, markers, pens, pencils, if possible computers with mathematical software such as Geogebra, Microsoft Excel, Math-lab or graph-application and internet can be used…

c) Prerequisites/Revision/Introduction

Students teacher will learn better in this lesson if they have a good background on: differentiation (Year 2: Unit 7), quadratic equations (year 1: unit 5), Trigonometry (year 1: unit 8), Logarithm (year 3: unit 4) .

d) Learning activities:

- Invite student-teachers to work in groups and do the activity 5.2.2 in their Mathematics books;
- Move around in the class for facilitating student teachers where necessary.
- Verify and identify groups with different working steps;
- Invite one member from each group with different working steps to present their work where they must explain the working steps;
- As a tutor, harmonize their findings from the presentation.;

- Use different probing questions and guide student teachers to explore the content and examples given in the student teachers' book and lead them to discover how to define increment of a function.
- After this step, guide students to do the application activity 5.2.2 and evaluate whether lesson objectives were achieved.

Answer for activity 5.2.2

a) i) $I_1 = \int f(x)\,dx = \int 5dx = 5x + c$

ii) $I_2 = \int g(x)\,dx = \int \frac{dx}{x} = \ln|x| + c$

b) $I = \int (f+g)(x)\,dx = \int \left(5 + \frac{1}{x}\right)dx = 5x + \ln|x| + c$

c) $I = I_1 + I_2 = 5x + \ln|x| + c$ **,** yes there are the same.

Application Activity 5.2.2

1. a. $\int (x^3 + 3\sqrt{x} - 7)dx = \frac{1}{4}x^4 + 2\sqrt{x^3} - 7x + C$

   b. $\int (4x - 12x^2 + 8x - 9)dx = 2x^2 - 4x^3 + 4x^2 - 9x + C$

   c. $\int (\frac{1}{x^2} + e^{-x} - \frac{2}{x})dx = -\frac{1}{x} - e^{-x} - 2\ln x + C$

2. It is not correct because the integral of a quotient is not the quotient of integrals.

$$\int \frac{x^3 - 2}{x^3}dx = \int (1 - 2x^{-3})dx = x + \frac{1}{x^2} + C$$

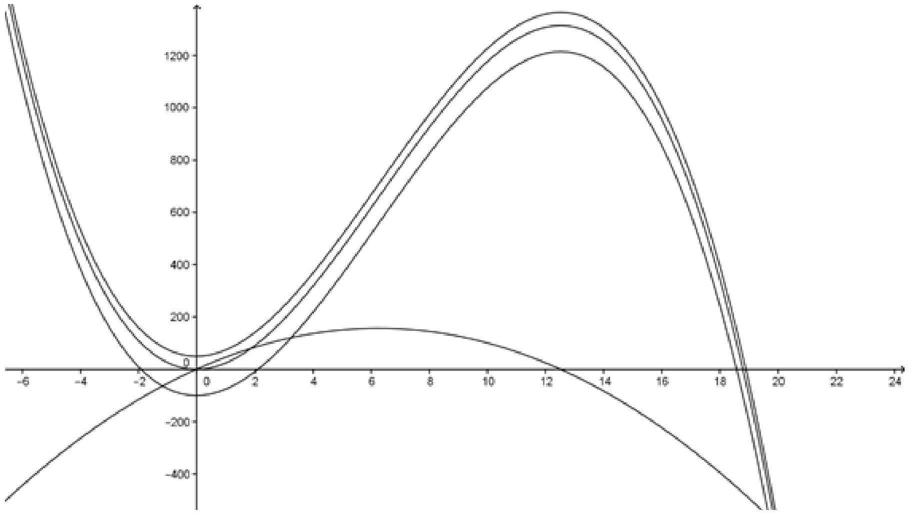
3. As $\int \frac{dy}{dx} = y$, Then $y = \int \frac{x^3 - 5}{x^2}dx = \int \left(x - 5x^{-2}\right)dx = \frac{1}{2}x^2 + \frac{5}{x} + c$

   Thus, $y = f(x) = \frac{1}{2}x^2 + \frac{5}{x} + c$; $f(1) = \frac{1}{2} + 5 + c \Leftrightarrow c = -5$

Therefore, $f(x) = \frac{1}{2}x^2 + \frac{5}{x} - 5$

4. $f(x) = \int (1 + 50x - 4x^2)dx = x + 25x^2 - \frac{4}{3}x^3 + C$

Figure: graph of the marginal cost and three of its corresponding possible total costs



Lesson 6: Basic integration formulas (or immediate integration)

a) Learning objectives

Calculate integrals using appropriate techniques.

b) Teaching resources

student teacher's book and other Reference books to facilitate research, scientific calculator, Manila paper, markers, pens, pencils, if possible computers with mathematical software such as Geogebra, Microsoft Excel, Math-lab or graph-application and internet can be used...

c) Prerequisites/Revision/Introduction

Students teacher will learn better in this lesson if they have a good background on: differentiation (Year 2: Unit 7), quadratic equations (year 1: unit 5), Trigonometry (year 1: unit 8), Logarithm (year 3: unit 4) .

d) Learning activities:

- Invite student-teachers to work in groups and do the activity 5.3.1 in their Mathematics books;
- Move around in the class for facilitating student teachers where necessary.
- Verify and identify groups with different working steps;
- Invite one member from each group with different working steps to present their work where they must explain the working steps;
- As a tutor, harmonize their findings from the presentation.;

- Use different probing questions and guide student teachers to explore the content and examples given in the student teachers' book and lead them to discover how to define increment of a function.
- After this step, guide students to do the application activity 5.3.1 and evaluate whether lesson objectives were achieved.

Answer for activity 5.3.1

Given: $(\arctan x)' = \frac{1}{1+x^2}$, it follows $\int \frac{dx}{1+x^2} = \arctan x + c$

If $\left(\frac{a^x}{\ln a}\right)' = a^x$, then $\int a^x dx = \frac{a^x}{\ln a} + c$

The indefinite integral of a function is a **set of all anti-derivatives of that function** .Therefore given any anti-derivative F of the function f, every possible anti-derivative of f can be written in the form of $F(x)+c$,where c is any constant.

Application Activity 5.3.1

1. $\int e^{3x+1} dx = \frac{1}{3} e^{3x+1} + c$

2. $\int 3^x dx = \frac{3^x}{\ln 3} + c$

3. $\int (10 + \sin x) dx = 10x - \cos x + c$

4. $\int (8 - x^5) dx = 8x - \frac{1}{6} x^6 + c$

Lesson 7: Integration by substitution or change of variable

a) Learning objectives

Calculate integrals using appropriate techniques.

b) Teaching resources

student teacher's book and other Reference books to facilitate research, scientific calculator, Manila paper, markers, pens, pencils, if possible computers with mathematical software such as Geogebra, Microsoft Excel, Math-lab or graph-application and internet can be used...

c) Prerequisites/Revision/Introduction

Students teacher will learn better in this lesson if they have a good background on: differentiation (Year 2: Unit 7), quadratic equations (year 1: unit 5), Trigonometry (year 1: unit 8), Logarithm (year 3: unit 4) .

d) Learning activities:

- Invite student-teachers to work in groups and do the activity 5.3.2 in their Mathematics books;
- Move around in the class for facilitating student teachers where necessary.
- Verify and identify groups with different working steps;
- Invite one member from each group with different working steps to present their work where they must explain the working steps;
- As a tutor, harmonize their findings from the presentation.;
- Use different probing questions and guide student teachers to explore the content and examples given in the student teachers' book and lead them to discover how to define increment of a function.
- After this step, guide students to do the application activity 5.3.2 and evaluate whether lesson objectives were achieved.

Answer for activity 5.3.2

$I=\int e^{5x+2}dx$; let $u=5x+2 \Rightarrow \frac{du}{dx}=5 \Rightarrow dx=\frac{1}{5}du$

$I=\frac{1}{5}\int e^{u}du=\frac{1}{5}e^{u}+c$

Finaly, $\int e^{5x+2}dx=\frac{1}{5}e^{5x+2}+c$

Application Activity 5.3.2

1. a) $\int (x^2+1)2xdx \Rightarrow$ Let $u=x^2+1 \Rightarrow \frac{du}{dx}=2x \Rightarrow 2xdx=du$

$\Rightarrow \int udu=\frac{1}{2}u^2+c \Rightarrow \frac{1}{2}(x^2+1)^2+c$

b) $\int x^2e^{x^3}dx=\frac{1}{3}e^{x^3}+c$

c) $\int (2x+1)e^{x^2+x+2}dx=e^{x^2+x+2}+c$

d) $\int e^{3\cos 2x}\sin 2x dx = -\frac{2}{3}e^{3\cos 2x} + c$

2. $v = \frac{100t}{\left(t^2+1\right)^3} ms^{-1}$, as we know: $v = \frac{dx}{dt} \Rightarrow dx = vdt$

$dx = \frac{100t}{\left(t^2+1\right)^3} dt$; $\int dx = 100\int \frac{tdt}{\left(t^2+1\right)^3}$; let $u = t^2+1 \Rightarrow du = 2tdt$

$tdt = \frac{1}{2}du$; $\Rightarrow x = 50\int u^{-3}du = -25u^{-2} + c$

$\Rightarrow x = -\frac{25}{\left(t^2+1\right)^2} + c$

Let us assume that at $t = 0s \; then \; x = 0m$, by replacing , we get

$0 = -\frac{25}{1} + c \Rightarrow c = 25$ and the new equation: $x = -\frac{25}{\left(t^2+1\right)^2} + 25$

at $t = 2s \Rightarrow x = 24m$

Lesson 8: Integration by parts

a) Learning objectives

Calculate integrals using appropriate techniques.

b) Teaching resources

student teacher's book and other Reference books to facilitate research, scientific calculator, Manila paper, markers, pens, pencils, if possible computers with mathematical software such as Geogebra, Microsoft Excel, Math-lab or graph-application and internet can be used...

c) Prerequisites/Revision/Introduction

Students teacher will learn better in this lesson if they have a good background on: differentiation (Year 2: Unit 7), quadratic equations (year 1: unit 5), Trigonometry (year 1: unit 8), Logarithm (year 3: unit 4) .

d) Learning activities:

- Invite student-teachers to work in groups and do the activity 5.3.3 in their Mathematics books;
- Move around in the class for facilitating student teachers where necessary.
- Verify and identify groups with different working steps;
- Invite one member from each group with different working steps to present their work where they must explain the working steps;
- As a tutor, harmonize their findings from the presentation.;
- Use different probing questions and guide student teachers to explore the content and examples given in the student teachers' book and lead them to discover how to define increment of a function.
- After this step, guide students to do the application activity 5.3.3 and evaluate whether lesson objectives were achieved.

Answer for activity 5.3.3

1. $\frac{d}{dx}f(x)=e^x\frac{d}{dx}(x-1)+(x-1)\frac{d}{dx}(e^x)$

$\Rightarrow e^x+xe^x-e^x=xe^x$

$\frac{d}{dx}f(x)=xe^x$

2. From (1), $\int xe^x dx=(x-1)e^x+c$

3. Let $u=x, v=e^x$,thus $\int uvdx=\int xe^x dx$, while $\int udx\int vdx=\int xdx\int e^x dx$

   From (2), we have: $\int xe^x dx=(x-1)e^x+c$

   Let us find $\int udx\int vdx$:

   $\int udx\int vdx=\int xdx\int e^x dx=\frac{1}{2}x^2\times e^x+c$

   Therefore, $\int uvdx\neq\int udx\int vdx$

   Finally, $\int udv=uv-\int vdu$

Answer for application activity 5.3.3

1. $I=\int x\cos 2xdx\Rightarrow I=\frac{1}{2}x\sin 2x+\frac{1}{4}\cos 2x+c$

2. $I=\int xe^{3x}dx \Rightarrow I=\frac{1}{3}xe^{3x}-\frac{1}{9}e^{3x}+c$

3. $I=\int x\sin 4x dx \Rightarrow I=-\frac{1}{4}x\cos 4x+\frac{1}{16}\sin 4x+c$

4. $\int xe^{-2x}dx$ Let $u=x \Rightarrow du=dx$ and $dv=e^{-2x}dx \Rightarrow v=-\frac{e^{-2x}}{2}$, then,

$\int xe^{-2x}dx=-\frac{xe^{-2x}}{2}+\frac{1}{2}\underbrace{\int e^{-2x}dx}_{I_1}$. For $I_1=\int e^{-2x}dx=-\frac{1}{2}e^{-2x}$ Thus,

$$\int xe^{-2x}dx=-\frac{xe^{-2x}}{2}-\frac{1}{4}e^{-2x}=-\frac{x}{2e^{2x}}-\frac{1}{4e^{2x}}+c, c\in\mathbb{R}$$

Lesson 9: Integration of rational function where numerator is expressed in terms of derivative of denominator

a) Learning objectives

Calculate integrals using appropriate techniques.

b) Teaching resources

student teacher's book and other Reference books to facilitate research, scientific calculator, Manila paper, markers, pens, pencils, if possible computers with mathematical software such as Geogebra, Microsoft Excel, Math-lab or graph-application and internet can be used...

c) Prerequisites/Revision/Introduction

Students teacher will learn better in this lesson if they have a good background on: differentiation (Year 2: Unit 7), quadratic equations (year 1: unit 5), Trigonometry (year 1: unit 8), Logarithm (year 3: unit 4) .

d) Learning activities:

- Invite student-teachers to work in groups and do the activity 5.4.1 in their Mathematics books;
- Move around in the class for facilitating student teachers where necessary.
- Verify and identify groups with different working steps;
- Invite one member from each group with different working steps to present their work where they must explain the working steps;

- As a tutor, harmonize their findings from the presentation.;
- Use different probing questions and guide student teachers to explore the content and examples given in the student teachers' book and lead them to discover how to define increment of a function.
- After this step, guide students to do the application activity 5.4.1 and evaluate whether lesson objectives were achieved.

Answer for activity 5.4.1

1. $I=\int \frac{xdx}{\left(1-x^2\right)^2} \Rightarrow$ let $u=1-x^2 \Rightarrow du=-2xdx \Rightarrow xdx=-\frac{1}{2}du$

$\Rightarrow I=-\frac{1}{2}\int u^{-2}du=\frac{1}{2u}+c$

$\Rightarrow I=\frac{1}{2\left(1-x^2\right)}+c$

2. $I=\int \frac{(2x-1)dx}{3x^2-3x+1} \Rightarrow$ let $u=3x^2-3x+1 \Rightarrow du=3(2x-1)dx \Rightarrow (2x-1)dx=\frac{1}{3}du$

$\Rightarrow I=\frac{1}{3}\int \frac{du}{u} \Rightarrow I=\frac{1}{3}\ln|u|+c$

$\Rightarrow \frac{1}{3}\ln\left|3x^2-3x+1\right|+c$

Application Activity 5.4.1

1. $I=\int \frac{(x+1)dx}{\left(x^2+2x+3\right)^2} \Rightarrow I=-\frac{1}{2\left(x^2+2x+3\right)}+c$

2. $I=\int \frac{xdx}{\left(1-x^2\right)^5} \Rightarrow I=\frac{1}{8\left(1-x^2\right)^4}+c$

3. $I=\int \frac{x^2dx}{\left(2x^3+3\right)^2} \Rightarrow I=-\frac{1}{6\left(2x^3+3\right)}+c$

$$I=\int\frac{x+1}{\left(x^2+2x+5\right)^3}dx\Rightarrow-\frac{1}{4\left(x^2+2x+5\right)^2}+c$$

Lesson 10: Integration of rational function where degree of numerator is greater or equal to the degree of denominator

a) Learning objectives

Calculate integrals using appropriate techniques.

b) Teaching resources

student teacher's book and other Reference books to facilitate research, scientific calculator, Manila paper, markers, pens, pencils, if possible computers with mathematical software such as Geogebra, Microsoft Excel, Math-lab or graph-application and internet can be used...

c) Prerequisites/Revision/Introduction

Students teacher will learn better in this lesson if they have a good background on: differentiation (Year 2: Unit 7), quadratic equations (year 1: unit 5), Trigonometry (year 1: unit 8), Logarithm (year 3: unit 4) .

d) Learning activities:

- Invite student-teachers to work in groups and do the activity 5.4.2 in their Mathematics books;
- Move around in the class for facilitating student teachers where necessary.
- Verify and identify groups with different working steps;
- Invite one member from each group with different working steps to present their work where they must explain the working steps;
- As a tutor, harmonize their findings from the presentation.;
- Use different probing questions and guide student teachers to explore the content and examples given in the student teachers' book and lead them to discover how to define increment of a function.
- After this step, guide students to do the application activity 5.4.2 and evaluate whether lesson objectives were achieved.

Answer for activity 5.4.2

1. $\frac{2x+4}{5x-3}=\frac{2}{5}+\frac{26}{5(5x-3)}\Rightarrow I=\frac{2}{5}x+\frac{26}{25}\ln|5x-3|+c$

2. $\frac{x^2-3x+2}{x^2+2}=1+\frac{1-3x}{x^2+1}\Rightarrow I=x+\arctan x-\frac{3}{2}\ln\left|x^2+1\right|+c$

3. $\frac{x^2+1}{x-1}=x+1+\frac{2}{x-1}\Rightarrow I=\frac{1}{2}x^2+x+2\ln\left|x-1\right|+c$

4. $\frac{x^3+2x-4}{x^2+2}=x-\frac{4}{x^2+2}\Rightarrow I=\frac{1}{2}x^2-2\sqrt{2}\arctan\left(\frac{\sqrt{2}}{2}x\right)+c$

Application Activity 5.4.2

1. $I=\int\left(\frac{x^3-2}{x^2+1}\right)dx\Rightarrow\frac{1}{2}x^2-\frac{1}{2}\ln\left|x^2+1\right|-\arctan x+c$

2. $\int\left(\frac{x^2-2}{x^2+x-2}\right)dx\Rightarrow x-\frac{1}{3}\ln\left|x-1\right|-\frac{2}{3}\ln\left|x+2\right|+c$

3. $\int\left(\frac{x^2+1}{6x-9x^2}\right)dx\Rightarrow-\frac{1}{9}x+\frac{1}{2}\ln\left|x\right|-\frac{13}{18}\ln\left|6-9x\right|+c$

4. $\int\left(\frac{x^3+1}{x^2+7x+12}\right)dx\Rightarrow\frac{1}{2}x^2-7x-26\ln\left|x+3\right|+63\ln\left|x+4\right|+c$

Lesson 11: The denominator is factorized into linear factors

a) Learning objectives

Calculate integrals using appropriate techniques.

b) Teaching resources

student teacher's book and other Reference books to facilitate research, scientific calculator, Manila paper, markers, pens, pencils, if possible computers with mathematical software such as Geogebra, Microsoft Excel, Math-lab or graph-application and internet can be used...

c) Prerequisites/Revision/Introduction

Students teacher will learn better in this lesson if they have a good background on: differentiation (Year 2: Unit 7), quadratic equations (year 1: unit 5), Trigonometry (year 1: unit 8), Logarithm (year 3: unit 4) .

d) Learning activities:

- Invite student-teachers to work in groups and do the activity 5.4.3.1 in their Mathematics books;
- Move around in the class for facilitating student teachers where necessary.
- Verify and identify groups with different working steps;
- Invite one member from each group with different working steps to present their work where they must explain the working steps;
- As a tutor, harmonize their findings from the presentation.;
- Use different probing questions and guide student teachers to explore the content and examples given in the student teachers' book and lead them to discover how to define increment of a function.
- After this step, guide students to do the application activity 5.4.3.1and evaluate whether lesson objectives were achieved.

Answer for activity 5.4.3.1

1. $\frac{x-2}{x^2+2x}=\frac{x-2}{x(x+2)}\Rightarrow\frac{A}{x}+\frac{B}{x+2}=\frac{Ax+2A+B}{x(x+2)}$

$$\Rightarrow\begin{cases}A+B=1\\2A=-2\end{cases}\Rightarrow\begin{cases}A=-1\\B=2\end{cases}$$

$$I=\int\frac{x-2}{x(x+2)}dx=-\int\frac{dx}{x}+2\int\frac{dx}{x+2}$$

$$I=2\ln|x+2|-\ln|x|+c$$

2. $\frac{x}{x^2+3x+2}=\frac{x}{(x+1)(x+2)}\Rightarrow I=\int\frac{x}{(x+1)(x+2)}dx\Rightarrow\frac{A}{x+1}+\frac{B}{x+2}=\frac{x}{(x+1)(x+2)}$

$$\Rightarrow Ax+2A+Bx+B=x$$

$$\Rightarrow\begin{cases}A+B=1\\2A+B=0\end{cases}\Rightarrow\begin{cases}A=-1\\B=2\end{cases}$$

$$\Rightarrow 2\int\frac{dx}{x+2}-\int\frac{dx}{x+1}\Rightarrow I=2\ln|x+2|-\ln|x+1|+c$$

3. $\frac{2}{x^2-4}\Rightarrow\frac{2}{(x-2)(x+2)}\Rightarrow\frac{A}{x-2}+\frac{B}{x+2}\Rightarrow\frac{Ax+2A+Bx-2B}{(x-2)(x+2)}=\frac{2}{(x-2)(x+2)}$

$$\Rightarrow \begin{cases} A+B=0 \\ 2A-2B=2 \end{cases} \Rightarrow \begin{cases} A=\frac{1}{2} \\ B=-\frac{1}{2} \end{cases}$$

$$\Rightarrow \frac{1}{2}\int \frac{dx}{x-2} - \frac{1}{2}\int \frac{dx}{x+2}$$

$$\Rightarrow \frac{1}{2}\ln|x-2| - \frac{1}{2}\ln|x+2| + c \text{ or } I = \ln\sqrt{\frac{x-2}{x+2}} + c$$

4. $$\frac{2x-3}{x^2-x-2} = \frac{2x-3}{(x-2)(x+1)} \Rightarrow \frac{A}{x-2} + \frac{B}{x+1} \Rightarrow \frac{Ax+A+Bx-2B}{(x-2)(x+1)} = \frac{2x-3}{(x-2)(x+1)}$$

$$\Rightarrow \begin{cases} A+B=2 \\ A-2B=-3 \end{cases} \Rightarrow \begin{cases} A=\frac{1}{3} \\ B=\frac{5}{3} \end{cases}$$

$$\Rightarrow I = \frac{1}{3}\int \frac{dx}{x-2} + \frac{5}{3}\int \frac{dx}{x+1}$$

$$\Rightarrow \frac{1}{3}\ln|x-2| + \frac{5}{3}\ln|x+1| + c$$

Application activity 5.4.3.1

1. $I = \int \frac{2dx}{x^2-1} \Rightarrow I = \ln\left|\frac{x-1}{x+1}\right| + c$

2. $I = \int \frac{xdx}{x^2+3x+2} \Rightarrow I = 2\ln|x+2| - \ln|x+1| + c$

3. $I = \int \frac{(x-2)dx}{2x-x^2} \Rightarrow I = -\ln|x| + c$

4. $I = \int \frac{xdx}{x^2+4x+4} \Rightarrow I = \ln|x+2| + \frac{2}{x+2} + c$

Lesson 12: The denominator is a quadratic factor

a) Learning objectives

Calculate integrals using appropriate techniques.

b) Teaching resources

student teacher's book and other Reference books to facilitate research, scientific calculator, Manila paper, markers, pens, pencils, if possible computers with mathematical software such as Geogebra, Microsoft Excel, Math-lab or graph-application and internet can be used...

c) Prerequisites/Revision/Introduction

Students teacher will learn better in this lesson if they have a good background on: differentiation (Year 2: Unit 7), quadratic equations (year 1: unit 5), Trigonometry (year 1: unit 8), Logarithm (year 3: unit 4) and every student teacher must remember this relation: $ax^2+bx+c=a\left[\left(x+\frac{b}{2a}\right)^2-\frac{b^2-4ac}{4a^2}\right]$.

d) Learning activities:

- Invite student-teachers to work in groups and do the activity 5.4.3.2 in their Mathematics books;
- Move around in the class for facilitating student teachers where necessary.
- Verify and identify groups with different working steps;
- Invite one member from each group with different working steps to present their work where they must explain the working steps;
- As a tutor, harmonize their findings from the presentation.;
- Use different probing questions and guide student teachers to explore the content and examples given in the student teachers' book and lead them to discover how to define increment of a function.
- After this step, guide students to do the application activity 5.4.3.2and evaluate whether lesson objectives were achieved.

Answer for activity 5.4.3.2

1. $\int\frac{dx}{x^2+3x+2}; a=1, b=3, c=2$; $x^2+3x+2=\left(x+\frac{3}{2}\right)^2-\left(\frac{1}{2}\right)^2$

$$\Rightarrow\int\frac{dx}{x^2+3x+2}=\int\frac{dx}{\left(x+\frac{3}{2}\right)^2-\left(\frac{1}{2}\right)^2}$$

Let $u=x+\frac{3}{2}\Rightarrow du=dx$

$$\Rightarrow \int \frac{du}{u^2-\left(\frac{1}{2}\right)^2}=\frac{1}{2\times\frac{1}{2}}\ln\left|\frac{u-\frac{1}{2}}{u+\frac{1}{2}}\right|+c$$

Finally, $\int \frac{dx}{x^2+3x+2}=\ln\left|\frac{x+\frac{3}{2}-\frac{1}{2}}{x+\frac{3}{2}+\frac{1}{2}}\right|+c=\ln\left|\frac{x+1}{x+2}\right|+c$

2. $\int \frac{dx}{x^2-4x+4}$, $a=1, b=-4, c=4$

$$x^2-4x+4=(x-2)^2$$

$\int \frac{dx}{x^2-4x+4}=\int \frac{dx}{(x-2)^2}$; let $u=x-2 \Rightarrow du=dx$

$$\int \frac{dx}{x^2-4x+4}=\int \frac{dx}{(x-2)^2}=\int \frac{du}{u^2}$$

$$\Rightarrow \int u^{-2}du=-\frac{1}{u}+c=-\frac{1}{x-2}+c$$

3. $\int \frac{dx}{x^2-6x+18}$, $x^2-6x+18=(x-3)^2+3^2$

$$\int \frac{dx}{x^2-6x+18}=\int \frac{dx}{(x-3)^2+(3)^2}=\frac{1}{3}\arctan\frac{x-3}{3}+c$$

Application Activity 5.4.3.2

1. $I=\int \frac{dx}{x^2+x+2}=\frac{2\sqrt{7}}{7}\arctan\frac{\sqrt{7}}{7}(2x+1)+c$

2. $\int \frac{xdx}{9x^2+6x+2}=\frac{1}{18}\ln\left|9x^2+6x+2\right|-\frac{1}{9}\arctan(3x+1)+c$

3. $I=\int \frac{6x^2-x+5}{(x-2)(2x^2+1)}dx=3\ln|x-2|-\frac{\sqrt{2}}{2}\arctan\sqrt{2}x+c$

4. $\int \frac{x-4}{(2x+1)(x^2+2)}dx = -\ln|2x+1| + \frac{1}{2}\ln|x^2+2| + c$

Lesson 13: Integral of the form: $\int \sin mx \cos nx dx$ or $\int \cos mx \cos nx dx$ or $\int \sin mx \sin nx dx$

a) Learning objectives

Calculate integrals using appropriate techniques.

b) Teaching resources

student teacher's book and other Reference books to facilitate research, scientific calculator, Manila paper, markers, pens, pencils, if possible computers with mathematical software such as Geogebra, Microsoft Excel, Math-lab or graph-application and internet can be used...

c) Prerequisites/Revision/Introduction

Students teacher will learn better in this lesson if they have a good background on: differentiation (Year 2: Unit 7), quadratic equations (year 1: unit 5), Trigonometry (year 1: unit 8), Logarithm (year 3: unit 4).

d) Learning activities:

- Invite student-teachers to work in groups and do the activity 5.5.1 in their Mathematics books;
- Move around in the class for facilitating student teachers where necessary.
- Verify and identify groups with different working steps;
- Invite one member from each group with different working steps to present their work where they must explain the working steps;
- As a tutor, harmonize their findings from the presentation.;
- Use different probing questions and guide student teachers to explore the content and examples given in the student teachers' book and lead them to discover how to define increment of a function.
- After this step, guide students to do the application activity 5.5.1 and evaluate whether lesson objectives were achieved.

Answer for activity 5.5.1

1. $f(x) = \sin 2x \cos x = \frac{1}{2}\sin x + \frac{1}{2}\sin 3x$

$$\int \sin 2x \cos x dx = -\frac{1}{2}\cos x - \frac{1}{6}\cos 3x + c$$

2. $f(x) = \sin x \sin 5x = \frac{1}{2}(\cos 4x - \cos 6x)$

$$\int \sin x \sin 5x dx = -\frac{1}{12}\sin 6x + \frac{1}{8}\sin 4x + c$$

3. $f(x) = \cos 2x \cos 3x = \frac{1}{2}(\cos 5x + \cos x)$

$$\int \cos 2x \cos 3x dx = \frac{1}{2}\sin x + \frac{1}{10}\sin 5x + c$$

Application Activity 5.5.1

1. $\int \sin 3x \cos 2x dx = -\frac{1}{10}\cos 5x - \frac{1}{2}\cos x + c$

2. $\int \sin 2x \cos 3x dx = \frac{1}{2}\cos x - \frac{1}{10}\cos 5x + c$

3. $\int \sin 3x \sin 3x dx = \frac{1}{2}x - \frac{1}{12}\sin 6x + c$

4. $\int \sin x \cos x dx = -\frac{1}{4} cox 2x + c$

5. $\int \cos 3x \cos 3x dx = \frac{1}{2}x + \frac{1}{12}\sin 6x + c$

Lesson 14: Integral of the form: $\int \sin^m x \cos^n x dx, (m, n \in \mathbb{Z}^+)$

a) Learning objectives

Calculate integrals using appropriate techniques.

b) Teaching resources

student teacher's book and other Reference books to facilitate research, scientific calculator, Manila paper, markers, pens, pencils, if possible computers with mathematical software such as Geogebra, Microsoft Excel, Math-lab or graph-application and internet can be used...

c) Prerequisites/Revision/Introduction

Students teacher will learn better in this lesson if they have a good background on: differentiation (Year 2: Unit 7), quadratic equations (year 1: unit 5), Trigonometry (year 1: unit 8), Logarithm (year 3: unit 4) every student teacher must remembers these formulas: a) $\sin^2\alpha+\cos^2\alpha=1$; b) $\sin 2\alpha=2\sin\alpha\cos\alpha$ and c) $\cos 2\alpha=\cos^2\alpha-\sin^2\alpha$.

d) Learning activities:

- Invite student-teachers to work in groups and do the activity 5.5.2 in their Mathematics books;
- Move around in the class for facilitating student teachers where necessary.
- Verify and identify groups with different working steps;
- Invite one member from each group with different working steps to present their work where they must explain the working steps;
- As a tutor, harmonize their findings from the presentation.;
- Use different probing questions and guide student teachers to explore the content and examples given in the student teachers' book and lead them to discover how to define increment of a function.
- After this step, guide students to do the application activity 5.5.2 and evaluate whether lesson objectives were achieved.

Answwer for activity 5.5.2

1. $I=\int \sin x\cos^2 xdx \Rightarrow u=\cos x \Rightarrow \sin xdx=-du$

$$\int \sin x\cos^2 xdx=-\int u^2du=-\frac{1}{3}u^3+c$$

$$\Rightarrow I=-\frac{1}{3}\cos^3 x+c$$

2. $\int \sin^2 x\cos^2 xdx=\int \frac{1}{2}(1-\cos 2x)\frac{1}{2}(1+\cos 2x)dx=\frac{1}{4}\int\left(1-\cos^2 2x\right)dx$

$$\Rightarrow \cos^2 x=\frac{1}{2}+\frac{1}{2}\cos 2x \Rightarrow \cos^2 2x=\frac{1}{2}+\frac{1}{2}\cos 4x$$

$$\int \sin^2 x\cos^2 xdx=\frac{1}{4}\int\left(1-\cos^2 2x\right)dx=\frac{1}{8}\int(1-\cos 4x)dx$$

$$I=\frac{1}{8}x-\frac{1}{32}\sin 4x+c$$

Application Activity 5.5.2

1. $I=\int\cos^3 x\sin x dx=-\frac{1}{4}\cos^4 x+c$

2. $I=\int\sin^4 2x\cos 2x dx=\frac{1}{10}\sin^5 2x+c$

3. $I=\int\sin^3 x dx=\frac{1}{12}\cos 3x-\frac{3}{4}\cos x+c$

4. $I=\int\cos^3 4x dx=\frac{1}{48}\sin 12x+\frac{3}{16}\sin 4x+c$

5. $\int\sin^3 x\cos^3 x dx=\frac{1}{192}\cos 6x-\frac{3}{64}\cos 2x+c$

Lesson 15: Riemann sum approximation and definition of define integrals

a) Learning objectives

Calculate integrals using appropriate techniques.

b) Teaching resources

Student teacher's book and other Reference books to facilitate research, scientific calculator, Manila paper, markers, pens, pencils, if possible computers with mathematical software such as Geogebra, Microsoft Excel, Math-lab or graph-application and internet can be used...

c) Prerequisites/Revision/Introduction

Students teacher will learn better in this lesson if they have a good background on: differentiation (Year 2: Unit 7), quadratic equations (year 1: unit 5), Trigonometry (year 1: unit 8), Logarithm (year 3: unit 4) .

d) Learning activities:

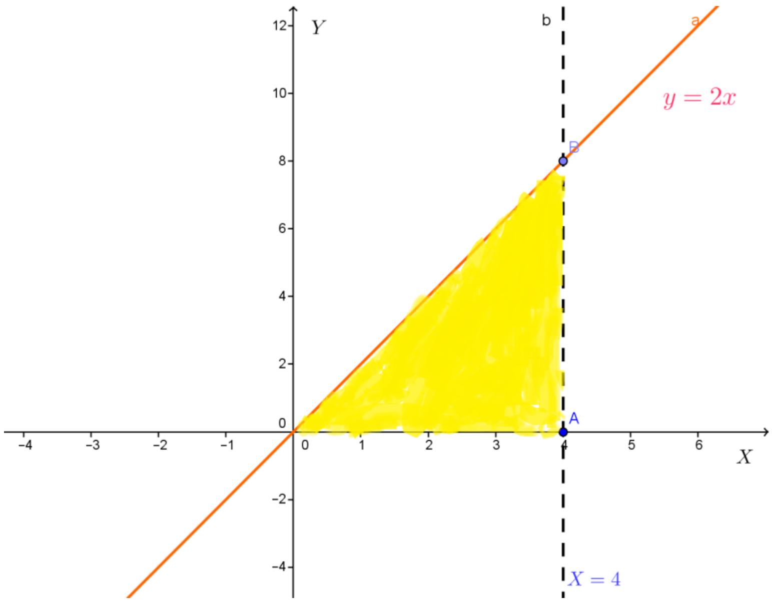
- Invite student-teachers to work in groups and do the activity 5.6.1 in their Mathematics books;
- Move around in the class for facilitating student teachers where necessary.
- Verify and identify groups with different working steps;
- Invite one member from each group with different working steps to present their work where they must explain the working steps;
- As a tutor, harmonize their findings from the presentation.;
- Use different probing questions and guide student teachers to explore the content and examples given in the student teachers' book and lead them to

discover how to define increment of a function.

