MATHEMATICS FOR TTCs

STUDENT'S BOOK

YEAR

2

OPTION:

Early Childhood and Lower Primary Education (ECLPE)

FOREWORD

Dear Student,

Rwanda Education Board (REB) is honored to present Year Two Mathematics book for Early Childhood and Lower Primary Education (ECLPE) Student Teachers. This book will serve as a guide to competence-based teaching and learning to ensure consistency and coherence in the learning of the Mathematics content. The Rwandan educational philosophy is to ensure that you achieve full potential at every level of education which will prepare you to be well integrated in society and exploit employment opportunities.

The government of Rwanda emphasizes the importance of aligning teaching and learning materials with the syllabus to facilitate your learning process. Many factors influence what you learn, how well you learn and the competences you acquire. Those factors include the relevance of the specific content, the quality of teachers' pedagogical approaches, the assessment strategies and the instructional materials available. In this book, we paid special attention to the activities that facilitate the learning process in which you can develop your ideas and make new discoveries during concrete activities carried out individually or with peers.

In competence-based curriculum, learning is considered as a process of active building and developing knowledge and meanings by the learner where concepts are mainly introduced by an activity, situation or scenario that helps the learner to construct knowledge, develop skills and acquire positive attitudes and values.

For efficiency use of this textbook, your role is to:

- Work on given activities which lead to the development of skills;
- Share relevant information with other learners through presentations, discussions, group work and other active learning techniques such as role play, case studies, investigation and research in the library, on internet or outside;
- Participate and take responsibility for your own learning;
- Draw conclusions based on the findings from the learning activities.

To facilitate you in doing activities, the content of this book is self explanatory so that you can easily use it yourself, acquire and assess your competences. The book is made of units as presented in the syllabus. Each unit has the following structure: the key unit competence is given and it is followed by the introductory activity before the development of mathematical concepts that are connected to real world problems or to other sciences.

The development of each concept has the following points:

- It starts by a learning activity: it is a hand on well set activity to be done by students in order to generate the concept to be learnt;
- Main elements of the content to be emphasized;
- Worked examples; and
- Application activities: those are activities to be done by the user to consolidate competences or to assess the achievement of objectives.

Even though the book has some worked examples, you will succeed on the application activities depending on your ways of reading, questioning, thinking and grappling ideas of calculus not by searching for similar-looking worked out examples.

Furthermore, to succeed in Mathematics, you are asked to keep trying; sometimes you will find concepts that need to be worked at before you completely understand. The only way to really grasp such a concept is to think about it and work related problems found in other reference books.

I wish to sincerely express my appreciation to the people who contributed towards the development of this book, particularly, REB staff, UR-CE Lecturers and TTC Tutors for their technical support. A word of gratitude goes to Head Teachers and TTCs principals who availed their staff for various activities.

Any comment or contribution would be welcome to the improvement of this text book for the next edition.

Dr. NDAYAMBAJE Irénée

Director General, REB

ACKNOWLEDGMENT

I wish to express my appreciation to the people who played a major role in the development of this Mathematics book for Year Two student teachers in Early Childhood and Lower Primary Education (ECLPE). It would not have been successful without active participation of different education stakeholders.

I owe gratitude to different universities and schools in Rwanda that allowed their staff to work with REB in the in-house textbooks production initiative.

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Finally, my word of gratitude goes to the Rwanda Education Board staffs who were involved in the whole process of in-house textbook Elaboration.

Joan MURUNGI

Head of CTLR Department

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UNIT 1

SEQUENCES AND SERIES

Key unit competence: Apply arithmetic and geometric sequences to solve problems in financial mathematics.

1.0 INTRODUCTORY ACTIVITY

Suppose that an insect population is growing in such a way that each new generation is 2 times as large as the previous generation. If there are 126 insects in the first generation, on a piece of paper, write down the number of insects that will be there in second, third, fourth,…n^{th} generation.

1.1 Generalities on sequences

ACTIVITY 1.1

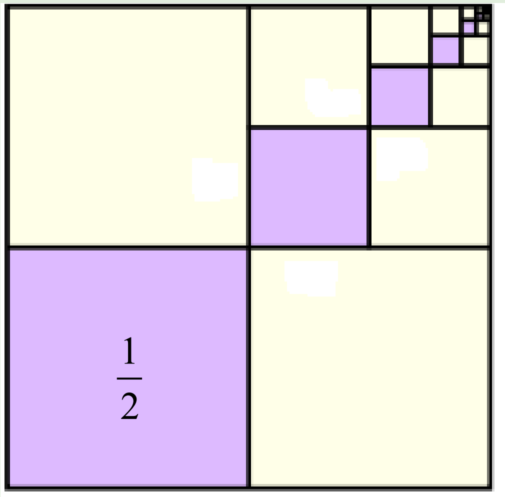
Fold once an A4 paper, what is the fraction that represents the part you are seeing?

Fold it twice, what is the fraction that represents the part you are seeing?

What is the fraction that represents the part you are seeing if you fold it ten times?

What is the fraction that represents the part you are seeing if you fold it *n* times?

Write a list of the fractions obtained starting from the first until the n[th] fraction.



CONTENT SUMMARY

Let us consider the following list of numbers: $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots \frac{1}{n}, \ldots$. The terms of this list are compared to the images of the function $f(x) = \frac{1}{x}$. The list never ends, as the ellipsis indicates. The numbers in this ordered list are called the **terms** of the sequence. In dealing with sequences, we usually use subscripted letters, such as u_1 to represent the first term, u_2 for the second term, u_3 for the third term, and so on such as in the sequence $f(n) = u_n = \frac{1}{n}$.

However, in the sequence such as $\{u_n\} : u_n = \sqrt{n-3}$, the first term is u_3 as the previous are not possible, in the sequence $\{u_n\} : u_n = 2n-5$, the first term is u_0.

Definition

A sequence is a function whose domain is the set of natural numbers.

The terms of a sequence are the range elements of the function.

It is denoted by $u_1, u_2, u_3, \ldots, u_{n-1}, u_n$ and shortly $\{u_n\}$. We can also write $\{u_1, u_2, u_3, \ldots, u_{n-1}, u_n\}$. The dots are used to suggest that the sequence continues indefinitely, following the obvious pattern.

The numbers $u_1, u_2, u_3, \ldots, u_{n-1}, u_n$ in a sequence are called **terms of the sequence**. The natural number *n* is called **term number** and value u_n is called a **general term** of a sequence and the term u_1 is the **initial term** or **the first term**.

As a sequence continues indefinitely, it can be denoted as $\{u_n\}_{n=1}^{\infty}$.

The number of terms of a sequence (possibly infinite) is called the **length of the sequence**.

Notice

- Sometimes, the term number, *n*, starts from 0. In this case terms of a sequence are $u_0, u_1, u_2, \ldots, u_{n-1}, u_n, \ldots$ and this sequence is denoted by $\{u_n\}_{n=0}^{+\infty}$. In this case the initial term is u_0.
- A sequence can be finite, like the sequence $2, 4, 8, 16, \ldots, 256$.

The empty sequence $\{\ \}$ is included in most notions of sequences, but may be excluded depending on the context.

Usually a numerical sequence is given by some formula $u_n = f(n)$, permitting to find any term of the sequence by its number *n*; this formula is called a **general term formula**.

A second way of defining a sequence is to assign a value to the first (or the first few) term(s) and specify the *n*th term by a formula or equation that involves one or more of the terms preceding it. Sequences defined this way are said to be defined **recursively,** and the rule or formula is called a **recursive formula.**

Example 1: The sequence $u_1 = 1, u_n = n.u_{n-1}$

Finite and infinite sequences:

Consider the sequence of odd numbers less than 11: This is 1, 3,5,7,9. This is a finite sequence as the list is limited and countable. However, the sequence made by all odd numbers is:

$1, 3, 5, 7, 9, \ldots 2n+1, \ldots$ This suggests the definition that an **infinite sequence** is a sequence whose terms are infinite and its domain is the set of positive integers.

Note that it is not always possible to give the numerical sequence by a general term formula; sometimes a sequence is given by description of its terms.

Examples 2:

1) Numerical sequences:

$0,\ 1,2,3,4,5,\ldots$ a sequence of natural numbers;

0,2,4,6,8,10..... a sequence of even numbers;

$1.4, 1.41, 1.414, 1.4142, \ldots$ a numerical sequence of approximate, defined more precisely by the values of $\sqrt{2}$.

For the last sequence it is impossible to give a general term formula, nevertheless this sequence is described completely.

2) List the first five term of the sequence $\{2^n\}_{n=1}^{+\infty}$

Solution

Here, we substitute $n = 1,2,3,4,5,\ldots$ into the formula 2^n . This gives

$2^1, 2^2, 2^3, 2^4, 2^5, \ldots$

Or, equivalently, $2, 4, 8, 16, 32, \ldots$

3) Express the following sequences in general notation

a) $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots$

b) $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots$

Solution

a) $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots$

Beginning by comparing terms and term numbers:

Term number	1	2	3	4	...
Term	$\frac{1}{2}$	$\frac{2}{3}$	$\frac{3}{4}$	$\frac{4}{5}$	...

In each term, the numerator is the same as the term number, and the denominator is one greater the term number.

Thus, the n^{th} term is $\frac{n}{n+1}$ and the sequence may be written as $\left\{\frac{n}{n+1}\right\}_{n=1}^{+\infty}$.

b) $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots$

Beginning by comparing terms and term numbers:

Term number	1	2	3	4	⋯
Term	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{16}$	⋯

Or

Term number	1	2	3	4	⋯
Term	$\frac{1}{2^1}$	$\frac{1}{2^2}$	$\frac{1}{2^3}$	$\frac{1}{2^4}$	⋯

In each term, the denominator is equal to 2 powers the term number. We observe that the n^{th} term is $\frac{1}{2^n}$ and the sequence may be written as $\left\{\frac{1}{2^n}\right\}_{n=1}^{+\infty}$.

4) A sequence is defined by

$$\{u_n\}: \begin{cases} u_0 = 1 \\ u_{n+1} = 3u_n + 2 \end{cases}$$

Determine u_1, u_2 *and* u_3

Solution

Since $u_0 = 1$ and $u_{n+1} = 3u_n + 2$, replace n by $0,1,2$ to obtain u_1, u_2, u_3 respectively.

$$n = 0, \quad u_{0+1} = u_1 = 3u_0 + 2 = 3\times 1 + 2 = 5 \quad \Rightarrow \quad n = 1, \quad u_{1+1} = u_2 = 3u_1 + 2 = 3\times 5 + 2 = 17$$

$$n = 2, \quad u_{2+1} = u_3 = 3u_2 + 2 = 3\times 17 + 2 = 53 \quad \Rightarrow \quad \text{Thus, } \begin{cases} u_1 = 5 \\ u_2 = 17 \\ u_3 = 53 \end{cases}$$

APPLICATION ACTIVITY 1.1

1. A sequence is given by $\{u_n\} : \begin{cases} u_0 = 1 \\ u_n = \dfrac{2n^2}{n^2+1} \end{cases}$

   Determine u_1, u_2 and u_3

2. List the first five term of the sequence $\left\{\sqrt{n+1} - \sqrt{n}\right\}_{n=1}^{+\infty}$

3. Express the following sequence in general notation

   1, 3, 5, 7, 9, 11, ...

1.2 Convergent or divergent sequences

ACTIVITY 1.2

Discuss the value of the value of each of the following sequences as n tends to $+\infty$ (plus infinity).

1. $\left\{\frac{3n^2-1}{n^2}\right\}$

2. $\left\{n^2\right\}$

CONTENT SUMMARY

A numerical sequence $\{u_n\}$ is said to be **convergent** if $\lim_{n\to\infty} u_n$ exist whereas if $\lim_{n\to\infty} u_n$ does not exist (or is infinity) the sequence is said to be **divergent.**

A number L *is called* **a limit** of a numerical sequence $\{u_n\}$ if $\lim_{n\to\infty} u_n = L$.

Or for the convergent sequence $\lim_{n\to\infty} u_n = L$ and for the divergent sequence $\lim_{n\to\infty} u_n = \infty$

Examples

1) Determine whether the sequence $\left\{\frac{n}{2n+1}\right\}_{n=1}^{+\infty}$ converges or diverges.

Solution

First we find the limit of this sequence as n tends to infinity

$$\lim_{n\to\infty}\frac{n}{2n+1} = \lim_{n\to\infty}\frac{n}{n\left(2+\frac{1}{n}\right)} = \frac{1}{2}$$

Thus, $\left\{\frac{n}{2n+1}\right\}_{n=1}^{+\infty}$ converges to $\frac{1}{2}$.

2) Determine whether the sequence $\{8-2n\}_{n=1}^{+\infty}$ converges or diverges.

Solution

First we find the limit of this sequence as n tends to infinity

$$\lim_{n\to\infty}(8-2n)=8-2(+\infty)=-\infty$$

Thus, $\{8-2n\}_{n=1}^{+\infty}$ **diverges.**

APPLICATION ACTIVITY 1.2

Which of the sequences converge, and which diverge? Find the limit of each convergent sequence.

1) $\left\{2+(0.1)^n\right\}$

2) $\left\{\dfrac{1-2n}{1+2n}\right\}$

3) $\left\{\dfrac{1-5n^4}{n^4+8n^3}\right\}$

4) $\left\{-1^n\right\}$

5) $\left\{\dfrac{2n}{\sqrt{3n+1}}\right\}$

6) $\dfrac{\sqrt{7}n^2+2}{n^3+8}$

1.3 Monotonic sequences

ACTIVITY 1.3

For each of the following sequences, state whether the terms are in ascending or descending order , both or neither order.

1) 1, 2, 3, 4, 5, 6, ...

2) $1,\frac{1}{2},\frac{1}{4},\frac{1}{8},\ldots$

3) 1, -1, 1, -1, 1, ...

4) 2, 2, 2, 2, 2, 2, ...

CONTENT SUMMARY

A sequence $\{u_n\}$ is said to be

- **Increasing** or in **ascending** order if $u_1 < u_2 < u_3 < \ldots < u_n < \ldots$
- **Non decreasing** if $u_1 \le u_2 \le u_3 \le \ldots \le u_n \le \ldots$
- **decreasing** or in **descending** order if $u_1 > u_2 > u_3 > \ldots > u_n > \ldots$
- **non increasing** $u_1 \ge u_2 \ge u_3 \ge \ldots \ge u_n \ge \ldots$

A sequence that is either non decreasing or non increasing is called **monotone**, and a sequence that is increasing or decreasing is called **strictly monotone**. Observe that a strictly monotone sequence is monotone, but not conversely.

In order, for a sequence to be **increasing**, all pairs of successive terms, u_n and u_{n+1}, must satisfy $u_n < u_{n+1}$, or equivalently, $u_n - u_{n+1} < 0$.

More generally, monotonic sequences can be classified as follows:

Difference between successive terms	Classification
$u_n - u_{n+1} < 0$	Increasing
$u_n - u_{n+1} > 0$	Decreasing
Otherwise	Non decreasing or Non increasing.

If the terms in the sequence are all positive, then we can divide both sides of the inequality $u_n < u_{n+1}$ by u_n to obtain $1 < \frac{u_{n+1}}{u_n}$ or equivalently $\frac{u_{n+1}}{u_n} > 1$

More, generally, monotonic sequences with positive terms can be classified as follows:

Difference between successive terms	Classification
$\frac{u_{n+1}}{u_n} > 1$	Increasing
$1 > \frac{u_{n+1}}{u_n}$	Decreasing
Otherwise	Non decreasing or Non increasing

Examples

1) Prove that the sequence $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots, \frac{n}{n+1}, \ldots$ is an increasing sequence.

Solution

Here, $u_n = \frac{n}{n+1}$ and $u_{n+1} = \frac{n+1}{n+2}$

Thus, for $n \geq 1$

$$u_n - u_{n+1} = \frac{n}{n+1} - \frac{n+1}{n+2}$$

$$= \frac{n^2 + 2n - n^2 - 2n - 1}{(n+1)(n+2)}$$

$$= -\frac{1}{(n+1)(n+2)} < 0$$

This proves that the given sequence is **increasing.**

Alternative method,

We can show that the sequence $\frac{1}{2},\frac{2}{3},\frac{3}{4},...,\frac{n}{n+1},...$ is increasing by examining the ratio of successive terms as follows

$$\frac{u_{n+1}}{u_n}=\frac{\frac{n+1}{n+2}}{\frac{n}{n+1}}=\frac{n+1}{n+2}\times\frac{n+1}{n}=\frac{n^2+2n+1}{n^2+2n}$$

Since the numerator exceeds the denominator, the ratio exceeds 1, that is

$\frac{u_{n+1}}{u_n}>1$ for $n\geq 1$. This proves that the sequence is increasing.

2) The sequence 4, 4, 4, 4, ... is both no decreasing no increasing.

3) The sequence -2, 2, -2, 2, -2, ... is not monotonic.

APPLICATION ACTIVITY 1.3

Which of the following sequences are in increasing, decreasing, non increasing, non decreasing, not monotonic

1. 1, 2, 3, ..., n, ...
2. $\left\{\frac{n}{n+1}\right\}$
3. $\left\{\frac{1}{2^n}\right\}$
4. 3, 3, 3, 3, ...
5. 1, -1, 1, -1, ...

1.4 Arithmetic sequence and its general term

ACTIVITY 1.4

1. In each of the following sequence, each term can be found by adding a constant number to the previous. Guess that constant number.

   a) Sequence $\{u_n\}$: 5,8,11,14,17,...

   b) Sequence $\{v_n\}$: 26,31,36, 41, 46,...

   c) Sequence $\{w_n\}$: 20,18,16,14,12, ...

2. In each of the following sequence, each term can be found by adding a constant number to the previous. Find u_0, u_1, u_2, u_3, u_4 and guess that constant number.

   d) $u_n = 3n + 2$

   e) $u_n = 16 - 6n$

CONTENT SUMMARY

Let u_1 be an initial term of a sequence. If we add d successively to the initial term to find other terms, the difference between successive terms of a sequence is always the same number and the sequence is called **arithmetic.**

This sequence has the following term u_1, $u_2 = u_1 + d$, $u_3 = u_2 + d = u_1 + 2d$, $u_4 = u_3 + d = u_1 + 3d$, ...$u_n = u_{n-1} + d = u_1 + (n-1)d$,...

An **arithmetic sequence** may be defined recursively as $u_n = u_1 + (n-1)d$ where $u_1 = a$ and d are real numbers. The number a is the first term, and the number d is called the **common difference.**

Example 1:

The following sequences are arithmetic sequences:

Sequence $\{u_n\}$: 5,8,11,14,17,...

Sequence $\{v_n\}$: $26, 31, 36, 41, 46, \ldots$

Sequence $\{w_n\}$: $20, 18, 16, 14, 12, \ldots$

Common difference

The fixed numbers that bind each sequence together are called the **common differences.** Sometimes mathematicians use the letter ***d*** when referring to these types of sequences.

d can be calculated by subtracting any two consecutive terms in an arithmetic sequence. That is $d = u_{n+1} - u_n$ or $d = u_n - u_{n-1}$.

Note:

If three consecutive terms are in arithmetic sequence, the double of the middle term is equal to the sum of extreme terms. That is for an arithmetic sequence u_{n-1}, u_n, u_{n+1}, we have $2u_n = u_{n-1} + u_{n+1}$.

Proof: If u_{n-1}, u_n, u_{n+1} form an arithmetic sequence, then

$u_{n+1} = u_n + d$ and $u_{n-1} = u_n - d$

Adding two equations, you get $u_{n+1} + u_{n-1} = 2u_n$

General term of an arithmetic sequence

The n^{th} term, u_n of an arithmetic sequence $\{u_n\}$ with common difference d and initial term u_1 is given by $u_n = u_1 + (n-1)d$, which is **the general term of an arithmetic sequence.** If the initial term is u_0 then the general term of an arithmetic sequence becomes $u_n = u_0 + nd$.

Generally, if u_p is any p^{th} term of a sequence then the n^{th} term is given by $u_n = u_p + (n-p)d$

Example 2:

1) $4, 6, 8$ are three consecutive terms of an arithmetic sequence because $2 \times 6 = 4 + 8 \Leftrightarrow 12 = 12$

It means that $u_{n-1} = 4$, $u_n = 6$, $u_{n+1} = 8$. Then $2u_n = u_{n+1} + u_{n-1}$

2) If the first and tenth terms of an arithmetic sequence are 3 and 30, respectively, find the fiftieth term of the sequence.

Solution

$u_1 = 3$ and $u_{10} = 30$

But $u_n = u_1 + (n-1)d$, $\Rightarrow$ $u_{10} = u_1 + (10-1)d$

Then $30 = 3 + (10-1)d \Leftrightarrow 30 = 3 + 9d \Rightarrow d = 3$

Now, $u_{50} = u_1 + (50-1)d = 3 + 49 \times 3 = 150$

Thus, the fiftieth term of the sequence is 150.

3) If the 3rd term and the 8th term of an arithmetic sequence are 5 and 15 respectively, find the common difference.

Solution

$u_3 = 5,\ u_8 = 15$

Using the general formula: $u_n = u_p + (n-p)d$

$u_3 = u_8 + (3-8)d$

$5 = 15 - 5d$

$\Leftrightarrow 5d = 15 - 5$

$\Rightarrow 5d = 10 \Rightarrow d = 2$

The common difference is 2.

Or starting by the highest term

$u_8 = u_3 + (8-3)\mathrm{d}$

$15 = 5 + 5d$

$10 = 5d$

$d = 2$

The common difference is 2

4) Consider the sequence $5, 8, 11, 14, 17, ..., 47$. Find the number of terms in this sequence

Solution

We see that $u_1 = 5$, $u_n = 47$ and $d = 3$.

And we know that $u_n = u_1 + (n-1)d$. This gives

$$47 = 5 + (n-1)3$$
$$\Leftrightarrow 42 = 3n - 3 \Rightarrow n = 15$$

This means that there are 15 terms in the sequence.

5) Consider the sequence which went from 20 to -26 with -2 as common difference. Find the number of terms.

Solution

We have

$$-26 = 20 + (n-1)(-2)$$
$$\Leftrightarrow -46 = -2n + 2 \Rightarrow n = 24$$

This means that there are 24 terms in the sequence.

6) Show that the following sequence is arithmetic. Find the first term and the common difference. $\{s_n\} = \{3n+5\}$

Solution:

The n^{th} term and the $(n-1)^{st}$ term of the sequence $\{s_n\}$ are $s_n = 3n+5$ *and* $s_{n-1} = 3(n-1)+5 = 3n+2$

Their difference d is $s_n - s_{n-1} = (3n+5)-(3n+2) = 3$.

The first term is $s_1 = 8$.

Since the difference of any two successive terms is the constant 3, the sequence $\{s_n\}$ is arithmetic and the common difference is 3.

APPLICATION ACTIVITY 1.4

1. If $\frac{1}{a+b}, \frac{1}{a+c}, \frac{1}{b+c}$ are 3 consecutive terms of an arithmetic progression, show that it will be the same for a^2, b^2, c^2.

2. Form an arithmetic progression of three numbers such that the sum of its terms is 30 and the sum of the squares of its terms is 332.

3. Calculate x so that the squares of $1+x,\ q+x,\ and\ q^2+x$ will be three consecutive terms of an arithmetic progression where q is any given number.

1.5. Arithmetic Means of an arithmetic sequence

ACTIVITY 1.5

Suppose that you need to form an arithmetic sequence of 7 terms such that the first term is 2 and the seventh term is 20. Write down that sequence given that those terms are $2, A, B, C, D, E, 20$.

CONTENT SUMMARY

If three or more than three numbers form an arithmetic sequence, then all terms lying between the first and the last numbers are called **arithmetic means.** If B is arithmetic mean between A and C, then $B=\frac{A+C}{2}$.

Let us see how to insert k terms between two terms u_1 and u_n to form an arithmetic sequence:

u_1	…	…	…	…	u_n

The terms to be inserted are called **arithmetic means** between two terms u_1 and u_n .

This requires to form an arithmetic sequence of $n=k+2$ terms whose the first term is u_1 and the last term is u_n. While there are several methods, we will use our n^{th} term formula $u_n = u_1 + (n-1)d$.

As u_1 and u_n are known, we need to find the common difference d taking $n=k+2$ where k is the number of terms to be inserted and 2 stands for the first and the last term.

Examples

1) Insert three arithmetic means between 7 and 23.

Solution

Here $k=3$ and then $n=k+2=5$, $u_1=7$ and $u_n=u_5=23$.

Then $$\begin{aligned} &u_5 = u_1 + (5-1)d \\ &\Leftrightarrow 23 = 7 + 4d \Rightarrow d = 4 \end{aligned}$$

Now, insert the terms using $d=4$, the sequence is $7,11,15,19,23$.

2) Insert five arithmetic means between 2 and 20.

Solution

Here $k=5$ and then $n=k+2=7$, $u_1=2$ and $u_n=u_7=20$.

Then $$\begin{aligned} &u_7 = u_1 + (7-1)d \\ &\Leftrightarrow 20 = 2 + 6d \Rightarrow d = 3 \end{aligned}$$

Now, insert the terms using $d=3$, the sequence is $2,5,8,11,14,17,20$.

APPLICATION ACTIVITY 1.5

1. Insert 4 arithmetic means between -3 and 7
2. Insert 9 arithmetic means between 2 and 32
3. Between 3 and 54, n terms have been inserted in such a way that the ratio of 8^{th} mean and $(n-2)^{th}$ mean is $\frac{3}{5}$. Find the value of n.
4. There are n arithmetic means between 3 and 54 terms such that 8^{th} mean is equal to $(n-1)^{th}$ mean as 5 to 9. Find the value of n.
5. Find the value of n so that $\frac{a^{n+1}+b^{n+1}}{a^n+b^n}$ will be the arithmetic mean between a and b.

1.6 Arithmetic series

ACTIVITY 1.6

Consider a finite arithmetic sequence $2,5,8,11,14,....$

a) What is its first term, the common difference d and the general term?

b) Determine the sum s_6 (in function of 6 and the first term 2) of the first 6 terms for this sequence taking that for each $u_n = u_1 + (n-1)d$ where for example $u_3 = u_1 + 2d$.

c) Try to generalize your results to determine the sum s_n for the first n terms of the arithmetic sequence $\{u_n\}$.

CONTENT SUMMARY

For finite arithmetic sequence $\{u_n\} = u_1, u_2, u_3, \ldots u_n$, the sum

$\sum_{r=1}^{n} u_r = u_1 + u_2 + u_3 + \ldots + u_n$ is called an **arithmetic serie.**

We denote the sum of the first n terms of the sequence by S_n.

Thus, $S_n = u_1 + u_2 + u_3 + \ldots + u_n = \sum_{r=1}^{n} u_r$

The sum of first n terms of a finite arithmetic sequence with initial term u_1 is given by

$$s_n = u_1 + u_2 + u_3 + \ldots + u_n$$
$$= u_1 + (u_1 + d) + (u_1 + 2d) + \ldots + (u_1 + (n-1)d)$$
$$= (\underbrace{u_1 + u_1 + \ldots u_1}_{n\ terms}) + (d + 2d + \ldots + (n-1)d)$$
$$= nu_1 + d[1 + 2 + 3 + \ldots + (n-1)] = nu_1 + d[\frac{n(n-1)}{2}]$$

$$s_n = nu_1 + \frac{n}{2}(n-1)d = \frac{n}{2}\left[2u_1 + (n-1)d\right]$$
$$= \frac{n}{2}\left[u_1 + u_1 + (n-1)d\right] = \frac{n}{2}(u_1 + u_n)$$

$$s_n = \frac{n}{2}(u_1 + u_n)$$

If the initial term is u_0, the formula becomes $S_n = \frac{n+1}{2}(u_0 + u_n)$

Example 1 : Calculate the sum of first 100 terms of the sequence $2, 4, 6, 8, \ldots$

Solution:

We see that the common difference is 2 and the initial term is $u_1 = 2$. We need to find $u_n = u_{100}$.

$$u_{100} = 2 + (100 - 1)2$$
$$= 2 + 198$$
$$= 200$$

Now,

$$S_{100} = \frac{100}{2}(u_1 + u_{100})$$
$$= 50(2 + 200)$$
$$= 10100$$

Example 2: Find the sum of first k even integers ($k \neq 0$).

Solution

$u_1 = 2$ and $d = 2$

$$u_n = u_k$$
$$u_k = 2 + (k-1)2$$
$$u_k = 2k$$

and

$$S_n = S_k$$
$$S_k = \frac{k}{2}(2 + 2k)$$
$$S_k = k(k+1)$$

Example 3: Find the sum: 60 + 64 + 68 + 72 + ... + 120

Solution:

This is the sum s_n of an arithmetic sequence u_n whose first term is $u_1 = 60$ and whose common difference is $d = 4$. The nth term is u_n.

We have $u_n = u_1 + (n-1)d$ and

$$120 = 60 + (n-1)4$$
$$60 = 4(n-1)$$
$$n = 16$$

Now, the sum is $u_{16} = 60 + 64 + 68 + \ldots + 120 = \frac{16}{2}(60 + 120) = 1440$.

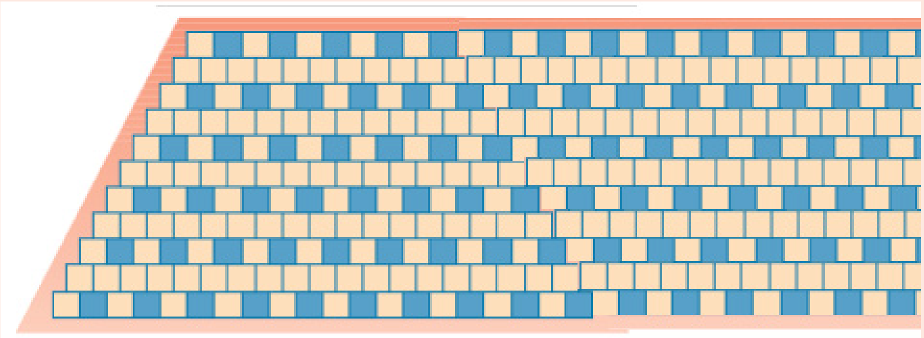
APPLICATION ACTIVITY 1.6

1) Consider the arithmetic sequence 8, 12, 16, 20, ... Find the expression for S_n

2) Find the sum of first twenty terms of the sequence 5, 9, 13, ...