- After this step, guide students to do the application activity 5.6.1 and evaluate whether lesson objectives were achieved.

Answer for activity 5.6.1

a)



The shape obtained is the Triangle that has three vertices $O(0,0)$, $A(4,0)$, $B(4,8)$

b) Area of Triangle: $A = \frac{1}{2}BH$

The base $B = 4$ unit of length B, and the height $H = 8$ unit of length :

$A = \frac{1}{2}4 \times 8 = 16UA$

The area of the triangle is $16UA$

c) The anti-derivative of $f(x) = 2x$ is $F(x) = x^2 + c$

$F(4) - F(0) = \left[\left(4^2 + c\right) - \left(0^2 + c\right)\right] = 16 + c - 0 - c = 16$

Comparison shows that the findings are the same.

Thus, the area of the triangle = $A = F(4) - F(0) = 16UA$, where UA=unit of area.

Application Activity 5.6.1

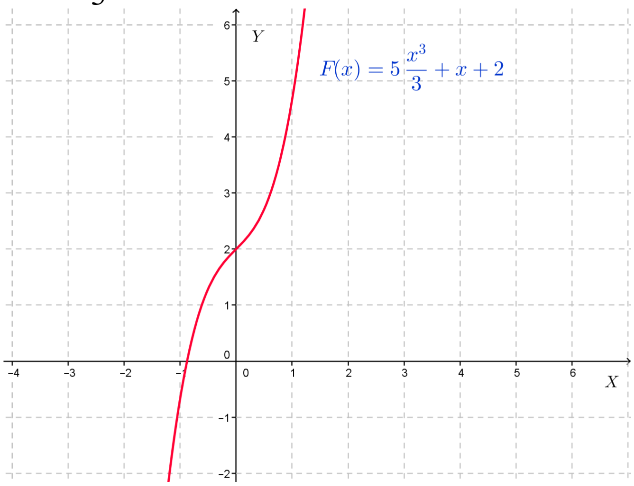
$$\frac{dF}{dx}=5x^2+1\Rightarrow F(x)=\frac{5}{3}x^3+x+c$$

$$F(0)=2\Rightarrow 2=\frac{5}{3}(0)^2+0+c\Rightarrow c=2$$

The new $F(x)=\frac{5}{3}x^3+x+2$

Then plot by the main points:

The graph of $F(x)=\frac{5x^3}{3}+x+2$



Lesson 16: Properties of definite integrals

a) Learning objectives

Extend the concepts of indefinite integrals to definite integrals .

b) Teaching resources

student teacher's book and other Reference books to facilitate research, scientific calculator, Manila paper, markers, pens, pencils, if possible computers with mathematical software such as Geogebra, Microsoft Excel, Math-lab or graph-application and internet can be used…

c) Prerequisites/Revision/Introduction

Students teacher will learn better in this lesson if they have a good background on: differentiation (Year 2: Unit 7), quadratic equations (year 1: unit 5), Trigonometry (year 1: unit 8), Logarithm (year 3: unit 4) .

d) Learning activities:

- Invite student-teachers to work in groups and do the activity 5.6.2 in their Mathematics books;
- Move around in the class for facilitating student teachers where necessary.
- Verify and identify groups with different working steps;
- Invite one member from each group with different working steps to present their work where they must explain the working steps;
- As a tutor, harmonize their findings from the presentation.;
- Use different probing questions and guide student teachers to explore the content and examples given in the student teachers' book and lead them to discover how to define increment of a function.
- After this step, guide students to do the application activity 5.6.2 and evaluate whether lesson objectives were achieved.

Answer for activity 5.6.2

1. a) $\int_1^2 f(x)dx = \int_1^2 (x^3+3)dx = \left[\frac{x^4}{4}+3x\right]_1^2 = \frac{27}{4}$;

b) $\int_1^2 g(x)dx = \int_1^2 (-2x^2)dx = \left[-\frac{2x^3}{3}\right]_1^2 = -\frac{14}{3}$;

c) $\int_1^2 (f+g)(x)dx = \int_1^2 (x^3+3-2x^2)dx = \left[\frac{x^4}{4}+3x-\frac{2x^3}{3}\right]_1^2 = \frac{25}{12}$

2. $\int_1^2 (f+g)(x)dx = \int_1^2 f(x)dx + \int_1^2 g(x)dx = \frac{27}{4} + (-\frac{14}{3}) = \frac{25}{12}$ they are the same

Application Activity 5.6.2

1. $I = \int_0^3 xdx = \left[\frac{x^2}{2}\right]_0^2 = \frac{9}{2}UA$

2. $I = \int_1^2 (x^2-x)dx = \left[\frac{x^3}{3}-\frac{x^2}{2}\right]_1^2 = \frac{5}{6}UA$

3. $I = \int_1^2 (3x^2-6x)dx = \left[\frac{3x^3}{3}-\frac{6x^2}{2}\right]_1^2 = -2UA$

4. $I = \int_{-1}^2 (x^3+3x^2-4)dx = \left[\left(\frac{x^4}{4}+\frac{3x^3}{3}-4x\right)\right]_{-1}^2 = \frac{3}{4}UA$

Lesson 17: Techniques of Integration of definite integrals

a) Learning objectives

Use integral calculus in solving problems from daily life.

b) Teaching resources

student teacher's book and other Reference books to facilitate research, scientific calculator, Manila paper, markers, pens, pencils, if possible computers with mathematical software such as Geogebra, Microsoft Excel, Math-lab or graph-application and internet can be used...

c) Prerequisites/Revision/Introduction

Students teacher will learn better in this lesson if they have a good background on: differentiation (Year 2: Unit 7), quadratic equations (year 1: unit 5), Trigonometry (year 1: unit 8), Logarithm (year 3: unit 4) .

d) Learning activities:

- Invite student-teachers to work in groups and do the activity 5.6.3 in their Mathematics books;
- Move around in the class for facilitating student teachers where necessary.
- Verify and identify groups with different working steps;
- Invite one member from each group with different working steps to present their work where they must explain the working steps;
- As a tutor, harmonize their findings from the presentation.;
- Use different probing questions and guide student teachers to explore the content and examples given in the student teachers' book and lead them to discover how to define increment of a function.
- After this step, guide students to do the application activity 5.6.3 and evaluate whether lesson objectives were achieved.

Answer for activity 5.6.3

1. $f(x) = e^{x^2}$

   i) $t = x^2$, then $x = 0 \Rightarrow t = 0$ and $x = 2 \Rightarrow t = 2^2 = 4$

   ii) $t = x^2 \Rightarrow dt = 2xdx$ and $dx = \frac{dt}{2x}$

   iii) $\int_0^2 2xe^{x^2} dx \Rightarrow \int_0^4 2xe^t \frac{dt}{2x} = \int_0^4 e^t dt = \quad e^t \Big|_0^4$

iv) It is clear that when we apply the substitution method, we also substitute boundaries to keep integral the same.

2. Evaluation of $\int_{1}^{e} x^2 \ln x dx$

Let evaluate this integral by parts.

Let $u = \ln x$ and $dv = x^2 dx$

$du = \frac{dx}{x}$ and $v = \int x^2 dx = \frac{x^3}{3}$

$$\int_{1}^{e} x^2 \ln x dx = \frac{x^3}{3} \ln x \Big|_0^e - \int_{0}^{e} \frac{x^3}{3} \frac{dx}{x} == \frac{x^3}{3} \ln x \Big|_0^e - \frac{1}{9} x^3 \Big|_0^e = \frac{e^3}{3} - \frac{e^3}{9} = \frac{2e^3}{9}$$

Application Activity 5.6.3

1. $\int_{0}^{\frac{\pi}{2}} \cos x e^{\sin x} dx$ using integration by substitution

Let $\begin{cases} t = \sin x \\ dt = \cos x dx \end{cases}$

When $x = 0,\ t = 0$

When $x = \frac{\pi}{2},\ t = 1$

$$I = \int_{0}^{\frac{\pi}{2}} \cos x e^{\sin x} dx = \int_{0}^{1} e^t dt = \left[e^t \right]_0^1 = e^1 - e^0 = e - 1$$

2. $\int_{0}^{1} \ln(1+x) dx$ using integration by parts

Let $u = \ln(1+x) \Rightarrow du = \frac{1}{1+x}$ *and* $dv = dx \Rightarrow v = x + c$

$$I = \int_{a}^{b} u dv = uv - \int_{a}^{b} v du \quad becomes$$

$$I = \int_{0}^{1} \ln(1+x) = [x \ln(1+x)]_0^1 - \int_{0}^{1} \frac{x}{1+x} dx$$

$$= \ln 2 - \int_0^1 \left(1 - \frac{1}{1+x}\right) dx = \ln 2 - \left[x - \ln(1+x)\right]_0^1 = \ln 2 - 1 + \ln 2 = -1 + 2\ln 2 = -1 + \ln 4$$

Lesson 18: Calculation of area of a plane surface

a) Learning objectives

Appreciate the importance of integral calculus in solving problems from daily life.

b) Teaching resources

student teacher's book and other Reference books to facilitate research, scientific calculator, Manila paper, markers, pens, pencils, if possible computers with mathematical software such as Geogebra, Microsoft Excel, Math-lab or graph-application and internet can be used...

c) Prerequisites/Revision/Introduction

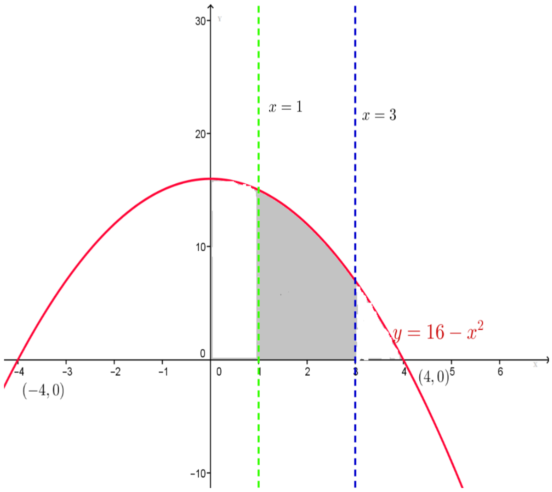
Students teacher will learn better in this lesson if they have a good background on: differentiation (Year 2: Unit 7), quadratic equations (year 1: unit 5), Trigonometry (year 1: unit 8), Logarithm (year 3: unit 4) .

d) Learning activities:

- Invite student-teachers to work in groups and do the activity 5.7.1 in their Mathematics books;
- Move around in the class for facilitating student teachers where necessary.
- Verify and identify groups with different working steps;
- Invite one member from each group with different working steps to present their work where they must explain the working steps;
- As a tutor, harmonize their findings from the presentation.;
- Use different probing questions and guide student teachers to explore the content and examples given in the student teachers' book and lead them to discover how to define increment of a function.
- After this step, guide students to do the application activity 5.7.1 and evaluate whether lesson objectives were achieved.

Answer for activity 5.7.1

a)



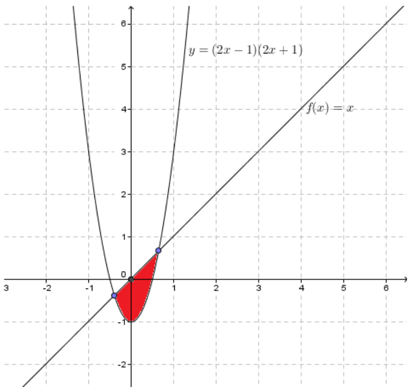
From the above figure, the definite integral which represents the measure of the area bounded by the curve $y = 16 - x^2$, *the x − axis and* $x = 1, x = 3$ is

$$\int_1^3 \left(16 - x^2\right) dx$$

b) $A = \int_1^3 \left(16 - x^2\right) dx = \left[16x - \frac{x^3}{3}\right] = \frac{70}{3} UA$

Application Activity 5.7.1

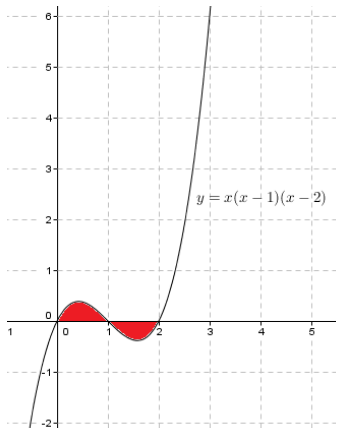
1. a)



$$A = \int_{-0.4}^{0.6} \left[x - (2x-1)(2x+1)\right] dx = \left[\frac{x^2}{2} - \frac{4x^3}{3} - x\right]_{-0.4}^{0.6} = \frac{2}{3} UA$$

;

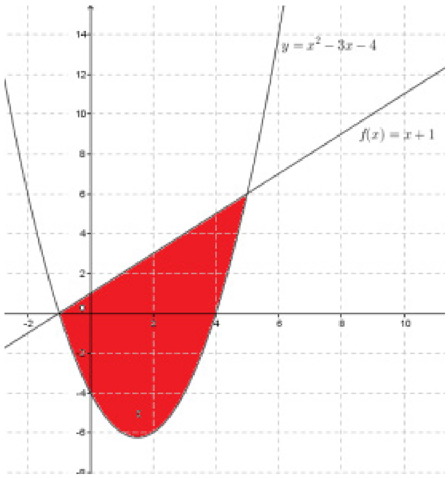
b)



$$A=\int_0^1 x(x-1)(x-2)\,dx-\int_1^2 x(x-1)(x-2)\,dx$$

$$=\left[\frac{x^4}{4}-x^3+x^2\right]_0^1-\left[\frac{x^4}{4}-x^3+x^2\right]_1^2=\frac{1}{2}UA$$

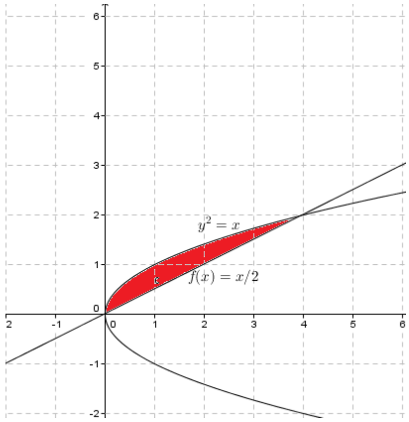
c)



$$A=\int_{-1}^{5}\left[(x-1)-\left(x^2-3x-4\right)\right]dx=\int_{-1}^{5}\left(-x^2+4x+5\right)dx$$

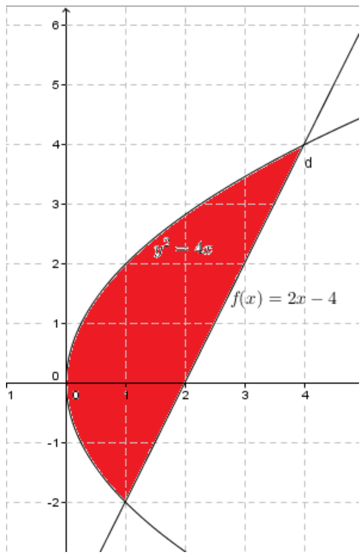
$$=\left[-\frac{x^3}{3}+2x^2+5x\right]_{-1}^{5}=33\frac{1}{3}UA$$

d)



$$\int_0^4 \left(\sqrt{x} - \frac{1}{2}x\right)dx = \int_0^4 \left(x^{\frac{1}{2}} - \frac{1}{2}x\right)dx = \left[\frac{x^{\frac{3}{2}}}{\frac{3}{2}} - \frac{x^2}{4}\right]_0^4 = \frac{4}{3}UA;$$

e)



$$\int_0^4 \left(\sqrt{4x} - (2x-4)\right)dx = \int_0^4 \left(2x^{\frac{1}{2}} - 2x + 4\right)dx = \left[\frac{4x^{\frac{3}{2}}}{3} - x^2 + 4x\right]_0^4 = 9UA$$

1. $y = 2a^2x^2 - x^4; a > 0$

Then

$$y^{'}(x)=0 \Leftrightarrow 4x\left(a^2-x^2\right) \Leftrightarrow x=0 \; or \; x=\pm a$$

$$y'=2a^2(2x)-4x^3=4a^2x-4x^3=4x\left(a^2-x^2\right)$$

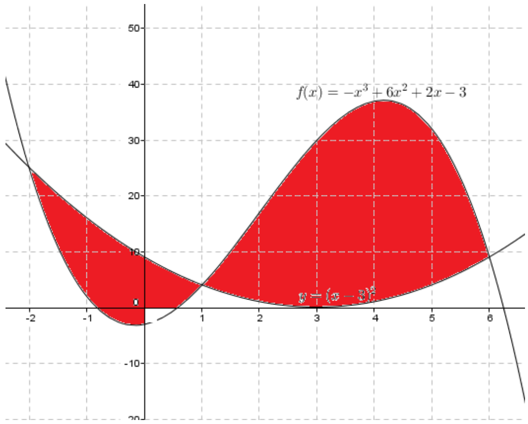
x	$-\infty$		$-a$		0		a		$+\infty$
y'		+	0	-	0	+	0	-	
$y=f(x)$	$-\infty$	↗	a^4	↘	0	↗	a^4	↘	$-\infty$

$$A=\int_{-a}^{a} f(x)dx=2\int_{0}^{a} f(x)dx$$

$$=2\int_{0}^{a}\left(2a^2x^2-x^4\right)dx=2(2a^2\int_{0}^{a}x^2dx-\int_{0}^{a}x^4dx)$$

$$=2\left(\frac{2a^2a^3}{3}-\frac{a^5}{5}\right)=\frac{14a^5}{15}UA$$

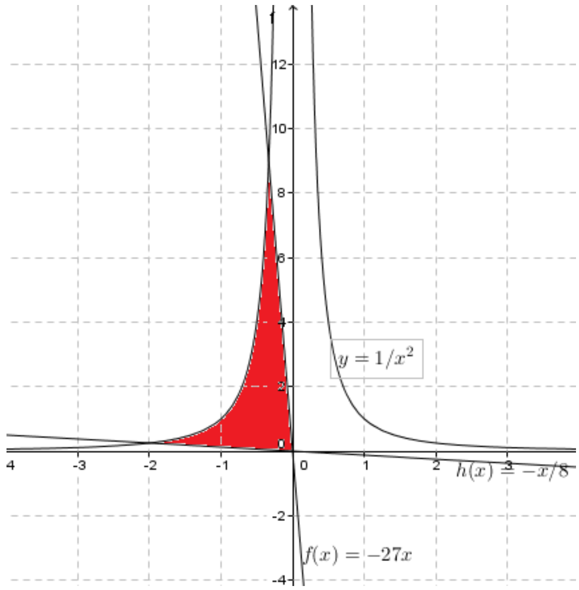
4)



$$\int_{-2}^{1}\left[(x-3)^2-\left(-x^3+6x^2+2x-3\right)\right]dx+\int_{1}^{6}\left[\left(-x^3+6x^2+2x-3\right)-(x-3)^2\right]dx$$

$$=\frac{1043}{6}$$

4)



$$A=\left[\int_{-2}^{\frac{-1}{3}}\left(x^{-2}\right)dx+\int_{\frac{-1}{3}}^{0}\left(-27x\right)dx\right]-\int_{-2}^{0}\left(-\frac{1}{8}x\right)dx$$

$$=\left(\left[-x^{-1}\right]_{-2}^{\frac{-1}{3}}+\left[\frac{-27x^{2}}{2}\right]_{\frac{-1}{3}}^{0}\right)-\left[\frac{-x^{2}}{16}\right]_{-2}^{0}=3.75UA$$

Lesson 19: Calculation of volume of a solid of revolution

a) Learning objectives

Appreciate the importance of integral calculus in solving problems from daily life.

b) Teaching resources

student teacher's book and other Reference books to facilitate research, scientific calculator, Manila paper, markers, pens, pencils, if possible computers with mathematical software such as Geogebra, Microsoft Excel, Math-lab or graph-application and internet can be used...

c) Prerequisites/Revision/Introduction

Students teacher will learn better in this lesson if they have a good background on: differentiation (Year 2: Unit 7), quadratic equations (year 1: unit 5), Trigonometry (year 1: unit 8), Logarithm (year 3: unit 4) .

d) Learning activities:

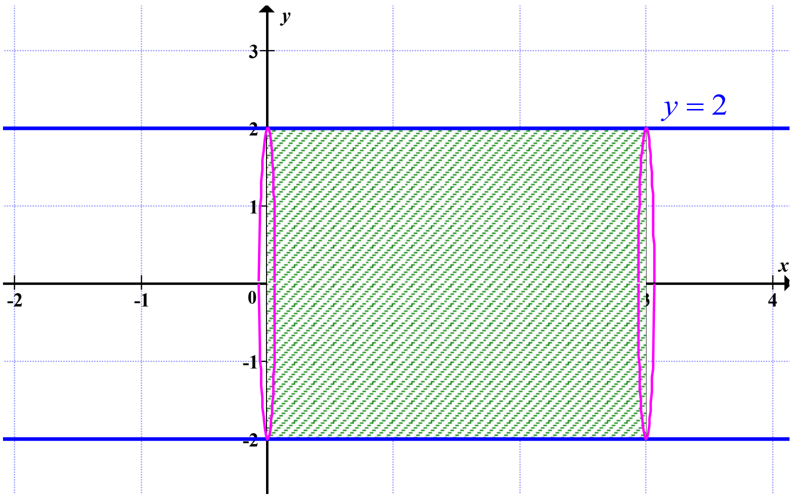
- Invite student-teachers to work in groups and do the activity 5.7.2 in their Mathematics books;
- Move around in the class for facilitating student teachers where necessary.
- Verify and identify groups with different working steps;
- Invite one member from each group with different working steps to present their work where they must explain the working steps;
- As a tutor, harmonize their findings from the presentation.;
- Use different probing questions and guide student teachers to explore the content and examples given in the student teachers' book and lead them to discover how to define increment of a function.
- After this step, guide students to do the application activity 5.7.2 and evaluate whether lesson objectives were achieved.

Answer for activity 5.7.2

1. $y = 2$ for $0 \le x \le 3$

The region enclosed by the curve $y = 2$ for $0 \le x \le 3$ and $x - axis$

a) The region for which the area is rotated about the *x*-axis



b) Solid of revolution obtained in (b) is a **cylinder** of radius 2 and height 3.

c) Volume of cylinder is $\pi r^2 h = \pi(2)^2(3) = 12\pi \, cubic \, units$.

$$= \int_0^3 \pi(2)^2 \, dx = \int_0^3 4\pi \, dx$$

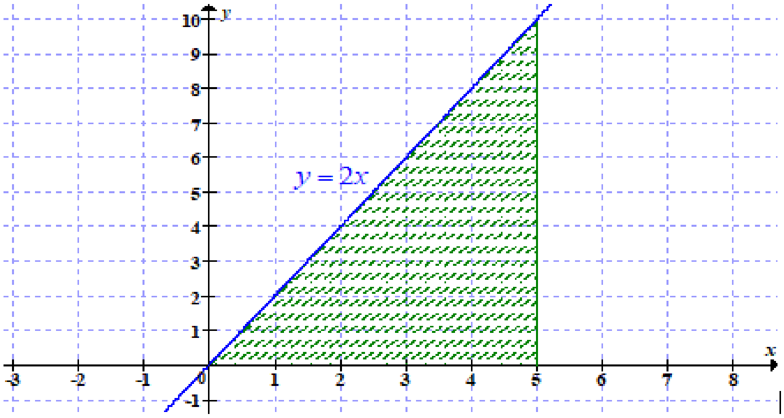
$= 4\pi[x]_0^3 = 12\pi \, cubic \, units$. The results obtained in are equal.

d) The volume of the solid of revolution bounded by the curve $f(x)$ about the $x-axis$ calculated from $x=a$ to $x=b$, is given $V=\pi\int_a^b f^2(x)dx$

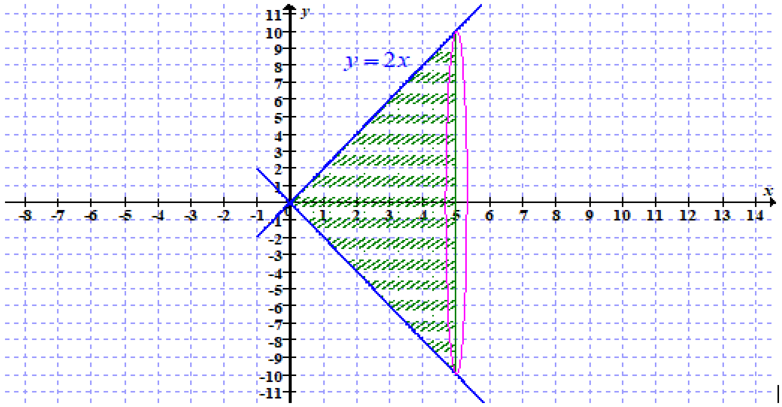
Application Activity 5.7.2

a) 1) $y=2x$ for $0\le x\le 5$

The region enclosed by the curve $y=2x$ for $0\le x\le 5$ and $x-axis$



b) The region for which the area in (a) is rotated 360^0 about the *x*-axis



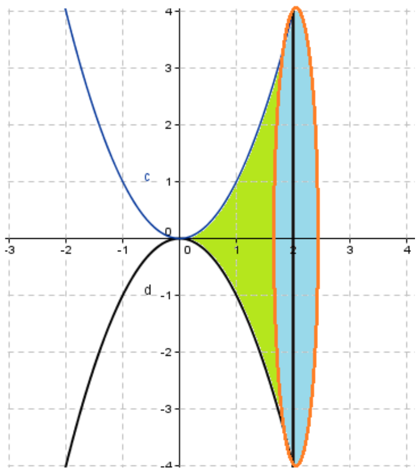
Solid of revolution obtained in (b) is a cone of radius 10 and height 5.

Volume of cone is

$$V=\int_0^5 \pi(2x)^2\,dx=\int_0^5 4\pi x^2\,dx$$

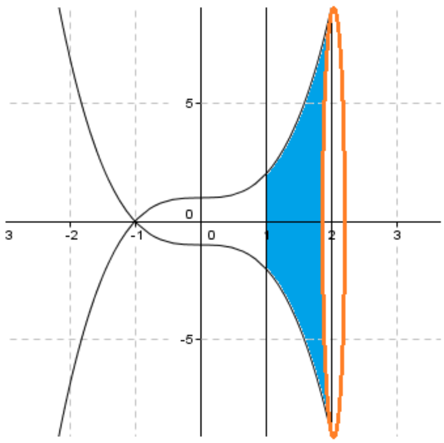
$$=4\pi\left[\frac{x^3}{3}\right]_0^5=\frac{500}{3}\pi\, cubic\, units$$

c) 1)



$$V=\int_0^2 \pi\left(x^2\right)^2 dx=\pi\int_0^2 x^4 dx=\pi\left[\frac{x^5}{5}\right]_0^2=\frac{32\pi}{5}\text{ cubic units}$$

2)

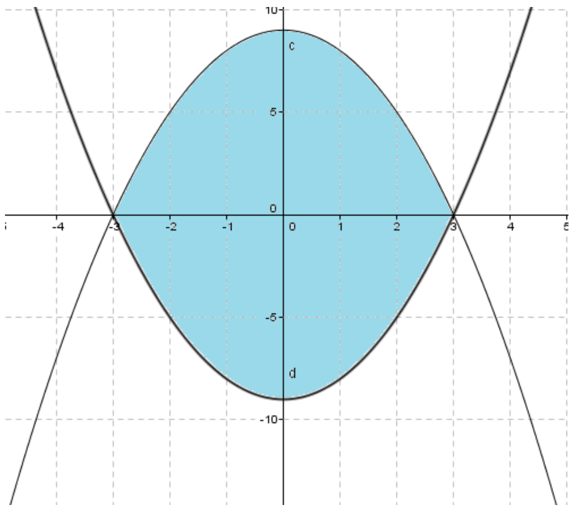


$$V=\int_1^2 \pi\left(1+x^3\right)^2 dx=\pi(\int_1^2 x^6 dx+2\int_1^2 x^3 dx+\int_1^2 dx$$

$$V=\left[\frac{2^7}{7}-\frac{1^7}{7}+2\left(\frac{2^4}{4}-\frac{1^4}{4}\right)+2-1\right]=\pi\left[\frac{128}{7}-\frac{1}{7}+1+2\left(4-\frac{1}{4}\right)\right]$$

$$V=\pi\left(\frac{14+105+254}{14}\right)=\frac{373\pi}{14}\text{ cubic units}$$

3)



$$V=\int_{-3}^{3}\pi\left(9-x^2\right)^2 dx=\pi\int_{-3}^{3}\left(9-x^2\right)^2 dx=\pi\int_{-3}^{3}\left(-x^2+9\right)^2 dx$$

$V=\dfrac{1296\pi}{5}$ cubic units

d) 1) $V=\int_{0}^{1}\pi\left(\sqrt[3]{y}\right)^2 dy=\pi\int_{0}^{1}y^{\frac{2}{3}}dy=\pi\left[\dfrac{y^{\frac{5}{3}}}{\frac{5}{3}}\right]_0^1=\dfrac{3\pi}{5}$ cubic units

2) $V=\int_{-1}^{3}\pi\left(\sqrt{y+1}\right)^2 dy=\pi\int_{-1}^{3}\left(y+1\right)dy=\pi(\int_{-1}^{3}ydy+\int_{-1}^{3}dy=\pi\left[\dfrac{y^2}{2}+y\right]_{-1}^{3}=8\pi$ cubic units.

Lesson 20: Calculation of the arc length of curved surface

a) Learning objectives

Use integral calculus in solving problems from daily life.

b) Teaching resources

student teacher's book and other Reference books to facilitate research, scientific calculator, Manila paper, markers, pens, pencils, if possible computers with mathematical software such as Geogebra, Microsoft Excel, Math-lab or graph-application and internet can be used...

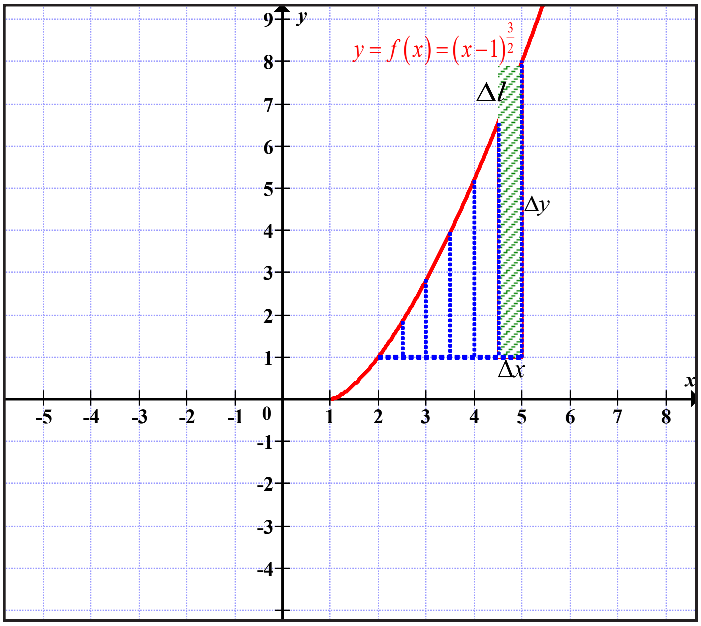
c) Prerequisites/Revision/Introduction

Students teacher will learn better in this lesson if they have a good background on: differentiation (Year 2: Unit 7), quadratic equations (year 1: unit 5), Trigonometry (year 1: unit 8), Logarithm (year 3: unit 4) .

d) Learning activities:

- Invite student-teachers to work in groups and do the activity 5.7.3 in their Mathematics books;
- Move around in the class for facilitating student teachers where necessary.
- Verify and identify groups with different working steps;
- Invite one member from each group with different working steps to present their work where they must explain the working steps;
- As a tutor, harmonize their findings from the presentation.;
- Use different probing questions and guide student teachers to explore the content and examples given in the student teachers' book and lead them to discover how to define increment of a function.
- After this step, guide students to do the application activity 5.7.3 and evaluate whether lesson objectives were achieved.

Answer for activity 5.7.3



1. $$(\Delta l)^2 = (\Delta x)^2 + (\Delta y)^2$$

$$\Delta l = \sqrt{(\Delta x)^2 + (\Delta y)^2}$$

2. $l=\sqrt{(\Delta x)^2+(\Delta y)^2} \Rightarrow dl=\sqrt{(dx)^2+(dy)^2}$

$$dl=\sqrt{(dx)^2+(dy)^2}$$

$$=\sqrt{(dx)^2\left(1+\frac{(dy)^2}{(dx)^2}\right)}$$

$$=\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

We recognize the ratio inside the square root as the derivative, $\frac{dy}{dx}=f'(x)$ then we can rewrite this as

$$dl=\sqrt{1+[f'(x)]^2}\,dx$$

But $f(x)=(x-1)^{\frac{3}{2}} \Rightarrow f'(x)=\frac{3}{2}(x-1)^{\frac{1}{2}}$, then

$$dl=\sqrt{1+\left[\frac{3}{2}(x-1)^{\frac{1}{2}}\right]^2}\,dx$$

$$=\sqrt{1+\frac{9}{4}(x-1)}\,dx$$

$$=\sqrt{\frac{4+9x-9}{4}}\,dx$$

$$=\sqrt{\frac{9x-5}{4}}\,dx$$

3. $\int dl=\int_2^5\sqrt{\frac{9x-5}{4}}\,dx$

$$\Rightarrow l=\int_2^5\sqrt{\frac{9x-5}{4}}\,dx$$

$$=\frac{1}{2}\int_2^5\sqrt{9x-5}\,dx$$

But $\frac{1}{2}\int_2^5\sqrt{9x-5}\,dx=\left[\frac{1}{27}(9x-5)^{\frac{3}{2}}\right]_2^5$

Then,

$$l=\frac{1}{27}\left[(9x-5)^{\frac{3}{2}}\right]_2^5$$

$$=\frac{1}{27}\left((45-5)^{\frac{3}{2}}-(18-5)^{\frac{3}{2}}\right)$$

$$=\frac{1}{27}\left((40)^{\frac{3}{2}}-(13)^{\frac{3}{2}}\right)$$

$$=\frac{1}{27}\left(\sqrt{(40)^3}-\sqrt{(13)^3}\right)$$

$$=\frac{1}{27}\left(\sqrt{40\times(40)^2}-\sqrt{13\times(13)^3}\right)$$

$$=\frac{1}{27}\left(\sqrt{4\times10\times(40)^2}-\sqrt{13\times(13)^2}\right)$$

$$=\frac{1}{27}\left(80\sqrt{10}-13\sqrt{13}\right)\ \textit{units}$$

Synthesis

Arc length of curve of function $f(x)$ in interval $]a,b[$ is given by

$$L=\int_a^b\sqrt{1+[f'(x)]^2}\,dx.$$

Application Activity 5.7.3

1. $\int_0^1\sqrt{1+\left[(3x^{\frac{3}{2}}-1)'\right]^2}\,dx=\int_0^1\left[1+\left(\frac{9}{2}x^{\frac{1}{2}}\right)^2\right]^{\frac{1}{2}}dx=\frac{85\sqrt{85}-8}{243}UL$; 2. $\int_1^8\sqrt{1+\left[\left(x^{\frac{2}{3}}\right)'\right]^2}\,dx=\int_1^8\sqrt{1+\left(\frac{2}{3}x^{\frac{-1}{3}}\right)^2}\,dx=\frac{80\sqrt{10}-13\sqrt{13}}{27}UL$;

3. $\int_{-1}^8\sqrt{1+\left[\left(x^{\frac{2}{3}}\right)'\right]^2}\,dx=\int_{-1}^8\sqrt{1+\left(\frac{2}{3}x^{\frac{-1}{3}}\right)^2}\,dx=\frac{14}{3}UL$

Lesson 21: Application of integrals in real life or other sciences

a) Learning objectives

Use integral calculus in solving problems from daily life.

b) Teaching resources

student teacher's book and other Reference books to facilitate research, scientific calculator, Manila paper, markers, pens, pencils, if possible computers with mathematical software such as Geogebra, Microsoft Excel, Math-lab or graph-application and internet can be used...

c) Prerequisites/Revision/Introduction

Students teacher will learn better in this lesson if they have a good background on: differentiation (Year 2: Unit 7), quadratic equations (year 1: unit 5), Trigonometry (year 1: unit 8), Logarithm (year 3: unit 4) .

d) Learning activities:

- Invite student-teachers to work in groups and do the activity 5.7.4 in their Mathematics books;
- Move around in the class for facilitating student teachers where necessary.
- Verify and identify groups with different working steps;
- Invite one member from each group with different working steps to present their work where they must explain the working steps;
- As a tutor, harmonize their findings from the presentation.;
- Use different probing questions and guide student teachers to explore the content and examples given in the student teachers' book and lead them to discover how to define increment of a function.
- After this step, guide students to do the application activity 5.7.4 and evaluate whether lesson objectives were achieved.

Answer for activity 5.7.4

1. Let $C(q)$ denote the total cost of producing q units. Then the marginal cost is the derivative

$$\frac{dC}{dq}=3(q-4)^2.$$

The increase in cost if production raised from $6 \text{ units to } 10 \text{ units}$ is given by the definite integral:

$$C(q)=\int_6^{10} 3(q-4)^2 dx=\left[(q-4)^3\right]_6^{10}=208\$.$$

2) The spring is compressed starting its natural length $(0m)$ and finish at $(0.25m)$ from the natural length, so the lower limit of the integral is (0) and the upper limit is (0)

Thus the work done= $\int_{0}^{0.25} 16x\,dx = 0.5J$

Application Activity 5.7.4

The total cost function is given by

$\int (7.5q^2 - 26q + 50)\,dq = 2.5q^3 - 13q^2 + 50q + k$.

5.6 Unit summary

1. The differential of a function

The rate of change $\frac{\Delta y}{\Delta x} = \frac{f(x+h) - f(x)}{h}$ means that $\Delta y = f'(x)\Delta x$.

When Δx becomes very small, the change in y can be approximated by the differential of y, that is, $\Delta y \approx dy$ and $\Delta x = dx$.

Therefore, the differential of a function $f(x)$ is the approximated increment of that function when the variation in x becomes very small. It is given by $dy = f'(x)dx$.

f(x) {\displaystyle f'(x)}

2. Anti-derivatives

Let $y = f(x)$ be a continuous function of variable x. An anti-derivative of $f(x)$ is any function $F(x)$ such that $F'(x) = f(x)$.

For any arbitrary C, $F(x) + C$ is also an anti-derivative of $f(x)$ because

$(F(x) + c)' = F'(x)$

3. Indefinite integral

Let $y = f(x)$ be a continuous function of variable x. The indefinite integral of $f(x)$ is the set of all its anti-derivatives. If $F(x)$ is any anti-derivative of function $f(x)$, then the indefinite integral of $f(x)$ is denoted and defined as follows:

$\int f(x)dx = F(x) + C$ where C is an arbitrary constant called the constant of integration.

Thus, $\int f(x)dx = F(x) + C$ if and only if $F'(x) = f(x)$.

- **Properties of indefinite integral**

Let $y = f(x)$ and $y = g(x)$ be continuous functions and k a constant. Integration obeys the following properties:

1. $\int kf(x)dx = k\int f(x)dx$: the integral of the product of a constant by a function is equal to the product of the constant by the integral of the function.

2. $\int [f(x) + g(x)]dx = \int f(x)dx + \int g(x)dx$: the integral of a sum of two functions is equal to the sum of the integrals of the terms.

- **Basic integration formulae**

1. If k is constant, $\int kdx = kx + C$

2. $\int u^n du = \frac{1}{n+1}u^{n+1} + C$,where $n \neq -1$, n is a rational number

3. If $b \neq -1$, and u a differentiable function , $\int u^b du = \frac{u^{b+1}}{b+1} + C$

4. $\int x^{-1}dx = \int \frac{1}{x}dx = \ln|x| + c$ for x nonzero

5. $\int e^x dx = e^x + c$, the integral of exponential function of base e

6. $If\ a > 0\ and\ a \neq 1, \int a^x dx = \frac{a^x}{\ln a} + c$

7. $\int \frac{1}{x-1}dx = \ln|x-1| + C$

8. $If\ a \neq 0, \int \frac{dx}{ax+b} = \frac{1}{a}\ln|ax+b| + C$

9. $|If\ a \neq 0, and\ n \neq -1, \int (ax+b)^n dx = \frac{(ax+b)^{n+1}}{a(n+1)} + C$

- **Integration involving trigonometric functions**

10. $\int \cos xdx = \sin x + C$

11. $\int \sin xdx = -\cos x + C$

12. $\int \frac{dx}{1+x^2} = Arc\tan x + C$

13. $\int \frac{dx}{\sqrt{1-x^2}} = Arc\sin x + C$

14. $\int \frac{dx}{\cos^2 x} = \tan x + C$

15. $\int \frac{dx}{\sin^2 x} = -\cot x + C$

16. $If\ \ a \neq 0, \int \sin(ax+b)dx = -\frac{1}{a}\cos(\text{ax}+\text{b}) + C$

17. $\int \sec^2 xdx = \tan x + C$

18. $\int \text{cosec}^2 \text{x}dx = -\cot x + C$

19. $\int \sec^2 (ax+b)dx = \frac{1}{a}\tan(\text{ax}+\text{b}) + C$

20. $\int \text{cosec}^2 (ax+b)dx = -\frac{1}{a}\cot(\text{ax}+\text{b}) + C$

21. $\int \cos(\text{ax}+\text{b})dx = \frac{1}{a}\sin(ax+b) + C$

4. Techniques of integration of indefinite integrals

- **Integration by substitution**

It is the method in which the original variables are expressed as functions of other variables.

Generally if we cannot integrate $\int h(x)dx$ directly, it is possible to find a new variable

u and function $f(u)$ for which $\int h(x)dx = \int f(u(x))dx = \int f(u)du$

- **Integration by parts**

If u and v are two functions of x, the product rule for differentiation can be used to integrate the product udv or vdu in the following way. Since $d(uv) = udv + vdu$ it comes that $\int d(uv) = \int udv + \int vdu$. This leads to: $uv = \int udv + \int vdu$. Thus

$$\int udv = u.v - \int v\mathrm{du}.$$

When using integration by parts, keep in mind that you are separating the integrand into two parts. One of these parts, corresponding to u will be differentiated and the other, corresponding to dv, will be integrated. Since you can differentiate easily both parts, you should choose a dv for which you know an anti-derivative to make easier the integration.

5. Definite integrals

Let f be a continuous function defined on a close interval $[a,b]$ and F be an anti-derivative of f

Thus, if $F(x)$ is an anti-derivative of $f(x)$, then

$$\int_a^b f(x)dx = \left[F(x)+c\right]_a^b = \left[\left(F(b)+c\right)-\left(F(a)+c\right)\right]$$

$$= \left[F(b)+c-F(a)-c\right] = F(b)-F(a)$$

- **Fundamental theorem of integral calculus:**

Let $F(x)$ and $f(x)$ be functions defined on an interval $[a,b]$. If $f(x)$ is continuous and $F'(x) = f(x)$, then $\int_a^b f(x)dx = \left[F(x)\right]_a^b = F(b)-F(a)$.

- **Properties of definite integral**

If $f(x)$ *and* $g(x)$ are continuous functions on a closed interval $[a,b]$ then:

1. $\int_a^b 0dx = 0$

2. $\int_a^b f(x)dx = -\int_b^a f(x)dx$ (Permutation of bounds)

3. $\int_a^b [\alpha f(x) \pm \beta g(x)]dx = \alpha \int_a^b f(x)dx \pm \beta \int_a^b g(x)dx,\ \alpha\ and\ \beta \in \mathbb{R}$ (Linearity)

4. $\int_a^a f(x)dx = 0$ (Bounds are equal)

5. $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx\ with\ a < c < b$ (Chasles relation)

6. $\forall x \in [a,b], f(x) \le g(x) \Rightarrow \int_a^b f(x)dx \le \int_a^b g(x)dx$ it follows that

$f(x) \ge 0 \Rightarrow \int_a^b f(x)dx \ge 0\ and \left|\int_a^b f(x)dx\right| \le \int_a^b |f(x)|dx$ (Positivity)

7. $\int_{-a}^a f(x)dx = \begin{cases} 2\int_0^a f(x)dx,\ if\ f(x),\ is\ even\ function \\ 0\ if\ f(x)\ is\ an\ odd\ function \end{cases}$

6. Techniques of integration of definite integrals

- **Integration by substitution**

The method in which we change the variable to some other variable is called "**Integration by substitution".**

When definite integral is to be found by substitution then change the lower and upper limits of integration. If substitution is $\varphi(x)$ and lower limit of integration is a and upper limit is b then new lower and upper limits will be $\varphi(a)$ and $\varphi(b)$ respectively.

- **Integration by parts**

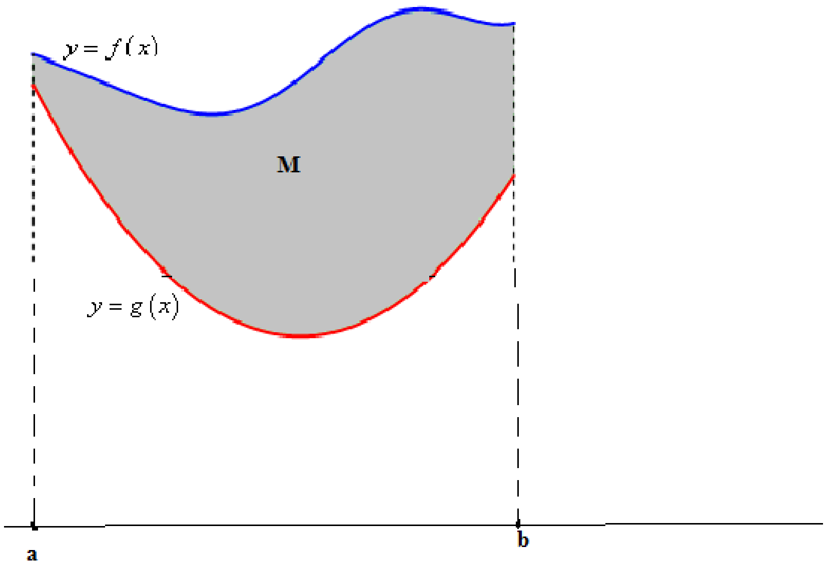
To compute the definite integral of the form $\int_a^b f(x)g(x)dx$ using integration by parts, simply set $u = f(x)\ and\ dv = g(x)dx$.

Then $du = f'(x)dx$ and $v = G(x),$ antiderivative of $g(x)$ so that the integration by parts becomes: $\int_a^b u\,dv = [uv]_a^b - \int_a^b v\,du$

7. Application of definite integrals

- **Calculation of area between two curves**

Suppose that a plane region M is bounded by the graphs of two continuous functions $y = f(x)$ *and* $y = g(x)$ and the vertical straight lines $x = a$ *and* $x = b$ as shown in figure below



From the above figure, we see that $f(x) \geq g(x)$ for $a \leq x \leq b$. Thus the area enclosed between those two functions, the vertical lines $x = a,\ x = b$ and the Horizontal line $y = 0$ is calculated as follows: $A = \int_a^b [f(x) - g(x)]dx$. It means that

$$A = \int_a^b \left[\begin{pmatrix} upper \\ function \end{pmatrix} - \begin{pmatrix} lower \\ function \end{pmatrix}\right]dx$$

- **Determination of the work done in Physics**

The **work** W done by a force F moving through x axis is given by $W = \int F(x)dx$.

Suppose that a force in the direction of the $x-axis$ moves an object from $x = a$ *to* $x = b$ on that axis and that force varies continuously with the position x of the object, that is $F = F(x)$ is a continuous function. The element of work done by the force in moving the object through a very short distance from x *to* $x + dx$ is

$dW = F(x)dx$, so the total work done by the force is $W = \int_{x=a}^{x=b} dW = \int_a^b F(x)dx$

- **Determination of cost function in Economics**

The cost C (respectively the revenue R, utility U and profit P) is related to the marginal cost M (respectively marginal revenue, marginal utility, and marginal profit) by the formula $C(x) = \int M(x)dx$, where x is the number of units produced. The marginal cost is the additional cost to produce one extra unit.

5.7. Additional information for the tutor

For the educative action of the tutor to be effective (in order to respond to all aspects of the student teachers' needs), it is worth mentioning that the tutor needs a wide range of skills, attitudes, a rich and deep understanding of the subject matter and the pedagogical processes to develop the understanding that is required from the student teacher. It is therefore, it is imperative for the tutor to not limit himself/herself to the only to the present book, but also to consider getting information from other relevant books, such as those mentioned in the reference.

Here the tutor has to emphasize the application of integral in solving problems related to area and volume in real life situations.

5.8. End unit assessment

This part provides the answers of end unit assessment activities designed in integrative approach to assess the key unit competence with cross reference to the textbook.

QUESTION ONE

a) $\int (9x^7 + \frac{1}{x-1} + \frac{2}{\cos^2 x} - \frac{1}{2}e^x)dx = \frac{9x^8}{8} + \ln|x-1| + 2\tan x - \frac{1}{2}e^x + c$

b) $\int \frac{1}{\sqrt{x+3}}dx$

Set $\sqrt{x+3} = t$ or equivalently $x+3 = t^2$. This implies $x = t^2 - 3$, then $dx = 2tdt$.

$$\int \frac{x}{\sqrt{x+3}}dx = \int \frac{(t^2-3)\times 2tdt}{t} = 2\int (t^2-3)dt = 2\frac{t^3}{3} - 6t + c$$

Since $t=\sqrt{x+3}$, we have $\int \frac{x}{\sqrt{x+3}}dx = 2\frac{\left(\sqrt{x+3}\right)^3}{3} - 6\sqrt{x+3} + c$

c) $\int \frac{1}{4}\sin 3x dx = -\frac{1}{12}\cos 3x + c$

QUESTION TWO

a) i) Total Cost Function$=\int \frac{x}{\sqrt{x^2+1600}}dx$

Set $\sqrt{x^2+1600}=t$ or equivalently $x^2+1600=t^2$. Thus $xdx=tdt$.

Total Cost Function$=\int \frac{x}{\sqrt{x^2+1600}}dx = \int \frac{tdt}{t} = \int dt = t+c$

Replacing t by $\sqrt{x^2+1600}$, gives the Total Cost Function

$$C(x)=\int \frac{x}{\sqrt{x^2+1600}}dx = \sqrt{x^2+1600}+c$$

Given that the Fixed Cost is $500FRW$ we get:

$$500=\sqrt{0^2+1600}+c$$
$$500=400+c$$
$$c=100$$

Total Cost Function $C(x)=\sqrt{x^2+1600}+100$

ii) An Average Cost $AC=\frac{C(x)}{x}=\frac{\sqrt{x^2+1600}+100}{x}=\sqrt{1+\frac{1600}{x^2}}+\frac{100}{x}$

b) $f(x)=4-\sqrt{x}$

i) The $y-$intercept is the point with abscissa $x=0$.

We have $y=4-\sqrt{0}=4 \Rightarrow A(0,4)$

The $x-$intercept, $y=0$. We have

$$0=4-\sqrt{x}\Rightarrow-\sqrt{x}=-4\Rightarrow x=16\Rightarrow B(16,0)$$

ii) The graph

iii) The shaded area in terms of a definite integral is expressed by

$$A=\int_0^{16} f(x)dx \Rightarrow A=\int_0^{16}\left(4-\sqrt{x}\right)dx$$

iv) $A=\int_0^{16}\left(4-\sqrt{x}\right)dx=\left[4x-\dfrac{x^{3/2}}{\frac{3}{2}}\right]_0^{16}=\left[(4\times 16)-\dfrac{16^{3/2}}{\frac{3}{2}}\right]$

$$=64-\frac{2}{3}\sqrt{4096}=64-\frac{2}{3}\times 64=64\left(1-\frac{2}{3}\right)=64\left(\frac{3-2}{3}\right)=\frac{64}{3}\ .$$

The area of the shaded region is $\frac{64}{3}$ *Square unit*.

5.9 Additional activities

5.9.1 Remedial activities:

1. The total cost C(x) in Rupees, associated with the production of x units of an item is given by: $C(x)=0.005x^3-0.02x^2+30x+5000$, find the marginal cost when 3 units are produced, where by marginal cost we mean the instantaneous

rate of change of total cost at any level output.

Answer:

Since marginal cost is the rate of change of total cost with respect to the output, we have: marginal cost: $MC = \frac{dC}{dx} = 0.005\left(3x^2\right) - 0.02\left(2x\right) + 30$; when x=3,

$$MC = 0.015\left(3^2\right) - 0.04\left(3\right) + 30 = 30.015$$

2. Calculate: $\int\left(1 + x + \frac{1}{x} + \text{sinx} + \text{e}^{2x}\right)dx$

Answer:

$$x + \frac{1}{2}x^2 + \ln x - \cos x + \frac{1}{2}e^{2x} + c$$

3. Calculate:

$$\int xe^x dx$$

Answer:

$$I = \int xe^x dx = xe^x - \int e^x dx = xe^x - e^x + c = e^x\left(x-1\right) + c$$

5.9.2 Consolidation activities

1. Determine the volume of the solid generated by rotating the region bounded by: $f\left(x\right) = x^2 - 4x + 5,\ x \in \left[1,4\right]$, and x-axis, about x-axis.

Answer:

$$V = \pi\int_1^4\left(x^2 - 4x + 5\right)^2 dx = \frac{78\pi}{5}UV$$

2. Determine the volume of the solid generated by rotating the region bounded by: $y = \sqrt{25 - x^2},\ y = 0,\ x \in \left[2,4\right]$, rotated about x-axis

Answer:

$$V = \pi\int_2^4\left(\sqrt{25 - x^2}\right)^2 dx = \pi\int_2^4 25dx - \int_2^4 x^2 dx = \frac{94\pi}{3}UV$$

5. 9.3 Extended activities

1. Determine the volume of the solid generated by rotating the region bounded by: $y=\sqrt{\sin x}$, $x\in[0,\pi]$, about x-axis

Answer:

$$V=\int_0^\pi\left(\sqrt{\sin x}\right)^2 dx=\pi\int_0^\pi \sin x dx = 2\pi UV$$

2. Find: $I=\int e^x\cos 2x dx$

Answer:

$$I=\frac{1}{5}e^x\left(\cos 2x+\sin 2x\right)+c$$

3. Evaluate the following integrals:

a) $\int\frac{xdx}{\sqrt{1+x}}$ b) $I=\int\frac{x^2dx}{\sqrt{1-x^6}}dx$ c) $I=\int\frac{x^2dx}{\sqrt{1-x^6}}dx$ d) $I=\int\frac{e^{p\arcsin x}}{\sqrt{1-x^2}}dx$

answers:

a) $\int\frac{(1+x)-1}{\sqrt{1+x}}dx\Rightarrow\int\frac{1+x}{\sqrt{1+x}}dx-\int\frac{dx}{\sqrt{1+x}}$

$$\Rightarrow\int(1+x)^{\frac{1}{2}}dx-\int(1+x)^{-\frac{1}{2}}dx\Rightarrow\frac{3}{2}(1+x)^{\frac{3}{2}}-2(1+x)^{\frac{1}{2}}+c$$

b) $I=\int\frac{x^2dx}{\sqrt{1-x^6}}dx$

Let $k=x^3\Rightarrow x^2dx=\frac{1}{3}dk$

$$I=\frac{1}{3}\int\frac{dk}{\sqrt{1-k^2}}=\frac{1}{3}\arcsin k+c$$

$$\Rightarrow I=\frac{1}{3}\arcsin x^3+c$$

c) $I=\int\frac{e^{p\arcsin x}}{\sqrt{1-x^2}}dx$

Answer: let $b = p\arcsin x \Rightarrow \frac{1}{p}db = \frac{dx}{\sqrt{1-x^2}}$

$$I = \frac{1}{p}\int e^b db = \frac{1}{p}e^b + c$$

$$\Rightarrow I = \frac{1}{p}e^{p\arcsin x} + c$$

d) $I = \int \frac{e^x+1}{e^x-1}dx$

Answer: $\frac{e^x+1}{e^x-1} = 1 + \frac{2}{e^x-1} = 1 + 2\left(\frac{1}{e^x\left(1-e^{-x}\right)}\right) = 1 + 2\left(\frac{e^{-x}}{1-e^{-x}}\right)$

$$I = \int dx + 2\int \frac{e^{-x}}{1-e^{-x}}dx$$

$$I = x + 2\ln\left|1-e^{-x}\right| + c$$

UNIT 6

DIFFERENTIAL EQUATION OF FIRST ORDER

6.1. Key unit competence:

Use differential equations to solve related problems that arise in daily life.

6.2 Prerequisites

Student-teachers will easily learn this unit, if they have a good background on: differentiation (Year 2: Unit 7), Quadratic equations (year 1: unit 5), Trigonometry (year 1: unit 8), Logarithm and exponential (year 3:unit 4), Integration(year 3: unit 5), Complex numbers(year 3: unit 1) .

6.3 Cross-cutting issues to be addressed:

- **Inclusive education**: promote the participation of all student teachers while teaching.
- **Peace and value Education**: During group activities, the teacher will encourage student teachers to help each other and to respect opinions of colleagues.
- **Gender**: During group activities try to form heterogeneous groups (with boys and girls) or when Student teachers start to present their findings encourage both (boys and girls) to present.
- **Environment and Sustainability:** During the lesson on population growth, guide Student teachers to discuss the effects of the high rate of population growth on the environment and sustainability.
- **Standardization culture**: During the lesson on application of differential equations in chemistry **(the quantity of a drug in the body)** guide student teachers to discuss advantages for respecting Doctor's instructions when taking drugs.

6.4 Guidance on introductory activity

- Invite student teachers to form groups and let them to work independently for some while on introductory activity 6.0 to understand the concept of differential equation.
- Walk around to provide various pieces of advice where necessary.

- After a given time, invite student teachers to present their findings and through their works help them to have an idea on the differential equation.
- Harmonize their works and emphasize that they had a differential equation representing a situation of the population for a country and that it can be solved to obtain the formula for estimating the population of that country at any time t.
- Invite student teachers to discuss positive measures that should be taken to address the problem of exponential growth of the population.
- Ask student teachers to discuss the importance of studying how to solve differential equations.
- Basing on student-teachers' experience, prior knowledge and abilities shown in answering the questions for this activity, use different questions to facilitate them to give their predictions and ensure that you arouse their curiosity on what is going to be leant in this unit after you harmonize their works and ensure that they got exact solution.

Answer for introductory activity 6.0:

The quantity $y(t)$ satisfies the exponential growth model: $\frac{dy}{y} = kdt$.

To find $y(t)$, let us integrate both sides of the equation $\frac{dy}{y} = kdt$;

$\int \frac{dy}{y} = k\int dt \Rightarrow \ln|y| = kt + c$ where c is an arbitrary constant.

We have the function $y = e^c e^{kt}$.

Taking the constant e^c, we find $y(t) = Ce^{kt}$.

This is an exponential function that is increasing when the constant k is positive.

Assuming an exponential growth model of the population y and constant growth rate k, at initial time $(t = 0)$, the population is $y(0) = Ce^{k\times 0} = C_0$.

If the population of a country is C_0 at time $t = 0$, this population with the growth rate k will be $y(t) = C_0 e^{kt}$ after the time t.

Therefore, given that the size of the Rwandan population is now (in the year 2018) estimated to $C_0 = 12,089,721$ with a growth rate of about $k = 2.3\% = 0.0237$ comparatively to the year 2017, the equation of Rwandan population becomes $y(t) = 12,089,721e^{0.0237t}$.

1. The national population at the beginning of the year 2020, 2030, 2040 and 2050:

From now in 2018 taken as initial time, in 2020 the time $t=2$, in 2030 the time $t=12$, in 2040 the time $t=22$, in 2050 the time $t=32$.

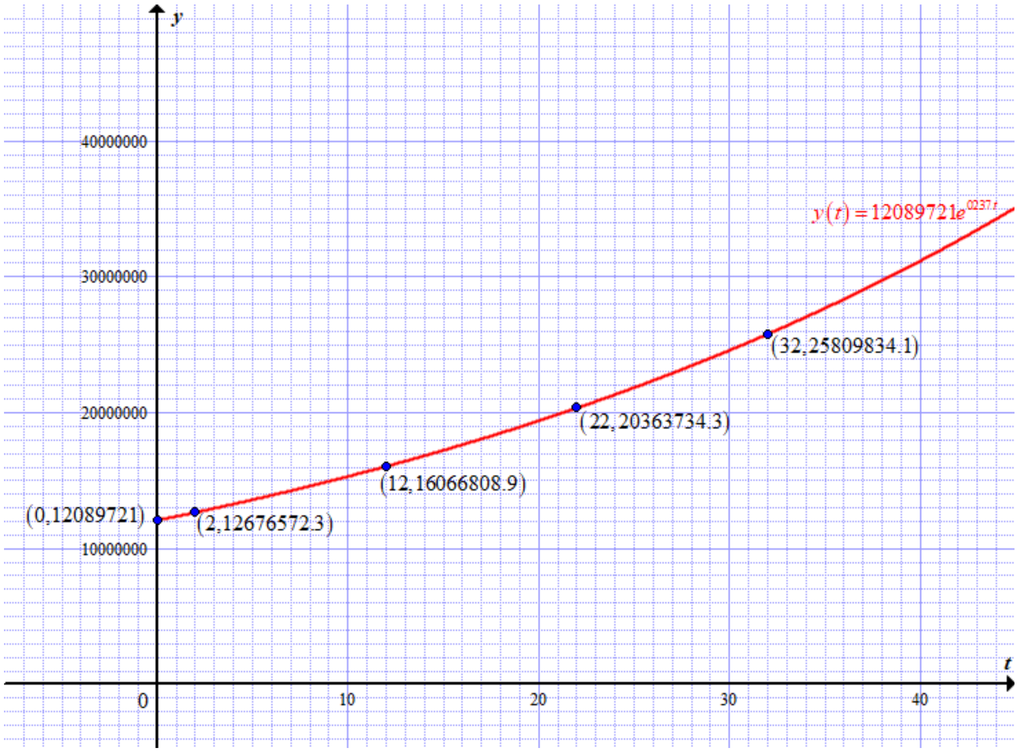
- Hence, the national population at the beginning of the year 2020 will be $y(2)=12,089,721e^{2(0.0237)}=12,676,572.28$ people.

- The national population at the beginning of the year 2030 will be $y(12)=12089721e^{12(0.0237)}=16,066,808.94$ people.

- The national population at the beginning of the year 2040 will be $y(22)=12089721e^{(0.0237)22}=20,363,734.28$ people.

- The national population at the beginning of the year 2050 will be $y(32)=12089721e^{32(0.0237)}=25,809,834.14$ people.

2. Graph representing the increasing population $y(t)=12,089,721e^{0.0237t}$



3. Our observation is that the national population is increasing with time but the surface where to live remains constant and we are not sure that the economy of the country is going to increase exponentially.

4. Pieces of advice: Police makers should adopt the family planning policy and

sensitize the population as well as integrate the family planning programs into school curricula at **all levels of education.**

The teacher should encourage student teachers to provide more ideas.

6.5. List of lessons/sub-heading

UNIT 6: DIFFERENTIAL EQUATION OF FIRST ORDER			
N0	**LESSON TITLE**	**LEARNING OBJECTIVES**	**NUMBER OF PERIODS**
0	Introductory activity	Arouse the curiosity of stent teacher.	1
1.	Definition and classification of differential equations.	Explore the concepts of differential equation and classify the types of differential equations.	1
2.	Differential equations with separable variables	Identify and solve an ordinary differential equation of first order is with separable variables.	2
3.	The solution of equation of the form $\frac{dy}{dx}=f(x)h(y)$	solve an ordinary differential equation of the form $\frac{dy}{dx}=f(x)h(y)$	1
4.	Simple homogeneous differential equations of first order	Use appropriate method to solve a simple homogeneous differential equation of first order.	2
5.	Linear differential equations of first order.	Use appropriate method to solve a linear differential equation of first order.	2
6.	Differential equations and the population growth.	Appreciate the use of differential equations of first order in solving problems related to the population growth.	1

7.	Differential equations and Crime investigation.	Appreciate the use of differential equations of first order in solving problems related to crime investigation.	2
8.	Differential equations and the quantity of a drug in the body.	Appreciate the use of differential equations of first order in solving problems related to quantity of a drug in the body.	2
9.	Differential equations in Economics and finance.	Appreciate the use of differential equations of first order in solving problems related to economics and finance.	2
10.	Differential equations in Electricity(series circuits)	Appreciate the use of differential equations of first order in solving problems related to electricity(series circuits)	2
11.	End assessment		2
Total periods			20

Lesson 1: Definitions and classification of differential equations

a) Learning objectives

Extend the concepts of differentiation and integration to ordinary differential equations.

b) Teaching resources:

student teacher's book and other Reference books to facilitate research, scientific calculator, Manila paper, markers, pens, pencils, if possible computers with mathematical software such as Geogebra, Microsoft Excel, Math-lab or graph-application and internet can be used...

c) Prerequisites/Revision/Introduction

Student-teachers will easily learn this unit, if they have a good background on: differentiation (Year 2: Unit 7), quadratic equations (year 1: unit 5), Trigonometry (year 1: unit 8), Logarithm and exponential (year 3:unit 4), integration(year 3: unit 5), complex numbers(year 3: unit 1) .

d) Learning activities:

This lesson will help student teachers to understand the concept of differential equation and their classifications through derivatives and the highest derivative degree.

- Form groups and ask student teachers to work out the activity 6.1.
- Let student teachers work independently for some while.
- Facilitate student teachers to derivate and to guess the power of the highest derivative.
- Ask randomly some groups to present their findings to the whole class;
- Guide student teachers to interact about the findings, to conclude on the new concept and to write a short summary.
- Let student teachers work out activity 6.1 under your guidance, and work individually application activity 6.1 to assess their competences.

Answers to activity 6.1

1. On differentiation $\frac{dy}{dx} = 4k \Rightarrow k = \frac{dy}{4dx}$

The given equation becomes $y = \frac{dy}{dx} x \; or \; y = y' x \Rightarrow y' = \frac{y}{x}$

Order of the derivative is 1.

2. $y = kx + bx^2$

Differentiate to get $\frac{dy}{dx} = k + 2bx$. Solving for k yields to $k = y' - 2bx$

Differentiating again: $\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2} = 2b$

Then $b = \frac{d^2y}{2dx^2} \; or \; b = \frac{y''}{2}$

Replace k and b by their values in $y = kx + bx^2$ to get:

$$y = \left[\frac{dy}{dx} - \left(\frac{d^2y}{dx^2}\right)x\right]x + \frac{1}{2}\frac{d^2y}{dx^2}x^2 \; or \; y = y'x - \frac{1}{2}y''x^2 \Rightarrow y'' - \frac{2y'}{x} + \frac{2y}{x^2} = 0$$

This is a differential equation of 2[nd] order.

3. $y = k\cos 2x - b\sin 2x$

Differentiate y with respect to x to get: $y' = -2k\sin 2x - 2b\cos 2x$

Differentiating again, we get:

$y'' = -4k\cos 2x + 4b\sin 2x = -4\left(k\cos 2x - b\sin 2x\right) = -4y$. Then,

$y'' = -4y \text{ or } y'' + 4y = 0$

This is the differential equation of 2nd order.