3) The sum of the terms in the sequence 1, 8, 15, ... is 396. How many terms does the sequence contain?

4) **Practical activity:** A ceramic tile floor is designed in the shape of a trapezium 10m wide at the base and 5m wide at the top as illustrated on the figure bellow.



The tiles, 10cm by 10cm, are to be placed so that each successive row contains one less tile than the preceding row. How many tiles will be required?

1.7 Harmonic sequence and its general term

ACTIVITY 1.7

Consider the following arithmetic sequence: $2, 4, 6, 8, 10, 12, 14, 16, \ldots 2^n, \ldots$

a) Form another sequence whose terms are the reciprocals of the terms of the given sequence.

b) What can you say about the new sequence? What is its first term? The third term and the general term? Is there a relationship between two consecutive terms?

CONTENT SUMMARY

Harmonic sequence is a sequence of numbers in which the reciprocals of the terms are in arithmetic sequence. It is of the following form:

$$\frac{1}{u}, \frac{1}{u+d}, \frac{1}{u+2d}, \ldots \frac{1}{u+(n-1)d}, \ldots$$ where u is not zero, *n-1* is a natural number.

Example of harmonic sequence is $\frac{1}{3}, \frac{1}{6}, \frac{1}{9}, \ldots$

If you take the reciprocal of each term from the above harmonic sequence, the sequence will become 3, 6, 9, ... which is an arithmetic sequence with a common difference of 3.

Another example of harmonic sequence is 6, 3, 2. The reciprocals of each term are $\frac{1}{6}, \frac{1}{3}, \frac{1}{2}$ respectively which is an arithmetic sequence with a common difference of $\frac{1}{6}$.

Remark

To find the term of **harmonic sequence**, convert the sequence into arithmetic sequence then do the calculations using the arithmetic formulae. Then take the reciprocal of the answer in arithmetic sequence to get the correct term in harmonic sequence.

Example 1:

The 2^{nd} term of an harmonic progression is $\frac{1}{6}$ and 6^{th} term is $-\frac{1}{6}$. Find 20^{th} term and n^{th} term.

Solution

In harmonic progression, $h_2 = \frac{1}{6}$ and $h_6 = -\frac{1}{6}$.

Thus, in the corresponding arithmetic progression $a_2 = 6$ and $a_6 = -6$

Or $a_6 = a_2 + 4d \Rightarrow 6 + 4d = -6$ or $d = -3$ knowing that $u_n = u_p + (n-p)d$ for u_p any p^{th} term of the sequence.

Hence $a_{20} = 6+18(-3) = -48 \Rightarrow h_{20} = -\frac{1}{48}$

$$a_n = 6+(n-2)(-3) = 12-3n \Rightarrow h_n = \frac{1}{12-3n}$$

Notice: Harmonic Means

If three terms a,b,c are in harmonic progression, the middle one is said to be Harmonic mean between the other two and $b = \frac{2ac}{a+c}$.

This can be shown as follows: If a,b,c are in harmonic progression,

$$\frac{1}{b} = (\frac{1}{a}+\frac{1}{c}):2 \Leftrightarrow 2ac = bc + ba \Leftrightarrow 2ac = b(a+c) \Rightarrow \text{b} = \frac{2ac}{a+c}$$

Example 2:

Insert 4 harmonic means between $\frac{2}{3}$ and $\frac{6}{19}$

Solution

Let the four harmonic means be h_1, h_2, h_3, h_4.

Then $\frac{2}{3}, h_1, h_2, h_3, h_4, \frac{6}{19}$ are in harmonic progression

$\Rightarrow \frac{3}{2}, \frac{1}{h_1}, \frac{1}{h_2}, \frac{1}{h_3}, \frac{1}{h_4}, \frac{19}{6}$ are in arithmetic progression. where $a_1 = \frac{3}{2}$ and $a_6 = \frac{19}{6}$

$a_6 = \frac{19}{6} \Leftrightarrow a_1 + 5d = \frac{19}{6}$ with d common difference.

$$\Rightarrow \frac{3}{2} + 5d = \frac{19}{6} \Leftrightarrow 5d = \frac{19}{6} - \frac{3}{2} \Leftrightarrow 5d = \frac{10}{6} \Rightarrow d = \frac{1}{3}$$

$$\Rightarrow \begin{cases} \frac{1}{h_1} = \frac{3}{2} + \frac{1}{3} = \frac{11}{6} \equiv 1^{st} \text{ term of arithmetic progression} \\ \frac{1}{h_2} = \frac{3}{2} + \frac{2}{3} = \frac{13}{6} \equiv 2^{nd} \text{ term of arithmetic progression} \\ \frac{1}{h_3} = \frac{3}{2} + \frac{3}{3} = \frac{15}{6} = \frac{5}{2} \equiv 3^{rd} \text{ term of arithmetic progression} \\ \frac{1}{h_4} = \frac{3}{2} + \frac{4}{3} = \frac{17}{6} \equiv 4^{th} \text{ term of arithmetic progression} \end{cases}$$

The four harmonic means are $\frac{6}{11}, \frac{6}{13}, \frac{2}{5}, \frac{6}{17}$

Example 3:

Find the nth term of the series $2\frac{1}{2} + 1\frac{7}{13} + 1\frac{1}{9} + \frac{20}{23} + \cdots$

Solution

The given series is $\frac{5}{2} + \frac{20}{13} + \frac{10}{9} + \frac{20}{23}, \cdots$ n

The reciprocal of the terms are $\frac{2}{5}, \frac{13}{20}, \frac{9}{10}, \frac{23}{20}, \cdots$

There are in arithmetic progression, with the first term $\frac{2}{5}$ and the common difference $\frac{13}{20} - \frac{2}{5} = \frac{1}{4}$

The given series in arithmetic progression:

nth term of arithmetic progression: $a_n = \frac{2}{5} + (n-1)\frac{1}{4} = \frac{8+5n-5}{20} = \frac{5n+3}{20}$

Hence nth term of the given harmonic progression is $h_n = \frac{1}{a_n}$ or $h_n = \frac{20}{5n+3}$

The n^{th} term of the series $2\frac{1}{2} + 1\frac{7}{13} + 1\frac{1}{9} + \frac{20}{23} + \cdots$ is $\frac{20}{5n+3}$

APPLICATION ACTIVITY 1.7

1. Find the 4^{th} and 8^{th} term of the harmonic series $6,4,3,\ldots$
2. Insert two harmonic means between 3 and 10.
3. If a, b, c are in harmonic progression, show that $\frac{a}{b+c},\frac{b}{c+a},\frac{c}{a+b}$ are also in harmonic progression.
4. Find the term number of harmonic sequence $\frac{\sqrt{5}}{3},\frac{\sqrt{5}}{4},\frac{1}{\sqrt{5}},\ldots,\frac{\sqrt{5}}{13}$
5. The harmonic mean between two numbers is 3 and the arithmetic mean is 4. Find the numbers.
6. Find the n^{th} term of the series $4+4\frac{2}{7}+4\frac{8}{13}+5+\cdots$

1.8 Geometric sequence and its general term

ACTIVITY 1.8

Take a piece of paper whose shape is the square.

1. Cut it into two equal parts.
2. Write down a fraction corresponding to one part according to the original piece of paper
3. Take one part obtained in step 2) and cut it by repeating step 1) and then step 2)
4. Continue until you remain with a small piece of paper that you are not able to cut it into two equal parts and write down the sequence of fractions obtained.
5. Observe the sequence of numbers you obtained and give the relationship between any two consecutive numbers.

CONTENT SUMMARY

Sequences of numbers that follow a pattern of multiplying a fixed number ***r*** from one term u_1 to the next are called **geometric sequences.**

The following sequences are geometric sequences:

Sequence $\{u_n\}$: $5,10,20,40,80,\ldots$

Sequence $\{v_n\}$: $2,1,\frac{1}{2},\frac{1}{4},\frac{1}{8},\ldots$

Sequence $\{w_n\}$: $1,-2,4,-8,16,\ldots$

Common ratio

We can examine these sequences to greater depth, we must know that the fixed numbers that bind each sequence together are called the **common ratios**, denoted by the letter ***r***. This means if u_1 is the first term, $u_2 = ru_1;\quad u_3 = r^2u_1; u_4 = r^3u_1;\ldots u_n = r^{n-1}u_1;\ldots$

The n^{th} term or the general term of a geometric sequence becomes $u_n = r^{n-1}u_1$. The common ratio r can be calculated by dividing any two consecutive terms in a geometric sequence. That is

$r = \frac{u_{n+1}}{u_n}$ or $r = \frac{u_n}{u_{n-1}}$ or $u_n = ru_{n-1}$.

Generally,

If u_p is the p^{th} term of the sequence, then the n^{th} term is given by $u_n = u_p r^{n-p}$.

Note that if three terms are consecutive terms of a geometric sequence, the square of the middle term is equal to the product of extreme terms. That is for a geometric sequence u_{n-1}, u_n, u_{n+1}, we have $u_n^2 = u_{n-1}\cdot u_{n+1}$.

Example 1: $6,12,24$ are consecutive terms of a geometric sequence because

$(12)^2 = 6\times 24 \Leftrightarrow 144 = 144$. Find b such that $8,b,18$ will be in geometric sequence.

Solution

$b^2 = 8 \times 18 = 144 \quad \Rightarrow \quad b = \pm\sqrt{144} = \pm 12$

Thus, $8, 12, 18$ or $8, -12, 18$ are in geometric sequence.

Example 2: The product of three consecutive numbers in geometric progression is 27. The sum of the first two and nine times the third is -79. Find the numbers.

Solution

Let the three terms be $\frac{x}{a}, x, ax$.

The product of the numbers is 27. So $\frac{x}{a} xax = 27 \Rightarrow x^3 = 27 \Rightarrow x = 3$

The sum of the first two and nine times the third is -79:

$$\frac{x}{a} + x + 9ax = -79 \Rightarrow \frac{3}{a} + 3 + 27a = -79 \quad \Rightarrow$$

$$27a^2 + 82a + 3 = 0 \Rightarrow a = -3 \ \ or \ \ a = -\frac{1}{27}$$

The numbers are: -1, 3, -9 or -81, 3, $-\frac{1}{9}$.

Example 3: If the first and tenth terms of a geometric sequence are 1 and 4, respectively, find the nineteenth term.

Solution

$u_1 = 1$ and $u_{10} = 4$

But $u_n = u_1 r^{n-1}$, then $4 = 1r^9 \Leftrightarrow r = \sqrt[9]{4} \ \ or \ \ r = 4^{\frac{1}{9}}$

Now,

$$u_{19} = u_1 r^{19-1}$$
$$= 1\left(4^{\frac{1}{9}}\right)^{18}$$
$$= 16$$

Thus, the nineteenth term of the sequence is 16.

Example 4: If the 2^{nd} term and the 9^{th} term of a geometric sequence are 2 and $-\frac{1}{64}$ respectively, find the common ratio.

Solution

$u_2 = 2,\ u_9 = -\frac{1}{64}$

Using the general formula: $u_n = u_p\, r^{n-p} \quad \Rightarrow \quad u_2 = u_9\, r^{2-9}$

$$2 = -\frac{1}{64} r^{-7}$$
$$\Leftrightarrow 128 = -\frac{1}{r^7}$$
$$\Leftrightarrow r^7 = -\frac{1}{128}$$
$$\Leftrightarrow r = \sqrt[7]{-\frac{1}{128}} \Rightarrow r = -\frac{1}{2}$$

The common ratio is $r = -\frac{1}{2}$.

Example 5: Find the number of terms in sequence $2, 4, 8, 16, \ldots, 256$.

Solution

This sequence is geometric with common ratio 2, $u_1 = 2$ and $u_n = 256$

But $u_n = u_1 r^{n-1}$, then $256 = 2 \times 2^{n-1} \Leftrightarrow 256 = 2^n$ or $2^8 = 2^n \Rightarrow n = 8$.

Thus, the number of terms in the given sequence is 8.

APPLICATION ACTIVITY 1.8

1. If the second and fifth terms of a geometric sequence are 6 and -48, respectively, find the sixteenth term.

2. If the third term and the 8th term of a geometric sequence are $\frac{1}{2}$ and $\frac{1}{128}$ respectively, find the common ratio.

3. The 4th term of a geometric sequence is square of its 2nd term, and the first term is -3. Determine its 7th term.

4. Find the fourth term from the end of geometric sequence $8, 4, 2, \cdots, \frac{1}{128}$

5. The fifth term of a geometric sequence is $\frac{81}{32}$ and the ratio of 3rd and 4th is $\frac{2}{3}$, write the geometric sequence and its 8th term.

6. If p^{th} terms of two sequences $5, 10, 20, \cdots$ and $1280, 640, 320, \cdots$, are equal, find the value of p.

1.9 Geometric Means

ACTIVITY 1.9

Suppose that you need to form a geometric sequence of 6 terms such that the first term is 1 and the sixth term is 243. Given that these terms are $1, A, b, C, D, 243$. Write down that sequence.

CONTENT SUMMARY

To insert k terms called **geometric means** between two terms u_1 and u_n is to form a geometric sequence of $n = k + 2$ terms whose the first term is u_1 and the last term is u_n.

While there are several methods, we will use our n$^{\text{th}}$ term formula $u_n = u_1 r^{n-1}$.

As u_1 and u_n are known, we need to find the common ratio r taking $n = k + 2$ where k is the number of terms to be inserted.

Example 1: Insert three geometric means between 3 and 48.

Solution

Here $k = 3$, then $n = 5, u_1 = 3$ and $u_n = u_5 = 48$

$u_5 = u_1 r^{n-1} \Leftrightarrow 48 = 3r^4 \Rightarrow r = 2$

Inserting three terms using common ratio $r = 2$ gives $3, 6, 12, 24, 48$

Example 2: Insert 6 geometric means between 1 and $-\frac{1}{128}$.

Solution

Here $k = 6$, then $n = 8, u_1 = 1$ and $u_n = u_8 = -\frac{1}{128}$

$$u_8 = u_1 r^{n-1} \quad \Leftrightarrow -\frac{1}{128} = 1r^7 \quad \Leftrightarrow r^7 = -\frac{1}{128} \quad \Leftrightarrow r^7 = -\frac{1}{(2)^7}$$

$$\Leftrightarrow r = \left[-\frac{1}{(2)^7}\right]^{\frac{1}{7}} = -\frac{1}{2}$$

Inserting 6 terms using common ratio $r = -\frac{1}{2}$ gives

$1, -\frac{1}{2}, \frac{1}{4}, -\frac{1}{8}, \frac{1}{16}, -\frac{1}{32}, \frac{1}{64}, -\frac{1}{128}$.

APPLICATION ACTIVITY 1.9

1. Insert 5 geometric means between $\frac{1}{4}$ and $\frac{1}{256}$.
2. Insert 5 geometric means between 2 and $\frac{2}{729}$
3. Find the geometric mean between
   a) 2 and 98
   b) $\frac{3}{2}$ and $\frac{27}{2}$
4. Suppose that $4, 36, 324$ are in geometric progression. Insert two more numbers in this sequence so that it again forms a geometric sequence.
5. The arithmetic mean of two numbers is 34 and their geometric mean is 16. Find the numbers.

1.10 Geometric series

ACTIVITY 1.10

During a competition of student teachers at the district level, 5 first winners were paid an amount of money in the way that the first got $100,000$Frw, the second earned the half of this money, the third got the half of the second's money, and so on until the fifth who got the half of the fourth's money.

a) Discuss and calculate the money earned by each student from the second to the fifth.

b) Determine the total amount of money for all the 5 student teachers.

c) Compare the money for the first and the fifth student and discuss the importance of winning at the best place.

d) Try to generalize the situation and guess the money for the student who passed at the n^{th} place if more students were paid. In this case, evaluate the total amount of money for n students.

CONTENT SUMMARY

A **Geometric series** is an infinite sum $\sum_{n=1}^{+\infty} u_n = u_1 + u_1 r + u_1 r^2 + \ldots + u_1 r^{n-1} + \ldots$ of the terms a geometric sequence.

If we have **a finite geometric sequence** $\{u_n\} = u_1, u_2, u_3, \ldots u_n$, the sum is $S_n = \sum_{n=1}^{n} u_n = u_1 + u_1 r + u_1 r^2 + \ldots + u_1 r^{n-1}$.

The sum of first *n* terms of a geometric sequence

The sum S_n of the first n terms of a geometric sequence $\{u_n\} = \{u_1 r^{n-1}\}$ is

$S_n = u_1 + r u_1 + \ldots + r^{n-1} u_1$ (1)

Multiply each side by r to obtain $rS_n = r u_1 + r^2 u_1 + \ldots + r^n u_1$ (2)

Subtracting equation (2) from equation (1) we obtain

$$S_n - rS_n = u_1 - u_1 r^n$$
$$(1-r) S_n = u_1 (1 - r^n)$$

Since $r \neq 1$, we can solve for S_n and find $S_n = u_1 \frac{(1-r^n)}{(1-r)}$

If the initial term u_1 and common ratio r is given, the sum $s_n = \frac{u_1(1-r^n)}{1-r}$ with $r \neq 1$

If the initial term is u_0, then the formula is $s_n = \frac{u_0(1-r^{n+1})}{1-r}$ with $r \neq 1$

If $r = 1$, $s_n = nu_1$.

Also, the product of first n terms of a geometric sequence with initial term u_1 and common ratio r is given by $P_n = (u_1)^n r^{\frac{n(n-1)}{2}}$

If the initial term is u_0 then $P_n = (u_0)^{n+1} r^{\frac{n}{2}(n+1)}$

Example 1: Find the sum of the first 20 terms of a geometric sequence if the first term is 1 and common ratio is 2.

Solution

Here $u_1 = 1, r = 2, n = 20$

Then, $s_{20} = \dfrac{1\left(1-2^{20}\right)}{1-2} = \dfrac{1-2^{20}}{-1} = 1048575$

Example 2: Consider the sequence $\{u_n\}$ defined by $u_0 = 0$ and $u_{n+1} = u_n + \dfrac{1}{2^n}$.
Consider another sequence $\{v_n\}$ defined by $v_n = u_{n+1} - u_n$.

a) Show that $\{v_n\}$ is a geometric sequence and find its first term and common ratio.

b) Express $\{v_n\}$ in term of n.

Solution

a) $u_0 = 0, \quad v_0 = u_1 - u_0 = 1$ $\qquad u_1 = u_0 + \dfrac{1}{2^0} = 1, \quad u_2 = u_1 + \dfrac{1}{2^1} = \dfrac{3}{2};$

$v_1 = u_2 - u_1 = \dfrac{1}{2}, \quad v_2 = u_3 - u_2 = \dfrac{1}{4} \Rightarrow \{v_n\}$ is a geometric sequence if $v_1^2 = v_0 \cdot v_2$.

$v_1^2 = \dfrac{1}{4}$ and $v_0 \cdot v_2 = \dfrac{1}{4}$. Thus, $\{v_n\}$ is a geometric sequence.

First term is $v_0 = 1$ and the common ratio is $r = \dfrac{v_1}{v_0} = \dfrac{1}{2}$

b) General term

$$v_n = v_0 r^n$$
$$= \frac{1}{2^n}$$

Thus, $\{v_n\}$ is defined by $v_n = \dfrac{1}{2^n}$

Example 3: Find the product of the first 20 terms of a geometric sequence if the first term is 1 and common ratio is 2.

Solution:

$P_n = (u_1)^n r^{\frac{n(n-1)}{2}}$ $\Rightarrow$ here $u_1 = 1, r = 2, n = 20$,

Then, $P_{20} = (1)^{20} 2^{\frac{20(19)}{2}} = 2^{190}$

APPLICATION ACTIVITY 1.10

1. Find the sum of the first 8 terms of the geometric sequence 32, -16, 8, ...
2. Find the sum of the geometric sequence with the first term 0.99 and the common ration is equal to the first term.
3. Find the first term and the common ratio of the geometric sequence for which $S_n = \frac{5^n - 4^n}{4^{n-1}}$
4. Find the product of the first 10 terms of the sequence in question 1.
5. Aloys wants to begin saving money for school. He decides to deposit $500 at the beginning of each quarter (January1, April 1, July1, and October1) in a saving account which pays an annual percentage of 6% compound quarterly. The interest for each quarter is posted on the last day of the quarter. Determine Aloys's account balance at the end of one year.

1.11 Infinity geometric series and its convergence

ACTIVITY 1.11

Given that the sum of n terms of a geometric series $\{u_n\} = \left\{5\left(\frac{1}{2}\right)^{n-1}\right\}$ is given by:

$$S_n = 5\frac{\left(1-\left(\frac{1}{2}\right)^n\right)}{\left(1-\left(\frac{1}{2}\right)\right)},$$

a) Evaluate $\lim_{n\to\infty} S_n = \lim_{n\to\infty}\left[5\frac{\left(1-\left(\frac{1}{2}\right)^n\right)}{\left(1-\left(\frac{1}{2}\right)\right)}\right]$

b) Extend your results considering the infinite geometric series

$\sum_{n=1}^{\infty} u_1 r^{n-1}$ and determine $\lim_{n\to\infty} S_n = \lim_{n\to\infty} u_1 \frac{(1-r^n)}{(1-r)}, \quad -1<r<1.$

CONTENT SUMMARY

The sum of a geometric series with first term u_1 and the common ratio r is the limit of this geometric sequence as n approaches ∞.

As $S_n = u_1\frac{(1-r^n)}{(1-r)}$, this limit is $\lim_{n\to\infty} u_1\frac{(1-r^n)}{(1-r)} = \lim_{n\to\infty} u_1\frac{1}{1-r} - \lim_{n\to\infty} u_1\frac{r^n}{1-r}$.

This limit is a real number if $|r|<1$ as in this case $|r^n|$ approaches o when $n\to\infty$

. Therefore, $\lim_{n\to\infty} u_1\frac{(1-r^n)}{(1-r)} = \lim_{n\to\infty} u_1\frac{1}{1-r} - \lim_{n\to\infty} u_1\frac{r^n}{1-r} = u_1\frac{1}{1-r} - 0$

Therefore, the sum of our geometric sequence becomes:

$S_\infty = \frac{u_1}{1-r}$ provided $-1<r<1$

As a conclusion, if $|r|<1$, the infinite geometric series $\sum_{n=1}^{\infty} u_1 r^{n-1}$ converges. Its sum is $\sum_{n=1}^{\infty} u_1 r^{n-1} = \frac{u_1}{1-r}$.

Examples 1: Given the geometric progression 16, 12, 9, ... Find the sum of terms up to infinity.

Solution:

Here $u_1 = 16, r = \frac{12}{16} = \frac{3}{4}$

Thus $-1 < r < 1$ and hence the sum to infinity will exists

$$S_\infty = \frac{u_1}{1-r} = \frac{16}{1-\frac{3}{4}} = 64$$

The sum to infinity is 64.

Examples 2: Express the recurring decimal $0.\overline{32}$ as a rational number.

Solution:

$0.\overline{32} = \frac{32}{10^2} + \frac{32}{10^4} + \frac{32}{10^6} + \ldots$ which is an infinite geometric series with first term $u_1 = 0.32$ and common ratio is $r = 0.01$.

Since $-1 < r < 1$, the sum to infinity exist and equal to $\frac{u_1}{1-r} = \frac{0.32}{1-0.01} = \frac{0.32}{0.99} = \frac{32}{99}$

Therefore, $0.\overline{32} = \frac{32}{99}$.

APPLICATION ACTIVITY 1.11

1) Consider the infinite geometric series $\sum_{n=1}^{\infty} 10\left(1-\frac{3x}{2}\right)^{n}$

   a. For what values of x does a sum to infinity exist?

   b. Find the sum of the series if $x = 1.3$

2) Show that the repeating decimal 0.9999.... equals 1.

3) Evaluate if the geometric series $\sum_{n=1}^{\infty} 2\left(\frac{2}{3}\right)^{n-1} = 2+\frac{4}{3}+\frac{8}{9}+...$ converges or diverges. If it converges, find its sum.

1.12 Application of sequences in real life

ACTIVITY 1.12

Carry out a research in the library or on internet and find out at least 3 problems or scenarios of the real life where sequences and series are applied.

CONTENT SUMMARY

There are many applications of sequences. Sequences are useful in our daily lives as well as in higher mathematics. For example; the monthly payments made to pay off an automobile or home loan with interest portion, the list of maximum daily temperatures in one area for a month are sequences. Sequences are used in calculating interest, population growth, half-life and decay in radioactivity ...

In economics and Finance, sequences and series can be used for example in solving problems related to:

a) **Final sum, the initial sum, the time period and the interest rate for an investment.**

The amount A after t years due to a principal P invested at an annual interest rate r compounded n times per year is $A = P.\left(1+\frac{r}{n}\right)^{n.t}$.

(i) If the compounding takes place Annually, n=1 and $A = P.\left(1+\frac{r}{1}\right)^{t}$

(ii) If the compounding takes place Monthly, n=12 and $A = P.\left(1+\frac{r}{12}\right)^{12t}$

(iii) If the compounding takes place Daily, n=365 and $A = P.\left(1+\frac{r}{365}\right)^{365t}$

In each case, the interest due is $A-P$.

Note that: From the compound interest formula $A = P\left(1+\frac{r}{n}\right)^{nt}$ If we let $k=\frac{n}{r}$,then $n = kr$ and $nt = krt$, and we may write the formula as

$$A = P\left(1+\frac{1}{k}\right)^{krt} = P\left[\left(1+\frac{1}{k}\right)^{k}\right]^{rt}.$$

For **continuously compound interest**, we may let n (the number of interest period per year) increase without bound towards infinity ($n \to \infty$), equivalently, by $k \to \infty$.

Using the definition of e, we see that $P\left[\left(1+\frac{1}{k}\right)^{k}\right]^{rt} \to P[e]^{rt}$ as $k \to \infty$. This result gives us the following formula:

$A = Pe^{rt}$ Where ***P*=Principal or initial value at *t* = 0**; r is the Interest rate expressed as a decimal; r is the number of years P is invested; A is the amount after t years.

The amount A after t years due to a principal P invested at an annual interest rate r compounded continuously is $A = P.e^{rt}$.

b) **Annual Equivalent Rate (AER) for part year investments and the nominal annual rate of return**.

For loan repayments the annual equivalent rate is usually referred to as the annual percentage rate (APR). If you take out a bank loan you will usually be quoted an APR even though you will be asked to make monthly repayments.

The **corresponding AER for any given monthly rate of interest** i_m can be found using the formula $AER = (i+i_m)^{12} - 1$.

The APR on loans is the same thing as the annual equivalent rate and so the same formula applies. The relationship between the daily interest rate i_d on a deposit account and the AER can be formulated as $AER = (i + i_d)^{365} - 1$.

c) **The Present value of investment**

The present value P of A money to be received after t years, assuming a per annum interest rate r compounded n times per year, is $P = A.\left(1+\frac{r}{n}\right)^{-n.t}$. If the interest is compounded continuously, $P = A.e^{-n.t}$

d) **Monthly repayments and the Annual Percentage Rate (APR) for a loans**

Suppose P is the deposit made at the end of each payment period for an annuity paying an interest rate of i per payment period. The amount A of the annuity after n deposits is

$$A = P.\left[\frac{(1+i)^n - 1}{i}\right]$$

Examples

1) A child building a tower with blocks uses 15 for the bottom row. Each row has 2 fewer blocks than the previous row. Suppose that there are 8 rows in the tower.

a) How many blocks are used for the top row?

b) What is the total number of blocks in the tower?

Solution

a) The number of blocks in each row forms an arithmetic sequence with $u_1 = 15$ and $d = -2$

$$n = 8,\ u_8 = u_1 + (8-1)(-2) \Rightarrow u_8 = 15+7(-2)$$
$$= 15-14$$
$$= 1$$

There is just one block in the top row due to $u_8 = 1$

b) Here we must find the sum of the terms of the arithmetic sequence formed with $u_1 = 15, n = 8, u_8 = 1 \Rightarrow S_8 = \frac{8}{2}(15+1) = 64$ Thus, there are 64 blocks in the tower.

2) An insect population is growing in such a way that each new generation is 1.5 times as large as the previous generation. Suppose there are 100 insects in the first generation.

a) How many will there be in the fifth generation?

b) What will be the total number of insects in the five generations?

Solution

a) The population can be written as a geometric sequence with $u_1 = 100$ as the first-generation population and common ratio $r = 1.5$. Then the fifth-generation population will be $u_5 = 100(1.5)^{5-1} = 506.25$. In the fifth-generation, the population will number about 506 insects.

b) The sum of the first five terms using the formula for the sum of the first n terms of a geometric sequence.

$$S_5 = \frac{100\left(1-(1.5)^5\right)}{1-1.5} = 1318.75$$

The total population for the five generations will be about 1319 insects.

3) Find the accumulated value of \$15,000 at 5% per year for 18 years using simple interest.

Solution

$P = 15000,\ r = 0.05, t = 18$

$$I = 15000(0.05)(18)$$
$$= 13500$$

A total of \$13,500 in interest will be earned.

Hence, the accumulated value in the account will be \$13,500+\$15,000= \$28,500.

4) Suppose 20,000Frw is deposit in a bank account that pays interest at a rate of 8% per year compound continuously. Determine the balance in the account in 5 years.

Solution:

Applying the formula for Continuously compound interest with $P = 20,000;\ r = 0.08;\ and\ t = 5$, we have

$A = Pe^{rt} = 20,000e^{0.08(5)} = 20,000\,\mathrm{e}^{0.4} = 29,836.49 Frw$.