Application Activity 6.1

a) $\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^4 - 4x + y = 1$; This is differential equation of the 2nd order and degree one

b) $\left(\frac{dy}{dx}\right)^3 - 2x = \cos y - 2\sin x$;

This is a differential equation of the 1st order and degree 3.

c) $(y'')^3 + (y') - 2y = x$;

This is a differential equation of the 2nd order and degree 3.

d) $y\frac{d^2y}{dx^2} = -\cos x$; This is a differential equation of the 2nd order and degree 1.

e) $x^2\left(\frac{d^2y}{dx^2}\right)^4 + y\left(\frac{dy}{dx}\right) + y^4 = 0$;

This is a differential equation of the 2nd order and degree 4.

NB: Ask student teachers to explain in words their answers based on the definition of order and degree of differential equations.

Lesson 2: Differential equations with separable variables

a) Learning objectives

Determine whether an ordinary differential equation of first order is with separable variables, homogeneous or linear.

b) Teaching resources:

student teacher's book and other Reference books to facilitate research, scientific calculator, Manila paper, markers, pens, pencils, if possible computers with mathematical software such as Geogebra, Microsoft Excel, Math-lab or graph-application and internet can be used...

c) Prerequisites/Revision/Introduction

Student-teachers will easily learn this unit, if they have a good background on: differentiation (Year 2: Unit 7), quadratic equations (year 1: unit 5), Trigonometry (year 1: unit 8), Logarithm and exponential (year 3:unit 4), integration(year 3: unit 5), complex numbers(year 3: unit 1) .

d) Learning activities:

This lesson will help student teachers to understand the concept of differential equation and their classifications through derivatives and the highest derivative degree.

- Form groups and ask student teachers to work out the activity 6.2.1
- Let student teachers work independently for some while.
- Facilitate student teachers to derivate and to guess the power of the highest derivative.
- Ask randomly some groups to present their findings to the whole class;
- Guide student teachers to interact about the findings, to conclude on the new concept and to write a short summary.
- Let student teachers work out activity 6.2.1 under your guidance, and work individually application activity 6.2.1 to assess their competences.

Answers to activity 6.2.1

1. $4y'-2x=0$

a) The above equation can be written as: $4\frac{dy}{dx}=2x$. and write dy as a subject of x and dx as follows $y=\frac{x}{2}dx$

Integrate both sides to deduce the value of the dependent variable y :

b) $\int dy=\frac{1}{2}\int xdx \Rightarrow y=\frac{1}{4}x^2+c$ Knowing that $y=\frac{dy}{dx}$, it is clear that

$4\frac{dy}{dx}-2x=0 \; and \; 4y'-2x=0$ are the same.

c) Replace the value of y in the given equation to get:

$4\left(\frac{x^2}{4}+c\right)' - 2x = 0 \Rightarrow 4\left(\frac{2x}{4}\right) - 2x = 0$ and it is clear that the equality remains correct.

2. a) $\sin x dx - \sin y \, dy = 0 \Rightarrow \sin y dy = \sin x dx$

By integrating both sides we get:

$$\int \sin y dy = \int \sin x dx \Leftrightarrow -\cos y = -\cos x + c$$

$$-\cos y = -\cos x + c \Leftrightarrow \cos y = \cos x - c$$

$$y = ar\cos(\cos x - c)$$

b) $x\frac{dy}{dx} = 1$

Separating variables yields to $dy = \frac{1}{x}dx$, $dy = \frac{1}{x}dx \Leftrightarrow dy = \frac{dx}{x}$

Integrating both sides:

$$\int dy = \int \frac{dx}{x} \Rightarrow y = \ln|x| + c$$

3. To solve $f(y)\frac{dy}{dx} = g(x)$ we separate variables to both sides of the equation and then integrate both sides to deduce the value of the dependent variable y.

Application Activity 6.2.1

1. The general solution for:

a) $\frac{dy}{dx} = x\cos x$; separate variables to get: $dy = x\cos x dx$

Direct integration gives $y = \int x\cos x dx$;

Integration by part gives : $y = x\sin x - \int \sin x dx = x\sin x + \cos x + c$

which is the required solution.

b) $x\dfrac{dy}{dx}=2-4x^3$.

Rearranging $x\dfrac{dy}{dx}=2-4x^3$ gives : $\dfrac{dy}{dx}=\dfrac{2-4x^3}{x}=\dfrac{2}{x}-4x^2$

Integrating both sides gives: $y=\int\left(\dfrac{2}{x}-4x^2\right)dx=2\ln x-\dfrac{4}{3}x^3+c$

Thus the required solution is $y=2\ln x-\dfrac{4}{3}x^3+c$

2. $(x+1)\dfrac{dy}{dx}=x\left(y^2+1\right)$

Separate the variables to get: $(x+1)\,dy=x\left(y^2+1\right)dx$ *or* $\dfrac{1}{y^2+1}dy=\dfrac{x}{x+1}dx$

Integrating both sides: $\int\dfrac{dy}{y^2+1}=\int\dfrac{xdx}{x+1}=\int\left(1-\dfrac{1}{x+1}\right)dx$

$\Rightarrow \arctan y=x-\ln|x+1|+c$

Putting the value of $y(0)=0$ $y(0)=0$

in the general solution to get: $\text{ın}^{-1}0=0-\ln(0+1)+c\Rightarrow c=$

Then,

$\tan^{-1}(y)=x-\text{h}\,|x+1|$

$\Rightarrow \tan^{-1}(y)=x\,\text{h}\ e-\text{h}\,|x+1|$

$\Rightarrow \tan^{-1}(y)=\text{h}\left(\dfrac{e^x}{|x+1|}\right)$

$\Rightarrow y=\tan\left[\text{h}\left(\dfrac{e^x}{|x+1|}\right)\right]$

3. (a)Let $y=y(x)$ be the required function and $P(x,y)$ any point on the curve.

The line OP has a slope $\dfrac{y}{x}$. The tangent to the curve at $P(x,y)$ that is perpendicular to the line OP has therefore the slope $\frac{-x}{y}$ $\sin ce\ \frac{y}{x}\left(\frac{-x}{y}\right)=-1$.

We know that the slope of the tangent line to the curve $y=y(x)\ at\ P(x,y)$ is defined by $\frac{dy}{dx}$. Therefore $\frac{dy}{dx}=-\frac{x}{y}$

is the required.

The initial value problem is: $\frac{dy}{dx}=-\frac{x}{y};\ y(0)=1$

(b) The differential: $\frac{dy}{dx}=-\frac{x}{y}$ is equivalent to $ydy=-xdx$ that is separable.

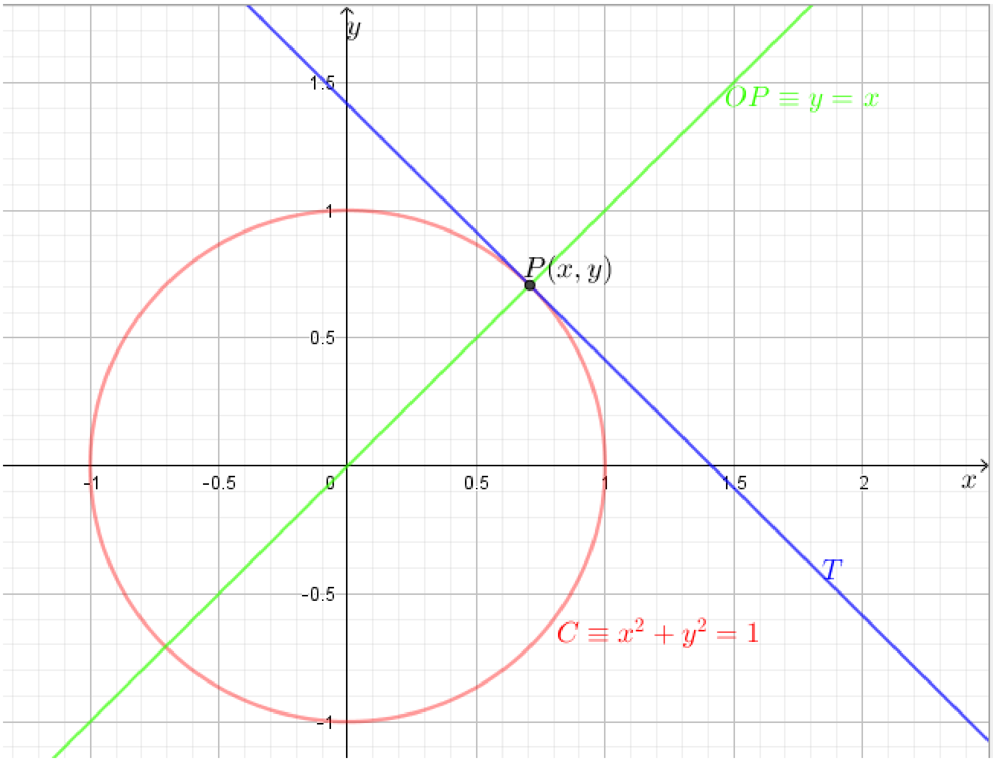
Let's solve the problem $ydy=-xdx \Rightarrow \int ydy=-\int xdx$

Simple integration gives: $\frac{1}{2}y^2+\frac{1}{2}x^2=c$

Replace the point $(0,1)$ to get $c=\frac{1}{2}$. So, $y^2+x^2=1$ is the final solution.

The curve representing this solution is the circle centered at $(0,1)$ and radius 1.

Graph of the differentiable function $y^2 + x^2 = 1$ **:**



This confirms the theorem of geometry that states that a tangent to a circle forms a right angle with the circle's radius, at the point of contact of the tangent.

4. Rearranging the given equation, $\frac{y^2-1}{3y}dy = dt \Leftrightarrow \left(\frac{y}{3} - \frac{1}{3y}\right)dy = dt$

Direct integration yields: $\frac{y^2}{6} - \frac{\ln y}{3} = t + c \text{ or } t = \frac{y^2}{6} - \frac{\ln y}{3} - c$

that is general equation.

Given that $y = 1$

When $t = \frac{13}{6}$, we have: $\frac{13}{6} = \frac{(1)^2}{6} - \frac{\ln 1}{3} - c \Rightarrow c = -2$

Hence particular solution is $t = \frac{y^2}{6} - \frac{\ln y}{3} + 2$

5. $\frac{dR}{d\theta}=\alpha \Rightarrow d\theta=\frac{dR}{\alpha R} \Rightarrow \int d\theta=\frac{1}{\alpha}\int\frac{dR}{R} \Rightarrow \theta=\frac{1}{\alpha}\ln|R|+c$

The general solution is: $\theta=\frac{1}{\alpha}\ln|R|+c$.

$R=R_0$ *when* $\theta=0^0C$, thus $0=\frac{1}{\alpha}\ln R_o+c \Rightarrow c=-\frac{1}{\alpha}\ln R_o$

Hence the particular solution is: $\theta=\frac{1}{\alpha}\ln|R|-\frac{1}{\alpha}\ln|R_0| \Rightarrow \theta=\frac{1}{\alpha}\ln\frac{R}{R_0}$

Finally, $R=R_0e^{\alpha\theta}$

(b) Substituting $\alpha=38\times10^{-4}$, $\theta=50$ and 24Ω into $R=R_0e^{\alpha\theta}$ gives the resistance at

50^0C

$R=24e^{38\times10^{-4}\times50}=29\Omega$

Lesson 3: The solution of equation of the form $\frac{dy}{dx}=f(x)h(y)$

a) Learning objectives

Determine whether an ordinary differential equation of first order is with separable variables, homogeneous or linear.

b) Teaching resources:

student teacher's book and other Reference books to facilitate research, scientific calculator, Manila paper, markers, pens, pencils, if possible computers with mathematical software such as Geogebra, Microsoft Excel, Math-lab or graph-application and internet can be used...

c) Prerequisites/Revision/Introduction

Student-teachers will easily learn this unit, if they have a good background on: differentiation (Year 2: Unit 7), quadratic equations (year 1: unit 5), Trigonometry (year 1: unit 8), Logarithm and exponential (year 3:unit 4), integration(year 3: unit 5), complex numbers(year 3: unit 1) .

d) Learning activities:

This lesson will help student teachers to understand the concept of differential equation and their classifications through derivatives and the highest derivative degree.

- Form groups and ask student teachers to work out the activity 6.2.2
- Let student teachers work independently for some while.
- Facilitate student teachers to derivate and to guess the power of the highest derivative.
- Ask randomly some groups to present their findings to the whole class;
- Guide student teachers to interact about the findings, to conclude on the new concept and to write a short summary.
- Let student teachers work out activity 6.2.2 under your guidance, and work individually application activity 6.2.2 to assess their competences.

Answers to activity 6.2.2

1. $4xy\frac{dy}{dx}=y^2-1\Rightarrow\int\frac{ydy}{y^2-1}=\frac{1}{4}\int\frac{dx}{x}\Rightarrow\ln\sqrt{y^2-1}=\frac{1}{4}\ln c|x|$ or $y^2=c\sqrt{x}+1$

2. $\frac{y^2+1}{x^2+1}=\frac{y}{x}\frac{dy}{dx}\Rightarrow\int\frac{ydy}{y^2+1}=\int\frac{xdx}{x^2+1}$ $\ln\sqrt{y^2+1}=\ln c\sqrt{x^2+1}\Rightarrow y^2=c\left(x^2+1\right)-1$

Application Activity 6.2.2

1. a) $\frac{dy}{dx}=2y\cos x\Rightarrow\ln|y|=2\sin x+c$;

   b) $(x+1)\frac{dy}{dx}=x\left(y^2+1\right)\Rightarrow\arctan y=x-\ln|x+1|+c$; for $x=1,\ y=0$; we get

   $\Rightarrow\arctan 0=1-\ln 2+c\Rightarrow c=\ln 2-1$, so $\Rightarrow\arctan y=x-\ln|x+1|+\ln 2-1$

   c) $(2y-1)\frac{dy}{dx}=3x^2+1\Rightarrow y^2-y=x^3+x+c$; for $x=1,\ y=2$; we get

   $\Rightarrow 4-2=1+1+c\Rightarrow c=0,\Rightarrow y^2-y=x^3+x$

   d) $\frac{d\theta}{dt}=2e^{3t-2\theta}=\frac{1}{2}e^{2\theta}=\frac{2}{3}e^{3t}+c$; for $\mathrm{t}=0,\ \theta=0$; we get: $3e^{2\theta}=4e^{3t}-1$

   e) $\left(xy^2+x\right)dx+\left(yx^2+y\right)dy=0\Rightarrow\frac{1}{2}\ln\left|y^2+1\right|=-\frac{1}{2}\ln\left|x^2+1\right|+c$

2. $x^3\frac{dy}{dx}=p-x \Rightarrow \int dy=\int\left(\frac{p-x}{x^3}\right)dx$

$$y=-\frac{p}{2x^2}+\frac{1}{x}+c;\ for\ y=0, x=2, x=6$$

$$\Rightarrow\begin{cases}-\frac{p}{8}+\frac{1}{2}+c=0\\ -\frac{p}{72}+\frac{1}{6}+c=0\end{cases}\Rightarrow\begin{cases}8c-p=-4\\ 72c-p=-12\end{cases}\Rightarrow\begin{cases}c=-\frac{1}{8}\\ p=3\end{cases}$$

Lesson 4: Simple homogeneous equations

a) Learning objectives

Determine whether an ordinary differential equation of first order is with separable variables, homogeneous or linear.

b) Teaching resources:

student teacher's book and other Reference books to facilitate research, scientific calculator, Manila paper, markers, pens, pencils, if possible computers with mathematical software such as Geogebra, Microsoft Excel, Math-lab or graph-application and internet can be used...

c) Prerequisites/Revision/Introduction

Student-teachers will easily learn this unit, if they have a good background on: differentiation (Year 2: Unit 7), quadratic equations (year 1: unit 5), Trigonometry (year 1: unit 8), Logarithm and exponential (year 3:unit 4), integration(year 3: unit 5), complex numbers(year 3: unit 1) .

d) Learning activities:

This lesson will help student teachers to understand the concept of differential equation and their classifications through derivatives and the highest derivative degree.

- Form groups and ask student teachers to work out the activity 6.3
- Let student teachers work independently for some while.
- Facilitate student teachers to derivate and to guess the power of the highest derivative.
- Ask randomly some groups to present their findings to the whole class;
- Guide student teachers to interact about the findings, to conclude on the

new concept and to write a short summary.

- Let student teachers work out activity 6.3 under your guidance, and work individually application activity 6.3 to assess their competences.

Answers to activity 6.3

$\frac{dy}{dx}=\frac{xy}{x^2-y^2}$; let $z=\frac{y}{x}\Rightarrow y=zx\Rightarrow\frac{dy}{dx}=z+x\frac{dz}{dx}$

$$\Rightarrow\frac{dy}{dx}=\frac{xy}{x^2-y^2}\Rightarrow\frac{dy}{dx}=\frac{\frac{xy}{x^2}}{\frac{x^2}{x^2}-\frac{y^2}{x^2}}=\frac{\frac{y}{x}}{1-\left(\frac{y}{x}\right)^2}=\frac{z}{1-z^2}$$

$$z+x\frac{dz}{dx}=\frac{z}{1-z^2}\Rightarrow x\frac{dz}{dx}=\frac{z^3}{1-z^2}\Rightarrow\frac{dx}{x}=\frac{\left(1-z^2\right)}{z^3}dz$$

$$\Rightarrow\int\frac{dx}{x}=\int\frac{dz}{z^3}-\int\frac{z^2dz}{z^3}$$

$\Rightarrow\ln|x|=-\frac{1}{2z^2}-\ln|z|+\ln c$, where $c>0$

$\Rightarrow\ln|x|=-\frac{x^2}{2y^2}-\ln|y|+\ln|x|+\ln c$, where $c>0$

$$\Rightarrow\frac{x^2}{2y^2}=-\ln|y|+\ln c=\ln\left|\frac{c}{y}\right|$$

$\Rightarrow x^2=2y^2\ln\left|\frac{c}{y}\right|$, where $c>0$

Application Activity 6.3

a) Letting $z=\frac{y}{x}\Leftrightarrow y=zx$, then $\frac{dy}{dx}=z+x\frac{dz}{dx}$. Plugging $y=zx$ and

$\frac{dy}{dx} = z + x\frac{dz}{dx}$ in the equation $\frac{dy}{dx} = \frac{x + xy}{xy + y^2}$, we get

$$x\frac{dz}{dx} = \frac{1}{z} - z \quad \Rightarrow \frac{z}{z^2 - 1}dz = \frac{dx}{x}$$

$\int \left(\frac{z}{z^2 - 1}\right)dz = \int \frac{dx}{x}$ integrate and simplify to get $y^2 - x^2 = c$

b) Letting $z = \frac{y}{x} \quad \Leftrightarrow y = zx$, then $\frac{dy}{dx} = z + x\frac{dz}{dx}$. Plugging

$y = zx$ and $\frac{dy}{dx} = z + x\frac{dz}{dx}$ in the equation $2x\frac{dy}{dx} = y^2 - x^2$, we get

$$x\frac{dz}{dx} = \frac{z^2 - 1}{2z} - z \quad \Rightarrow \frac{2z}{z^2 + 1}dz = -\frac{dx}{x}$$

$\int \left(\frac{2z}{z^2 + 1}\right)dz = -\int \frac{dx}{x}$ integrate to get $\text{n}\left|z^2 + 1\right| = -\ln x + c$. Replacing z by its value and simplify to get $y^2 + x^2 = cx$

Lesson 5: Linear differential equations of the first order

a) Learning objectives

Determine whether an ordinary differential equation of first order is with separable variables, homogeneous or linear.

b) Teaching resources:

student teacher's book and other Reference books to facilitate research, scientific calculator, Manila paper, markers, pens, pencils, if possible computers with mathematical software such as Geogebra, Microsoft Excel, Math-lab or graph-application and internet can be used...

c) Prerequisites/Revision/Introduction

Student-teachers will easily learn this unit, if they have a good background on: differentiation (Year 2: Unit 7), quadratic equations (year 1: unit 5), Trigonometry (year 1: unit 8), Logarithm and exponential (year 3:unit 4), integration(year 3: unit 5), complex numbers(year 3: unit 1) .

d) Learning activities:

This lesson will help student teachers to understand the concept of differential equation and their classifications through derivatives and the highest derivative degree.

- Form groups and ask student teachers to work out the activity 6.4
- Let student teachers work independently for some while.
- Facilitate student teachers to derivate and to guess the power of the highest derivative.
- Ask randomly some groups to present their findings to the whole class;
- Guide student teachers to interact about the findings, to conclude on the new concept and to write a short summary.
- Let student teachers work out activity 6.4 under your guidance, and work individually application activity 6.4 to assess their competences.

Answers to activity 6.4

Given the differential equation $\frac{dy}{dx}+2xy=x$ or $y'+2xy=x$ (1)

1. $\frac{dy}{dx}+2xy=x$ (1)

   $I(x)=e^{\int 2xdx}=e^{x^2}$. For convenience, we set the integration constant to 0.

2. Multiplying both sides in the differential equation (1) by $I(x)=e^{x^2}$ to get:

   $e^{x^2}(y'+2xy)=xe^{x^2}$

   Or $\frac{d}{dx}\left(e^{x^2}y(x)\right)=xe^{x^2}$

3. Integrating both sides and divide by integrating factor $I(x)$ to get:

   $y(x)e^{x^2}=\frac{1}{2}e^{x^2}+c$

4. Solve for $y(x)$ to get $y(x)=\frac{1}{2}+ce^{-x^2}$.

5. Replacing $y(x)=\frac{1}{2}+ce^{-x^2}$ and $y'=-2cxe^{-x^2}$ in (1) we find that $y(x)$ is a solution of (1)

$$y'+2xy=-2cxe^{-x^2}+2x\left(\frac{1}{2}+ce^{-x^2}\right)=-2cxe^{-x^2}+x+2cxe^{-x^2}=x$$

Application Activity 6.4

1. a) $y'+\frac{y}{x}=1$. This is the form of $\frac{dy}{dx}+py=q;\ p=\frac{1}{x};\ q=1;\ y=uv$;

$$u=\int qe^{\int pdx}dx;\ v=e^{-\int pdx}$$

$$u=\int e^{\int\frac{1}{2}dx}dx=\int e^{\ln x}dx=\int xdx=\frac{1}{2}x^2+c\Rightarrow u=\frac{1}{2}x^2+c$$

$$v=e^{-\int\frac{1}{x}dx}=e^{-\ln x}=\frac{1}{e^{\ln x}}=\frac{1}{x}$$

$y(x)=uv=\left(\frac{1}{2}x^2+c\right)\times\frac{1}{x}=\frac{1}{2}x+\frac{c}{x}$, where c is a constant

b) $y'+xy=x$, This is the form of $\frac{dy}{dx}+py=q,\ p=x,q=x$

$y=uv$, $u=\int qe^{\int pdx}dx=\int xe^{\int xdx}dx=\int xe^{\frac{1}{2}x^2}$, let $t=\frac{1}{2}x^2\Rightarrow dt=xdx$, then,

$\int e^t dt=e^t+c, u=e^{\frac{1}{2}x^2}+c\ and\ v=e^{-\int pdx}=e^{-\int xdx}=e^{-\frac{1}{2}x^2}$.

$y(x)=uv=\left(e^{\frac{1}{2}x^2}+c\right)e^{-\frac{1}{2}x^2}=1+ce^{-\frac{1}{2}x}$, where c is a constant

c) $y'+\frac{y}{x}=x$ This is the form of $\frac{dy}{dx}+py=q$,where $p=\frac{1}{x},q=x$

$y=uv$; $u=\int qe^{\int pdx}dx\Rightarrow u=\int xe^{\int\frac{1}{2}dx}dx=\int xe^{\ln x}dx=\int x^2dx=\frac{1}{3}x^3+c$

$$v=e^{-\int pdx}=e^{-\int\frac{1}{x}dx}=e^{-\ln x}=e^{\ln\left(\frac{1}{x}\right)}=\frac{1}{x}$$

Then, $y=uv=\left(\frac{1}{3}x^3+c\right)\frac{1}{x}=\frac{1}{3}x^2+\frac{c}{x}$, where c is a constant

d) $y'+2y=e^{x}\Rightarrow y=\frac{1}{3}e^{x}+ce^{-2x}$, where c is a constant

e) $y'-2xy=e^{x^2}\Rightarrow y=xe^{x^2}+ce^{x^2}$, where c is an arbitrary constant.

f) $y'+\frac{3}{x}y=\frac{\sin x}{x^3}\Rightarrow y=-\frac{\cos x}{x^3}+\frac{c}{x^3}$

2. $CR\frac{dV}{dt}=E-V=E\Rightarrow\frac{dV}{E-V}=\frac{1}{CR}dt\Rightarrow\int\frac{dV}{E-V}=\frac{1}{CR}\int dt$

$-\ln\left(E-V\right)=\frac{t}{CR}+k$; for $t=0;V=0\Rightarrow-\ln\left(E-0\right)=o+k$

$-\ln E=k\Rightarrow k=\ln\left(\frac{1}{E}\right)\Rightarrow-\ln\left(E-V\right)=\frac{t}{CR}+\ln\left(\frac{1}{E}\right)$

$\Rightarrow\ln\left(\frac{1}{E-V}\right)-\ln\frac{1}{E}=\frac{t}{CR}$

$\Rightarrow\ln\left(\frac{E}{E-V}\right)=\frac{t}{CR}\Rightarrow\frac{E-V}{E}=e^{-\frac{t}{CR}}$

$\Rightarrow V=E\left(1-e^{-\frac{t}{CR}}\right)$;When $E=25V,C=20\times10^{-6}F,R=200\times10^{3}\Omega;\ and\ t=3s$
.

we get $V\approx13.2Volts$.

Lesson 6: Differential equations and the population growth

a) Learning objectives

Appreciate the use of differential equations in solving problems occurring from daily life.

b) Teaching resources:

student teacher's book and other Reference books to facilitate research, scientific calculator, Manila paper, markers, pens, pencils, if possible computers with mathematical software such as Geogebra, Microsoft Excel, Math-lab or graph-application and internet can be used...

c) Prerequisites/Revision/Introduction

Student-teachers will easily learn this unit, if they have a good background on: differentiation (Year 2: Unit 7), quadratic equations (year 1: unit 5), Trigonometry (year 1: unit 8), Logarithm and exponential (year 3:unit 4), integration(year 3: unit 5), complex numbers(year 3: unit 1) .

d) Learning activities:

This lesson will help student teachers to understand the concept of differential equation and their classifications through derivatives and the highest derivative degree.

- Form groups and ask student teachers to work out the activity 6.5.1
- Let student teachers work independently for some while.
- Facilitate student teachers to derivate and to guess the power of the highest derivative.
- Ask randomly some groups to present their findings to the whole class;
- Guide student teachers to interact about the findings, to conclude on the new concept and to write a short summary.
- Let student teachers work out activity 6.5.1 under your guidance, and work individually application activity 6.5.1 to assess their competences.

Answers to activity 6.5.1

1. Differential equation expressing this model is $\frac{dP}{dt} = KP$
2. Separating variables and integrating both sides we get:

   $$\int \frac{dP}{P} = K\int dt \Rightarrow \ln P = Kt + c \Rightarrow P = ce^{Kt}$$

   If the initial population at time $t = 0 \; is \; P_0$, and $K = 0.05 \; then \; P_0 = ce^{(0.05\times 0)} = c$

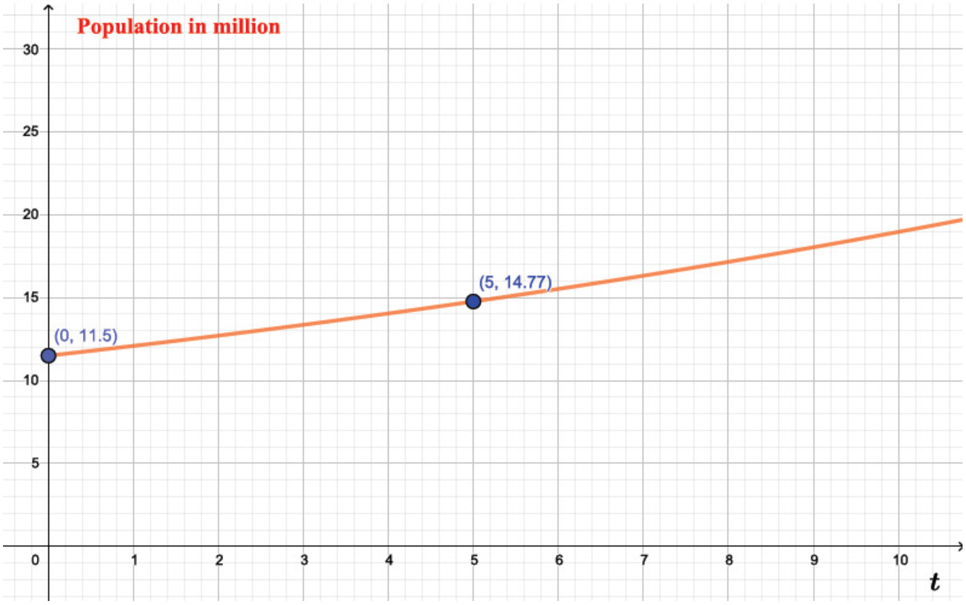
   Therefore, $P_0 = c$ and we have $P = P_0 e^{0.05t}$
3. If the population $P_0 = 11,500,000$; respects the same variation $P = P_0 e^{0.05t}$ then

   $P = 11,500,000e^{0.05t}$.

After 5 years, this population will be $P = 11,500,000e^{0.05\times 5} = 14,766,292.29 \; people$

that is 14,766,292 people.

Graph showing the population growth



The population is exponentially increasing and the policy makers of that town should think about family planning, environment protection, etc.

Application Activity 6.5.1

1. Let P be the number of population $\frac{dP}{dt} \approx P \quad \Rightarrow \frac{dP}{dt} = \lambda P$

   Separating variables to get $\frac{dP}{P} = \lambda dt$, then integrating both sides:

   $\int \frac{dP}{P} = \int \lambda dt \quad \Rightarrow \ln P = \lambda t + c$

   $\Rightarrow P = e^{\lambda t + c} \Leftrightarrow P = e^c e^{\lambda t} \Leftrightarrow P = \alpha e^{\lambda t}$.

   For $t = 0, P_0 = \alpha e^0 \Rightarrow P_0 = \alpha$; Hence $P = P_0 e^{\lambda t}$ because $\alpha = P_0$

   $P = 2P_0$ in $t = 20$ years with $P = P_0 e^{\lambda t}$,

   Thus $2P_0 = P_0 e^{20\lambda} \Rightarrow 2 = e^{20\lambda} \Rightarrow 20\lambda = \ln 2 \Rightarrow \lambda = \frac{\ln 2}{20}$

   If P_0 is the initial population, then $t = ?$ if we have $3P_0 = P \Rightarrow P = 3P_0$ then

   $t = ?$ with $P = P_0 e^{\frac{\ln 2}{20} t} \Rightarrow 3 = e^{\frac{\ln 2}{20} t}$

$$n3=\frac{\ln 2}{20}t \quad \Rightarrow t=\frac{20\ln 3}{\ln 2}=31.69925001$$ years

The population will triple after approximately 32 years; t=31 which is equivalent to 8 months11days 17hours 31min and 12s.

2. Let P be the population of bacteria $\frac{dP}{dt}\approx P \quad \Rightarrow \frac{dP}{dt}=\alpha P$ where α is the proportionality coefficient $\frac{dP}{dt}=\alpha P \Rightarrow \frac{dP}{P}=\alpha dt \Rightarrow \int \frac{dP}{P}=\alpha \int dt$

   $\ln P=\alpha t+c$

   $P=Ae^{\alpha t}$ where $A=e^{c}$

   The number of bacteria is increasing from 1000 to 3000 in 10 hours

   For t=0h; P=1000; for t=10h, P=3000

   . $P=1000;\ P=Ae^{\alpha t} \Rightarrow 1000=Ae^{0} \Rightarrow A=1000$; hence $P=1000e^{\alpha t}$; for P=3000 and t=10h $\Rightarrow 3000=1000e^{10\alpha} \Rightarrow \alpha=\frac{\ln 3}{10}=0.109861228$

   If A=1000 and $\alpha=0.109861228$; then $P=1000e^{0.109861228t}$

b)if t=5h; then $P=1000e^{0.109861228\times 5}=1732.05807$

   $P=1,732.05807$ bacteria.

   Hence, after 5 hours the number of bacteria is about 1732.

Lesson 7: Differential equations and Crime investigation

a) Learning objectives

Appreciate the use of differential equations in solving problems occurring from daily life.

b) Teaching resources:

student teacher's book and other Reference books to facilitate research, scientific calculator, Manila paper, markers, pens, pencils, if possible computers with mathematical software such as Geogebra, Microsoft Excel, Math-lab or graph-application and internet can be used...

c) Prerequisites/Revision/Introduction

Student-teachers will easily learn this unit, if they have a good background on: differentiation (Year 2: Unit 7), quadratic equations (year 1: unit 5), Trigonometry (year 1: unit 8), Logarithm and exponential (year 3:unit 4), integration(year 3: unit 5), complex numbers(year 3: unit 1) .

d) Learning activities:

This lesson will help student teachers to understand the concept of differential equation and their classifications through derivatives and the highest derivative degree.

- Form groups and ask student teachers to work out the activity 6.5.2
- Let student teachers work independently for some while.
- Facilitate student teachers to derivate and to guess the power of the highest derivative.
- Ask randomly some groups to present their findings to the whole class;
- Guide student teachers to interact about the findings, to conclude on the new concept and to write a short summary.
- Let student teachers work out activity 6.5.2 under your guidance, and work individually application activity 6.5.2 to assess their competences.

Answers to activity 6.5.2

1. The differential equation expressing the model is $\frac{dT}{dt} = -K\left(T - T_0\right)$ where T_o=temperature of environment.