5) Find the amount of an annuity after 5 deposits if a deposit of \$100 is made

each year, at 4% compounded annually. How much interest is earned?

Solution:

The deposit is $P=\$100$. The number of deposits is $n=5$ and the interest per payment period is $i=0.04$. Using Formula, the amount A after 5 deposits is

$$A=P.\left[\frac{(1+i)^n-1}{i}\right]=100\left[\frac{\left((1+0.04)^5-1\right)}{0.04}\right]=541.63$$

The interest earned is the amount after 5 deposits less the 5 annual payments of $100 each:

Interest earned $=A-500=543.63-500=41.63$. It is $41.63.

6) Mary decides to put aside $100 every month in a credit union that pays 5% compounded monthly. After making 8 deposits, how much money does Mary have?

Solution:

This is an annuity with $P=\$100, n=8$ deposits, and interest $i=\dfrac{0.05}{12}$ per payment period. Using the formula, the amount A after 8 deposits is

$$A=P.\left[\frac{(1+i)^n-1}{i}\right]=100\left[\frac{\left(\left(1+\frac{0.05}{12}\right)^5-1\right)}{\frac{0.05}{12}}\right]=811.76$$

Mary has $811.76 after making 8 deposits.

7) To save for her daughter's education, Martha decides to put $50 aside every month in a bank guaranteed-interest account paying 4% interest compounded monthly.

She begins this savings program when her daughter is 3 years old. How much will she have saved by the time she makes the 180th deposit? How old is her daughter at this time?

Solution:

This is an annuity with $P=\$50$, $\text{n}=180$ deposits, and $i=\dfrac{0.04}{12}$. The amount A saved is

$$A = P.\left[\frac{(1+i)^n - 1}{i}\right] = 50\left[\frac{\left(\left(1+\frac{0.04}{12}\right)^{180} - 1\right)}{\frac{0.04}{12}}\right] = 12,304.52$$

Since there are 12 deposits per year, when the 180th deposit is made $\frac{180}{12} = 15$ have passed and Martha's daughter is 18 years old.

APPLICATION ACTIVITY 1.12

1. If Linda deposits \$1300 in a bank at 7% interest compounded annually, how much will be in the bank 17 years later?
2. The population of a city in 1970 was 153,800. Assuming that the population increases continuously at a rate of 5% per year, predict the population of the city in the year 2000.
3. To save for retirement, Manasseh, at age 35, decides to place 2000Frw into an Individual Retirement Account (IRA) each year for the next 30 years. What will the value of the IRA be when Joe makes his 30th deposit? Assume that the rate of return of the IRA is 4% per annum compounded annually.
4. A private school leader received permission to issue 4,000,000Frw in bonds to build a new high school. The leader is required to make payments every 6 months into a sinking fund paying 4% compounded semiannually. At the end of 12 years the bond obligation will be retired. What should each payment be?

1.13 END UNIT ASSESSMENT 1

1) Find first four terms of the sequence

a) $\left\{\frac{1-n}{n^2}\right\}$ b) $\left\{\frac{(-1)^{n+1}}{2n-1}\right\}$ c) $\left\{2+(-1)^n\right\}$

2) Find the formula for the n^{th} term of the sequence

a) 1, -1, 1, -1, 1, ...

b) 0, 3, 8, 15, 24, ...

c) 1, 5, 9, 13, 17, ...

3) Which of the following sequences converge, and which diverge? Find the limit of each convergent sequence.

a) $\left\{\sqrt{\dfrac{2n}{n+1}}\right\}$ b) $\left\{\dfrac{n}{2^n}\right\}$ c) $\left\{8^{\frac{1}{n}}\right\}$

4) A mathematical child negotiates a new pocket money deal with her unsuspecting father in which she receives 1 pound on the first day of the month, 2 pounds on the second day, 4 pounds on the third day, 8 pounds on the fourth day, 16 pounds on the fifth day, ... until the end of the month. How much would the child receive during the course of a month of 30 days? (Give your answer to the nearest million pounds).

5) A culture of bacteria doubles every 2 hours. If there are 500 bacteria at the beginning, how many bacteria will there be after 24 hours?

6) You complain that the hot tub in your hotel suite is not hot enough. The hotel tells you that they will increase the temperature by 10% each hour. If the current temperature of the hot tub is 75º F, what will be the temperature of the hot tub after 3 hours, to the *nearest tenth* of a degree?

7) The sum of the interior angles of a triangle is 180º, of a quadrilateral is 360º and of a pentagon is 540º. Assuming this pattern continues, find the sum of the interior angles of a dodecagon (12 sides).

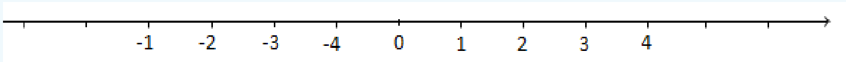
UNIT 2

POINTS, STRAIGHT LINES AND CIRCLES IN 2 DIMENSIONS

Key Unit competence: Determine equations of lines and circles.

2.0 INTRODUCTORY ACTIVITY 2

RUKUNDO uses a number line to graph the points -4, -2, 3, and 4. Her classmate ISIMBI, notices that -4 is closer to zero than -2 as shown in the figure below. She told him that he is mistaken. Rukundo replied that his is not sure about his diagram because he did not have time to repeat his course yesterday evening.



a) Use what you know about a vertical number line to determine if RUKUNDO made a mistake or not. Support your explanation with a number line diagram

b) What is a number line

c) What is Cartesian plane?

2.1. Cartesian coordinates of a point

ACTIVITY 2.1

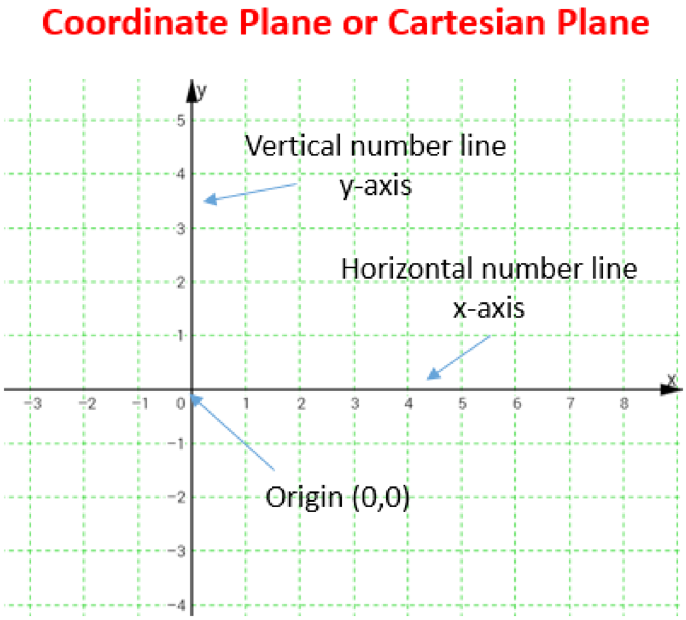
Consider the points $A(1,2)$, $B(5,4)$

1. Represents these points in xy plane
2. Draw a line segment from A to B

CONTENT SUMMARY

A point is represented by Cartesian coordinates (also called rectangular coordinates). In two dimensions, Cartesian coordinates are a pair of numbers that specify signed distances from the coordinates axes. They are specified in terms of the x coordinates and the y coordinates. The origin is the intersection of the two axes.

The position of a point on the Cartesian plane is represented by a pair of numbers. The pair is called an ordered pair or coordinates (x, y). The first number, x, is called the x-coordinate and the second number, y, is called the y-coordinate.



The origin is indicated by the ordered pair or coordinates (0, 0).

To get to the point (x, y) in cartesian plane, we start from the origin. If x is positive then we move x units right from the origin otherwise if x is negative then we move x units left from the origin. Then, if y is positive, we move y units up otherwise if y is negative, we move y units down

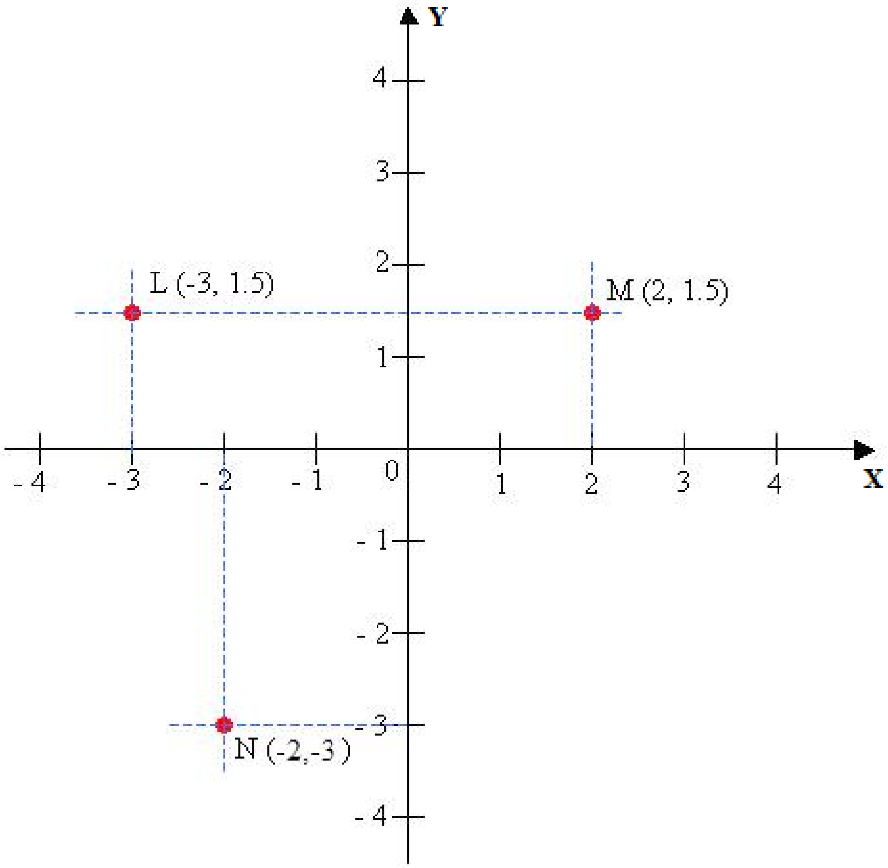
Example:

Represent the points $M(2,1.5)$, $L(-3,1.5)$ and $N(-2,-3)$ in Cartesian plane.

Solution

- Point M has coordinates $M(2,1.5)$. To get to point M, we move 2 units to the right of x-axis from the origin (positive side) and 1.5 units up on y-axis from the origin (positive side).

- Point L is represented by the coordinates $L(-3,1.5)$. To get to point L, we move 3 units to the left of x-axis from the origin (negative side) and 1.5 units up on y-axis from the origin (positive side)
- Point N has coordinates $N(-2,-3)$. To get to point N, we move 2 units to the left of x-axis from the origin of(negative) and 3 units down on y-axis from the origin (negative side).



Points on Coordinate Plane

APPLICATION ACTIVITY 2.1

Represent the points $A(-3,2)$, $B(4,2)$, $C(-3,-2)$, $D(3,-1)$ in xy-plane

2.2. Distances between two points

ACTIVITY 2.2

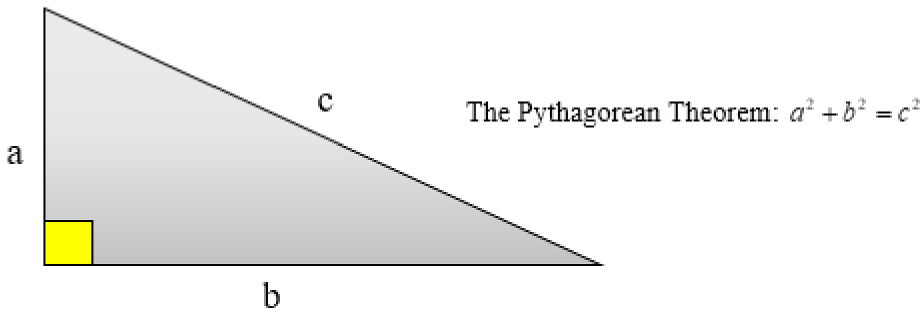
Kalisa and Mugisha live in the same Sector but two different hills. If Kalisa's house is located at $A(1,2)$ from the Sector and Mugisha's house is at $A(4,6)$.

a) If the office of the sector is considered as the origin, present this situation on Cartesian plan.

b) Use the ruler to calculate the distance between Kalisa's house and Mugisha's.

CONTENT SUMMARY

Recall from the Pythagorean Theorem that, in a right triangle, the hypotenuse ***c*** and sides ***a*** and ***b*** are related by $a^2+b^2=c^2$. Conversely, if $a^2+b^2=c^2$ the triangle is a right (see figure below).



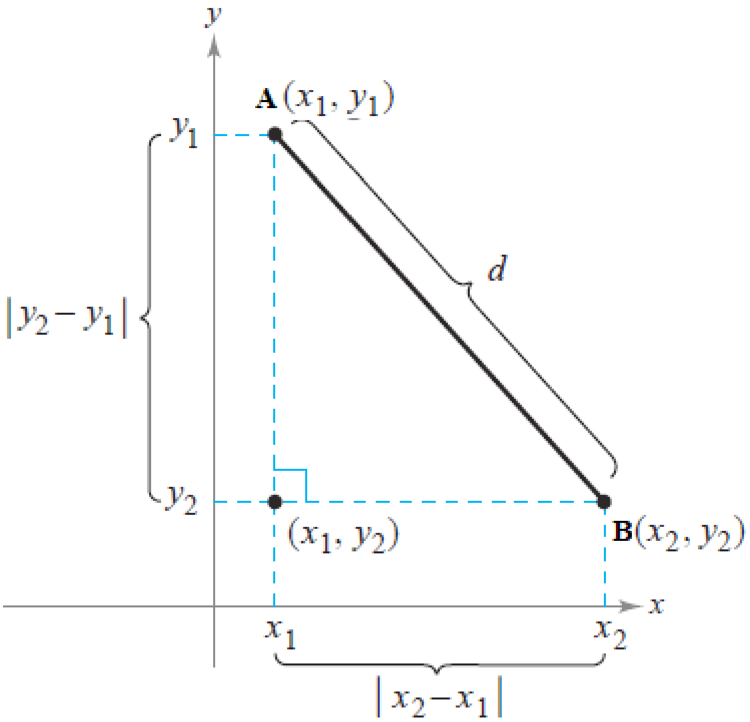
If *A* and *B* are two points on xy - coordinates, we can form a vector $\overrightarrow{AB}$ and the distance between these two points denoted $d(A,B)$ is given by $\left\|\overrightarrow{AB}\right\|$.

Suppose you want to determine the distance "*d*" between the two points (x_1,y_1) and (x_2,y_2) in the plane.

If the points lie on a horizontal line, then $y_1=y_2$ and the distance between the points is $d=|x_2-x_1|$.

If the points lie on a vertical line, then $x_1 = x_2$ and the distance between the point is $d = |y_2 - y_1|$.

If the two points do not lie on a horizontal or on a vertical line, they can be used to form a right triangle, as shown in the figure below.



The length of the vertical side of the triangle is $|y_2 - y_1|$, and the length of the horizontal side is $|x_2 - x_1|$. By Pythagorean Theorem, it follows that:

and $d(A,B) = \|\overrightarrow{AB}\| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.

Thus, if $A(x_1, y_1)$ and $B(x_2, y_2)$ are points of plane, then

The distance "d " between the points $A(x_1, y_1)$ and $B(x_2, y_2)$ in the Cartesian plane, is given by: $d(A,B) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

Examples:

1) Consider the points $A(1,4), B(-2,-3)$ in **Cartesian** plane. Find the distance between the point A and B.

Solution

The distance is

$$d(A,B)=\left\|\overrightarrow{AB}\right\|=\sqrt{(-2-1)^2+(-3-4)^2}$$
$$=\sqrt{9+49}$$
$$=\sqrt{58} \quad units$$

2) Consider the points $C(k,-2)$ and $D(0,1)$ in cartesian plane. Find the distance between the point A and B.

Solution

$$d(C,D)=\sqrt{k^2+9}$$

$$\sqrt{k^2+9}=5$$
$$\Leftrightarrow k^2+9=25$$
$$\Leftrightarrow k^2=16 \Rightarrow k=\pm 4$$

Thus the values of k are -4 and 4

APPLICATION ACTIVITY 2.2

1. Calculate the distance between the points given below:
   a) $S(-2;-5)$ and $Q(7;-2)$
   b) $A(2;7)$ and $B(-3;5)$
   c) $A(x;y)$ and $B(x+4;y-1)$
2. The length of $CD = 5$. Find the missing coordinate if:
   a) $C(6;-2)$ and $D(x;2)$.
   b) $C(4;y)$ and $D(1;-1)$.

2.3. Midpoint of a line segment

ACTIVITY 2.3

Three friends: Pascal, Steve, and Benjamin live on the same side of a street from Nyabugogo to Ruyenzi. Steve's house is halfway between Pascal's and Benjamin's houses. If the locations of Pascal can be given by the coordinates (2,5) , Benjamin's house at (4,7)

a) Draw a Cartesian plan and locate Pascal's and Benjamin's houses

b) Show the line segment joining the locations of Pascal's and Benjamin's houses;

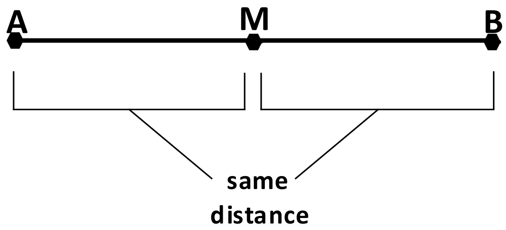
c) Given that Steve's house is in the half way, locate his house, estimate the coordinate of that location and explain how to find it.

CONTENT SUMMARY

A line segment from point A to point B, denoted $[AB]$ is the set of all points on the part of the line that joins A and B, including A and B. The midpoint of this segment is point M *such that the distance* $[AM]=[MB]$ and it is given by given by $M=\frac{1}{2}(A+B)$.

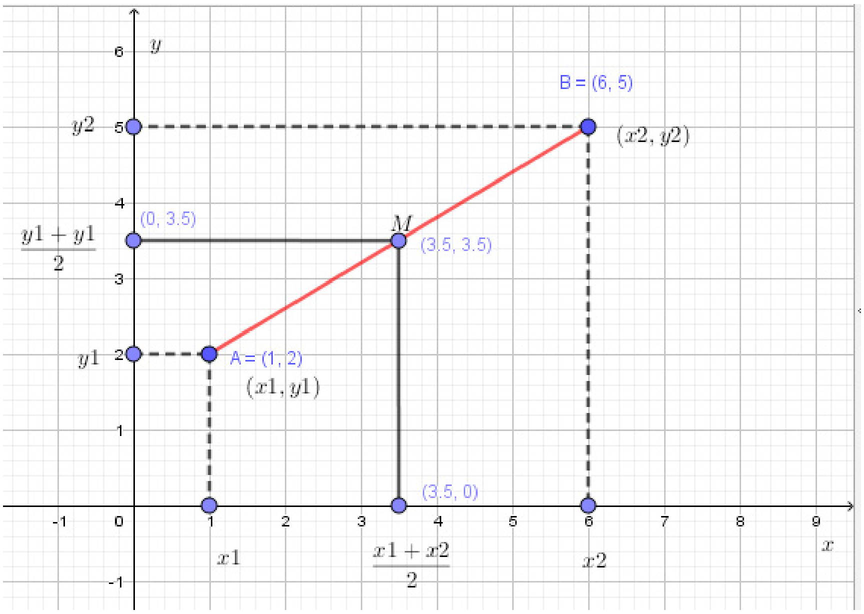
The midpoint of a segment is a point on that line segment which maintains the same distance from both of the endpoints of that line segment.

For example, consider segment to the left.



It has endpoints named A and B. The midpoint of the segment is labeled M. It is at the same distance from each of the endpoints.

Suppose that the coordinates of the line segment $\overline{AB}$, are $A(x_1, y_1)$ and $B(x_2, y_2)$



The coordinates of the midpoint, are obtained using the midpoint formula, such that $\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$ are coordinates of midpoint

Example:

1) *Find the midpoint of the segment joining points* $A(3,0)$ *and* $B(1,8)$.

Solution

The midpoint is $M = \frac{1}{2}(A+B) = \frac{1}{2}(4,8) = (2,4)$.

2) If $(-3,5)$ is the midpoint of $(2,6)$ and (a,b), find the value of a and b.

Solution

$$(-3,5) = \frac{1}{2}(2+a, 6+b) \Leftrightarrow (-3,5) = \left(\frac{2+a}{2}, \frac{6+b}{2}\right) \Rightarrow \begin{cases} \frac{2+a}{2} = -3 \\ \frac{6+b}{2} = 5 \end{cases} \Leftrightarrow \begin{cases} 2+a=-6 \\ 6+b=10 \end{cases} \Rightarrow \begin{cases} a=-8 \\ b=4 \end{cases}$$

APPLICATION ACTIVITY 2.3

Micheal and Sarah live in different cities and one day they decided to meet up for lunch. Because they both wanted to travel as little as possible they decided to meet at a point halfway between their homes. If their positions are given by $(3100, 500)$ and $(5120, 125)$.

Which of the following coordinates represents the place where they should meet?

a) $(4110, 312.5)$

b) $(4110, 375)$

c) $(2020, 375)$

d) $(8220, 625)$

2.4. Vector in 2D and dot product

2.4.1. Vectors in 2D

ACTIVITY 2.4.1

In xy plane:

1. Represent the points $A(1,2)$ and $B(-3,1)$
2. Draw arrow from point A to point B
3. Refer to the content of S2 and represent the Vector $\vec{AB}$

CONTENT SUMMARY

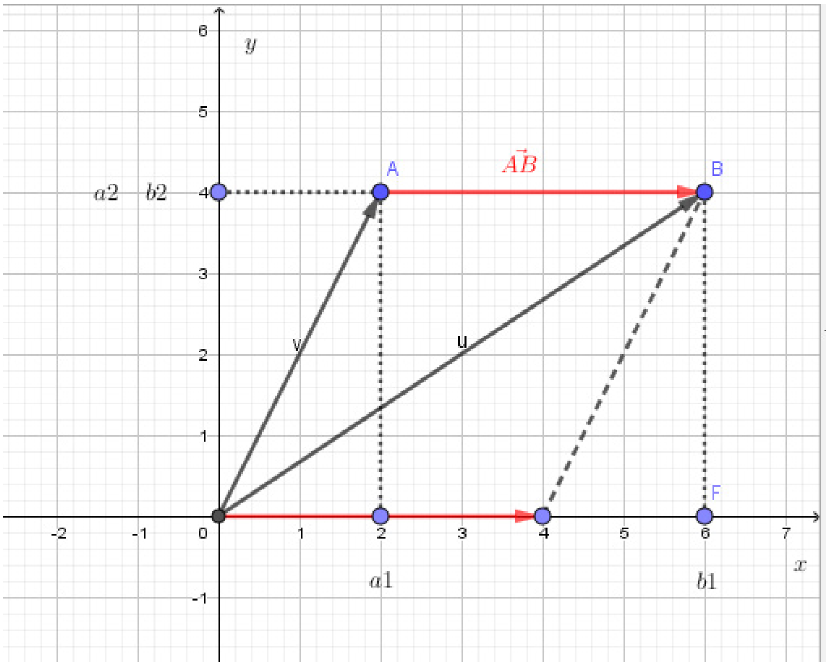
Definitions and operations on vectors

A vector is a directed line segment; it is a quantity that has both magnitude and direction. A vector is represented by an arrow.

The length of the arrow represents the **magnitude** of the vector, and the arrowhead indicates the **direction** of the vector. That is to say, a vector has a given length and a given direction.

The vector joining point A and point B is denoted by $\overrightarrow{AB}$ and to find it we subtract the coordinates of point A from the coordinates of point B.

For example the vector $\overrightarrow{AB}$ defined by two points $A(a_1, a_2)$ and $B(b_1, b_2)$



On the figure, it is clear that the vector

$$\vec{AB} = \vec{OB} - \vec{OA} = (b_1, b_2) - (a_1, a_2) = (b_1 - a_1, b_2 - a_2) = \begin{pmatrix} b_1 - a_1 \\ b_2 - a_2 \end{pmatrix}$$

Therefore, $\overrightarrow{AB} = (b_1 - a_1, b_2 - a_2)$ which can also be written in the form of column vector $\vec{AB} = \begin{pmatrix} b_1 - a_1 \\ b_2 - a_2 \end{pmatrix}$

The point A is called the **initial point** or tail of $\overrightarrow{AB}$ and B is called the **terminal point** or **tip**. The **zero vector** is $(0,0)$ denoted by $\vec{0}$.

Examples

1) The vector defined by point $A(1,2)$ and $B(4,3)$ is $\overrightarrow{AB} = (3,1)$

2) Vector $\overrightarrow{CD} = (-4,2)$ is defined by point $C(-3,4)$ and D. Find the coordinates of point D.

Solution

Let point D be $D(x, y)$, then $\overrightarrow{CD} = (x+3, y-4) = (-4, 2)$

$x+3=-4 \Rightarrow x=-7$

$y-4=2 \Rightarrow y=6$

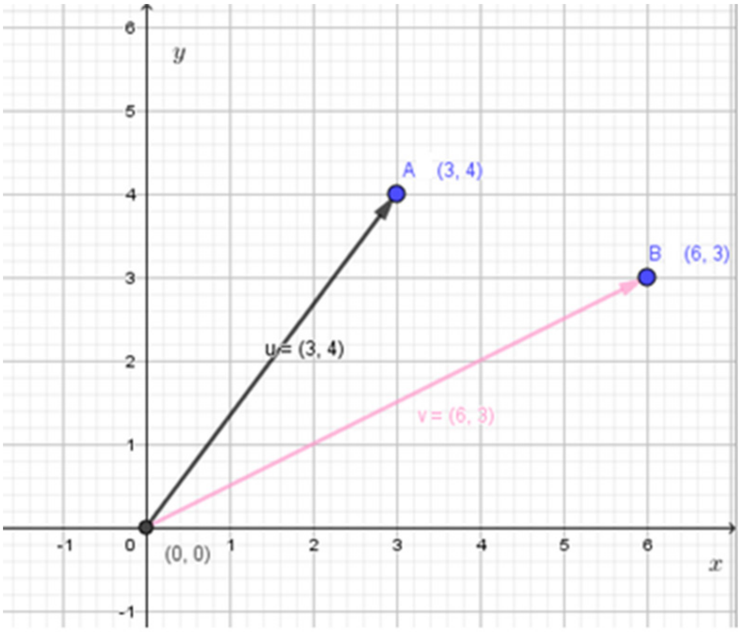
Thus, point D *has the coordinates (-7, 6).*

Position Vector

In a Cartesian plane, an **algebraic vector** $\vec{v}$ is represented as $\vec{v} = (a, \mathrm{b})$ where a and b are real numbers (scalars) called the **components** of the vector $\vec{v}$.

We use a rectangular coordinate system to represent algebraic vectors in the plane.

If $\vec{v} = (a, \mathrm{b})$ is an algebraic vector whose initial point is at the origin, then $\vec{v}$ is called a position vector. See the figure below.

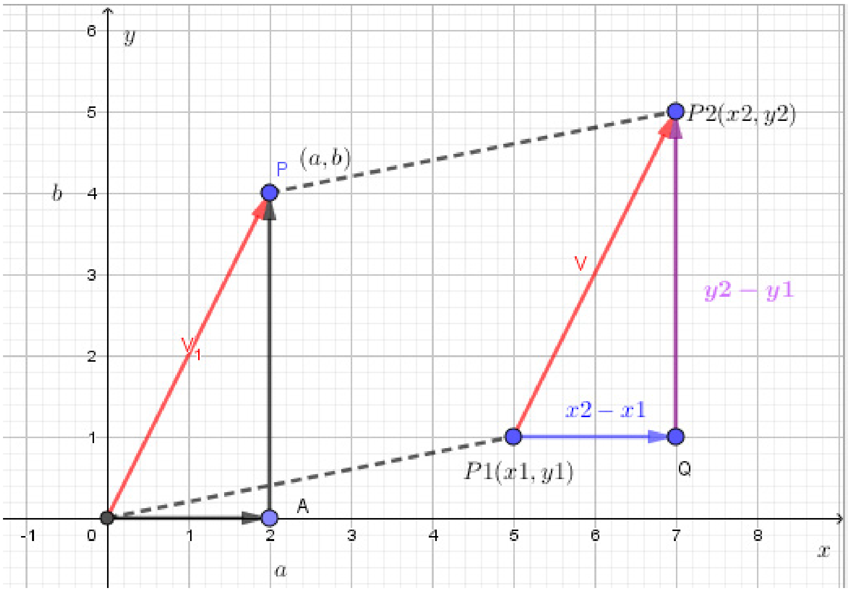


Notice that the terminal point of the position vector $\vec{v} = (a, \mathrm{b})$ is $\mathrm{A}(a, \mathrm{b})$. On the figure we have $\vec{u} = (3, 4)$ and $\vec{v} = (6, 3)$.

The next result states that any vector whose initial point is not at the origin is equal to a unique position vector.

Suppose that $\vec{v}$ is a vector with **initial point** $P_1(x_1, y_1)$ not necessarily the origin, and **terminal point** $P_2(x_2, y_2)$.

If $\vec{v} = \overrightarrow{P_1P_2}$, then $\vec{v}$ is equal to the vector $\vec{v} = \overrightarrow{P_1P_2} = (x_2 - x_1, y_2 - y_1)$. See the figure below.



Triangle OPA and triangle P_1P_2Q are congruent. This is because the line segments have the same magnitude. So, $d(0, \text{P}) = d(\text{P}_1, P_2)$; and they have the same direction, so the angle $\angle POA = \angle P_2P_1Q$.

Since the triangles are right triangles, we have angle–side–angle congruence. It follows that corresponding sides are equal. As a result, $x_2 - x_1 = a$ and $y_2 - y_1 = b$, and so $\vec{v}$ may be written as

$$\vec{v} = \overrightarrow{P_1P_2} = (a, b) = (x_2 - x_1, y_2 - y_1).$$

Because of this result, we can replace any algebraic vector by a unique position vector with the initial point at the origin $O(0,0)$ and vice versa.

On the figure above we have: $\vec{V_1} = (2, 4)$; which is the position vector of V

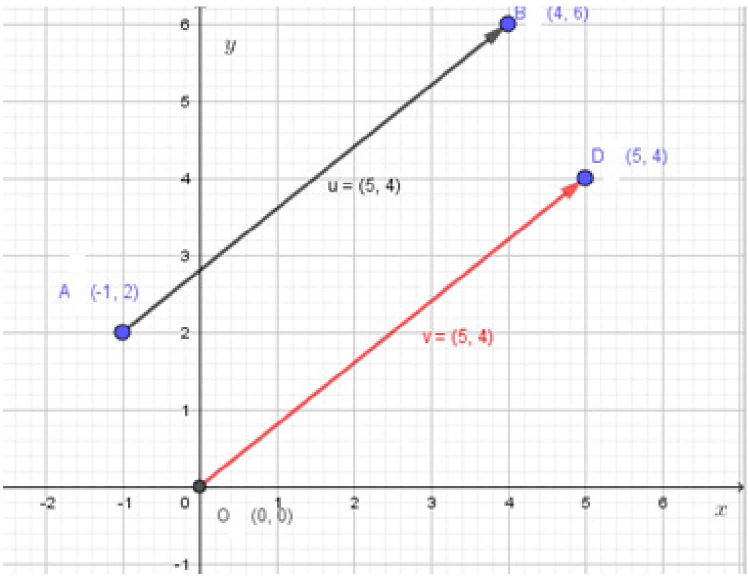
$P_1(5,1); P_2(7,5)$ *and* $\vec{V} = \overrightarrow{P_1P_2} = (7-5, 5-1) = (2,4)$. This shows that the vector $\vec{V}_1 = \vec{V} = (2,4)$ which means that $\vec{V}_1 = (2,4)$ is the position vector of the vector $\vec{V}$.

Example:

Find the position vector of the vector $\vec{v} = \overrightarrow{P_1P_2}$ if $P_1(-1,2)$ and $P_2 = (4,6)$.

Solution

$$\vec{v} = \overrightarrow{P_1P_2} = P_2 - P_1 = (4-(-1), 6-2) = (5,4).$$



Components of a vector

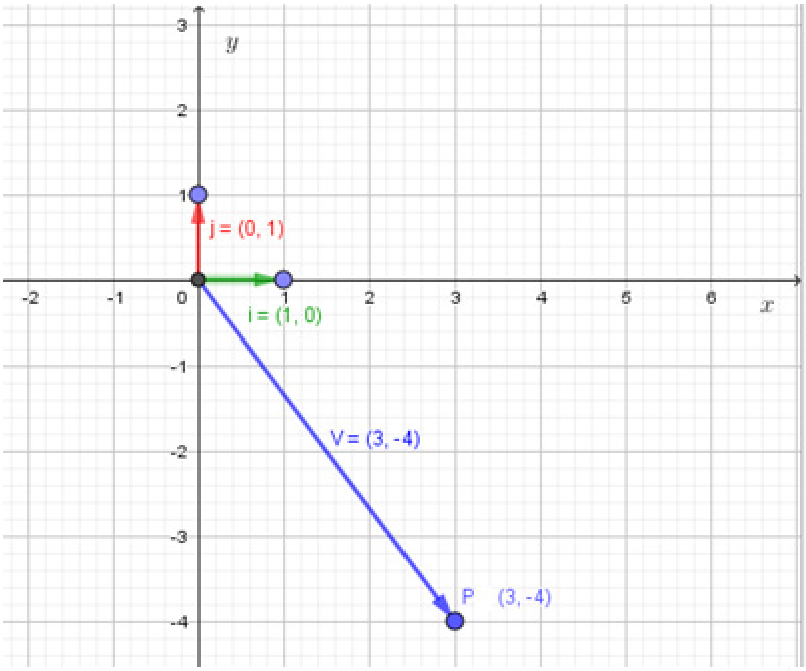
We now present an alternative representation of a vector in the plane that is common in the physical sciences. Let $\vec{i}$ denote the unit vector whose direction is along the positive *x*-axis; let $\vec{j}$ denote the unit vector whose direction is along the positive *y*-axis. Then $\vec{i} = (1,0)$ and $\vec{j} = (0,1)$.

Any vector $\vec{v} = (a,b)$ can be written using the unit vectors $\vec{i}$ and $\vec{j}$ as follows:

$$\vec{v} = (a,b) = a(1,0) + b(0,1) = a\,\vec{i} + b\,\vec{j}$$

We call a and b the **horizontal** and **vertical components** of $\vec{v}=(a,b)$, respectively.

For example, if $\vec{v}=(3,-4)=3\vec{i}-4\vec{j}$, then 3 is the **horizontal component** and -4 is the **vertical component**.



Note: It is now easy to show that for any two points $A(a_1,a_2)$ and $B(b_1,b_2)$, the vector $\vec{AB}=\vec{OB}-\vec{OA}=(b_1,b_2)-(a_1,a_2)=(b_1-a_1,b_2-a_2)$

Because

$$\vec{OA}=a_1\vec{i}+a_2\vec{j},\ \vec{OB}=b_1\vec{i}+b_2\vec{j}$$

$$\vec{AB}=\vec{OB}-\vec{OA}=\left(b_1\vec{i}+b_2\vec{j}\right)-\left(a_1\vec{i}+a_2\vec{j}\right)$$

$$=b_1\vec{i}+b_2\vec{j}-a_1\vec{i}-a_2\vec{j}$$

$$=b_1\vec{i}-a_1\vec{i}+b_2\vec{j}-a_2\vec{j}$$

$$=(b_1-a_1)\vec{i}+(b_2-a_2)\vec{j}$$

$$\overrightarrow{AB}=(b_1-a_1)\vec{i}+(b_2-a_2)\vec{j}=(b_1-a_1,b_2-a_2)$$

Therefore, $\overrightarrow{AB}=(b_1-a_1,b_2-a_2)$.

APPLICATION ACTIVITY 2.4.1

1) In xy plane, present the following vectors:

   a. $\vec{u}=(5,6)$, $\vec{v}=(-1,4)$

   b. $\vec{u}=(3,5)$, $\vec{v}=(-1,6)$

2) Plot a vector with initial point P(1,1) and terminal point Q(8,5) in the Cartesian plane and illustrate its position vector.

2.4.2 Dot product

ACTIVITY 2.4.2

1. In xy plane

   a. represent the points $A(1,2)$ and $B(-3,1)$

   b. find the distance between point A to point B

2. Given that the product of two unit vectors is such that $\vec{i}.\vec{i}=1$, $\vec{i}.\vec{j}=0$ and $\vec{j}.\vec{j}=1$, apply the distributivity of multiplication under addition to evaluate the following products:

a) $(1,2)(-3,1)=\left(1\vec{i}+2\vec{j}\right)\left(-3\vec{i}+1\vec{j}\right)$

b) $(-4,2).(1,2)$

c) From your results, deduce how to calculate $(a_1,a_2).(b_1,b_2)$.

CONTENT SUMMARY

Scalar product and properties

The scalar product or dot product (or sometimes inner product) is an algebraic operation that takes two coordinate vectors and returns a single number.

Algebraically, it is the sum of the products of the corresponding coordinates of the two vectors.

That is, the ***scalar product*** of vectors $\vec{u}=(a_1,a_2)$ and $\vec{v}=(b_1,b_2)$ of plane is defined by $\vec{u}\cdot\vec{v}=a_1b_1+a_2b_2$

Example:

$(2,4).(10,4)=2.10+4.4=20+16=36$ or

$$(2,4).(10,4)=\left(2\vec{i}+4\vec{j}\right)\left(10\vec{i}+4\vec{j}\right)$$

$$=2.10\vec{i}\,\vec{i}+2.4\vec{i}\,\vec{j}+4.10\vec{i}\,\vec{j}+4.4\vec{j}\,\vec{j}$$

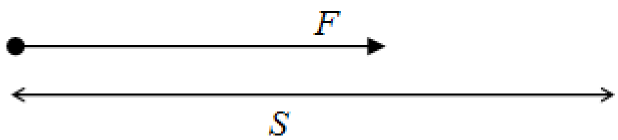
$$=20+0+0+16=36$$

They give the same result.

We can illustrate this scalar product in terms of work done by a force on the body.

Suppose that a person is holding a heavy weight at rest. This person may say and feel he is doing hard work but in fact none is being done on the weight in the scientific sense. Work is done when a force moves its point of application along the direction of its line of action.

If the constant force $\vec{F}$ and the displacement $\vec{S}$ are in the same direction ,we define the work W done by the force on the body by $W=\vec{F}\cdot\vec{S}$



Properties of scalar product

a) If $\vec{u}=\vec{0}$ or $\vec{v}=\vec{0}$, then $\vec{u}\cdot\vec{v}=\vec{0}$

b) If $\vec{u}\parallel\vec{v}$ and $\vec{u},\vec{v}$ have same direction, then $\vec{u}\cdot\vec{v}>0$

c) If $\vec{u}\parallel\vec{v}$ and $\vec{u},\vec{v}$ have opposite direction, then $\vec{u}\cdot\vec{v}<0$

d) If $\vec{u}\perp\vec{v}$, then $\vec{u}\cdot\vec{v}=0$

e) $\vec{u}\cdot\vec{v}=\vec{v}\cdot\vec{u}$

f) $\vec{u}\cdot\left(a\vec{v}+b\vec{w}\right)=a\vec{v}\cdot\vec{u}+b\vec{w}\cdot\vec{u}$ $\quad\left(a\vec{u}+b\vec{v}\right)\cdot\vec{w}=a\vec{u}\cdot\vec{w}+b\vec{v}\cdot\vec{w}$

g) $\vec{u}\cdot\vec{u}>0$, $\vec{u}\neq\vec{0}$

We define the square of $\vec{u}$ to be $\vec{u}\cdot\vec{u}=\left(\vec{u}\right)^2$

Example

The scalar product of the vector $\vec{u}=(2,4)$ and vector $\vec{v}=(-5,0)$ is $\vec{u}\cdot\vec{v}=2(-5)+0=-10$

The square of the vector $\vec{u}=(10,4)$ is $\left(\vec{u}\right)^2=10(10)+4(4)=100+16=116$

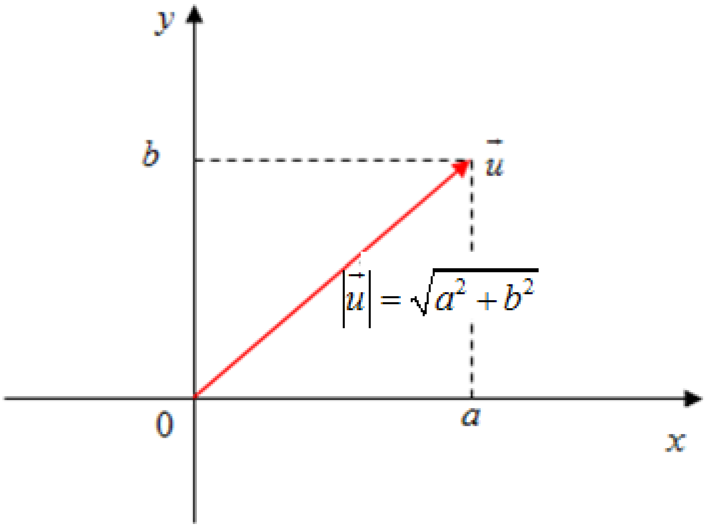
Notice

Two vectors are perpendicular if their scalar product is zero.

Magnitude or Modulus or norm of a vector

The magnitude of the vector $\vec{u}$ noted by $\|\vec{u}\|$ is defined as its length and is given by the square root of the sum of the squares of its components.

The magnitude of a vector $\vec{u}$ is also noted by $|\vec{u}|$.



That is $\|\vec{u}\| = \sqrt{(\vec{u})^2}$ or $\|\vec{u}\|^2 = (\vec{u})^2$. Thus if $\vec{u} = (a,b)$

$\vec{u} \cdot \vec{u} = (\vec{u})^2$

$$\vec{u} = (a,b),\ \vec{u}.\vec{u} = (a,b)(a,b) = a^2 + b^2$$

$$\left(\vec{u}\right)^2 = a^2 + b^2$$

$$\|u\|^2 = a^2 + b^2$$

$$\|u\| = \sqrt{a^2 + b^2}$$

Consequences

a) If $\vec{u} = \vec{0}$ then $\|\vec{u}\| = 0$

b) $\|k\vec{u}\| = |k|\|\vec{u}\|$, k is a real number.

c) Geometrically, the scalar product of two vectors $\vec{u} = (a,b)$ and $\vec{v} = (c,d)$ of plane is given by $\vec{u} \cdot \vec{v} = \|\vec{u}\|\|\vec{v}\|\cos\theta$. Where θ is the angle between vectors $\vec{u}$ and $\vec{v}$. From this relation we have $\cos\theta = \frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\|\vec{v}\|}$ or

$$\cos\theta = \frac{a\cdot c + b\cdot d}{\sqrt{a^2+b^2}\sqrt{c^2+d^2}}.$$

Examples

1) Find the norm of the vector $\vec{v} = (3,4)$.

The norm is $\|\vec{v}\| = \sqrt{9+16} = 5$

2) Find the norm of the vector $\vec{u} = (-1,4)$

The norm is $\|\vec{u}\| = \sqrt{1+16} = \sqrt{17}$

APPLICATION ACTIVITY 2.4.2

1. Find the norm(magnitude) of the following vectors:

   a) $\vec{u} = (-3,4)$

   b) $\vec{v} = (3,1)$

2. Consider the following points $A(3,4)$, $B(-2,3)$ and the vectors $\vec{u} = (4,5)$, $\vec{v} = (-3,1)$ in Cartesian plane. Find

   a) vector $\overrightarrow{AB}$

   b) vector $\vec{w} = 2\overrightarrow{AB} - 3\vec{u} + \vec{v}$

   c) the norm of vector $\overrightarrow{AB}$ and norm of vector $\vec{w}$

   d) the scalar product of vector $\vec{u}$, vector $\vec{v}$

   e) the scalar product of vector $\vec{v}$ and vector $\vec{w}$

2.5. Equation of a straight line

2.5.1. Determination of equation of a straight line passing through a point and parallel to a direction vector

ACTIVITY 2.5.1

Given the vector $\vec{v} = (2,-3)$.

a) Determine the form of a vector $\vec{w} = (x, y)$ parallel to $\vec{v}$ and passing to the point $P(1,6)$.

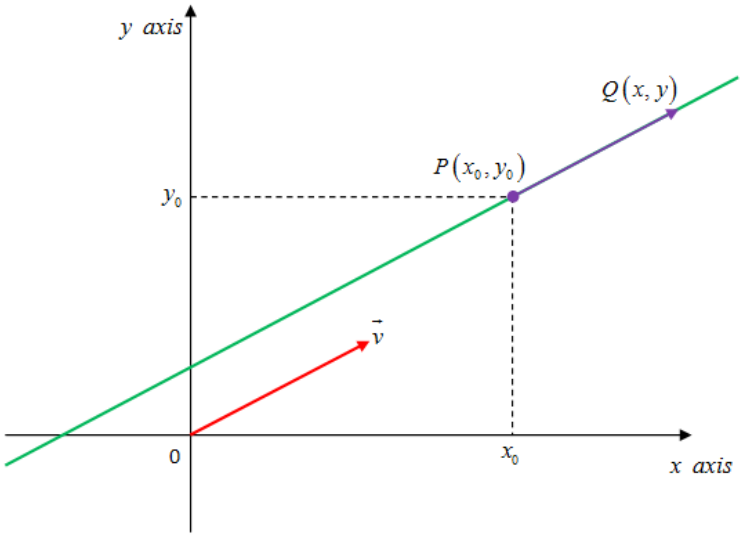
b) Give 2 examples of such vectors $\vec{w}$.

c) If D is a line with the direction vector $\vec{w}$, what should be the equation of D?

CONTENT SUMMARY

To determine the equation of the line passing through the point $P(x_0, y_0)$ and parallel to the direction vector, $\vec{v} = (a,b)$, we will use our knowledge that parallel vectors are scalar multiples.

The vector $\overrightarrow{0P}$ is called the **position vector** of *P*.



Thus, the vector through $P(x_0, y_0)$ and any other point $Q(x, y)$ on the line is the product of a scalar and the direction vector $\vec{v} = (a, b)$.

$\overrightarrow{PQ} = r\vec{v}$ with r a parameter and $\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP}$

Hence, the **vector equation** of the line that is parallel to the vector $\vec{v} = (a, b)$ and which passes through the point P with position vector $\overrightarrow{0P} = (x_0, y_0)$ is given as

$\overrightarrow{OQ} = \overrightarrow{OP} + r\vec{v}$ where $Q(x, y)$ is any point of the line.

$(x, y) = (x_0, y_0) + r(a, b)$ or $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} + r\begin{pmatrix} a \\ b \end{pmatrix}$

The **parametric equations** of the line that is parallel to the vector $\vec{v} = (a, b)$ and which passes through the point P with position vector $\overrightarrow{0P} = (x_0, y_0)$ are given by:

$$\begin{cases} x = x_0 + ra \\ y = y_0 + rb \end{cases}$$

The **symmetric equation** or **Cartesian equation is** found after eliminating parameter ***r***.

$$\frac{x - x_0}{a} = \frac{y - y_0}{b}$$

This can be expanded to the standard form $bx - ay = bx_0 - y_0$.

Example

Find the vector equation of the straight line that is parallel to the vector $\vec{u} = (2, -1)$ and which passes through the point with position vector $\vec{v} = (3, 2)$.

Solution

The vector equation is $(x, y) = (3, 2) + r(2, -1)$, r is a parameter

The parametric equations:

$$\begin{cases} x = 3 + 2r \\ y = 2 - r \end{cases}$$

The Cartesian equation:

$$\frac{x-3}{2} = \frac{y-2}{-1} \text{ or } 2y - 4 = -x + 3 \text{ or } x + 2y = 7$$

In general

The **vector equation** of the line can be rewritten as

$(x, y) = (x_0, y_0) + r(a, b)$ or $x\vec{i} + y\vec{j} = x_0\vec{i} + y_0\vec{j} + r(a\vec{i} + b\vec{j})$ where $\{\vec{i} = (1,0), \vec{j} = (0,1)\}$ form the standard basis of IR^2.

APPLICATION ACTIVITY 2.5.1

1. Find the vector, parametric and Cartesian equation of the line passing through the point with position vector $(2,-3)$ and parallel to the line $(x, y) = (3,5) + r(1,6)$
2. Find the Cartesian equations of the lines whose vector equations are given below. Give your answers in the form $y = mx + c$

   a. $(x, y) = (-1, 2) + r(-2, 3)$

   b. $x\vec{i} + y\vec{j} = 3\vec{i} + 2\vec{j} + r(3\vec{i} - \vec{j})$

2.5.2. Determination of an equation of a straight line given 2 points and direction vector.

ACTIVITY 2.5.2

Consider the line passing through points $A(1,4)$ and $B(3,-2)$.

a) Determine the vector $\overrightarrow{AB}$

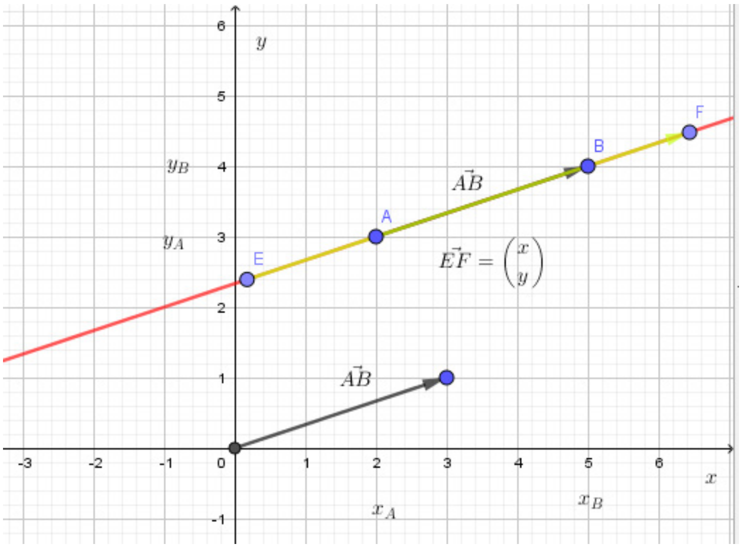
b) Determine the form of a vector $\vec{w} = (x, y)$ parallel to $\overrightarrow{AB}$ and passing to the point $B(3,-2)$

c) Give 2 examples of vectors like $\vec{w}$.

c) If D is a line with the direction vector $\vec{w}$, what should be the equation of D?

CONTENT SUMMARY

Two points $A(x_A, y_A)$ *and* $B(x_B, y_B)$ form a vector $\overrightarrow{AB} = \begin{pmatrix} x_B - x_A \\ y_B - y_A \end{pmatrix}$. A vector parallel to $\overrightarrow{AB} = \begin{pmatrix} x_B - x_A \\ y_B - y_A \end{pmatrix}$ and passing to the point $B(x_B, y_B)$ is of the form $\vec{w} = (x, y) = r\overrightarrow{AB} + A$



Therefore,

The vector equation of the line is:

$(x,y)=\overrightarrow{0A}+r\overrightarrow{AB}$ or $\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} x_A \\ y_A \end{pmatrix}+r\begin{pmatrix} x_B-x_A \\ y_B-y_A \end{pmatrix}$, where $\overrightarrow{AB}$ is the direction vector.

The parametric equations of the line are:

$$\begin{cases} x=x_A+r(x_B-x_A) \\ y=y_A+r(y_B-y_A) \end{cases}$$

The Cartesian equation is obtained in this way as follows:

$$x=x_A+r(x_B-x_A),\quad r=\frac{x-x_A}{x_B-x_A}$$

$$y=y_A+r(y_B-y_A),\quad r=\frac{y-y_A}{y_B-y_A}$$

Equalizing the value of r we find the Cartesian equation:

$$\frac{x-x_A}{x_B-x_A}=\frac{y-y_A}{y_B-y_A}$$

This can be expanded to the standard form

$$x(y_B-y_A)-y(x_B-x_A)=x_A(y_B-y_A)-y_A(x_B-x_A).$$

Example

Find the vector equation of the straight line passing through $A(3,2)$ and $B(4,1)$

Solution

The direction vector is $\overrightarrow{AB}=(1,-1)$, r is a parameter

The vector equation is $(x,y)=\overrightarrow{0A}+r\overrightarrow{AB}$ or $(x,y)=(3,2)+r(1,-1)$

The parametric equations: is $\begin{cases} x=3+r \\ y=2-r \end{cases}$

The Cartesian equation, is $\frac{x-3}{1}=\frac{y-2}{-1}$ or $y-2=-x+3$ or $x+y=5$

APPLICATION ACTIVITY 2.5.2

1) Determine the equation of a straight line passing through $P(2,4)$ and $B(3,-2)$.

2) Find the vector equation of the straight line that passes through the point with position vector $2\vec{i}+3\vec{j}$ and which is perpendicular to the line $x\vec{i}+y\vec{j}=3\vec{i}+2\vec{j}+r\left(\vec{i}-2\vec{j}\right)$.

2.5.3 Equation of a straight line given its gradient

ACTIVITY 2.5.3

Determine the slope and the -intercept in the equations below. Hence draw the lines in the same graph.

a) $y=\frac{3}{2}x-2$

b) $y=-3x+\frac{5}{2}$

CONTENT SUMMARY

Consider a line having gradient and passing through the point $P_1\left(x_1,y_1\right)$.

Suppose that the point $P\left(x,y\right)$ is an other point on the line. Then the gradient of the line is the rate at which the line rises (or falls) vertically for every unit across to the right. It is defined by the change in $m=\frac{y-y_1}{x-x_1}$

Thus, the equation of the line in point-slope form is defined by $y-y_1=m\left(x-x_1\right)$

From the equation above, if we take any other point $P_2\left(x_2,y_2\right)$ that lies on this line, then:

Vector equation

$\overrightarrow{OP} = \overrightarrow{OP_1} + r\overrightarrow{P_1P_2}$ or $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + r\begin{pmatrix} x_2 - x_1 \\ y_2 - y_1 \end{pmatrix}$,

is the direction vector.

Parametric equations

$$\begin{cases} x = x_1 + r(x_2 - x_1) \\ y = y_2 + r(y_2 - y_1) \end{cases}$$

Cartesian equation

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_2}$$

Example :

Write the equation of the line that has slope that passes through the

Solution

$y - y_1 = m(x - x_1)$ use point-slope form

$\text{y} - 7 = -3(x - (-1))$ substitute $(-1, 7)$ for (x_1, y_1) and -3 for m.

$y - 7 = -3(x + 1)$

Note:

The general equation of the line is $Ax + By + C = 0$, where A, B, C are constantes,

And, $A \neq 0, B \neq 0$.

- If $A = 0$ the line is horizontal.
- If $B = 0$ the line is vertical.
- If $C = 0$ the line passes through the origin.

On the other hand the line has slope $m = \frac{-A}{B}$ and the intercept of , $b = \frac{-c}{B}$

- If the two lines are parallel, then their slopes/gradients are equal.

  Therefore $m_1 = m_2$

Thus, the equations $Ax+By+C=0$ and $Ax+By+D=0$ are parallel.

- If two lines are perpendicular, then the product of their slopes/gradients is equal to -1.

  Therefore $m_1 \times m_2 = -1$.

  Thus, the equations $Ax+By+C=0$ and $Bx-Ay+D=0$ are perpendicular.

Example :

1) Find the equation for the line that contains the point $(5,1)$ and is parallel to

$$y=\frac{3}{5}x+3.$$

Step1: Identify the slope of the given line

$$y=\frac{3}{5}x+3 \Rightarrow m=\frac{3}{5}$$

Step2: Write the equation of the line trougth point $(5,1)$ and $m=\frac{3}{5}$

$$y-y_1=m(x-x_1)$$

$$y-1=\frac{3}{5}(x-5)$$

$$y=\frac{3}{5}x-3+1$$

$$y=\frac{3}{5}x-2$$

2. Find the equation for the line that contains the point $(0,-2)$ and is perpendicular to $y=5x+3$?

Step1: Identify the slope of the given line and write its negative reciprocal.

$$y=5x+3 \quad \Rightarrow m=5 \text{ and it has negative reciprocal } \frac{-1}{5}$$

Step2: Write the equation of the line trougth point $(0,-2)$ and $m=\frac{-1}{5}$

$$y-y_1=m(x-x_1)$$

$$y-(-2)=\frac{-1}{5}(x-0)$$

$$y=\frac{-1}{5}x-2$$

APPLICATION ACTIVITY 2.5.3

1. Write the equation in point-slope form for the line through the given point that has the given slope

   a) $(3,-4);m=6$

   b) $(-2,-7);m=\frac{-3}{2}$

   c) $(-5,2);m=0$

   d) $(4,2);m=\frac{-5}{3}$

   e) $(4,0);m=1$

   f) $(1,-8);m=\frac{-1}{5}$

2. Is $y-5=2(x-1)$ an equation of a line passing through $(4,11)$? Explain.

3. Write an equation of the line that contains the point $(-3,-5)$ and the same slope as $y+2=7(x+3)$

2.6. Problems on points and straight lines in 2D

2.6.1 Perform operations to determine the intersection, perpendicularity or parallelism of two lines

ACTIVITY 2.6.1

1. In the same Cartesian plane plot the straight lines containing the following points

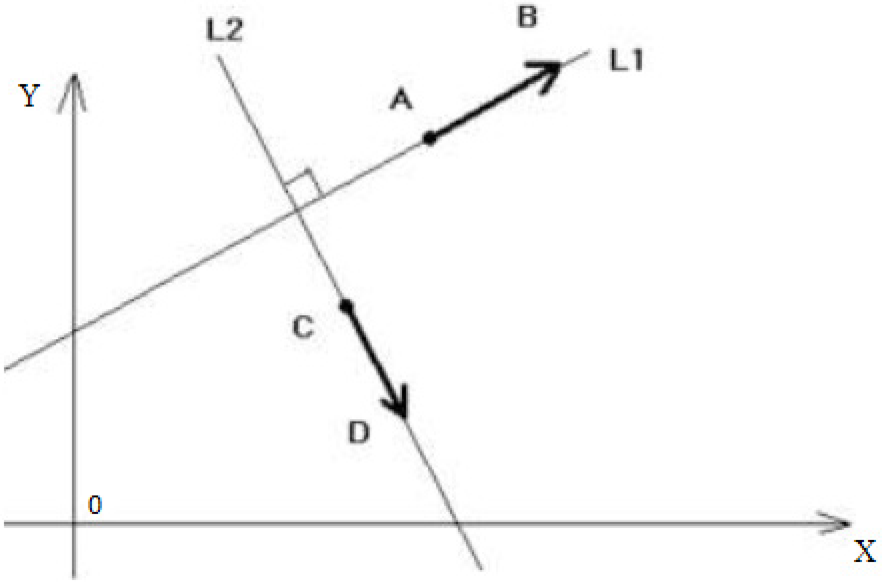
   $(0,4);\ (\frac{1}{2},2)$ and $(\frac{1}{2},2);\ (4,1)$

   What is your opinion about the two lines in the Cartesian plane?

2. In the same Cartesian plane plot the straight lines containing the following points $(-3,-1);\ (0,3)$ and $(-2,-4);\ (1,0)$

CONTENT SUMMARY

a) Perpendicularity and intersection of two lines



Here L_1 and L_2 are perpendicular

To write a straight line perpendicular to a given straight line we proceed as follows:

Step I: Interchange the coefficients of x and y in equation $ax+by+c=0$.