2. $\frac{dT}{dt} = -K\left(T - T_0\right) \Rightarrow \frac{dT}{dt} + KT = KT_0$.This is a first order linear differential

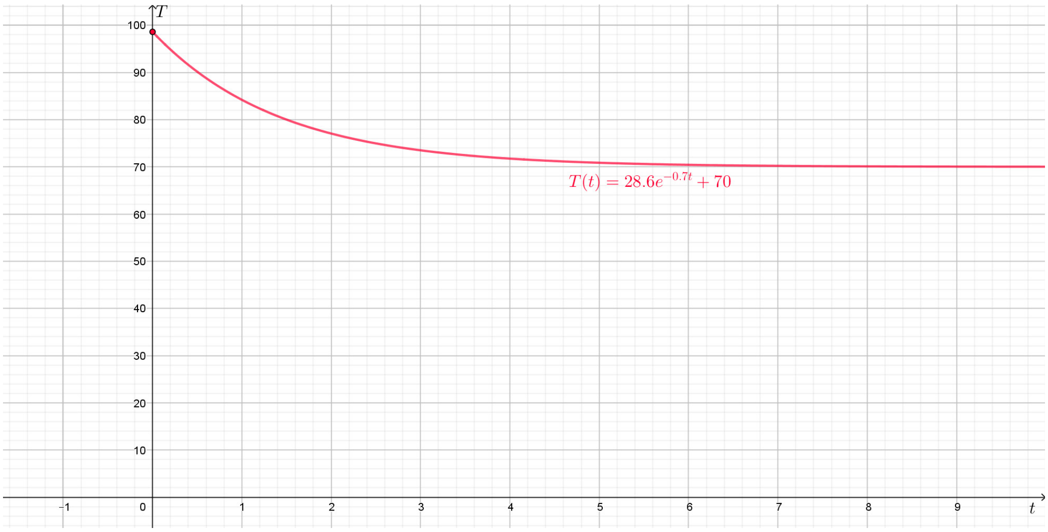
equation, its solution

$T\left(t\right) = T_0 + Be^{Kt}$, where B is a constant. Since $K = \ln\left(\frac{1}{2}\right); T\left(t\right) = T_0 + Be^{\ln\left(\frac{1}{2}\right)t}$

3. At t=oh; T=98.6°F , $T_0 = 70^0 F$ our equation: $T\left(t\right) = T_0 + Be^{Kt}$ becomes
   $98.6 = 70 + Be^0 \Rightarrow B = 28.6$

   At t time, when $K = \ln\left(\frac{1}{2}\right); T\left(t\right) = 70 + 28.6e^{-0.7t}$

   The graph of $T\left(t\right) = 70 + 28.6e^{-0.7t}$



As the time elapses the temperature decreases and converges to 70 °F which is the temperature of the room.

Application Activity 6.5.2

According to Newton's law of cooling, the body will radiate heat energy into the room at a rate proportional to the difference in temperature between the body and the room.

If $T(t)$ is the body temperature at time t, then for some constant of proportionality K,

a) $\frac{dT}{dt} = K\left[T(t) - T_0\right] \Rightarrow \frac{dT}{dt} = K\left[T(t) - 70\right]$

Separating variables and integrating both sides,

$$\int \frac{1}{T-70} dT = K\int dt \Rightarrow \ln|T-70| = Kt + C$$

$$\Rightarrow T - 70 = Be^{Kt} \Rightarrow T(t) = 70 + Be^{Kt}$$

Constants K and B can be determined provided the following information is available:

Time of arrival of the police personnel, the temperature of the body just after his arrival and the temperature of the body after certain interval of time

b) The officer arrived at 10.40 p.m. while the body temperature was 94.4 degrees. This means that if the officer considers 10:40 p.m. as t=0 then $T(0) = 94.4 = 70 + B \Rightarrow B = 24.4$ and so giving $T(t) = 70 + 24.4e^{Kt}$.

Taking $K = 28\times10^{-4}$, the officer has now temperature function

$T(t) = 70 + 24.4e^{28\times10^{-4}t}$

In order to find when the last time t the body was 98.6 (presumably the time of death),

one has to solve for time the equation

$$T(t) = 98.6 = 70 + 24.4e^{28\times10^{-4}t} \Rightarrow 28\times10^{-4}t = \ln\frac{28.6}{24.4} \Rightarrow t = 56.72270912 \text{ min}$$

We find approximately: $t = 57$ min.

The death occurred approximately 57 minutes before the first measurement at 10.40 p.m, that is at 9.43 p.m. approximately.

Lesson 8: Differential equations and quantity of a drug in the body

a) Learning objectives

Appreciate the use of differential equations in solving problems occurring from daily life.

b) Teaching resources:

student teacher's book and other Reference books to facilitate research, scientific calculator, Manila paper, markers, pens, pencils, if possible computers with mathematical software such as Geogebra, Microsoft Excel, Math-lab or graph-application and internet can be used...

c) Prerequisites/Revision/Introduction

Student-teachers will easily learn this unit, if they have a good background on: differentiation (Year 2: Unit 7), quadratic equations (year 1: unit 5), Trigonometry (year 1: unit 8), Logarithm and exponential (year 3:unit 4), integration(year 3: unit 5), complex numbers(year 3: unit 1) .

d) Learning activities:

This lesson will help student teachers to understand the concept of differential equation and their classifications through derivatives and the highest derivative degree.

- Form groups and ask student teachers to work out the activity 6.5.3
- Let student teachers work independently for some while.
- Facilitate student teachers to derivate and to guess the power of the highest derivative.

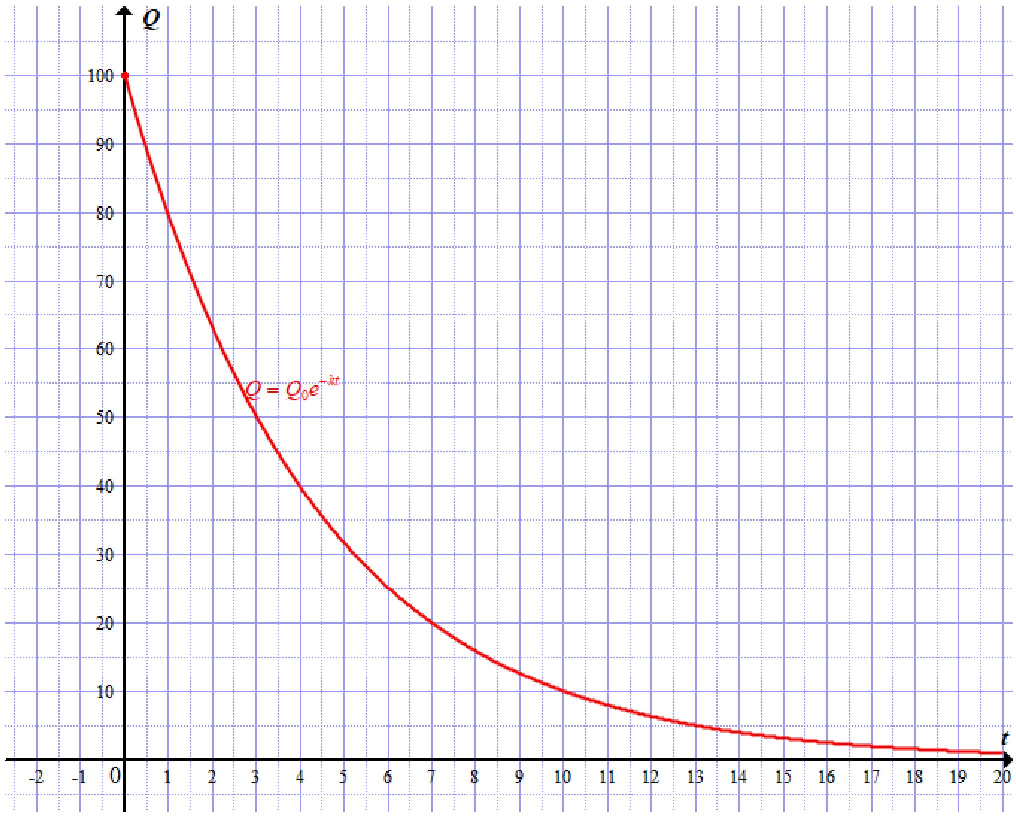
- Ask randomly some groups to present their findings to the whole class;
- Guide student teachers to interact about the findings, to conclude on the new concept and to write a short summary.
- Let student teachers work out activity 6.5.3 under your guidance, and work individually application activity 6.5.3 to assess their competences.

Answers to activity 6.5.3

1. The equation for modeling the situation is $\frac{dQ}{dt} = -kQ$
2. $\frac{dQ}{Q} = -kdt \Rightarrow \int \frac{dQ}{Q} = -k\int dt \Rightarrow \ln|Q| = -kt + c \Rightarrow Q = Ae^{-kt}$

   The solution of this equation is $Q = Q_0 e^{-kt}$, where $A = Q_0$
3. When $t = 0$, the drug provided was $Q_0 = 100mg$ then $Q = 100e^{-kt}$

The graph of the equation $Q = 100e^{-kt}$



The graphs show that the drug in the human body decreases to 0 when the time of taking medicine increases.

4. When a patient does not respect the doctor's prescriptions he/she could suffer from the effects of medicine.

Application Activity 6.5.3

a) Since the half-life is 15 hours, we know that the quantity remaining $Q = \frac{1}{2}Q_0$ when t = 15 h. We substitute into the solution to the differential equation $Q = Q_0 e^{-kt}$

, and solve for k: $Q = Q_0 e^{-kt}$ $\frac{1}{2}Q_0 = Q_0 e^{-15k}$ $\Rightarrow -15k = \ln\left(\frac{1}{2}\right)$ $\Rightarrow k = 0.046$

b) To find the time when 10% of the original dose remains in the body, we substitute $10\% Q_0$

For the quantity remaining, Q ; and solve for the time, t

$$\Rightarrow \frac{10}{100}Q_0 = Q_0 e^{-0.046t}$$

$-0.046t = \ln\left(\frac{10}{100}\right) \Rightarrow t \approx 50$ hours

There will be 10% of the drug still in the body at $t = 49.84$, or after about 50 hours

Lesson 9: Differential equations in Economics and finance

a) Learning objectives

Appreciate the use of differential equations in solving problems occurring from daily life.

b) Teaching resources:

student teacher's book and other Reference books to facilitate research, scientific calculator, Manila paper, markers, pens, pencils, if possible computers with mathematical software such as Geogebra, Microsoft Excel, Math-lab or graph-application and internet can be used...

c) Prerequisites/Revision/Introduction

Student-teachers will easily learn this unit, if they have a good background on:

differentiation (Year 2: Unit 7), quadratic equations (year 1: unit 5), Trigonometry (year 1: unit 8), Logarithm and exponential (year 3:unit 4), integration(year 3: unit 5), complex numbers(year 3: unit 1) .

d) Learning activities:

This lesson will help student teachers to understand the concept of differential equation and their classifications through derivatives and the highest derivative degree.

- Form groups and ask student teachers to work out the activity 6.5.4
- Let student teachers work independently for some while.
- Facilitate student teachers to derivate and to guess the power of the highest derivative.
- Ask randomly some groups to present their findings to the whole class;
- Guide student teachers to interact about the findings, to conclude on the new concept and to write a short summary.
- Let student teachers work out activity 6.5.4 under your guidance, and work individually application activity 6.5.4 to assess their competences.

Answers to activity 6.5.4

a) In general we have $\frac{dP}{dt}=k\left(Q_d-Q_s\right)$; if k=0.08 Then our equation becomes

$$\frac{dP}{dt}=0.08\left(Q_d-Q_s\right)$$

b) $\frac{dP}{dt}=k\left(Q_d-Q_s\right)\Rightarrow\frac{dP}{dt}=0.08\left(280-4P-\left(-35+8P\right)\right)$

$\Rightarrow\frac{dP}{dt}=-0.96P+25.2$ which is a linear first-order differential equation.

$$\int\frac{dP}{25.2-0.96P}=\int dt\Rightarrow-\frac{1}{0.96}\ln\left|25.2-0.96P\right|=t+c$$

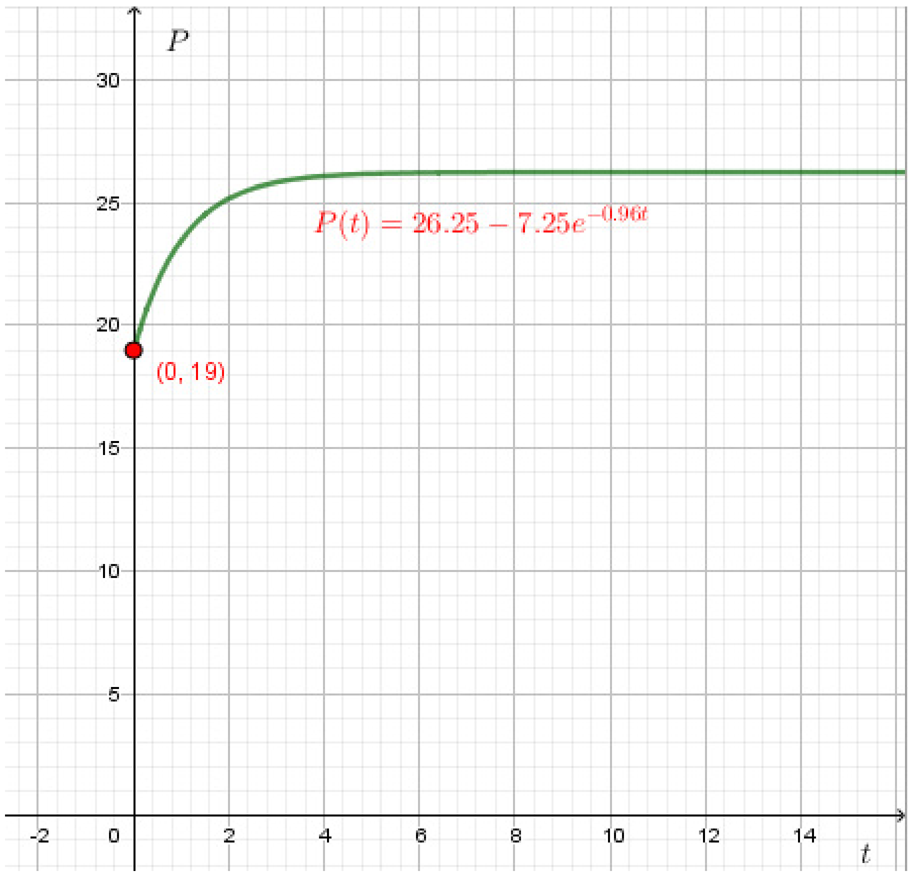
$$\Rightarrow\ln\left|25.2-0.96P\right|=-0.96t+c\Rightarrow 25.2-0.96P=Ae^{-0.96t}$$

$$\Rightarrow P=26.25-\frac{A}{0.96}e^{-0.96t}$$

c) Considering the initial condition $P(0)=19$, we find $A=7.25$

Therefore,

Graph of $P(t) = 26.25 - 7.25e^{-0.96t}$ **.**



d) At t=1, $P(1) = 26.25 - 7.25e^{-0.96\times 1} = 23.25$ and

$$\lim_{t\to\infty} P(t) = \lim_{t\to\infty}\left(26.25 - 7.25e^{-0.96\times\infty}\right) = 26.25$$

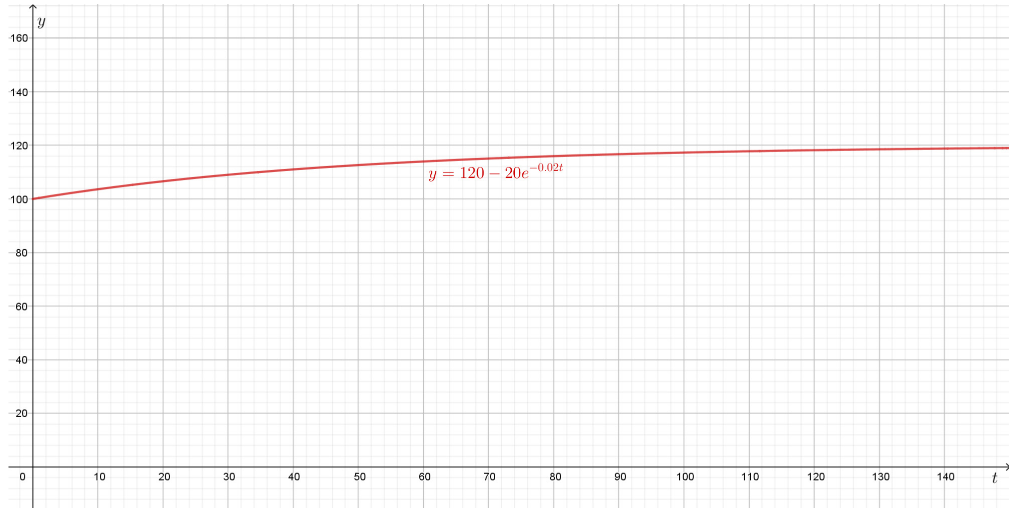
If you compare those two situations, you find that the price is decreasing and tends to 26.25 as t gets larger.

Application Activity 6.5.4

1. $\frac{dP}{dt} = r\left(Q_d - Q_s\right) \Rightarrow \frac{dP}{dt} = 0.04\left(50 - 0.2P - \left(-10 + 0.3P\right)\right) = 2.4 - 0.02P$

$\frac{dP}{dt} + 0.02P = 2.4$ which is a linear first-order differential equation.

$P(t) = 120 - 20e^{-0.02t}$; when P=100 and t=0; the graph



We can say that this market is stable as the coefficient of t in the exponential function is the negative number -0.02

However, the graph shows that the convergence of $P(t)$ on its equilibrium value of 120 is relatively slow.

This time, price gradually approaches its equilibrium value *from one direction only*. A similar time path will occur in other similar market models with continuous price adjustment, although if the initial value is above the equilibrium then price will, obviously, approach this equilibrium from above rather than from below.

Lesson 10: Differential equations in Electricity (series circuits)

a) Learning objectives

Appreciate the use of differential equations in solving problems occurring from daily life..

b) Teaching resources:

student teacher's book and other Reference books to facilitate research, scientific calculator, Manila paper, markers, pens, pencils, if possible computers with mathematical software such as Geogebra, Microsoft Excel, Math-lab or graph-application and internet can be used...

c) Prerequisites/Revision/Introduction

Student-teachers will easily learn this unit, if they have a good background on: differentiation (Year 2: Unit 7), quadratic equations (year 1: unit 5), Trigonometry (year 1: unit 8), Logarithm and exponential (year 3:unit 4), integration(year 3: unit 5), complex numbers(year 3: unit 1) .

d) Learning activities:

This lesson will help student teachers to understand the concept of differential equation and their classifications through derivatives and the highest derivative degree.

- Form groups and ask student teachers to work out the activity 6.5.5
- Let student teachers work independently for some while.
- Facilitate student teachers to derivate and to guess the power of the highest derivative.
- Ask randomly some groups to present their findings to the whole class;
- Guide student teachers to interact about the findings, to conclude on the new concept and to write a short summary.
- Let student teachers work out activity 6.5.5 under your guidance, and work individually application activity 6.5.5 to assess their competences.

Answers to activity 6.5.5

a) The situation is modelled by $L\frac{di}{dt}+Ri=E(t)\Rightarrow\frac{di}{dt}+\frac{R}{L}i=\frac{E(\text{t})}{L}$

b) Since L, R and E are constant, the equation is linear of first-order in i . The integrating factor is

let $i=uv\Rightarrow\frac{di}{dt}=u\frac{dv}{dt}+v\frac{du}{dt}$

$$\Rightarrow u\frac{dv}{dt}+v\frac{du}{dt}+\frac{R}{L}uv=\frac{E}{L}$$

$$\Rightarrow u\left(\frac{dv}{dt}+\frac{R}{L}v\right)=0\Rightarrow\frac{dv}{dt}=-\frac{R}{L}v\Rightarrow\int\frac{dv}{v}=-\frac{R}{L}\int dt$$

$$\Rightarrow\ln|v|=-\frac{R}{L}t\Rightarrow v=e^{-\frac{R}{L}t}$$

$$\Rightarrow v\frac{du}{dt}=\frac{E}{L}\Rightarrow du=\frac{1}{v}\frac{E}{L}dt\Rightarrow\int du=\frac{E}{L}\int e^{\frac{R}{L}t}dt$$

$$u=\frac{E}{L}\times\frac{L}{R}e^{\frac{R}{L}t}+c=\frac{E}{R}e^{\frac{R}{L}t}+c$$

So $i=uv=\left(\frac{E}{R}e^{\frac{R}{L}t}+c\right)e^{-\frac{R}{L}t}=\frac{E}{R}+ce^{-\frac{R}{L}t}$; for $t=0,\ and\ i=0\Rightarrow 0=\frac{E}{R}+c\Rightarrow c=-\frac{E}{R}$

Since the current was zero before switching on, it means

$$t=0,\ and\ i=0 \Rightarrow 0=\frac{E}{R}+c \Rightarrow c=-\frac{E}{R}$$

Therefore $i=\frac{E}{R}\left(1-e^{-\frac{R}{L}t}\right)$

c) $\lim_{t\to\infty} i(t)=\lim_{t\to\infty}(\frac{E}{R}-\frac{E}{R}e^{-\frac{E}{R}\times\infty})=\frac{E}{R}$, this looks like the current of the circuit is

composed of the resistor R and the generator

Application Acctivity 6.5.5

a) The situation is modelled by $L\frac{di}{dt}+Ri=E(t) \Rightarrow \frac{di}{dt}+\frac{R}{L}i=\frac{E(\text{t})}{L}$

b) Since L, R and E are constant, is linear equation of first-order in i with integrating factor

The solution is $i=\frac{E}{R}\left(1-e^{-\frac{R}{L}t}\right)$

Taking $E=60V;R=12\Omega;L=4H \Rightarrow i=\frac{60}{12}\left(1-e^{-\frac{12}{4}t}\right)=5\left(1-e^{-3t}\right)$ at t=1s;

$i=4.75\,Amperes$

c) When t gets larger, we have $\lim_{t\to\infty} i(t)=\lim_{t\to\infty}(\frac{E}{R}-\frac{E}{R}e^{-\frac{E}{R}\times\infty})=\frac{E}{R}$, this looks like the current

of the circuit is composed of the resistor R and the generator.

6.6. Unit summary

6.6.1. Definition and classification of differential equations

An ordinary differential equation (ODE) for a dependent variable *y* (unknown) in terms of an independent variable *x* is any equation which involves first or higher order derivatives of *y* with respect to *x*, and possibly *x* and *y*.

The general differential equation of the 1st order is $F\left(x,y,\frac{dy}{dx}\right)=0 \; or \; \frac{dy}{dx}=f(x,y)$

The order of a differential equation is the highest derivative present in the differential equation.

The degree of an ordinary differential equation is the algebraic degree of its highest ordered derivative after simplification.

6.6.2. First order of Differential equations with separable variables

A separable differential equation is an equation of the form $\frac{dy}{dx}=f(x)h(y)$.

Before integrating both sides, such equation can be rewritten so that all terms involving y are on one side of the equation and all terms involving x are on the other. That is $\frac{dy}{h(y)}=f(x)dx \; and \int\frac{dy}{h(y)}=\int f(x)dx+c$.

6.6.3. Linear differential equations of the first order

If p and q are functions in x or constants the general linear equation of first order can take the form $\frac{dy}{dx}+py=q$

To solve such equation, determine an integrating factor $I(x)=e^{\int pdx}$ taking the integrating constant $c\;=0$ and then find $dy(x)=\frac{\int I(x)q(x)+c}{I(x)}$.

Or let $y=uv$ where u and v are functions in x to be determined in the following ways: $v=e^{-\int pdx}$

by taking the constant $c=0$ and $u=\int qe^{\int pdx}dx$

The solution of the equation $\frac{dy}{dx}+py=q$ becomes $y=uv$ where $u=\int qe^{\int pdx}dx$ and $v=e^{-\int pdx}$.

6.6.4. Application of differential equations of first order

a) Differential equations are applied in the population growth:

If P is the population of a country, its variation is related to $\frac{dP}{dt} = KP$, where K is the timely growth rate.

b) Differential equations are applied in crime investigation:

The time of death of a murdered person can be determined by the police by measuring the variation of the temperature T with the help of the differential

equation: $\frac{dP}{dt} = K(T - T_e)$

where Te is the temperature of the environment surrounding the murdered person and K the constant of proportionality.

c) Differential equations are applied to determine the quantity of a drug in the body

The quantity of drug Q in the body of a patient in the time t is modelled by

$\frac{dQ}{dt} = -kQ$ where k is a constant that depends on the specific type of drug.

d) Differential equations in economics and finance

If r represents the rate of adjustment of *P* in proportion to excess demand,

we can write $\frac{dP}{dt} = r(Q_d - Q_s)$. Where Q_d and Q_s are respectively the

demand and supply functions.

e) Differential equations are applied in the Series Circuits

Given that the voltage drops across the resistor, inductor, and capacitor are

respectively $RI; L\frac{dI}{dt};\text{and } \frac{Q}{C}$, where Q is the charge.

Kirchhoff's second law saying that the voltage $V(t)$ in the circuit is the sum of the voltage drop across the components of the circuit is used to determine all variables of the circuit:

- For a series circuit containing only a resistor R and an inductor L, we have

$$Ri + L\frac{di}{dt} = E(t)$$

- For a series circuit containing only a resistor R and capacitor with capacitance

C, we have : $R\frac{dQ}{dt} + \frac{Q}{C} = E(t)$ where Q is the electrical charge at the capacitor.

$$L\frac{d^2i}{dt^2} + R\frac{di}{dt} + i\frac{1}{C} = \frac{dE(t)}{dt}$$

This is a non-homogeneous second order differential equation because $E(t) \neq 0$
.

6.7. Additional information for the tutor

For the educative action of the tutor to be effective (in order to respond to all aspects of the student teachers' needs), it is worth mentioning that the tutor needs a wide range of skills, attitudes, a rich and deep understanding of the subject matter and the pedagogical processes to develop the understanding that is required from the student teacher. It is therefore, imperative for the tutor to not limit himself/herself to the only to the present book, but also to consider getting information from other relevant books, such as those mentioned in the reference.

Here the tutor has to emphasize the application of differential equations in solving problems in our real life situations.

6.8. End unit assessment

1. a) $\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^4 - 4x + y = 1$; Answer: order:2; degree: 1

   b) $\left(\frac{dy}{dx}\right)^3 - 2x = \cos y - 2\sin x$; answer: order: 1; degree: 3

   c) $(y'')^3 + (y') - 2y = x$; answer: order: 2; degree: 3

   d) $y\frac{d^2y}{dx^2} = -\cos x$; answer: order: 2; degree: 1

   e) $x^2\left(\frac{d^2y}{dx^2}\right)^4 + y\left(\frac{dy}{dx}\right) + y^4 = 0$; answer: order: 2; degree: 4

2. a) $\frac{dy}{dx}=x+1 \Rightarrow dy=(x+1)dx$. Integrating both sides you get $y=\frac{x^2}{2}+x+c$.

Using initial condition then $y=\frac{x^2}{2}+x+1$

b) $\frac{dy}{y^2}=dx$. Integrating both sides you get $y=\frac{-1}{x+c}$. Using initial condition then

$y=\frac{-1}{x-1}$

c) $y'=x^2y \Rightarrow \frac{dy}{dx}=x^2y$

$\frac{dy}{y}=x^2dx \quad \Leftrightarrow y=ce^{\frac{x^3}{3}}$. Using initial condition then $y=e^{\frac{x^3}{3}}$

3. $\frac{d\theta}{dt}=k\theta \Rightarrow \theta=Ae^{kt};\ at\ \theta=40^0C \Rightarrow t=8.67\,\text{min} \approx 9\,\text{min}$

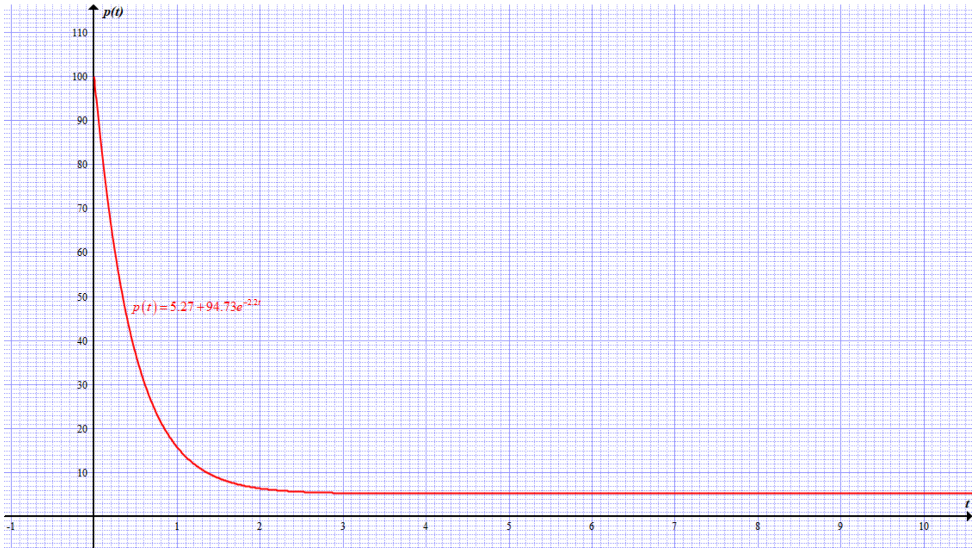
4. a) We have $\frac{dP}{dt}=0.2(Q_d-Q_s)=0.2(35-5P+23-6P)=11.6-2.2P$

Which is a first order linear differential equation $p=2.2;\quad q=11.6$ and the initial condition is $P(0)=100$.

Let us determine an integrating factor: $\frac{dP}{dt}+2.2P=11.6 \Rightarrow P(t)=5.3+ce^{-2.2t}$.

Applying the initial condition is $P(0)=100 \Rightarrow c=94.7 \Rightarrow P(t)=5.3+94.7e^{-2.2t}$.

b) The graph of $P(t)=5.3+94.7e^{-2.2t}$



This market is stable to the price of 5.27 when t becomes larger.

6.9. Additional activities

The tutor's guide suggests additional questions and answers to assess the key unit competence.

6.9.1 Remedial activities:

Solve the following differential equations

a) $\frac{dy}{dx}=\frac{1}{x^2+2x+1}$

b) $\frac{dy}{dx}=\frac{x}{x^2+1}$

c) $\frac{dy}{dx}=2x^3-x;\ y(0)=1$

Solution:

(a) $y=\arctan(x+1)+c$; b) $y=\frac{1}{2}\ln\left|x^2+1\right|+c$; c) $y=\frac{1}{2}x^2\left(x^2-1\right)+1$

6.9.2: Consolidation activities:

Solve the following differential equations:

1. $(x-1)y'=2x^3y$; 2. $y'=(1+x)\left(1+y^2\right)$; 3. $5y'=e^xy^4$

Solutions:

1. $ln|y| = 2\left(\frac{x^3}{3}+\frac{x^2}{2}+x+\ln|x-1|\right)+c$, 2. $y=\tan\left(x+\frac{x^2}{2}+c\right)$; 3. $-\frac{5}{3y^3}=e^x+c$

6.9.3 Extended activities:

Suggestion of Questions and Answers for gifted and talented Student teachers.

1. Solve the following IVP:

$$\cos x\frac{dy}{dx}+y\sin x=-1+2\cos^3 x\sin x,\ y\left(\frac{\pi}{4}\right)=3\sqrt{2},0\le x<\frac{\pi}{2},$$

Solution

$$\cos x\frac{dy}{dx}+y\sin x=-1+2\cos^3 x\sin x,\ y\left(\frac{\pi}{4}\right)=3\sqrt{2},\ \Rightarrow\frac{dy}{dx}+y\frac{\sin x}{\cos x}=-\frac{1}{\cos x}+2\cos^2 x\sin x$$

$\frac{dy}{dx}+y\tan x=2\cos^2 x\sin x-\sec x$; let $y=uv\Rightarrow\frac{dy}{dx}=u\frac{dv}{dx}+v\frac{du}{dx}$

$$u\left(\frac{dv}{dx}+\frac{\sin x}{\cos x}v\right)=0\Rightarrow\int\frac{dv}{v}=-\int\frac{\sin x}{\cos x}dx\Rightarrow v=\cos x$$

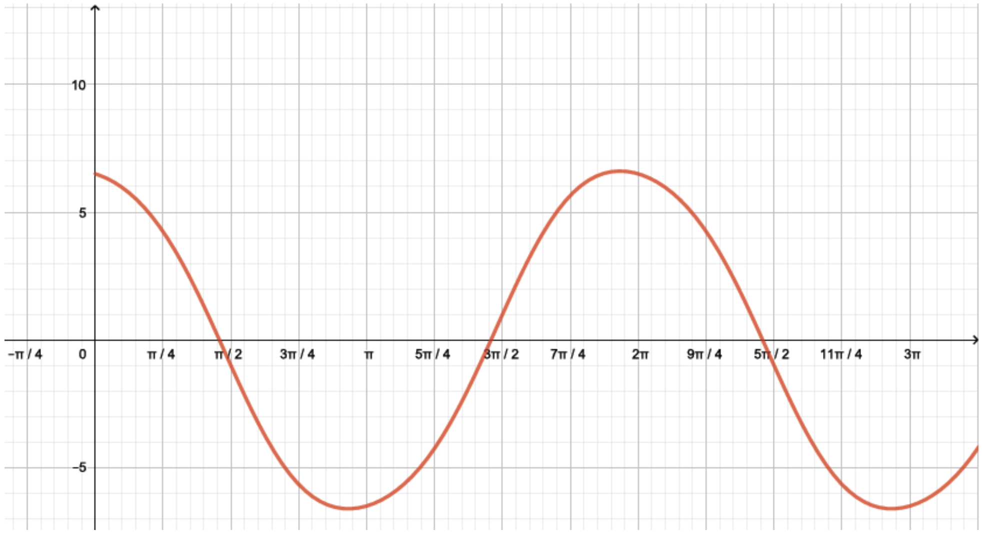
$$v\frac{du}{dx}=-1+2\cos^2 x\sin x\Rightarrow\int du=-\int\frac{dx}{\cos^2 x}+\int\sin 2xdx$$

$$\Rightarrow u=-\tan x-\frac{1}{2}\cos 2x+c$$

$$\Rightarrow y=\left(-\frac{\sin x}{\cos x}-\frac{1}{2}\cos 2x+c\right)\cos x=-\sin x-\frac{1}{2}\cos x\cos 2x+c\cos x$$

$$y\left(\frac{\pi}{4}\right)=3\sqrt{2}\quad\Rightarrow c=7\Rightarrow y=-\sin x-\frac{1}{2}\cos x\cos 2x+7\cos x$$

Graph of $y=-\sin x-\frac{1}{2}\cos x\cos 2x+7\cos x$



2. A local company has 50 employees and one Director General. One day, a rumor (prolonged noise) began to spread among the employees that the DG was promoted to be a CEO of another company while he was in the mission to America. It is reasonable to assume that the rate of the spread of the rumor is proportional to the number of possible encounters between employees $y(t)$ who have heard the rumor and those $50-y(t)$ who have not after t days.

a) Given that $\frac{dy}{dt}=ky(50-y)$ where k is a constant of proportionality and $50-y(t)$ the number of possible meetings = (the number who've heard the rumor) times (the number who haven't);

Solve the equation considering that at the beginning 5 people participated in the same first meeting and had heard the rumor.

b) If 10 people had heard the rumor after one day, deduce $y(t)$ and plot its graph.

c) Use the calculation and the graph to estimate when 40 people will have heard the rumor.

d) What will happen as the time becomes larger?