Step II: Interchange the sign between the terms in x and y of equation i.e., If the coefficient of x and y in the given equation are of the same signs make them of opposite signs and if the coefficient of x and y in the given equation are of the opposite signs make them of the same sign.

Step III: Replace the given constant of equation $ax+by+c=0$ by an arbitrary constant.

For example, the equation of a line perpendicular to the line $7x+2y+5=0$ is $2x-7y+c=0$ again, the equation of a line, perpendicular to the line $9x-3y=1$ is $3x+9y+k=0$

Note:

Assigning different values to k in $bx-ay+k=0$, we shall get different parallel straight lines each of which is perpendicular to the line $ax+by+c=0$. Thus we can have a family of straight lines perpendicular to a given straight line.

b) Intersection point of two lines

Let the equations of two intersecting straight lines be

$a_1x+b_1y+c_1=0$ ....(i)
and
$a_2x+b_2y+c_2=0$.......(ii)

Suppose the above equations of two intersecting lines intersect at $P(x_1,y_1)$.

Then (x_1,y_1) will satisfy both the equations *(i)* and *(ii).*

Therefore, $a_1x_1+b_1y_1+c_1=0$ and $a_2x_1+b_2y_1+c_2=0$

Solving the above two equations by using the method of cross-multiplication, we get,

$$\frac{x_1}{b_1c_2-b_2c_1}=\frac{y_1}{c_1a_2-c_2a_1}=\frac{1}{a_1b_2-a_2b_1}$$

Therefore, $x_1 = \dfrac{b_1c_2 - b_2c_1}{a_1b_2 - a_2b_1}$ and $y_1 = \dfrac{c_1a_2 - c_2a_1}{a_1b_2 - a_2b_1}$, $a_1b_2 - a_2b_1 \neq 0$

$$\left(\frac{b_1c_1 - b_2c_1}{a_1b_2 - a_2b_1}, \frac{c_1a_2 - c_2a_1}{a_1b_2 - a_2b_1}\right),\ a_1b_2 - a_2b_1 \neq 0$$

Notes:

To find the coordinates of the point of intersection of two non-parallel lines, we solve the given equations simultaneously and the values of x and y so obtained determine the coordinates of the point of intersection.

If $a_1b_2 - a_2b_1 = 0$ then $a_1b_2 = a_2b_1$

$$\Rightarrow \frac{a_1}{b_1} = \frac{a_2}{b_2}$$

$\Rightarrow -\dfrac{a_1}{b_1} = -\dfrac{a_2}{b_2}$ **i.e., the slope of line (i) equals to the slope of line (ii)**

Therefore, in this case the straight lines (i) and (ii) are parallel and hence they do not intersect at any real point.

Example

Find the coordinates of the point of intersection of the lines $2x - y + 3 = 0$ and $x + 2y - 4 = 0$

Solution:

We know that the co-ordinates of the point of intersection

of the lines $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$ are

$$\left(\frac{b_1c_1 - b_2c_1}{a_1b_2 - a_2b_1}, \frac{c_1a_2 - c_2a_1}{a_1b_2 - a_2b_1}\right),\ a_1b_2 - a_2b_1 \neq 0$$

Given equations are

$2x - y + 3 = 0$....(i)

$x + 2y - 4 = 0$.....(ii)

Here $a_1 = 2,\ b_1 = -1, c_1 = 3,\ a_2 = 1,\ b_2 = 2$ and $c_2 = -4$

$$\left(\frac{(-1).(-4)-(2)(3)}{(2).(2)-(1).(-1)}, \frac{(3)(1)-(-4)(2)}{(2)(2)-(1)(-1)}\right)$$

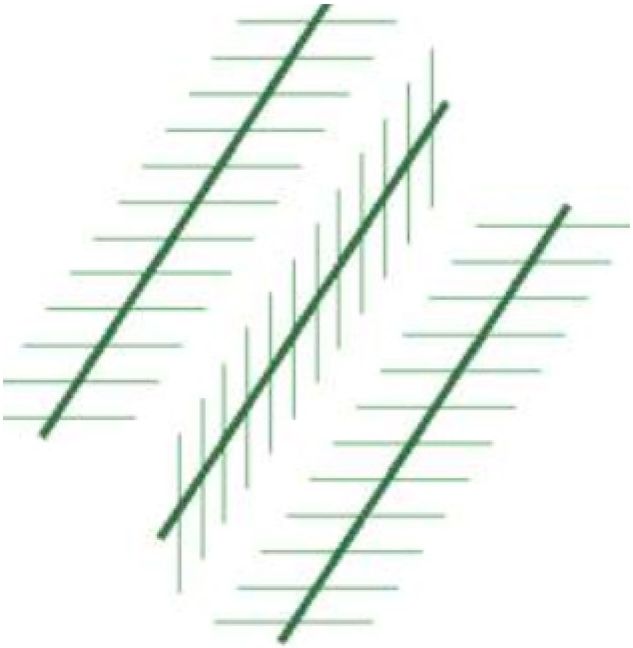
$$\Rightarrow \left(\frac{4-6}{4+1}, \frac{3+8}{4+1}\right)$$

$$\Rightarrow \left(\frac{-2}{5}, \frac{11}{5}\right)$$

Therefore, the co-ordinates of the point of intersection of the lines $2x - y + 3 = 0$ and $x + 2y - 4 = 0$ are $(\frac{-2}{5}, \frac{11}{5})$

Alternatively, by solving simultaneous equations $2x - y + 3 = 0$....(i) and $x + 2y - 4 = 0$.....(ii) using different methods you get the same answer. $(\frac{-2}{5}, \frac{11}{5})$.

c) Equation of a Line Parallel to a given line



These are three parallel antennas

Let, $ax + by + c = 0 \quad (b \neq 0)$ be the equation of the given straight line.

Now, convert the equation $ax+by+c=0$ to its slope-intercept form.

$ax+by+c=0$

$\Rightarrow by=-ax-c$

Dividing both sides by b , $[b \neq 0]$ we get, $y=-\frac{a}{b}x-\frac{c}{b}$ which is the slope-intercept form.

Now comparing the above equation to slope-intercept form $(y=mx+b)$ we get,

The slope of the line $ax+by+c=0$ is $(-\frac{a}{b})$.

Since the required line is parallel to the given line, the slope of the required line is also $(-\frac{a}{b})$.

Let ***k*** (an arbitrary constant) be the intercept of the required straight line. Then the equation of the straight line is $y=-\frac{a}{b}x+k$

$\Rightarrow by=-ax+bk$

$\Rightarrow ax+by=\lambda$, Where $\lambda=bk$ λ = another arbitrary constant.

Note:

(i) Assigning different values to λ in $ax+by=\lambda$, we shall get different straight lines each of which is parallel to the line $ax+by+c=0$. Thus, we can have a family of straight lines parallel to a given line.

(ii) To write a line parallel to a given line we keep the expression containing x and y same and simply replace the given constant by a new constant λ. The value of λ can be determined by some given conditions.

To get it more clear let us compare the equation $ax+by=\lambda$ with equation $ax+by+c=0$. It follows that to write the equation of a line parallel to a given straight line we simply need to replace the given constant by an arbitrary constant, the terms with x and y remain unaltered. For example, the equation of a straight line parallel to the straight line;

$7x-5y+9=0$ is $7x-5y+\lambda=0$ where λ is an arbitrary constant.

Example

Find the equation of the straight line which is parallel to $5x-7y=0$ and passing through the point $(2,-3)$.

Solution:

The equation of any straight line parallel to the line $5x-7y=0$ is $5x-7y+\lambda=0$ .... (i) [Where λ is an arbitrary constant].

If the line (i) passes through the point $(2,-3)$ then we shall have,

$$5.2-7(-3)+\lambda=0$$
$$\Rightarrow 10+21+\lambda=0$$
$$\Rightarrow 31+\lambda=0$$
$$\Rightarrow \lambda=-31$$

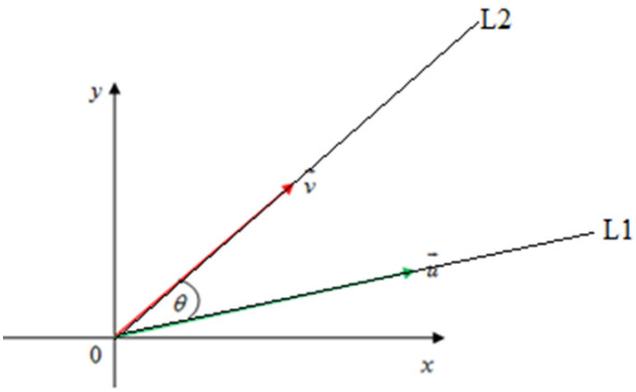
Therefore, the equation of the required straight line is $5x-7y+31=0$

d) Angles between two lines

Geometrically, the scalar product of two vectors $\vec{u}=(a,b)$ and $\vec{v}=(c,d)$ of plane is given by $\vec{u}\cdot\vec{v}=\|\vec{u}\|\|\vec{v}\|\cos\theta$. Where θ is the angle between vectors $\vec{u}$ and $\vec{v}$. From this relation we have $\cos\theta=\frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\|\vec{v}\|}$ or $\cos\theta=\frac{a\cdot c+b\cdot d}{\sqrt{a^2+b^2}\sqrt{c^2+d^2}}$.

We deduce that the angle θ between two lines l_1 and l_2 with direction vectors $\vec{u}=(a,b)$ and $\vec{v}=(c,d)$ respectively is given by $\theta=\cos^{-1}\left(\frac{a\cdot c+b\cdot d}{\sqrt{a^2+b^2}\sqrt{c^2+d^2}}\right)$.

Where $\cos^{-1}$ denote the inverse function of cosine.

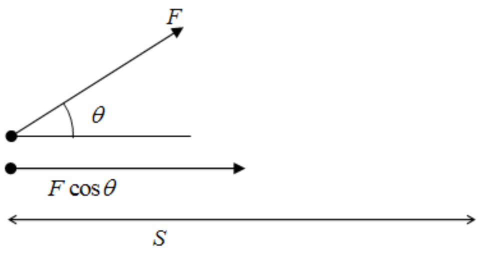


We can illustrate the scalar product in terms of work done by the force on the body:

We saw that if the constant force F and the displacement S are in the same direction, the work W done by the force on the body is $W = \vec{F}.\vec{S}$

If the force does not act in the direction in which motion occurs but an angle θ to it , then the work done is defined as the product of the component of the force in the direction of motion and the displacement in that direction.

$$W = \|\vec{F}\|.\|\vec{S}\| cos\theta$$



Notice

- Two lines are perpendicular if the angle between them is a multiple of a right angle
- Two lines are parallel and with the same direction if the angle between them is a multiple of a zero angle
- Two lines are parallel and with the opposite direction if the angle between them is a multiple of a straight angle

Example:

Find the angle between vectors $\vec{u}=(3,0)$ *and* $\vec{v}=(5,5)$

Solution

Let α be the angle between these two vectors

$$\alpha=\cos^{-1}\left(\frac{3\cdot5+0\cdot5}{\sqrt{3^3+0^2}\sqrt{5^2+5^2}}\right)=\cos^{-1}\left(\frac{\sqrt{2}}{2}\right)$$

$$\alpha=45^\circ$$

APPLICATION ACTIVITY 2.6.1

1. Find the equation of a straight line that passes through the point $(-2,3)$ and perpendicular to the straight line $2x+4y+7=0$

2. Find the equation of the straight line which passes through the point of intersection of the straight lines $x+y+9=0$ and $3x-2y+2=0$ and is perpendicular to the line

   $4x+5y+1=0$

3. Find the equation of the straight line passing through the point $(5,-6)$ and parallel to the straight line $3x-2y+10=0$.

4. Calculate the dot product and the angle formed by the following vectors: $\vec{u}=(3,4)$ and $\vec{v}=(-8,6)$

2.6.2. Distance between two lines and the distance between a point and line

ACTIVITY 2.6.2

1. In one Cartesian plane plot the straight lines containing the following points

$(0,4);\ (\frac{1}{2},2)$ and $(\frac{1}{2},2);\ (4,1)$

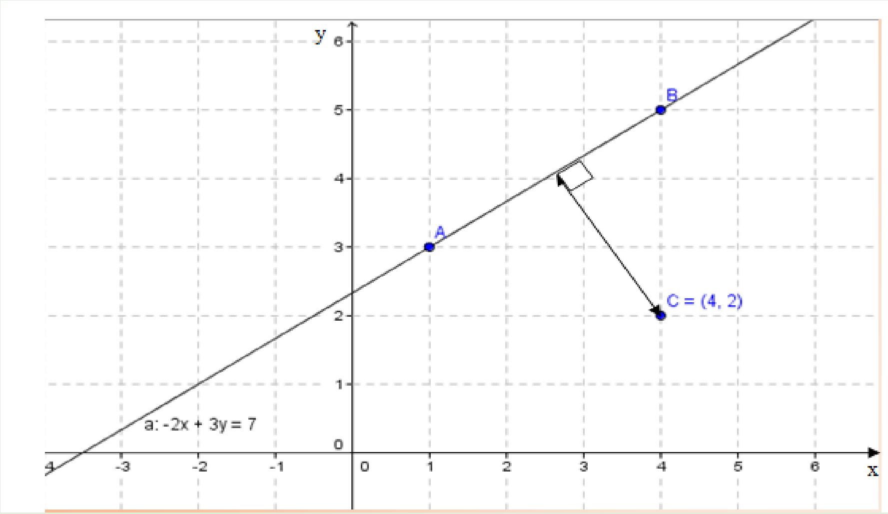
What is the distance between these two lines?

2. Also in one Cartesian plane plot the straight lines containing the following points

$(-3,-1);\ (0,3)$ and $(-2,-4);\ (1,0)$

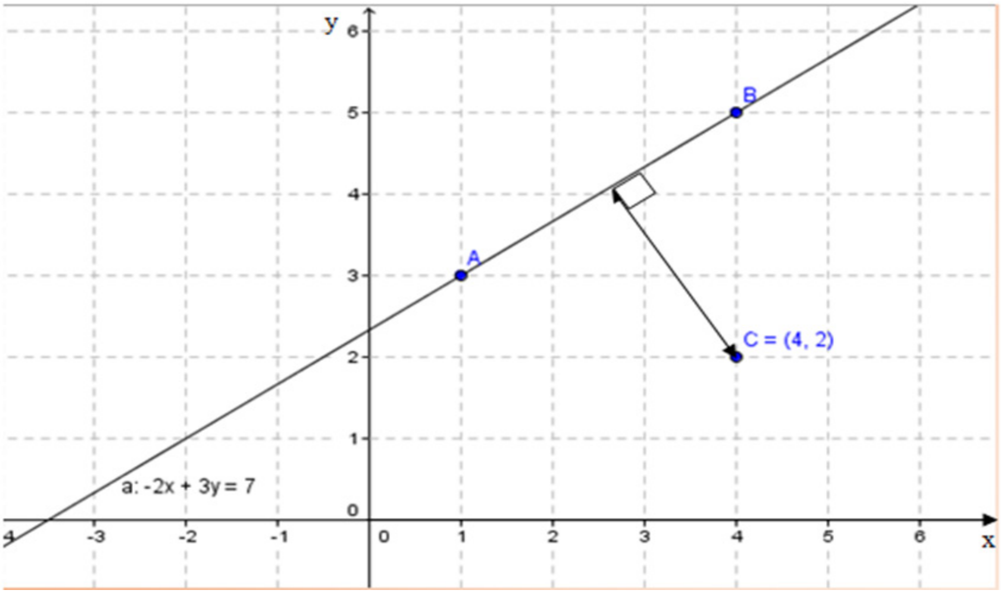
What is the distance between these two lines?

3. Find the shortest distance between the point $C(4,2)$ and the line $a: -2x+3y=7$



CONTENT SUMMARY

Distance between a point and a line



The perpendicular distance from a point $D(x_1, y_1)$ to the line $ax + by + c = 0$ is given by

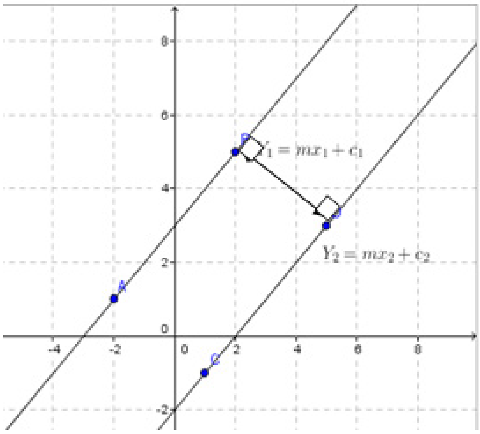
$$\frac{ax_1 + by_1 + c}{\sqrt{a^2 + b^2}}$$

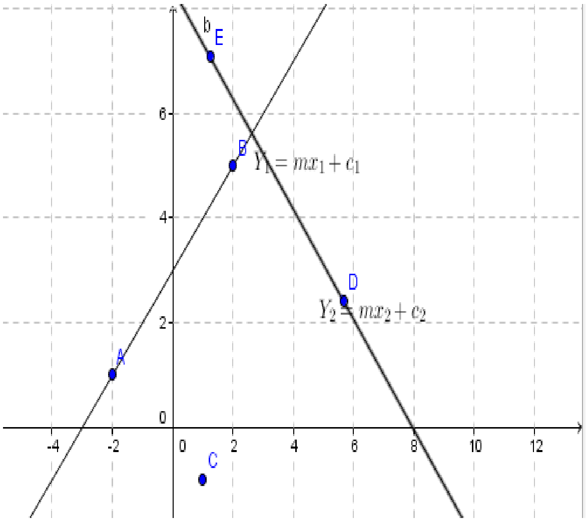
Example

Find the shortest distance from the line $2x + y - 1 = 0$ to the point $(3,2)$

Solution : The distance is $\frac{2(3)+1(2)-1}{\sqrt{(2)^2+(1)^2}} = \frac{7}{\sqrt{5}} = \frac{7\sqrt{5}}{5}$.

Distance between two parallel lines and two intersecting lines





$y_1 = mx_1 + c_1$ and $y_2 = mx_2 + c_2$ are Parallel $y_1 = mx_1 + c_1$ and $y_2 = mx_2 + c_2$ are intersecting

If they intersect, the distance is **0** at the point of intersection.

If they're parallel: the distance between them will be given by the following formula

$$d = \frac{|c_1 - c_2|}{\sqrt{1+m^2}}$$

Example

What is the distance between the lines $3x + 4y = 9$ and $6x + 8y = 15$?

Solution:

First of all we find the slopes of the two lines. We convert the equations into slope intercept form

$$3x+4y=9$$
$$\Rightarrow 4y=-3x+9$$
$$\Rightarrow y=\frac{-3}{4}x+\frac{9}{4}$$

$$6x+8y=15$$
$$\Rightarrow 8y=-6x+15$$
$$\Rightarrow y=\frac{-6}{8}x+\frac{15}{8}$$
$$\Rightarrow y=\frac{-3}{4}x+\frac{15}{8}$$

The slope m of the lines is $\frac{-3}{4}$

Hence both lines are parallel

y Intercept of the first line is $\frac{9}{4}$

y Intercept of the second line is $\frac{15}{8}$

Difference between the y- intercepts is $\left|\frac{9}{4}-\frac{15}{8}\right|$

$$=\left|\frac{18}{8}-\frac{15}{8}\right|$$
$$=\frac{3}{8}$$

$$\sqrt{1+m^2}=\sqrt{1+\frac{9}{16}}=\sqrt{\frac{25}{16}}=\frac{5}{4}$$

Distance between the lines $=\frac{3}{8}\div\frac{5}{4}=\frac{3}{8}\times\frac{4}{5}=\frac{3}{10}$.

Alternatively, the distance between two parallel lines

$L_1 \equiv ax+by+c_1=0$ *and* $L_2 \equiv ax+by+c_2$ is given by $\frac{|c_1-c_2|}{\sqrt{a^2+b^2}}$. Using the previous example, $3x+4y=9$ and $6x+8y=15$ the distance between the two lines is calculated taking $3x+4y=9$ and $3x+4y=\frac{15}{2}$.

Therefore, the distance is

$$\frac{|c_1-c_2|}{\sqrt{a^2+b^2}}=\frac{\left|9-\frac{15}{2}\right|}{\sqrt{3^2+4^2}}=\frac{\frac{3}{2}}{5}=\frac{3}{10}.$$

APPLICATION ACTIVITY 2.6.2

1. What is the distance between the lines
   $3x+4y=9$ and $6x+8y=-15$
2. What will be the distance between the lines
   $3x+4y=5$ *and* $6x-8y=45$
3. What is the distance between the lines
   $3x+4y=9$ *and* $6x-8y=18$
4. The distance between the straight lines
   $3x+4y=15$ *and* $6x+8y=9$ is?
5. How far is the line $3x-4y+15=0$ from the origin?
6. The distance between the lines $3x+4y+5=0$ and $6x-8y+5=0$ is?
7. What is the distance of the point $(-2,3)$ from the line from the line $x-y=5$?
8. What is distance from the point $(-3,-4)$ to the line $3x-4y-1=0$?
9. What is distance from the point $(2,-1)$ to the line $3x+4y=6$?

10. *Calculate the angles of the triangle with vertices:* $A=(6,0),\ B=(3,5)$ *and* $C=(-1,-1)$.

11. *Find the value of k if the angle between* $\vec{u}=(k,3)$ *and* $\vec{v}=(4,0)$ *is* 45°

12. Given the vectors $\vec{u}=(2,k)$ and $\vec{v}=(3,-2)$, calculate the value of k so that the vectors $\vec{u}$ and $\vec{v}$ are:

    a) Perpendicular.

    b) Parallel.

    c) Make an angle of 60°.

2.7. Determination of equation of a circle in 2D

2.7.1. Cartesian equation

ACTIVITY 2.7.1

In xy plane,

1. Use a compass to draw a circle centered at point $C(2,1)$ and passing through the point $P(3,0)$
2. Draw a line r that is a segment from C to P and find its length;
3. Use the distance between two points formula to calculate the distance $d(c,p)=r$
4. Deduce the distance between the points $M(x,y)$ and the center $C(2,1)$ to calculate the radius r of the circle

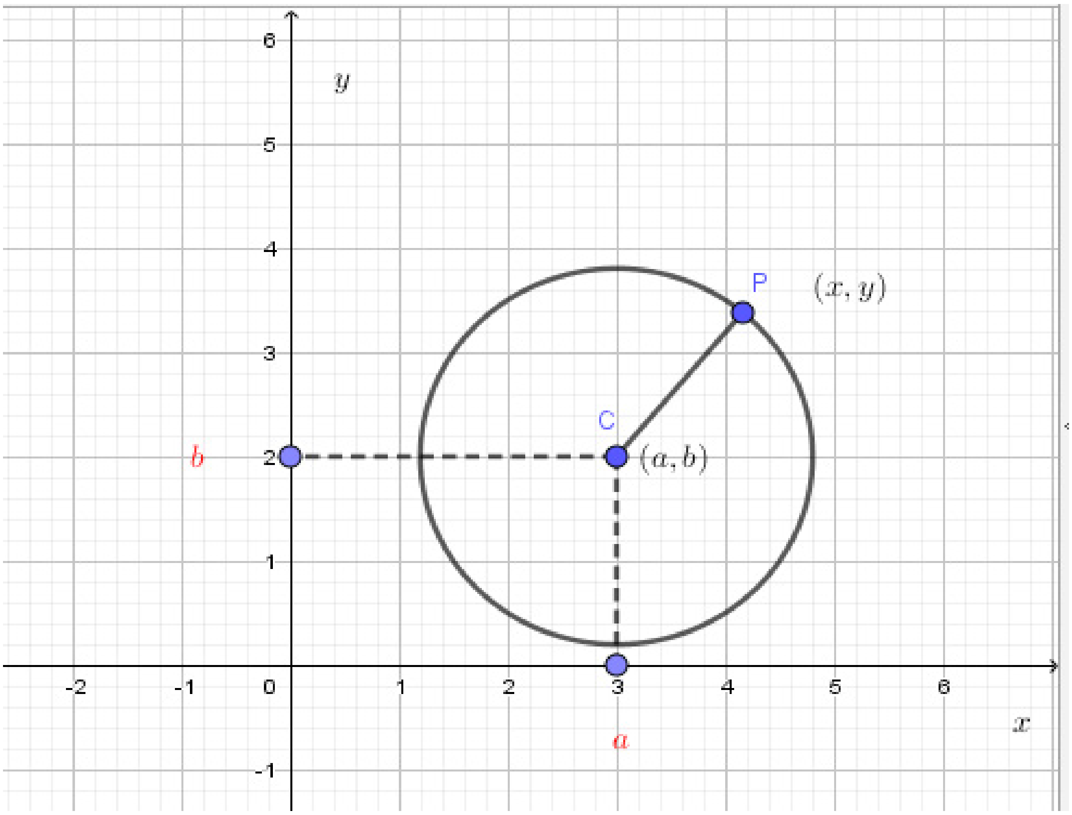
CONTENT SUMMARY

The circle is the locus of all points (x,y) which are equidistant from a fixed given point. The fixed point is called the centre of the circle and the distance from the centre to ant point is called the radius of the circle.

Let $P(x, y)$ be one point of the circle of Centre $C(a,b)$ and radius r. The distance between the center and the point P is: $d = r = \sqrt{(x-a)^2 + (y-b)^2}$. Squaring the two sides of the equation we find the equation $(x-a)^2 + (y-b)^2 = r^2$

This is the equation of tour circle of centre $C(a,b)$ and radius *r.*

$$C \equiv (x-a)^2 + (y-b)^2 = r^2$$



Expanding this equation we have $C \equiv x^2 + y^2 - 2ax - 2by + a^2 + b^2 = r^2$.
Supposing that , and that ; then
General equation of the circle $C \equiv x^2 + y^2 + kx + ly + m = 0$

In this case the centre is given by $C = \left(-\frac{k}{2}, -\frac{l}{2}\right)$ and the radius is given by $r = \frac{1}{2}\sqrt{k^2 + l^2 - 4m}$.

If the coordinates of the point C are $(0,0)$, then the centre is at origin. In this case the equation of the circle $C \equiv x^2 + y^2 = r^2$.

Example 1: Find the equation of the circle centred at origin and with radius 4

Solution:

Centre is $C(0,0)$, radius is $r = 4$. Then the equation is

$(x-0)^2 + (y-0)^2 = 4^2$ or $x^2 + y^2 = 16$

Example 2: Find the equation of the circle centred at point $A(3,5)$ and passing through point $B(3,0)$

Solution:

Centre is $C(3,5)$, radius is the distance from point A to point B

$r = \sqrt{0+25} = 5$. Then the equation is

$(x-3)^2 + (y-5)^2 = 5^2$ or $x^2 + y^2 - 6x - 10y = -9$

Example 3: Find the centre and the radius of the circle with equation $(x+4)^2 + (y-3)^2 - 16 = 0$

Solution:

Compare the given equation with $(x-a)^2 + (y-b)^2 = r^2$ we can rewrite it as $(x-(-4))^2 + (y-3)^2 = 4^2$. Then the centre is $(4,3)$ and the radius is $r = 4$

Example 4: Find the centre and the radius of the circle with equation $x^2 + y^2 - 4x - 2y = 4$

Solution:

$x^2+y^2-4x-2y=4 \Leftrightarrow x^2+y^2-4x-2y-4=0$.

The centre is given by $C\left(-\frac{-4}{2},-\frac{-2}{2}\right)=(2,1)$ and the radius is

$r=\frac{1}{2}\sqrt{(-4)^2+(-2)^2-4\times(-4)}=\frac{1}{2}\sqrt{36}=3$

Another method:

Rearranging the given equation

$x^2+y^2-4x-2y=4$

$\Leftrightarrow x^2-4x+4-4+y^2-2y+1-1=4$

$\Leftrightarrow (x^2-4x+4)+(y^2-2y+1)=4+5$

$\Leftrightarrow (x-2)^2+(y-1)^2=3^2$

The centre is $(2,1)$ and the radius is 3.

APPLICATION ACTIVITY 2.7.1

1. Find the equation of the circle with centre $(3,2)$ and radius 4.
2. Find the equation of the circle whose diameter is a line segment joining the points $(3,-2)$ and $(3,6)$
3. Find the centre and the radius of the circle with equation $x^2+y^2-4x+8y=12$
4. Find the centre and the radius of the circle with equation $3x^2+3y^2-18x-12y-27=0$

2.7.2. Parametric equation

ACTIVITY 2.7.2

a) Find the centre and the radius of circle $x^2 + y^2 = 19$

b) Given that (a,b) is the center of the circle and r its radius, use the result from (a) and complete the following formula:

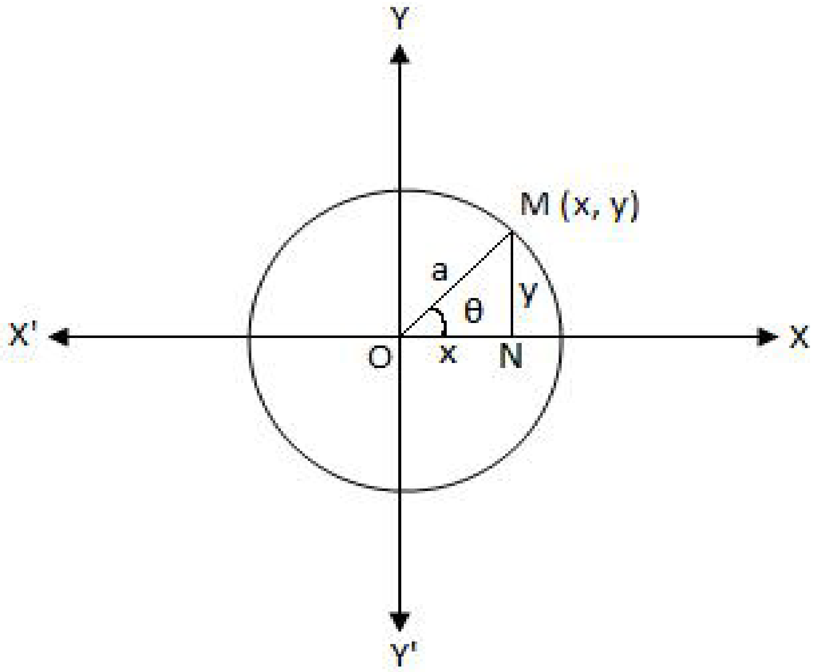
$$\begin{cases} x = a + r\cos\theta \\ y = b + r\sin\theta \end{cases}$$

c) What do you notice from solution (b) ?

CONTENT SUMMARY

From the figure, a circle whith centre at the origin and radius , the equation of a circle is given by PYTHAGORAS' THEOREM

$x^2 + y^2 = a^2 \quad or \quad x^2 + y^2 = r^2; (r = a)$

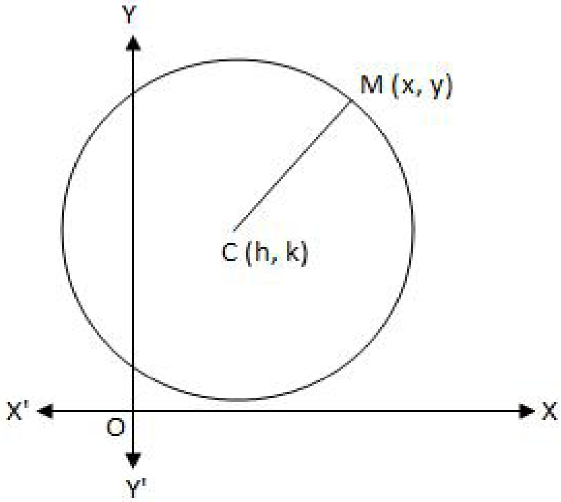


Expressing the equation of a circle using a parameter , we write parametric equations

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases} for\ 0 \le \theta \le 2\pi$$

For a circle with centre at the point (h, k) and radius *r*, the equation of a circle

$(x-h)^2 + (y-k)^2 = r^2$



Hence, we write parametric equations

$$\begin{cases} x = h + r\cos\theta \\ y = k + r\sin\theta \end{cases} \qquad for\ 0 \le \theta \le 2\pi$$

Noted: the angle θ is given by

- $\begin{cases} \cos\theta = \dfrac{x-h}{r} \\ \sin\theta = \dfrac{y-k}{r} \end{cases}$ with $C(h, k)$ and r
- $\begin{cases} \cos\theta = \dfrac{x}{r} \\ \sin\theta = \dfrac{y}{r} \end{cases}$ with $C(0, 0)$ and r

Therefore, the formula obtained from solution (b) is called the parametric equation of a circle of radius r and centred at the point P with coordinate (a, b).

Example:

Find the centre and the radius of the circles with equations. Hence, write parametric equations for each.

$a)\ (x-2)^2+y^2=25$

$b)\ x^2+y^2-4x-2y=4$

$c)\ x^2+y^2=4$

Solution

a. i) $(x-2)^2+y^2=25$

Comparing with $(x-a)^2+(y-b)^2=r^2$

$(x-2)^2+(y-0)^2=5^2$. Therefore; the centre $C(2,0)$ and $r=5$

ii) The parametric equations are:

$$\begin{cases} x=2+5\cos\theta \\ y=5\sin\theta \end{cases} \quad \text{for } 0\le\theta\le 2\pi$$

b. i) $x^2+y^2-4x-2y=4$

By rearranging like terms and completing terms

$$(x^2-4x+4)+(y^2-2y+1)=4+4+1$$

$$(x-2)^2+(y-1)^2=3^2$$

Therefore; the centre $C(2,1)$ and $r=3$

ii) The parametric equations are:

$$\begin{cases} x=2+3\cos\theta \\ y=1+3\sin\theta \end{cases} \quad \text{for } 0\le\theta\le 2\pi$$

c. i) $x^2 + y^2 = 4$

Comparing with $(x-a)^2 + (y-b)^2 = r^2$

$(x-0)^2 + (y-0)^2 = 2^2$

Therefore; the centre $C(0,0)$ and $r = 2$

ii) The parametric equations are:

$$\begin{cases} x = 2\cos\theta \\ y = 2\sin\theta \end{cases} \quad for \quad 0 \le \theta \le 2\pi$$

APPLICATION ACTIVITY 2.7.2

1. State the equation of the circle that can satisfy the given conditions. Hence, write parametric equations for each.

   a) $c(0,0)$, $\mathrm{r} = 5$

   b) C(4,-2);r=8

   c) $c(-4,-2)$ and passes through $P(1,3)$

2. Find the equation of the circle passing through the points $(0,1),(4,3)$ and $(1,-1)$.

3. Given the equations of a circle

   $x^2 + y^2 - 10x - 8y + 5 = 0$ (1)

   $x^2 + y^2 - 6x + 8y + 25 = 0$ (2)

   a) Express each equation below in the form $(x-a)^2 - (y-b)^2 = r^2$

   b) Write the coordinates centre and find the radius of the circle.

   c) Find parametric equations.

2.7.3 Intersection of a line and a circle

ACTIVITY 2.7.3

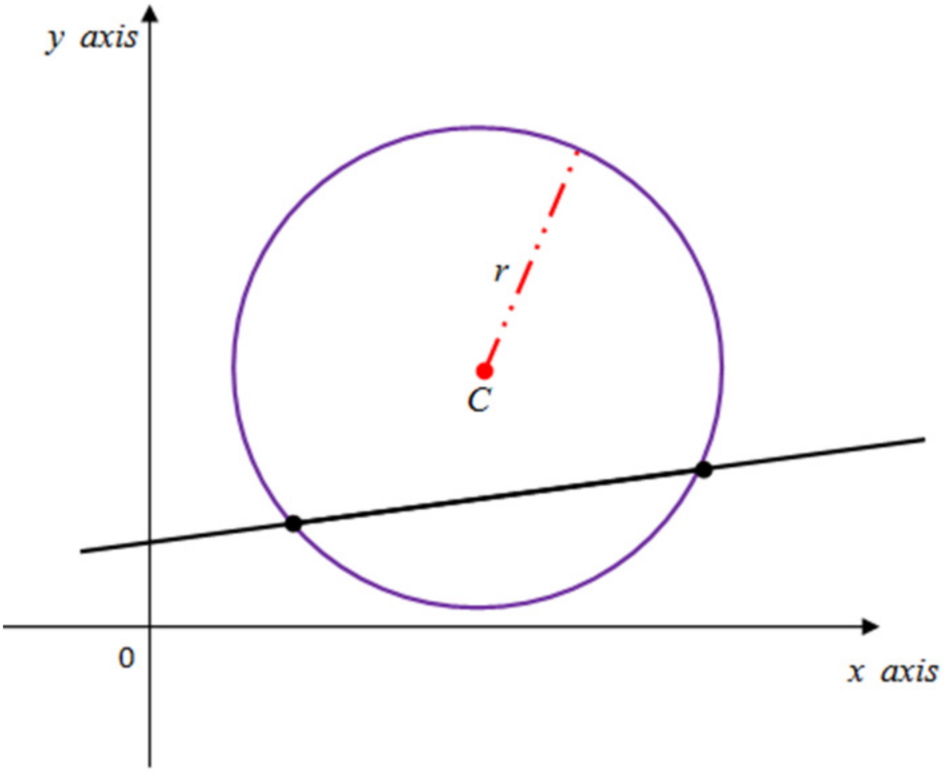
Consider the circle with equation $x^2 + y^2 - 10x - 6y + 25 = 0$ and the lines $y = x$, $x + y = 2$ and $y = 6$. In Cartesian plane

1. Sketch that circle and these lines
2. Find the intersection of the circle and the lines.
3. What can you say about the intersection of the circle and each line?

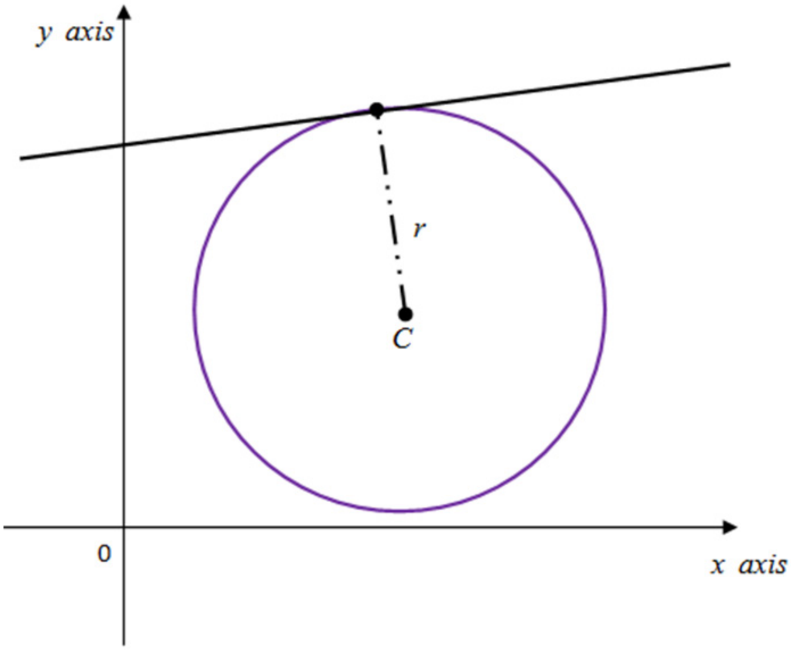
CONTENT SUMMARY

Intersection of a line and a circle occurs in three possible situations:

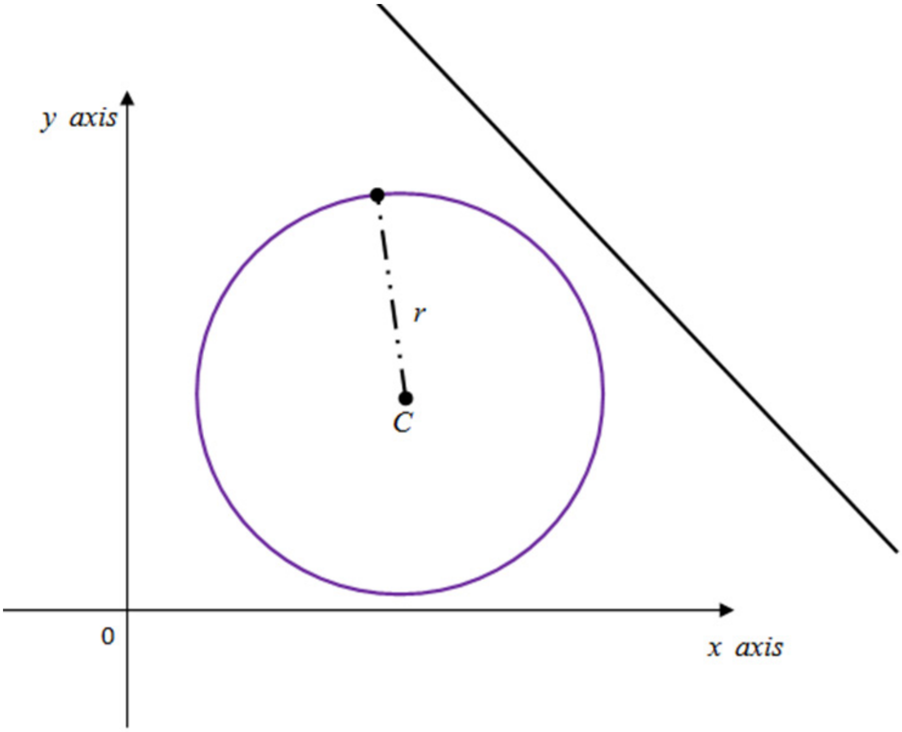
- **The line cuts the circle in two distinct points**



- **The line touches the circle at one point (tangent to the circle)**



- The **line neither cuts nor touches the circle.**



Given the equations of a line and a circle we can solve the two equations simultaneously.

If

- There are two distinct roots then the line cuts the circle in two distinct points and the intersection is the set of these two points. In this case part of the line is a chord of the circle and the distance from the centre of the circle to the line is less than the radius of the circle.
- There is a repeated root then the line touches the circle and the intersection is this root. In this case the line is tangent to the circle and the distance from the centre of the circle to the line is equal to the radius of the circle.
- There is no real root then the line neither cuts nor touches the circle. In this case there is no intersection and the distance from the centre of the circle to the line is greater than the radius of the circle.

Examples 1:

Show that part of the line $3y = x + 5$ is a chord of the circle $x^2 + y^2 - 6x - 2y - 15 = 0$ and find the length of this chord .

Solution

Method 1

Solve simultaneously the system

$$\begin{cases} 3y = x + 5 \\ x^2 + y^2 - 6x - 2y - 15 = 0 \end{cases}$$

$$x = 3y - 5$$

$$(3y - 5)^2 + y^2 - 6(3y - 5) - 2y - 15 = 0$$

$$9y^2 - 30y + 25 + y^2 - 18y + 30 - 2y - 15 = 0$$

$$y^2 - 5y + 4 = 0$$

Either $y = 4$ or $y = 1$

If $y = 4,\ x = 7$

$y = 1,\ x = -2$

Thus, the line cuts the circle in 2 distinct points $(7,4)$ and $(-2,1)$

The length of the cord is the distance between points $(7,4)$ and $(-2,1)$

i.e the length of the chord is

$$\begin{aligned} l &= \sqrt{81+9} \\ &= \sqrt{90} \\ &= 3\sqrt{10} \ \textit{units} \end{aligned}$$

Method 2

Circle: $x^2 + y^2 - 6x - 2y - 15 = 0$

Line: $3y = x + 5 \Leftrightarrow x - 3y + 5 = 0$

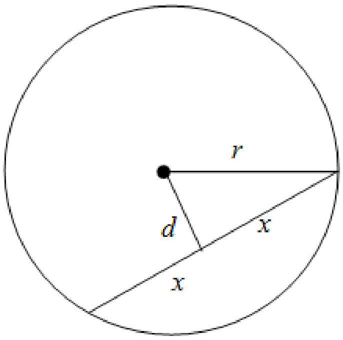
The centre of the circle is $\left(-\frac{-6}{2}, -\frac{-2}{2}\right) = (3,1)$

The radius is $\frac{1}{2}\sqrt{36+4+60} = 5$

The distance from the centre of the circle to the line is

$$\begin{aligned} \frac{3-3+5}{\sqrt{1+9}} &= \frac{5}{\sqrt{10}} \\ &= \frac{5\sqrt{10}}{10} \\ &= \frac{1}{2}\sqrt{10} \ \textit{units} \end{aligned}$$

Since this distance is less than the radius of the circle, then the line cuts the circle in two points and hence part of the line is a chord of the circle.



From the diagram:

r is the radius of the circle equals to 5

d is the distance from the centre of the circle to the line equals $\frac{1}{2}\sqrt{10}$

2x is the length of the chord

Now,

$$\begin{aligned} 2x &= 2\sqrt{r^2 - d^2} \\ &= 2\sqrt{25 - \frac{10}{4}} \\ &= \sqrt{90} \\ &= 3\sqrt{10} \end{aligned}$$

Example 2:

Find the intersection of the circle $x^2 + y^2 - 14x + 12y + 69 = 0$ and the line $x + y = 1$

Solution

Solve simultaneously the system

$$\begin{cases} x^2 + y^2 - 14x + 12y + 69 = 0 & (1) \\ x + y = 1 & (2) \end{cases}$$

From (2): $y=1-x \quad (3)$

(3) in (1):

$$x^2+(1-x)^2-14x+12(1-x)+69=0$$

$$\Leftrightarrow x^2+1-2x+x^2-14x+12-12x+69=0$$
$$\Leftrightarrow 2x^2-28x+82=0$$
$$\Leftrightarrow x^2-14x+41=0$$

$\Delta=196-164=32$

Since $\Delta>0$, there are two points of intersection.

$$x_1=\frac{14+4\sqrt{2}}{2}=7+2\sqrt{2} \Rightarrow y_1=1-x_1=-6-2\sqrt{2}$$
$$x_2=\frac{14-4\sqrt{2}}{2}=7-2\sqrt{2} \Rightarrow y_2=1-x_2=-6+2\sqrt{2}$$

The points of intersection are $\left(7+2\sqrt{2},-6-2\sqrt{2}\right)$ and $\left(7-2\sqrt{2},-6+2\sqrt{2}\right)$

APPLICATION ACTIVITY 2.7.3

Find the intersection point of the circle and the lines.

Consider the following equation of circle and the line:

(1) $x^2+y^2-14x+12y+69=0$ and line $y=-2$

(2) $x^2+y^2-14x+12y+69=0$ and line $y=x$

(3) $x^2+y^2-14x+12y+69=0$ and line $x+y=1$

2.8 END UNIT ASSESSMENT 2

1. Determine the distance between the pairs of the points below:

   a) $(2,1);(8,7)$ b) $(-3,-8);(-3,-2)$

   c) $(5,-6);(0,-2)$; d) $(0,0);(x,y)$

2. Find the centre and radius of each of the following circles

   a) $x^2+y^2=10$ b) $(x-3)^2+y^2=25$

   c) $(x-2)^2+(y+1)^2=18$ d) $x^2+y^2+2x-2y=2$

3. Three points A, B and C have coordinates $(1,0),(4,4)$ and $(13,5)$ respectively. By how many units does the length of AC exceed the length of AB?

4. Find the coordinates of the midpoints of the straight lines joining each of the following pairs of points:

   a. $(3,4)$ and $(7,10)$

   b. $(2,8)$ and $(10,4)$

   c. $(0,5)$ and $(6,3)$

   d. $(4,3)$ and $(1,5)$

5. $L(5,1)$ is the midpoint of the straight line joining point $C(p,-3)$ to point $D(7,q)$. Find p and q.

6. Find the Cartesian equations of the lines whose vector equations are given below. Give your answers in the form $y = mx + c$

a) $(x, y) = (3, 2) + r(1, 4)$

b) $(x, y) = (-1, 2) + r(-2, 3)$

7. Find the vector equations of the lines whose Cartesian equations are given below

a) $\dfrac{x-2}{3} = \dfrac{y-1}{4}$ b) $\dfrac{x-5}{1} = \dfrac{y+3}{-4}$

8. State which of the following are equations of circles

a) $x^2 + 2xy + y^2 = 4$ b) $x^2 + y^2 = 25$

c) $x^2 + y^2 = 19$ d) $2x^2 + 3x - y^2 + 2y = 16$

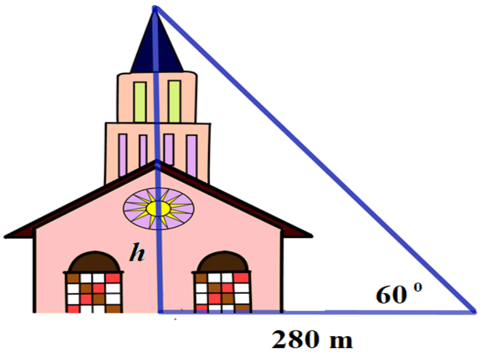
UNIT 3

APPLICATION OF TRIGONOMETRIC CONCEPTS IN SOLVING PROBLEMS

Key Unit competence: Apply trigonometric concepts in solving problems on triangles and real life situation.

3.0 INTRODUCTORY ACTIVITY 3

The angle of elevation of the top of the Cathedral from a point 280 m away from the base of its steeple on level ground is 60^0. By using trigonometric concepts, find the height of the cathedral.



Trigonometry studies relationship involving lengths and angles of a triangle. The techniques in trigonometry are used for finding relevance in navigation particularly satellite systems and astronomy, naval and aviation industries, land surveying and in cartography (creation of maps). Even if those are the scientific applications of the concepts in trigonometry, most of the mathematics we study would seem to have little real-life application. Trigonometry is really relevant in our day to day activities.

3.1 Angle and its measurements

ACTIVITY 3.1

Discuss what an angle is. Sketch different types of angles and name them (acute, obtuse,etc). Use the protractor to measure the angles to verify the sizes.

1. Copy the diagram and rotate vector $\overrightarrow{0A}$ by an angle of

   a) 30 degrees b) -45 degrees c) 120 degrees

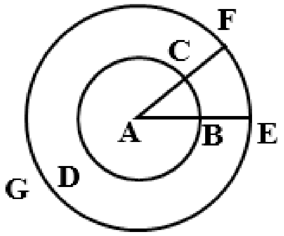
CONTENT SUMMARY

An angle is the opening that two straight lines form when they meet ;The measure of an angle is a number which indicates the amount of rotation that separates the rays of the angle. There is one immediate problem with this, as pictured below.

Which amount of rotation are we attempting to quantify? What we have just discovered is that we have at least two angles described by this diagram.

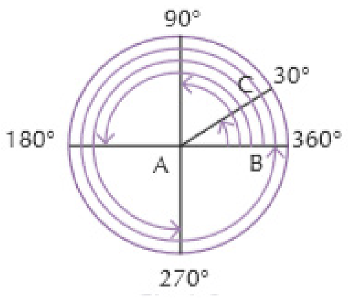
Recall that if an angle measures strictly between 0° and 90° it is called an **acute angle** and if it measures strictly between $90^{\circ}$and 180° it is called an **obtuse** angle.

In the figure below, when the straight line FA meets the straight line EA, they form the angle that we call angle FAE. We may also call it “the angle at the point A,” or simply “angle A.” The two straight lines that form an angle are called its sides. And the size of the angle does not depend on the lengths of its sides.

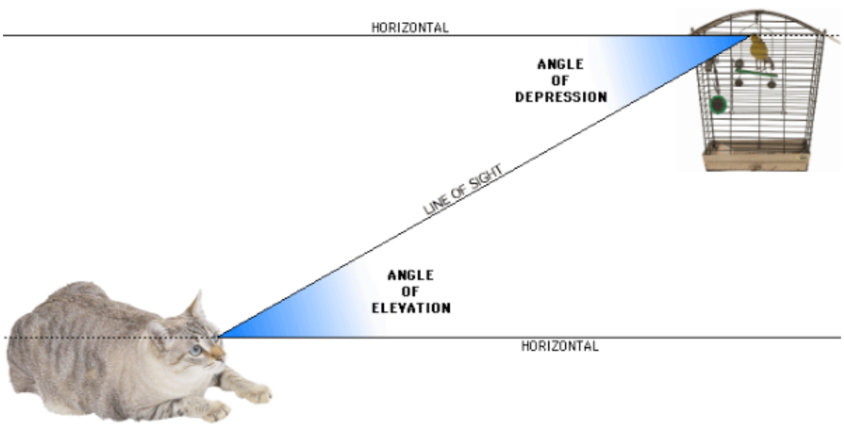


3.1.1 Degree measure

To measure an angle in degrees, we take the circumference of a circle and divide it into 360 equal parts. We call each of those equal parts a "degree." Its symbol is a small o: 1° = "1 degree." From figure below, angle BAC is 30° means that its sides enclose 30 of those equal divisions. Arc BC is $\frac{30}{360}$ of the entire circumference.

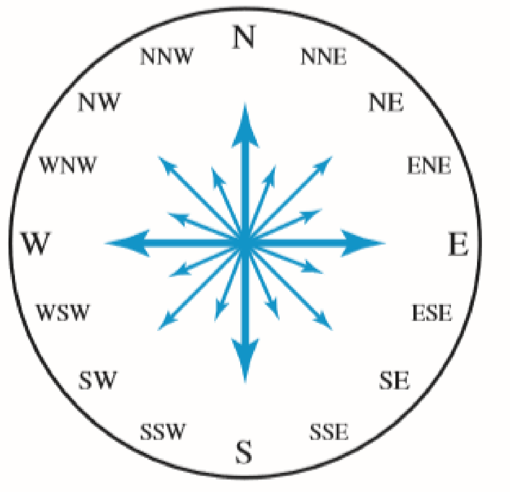


In observation, the angle below horizontal that an observer must look to see an object that is lower than the observer is called the angle of depression (see the figure below), while the angle of elevation of an object as seen by an observer is the angle between the horizontal and the line from the object to the observer's eye.



Example:

Refer to the 16 compass bearings shown in the figure below, North corresponds to an angle of 0°, and other angles are measured clockwise from north.



Find the angle in degrees that describes the compass bearing.

(a) NE (northeast) (b) NNE (north-northeast) (c) WSW (west-southwest)

Solution

From the given compass bearing, we note that between two consecutive bearings, there are 22.50 $\left(\text{since } \frac{360^0}{16} = 22.5^0\right)$. Then, measurements in degrees

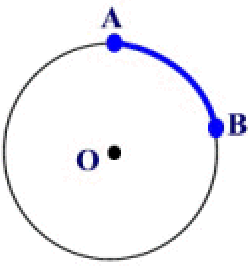
a) for NE (northeast) is $22.5^0 \times 2 = 45^0$

b) for NNE (north-northeast) is $22.5^0 \times 1 = 22.5^0$

c) for WSW (west-southwest) is $22.5^0 \times 11 = 247.5^0$

3.1.2 Radian measure

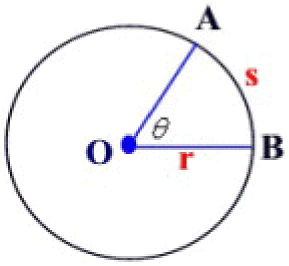
An arc of a circle is a "portion" of the circumference of the circle.



The **length of an arc** is simply the length of its "portion" of the circumference. Actually, the circumference itself can be considered as an arc length

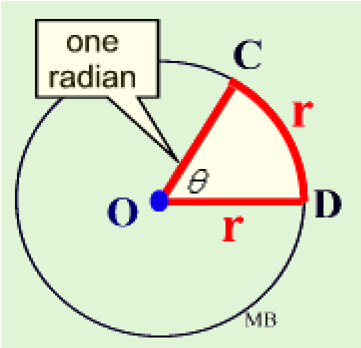
The length of an arc (or arc length) is traditionally symbolized by **S.**

In the diagram below, it can be said that " $\overset{\frown}{AB}$ subtends angle θ ".



If the length of an arc of a circle, **S**, (think of straightening it out), is the same as the length of the circle's radius, **r**, a specific situation occurs. The angle, θ, created by this situation is called a radian.

A radian is the measure of an angle θ that, when drawn as a central angle, subtends an arc whose length equals the length of the radius of the circle.



When radius **r** = arc length **r**, the angle θ measures 1 radian

Radian measure is another way of expressing the measure (size) of an angle. It is considered to be a "pure" measure since it is based upon the radius of the circle being wrapped along the circumference. While at first it may seem easier to work with degrees, you will find that there is a mathematical simplicity to the use of radians

Like degrees, radian measures the amount of the rotation from the initial side to the terminal side of an angle.

Example:

Through how many radians does the minute hand of a clock rotate in 50 min? In 7 hours?

Solution

It is known that, the minute hand of a clock rotates circumference of a circle in 60min, through $2\pi\ radians$. Here, we are required to find the radian travelled by minute hand of a clock for a given time.

a) Minute hand of a clock rotates the circle in 50min, through $\frac{2\pi \times 50}{60} radians$ or $\frac{5\pi}{3}\ radians$

b) Minute hand of a clock rotates the circle in 7 hours or 420 minutes, through $\frac{2\pi \times 420}{60} radians$ or $14\pi\ radians$.

APPLICATION ACTIVITY 3.1

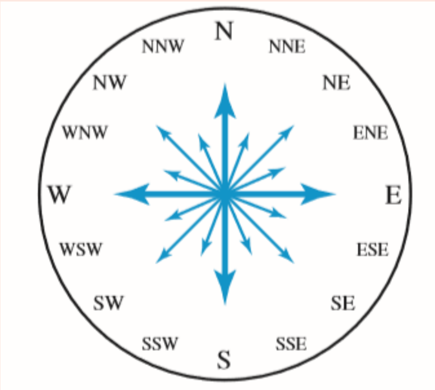
1. Find the circumference of the circle with the given radius r. State the correct unit

   a) $r = 5\,cm$ b) $r = 4.6\,m$

2. Find the radius of the circle with the given circumference C.

   a) $c = 14\,m$ b) $c = 16m$

3. Referring to the 16 compass bearings shown in the figure, North corresponds to an angle of 0°, and other angles are measured clockwise from north.



Find the angle in degrees that describes the compass bearing

a) SSW (south-southwest)

b) WNW (west-northwest)

c) NNW (north-northwest)

d) Which compass direction is closest to a bearing of 121°?

e) Which compass direction is closest to a bearing of 219°?

f) Through how many radians does the hour hand of a clock rotate in 8h? In 1 week?

4. Express in radians the smaller angle made by the hands of a clock at 4:00, at 2:30, and at 10:12.

3.2 Units conversion

ACTIVITY 3.2

Carefully draw a large circle on a piece of paper, either by tracing around a circular object or by using a compass. Identify the center of the circle (O) and draw a radius horizontally from O toward the right, intersecting the circle at point A. Then cut a piece of thread or string the same size as the radius. Place one end of the string at A and bend it around the circle counter clockwise, marking the point B on the circle where the other end of the string ends up. Draw the radius from O to B. As the measure of angle AOB is one radian,

1. What is the circumference of the circle, in terms of its radius r?
2. How many radians must be there in a complete circle?
3. If we cut a piece of thread 3 times as big as the radius, would it extend halfway around the circle? Why or why not?
4. How many radians are in a straight angle?

CONTENT SUMMARY

According to the definition of a radian, the number of radians in a circle is equal to the number of the radius can be laid off along the circumference.

Since the circumference of the circle is $c = 2\pi r$, there are $\frac{2\pi r}{r} = 2\pi$ radians in any circle.

But the angle of degrees in a circle is 360^0. Hence $2\pi\, radians = 360^0$ or $\pi\, radians = 180^0$.