Solution

$$\frac{dy}{dt}=ky\left(50-y\right)$$

It is a first order separable differential equation: $\frac{dy}{y\left(50-y\right)}=kdt$;

$$\frac{A}{y}+\frac{B}{50-y}=\frac{1}{y\left(50-y\right)}$$

$$\Rightarrow 50A-Ay+By=1\Rightarrow\begin{cases}-A+B=0\\50A=1\end{cases}\Rightarrow\begin{cases}A=B=\frac{1}{50}\\B=\frac{1}{50}\end{cases}$$

$$\Rightarrow\frac{dy}{y(50-y)}=kt\Rightarrow\int\frac{dy}{y\left(50-y\right)}=k\int dt\Rightarrow\frac{1}{50}\int\frac{dy}{y}+\frac{1}{50}\int\frac{dy}{50-y}=k\int dt$$

$$\frac{1}{50}\ln\left|y\right|-\frac{1}{50}\ln\left|50-y\right|=kt+c\Rightarrow\frac{1}{50}\ln\left|\frac{y}{50-y}\right|=kt+c$$

$\Rightarrow\frac{y}{50-y}=Ae^{kt}$; At the beginning 5 people participated in the same meeting

and had heard the rumor, This means that

when t=0, y=5; $\Rightarrow A=\frac{1}{9}\Rightarrow y\left(t\right)=\frac{50e^{50kt}}{9+e^{50kt}}$

b) If 10 people had heard the rumor after one day,

$$y\left(1\right)=10=\frac{50e^{50k\times1}}{9+e^{50k\times1}}\Rightarrow k=0.0162\Rightarrow y=\frac{50e^{0.81t}}{9+e^{0.81t}}$$

Figure: graph of $y(t)=\dfrac{50e^{0.81t}}{9+e^{0.81t}}$

c)When 40 people will have heard the rumor:

$$y(t)=40=\frac{50e^{0.81t}}{9+e^{0.81t}}\Rightarrow 360+40e^{0.81t}=50e^{0.81t}\Rightarrow e^{0.81t}=36$$

$\Rightarrow \ln e^{0.81t}=\ln 36\Rightarrow t=\dfrac{\ln 36}{0.81}=4.42$ days=4days10hrs04min48sec. This means

that after 5 days, 40 employees had heard the rumor.

d)As time becomes larger, we have $\lim\limits_{t\to\infty}y(t)=\lim\limits_{t\to\infty}(\dfrac{50e^{\infty}}{9+e^{\infty}})=\dfrac{\infty}{\infty}=IF$

$\lim\limits_{t\to\infty}y(t)=\lim\limits_{t\to\infty}(\dfrac{e^{0.81t}}{e^{0.81t}})\left(\dfrac{50}{9e^{-0.81t}+1}\right)=50$, this means that all 50 employees will

hear the rumor. $f(y)\geq g(y)$

3. Evaluate the following differential equations:

a) $\dfrac{dy}{dx}=\dfrac{xy+y}{xy+x}$

b) $xyy'=y+2, y(2)=0$

c) $x^2(y-1)dx+y^2(x-1)dy=0$

d) $x(e^{2y}-1)dy+(x^2-1)e^y dx=0$

Answer:

a) $\dfrac{xy+y}{xy+x}=\dfrac{y(x+1)}{x(y+1)}\Rightarrow\dfrac{dy}{dx}=\left(\dfrac{x+1}{x}\right)\left(\dfrac{y}{y+1}\right)$

$\Leftrightarrow\left(\dfrac{y+1}{y}\right)dy=\left(\dfrac{x+1}{x}\right)dx\Rightarrow\left(1+\dfrac{1}{y}\right)dy=\left(1+\dfrac{1}{x}\right)dx$

$\Rightarrow\int dy+\int\dfrac{dy}{y}=\int dx+\int\dfrac{dx}{x}$

$\Rightarrow y+\ln|y|=x+\ln|x|+c$

$\ln y-\ln x=x-y+c\Rightarrow\ln\left|\dfrac{y}{x}=x-y+c\right|$

b) $xy\dfrac{dy}{dx}=y+2\Rightarrow\left(\dfrac{y}{y+2}\right)dy=\dfrac{dx}{x}\Rightarrow\left(1-\dfrac{2}{y+2}\right)dy=\dfrac{dx}{x}$

$\Rightarrow\int dy-2\int\dfrac{dy}{2+y}=\int\dfrac{dx}{x}\Rightarrow y-2\ln|2+y|=\ln|x|+c$

When $x=2,\ y=0\Rightarrow 0-2\ln 2=\ln 2+c\Rightarrow c=-3\ln 2$

Finally $y-2\ln|2+y|=\ln|x|-3\ln 2$

$y-\ln(y+2)^2=\ln\left(\dfrac{x}{8}\right)$

c) $\dfrac{1}{2}(x^2+y^2)+(x+y)+\ln|(x-1)(y-1)|=c$

d) $e^y+e^{-y}+\dfrac{1}{2}x^2-\ln|x|=c$

UNIT 7

DIFFERENTIAL EQUATION OF SECOND ORDER

7.1 Key unit competence:

Use differential equations to solve related problems that arise in a variety of practical contexts.

7.2 Prerequisites

student teachers will easily learn this unit, if they have a good background on: differentiation (Year 2: Unit 7), quadratic equations (year 1: unit 5), Trigonometry (year 1: unit 8), Logarithm and exponential (year 3:unit 4), integration(year 3: unit 5), complex numbers(year 3: unit 1) .

7.3 Cross-cutting issues to be addressed:

- **Inclusive education**: promote the participation of all student teachers while teaching.
- **Peace and value Education**: During group activities, the teacher will encourage student teachers to help each other and to respect opinions of colleagues.
- **Gender**: During group activities try to form heterogeneous groups (with boys and girls) or when student teachers start to present their findings encourage both (boys and girls) to present.
- **Environment and Sustainability:** During the lesson on population growth, guide student teachers to discuss the effects of the high rate of population growth on the environment and sustainability.
- **Standardization culture**: During the lesson on application of differential equations in chemistry (**the quantity of a drug in the body)** guide student teachers to discuss advantages for respecting Doctor's instructions when taking drugs.

7.4 Guidance on introductory activity

- Invite student teachers to form groups and let them to work independently for some while on introductory activity 7.0 to understand the concept of differential equation.

- Walk around to provide various pieces of advice where necessary.
- After a given time, invite student teachers to present their findings and through their works help them to have an idea on the differential equation.
- Harmonize their works and emphasize that they had a differential equation representing a situation of the population for a country and that it can be solved to obtain the formula for estimating the population of that country at any time t.
- Invite student teachers to discuss positive measures that should be taken to address the problem of exponential growth of the population.
- Ask student teachers to discuss the importance of studying how to solve differential equations.
- Basing on student teachers' experience, prior knowledge and abilities shown in answering the questions for this activity, use different questions to facilitate them to give their predictions and ensure that you arouse their curiosity on what is going to be leant in this unit after you harmonize their works and ensure that they got exact solution.

Answer for introductory activity 7.0:

Every student teacher will write his/her examples of second order differential equations with degree one or greater than one.

Eg: 1) order 2, and degree 1: $x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+y=0$; 2) order 2, and degree 3:

$$\left(\frac{d^2y}{dx^2}\right)^3+\left(\frac{dy}{dx}\right)^4=e^x$$

7.5. List of lessons/sub-heading

UNIT 7: DIFFERENTIAL EQUATION OF SECOND ORDER			
No	**Lesson title**	**Learning objectives**	**Number of periods**
0	Introductory activity	Arouse the curiosity of stutent teachers on the content of this unit.	1
1.	Second order differential equations	Define and identify an ordinary linear differential equation of second order.	1

2.	Linear independence and superposition principle	Establish the linear independence of two solutions of an ordinary linear differential equation of second order.	1
3.	Characteristic equation of an homogeneous second order differential equations	Establish the characteristic equation of an ordinary linear differential equation of second order.	2
4.	Homogeneous differential equation of second order whose characteristic equation has two distinct real roots.	Solve an homogeneous differential equation of second order whose Characteristic equation has two distinct real roots.	2
5.	Homogeneous differential equation of second order whose Characteristic equation has a real double root/ repeated root	Solve an homogeneous differential equation of second order whose Characteristic equation has a real double root/repeated root.	2
6.	Homogeneous differential equation of second order whose characteristic equation has complex roots	Solve an homogeneous differential equation of second order whose characteristic equation has complex roots.	2
7.	Non-homogeneous differential equation of second order whose the right hand side is a polynomial function.	Solve an non-homogeneous differential equation of second order whose right hand side is a polynomial function	2
8.	Non-homogeneous differential equation of second order whose the right hand side is an exponential function.	Solve an non-homogeneous differential equation of second order whose the right hand side is an exponential function.	2

9.	Applications of second order linear homogeneous differential equation	Use differential equations to model and solve problems in physics (simple harmonic motion, electricity), economics (point elasticity).	2
10.	End assessment		1
Total periods			18

Lesson 1: Second order differential equations

a) Learning objectives

Solve an ordinary linear differential equation of second order.

b) Teaching resources:

student teacher's book and other Reference books to facilitate research, scientific calculator, Manila paper, markers, pens, pencils, if possible computers with mathematical software such as Geogebra, Microsoft Excel, Math-lab or graph-application and internet can be used...

c) Prerequisites/Revision/Introduction

student teachers will easily learn this unit, if they have a good background on: differentiation (Year 2: Unit 7), quadratic equations (year 1: unit 5), Trigonometry (year 1: unit 8), Logarithm and exponential (year 3:unit 4), integration(year 3: unit 5), complex numbers(year 3: unit 1) .

d) Learning activities:

This lesson will help student teachers to understand the concept of differential equation and their classifications through derivatives and the highest derivative degree.

- Form groups and ask student teachers to work out the activity 7.1.
- Let student teachers work independently for some while.
- Facilitate student teachers to derivate and to guess the power of the highest derivative.
- Ask randomly some groups to present their findings to the whole class;
- Guide student teachers to interact about the findings, to conclude on the new concept and to write a short summary.
- Let student teachers work out activity 7.1 under your guidance, and work individually application activity 7.1 to assess their competences.

Answers to activity 7.1

a) $2\frac{d^2y}{dt^2}+0.1y=0$?

$y=A\cos\sqrt{\frac{1}{20}}t+B\sin\sqrt{\frac{1}{20}}t$,

$y'=-A\sqrt{\frac{1}{20}}\sin\sqrt{\frac{1}{20}}t+B\sqrt{\frac{1}{20}}\cos\sqrt{\frac{1}{20}}t$ and

$y''=-\frac{1}{20}\left(A\cos\sqrt{\frac{1}{20}}t+B\sin\sqrt{\frac{1}{20}}t\right)$

$$2\frac{d^2y}{dt^2}+0.1y=-\frac{2}{20}\left(A\cos\sqrt{\frac{1}{20}}t+B\sin\sqrt{\frac{1}{20}}t\right)+0.1\left(A\cos\sqrt{\frac{1}{20}}t+B\sin\sqrt{\frac{1}{20}}t\right)$$

$$=-0.1\left(A\cos\sqrt{\frac{1}{20}}t+B\sin\sqrt{\frac{1}{20}}t\right)+0.1\left(A\cos\sqrt{\frac{1}{20}}t+B\sin\sqrt{\frac{1}{20}}t\right)=0$$

Therefore, $y=A\cos\sqrt{\frac{1}{20}}t+B\sin\sqrt{\frac{1}{20}}t$ is a general solution of $2\frac{d^2y}{dt^2}+0.1y=0$

b) We compare these equations basing on the coefficient of the derivative for the dependent variable y.

Equation	**Order of the highest derivative**	**Coefficient of the second derivative**	**Coefficient of the first derivative**	**Coefficient of y**	**Function in the second side**
$\frac{d^2y}{dx^2}+p(x)\frac{dy}{dx}+q(x)y=r(x)$	2	1	$p(x)$	$q(x)$	$r(x)$
$\frac{d^2y}{dt^2}+0.1y=0$	2	1	0	0.1	0

Application Activity 7.1

a) $\frac{d^2y}{dx^2}-9y=0$; b) $x\frac{d^2y}{dx^2}+\cos x=0$; c) $x\frac{d^2y}{dx^2}+\frac{dy}{dx}+xy=0$

The given equations are all linear homogeneous differential equations.

Lesson 2: Linear independence and superposition principle

a) Learning objectives

Solve an ordinary linear differential equation of second order.

b) Teaching resources:

student teacher's book and other Reference books to facilitate research, scientific calculator, Manila paper, markers, pens, pencils, if possible computers with mathematical software such as Geogebra, Microsoft Excel, Math-lab or graph-application and internet can be used...

c) Prerequisites/Revision/Introduction

student teachers will easily learn this unit, if they have a good background on: differentiation (Year 2: Unit 7), quadratic equations (year 1: unit 5), Trigonometry (year 1: unit 8), Logarithm and exponential (year 3:unit 4), integration(year 3: unit 5), complex numbers(year 3: unit 1) and how to calculate determinant of matrix of order 2 (year 2: unit 3).

d) Learning activities:

This lesson will help student teachers to understand the concept of differential equation and their classifications through derivatives and the highest derivative degree.

- Form groups and ask student teachers to work out the activity 7.2.1
- Let student teachers work independently for some while.
- Facilitate student teachers to derivate and to guess the power of the highest derivative.
- Ask randomly some groups to present their findings to the whole class;
- Guide student teachers to interact about the findings, to conclude on the new concept and to write a short summary.
- Let student teachers work out activity 7.2.1 under your guidance, and work individually application activity 7.2.1to assess their competences.

Answers to activity 7.2.1

1. For the function $y = e^{-x}$, facilitate student teachers in calculating its successive derivatives up to order 2 and then substitute them in $\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y$ to check if $\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = 0$ and then give conclusion.

2. Let student teachers multiply e^{-x} by any constant let a and e^{-2x} be another constant let b and taking the sum of results. The corresponding results is $y = ae^{-x} + be^{-2x},\ a,b \in \mathbb{R}$.

3. Guide student teachers in verifying if $y = ae^{-x} + be^{-2x},\ a,b \in \mathbb{R}$ is a solution of

$$\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = 0.$$

- Invite the group representatives to present their works and request their classmates to follow carefully.
- Harmonize their work emphasizing the key terms that e^{-x} and e^{-2x} are linearly independent and the superposition principle leads to $y = ae^{-x} + be^{-2x},\ a,b \in \mathbb{R}$. Ask student teachers to go through the example 7.2.1 and work out application activities 7.2.1.

Application Activity 7.2.1

1. Let $y_1 = 1 + \cos x \Rightarrow y' = -\sin x\ and\ y'' = -\cos x$ and

$y_2 = 1 + \sin x \Rightarrow y' = \cos x\ and\ y'' = -\sin x$.

For $y_1 = 1 + \cos x$, $y'' + y = (1 + \cos x)'' + (1 + \cos x) = -\cos x + 1 + \cos x = 1$ as required.

For $y_2 = 1 + \sin x$, $y'' + y = (1 + \sin x)'' + (1 + \sin x) = -\sin x + 1 + \sin x = 1$ as required.

Therefore $y_1 = 1 + \cos x$ and $y_2 = 1 + \sin x$ are solutions of $y'' + y = 1$.

The sum of $y_1 = 1 + \cos x$ and $y_2 = 1 + \sin x$ is $y = 2 + \sin x + \cos x$, and then

$$y'' + y = (2 + \sin x + \cos x)'' + (2 + \sin x + \cos x)$$

$$= (\cos x - \sin x)' + (2 + \sin x + \cos x)$$

$$= -\sin x - \cos x + 2 + \sin x + \cos x = 2 \neq 1.$$

Therefore, $y = 1 + \cos x$ and $y = 1 + \sin x$ are solution of $y'' + y = 1$ but their sum is not a solution. In fact, equation $y'' + y = 1$ is not homogeneous linear differential equation; (the superposition principle holds on homogeneous linear differential equations)

2.

a) $\cos^2 x$ and $\sin^2 x$ are linear independent since $\dfrac{\sin^2 x}{\cos^2 x} = \tan^2 x \neq c$.

Or two functions are linear independent if their Wronskian is different from zero.

For our case $W(x) = \begin{vmatrix} \sin^2 x & \cos^2 x \\ 2\sin x\cos x & -2\cos x\sin x \end{vmatrix}$

$$= -2\cos x\sin x\sin^2 x - 2\cos x\sin x\cos^2 x$$

$$= -2\cos x\sin x\left(\sin^2 x + \cos^2 x\right) = -2\cos x\sin x \neq 0$$

b) Functions e^{-x} and e^{2x} are linear independent since $\dfrac{e^{-x}}{e^{2x}} = e^{-3x} \neq c$.

Or using Wronskian, $W(x) = \begin{vmatrix} e^{-x} & e^{2x} \\ -e^{-x} & 2e^{2x} \end{vmatrix} = 2e^{-x}\,\mathrm{e}^{2x} + e^{-x}\,\mathrm{e}^{2x} = 3\mathrm{e}^{x} \neq 0$

c) e^{ax} and $5e^{ax}$ are not linear independent since $\dfrac{e^{ax}}{5e^{ax}} = 5 = \text{constant}$.

Or using Wronskian, $W(x) = \begin{vmatrix} e^{\alpha x} & 5e^{\alpha x} \\ \alpha e^{\alpha x} & 5\alpha e^{\alpha x} \end{vmatrix} = 5\alpha e^{\alpha x} - 5\alpha e^{\alpha x} = 0$

d) $5\sin x\cos x$ and $4\sin 2x$ are not linear independent since

$$\frac{5\sin x\cos x}{4\sin x\cos x} = \frac{\frac{5}{2}\sin 2x}{2\sin 2x} = \frac{5}{4} = \text{constant}$$

Alternatively using Wronskian, $W(x) = \begin{vmatrix} 5\sin x\cos x & 4\sin 2x \\ (5\sin x\cos x)' & (4\sin 2x)' \end{vmatrix} = \begin{vmatrix} \frac{5}{2}\sin 2x & 4\sin 2x \\ \left(\frac{5}{2}\sin 2x\right)' & 8\cos 2x \end{vmatrix}$

$$= \begin{vmatrix} \frac{5}{2}\sin 2x & 4\sin 2x \\ 5\cos 2x & 8\cos 2x \end{vmatrix} = 20\sin 2x\cos 2x - 20\cos 2x\sin 2x = 0$$

e) $e^{ax}\cos 2x$ and $e^{ax}\sin 2x$ are linear independent since $\dfrac{e^{ax}\cos 2x}{e^{ax}\sin 2x} = \cot 2x \neq c$

Alternatively, using Wronskian, we have

$$W(x) = \begin{vmatrix} e^{\alpha x}\cos 2x & e^{\alpha x}\sin 2x \\ \left(e^{\alpha x}\cos 2x\right)' & \left(e^{\alpha x}\cos 2x\right)' \end{vmatrix}$$

$$W(x) = \begin{vmatrix} e^{ax}\cos 2x & e^{ax}\sin 2x \\ ae^{ax}\cos 2x - 2e^{ax}\sin 2x & ae^{ax}\sin 2x + 2e^{ax}\cos 2x \end{vmatrix}$$

$$\Rightarrow ae^{2ax}\sin 2x\cos 2x + 2e^{2ax}\cos^2 2x - ae^{2ax}\cos 2x\sin 2x + 2e^{2ax}\sin^2 2x$$

$$\Rightarrow 2e^{2ax} \neq 0$$

f) $\ln x$ and $\ln\sqrt{x}$ are not linear independent since $\dfrac{\ln x}{\ln\sqrt{x}} = \dfrac{\ln x}{\frac{1}{2}\ln x} = 2 = \text{constant}$.

Alternatively, from Wronskian, we have

$$W(x) = \begin{vmatrix} \ln x & \frac{1}{2}\ln x \\ \frac{1}{x} & \frac{1}{2x} \end{vmatrix} \Rightarrow \frac{\ln x}{2x} - \frac{\ln x}{2x} = 0$$

g) e^{ax} and xe^{ax} are linear independent since $\dfrac{e^{ax}}{xe^{ax}} = \dfrac{1}{x} \neq \text{constant}$.

Using the Wronskian, $W(x) = \begin{vmatrix} e^{\alpha x} & x e^{\alpha x} \\ \left(e^{\alpha x}\right)' & \left(x e^{\alpha x}\right)' \end{vmatrix} = \begin{vmatrix} e^{\alpha x} & x e^{\alpha x} \\ \alpha e^{\alpha x} & (\alpha x + 1)e^{\alpha x} \end{vmatrix}$

$$= (\alpha x + 1)e^{2\alpha x} - \alpha x e^{2\alpha x} = e^{2\alpha x} \neq 0$$

h) $\dfrac{2\sin^2 x}{1-\cos^2 x} = \dfrac{2\sin^2 x}{\sin^2 x} = 2 = c^{te}$, therefore $2\sin^2 x$ and $1-\cos^2 x$ are not linearly independent.

From Wronskian, $W(x) = \begin{vmatrix} 2\sin^2 x & 1-\cos^2 x \\ \left(2\sin^2 x\right)' & \left(1-\cos^2 x\right)' \end{vmatrix} = \begin{vmatrix} 2\sin^2 x & \sin^2 x \\ 4\sin x\cos x & 2\sin x\cos x \end{vmatrix}$

$$= 4\sin x\cos x\sin^2 x - 4\sin x\cos x\sin^2 x = 0$$

Lesson 3: Characteristic equation of a second order differential equations

a) Learning objectives

Solve an ordinary linear differential equation of second order.

b) Teaching resources:

student teacher's book and other Reference books to facilitate research, scientific calculator, Manila paper, markers, pens, pencils, if possible computers with mathematical software such as Geogebra, Microsoft Excel, Math-lab or graph-application and internet can be used...

c) Prerequisites/Revision/Introduction

student teachers will easily learn this unit, if they have a good background on: differentiation (Year 2: Unit 7), quadratic equations (year 1: unit 5), Trigonometry (year 1: unit 8), Logarithm and exponential (year 3:unit 4), integration(year 3: unit 5), complex numbers(year 3: unit 1).

d) Learning activities:

This lesson will help student teachers to understand the concept of differential equation and their classifications through derivatives and the highest derivative degree.

- Form groups and ask student teachers to work out the activity 7.2.2
- Let student teachers work independently for some while.
- Facilitate student teachers to derivate and to guess the power of the highest derivative.
- Ask randomly some groups to present their findings to the whole class;
- Guide student teachers to interact about the findings, to conclude on the new concept and to write a short summary.
- Let student teachers work out activity 7.2.2 under your guidance, and work individually application activity 7.2.2 to assess their competences.

Answers to activity 7.2.2

1. $y' - ky = 0 \Leftrightarrow \frac{dy}{dx} = ky \Leftrightarrow \frac{dy}{y} = kdx$

2. $\Leftrightarrow \int \frac{dy}{y} = \int kdx \Leftrightarrow \ln|y| = kx \Leftrightarrow y = e^{kx}$.

3.

$$y''-3y'-4y=0$$

Given $y=e^{kx}$, it follows $y'=ke^{kx}$ and $y''=k^2e^{kx}$

Plug these values in $y''-3y'-4y=0$ to have $k^2e^{kx}-3ke^{kx}-4e^{kx}=0$.

This is equivalent to $\left(k^2-3k-4\right)e^{kx}=0.$

This relation is true if and only if $k^2-3k-4=0$ since $e^{kx}\neq 0$.

Thus, the solution of $y'-ky=0$ is also a solution of $y''-3y'-4y=0$ if k is a root of $k^2-3k-4=0$.

Therefore the solution of the form e^{kx} is a solution of $y''-3y'-4y=0$.

Application Activity 7.2.2

a) For $y=\cos 2x$,

$y''+4y=\left(\cos 2x\right)''+4\cos 2x=\left(-2\sin 2x\right)'+4\cos 2x$

$=-4\cos 2x+4\cos 2x=0,$

hence $y=\cos 2x$ is a solution of $y''+4y=0.$

Similarly, for $y=2\sin x\cos x$,

$y''+4y=\left(2\sin x\cos x\right)''+8\sin x\cos x=\left(\sin 2x\right)''+4\sin 2x$

$=\left(2\cos 2x\right)'+4\sin 2x=\left(-4\sin 2x\right)+4\sin 2x=0,$

Thus, $y=\sin 2x$ is a solution of $y''+4y=0.$

Since $\cos 2x$ and $2\sin x\cos x$ are linearly independent, they form a basis and the general solution is $y=c_1\cos 2x+2c_2\sin x\cos x$.

Characteristic equation of $y''+4y=0$ is $\lambda^2+4=0$

Roots $\lambda^2+4=0$: $\lambda=2i$ or $-2i$.

b) For $y=e^x$,

$$y''-2y'+y=\left(e^x\right)''-2\left(e^x\right)'+e^x=\left(e^x\right)'-2e^x+e^x=e^x-2e^x+e^x=0$$

thus $y = e^x$ is a solution of $y'' - 2y' + y = 0$.

For $y = 3e^x$,

$$y'' - 2y' + y = \left(3e^x\right)'' - 2\left(3e^x\right)' + 3e^x$$

$$= 3\left(e^x\right)' - 2\left(3e^x\right) + 3e^x = 3e^x - 6e^x + 3e^x = 0;$$

hence $y = 3e^x$ is a solution of $y'' - 2y' + y = 0$.

As e^x and $3e^x$ are not linearly independent, they do not form a basis.

Characteristic equation of $y'' - 2y' + y = 0$ is $\lambda^2 - 2\lambda + 1 = 0$

Roots $\lambda^2 - 2\lambda + 1 = 0$: $\Delta = 4 - 4 = 0$ thus $\lambda = 2$.

c) If $y = e^{-\frac{x}{2}}$, then

$$4y'' + 4y' + y = 4\left(e^{-\frac{x}{2}}\right)'' + 4\left(e^{-\frac{x}{2}}\right)' + e^{-\frac{x}{2}}$$

$$= -2\left(e^{-\frac{x}{2}}\right)' - 2e^{-\frac{x}{2}} + e^{-\frac{x}{2}} = e^{-\frac{x}{2}} - 2e^{-\frac{x}{2}} + e^{-\frac{x}{2}} = 0,$$

Therefore, $e^{-\frac{x}{2}}$ is a solution of $4y'' + 4y' + y = 0$.

Similarly, if $y = xe^{-\frac{x}{2}}$, then

$$4y'' + 4y' + y = 4\left(xe^{-\frac{x}{2}}\right)'' + 4\left(xe^{-\frac{x}{2}}\right)' + xe^{-\frac{x}{2}}$$

$$= 4\left(e^{-\frac{x}{2}} - \frac{1}{2}xe^{-\frac{x}{2}}\right)' + 4\left(e^{-\frac{x}{2}} - \frac{1}{2}xe^{-\frac{x}{2}}\right) + xe^{-\frac{x}{2}}$$

$$= 4\left(-\frac{1}{2}e^{-\frac{x}{2}} - \frac{1}{2}e^{-\frac{x}{2}} + \frac{1}{4}xe^{-\frac{x}{2}}\right) + 4\left(e^{-\frac{x}{2}} - \frac{1}{2}xe^{-\frac{x}{2}}\right) + xe^{-\frac{x}{2}}$$

$$= -2e^{-\frac{x}{2}} - 2e^{-\frac{x}{2}} + xe^{-\frac{x}{2}} + 4e^{-\frac{x}{2}} - 2xe^{-\frac{x}{2}} + xe^{-\frac{x}{2}} = 0$$

Therefore, $xe^{-\frac{x}{2}}$ is a solution of $4y''+4y'+y=0.$

Since $e^{-\frac{x}{2}}$ and $xe^{-\frac{x}{2}}$ are both solutions of $4y''+4y'+y=0,$ and linearly independent, they form a basis of solution of $4y''+4y'+y=0,$ and corresponding general solution is $y=c_1e^{-\frac{x}{2}}+c_2xe^{-\frac{x}{2}}.$

Characteristic equation of $4y''+4y'+y=0$ is $4\lambda^2+4\lambda+1=0$

Roots $4\lambda^2+4\lambda+1=0$: $\Delta=16-16=0$ thus $\lambda=-\frac{4}{8}=-\frac{1}{2}.$

Lesson 4: Solving DE whose Characteristic equation has two distinct real roots

a) Learning objectives

Solve an ordinary linear differential equation of second order.

b) Teaching resources:

student teacher's book and other Reference books to facilitate research, scientific calculator, Manila paper, markers, pens, pencils, if possible computers with mathematical software such as Geogebra, Microsoft Excel, Math-lab or graph-application and internet can be used...

c) Prerequisites/Revision/Introduction

student teachers will easily learn this unit, if they have a good background on: differentiation (Year 2: Unit 7), quadratic equations (year 1: unit 5), Trigonometry (year 1: unit 8), Logarithm and exponential (year 3:unit 4), integration(year 3: unit 5), complex numbers(year 3: unit 1).

d) Learning activities:

This lesson will help student teachers to understand the concept of differential equation and their classifications through derivatives and the highest derivative degree.

- Form groups and ask student teachers to work out the activity 7.3.1
- Let student teachers work independently for some while.
- Facilitate student teachers to derivate and to guess the power of the highest derivative.
- Ask randomly some groups to present their findings to the whole class;

- Guide student teachers to interact about the findings, to conclude on the new concept and to write a short summary.
- Let student teachers work out activity 7.3.1 under your guidance, and work individually application activity 7.3.1 to assess their competences.

Answers to activity 7.3.1

Characteristic equation: $\lambda^2 + 7\lambda + 6 = 0$.

Simple factorization yields to $(\lambda + 6)(\lambda + 1) = 0$. Thus $\lambda = -6$ or $\lambda = -1$. You can verify that $y = e^{-x}$ and $y = e^{-6x}$ are solutions of $y'' + 7y' + 6y = 0$ and they are linearly independent.

Therefore, the general solution is $y = c_1 e^{-x} + c_2 e^{-6x}$.

Application Activity 7.3.1

1. General solutions

a) $y = c_1 + c_2 e^{3x}$

b) $y = c_1 e^{2\sqrt{2}x} + c_2 e^{-2\sqrt{2}x}$

c) $y = c_1 e^{-6x} + c_2 e^{-x}$

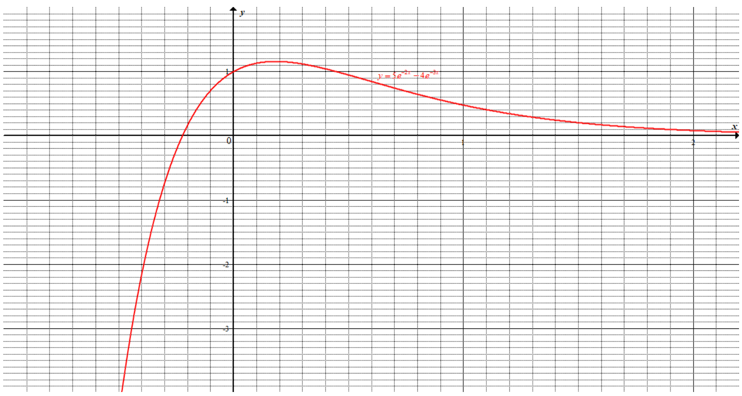
d) $y = c_1 e^{x} + c_2 e^{-2x}$

e) $y = c_1 e^{2x} + c_2 e^{3x}$

f) $y = c_1 e^{-2x} + c_2 e^{\frac{x}{2}}$

2. Particular solutions

a) $y = 5e^{-2x} - 4e^{-3x}$



b) $y = -e^{-2x} + 3e^{\frac{1}{3}x}$

Lesson 5: Solving DE whose Characteristic equation has a real double /repeated roots

a) Learning objectives

Solve an ordinary linear differential equation of second order.

b) Teaching resources:

student teacher's book and other Reference books to facilitate research, scientific calculator, Manila paper, markers, pens, pencils, if possible computers with mathematical software such as Geogebra, Microsoft Excel, Math-lab or graph-application and internet can be used...

c) Prerequisites/Revision/Introduction

student teachers will easily learn this unit, if they have a good background on: differentiation (Year 2: Unit 7), quadratic equations (year 1: unit 5), Trigonometry (year 1: unit 8), Logarithm and exponential (year 3:unit 4), integration(year 3: unit 5), complex numbers(year 3: unit 1)

d) Learning activities:

This lesson will help student teachers to understand the concept of differential equation and their classifications through derivatives and the highest derivative degree.

- Form groups and ask student teachers to work out the activity 7.3.2.
- Let student teachers work independently for some while.
- Facilitate student teachers to derivate and to guess the power of the highest derivative.

- Ask randomly some groups to present their findings to the whole class;
- Guide student teachers to interact about the findings, to conclude on the new concept and to write a short summary.
- Let student teachers work out activity 7.3.2 under your guidance, and work individually application activity 7.3.2 to assess their competences.