Thus, $1\, radian = \frac{180^0}{\pi} \simeq \frac{180^0}{3.14159} \simeq 57.2958^0 \simeq 57^0 17.75'$ with $1^0 = 60' = 60$ minutes.

Also, $1^0 = \frac{\pi}{180^0} radian \simeq 0.017453\, radian$.

- To convert radians to degrees, you multiply the given radians by $\left(\frac{180^0}{\pi\, radians}\right)$
- To convert degrees to radians, you multiply the given degrees by $\left(\frac{\pi\, radians}{180^o}\right)$.

By convention, for angles measurement, when no unit of measure is indicated, note that an angle is expressed in radians. Thus if we read $\theta = \frac{\pi}{3}$, we understand that $\theta = \frac{\pi}{3} radians = 60^0$.

From definition of radian, if θ is a central angle in a circle of radius r, and if θ is measured in radians, then the length s of the intercepted arc is given by $s = r\theta$.

Example 1:

a) How many radians are in 90 degrees?

b) How many degrees are in $\frac{\pi}{6}$ radians?

c) Find the length of an arc intercepted by a central angle of $\frac{\pi}{2}$ radian in a circle of radius 10 metres

d) Find the radian measure of a central angle that intercepts an arc of length s in a circle of radius r.

Solution

a) Since π radians and 180^0 both measure a straight angle, to convert degrees to radians, we can use the conversion factor $\frac{\pi\, radians}{180^o}=1$. Let us multiply both sides by 90^0:

$$90^0\left(\frac{\pi\, radians}{180^o}\right)=90^0 \text{ or } \frac{\pi}{2} radians=90^0.$$

Hence, in 90^0 there is $\frac{\pi}{2} radians$.

b) In this case, to convert radians to degrees, we use the conversion factor

$$\frac{180^0}{\pi\, radians}=1.$$

And we get: $\frac{\pi}{6} radians=\frac{\pi}{6} radians\left(\frac{180^0}{\pi\, radians}\right)=30^0$

Finally, $\frac{\pi}{6} radians=30^0$.

c) A central angle of 1 radian intercepts an arc of length 1 radius, which is 10 metres. Therefore, a central angle of $\frac{1}{2}$ radian intercepts an arc of length $\frac{1}{2}$ radius, which is 5 metres.

d) Let x be the radian measure of a central angle that intercepts an arc of length s in a circle of radius r .

We get the following problem with ratios: $\frac{x\, radians}{s\, units}=\frac{1\, radian}{r\, units}$ or $xr=s$ or $x=\frac{s}{r}$.

Thus, the radian measure of a central angle that intercepts an arc of length s in a circle of radius r is $x=\frac{s}{r}$.

Example 2

Express 6.75^0 in terms of radians

Solution 2A:

Since $180^0 = \pi$, let us use conversion factor $\frac{\pi \, radians}{180^o} = 1$.

Thus, $6.75^0 = 6.75^0 \left(\frac{\pi \, radians}{180^o} \right) = \frac{675\pi \, radians}{18,000^o} = \frac{3\pi}{80}$.

Replacing π by 3.1416, we get $6.75^0 = 0.1178$ correct to four decimal places.

Solution 2B:

Some scientific calculators have **DRG** key which converts from degrees to radians to grads. With the calculator in degree mode, enter 6.75 and then press **DRG** key. The result is $0.1178097 \, radians$.

Example 3

A circle has a radius of $50\,cm$.

a) How long is the arc intercepted by a central angle of 72^0?

b) How large is the central angle that intercepts an arc of $60\,cm$?

Solution

a) As the length s of the intercepted arc is given by $s = r\theta$, let us first convert

72^0 into radians. $72^0 = 72\left(\frac{\pi}{180}\right) = \frac{2\pi}{5}$;

Using $s = r\theta$, with $\theta = \frac{2\pi}{5}$ and $r = 50\,cm$, we get $s = 50\left(\frac{2\pi}{5}\right) = 20\pi \, cm$.

b) Using $s = r\theta$, where $s = 60$ and $r = 50\,cm$, we get $60 = 50\theta$ or $\theta = \frac{6}{5} radians$.

APPLICATION ACTIVITY 3.2

1. Without using a calculator express in degrees the following angles

   a) $\frac{\pi}{4}$ b) $\frac{2\pi}{3}$ c) $\frac{\pi}{6}$ d) $\frac{\pi}{8}$

   e) $\frac{5\pi}{12}$ f) $\frac{3\pi}{8}$ g) $\frac{2\pi}{5}$ h) $\frac{5\pi}{4}$

2. Express in radians, leaving the result in terms of π (do not use a calculator).

   a) 150^0 b) 225^0 c) 45^0 d) 90^0 e) 30^0

3. One angle of a triangle is $\frac{3\pi}{10}$. Another angle is 24^0. Express the third angle in radians.

4. A simple pulley with the given radius r used to lift heavy objects is positioned 10 feet above the ground level. Given that the pulley rotates θ^0, determine the height to which the object is lifted. (a) $r = 4\,cm, \theta = 720^0$ (b) $r = 2m, \theta = 180^0$

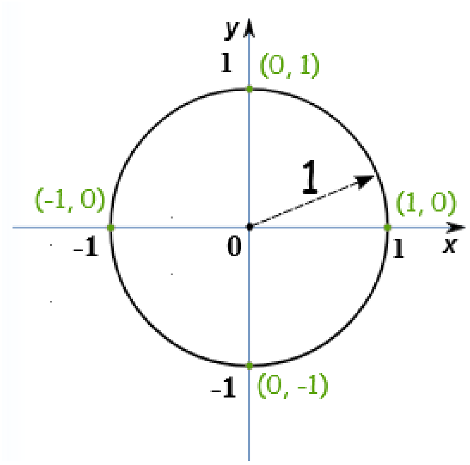
3.3 Unit Circle

ACTIVITY 3.3

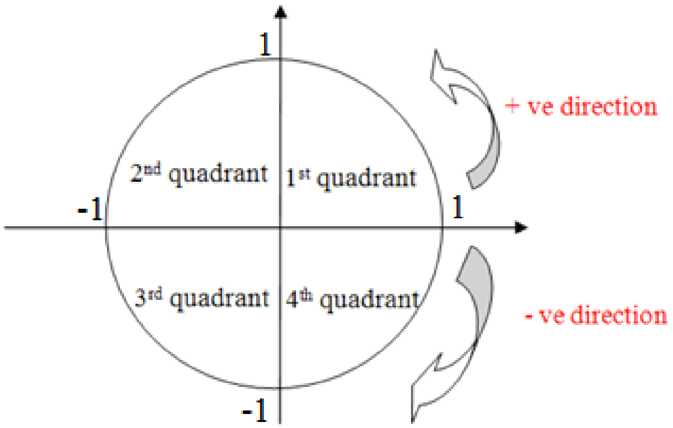
Imagine a point on the edge of a wheel. As the wheel turns, how is the point above the centre? Represent this using a drawing.

CONTENT SUMMARY

The unit circle is the circle with centre (0, 0) and radius 1 unit.

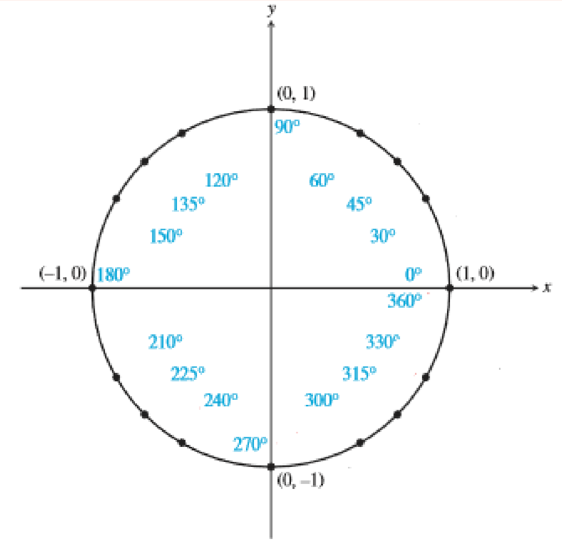


A unit circle is a circle of radius one centered at the origin (0,0) in the Cartesian coordinate system in the Euclidian plane. In the unit circle, the coordinate axes delimit four quadrants that are numbered in an anticlockwise direction. Each quadrant measures 90 degrees, means that the entire circle measures 360 degrees or 2π radians.



APPLICATION ACTIVITY 3.3

Study this diagram by finding out the radians related to the given angle in degrees on this unit circle. Can you find the coordinates of these points related to the given angles?

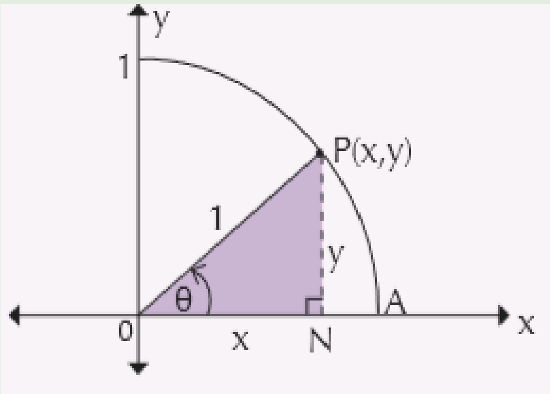


3.4 Trigonometric ratio of special angles

3.4.1 Trigonometric ratios

ACTIVITY 3.4.1

Consider a point $P(x, y)$ which lies on the unit circle in the first quadrant. OP makes an angle θ with the *x-axis* as shown in the following figure:



Draw a figure similar to figure above on the Cartesian plane. The radius is 1 unit. Pick any point $P(x, y)$. What is the value of $\frac{x}{1}$ and $\frac{y}{1}$?

Pick different points on the circumference and calculate $\frac{x}{1}$ and $\frac{y}{1}$.

What do you notice?

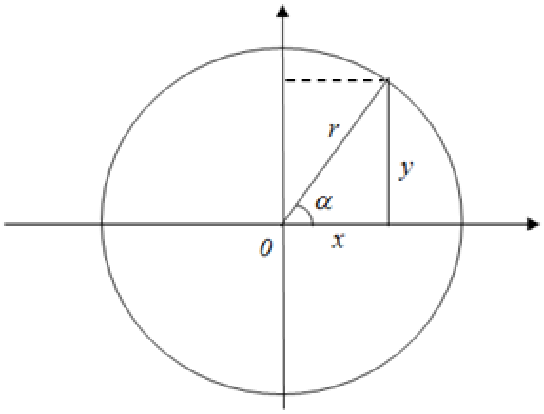
CONTENT SUMMARY

From centre of unit circle, you can draw different right-angled triangle. The x is the side adjacent to the angle α, y is the side opposite to the angle α, while the radius of 1 unit is the hypotenuse.

Using right-angled triangle, we get the important trigonometric ratios:

$\frac{Adjacent\ side}{Hypotenuse\ side} = \frac{x}{1} = x$ that is defined as cosine of α and noted $\cos\alpha$.

$\frac{Oposite\ side}{Hypotenuse\ side} = \frac{y}{1} = y$ that is defined as sine of α and noted $\sin\alpha$.



Generally, from any right triangle, we have and define the following six ratios:

- The three primary trigonometric values
  - The ratio $\frac{x}{r}$ is called cosine of the angle α, noted $\cos\alpha$.
  - The ratio $\frac{y}{r}$ is called sine of the angle α, noted $\sin\alpha$.

- The ratio $\frac{y}{x}$ is called tangent of the angle α, noted $\tan\alpha = \frac{\sin\alpha}{\cos\alpha}$

- The three reciprocal trigonometric values

    - Secant of the angle α, noted $\sec\alpha$ is the ratio $\frac{r}{x} = \frac{1}{\cos\alpha}$

    - Cosecant of the angle α, noted $\csc\alpha$ is the ratio $\frac{r}{y} = \frac{1}{\sin\alpha}$

    - Cotangent of the angle α, noted $\cot\alpha$ is the ratio $\frac{x}{y} = \frac{\cos\alpha}{\sin\alpha}$

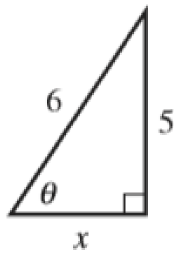
Example 1:

Let θ be an acute angle such that $\sin\theta = \frac{5}{6}$.

Evaluate the other five trigonometric functions of θ

Solution:

Sketch a triangle showing an acute angle θ. Label the opposite side 5 and the hypotenuse 6. Since $\sin\theta = \frac{5}{6}$, this must be our angle! Now we need the other side of the triangle (labeled x in the figure).

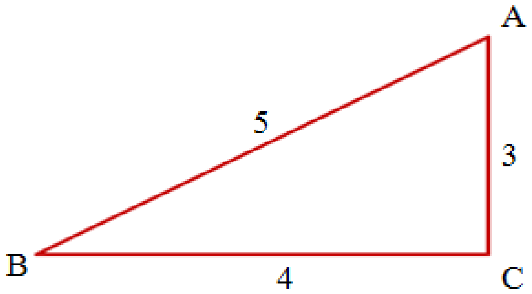


From the Pythagorean theorem, in this triangle with $\sin\theta = \frac{5}{6}$ it follows that $x^2 + 5^2 = 6^2$, so $x = \sqrt{36-25} = \sqrt{11}$. Applying the definitions,

$\sin\theta = \frac{opp}{hyp} = \frac{5}{6} \simeq 0.833$	$\csc\theta = \frac{hyp}{opp} = \frac{6}{5} \simeq 1.2$
$\cos\theta = \frac{adj}{hyp} = \frac{\sqrt{11}}{6} \simeq 0.553$	$\sec\theta = \frac{hyp}{adj} = \frac{6}{\sqrt{11}} \simeq 1.809$
$\tan\theta = \frac{opp}{adj} = \frac{5}{\sqrt{11}} \simeq 1.508$	$\cot\theta = \frac{adj}{opp} = \frac{\sqrt{11}}{5} \simeq 0.663$

Example 2:

For each angle, calculate the reciprocal trigonometric values



Solution:

Angle	$cosecant = \frac{1}{sine} = \frac{hypotenuse}{opposite}$	$secant = \frac{1}{cosine} = \frac{hypothenuse}{adjacent}$	$cotangent = \frac{1}{tangent}$
A	$\frac{5}{4}$	$\frac{5}{3}$	$\frac{3}{4}$
B	$\frac{5}{3}$	$\frac{5}{4}$	$\frac{4}{3}$
$C = 90^0$	$\frac{5}{5} = 1$	$\frac{5}{0}$ *which does not exist*	$\frac{0}{5} = 0$

Example 3:

A positive angle, θ, is in the second quadrant. If $\cos\theta = -\frac{3}{4}$, find the values of the other primary trigonometric values.

Solution

Let h, x *and* y be hypotenuse, adjacent and opposite side respectively.

$$\cos\theta = -\frac{3}{4} \Leftrightarrow \frac{x}{h} = -\frac{3}{4}.$$

Since $h > 0$, thus $h = 4$ and $x = -3$.

$$h^2 = x^2 + y^2 \Rightarrow 16 = 9 + y^2$$

$$\Leftrightarrow y^2 = 7 \Rightarrow y = \pm\sqrt{7}$$

As θ is in the second quadrant, $y > 0 \Rightarrow y = \sqrt{7}$.

Hence the other primary trigonometric values are $\sin\theta = \frac{y}{h} = \frac{\sqrt{7}}{4}$ and

$$\tan\theta = \frac{x}{y} = -\frac{\sqrt{7}}{3}.$$

Example 4:

Simplify $\frac{\csc x}{\sec x}$

Solution

$$\frac{\csc x}{\sec x} = \frac{\frac{1}{\sin x}}{\frac{1}{\cos x}} = \frac{\cos x}{\sin x} = \cot x$$

APPLICATION ACTIVITY 3.4.1

1. There are twice as many trigonometric functions as there are triangle sides which define them, so we can already explore some ways in which the trigonometric functions relate to each other. Doing this Exploration will help you learn which ratios are which.

   a) Each of the six trig functions can be paired to another that is its reciprocal. Find the three pairs of reciprocals.

   b) Which trigonometric function can be written as the quotient $\frac{\sin\theta}{\cos\theta}$?

   c) Which trigonometric function can be written as the quotient $\frac{\csc\theta}{\cot x}$?

   d) What is the (simplified) product of all six trigonometric functions multiplied together?

   e) What are the two trigonometric functions that are always less than 1 for any acute angle θ? [Hint: What is always the longest side of a right triangle?].

2. If $\sin\theta = 0.001$, find $\csc\theta$. If in addition, $\cos\theta < 0$, in what quadrant is θ?

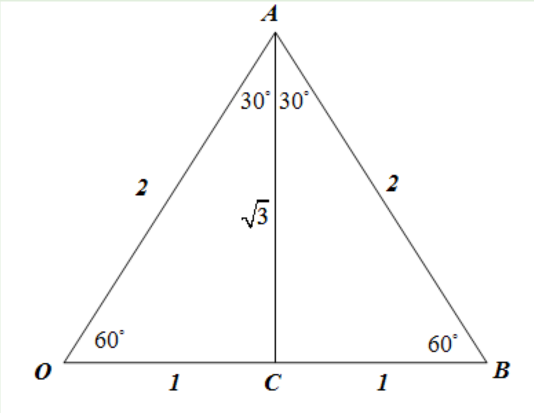
3. Find the exact values of the trigonometric functions of θ if $tan\theta = \frac{5}{6}$. In what quadrant is θ?

3.4.2 Trigonometric ratio of special angles $30^\circ, 45^\circ, 60^\circ$

ACTIVITY 3.4.2

1. Draw an isosceles right -angled triangle where the two equal sides are 1 unit in length. Use Pythagoras' theorem to calculate the hypotenuse. Deduce the three primary trigonometric ratios of 45^0 (cosine, sine and tangent).

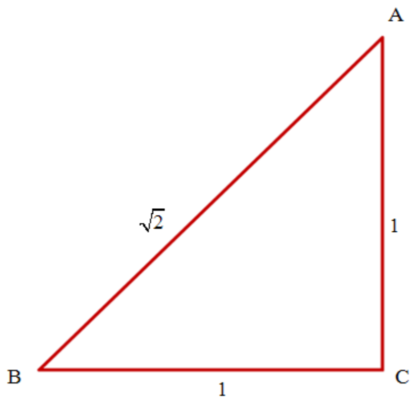
2. Consider the following diagram



- From $\triangle OAC$, find the six trigonometric values of 60°
- From $\triangle OAC$, find the six trigonometric values of 30°

CONTENT SUMMARY

As the angles $30^\circ, 45^\circ, 60^\circ$ are often used, it is better to keep in your mind their trigonometric ratios in fraction form.

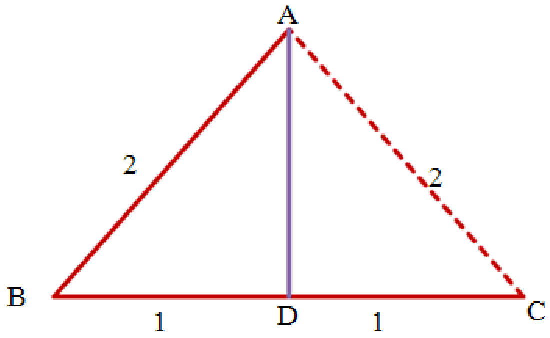


In the above figure, ABC is an isosceles right angled triangle with $BC = CA = 1$. Hence $AB = \sqrt{2}$ and $\angle A = \angle B = 45^0$.

From definition of trigonometric ratios, we have

$$\sin 45^0 = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2} = \cos 45^0$$

$$\tan 45^0 = 1$$



In the above figure, ABC is an equilateral triangle with $side\, 2$. AD is perpendicular bisector of BC, which implies $BD = 1$ and $AD = \sqrt{3}$. $\angle B = 60^0$ and $\angle BAD = 30^0$.

Then

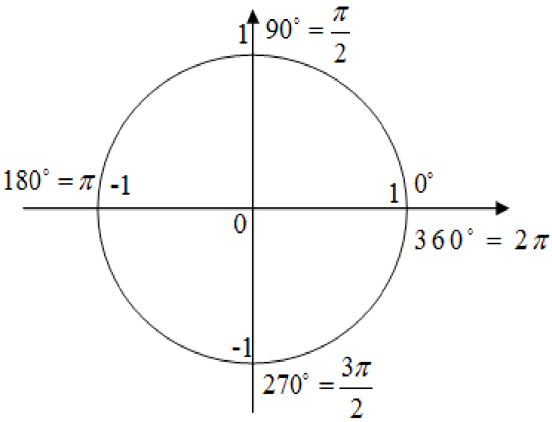
$$\sin 60^\circ = \cos 30^\circ = \frac{\sqrt{3}}{2}$$

$$\sin 30^\circ = \cos 60^\circ = \frac{1}{2}$$

$$\tan 60^\circ = \sqrt{3}, \tan 30^\circ = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$$

Other remarkable angles:

Consider the following unit trigonometric circle



From the figure above, let angle to consider be named as θ , thus we have:

Angle	0^0	90^0	180^0	270^0	360^0
$Sin\,\theta$	**0**	**1**	**0**	**-1**	**0**
$Cos\,\theta$	**1**	**0**	**-1**	**0**	**1**

APPLICATION ACTIVITY 3.4.2

1. Fill in the following table from your memory

θ	0^0	30^0	45^0	60^0	90^0
$\sin\theta$					
$\cos\theta$					
$\tan\theta$					

2. Compute the exact value of

   a) $\sin 30^0 + \sin 60^0 - \sin 90^0$

   b) $\sin^2 45^0 + \cos^2 45^0 - \tan 45^0$

3. Without using calculator, find out the following trigonometric ratios

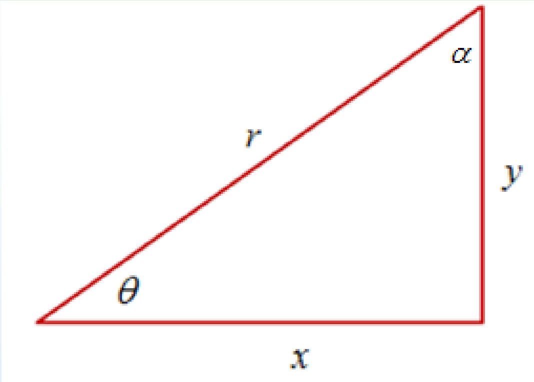
   a) $\cot 45^0$ b) $\cot 30^0$ c) $\cot 60^0$ d) $\tan 180^0$ e) $\tan 270^0$

   f) $\cot 0^0$ g) $\cot 90^0$ h) $\cot 180^0$ i) $\cot 270^0$

3.5 Trigonometric identities

ACTIVITY 3.5

Here is a right triangle



In this triangle, find out

a) $\sin\theta, \cos\theta, \sin\alpha$ and $\cos\alpha$

b) Remember that in this triangle, Pythagoras' theorem states $x^2 + y^2 = r^2$. Dividing each term by r^2 yields $\frac{x^2}{r^2} + \frac{y^2}{r^2} = 1$.

c) Express $\frac{x^2}{r^2} + \frac{y^2}{r^2} = 1$ in terms of $\sin\theta$ and $\cos\theta$

d) Find the value of $\cos^2\alpha + \sin^2\alpha$

CONTENT SUMMARY

In mathematics, an identity is an equation which is always true. There are many trigonometric identities, but the one you are most likely to see and use is:

$$\cos^2\theta + \sin^2\theta = 1$$

This relation is called the fundamental formula of trigonometry and is the most frequently used identity in trigonometry.

Dividing on both side of the above formula by $\cos^2\theta$ and $\sin^2\theta$, we get two other identities.

$$1+\tan^2\theta=\sec^2\theta$$

$$1+\cot^2\theta=\csc^2\theta$$

Example 2

Simplify $\left(\frac{1}{\tan x}+\frac{1}{\cot x}\right)\sin x\cos x$

Solution

$$\left(\frac{1}{\tan x}+\frac{1}{\cot x}\right)\sin x\cos x=\left(\frac{\cos x}{\sin x}+\frac{\sin x}{\cos x}\right)\sin x\cos x$$

$$=\left(\frac{\cos x\cos x+\sin x\sin x}{\sin x\cos x}\right)\sin x\cos x$$

$$=\cos x\cos x+\sin x\sin x$$

$$=\cos^2 x+\sin^2 x$$

$$=1$$

APPLICATION ACTIVITY 3.5

1. Prove the following trigonometric identities

1) $\frac{\cot\theta}{1+\sin\theta}=\frac{1-\sin\theta}{\sin\theta\cos\theta}$

2) $\frac{\sin\theta}{1+\cos\theta}+\frac{1+\cos\theta}{\sin\theta}=2\csc\theta$

3.6 Reduction to functions of positive acute angles

3.6.1 Equivalent angles

ACTIVITY 3.6.1

1. Which of the following angles are coterminal with 120° ?

   a) -270° b) 600° c) -240° d) 840°

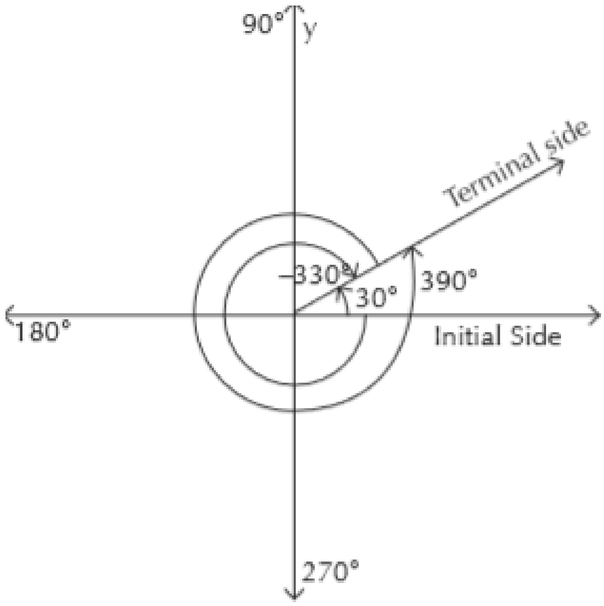
2. Write two angles coterminal with each angle

   a) 180° b) -141° c) 129° d) 0°

3. What is the relationship between trigonometric values of two coterminal angles?

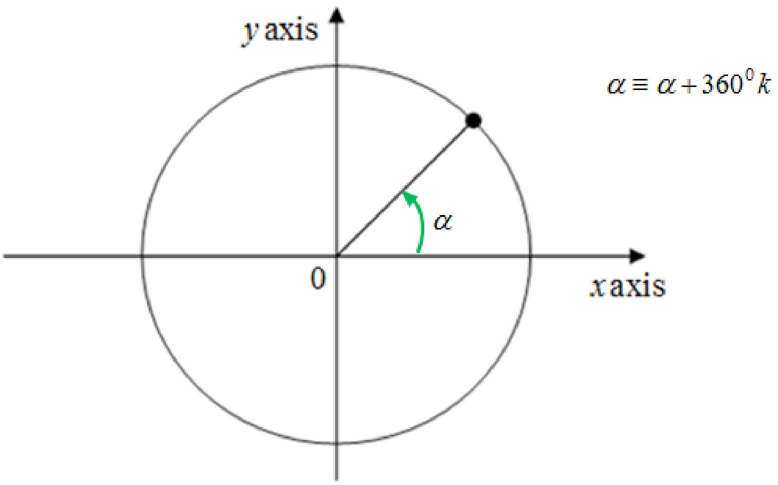
CONTENT SUMMARY

Coterminal angles are angles in standard position (angles with the initial side on the positive x-axis) that have a common terminal side. For example, 30°, –330° and 390° are all coterminal.



Two angles are equivalent if their difference is $2k\pi,\ k\in\mathbb{Z}$ (or $360^{\circ}k,\ k\in\mathbb{Z}$). This means that the angle α and $\alpha+2k\pi$ are equivalents angles.

Note that equivalent angles are **coterminal**.



The equivalent angles have the same trigonometric values. α *and* $\alpha + 360^\circ k, k \in \mathbb{Z}$ are equivalent

since

$$\left.\begin{aligned} \sin(\alpha + 360^\circ k) &= \sin\alpha \\ \cos(\alpha + 360^\circ k) &= \cos\alpha \\ \tan(\alpha + 360^\circ k) &= \tan\alpha \\ \cot(\alpha + 360^\circ k) &= \cot\alpha \end{aligned}\right\} k \in \mathbb{Z}$$

Example 1

a) The angle 60° and 780° are equivalent since 780° can be written as

$60^\circ + 2 \times 360^\circ$, so that $780^0 = 60^0 + \underbrace{2}_{k} \times \underbrace{360^0}_{2\pi}$

b) The angle 30° and -1050° are equivalent since $-1050^0 = 30^0 - \underbrace{3}_{k} \times \underbrace{360^0}_{2\pi}$

c) The angle $\frac{2\pi}{3}$ and $\frac{14\pi}{3}$ are equivalent since $\frac{14\pi}{3} = \frac{2\pi}{3} + 2(2\pi)$

If α and β are equivalent we write $\alpha \equiv \beta$. Since 60° and 780° are equivalent, we may write $60^\circ \equiv 780^\circ$.

APPLICATION ACTIVITY 3.6.1

1. Find a positive and a negative angle coterminal with a $55°$ angle.
2. Find a positive and a negative angle coterminal with a $\frac{\pi}{3}$ angle.