Answers to activity 7.3.2

a) Let y be the function defined by $y(x)=\frac{e^{mx}-e^{nx}}{m-n}$, thus

$$y''+py'+qy=\left(\frac{e^{mx}-e^{nx}}{m-n}\right)''+p\left(\frac{e^{mx}-e^{nx}}{m-n}\right)'+q\frac{e^{mx}-e^{nx}}{m-n}$$

$$=\left(\frac{me^{mx}-ne^{nx}}{m-n}\right)'+p\frac{me^{mx}-ne^{nx}}{m-n}+q\frac{e^{mx}-e^{nx}}{m-n}$$

$$=\frac{m^2e^{mx}-n^2e^{nx}}{m-n}+p\frac{me^{mx}-ne^{nx}}{m-n}+q\frac{e^{mx}-e^{nx}}{m-n}$$

$$=\frac{m^2e^{mx}+pme^{mx}+qe^{mx}}{m-n}-\frac{n^2e^{nx}+pne^{nx}+qe^{nx}}{m-n}$$

$$=\frac{e^{mx}}{m-n}\left(m^2+pm+q\right)-\frac{e^{nx}}{m-n}\left(n^2+pn+q\right)=0$$

From characteristic equation of $y''+py'+qy=0$, $m^2+pm+q=0=n^2+pn+q$.

Thus, $y''+py'+qy=\frac{e^{mx}}{m-n}\left(m^2+pm+q\right)-\frac{e^{nx}}{m-n}\left(n^2+pn+q\right)=0-0=0$ and then

$y=f(x)=\frac{e^{mx}-e^{nx}}{m-n}$ Hence it is a solution of $y''+py'+qy=0$.

b) $\lim_{m\to n}f(x)=\lim_{m\to n}\frac{e^{mx}-e^{nx}}{m-n}=\frac{e^{nx}-e^{nx}}{m-n}=\frac{0}{0}$ which is indeterminate form.

Remove this indeterminate by Hospital rule:

$$\lim_{m\to n}f(x)=\lim_{m\to n}\frac{\left(e^{mx}-e^{nx}\right)'}{(m-n)'}=\lim_{m\to n}\frac{xe^{mx}-0}{1-0}=xe^{nx}$$ as required.

Let $y = \lim_{m \to n} f(x) = xe^{nx}$, thus

$$y'' + py' + qy = \left(xe^{nx}\right)'' + p\left(xe^{nx}\right)' + qxe^{nx}$$

$$= \left(e^{nx} + nxe^{nx}\right)' + p\left(e^{nx} + nxe^{nx}\right) + qxe^{nx}$$

$$= ne^{nx} + ne^{nx} + n^2xe^{nx} + pe^{nx} + pnxe^{nx} + qxe^{nx}$$

$$= xe^{nx}\left(n^2 + pn + q\right) + e^{nx}\left(2n + p\right)$$

$= e^{nx}\left(2n + p\right)$ from auxiliary equation $n^2 + pn + q = 0$.

As n, is a double root of $n^2 + pn + q = 0$, then $n = -\frac{p}{2}$;

therefore $y'' + py' + qy = e^{nx}\left(2n + p\right) \Leftrightarrow y'' + py' + qy = 0$.

Hence, the function $y = xe^{mx}$ is a solution of $y'' + py' + qy = 0$ when $m \to n$ and $f(x) = \frac{e^{mx} - e^{nx}}{m - n}$ is a solution of the equation $y'' + py' + qy = 0$ (generally if m is a repeated root of the auxiliary equation $y = xe^{mx}$ is a solution of the given differential equation).

c) Let us check whether $y = \lim_{m \to n} f(x)$ and $y = e^{nx}$ are linearly independent ;

Since $\frac{\lim_{m \to n} f(x)}{e^{nx}} = \frac{xe^{nx}}{e^{nx}} = x \neq c^{te}$, $y = \lim_{m \to n} f(x) = xe^{nx}$ and $y = e^{nx}$ are linearly independent that leads to general solution $y = c_1e^{nx} + c_2xe^{nx}$, n being a double root of auxiliary equation.

For $y'' - 2y' + y = 0$, the characteristic equation is $n^2 - 2n + 1 = 0$ whose double root $n = 1$ Therefore the general solution is $y = c_1e^x + c_2xe^x$.

Application Activity 7.3.2

1. General solutions:

   b) $y = \left(c_1 + c_2x\right)e^{-4x}$

c) $y=(c_1+c_2x)e^{\frac{3}{2}x}$

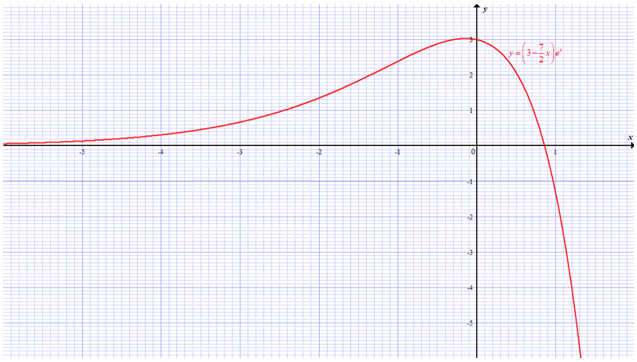
d) $y=(c_1+c_2x)e^{-\frac{x}{2}}$

e) $y=(c_1+c_2x)e^{\frac{1}{6}x}$

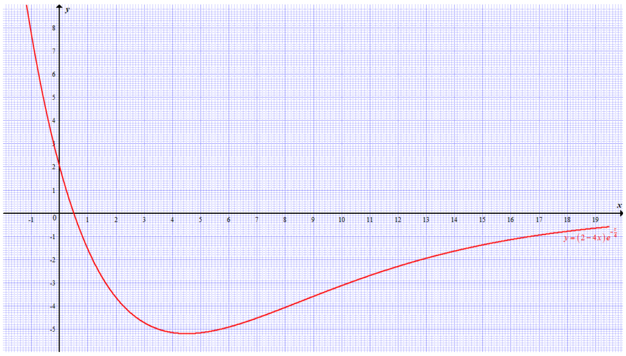
f) $y=(c_1+c_2x)e^{\frac{\pi}{2}x}$

2. Particular solutions

c) $y=\left(3-\dfrac{7}{2}x\right)e^{x}$



d) $y=(2-4x)e^{-\frac{1}{4}x}$



Lesson 6: Solving DE whose Characteristic equation has complex roots

a) Learning objectives

Solve an ordinary linear differential equation of second order.

b) Teaching resources:

student teacher's book and other Reference books to facilitate research, scientific calculator, Manila paper, markers, pens, pencils, if possible computers with mathematical software such as Geogebra, Microsoft Excel, Math-lab or graph-application and internet can be used...

c) Prerequisites/Revision/Introduction

student teachers will easily learn this unit, if they have a good background on: differentiation (Year 2: Unit 7), quadratic equations (year 1: unit 5), Trigonometry (year 1: unit 8), Logarithm and exponential (year 3:unit 4), integration(year 3: unit 5), complex numbers(year 3: unit 1)

d) Learning activities:

This lesson will help student teachers to understand the concept of differential equation and their classifications through derivatives and the highest derivative degree.

- Form groups and ask student teachers to work out the activity 7.3.3.
- Let student teachers work independently for some while.
- Facilitate student teachers to derivate and to guess the power of the highest derivative.
- Ask randomly some groups to present their findings to the whole class;
- Guide student teachers to interact about the findings, to conclude on the new concept and to write a short summary.
- Let student teachers work out activity 7.3.3 under your guidance, and work individually application activity 7.3.3 to assess their competences.

Answers to activity 7.3.3

1. $y_1 = e^{(1+2i)x} = e^x e^{2ix}$ and $y_2 = e^{(1-2i)x} = e^x e^{-2ix}$

2. We know that $e^{\theta i} = \cos\theta + i\sin\theta$ and $e^{-\theta i} = \cos\theta - i\sin\theta$

From this we have $e^{2ix} = \cos 2x + i\sin 2x$ and $e^{-2ix} = \cos 2x - i\sin 2x$, then

$y_1 = e^{(1+2i)x} = e^x e^{2ix} = e^x\left(\cos 2x + i\sin 2x\right)$ and $y_2 = e^{(1-2i)x} = e^x e^{-2ix} = e^x\left(\cos 2x - i\sin 2x\right)$.

3. Remember that **Euler formulae** are $\cos\theta = \dfrac{e^{i\theta}+e^{-i\theta}}{2}$ and $\sin\theta = \dfrac{e^{i\theta}-e^{-i\theta}}{2i}$.

By combining y_1 and y_2 we get

$y_1 + y_2 = e^x\left(\cos 2x + i\sin 2x\right) + e^x\left(\cos 2x - i\sin 2x\right) = 2e^x \cos 2x$

Or $e^x \cos 2x = \dfrac{y_1 + y_2}{2}$ is a real valued solution.

Also , $y_1 - y_2 = e^x\left(\cos 2x + i\sin 2x\right) - e^x\left(\cos 2x - i\sin 2x\right) = 2ie^x \sin 2x$

From $e^x \sin 2x = \dfrac{y_1 - y_2}{2i}$, omitting i, we get $e^x \sin 2x = \dfrac{y_1 - y_2}{2}$ which is another real valued solution.

Hence, real basis is formed by $y_1 = e^x \cos 2x$ and $y_2 = e^x \sin 2x$.

Therefore the general solution is $y = c_1 e^x \cos 2x + c_2 e^x \sin 2x$

Or $y = e^x\left(c_1 \cos 2x + c_2 \sin 2x\right)$.

Application Activity 7.3.3

1.

a) $y = c_1 \cos 5x + c_2 \sin 5x$

b) $y = e^{2x}\left(c_1 \cos x + c_2 \sin x\right)$

c) $y = e^{-2x}\left(c_1 \cos 3x + c_2 \sin 3x\right)$

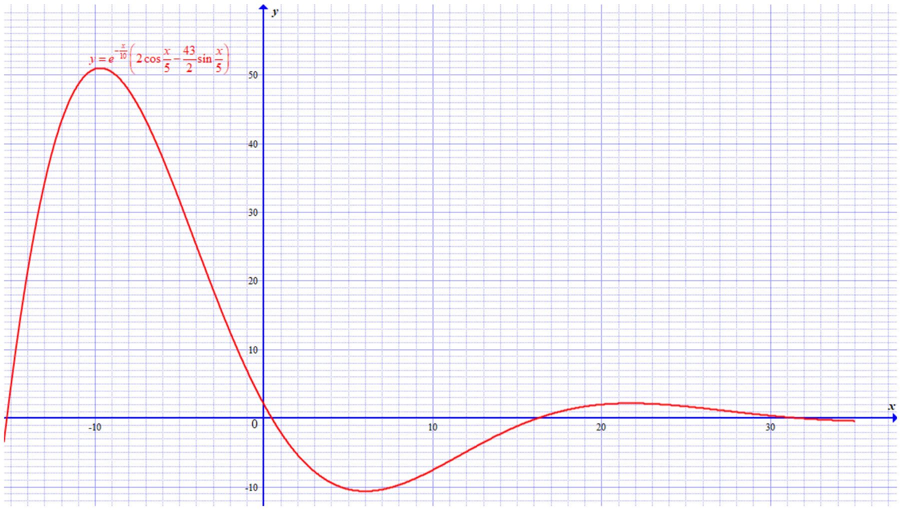
d) $y = e^{-3x}\left(c_1 \cos\sqrt{2}x + c_2 \sin\sqrt{2}x\right)$

e) $y = e^x\left(c_1 \cos 3x + c_2 \sin 3x\right)$

f) $y = e^{-\frac{x}{10}}\left(c_1 \cos\dfrac{2}{5}x + c_2 \sin\dfrac{2}{5}x\right)$

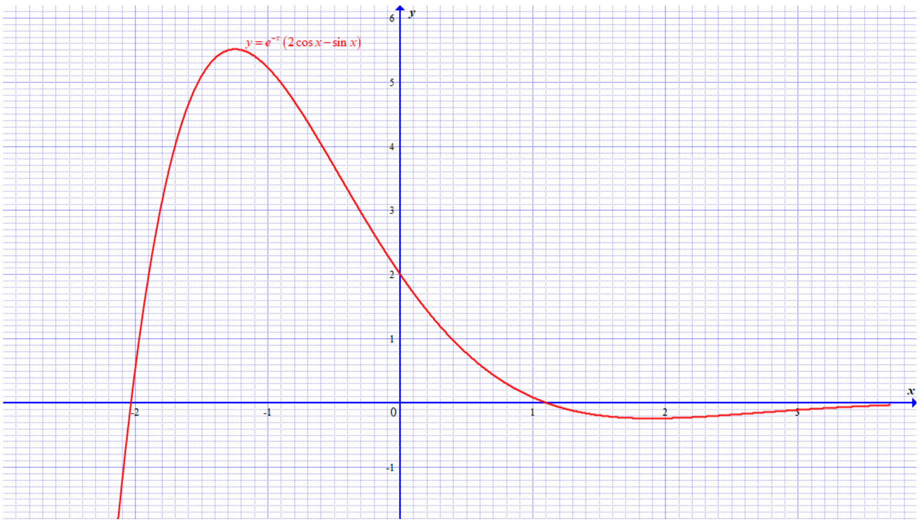
2.

a) $y = e^{-\frac{x}{10}}\left(2\cos\dfrac{x}{5} - \dfrac{43}{2}\sin\dfrac{x}{5}\right)$



b) $y=e^{-x}\left(2\cos x-\sin x\right)$

c)



3.

Case	Roots	Basis	General solution
1	Distinct real roots: λ_1 and λ_2	$e^{\lambda_1 x}$ and $e^{\lambda_2 x}$	$y=c_1e^{\lambda_1 x}+c_2e^{\lambda_2 x}$
2	Real double : λ	$e^{\lambda x}$ and $xe^{\lambda x}$	$y=(c_1+c_2x)e^{\lambda x}$
3	Complex conjugate $\alpha \pm i\omega$	$e^{\alpha x}\cos\omega x$ and $e^{\alpha x}\sin\omega x$	$y=e^{\alpha x}\left(A\cos\omega x+B\sin\omega x\right)$

Non-homogeneous linear equations with constant coefficients

Lesson 7: The right hand side is a polynomial function

a) Learning objectives

Solve an ordinary linear differential equation of second order.

b) Teaching resources:

student teacher's book and other Reference books to facilitate research, scientific calculator, Manila paper, markers, pens, pencils, if possible computers with mathematical software such as Geogebra, Microsoft Excel, Math-lab or graph-application and internet can be used...

c) Prerequisites/Revision/Introduction

student teachers will easily learn this unit, if they have a good background on: differentiation (Year 2: Unit 7), quadratic equations (year 1: unit 5), Trigonometry (year 1: unit 8), Logarithm and exponential (year 3:unit 4), integration(year 3: unit 5), complex numbers(year 3: unit 1)

d) Learning activities:

This lesson will help student teachers to understand the concept of differential equation and their classifications through derivatives and the highest derivative degree.

- Form groups and ask student teachers to work out the activity 7.4.1.
- Let student teachers work independently for some while.
- Facilitate student teachers to derivate and to guess the power of the highest derivative.
- Ask randomly some groups to present their findings to the whole class;
- Guide student teachers to interact about the findings, to conclude on the new concept and to write a short summary.
- Let student teachers work out activity 7.4.1 under your guidance, and work individually application activity 7.4.1 to assess their competences.

Answers to activity 7.4.1

$\frac{d^2y}{dx^2}-4\frac{dy}{dx}=5y$, this equation is the same as: $\frac{d^2y}{dx^2}-4\frac{dy}{dx}-5y=0$ and its solution

is: $y=c_1e^{5x}+c_2e^{-x}$ which is homogeneous linear equation with right hand side is zero.

1. $\frac{d^2y}{dx^2}-4\frac{dy}{dx}-5y=x^2$ this equation is non-homogeneous linear equation

Its solution, first we need the solution for homogeneous equation, after you find its particular solution because it has right side.

It means: the general solution will be: $y=\overline{y}+y^*$

So, $\frac{d^2y}{dx^2}-4\frac{dy}{dx}-5y=0 \Rightarrow \overline{y}=c_1e^{5x}+c_2e^{-x}$; where $\overline{y}$ is called the complementary solution (or the homogeneous solution)

And $y^*=Ax^2+Bx+c \Rightarrow y^{*'}=2Ax+B \Rightarrow y^{*''}=2A$

By replacing in this equation $\frac{d^2y}{dx^2}-4\frac{dy}{dx}-5y=x^2$,we get:

$$2A-8Ax-4B-5Ax^2-5Bx-5C=x^2$$

$$\Rightarrow \begin{cases} -5A=1 \\ -8A-5B=0 \\ 2A-4B-5C=0 \end{cases} \Rightarrow \begin{cases} A=-\frac{1}{5} \\ B=\frac{8}{5} \\ C=-\frac{34}{25} \end{cases} \Rightarrow y^*=-\frac{1}{5}x^2+\frac{8}{5}x-\frac{34}{25}$$

The general solution: $y=\overline{y}+y^*=c_1e^{5x}+c_2e^{-x}-\frac{1}{5}x^2+\frac{8}{5}x-\frac{34}{25}$

Application Activity 7.4.1

1. $y''-2y'+y=x+1 \Rightarrow y=(c_1+c_2x)e^x+x+3$

2. $y''-3y'=3x^2+2x+1 \Rightarrow y=c_1+c_2e^{3x}-\frac{1}{3}x^3-\frac{2}{3}x^2-\frac{7}{9}x$

3. $y''=2x+4 \Rightarrow y=c_1+c_2x+\frac{1}{3}x^3+2x^2$

Lesson 8: The right hand side is a product of the form: $r(x) = Pe^{\alpha x}$

a) Learning objectives

Solve an ordinary linear differential equation of second order.

b) Teaching resources:

student teacher's book and other Reference books to facilitate research, scientific calculator, Manila paper, markers, pens, pencils, if possible computers with mathematical software such as Geogebra, Microsoft Excel, Math-lab or graph-application and internet can be used...

c) Prerequisites/Revision/Introduction

student teachers will easily learn this unit, if they have a good background on: differentiation (Year 2: Unit 7), quadratic equations (year 1: unit 5), Trigonometry (year 1: unit 8), Logarithm and exponential (year 3:unit 4), integration(year 3: unit 5), complex numbers(year 3: unit 1)

d) Learning activities:

This lesson will help student teachers to understand the concept of differential equation and their classifications through derivatives and the highest derivative degree.

- Form groups and ask student teachers to work out the activity 7.4.2
- Let student teachers work independently for some while.
- Facilitate student teachers to derivate and to guess the power of the highest derivative.
- Ask randomly some groups to present their findings to the whole class;
- Guide student teachers to interact about the findings, to conclude on the new concept and to write a short summary.
- Let student teachers work out activity 7.4.2 under your guidance, and work individually application activity 7.8 to assess their competences.

Answers to activity 7.4.2

1. Characteristic equation: $m^2 - 2m + 1 = 0$

$$\Delta = 4 - 4 = 0$$

$$m_1 = m_2 = \frac{2-0}{2} = 1$$

$$\overline{y} = c_1 e^x + c_2 e^x$$

2. The right hand side can be written as $e^x = 1e^{1x}$.

$P = 1$ and $\alpha = 1$

$\alpha = 1$, is double root of characteristic equation, so $k = 2$

$y^* = Ax^2e^x$, $Q(x) = A$ as $P = 1$ in right hand side, $Q(x)$ has degree zero.

3. $y^{*/} = 2Axe^x + Ax^2e^x$

$y^{*//} = 2Ae^x + 2Axe^x + 2Axe^x + Ax^2e^x$

$\Rightarrow 2Ae^x + 2Axe^x + 2Axe^x + Ax^2e^x - 4Axe^x - 2Ax^2e^x + Ax^2e^x = e$

$\Rightarrow 2A + 2Ax + 2Ax + Ax^2 - 4Ax - 2Ax^2 - Ax^2 = 1$

$\Rightarrow -2Ax^2 + 2Ax^2 + x(2A + 2A - 4A) - 2A = 1$

$2A = 1 \Rightarrow A = \frac{1}{2}$

Thus, $y^* = \frac{1}{2}x^2e^x$

Synthesis

If the right hand side of the equation $y'' + py' + qy = r(x)$ is $r(x) = Pe^{\alpha x}$ where P is a polynomial, we take the particular solution to be

$y^* = x^kQ_n(x)e^{\alpha x}$, $Q_n = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + ... + a_n$

Here: k - is the number of roots of the associated homogeneous equation equals to α.

α - Coefficient of x in $e^{\alpha x}$ in the right hand side

n - Degree of $Q(x)$, the same as degree of $P(x)$ in right hand side.

3 cases arise

- If α is not a root of characteristic equation $k = 0$
- If α is a simple root of characteristic equation $k = 1$
- If α is a double root of characteristic equation $k = 2$

Note that the simple root or double root in the last 2 cases must be real numbers.

Application Activity 7.4.2

1. $y = c_1e^{-3x} + c_2xe^{-3x} + \frac{5e^{3x}}{36}$

2. $c_1e^{x} + c_2e^{2x} + \frac{e^{3x}}{2}$

3. $y = c_1e^{-x} + c_2e^{-2x} + \frac{1}{4}e^{2x}$

Lesson 9: Applications of second order linear homogeneous differential equation

a) Learning objectives

Solve an ordinary linear differential equation of second order.

b) Teaching resources:

student teacher's book and other Reference books to facilitate research, scientific calculator, Manila paper, markers, pens, pencils, if possible computers with mathematical software such as Geogebra, Microsoft Excel, Math-lab or graph-application and internet can be used...

c) Prerequisites/Revision/Introduction

student teachers will easily learn this unit, if they have a good background on: differentiation (Year 2: Unit 7), quadratic equations (year 1: unit 5), Trigonometry (year 1: unit 8), Logarithm and exponential (year 3:unit 4), integration(year 3: unit 5), complex numbers(year 3: unit 1)

d) Learning activities:

This lesson will help student teachers to understand the concept of differential equation and their classifications through derivatives and the highest derivative degree.

- Form groups and ask student teachers to work out the activity 7.5
- Let student teachers work independently for some while.
- Facilitate student teachers to derivate and to guess the power of the highest derivative.
- Ask randomly some groups to present their findings to the whole class;
- Guide student teachers to interact about the findings, to conclude on the

new concept and to write a short summary.

- Let student teachers work out activity 7.5 under your guidance, and work individually application activity 7.5 to assess their competences.

Answers to activity 7.5

a) $m\frac{d^2x}{dt^2} = -kx \Leftrightarrow m\frac{d^2x}{dt^2} + kx = 0$

Identifying the coefficients to $a\frac{d^2x}{dt^2} + b\frac{dx}{dt} + cx = 0$ according the order of

derivative, we get $a = m,\ b = 0$ and $c = k$.

b) The simple harmonic motion has equation

$$m\frac{d^2x}{dt^2} + kx = 0$$

Characteristic equation: $m\lambda^2 + k = 0 \Leftrightarrow \lambda = \pm i\sqrt{\frac{k}{m}}$

General solution is

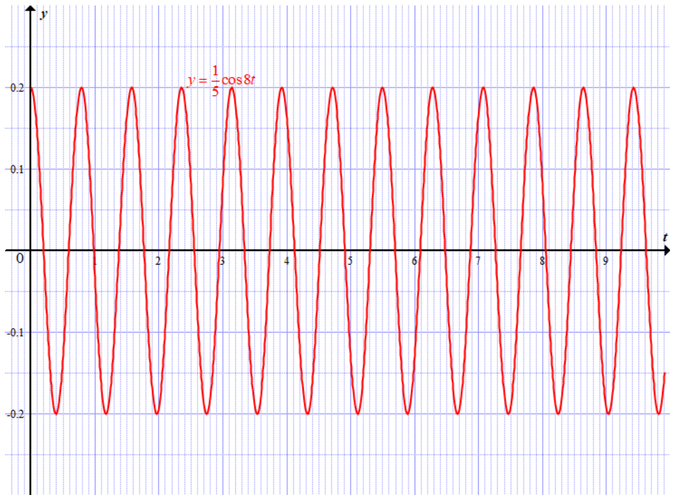
$$x(t) = A\cos\sqrt{\frac{k}{m}}t + B\sin\sqrt{\frac{k}{m}}t \xrightarrow{\omega=\sqrt{\frac{k}{m}}} x(t) = A\cos\omega t + B\sin\omega t.$$

c) For $m = 2$, $k = 128$ we have $2\frac{d^2x}{dt^2} + 128x = 0$ and then $x(t) = c_1\cos 8t + c_2\sin 8t$.

Given the initial conditions $\begin{cases} x(0) = 0.2 \\ x'(0) = 0 \end{cases}$, we find $c_1 = \frac{1}{5}$ and $c_2 = 0$.

Therefore, $x(t) = \frac{1}{5}\cos 8t$.

Graph of $x(t) = \frac{1}{5}\cos 8t$



Application Activity 7.5

1. The motion of that mass is modeled by the equation

$m\frac{d^2y}{dt^2}+c\frac{dy}{dt}+ky=0$, introducing the data given, $2\frac{d^2y}{dt^2}+40\frac{dy}{dt}+128y=0$

$\Leftrightarrow \frac{d^2y}{dt^2}+20\frac{dy}{dt}+64y=0$ whose characteristic equation is
$\lambda^2+20\lambda+64=0 \Leftrightarrow (\lambda+4)(\lambda+16)=0$ and $\lambda_1=-4 \quad and \quad \lambda_2=-6$

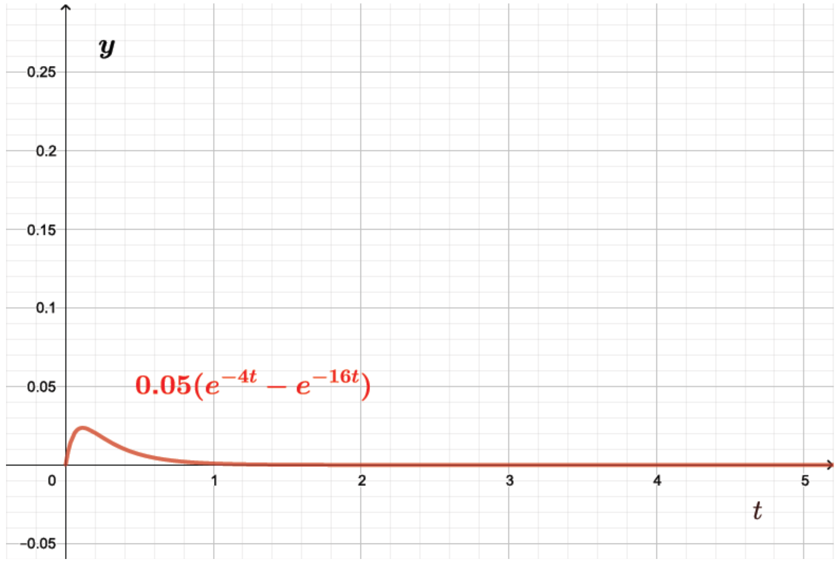
As $\lambda_1=-4 \quad and \quad \lambda_2=-6$ are all real and different from each other, we have an over-damping spring and the general solution is $y(t)=C_1e^{-4t}+C_2e^{-16}$

The initial conditions are $y(0)=0 \quad and \quad y'(0)=0.6$

i.e $0=C_1+C_2$ and $0.6=-4C_1-16C_2$. This two equations give

$C_1=0.05 \quad and \quad C_2=-0.05$. Therefore, $y(t)=0.05\left(e^{-4t}-e^{-16t}\right)$.

Graph for $y(t)=0.05\left(e^{-4t}-e^{-16t}\right)$



The spring is over-damped, it will oscillate only once.

2. a) If $R=40$, $L=1$, $C=16.10^{-4}$ and $E(t)=100\cos 10t$

, $L\dfrac{d^2Q}{dt^2}+R\dfrac{dQ}{dt}+\dfrac{1}{C}Q=E(t)$ will be written as

$\dfrac{d^2Q}{dt^2}+40\dfrac{dQ}{dt}+\dfrac{10^4}{16}Q=100\cos 10t$ or $\dfrac{d^2Q}{dt^2}+40\dfrac{dQ}{dt}+625Q=100\cos 10t$

b) What type of equation obtained if you consider $E(t)$ for $t=\dfrac{\pi}{20}$?

For $t=\dfrac{\pi}{20}$, $E(t)=100\cos 10t=0$.

Therefore, $\dfrac{d^2Q}{dt^2}+40\dfrac{dQ}{dt}+625Q=0$,
this is a second order homogeneous differential equation.

c) Determine the general solution for the equation obtained in (b).

$\dfrac{d^2Q}{dt^2}+40\dfrac{dQ}{dt}+625Q=0$ and $\lambda^2+40\lambda+625=0$, $\lambda_1=-20+15i$ and

$\lambda_1=-20-15i$

The general solution is $Q(t)=e^{-20t}\left(C_1\cos 15t+C_2\sin 15t\right)$.

7.6. Unit summary

1. Second order linear homogeneous differential equations

   A second order linear homogeneous differential equation is of the form

   $a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=0$ where *a*, *b*, *c* are constants (and $a\neq 0$).

2. Linearity of solutions of linear homogeneous differential equations:

   If y_1 is solution of the homogeneous linear differential equation

   $a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=0$, the second solution y_2 is such that $y_1(x)$ and $y_2(x)$ are linearly independent where the determinant called the Wronskian of y_1 and y_2 denoted and defined by:

   $$W(y_1,y_2)=\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}=y_1y_2'-y_1'y_2 \text{ is not zero.}$$

3. Superposition principle:

   If y_1 and y_2 are two solutions of the homogeneous linear differential equation

   $a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=0$, then any other linear combination $y=Ay_1+By_2$ of these two solutions is a solution of the equation.

4. Characteristic equation of a second order differential equation

   The equation $a\lambda^2+b\lambda+c=0$ in is called the auxiliary or characteristic equation of the second order differential equation $a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=0$.

5. Solution of a second order differential equation $a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=0$

- If the Characteristic equation $a\lambda^2+b\lambda+c=0$ has two distinct real roots λ_1 and λ_2, the corresponding general solution is $y=c_1e^{\lambda_1 x}+c_2e^{\lambda_2 x}$.

- If the Characteristic equation $a\lambda^2+b\lambda+c=0$ has a real double root/repeated root $\lambda_1=\lambda_2=\lambda$, we have $y_1=e^{\lambda x}$ and the second linearly independent solution will be $y_2=xe^{\lambda x}$.

- If the characteristic equation $a\lambda^2+b\lambda+c=0$ has complex roots, $\lambda_1=\alpha+i\beta$ and $\lambda_2=\alpha-i\beta$, the general solution $y=e^{\alpha x}\left(c_1e^{i\beta x}+c_2e^{-i\beta x}\right)$ is modified using Euler's formula and it becomes $y=e^{\alpha x}\left(A\cos\beta x+B\sin\beta x\right)$.

6. Applications of second order linear homogeneous differential equation

6.1 The oscillatory movement of a mass on a spring

- At the end of a simple spring with the spring constant k, the mass m makes a simple harmonic motion with the equation $m\frac{d^2x}{dt^2}+kx=0$;
- The mass m at the end of a simple spring with the spring constant k in the milieu of the damping constant C makes a motion modeled by the equation

$$m\frac{d^2x}{dt^2}+C\frac{dx}{dt}+kx=0.$$

6.2 The Electric charge in a RLC series circuit

The voltage drops across the resistor, inductor, and capacitor are respectively RI, $L\frac{dI}{dt}$ and $\frac{Q}{C}$.

The Kirchhoff's voltage law says that the sum of these voltage drops is equal to the supplied voltage $E(t)$ and expressed in the following equation:

$$L\frac{d^2Q}{dt^2}+R\frac{dQ}{dt}+\frac{1}{C}Q=E(t).$$

Given that the current $i=\frac{dQ}{dt}$, the differential equation becomes

$$L\frac{d^2i}{dt^2}+R\frac{di}{dt}+\frac{1}{C}i=\frac{dE(t)}{dt}$$

This is a non-homogeneous second order differential equation because $E(t)\neq 0$.

7. Second order differential equations

The general second order linear differential equation is of the form

$$\frac{d^2y}{dx^2}+p(x)\frac{dy}{dx}+q(x)y=r(x)$$

Let $y''+py'+qy=0$ be an homogeneous linear equation of second order (right hand side is equal to zero) where p and q are constants.

The equation $m^2+pm+q=0$ is called the **characteristic auxiliary equation**

- If characteristic equation has two distinct real roots then, $y=c_1e^{m_1x}+c_2e^{m_2x}$ is the general solution of $y''+py'+qy=0$
- If characteristic equation has a real double root then, $y=c_1e^{mx}+c_2xe^{mx}$ is the general solution of $y''+py'+qy=0$
- If characteristic equation has complex roots then, $y=e^{ax}\left(c_1\cos bx+c_2\sin bx\right)$ is the general solution of $y''+py'+qy=0$

Let $y''+py'+qy=r(x)$ (1) be a non-homogeneous linear equation of second order (right hand side is different of zero) where p and q are real numbers.

If the right hand side of the equation $y''+py'+qy=r(x)$ is $r(x)=Pe^{\alpha x}$ where P is a polynomial, we take the particular solution to be

$$y^*=x^kQ_n(x)e^{\alpha x},\ Q_n=a_0x^n+a_1x^{n-1}+a_2x^{n-2}+.....+a_n$$

Here: k - is the number of roots of the associated homogeneous equation equals to α.

n - Degree of $Q(x)$, the same as degree of $P(x)$ in right hand side.

α - Coefficient of x in $e^{\alpha x}$ in the right hand side

- If α is not a root of characteristic equation $k=0$
- If α is a simple root of characteristic equation $k=1$
- If α is a double root of characteristic equation $k=2$

Note that the simple root or double root in the last 2 cases must be real numbers.

If the right hand side of the equation $y''+py'+qy=r(x)$ is

$r(x)=Pe^{\alpha x}\cos\beta x+Qe^{\alpha x}\sin\beta x$ where P

Alternative method: Variation of parameters

Assume that the general solution of the characteristic equation associated to the equation $y''+py'+qy=r(x)$ is found to be $\overline{y}=c_1y_1(x)+c_2y_2(x)$.

To get particular solution:

From $\overline{y}=c_1y_1(x)+c_2y_2(x)$,

- Determine $W(y_1,y_2)$ known as **Wronskian** of two functions y_1 and y_2 defined by

$$W(y_1,y_2)=\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

- Find out $v_1=\int\frac{-y_2r(x)}{W(y_1,y_2)}$, and $v_2=\int\frac{y_1r(x)}{W(y_1,y_2)}$

  where $r(x)$ is the right hand side of the given equation.

  Here $W(y_1,y_2)$ must be different from zero as y_1 and y_2 are linearly independent.

- Find out particular solution y^* as $y^*=v_1(x)y_1(x)+v_2(x)y_2(x)$.

The **general solution** is $y=\overline{y}+y^*$

7.7 Additional information for the teacher

For the educative action of the tutor to be effective (in order to respond to all aspects of the student teachers' needs), it is worth mentioning that the tutor needs a wide range of skills, attitudes, a rich and deep understanding of the subject matter and the pedagogical processes to develop the understanding that is required from the student teacher. It is therefore, it is imperative for the tutor to not limit himself/herself to the only to the present book, but also to consider getting information from other relevant books, such as those mentioned in the reference.

Here the tutor has to emphasize on the application of differential equation, for example:

Height of the falling object above the ground

Consider a mass m falling under the influence of constant gravity, such as approximately found on the Earth's surface.

Newton's law results in the equation $F = m.a$ where m is the mass of the object and F the sum of all forces acting on that mass. If we suppose that the resistance of the air is absent,

We have $m\frac{d^2y}{dt^2} = -mg$ where y is the height of the object above the ground, m is the mass of the object, and $g = 9.8m/\sec^2$ is the constant gravitational acceleration. As Galileo suggested,

the mass cancels from the equation, and $\frac{d^2y}{dt^2} = -g$

Here, the right-hand-side of the ode is a constant. The first integration, obtained by anti-differentiation, yields $\frac{d\ y}{dt} = A - gt$ with A the first constant of integration; and

the second integration yields $y = B + At - \frac{1}{2}gt^2$ with B the second constant of integration.

The two constants of integration A and B can then be determined from the initial conditions.

If we know that the initial height of the mass is H_0 , and the initial velocity is V_0, then the initial conditions are $y(0) = H_0$ and $y'(0) = V_0$

Substitution of these initial conditions into the equations for $\frac{dx}{dt}$ and x allows us to solve for A and B. The unique solution that satisfies both the ode and the initial conditions is given by

$$y = H_0 + V_0t - \frac{1}{2}gt^2$$

This is the formula that gives the height at which arrives a falling object left with the initial velocity V_0 downwards from the height H_0 after the time t as it was learnt in physics Senior2.

7.8 END UNIT ASSESSMENT

1. a) $y=e^{5x}\Rightarrow y'=5e^{5x}\Rightarrow y''=25e^{5x}\Rightarrow y^{(iii)}=125e^{5x}\Rightarrow y^{(iv)}=625e^{5x}$;

   b) i) $y'-5y=5e^{5x}-5e^{5x}=0$; ii) $y''-4y'-5y=25e^{5x}-20e^{5x}-5e^{5x}=0$

2. $y''+2y'+y=x+2\Rightarrow y=c_1e^{-x}+c_2xe^{-x}+x$

3. $y''+2y'+5y=0\Rightarrow y=e^{-x}\left(c_1\cos 2x+c_2\sin 2x\right)$

4. $y''-4y'+4y=0\Rightarrow y=\left(c_1+c_2x\right)e^{2x}$

5. $y''+y'-2y=0\Rightarrow y=c_1e^{x}+c_2e^{-2x}$

6. $y''+y=e^{x}\Rightarrow c_1\cos x+c_2\sin x+\frac{1}{2}e^{x}$

7.

a) $\frac{d^2y}{dx^2}+\frac{dy}{dx}-6y=0,\ y(0)=10, y'(0)=0$

The Differential equation is of order 2 and degree one.

b) Its characteristic equation is $\lambda^2+\lambda-6=0$

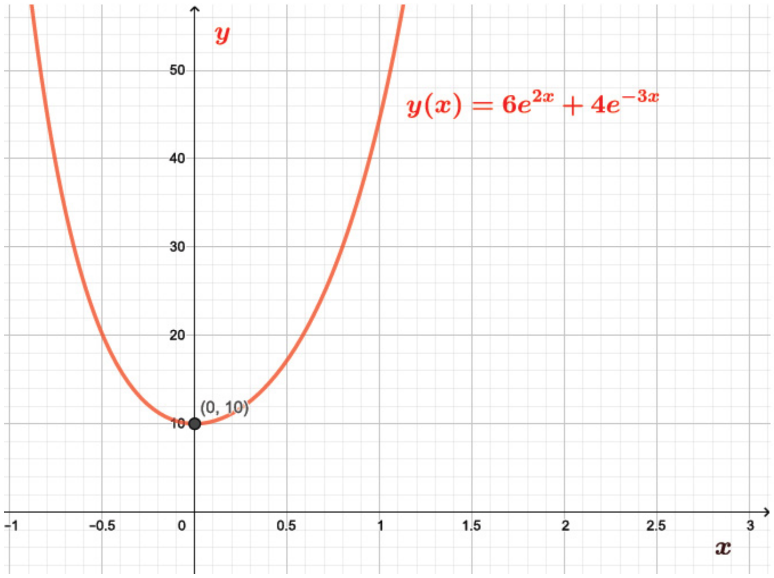
$\lambda_{1,2}=\frac{-1\pm\sqrt{1^2+4.6}}{2}=\frac{-1\pm 5}{2}$, then $\lambda_1=2$ and $\lambda_2=-3$

The general solution $y(x)=c_1e^{2x}+c_2e^{-3x}$

c) Applying the initial conditions$y(0)=10, y'(0)=0$, we get $c_1=6$ and $c_2=4$

Therefore, the particular solution is $y(x)=6e^{2x}+4e^{-3x}$.

Graphical presentation



This function is decreasing on $]-\infty, 0[$ but increasing on $]0, +\infty[$ and its minimum is the point $(0,10)$.

7.9 Additional activities

7.9.1 Remedial activities

1. Solve the following differential equations

a) $4\frac{d^2y}{dx^2} - 25y = 0$

b) $\frac{d^2y}{dx^2} + 2\pi y' + \pi^2 y = 0$

c) $\frac{d^2y}{dx^2} + 25y = 0; y(0) = 4.6,\ y'(0) = -1.2$

d) $y'' - y = 0,\ y(0) = 2,\ y'(0) = -2$

e) $y'' + 0.54y' + (0.0729 + \pi)y = 0,\ y(0) = 0,\ y'(0) = 1$

Solution

a) For $4\frac{d^2y}{dx^2} - 25y = 0$, characteristic equation is $4\lambda^2 - 25 = 0$;

Roots are $\lambda = \frac{5}{2}$ and $\lambda = -\frac{5}{2}$; the corresponding general equation is

$y = c_1 e^{\frac{5}{2}x} + c_2 e^{-\frac{5}{2}x}$.

b) For $\frac{d^2y}{dx^2} + 2\pi y' + \pi^2 y = 0$; $y(0) = 0,\ y'(0) = 13.137$, characteristic equation is

$\lambda^2 + 2\pi\lambda + \pi^2 = 0$ where we have a repeated root $\lambda = -\pi$; the corresponding general solution is $y = c_1 e^{-\pi x} + c_2 x e^{-\pi x}$.

c) For $\frac{d^2y}{dx^2} + 25y = 0; y(0) = 4.6,\ y'(0) = -1.2$, characteristic equation is

$\lambda^2 + 25 = 0$;Roots are $\lambda = 5i$ and $\lambda = -5i$; the corresponding general solution is $y = c_1 \cos 5x + c_2 \sin 5x$

$y(0) = 4.6 \Leftrightarrow c_1 \cos 0 + c_2 \sin 0 = 4.6 \Leftrightarrow c_1 = 4.6$

$y' = -5c_1 \sin 5x + 5c_2 \cos 5x$ and

$y'(0) = -1.2 \Leftrightarrow -5c_1 \sin 5x + 5c_2 \cos 5x = -1.2 \Leftrightarrow c_2 = -0.24$

Therefore, the particular solution is $y = 4.6\cos 5x - 0.24\sin 5x$

For $y'' - y = 0,\ y(0) = 2,\ y'(0) = -2$, characteristic equation is $\lambda^2 - 1 = 0$ and the roots are $\lambda = 1$ and $\lambda = -1$. Thus, the corresponding general solution is $y = c_1 e^x + c_2 e^{-x}$.

$y(0) = 2 \Leftrightarrow c_1 + c_2 = 2$; $y' = c_1 e^x - c_2 e^{-x}$ and $y'(0) = -2 \Leftrightarrow c_1 - c_2 = -2$.

Solving simultaneously $c_1 + c_2 = 2$ and $c_1 - c_2 = -2$ yields $c_1 = 0,\ c_2 = 1$ and then, the particular solution is $y = 2e^{-x}$.

d) For $y'' + 0.54y' + (0.0729 + \pi)y = 0,\ y(0) = 0,\ y'(0) =$, the characteristic equation is $\lambda^2 + 0.54\lambda + (0.0729 + \pi) = 0$ and the roots are $\lambda = -0.27 + \sqrt{\pi}i$ and $\lambda = -0.27 - \sqrt{\pi}i$. Then, the corresponding general solution is

$y = e^{-0.27x}\left(c_1 \cos\sqrt{\pi}x + c_2 \sin\sqrt{\pi}x\right)$.

$y(0) = 0 \Leftrightarrow c_1 = 0$ thus $y' = -0.27e^{-0.27x}c_2 \sin\sqrt{\pi}x + \sqrt{\pi}e^{-0.27x}c_2 \cos\sqrt{\pi}x$ and

$$y'(0)=1 \Leftrightarrow \sqrt{\pi}c_2=1 \Leftrightarrow c_2=\frac{1}{\sqrt{\pi}}.$$

Therefore, the particular solution is $y=\frac{1}{\sqrt{\pi}}e^{-0.27x}\sin\sqrt{\pi}x.$

2. Find an ordinary differential equation $a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=0$ whose solution is formed by:

(a) $e^{\sqrt{5}x}$ *and* $xe^{\sqrt{5}x}$ (b) $\cos 2\pi x$ *and* $\sin 2\pi x$ (c) e^{x} *and* e^{-4x}

(d) $e^{(-2+i)x}$ *and* $e^{(-2+i)x}$ (e) $e^{3x}\cos 2x$ *and* $e^{3x}\sin 2x$

Solution

The standard form of a quadratic equation in λ is $\lambda^2-s\lambda+p=0$ where "p" and "p" are the sum and product of its roots respectively.

(a) From the basis $e^{\sqrt{5}x}$, $xe^{\sqrt{5}x}$, we note that $\sqrt{5}$ is a repeated root of characteristic equation $\lambda^2-s\lambda+p=0$ where $s=2\sqrt{5}$ and $p=\sqrt{5}=5$.

Hence, the corresponding differential equation is $\frac{d^2y}{dx^2}-2\sqrt{5}\frac{dy}{dx}+5y=0$.

(b) From the basis $\cos 2\pi x, \sin 2\pi x$, we note that $2\pi i$ and $-2\pi i$ are two conjugate roots of characteristic equation $\lambda^2-s\lambda+p=0$ where $s=0$ and $p=4\pi^2$.

Hence, the corresponding differential equation is $\frac{d^2y}{dx^2}+4\pi^2y=0$.

(c) From the basis e^{x}, e^{-4x}, we note that 1 and -4 are two distinct roots of characteristic equation $\lambda^2-s\lambda+p=0$ where $s=3$ and $p=-4$.

Hence, the corresponding differential equation is $\frac{d^2y}{dx^2}+3\frac{dy}{dx}-4y=0$.

(d) From the basis $e^{(-2+i)x}, e^{(-2-i)x}$, we note that $-2+i$ and $-2-i$ are two conjugate root of characteristic equation $\lambda^2-s\lambda+p=0$ where $s=-4$ and $p=5$.

Hence, the corresponding differential equation is $\frac{d^2y}{dx^2}+4\frac{dy}{dx}+5y=0$.

(e) From the basis $e^{3x}\cos 2x, e^{3x}\sin 2x$, we note that $3+2i$ and $3-2i$ are two conjugate root of characteristic equation $\lambda^2 - s\lambda + p = 0$ where $s = 6$ and $p = 13$.

Hence, the corresponding differential equation is $\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 13y = 0$.

7.9.2 Consolidation activities:

Solve:

1. $y'' + 5y' + 4y = 3 - 2x$;

2. $y'' + 4y = xe^{2x}$;

3. $2y'' + 3y' + ky = 0$: find the value of k and you solve it : a) when $\Delta < 0$; b) $\Delta > 0$; c) $\Delta = 0$

Solutions:

1. $y = c_1 e^{-x} - 4c_2 e^{-4x} - \frac{1}{2}x + \frac{11}{8}$

2. $y = c_1 \cos 2x + c_2 \sin 2x + \frac{1}{8}xe^{2x} - \frac{1}{16}e^{2x}$

3. $2y'' + 3y' + ky = 0 \Rightarrow \Delta = 9 - 8k$;

d) when $\Delta < 0$; $\Rightarrow 9 - 8k < 0 \Rightarrow k > \frac{9}{8}$ or $k \in \left]\frac{9}{8}, \infty\right[$; let $k = \frac{25}{8}$ so

$\Delta = -16 = 16i^2 \Rightarrow \sqrt{\Delta} = \pm 4i$; from here we get the solution:

$y = e^{-\frac{3}{4}x}(c_1 \cos x + c_2 \sin x)$

e) when $\Delta > 0$; $\Rightarrow 9 - 8k > 0 \Rightarrow k < \frac{9}{8}$ or $k \in \left]-\infty, \frac{9}{8}\right[$; let $k = \frac{5}{8}$ so

$\Delta = 4 \Rightarrow \sqrt{\Delta} = \pm 2$; from here we get the solution: $y = c_1 e^{-\frac{1}{4}x} + c_2 e^{-\frac{5}{4}x}$

f) when $\Delta = 0$; $\Rightarrow 9 - 8k = 0 \Rightarrow k = \frac{9}{8}$; take $k = \frac{9}{8}$ so

$\Delta = 0 \Rightarrow \sqrt{\Delta} = 0$; from here we get the solution: $y = c_1 e^{-\frac{3}{4}x} + c_2 x e^{-\frac{3}{4}x}$

7.9.3 Extended activities:

For each of the following equations, determine the particular solution for initial value problems and give its graphical representation

a) $\frac{d^2y}{dx^2}-\frac{dy}{dx}-6y=0; \quad y(0)=1,\ y'(0)=0$

b) $2\frac{d^2y}{dx^2}+\frac{dy}{dx}-10y=0; \quad y(0)=0,\ y'(0)=1$

c) $\frac{d^2y}{dx^2}-4\frac{dy}{dx}+4y=0; \quad y(0)=0,\ y'(1)=1$

d) $\frac{d^2y}{dx^2}+4\frac{dy}{dx}+13y=0; \quad y(0)=0,\ y'\left(\frac{\pi}{2}\right)=1$

Solutions:

a) $\frac{d^2y}{dx^2}-\frac{dy}{dx}-6y=0; \quad y(0)=1,\ y'(0)=0$:

Characteristic equation is $\lambda^2-\lambda-6=0$;

Roots are $\lambda=3$ and $\lambda=-2$; the corresponding general solution is

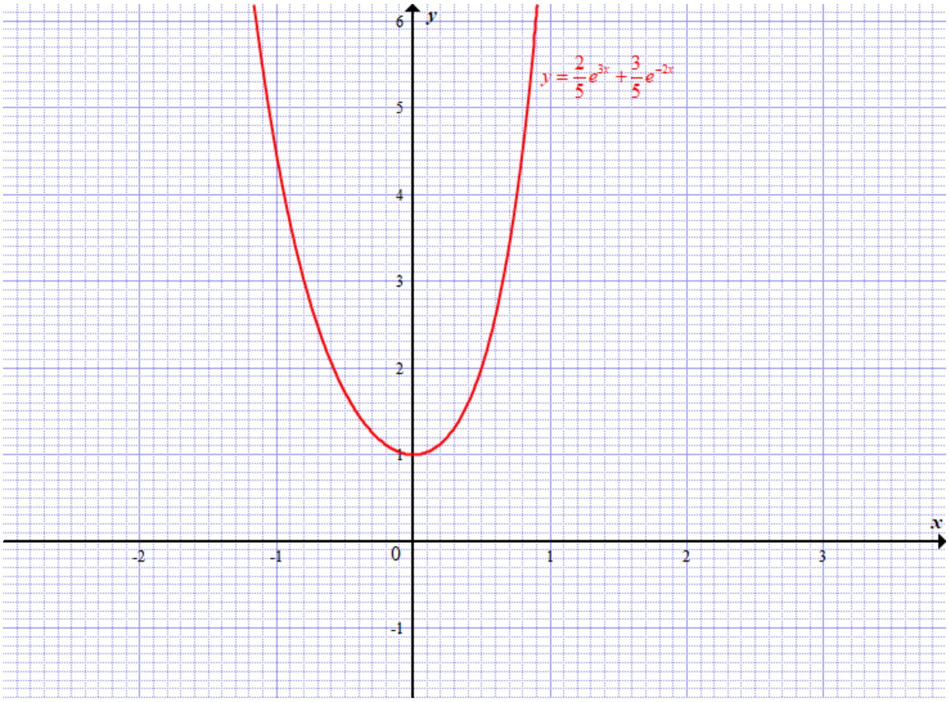
$y=c_1e^{3x}+c_2e^{-2x}$.

$y(0)=1 \Leftrightarrow c_1+c_2=1$

$y'=3c_1e^{3x}-2c_2e^{-2x}$ and $y'(0)=0 \Leftrightarrow 3c_1-2c_2=0$.

Solving simultaneously $c_1+c_2=1$ and $3c_1-2c_2=0$ yields $c_1=\frac{2}{5}$, $c_2=\frac{3}{5}$ and

then, the particular solution is $y=\frac{2}{5}e^{3x}+\frac{3}{5}e^{-2x}$.



b) $2\frac{d^2y}{dx^2}+\frac{dy}{dx}-10y=0; \quad y(0)=0,\ y'(0)=1$

Characteristic equation is $2\lambda^2+\lambda-10=0$;

Roots are $\lambda=2$ and $\lambda=-\frac{5}{2}$; the corresponding general solution is

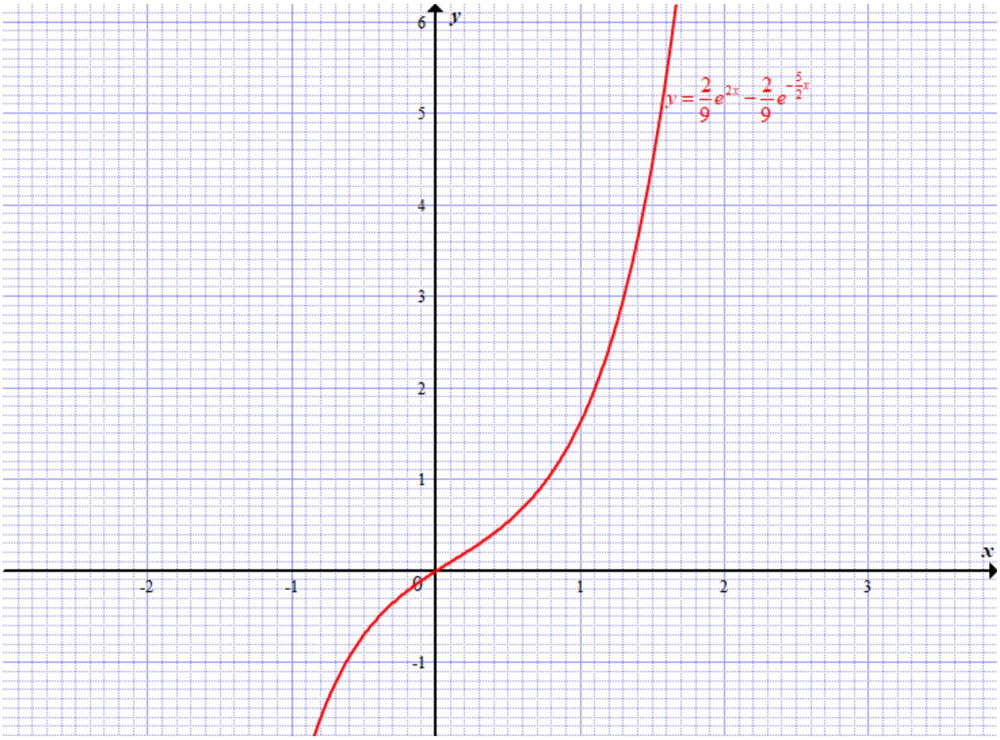
$y=c_1e^{2x}+c_2e^{-\frac{5}{2}x}$.

$y(0)=0 \Leftrightarrow c_1+c_2=0$

$y'=2c_1e^{2x}-\frac{5}{2}c_2e^{-\frac{5}{2}x}$ and $y'(0)=1 \Leftrightarrow 2c_1-\frac{5}{2}c_2=1$.

Solving simultaneously $c_1+c_2=0$ and $2c_1-\frac{5}{2}c_2=1$ yields $c_1=\frac{2}{9},\ c_2=-\frac{2}{9}$.

Therefore, the particular solution is $y=\frac{2}{9}e^{2x}-\frac{2}{9}e^{-\frac{5}{2}x}$



c) $\frac{d^2y}{dx^2}-4\frac{dy}{dx}+4y=0;\quad y(0)=0,\ y'(1)=1$:

Characteristic equation is $\lambda^2-4\lambda+4=0$;

There is a repeated root $\lambda=2$; the corresponding general solution is

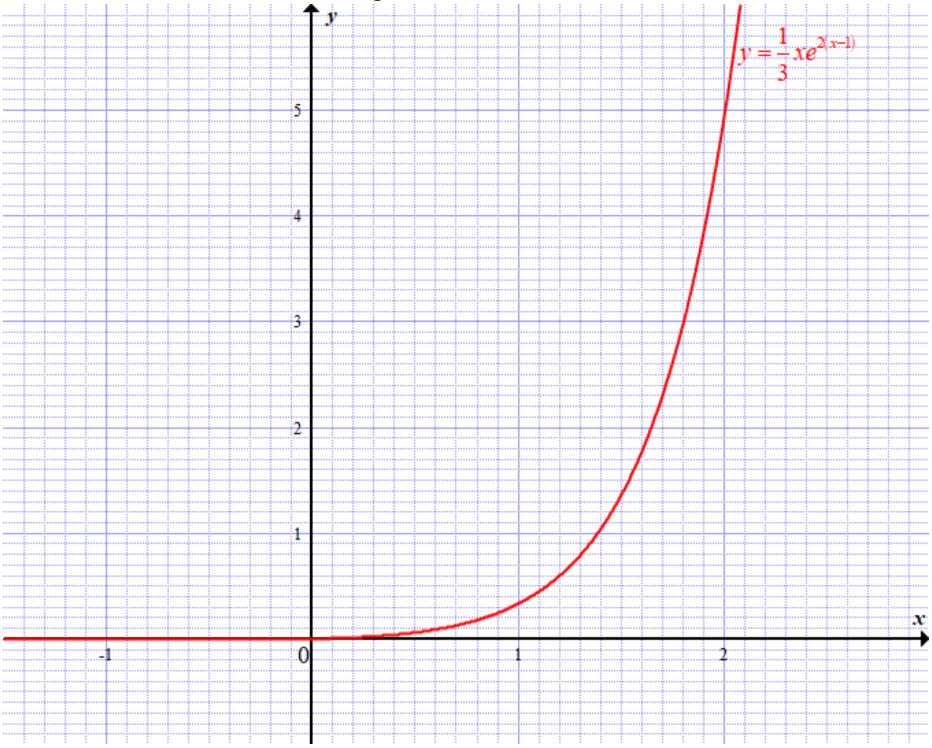
$y=(c_1+c_2x)e^{2x}$.

$y(0)=0\Leftrightarrow c_1=0;\ y'=(2c_1+c_2+2xc_2)e^{2x}$ and

$y'(1)=1\Leftrightarrow(2c_1+c_2+2c_2)e^2=1$

$\Rightarrow 3c_2e^2=1\Rightarrow c_2=\frac{1}{3}e^{-2}$;

The particular solution is $y = \frac{1}{3}xe^{2(x-1)}$.



d) $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 13y = 0; \quad y(0) = 0,\ y'\left(\frac{\pi}{2}\right) = 1$:

Characteristic equation is $\lambda^2 + 4\lambda + 13 = 0$;

Roots are $\lambda = -2 + 3i$ and $\lambda = -2 - 3i$; the corresponding general solution is

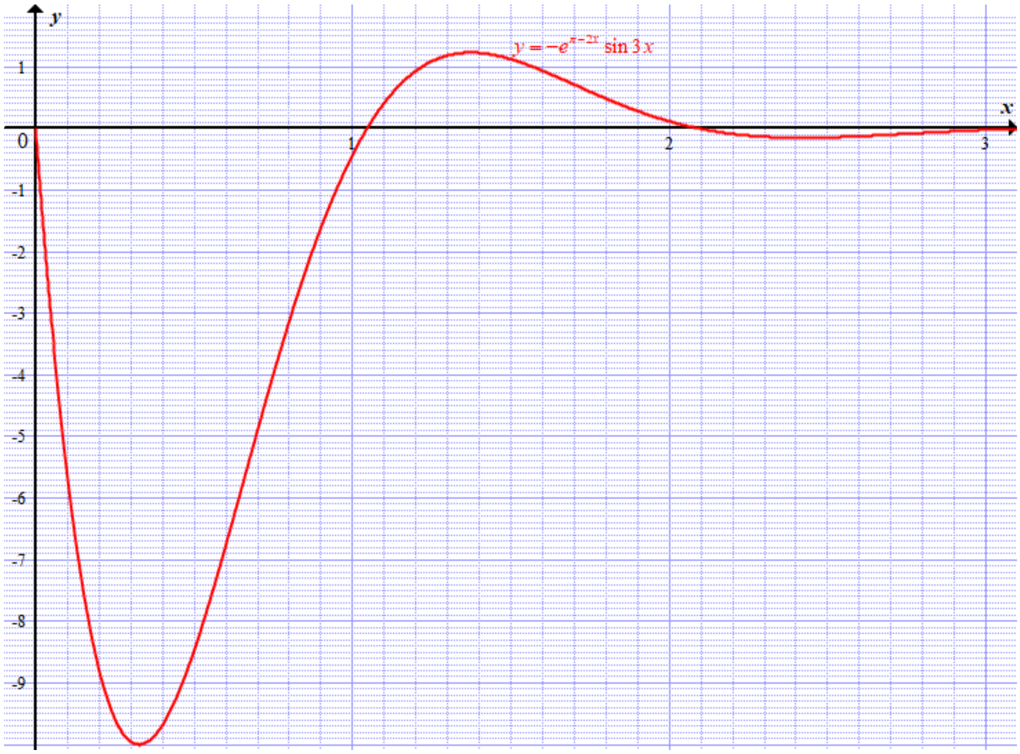
$y = e^{-2x}\left(c_1 \cos 3x + c_2 \sin 3x\right)$.

$y(0) = 0 \Leftrightarrow c_1 = 0$

$y' = -2e^{-2x}\left(c_1 \cos 3x + c_2 \sin 3x\right) + 3e^{-2x}\left(-c_1 \sin 3x + c_2 \cos 3x\right.$ and

$y'\left(\frac{\pi}{2}\right) = 1 \Leftrightarrow 2e^{-\pi}c_2 = 1 \Leftrightarrow c_2 = \frac{1}{2}e^{\pi}$.

Therefore, the particular solution is $y = \frac{1}{2}e^{\pi - 2x}\sin 3x$.



REFERENCES

1. J. Sadler, D. W. S. Thorning: Understanding Pure Mathematics, Oxford University Press 1987.
2. Arthur Adam Freddy Goossens: Francis Lousberg. Mathématisons 65. DeBoeck,3e edition 1991.
3. Charles D. Hodgman, M.S., Samuel M. Selby, Robert C.Weast. Standard Mathematical Table. Chemical Rubber Publishing Company, Cleveland, Ohio 1959.
4. David Rayner, Higher GCSE Mathematics, Oxford University Press 2000
5. Direction des Progammes de l'Enseignement Secondaire. Géometrie de l'Espace 1er Fascule. Kigali, October1988
6. Direction des Progammes de l'Enseignement Secondaire. Géometrie de l'Espace 2ème Fascule. Kigali, October1988
7. Frank Ebos, Dennis Hamaguchi, Barbana Morrison & John Klassen, Mathematics Principles & Process, Nelson Canada A Division of International Thomson Limited 1990
8. George B. Thomas, Maurice D. Weir & Joel R. Hass, Thomas' Calculus Twelfth Edition, Pearson Education, Inc. 2010

9. Geoff Mannall & Michael Kenwood, Pure Mathematics 2, Heinemann Educational Publishers 1995

10. H.K. DASS...Engineering Mathematics. New Delhi, S. CHAND&COMPANY LTD, thirteenth revised edition 2007.

11. Hubert Carnec, Genevieve Haye, Monique Nouet, ReneSeroux, Jacqueline Venard. Mathématiques TS Enseignement obligatoire. Bordas Paris 1994.

12. James T. McClave, P.George Benson. Statistics for Business and Economics. USA, Dellen Publishing Company, a division of Macmillan, Inc 1988.

13. J CRAWSHAW, J CHAMBERS: A concise course in A-Level statistics with worked examples, Stanley Thornes (Publishers) LTD, 1984.

14. Jean Paul Beltramonde, VincentBrun, ClaudeFelloneau, LydiaMisset, Claude Talamoni. Declic 1re S Mathématiques. Hachette-education, Paris 2005.

15. JF Talber & HH Heing, Additional Mathematics 6th Edition Pure & Applied, Pearson Education South Asia Pte Ltd 1995

16. J.K. Backhouse, SPTHouldsworth B.E.D. Copper and P.J.F. Horril. Pure Mathematics 2. Longman, third edition 1985, fifteenth impression 1998.

17. Mukasonga Solange. Mathématiques 12, AnalyseNumérique. KIE, Kigali 2006.

18. N. PISKOUNOV, Calcul Différential et Integral tom II 9ème édition. Editions MIR. Moscou, 1980.

19. Paule Faure- Benjamin Bouchon, Mathématiques Terminales F. Editions Nathan, Paris 1992.

20. Peter Smythe: Mathematics HL & SL with HL options, Revised Edition, Mathematics Publishing Pty. Limited, 2005.

21. Robert A. Adms & Christopher Essex, Calculus A complete course Seventh Edition, Pearson Canada Inc., Toronto, Ontario 2010

22. Seymour Lipschutz. Schaum's outline of Theory and Problems of Finite Mathematics. New York, Schaum Publisher, 1966

23. Seymour Lipschutz. Schaum's outline of Theory and Problems of linear algebra. McGraw-Hill 1968.

24. Shampiyona Aimable : Mathématiques 6. Kigali, Juin 2005.

25. Yves Noirot, Jean–Paul Parisot, Nathalie Brouillet. Cours de Physique Mathématiques pour la Physique. Paris, DUNOD, 1997.

26. Swokowski, E.W. (1994). Pre-calculus: Functions and graphs, Seventh edition. PWS Publishing Company, USA.

27. Allan G. B. (2007). Elementary statistics: a step by step approach, seventh

edition, *Von Hoffmann Press,* New York.

28. David R. (2000). Higher GCSE Mathematics, revision and Practice. Oxford University Press, UK.

29. Ngezahayo E.(2016). Subsidiary Mathematics for Rwanda secondary Schools, Learners' book 4, Fountain publishers, Kigali.

30.Ngezahayo E.(2016). Advanced Mathematics for Rwanda secondary Schools, Learners' book 4, Fountain publishers, Kigali.

31. Ngezahayo E.(2017). Subsidiary Mathematics for Rwanda secondary Schools, Learners' book 5, Fountain publishers, Kigali.

32. Ngezahayo E.(2017). Advanced Mathematics for Rwanda secondary Schools, Learners' book 5, Fountain publishers, Kigali.

33. Ngezahayo E.(2018). Advanced Mathematics for Rwanda secondary Schools, Learners' book 6, Fountain publishers, Kigali.

34. REB. (2015). Subsidiary Mathematics Syllabus, MINEDUC, Kigali, Rwanda.

35. REB. (2019). Mathematics Syllabus for TTC-Option of LE, MINEDUC, Kigali Rwanda.

36. Peter S. (2005). Mathematics HL&SL with HL options, Revised edition. Mathematics Publishing PTY. Limited.

37. Elliot M. (1998). Schaum's outline series of Calculus. MCGraw-Hill Companies, Inc. USA.

38. Frank E. et All. (1990). Mathematics. Nelson Canada, Scarborough, Ontario (Canada)

39. Gilbert J.C. et all. (2006). Glencoe Advanced mathematical concepts, MCGraw-Hill Companies, Inc. USA.

40. Robert A. A. (2006). Calculus, a complete course, sixth edition. Pearson Education Canada, Toronto, Ontario (Canada).

41. Sadler A. J & Thorning D.W. (1997). Understanding Pure mathematics, Oxford university press, UK.

42. J. CRAWSHAW and J. CHAMBERS 2001. A concise course in Advanced Level Statistics with worked examples 4th Edition. Nelson Thornes Ltd, UK.

43. Ron Larson and David C (2009). Falvo. Brief Calculus, An applied approach. Houghton Mifflin Company.

44. Michael Sullivan, 2012. Algebra and Trigonometry 9th Edition. Pearson Education, Inc

45. Swokowski & Cole. (1992). Preaclaculus, Functions and Graphs. Seventh edition.

46. Glencoe. (2006). Advanced mathematical concepts, Precalculus with Applications.

47. Seymour Lipschutz, PhD. & Marc Lipson, PhD. (2007). Discrete mathematics. 3rd edition.

48. K.A. Stroud. (2001). Engineering mathematics. 5th Edition. Industrial Press, Inc, New York

49. John bird. (2005). Basic engineering mathematics. 4th Edition. Linacre House, Jordan Hill, Oxford OX2 8DP