3.6.2 Negative angle or Opposite angle

ACTIVITY 3.6.2

In the following diagrams, the line segment is the terminal side of a given angle in standard position.

a) Using protractor, find the given angle and its principal trigonometric values.

b) The line segment is symmetrical about the x-axis, copy each graph and draw the remaining part (reflection on the x-axis).

c) Indicate the value of new angle in standard position and give its principal trigonometric values.

d) Complete the given table and give your observations

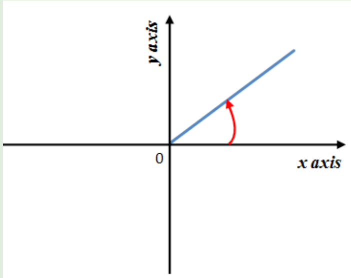


Figure 1

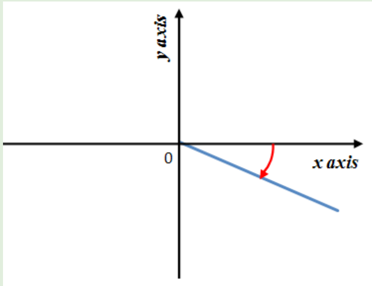


Figure 2

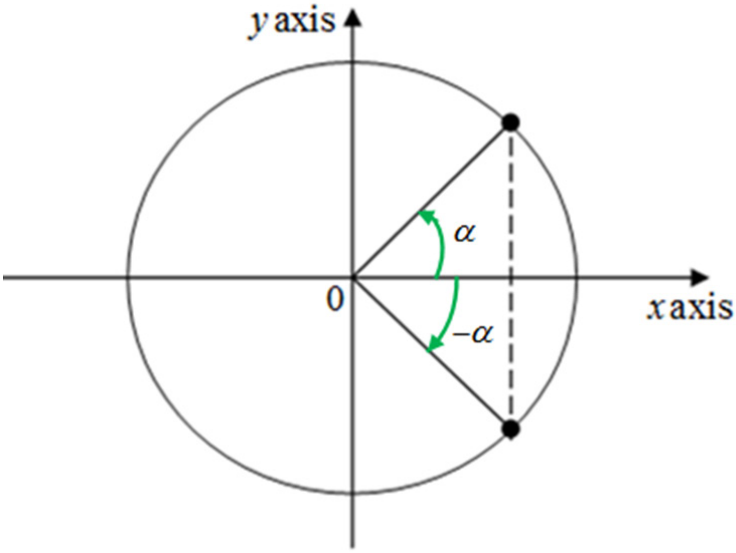
	The given angle (in degree)		New angle (in degree) after reflection		Observations from trigonometric values of both angles
	Figure 1	Figure 2	Figure 1	Figure 2	
sin					
cos					
tan					

CONTENT SUMMARY

From set theory, angle $-\alpha$ is opposite of angle α

Geometrically

The terminal sides of angle and its opposite angles are **symmetric to x-axis.**



For two angles, α and $-\alpha$, the following identities are true

$\sin(-\alpha) = -\sin\alpha$

$\cos(-\alpha) = \cos\alpha$

$\tan(-\alpha) = -\tan\alpha$

$\cot(-\alpha) = -\cot\alpha$

Examples

a) -10° is opposite of 10°

b) $-\frac{\pi}{4}$ is opposite of $\frac{\pi}{4}$

APPLICATION ACTIVITY 3.6.2

1. If $\sin\theta = -0.1903$, find $\sin(-\theta)$
2. If $\cos(\theta) = 0.0133$, find $\cos(-\theta)$
3. If $\sec(\theta) = -1.753$, find $\cos(-\theta)$
4. If $\csc(-\theta) = \sqrt{3}$, find $\sin\theta$
5. If $\cos(-\theta) = \frac{1}{7}$, find $\sec\theta$
6. If $\cot(\theta) = -5.4219$, find $\tan(-\theta)$

3.6.3 Complementary angles

ACTIVITY 3.6.3

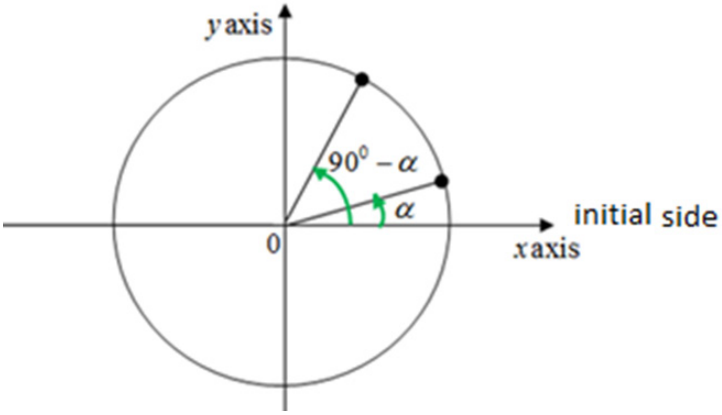
Give examples of 3 pairs of two angles whose sum is a right angle.

Present those paired angles on trigonometric circle and compare their trigonometric ratios cosine and sine. What can you conclude?

CONTENT SUMMARY

Two angles are said to be complementary if their sum is 90° (or $\frac{\pi}{2}$).

The angles α and $90^\circ - \alpha$ are complementary.



For two complementary angles α and $90^\circ - \alpha$, the following identities are true

$$\sin\left(90^\circ - \alpha\right) = \cos\alpha$$
$$\cos\left(90^\circ - \alpha\right) = \sin\alpha$$
$$\tan\left(90^\circ - \alpha\right) = \cot\alpha$$
$$\cot\left(90^\circ - \alpha\right) = \tan\alpha$$

Example 1

The angles 30° and 60° are complementary

Example 2

If $\cos\theta = 0.975$, find $\sin\left(\theta - \frac{\pi}{2}\right)$.

Solution:

As $\sin(-\theta) = -\sin\theta$, thus $\sin\left(\theta - \frac{\pi}{2}\right) = -\sin\left(\frac{\pi}{2} - \theta\right)$, from identity of negative angles

$= -\cos\theta$, from identity of complementary angles

$= -0.975$, from the given value of $\cos\theta$.

APPLICATION ACTIVITY 3.6.3

1. If $\sin\theta = 0.00213$, find $\cos\left(\frac{\pi}{2} - \theta\right)$
2. If $\tan\left(\frac{\pi}{2} - \theta\right) = -0.11221$, find $\cot\theta$
3. If $\cot(-\theta) = 1.1482$, find $\tan\left(\theta - \frac{\pi}{2}\right)$
4. If $\cos(\theta) = 0.5$, find $\csc\left(\theta - \frac{\pi}{2}\right)$

5. If $\sin\left(\frac{\pi}{2}-\theta\right)=\frac{1}{2}$, find $\sec\theta$

6. If $\sec\left(\theta-\frac{\pi}{2}\right)=7$, find $\csc\theta$

3.6.4 Supplementary angles

ACTIVITY 3.6.4

In the following diagrams, the line segment is the terminal side of a given angle in standard position.

a) Using protractor, find the given angle and its principal trigonometric values.

b) The line segment is symmetrical about the y-axis, copy each graph and draw the remaining part (reflection on the y-axis).

c) Indicate the value of new angle in standard position and give its principal trigonometric values.

d) Complete the given table and give your comments

	The given angle(in degree)		New angle(in degree) after reflection		Observations from trigonometric values of both angles
	Figure 1	**Figure 2**	**Figure 1**	**Figure 2**	
sin					
cos					
tan					

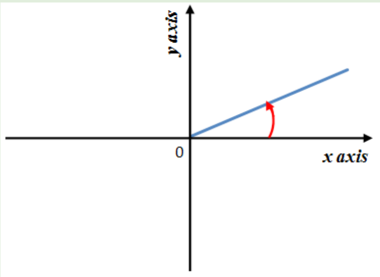


Figure 1

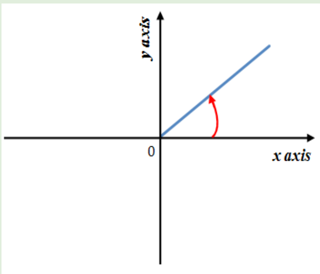


Figure 2

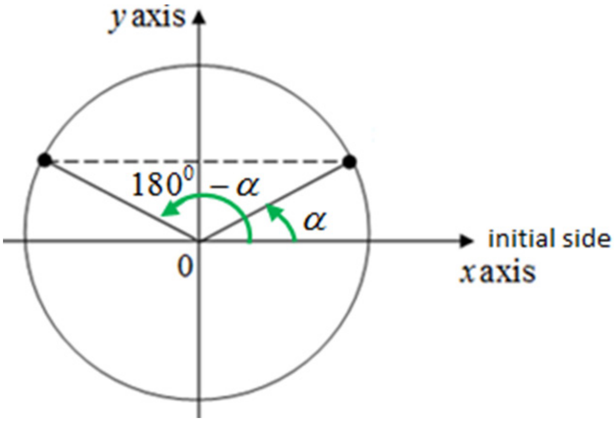
CONTENT SUMMARY

Two angles are said to be supplementary if their sum is 180° (or π).

From this definition, we note that α and $180^\circ - \alpha$ are supplementary.

Geometrically

Two angles in standard position are supplementary, if their terminal sides are **symmetric to y-axis**.



For two supplementary angles α and $180^\circ - \alpha$, the following identities are true

$$\sin\left(180^0 - \alpha\right) = \sin\alpha$$

$$\cos\left(180^0 - \alpha\right) = -\cos\alpha$$

$$\tan\left(180^0 - \alpha\right) = -\tan\alpha$$

$$\cot\left(180^0 - \alpha\right) = -\cot\alpha$$

Example

a) *The angles* 160° *and* 20° *are supplementary*

b) *The angles* 110° *and* 70° *are supplementary*

APPLICATION ACTIVITY 3.6.4

1. If $\sin\theta = 0.0312$, find $\sin(\pi-\theta)$
2. If $\tan(\pi-\theta) = -0.11221$, find $\tan\theta$
3. If $\cot(\theta) = -2.5148$, find $\cot(\theta-\pi)$
4. If $\sec(\theta) = 0.5$, find $\cos(\theta-\pi)$
5. If $\csc(\pi-\theta) = \frac{1}{2}$, find $\csc\theta$
6. If $\sec(\theta-\pi) = 3$, find $\sec\theta$

3.7 Transformation formulae

3.7.1 Sum and difference formula

ACTIVITY 3.7.1

There is a powerful instinct in all of us to believe that functions obey the following

law of additivity: $f(x+y) = f(x) + f(y)$ but it is not always true for some of them.

1) If we Let $x = \pi$ and $y = \frac{\pi}{2}$, then:
   a) Find $\sin(x+y)$ and $\sin(x)+\sin(y)$
   b) Does $\sin(x+y) = \sin(x)+\sin(y)$?

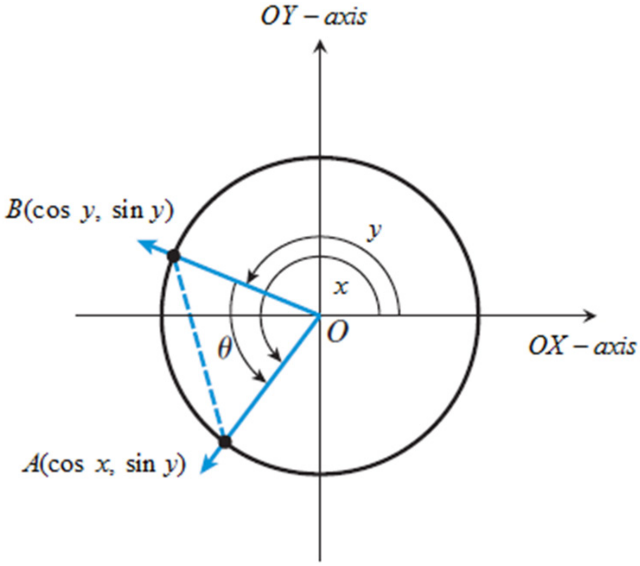
2) Let $x = \pi$ and $y = \frac{\pi}{2}$
   a) Find $\sin(x-y)$ and $\sin(x)-\sin(y)$
   b) Does $\sin(x-y) = \sin(x)-\sin(y)$?
3) Find your own values of x and y that will confirm

$$\tan(x+y) \neq \tan(x)+\tan(y) \quad ?$$

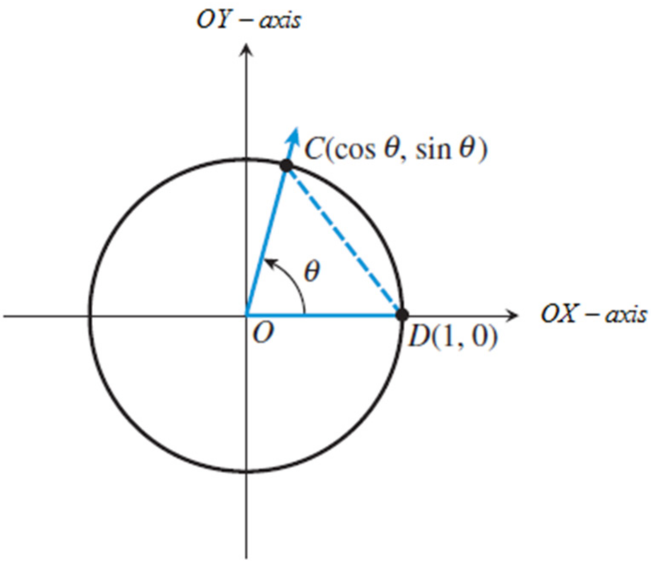
CONTENT SUMMARY

a) Cosine of a Sum or Difference

Consider the figure, which show angles x and y in standard position on the unit circle, determining point A and B with coordinates $(\cos x, \sin x)$ and $(\cos y, \sin y)$, respectively.



The following figure also show the triangle *ABO* rotated so that the angle $\theta = x - y$ is in standard position. The angle θ determines point C with coordinates $(\cos\theta, \sin\theta)$.



The chord opposite angle θ has the same length in both circles above, even though the coordinatization of the endpoints is different. Use the difference formula to find the length in each case, and set the formulas equal to each other:

Consider the figure, which show angles x and y in standard position on the unit circle, determining point A and B with coordinates $(\cos x, \sin x)$ and $(\cos y, \sin y)$, respectively.

Square both sides to eliminate the radical and expand the binomials to get:

$$\cos^2 x - 2\cos x\cos y + \cos^2 y + \sin^2 x - 2\sin x\sin y + \sin^2 y = \cos^2\theta - 2\cos\theta + 1 + \sin^2\theta$$

$$(\cos^2 x + \sin^2 x) + (\cos^2 y + \sin^2 y) - 2\cos x\cos y - 2\sin x\sin y = (\cos^2\theta + \sin^2\theta) + 1 - 2\cos\theta$$

$$2 - 2\cos x\cos y - 2\sin x\sin y = 2 - 2\cos\theta$$

$$\Rightarrow \quad \cos\theta = \cos x\cos y + \sin x\sin y$$

Finally, since $\theta = x - y$ as mentioned above, we write:

$$\cos(x-y) = \cos x\cos y + \sin x\sin y$$

Now that we have the formula for the cosine of a difference, we can get the formula

for the cosine of a sum almost for free by using the odd-even identities.

$$\begin{aligned} \cos(x+y) &= \cos(x-(-y)) \\ &= \cos x\cos(-y) + \sin x\sin(-y) && \text{Cosine difference identity} \\ &= \cos x\cos y + \sin x\sin(-y) && \text{Odd}-\text{even identity} \\ \cos(x+y) &= \cos x\cos y - \sin x\sin y \end{aligned}$$

We can now combine the sum and difference formulas for cosine as follows:

1. Cosine of a Sum or Difference is given by:

$$\cos(x+y) = \cos x\cos y - \sin x\sin y$$

$$\cos(x-y) = \cos x\cos y + \sin x\sin y$$

(Note that the sign switch in either case)

Example 1:

Find the exact value of $\cos 15^0$ without using a calculator.

Solution:

The trick is to write $\cos 15^0$ **as** $\cos 15^0 = \cos\left(45^0 - 30^0\right)$**, then we can use the knowledge of special angles.**

$$\cos 15^0 = \cos\left(45^0 - 30^0\right)$$

$$= \cos 45^0 \cos 30^0 + \sin 45^0 \sin 30^0 \qquad \text{Cosine difference identity}$$

$$= \cos 45^0 \cos 30^0 + \sin 45^0 \sin 30^0$$

$$= \left(\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{3}}{2}\right) + \left(\frac{\sqrt{2}}{2}\right)\left(\frac{1}{2}\right)$$

$$= \frac{\sqrt{6}+\sqrt{2}}{4}$$

b) Sine of a Sum or Difference

We can use the function identities to get the formula for the sine of a sum from the formula for the cosine of a difference

$$\sin\left(x+y\right) = \cos\left(\frac{\pi}{2} - (x+y)\right)$$

$$= \cos\left(\left(\frac{\pi}{2} - x\right) - y\right)$$

$$= \cos\left(\frac{\pi}{2} - x\right)\cos y + \sin\left(\frac{\pi}{2} - x\right)\sin y \qquad \text{Cosine difference identity}$$

$$= \sin x \cos y + \cos x \sin y$$

Then we can use the odd-even identities to get the formula for the sine of a difference

from the formula for the sine of a sum.

$$\sin\left(x-y\right) = \sin\left(u + (-y)\right)$$

$$= \sin\left(x\right)\cos\left(-y\right) + \cos\left(x\right)\sin\left(-y\right)$$

$$= \sin x \cos y + \cos x\left(-\sin y\right)$$

$$= \sin x \cos y - \cos x \sin y$$

We can now combine the sum and difference formulas for sine as follows:

2) Sine of a Sum or Difference, is given by:

$$\sin(x+y) = \sin x \cos y + \cos x \sin y$$
$$\sin(x-y) = \sin x \cos y - \cos x \sin y$$

(Note that the signs does not switch in either case)

Example 2:

Write each of the following as the cosine and / or sine of angle.

Given that $\sin 22^0 \cos 13^0 + \cos 22^0 \sin 13^0$

Solution:

$$\sin 22^0 \cos 13^0 + \cos 22^0 \sin 13^0 = \sin\left(22^0 + 13^0\right)$$
$$= \sin 35^0$$

c) Tangente or Cotangente of a Sum or Difference

We can derive a formula for $\tan(x \pm y)$ directly from the corresponding formulas for

sine and cosine, as follows:

$$\tan(x+y) = \frac{\sin(x+y)}{\cos(x+y)} = \frac{\sin x \cos y + \cos x \sin y}{\cos x \cos y - \sin x \sin y}$$
$$\tan(x-y) = \frac{\sin(x-y)}{\cos(x-y)} = \frac{\sin x \cos y - \cos x \sin y}{\cos x \cos y + \sin x \sin y}$$

Dividing the numerator and the denominator by $\cos x \cos y$ we get the tangent of the sum as follows:

$$\frac{\dfrac{\sin x \cos y}{\cos x \cos y} + \dfrac{\cos x \sin y}{\cos x \cos y}}{\dfrac{\cos x \cos y}{\cos x \cos y} - \dfrac{\sin x \sin y}{\cos x \cos y}} = \frac{\tan x + \tan y}{1 - \tan x . \tan y}$$

Similarly, we get the tangent of a difference as follows:

$$\frac{\dfrac{\sin x \cos y}{\cos x \cos y} - \dfrac{\cos x \sin y}{\cos x \cos y}}{\dfrac{\cos x \cos y}{\cos x \cos y} + \dfrac{\sin x \sin y}{\cos x \cos y}} = \frac{\tan x - \tan y}{1 + \tan x . \tan y}$$

Tangent of a Sum or Difference, is given by:

$$\tan(x+y) = \frac{\tan x + \tan y}{1 - \tan x \,.\, \tan y}$$

$$\tan(x-y) = \frac{\tan x - \tan y}{1 + \tan x \,.\, \tan y}$$

Cotangent of a Sum or Difference is given by:

$$\cot(x+y) = \frac{\cot x \,.\, \cot y - 1}{\cot x + \cot y}$$

$$\cot(x-y) = \frac{\cot x \,.\, \cot y + 1}{\cot x - \cot y}$$

Using the same process, you can easily find $\cot(x+y)$ and $\cot(x-y)$

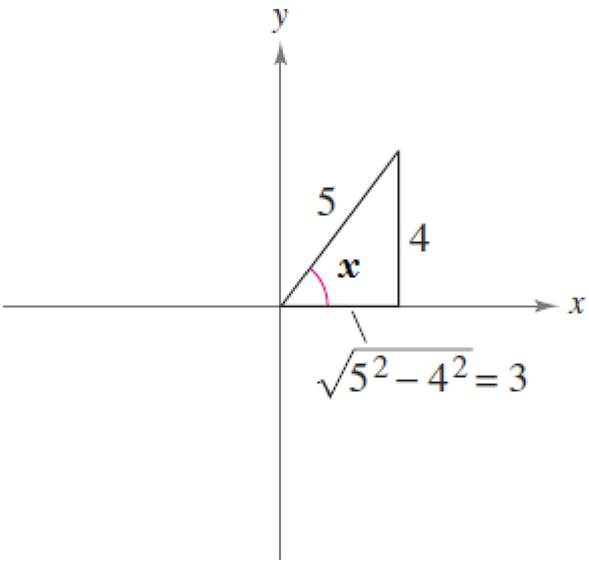
Example 3:

Find the exact value of $\sin(x+y)$ without using a calculator, given that:

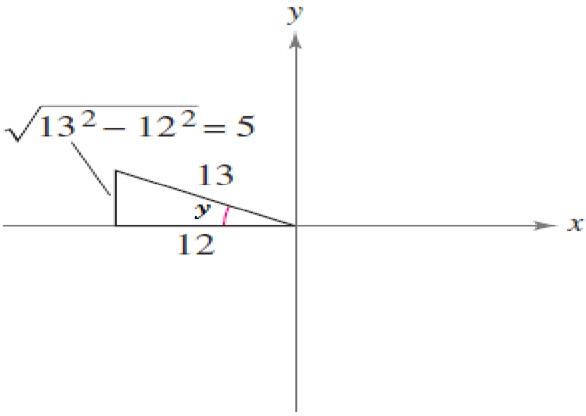
$\sin x = \frac{4}{5}$ Where $0 < x < \frac{\pi}{2}$ and $\cos y = \frac{-12}{13}$ where $\frac{\pi}{2} < y < \pi$

Solution:

- Because $\sin x = \frac{4}{5}$ and x is in quadrant I, $\cos y = \frac{3}{5}$ as shown in the figure below.



- Because $\cos y = -\dfrac{12}{13}$ and y is in quadrant II, $\sin y = \dfrac{5}{13}$ as shown in the figure below.



We calculate $\sin(x+y)$ as follows:

$$\sin(x+y) = \sin x \cos y + \cos x \sin y$$

$$= \left(\frac{4}{5}\right)\left(\frac{-12}{13}\right) + \left(\frac{3}{5}\right)\left(\frac{5}{13}\right) = \frac{-48}{65} + \frac{15}{65} = \frac{-33}{65}$$

APPLICATION ACTIVITY 3.7.1

1. Simplify $2\sin\theta\sin 4\theta + 2\cos\theta\cos 4\theta$
2. Use addition and subtraction formulas to find

   a. $\sin 75^0$

   b. $\cos\frac{13\pi}{6}$

   c. $\tan 330^0$

3. Evaluate

   a) $\tan 75^0$

   b) $\sin 15^0$

   c) $\sin 47^0 \cos 13^0 + \cos 47^0 \sin 13^0$

   d) $\cos 70^0 \cos 10^0 + \sin 70^0 \sin 10^0$

   e) $\frac{\tan 69^0 + \tan 66^0}{1 - \tan 69^0 \tan 66^0}$

3.7.2 Double angle formulae

ACTIVITY 3.7.2

For each of the following relations, replace y by x and give your result.

1) $\cos(x+y) = \cos x\cos y - \sin x\sin y$

2) $\sin(x+y) = \sin x\cos y + \cos x\sin y$

3) $\tan(x+y) = \frac{\tan x + \tan y}{1 - \tan x\tan y}$

4) $\cot(x+y) = \frac{\cot x\cot y - 1}{\cot y + \cot x}$

CONTENT SUMMARY

We start by recalling the addition formulae as given in the learning activity above. We consider what happens if we let y equal to x. Then:

The double angle formulae for $\cos 2x$, $\sin 2x$, $\tan 2x$ and $\cot 2x$

Will be:

1) $\cos 2x = \cos^2 x - \sin^2 x$ (1)
2) $\sin 2x = 2\sin x \cos x$
3) $\tan 2x = \dfrac{2\tan x}{1-\tan^2 x}$
4) $\cot 2x = \dfrac{\cot^2 x - 1}{1-\tan^2 x}$

Replacing $\cos^2 x$ by $1-\sin^2 x$ in (1) we get

$$\cos 2x = 1 - 2sn^2 x$$
$$\cos 2x - 1 = -2\sin^2 x$$
$$2\sin^2 x = 1 - \cos 2x$$
$$\sin^2 x = \frac{1-\cos 2x}{2}$$

Replacing $\sin^2 x$ by $1-\cos^2 x$ we get

$$\cos 2x = 2\cos^2 x - 1$$
$$2\cos^2 x = 1 + \cos 2x$$
$$\cos^2 x = \frac{1+\cos 2x}{2}$$

Example 1:

From double angle formulae and fundamental relation of trigonometry, prove that $\sin 2x = \dfrac{2\tan x}{1+\tan^2 x}$

Solution

$$\sin 2x = 2\sin x\cos x$$

Dividing the right hand side by $\sin^2 x + \cos^2 x$, gives

$$\sin 2x = \frac{2\sin x\cos x}{\sin^2 x + \cos^2 x}$$

Dividing each term of the right hand side by $\cos^2 x$, we get

$$\sin 2x = \frac{\frac{2\sin x\cos x}{\cos^2 x}}{\frac{\sin^2 x + \cos^2 x}{\cos^2 x}} \Leftrightarrow \sin 2x = \frac{\frac{2\sin x}{\cos x}}{\frac{\sin^2 x}{\cos^2 x} + 1}$$

$$\Leftrightarrow \sin 2x = \frac{2\tan x}{\tan^2 x + 1}$$

Hence $\sin 2x = \frac{2\tan x}{1+\tan^2 x}$ as required.

Example 2:

Express $\cos 4x$ in function of $\sin x$ only

Solution

$$\cos 4x = \cos 2(2x) = 1 - 2\sin^2 2x$$

$$= 1 - 2(2\sin x\cos x)^2$$

$$= 1 - 2(4\sin^2 x\cos^2 x)$$

$$= 1 - 8\sin^2 x\cos^2 x$$

$$= 1 - 8\sin^2 x(1 - \sin^2 x)$$

$$= 1 - 8\sin^2 x + 8\sin^4 x$$

Alternatively, using $\cos 4x = (\cos 2x + \cos 2x)$ we get the same result.

APPLICATION ACTIVITY 3.7.2

1. Express $\sin 4x$ in function of $\sin x$ and $\cos x$
2. Express $\cos 8x$ in function of $\sin x$
3. Evaluate each of the following without using a calculator $2\sin 15^0 \cos 15^0$
4. If $\cos A = \sqrt{\frac{\sqrt{2}+1}{2\sqrt{2}}}$, find $\cos 2A$
5. If $\sin A = \frac{\sqrt{5}}{5}$, find $\sin 2A, \cos 2A$ and $\tan 2A$ if A
   a) is acute
   b) is obtuse

3.7.3 Half angle formulae

ACTIVITY 3.7.3

1. Knowing that $\cos 2x = 1 - 2\sin^2 x$ .Let $\theta = 2x$ then deduce the value of $\sin\frac{\theta}{2}$
2. Knowing that $\cos 2x = 2\cos^2 x - 1$. Let $\theta = 2x$ then deduce the value of $\cos\frac{\theta}{2}$
3. Using results in 1 and 2, deduce the value of $\tan\frac{\theta}{2}$. (Recall that $\tan x = \frac{\sin x}{\cos x}$)

CONTENT SUMMARY

The power-reducing identities can be used to extend our stock of "special" angles whose trigonometric ratios can be found without a calculator. As usual, we are not suggesting that this algebraic procedure is any more practical than using a calculator, but we are suggesting that this sort of exercise helps you to understand how the functions behave.

A little alteration of the power-reducing identities results in the *Half- Angle Identities*, which can be used directly to find trigonometric functions of $\frac{x}{2}$ in terms of trigonometric functions of x. After exploration, we suggests that there is an unavoidable ambiguity of sign involved with the square root that must be resolved in particular cases by checking the quadrant in which $\frac{\theta}{2}$ lies.

Half angle formulas:

$$\sin\frac{x}{2}=\pm\sqrt{\frac{1-\cos x}{2}}$$

$$\cos\frac{x}{2}=\pm\sqrt{\frac{1+\cos x}{2}}$$

$$\tan\frac{x}{2}=\pm\sqrt{\frac{1-\cos x}{1+\cos x}}$$

Where the $+$ *or* $-$ sign is determined by the quadrant of the angle $\frac{x}{2}$

Example 1:

Use the half angle formula, to find the exact value of $\cos 15^0$

Solution:

Let $x=15^0$ and notice that it is in first quadrant, then $\cos 15^0$ must be positive

$$\cos 15^0=\cos\left(\frac{1}{2}\left(30^0\right)\right)$$

$$=\sqrt{\frac{1+\cos 30^0}{2}}$$

$$\Rightarrow \qquad = \sqrt{\frac{1+\frac{\sqrt{3}}{2}}{2}}$$

t

$$= \frac{\sqrt{2+\sqrt{3}}}{2}$$

Example 2

If $\cos A = -\frac{7}{25}$, find the values of $\sin\frac{1}{2}A$, $\cos\frac{1}{2}A$ and $\tan\frac{1}{2}A$

Solution:

- $\sin\frac{1}{2}A = \pm\sqrt{\frac{1-\cos A}{2}}$

$$\Rightarrow \sin\frac{1}{2}A = \pm\sqrt{\frac{1-\left(-\frac{7}{25}\right)}{2}} \Rightarrow \sin\frac{1}{2}A = \pm\sqrt{\frac{1+\frac{7}{25}}{2}} \Rightarrow \sin\frac{1}{2}A = \pm\sqrt{\frac{32}{50}}$$

Finally, $\sin\frac{1}{2}A = \pm\sqrt{\frac{16}{25}} = \pm\frac{4}{5}$

- $\cos\frac{1}{2}A = \pm\sqrt{\frac{1+\cos A}{2}}$

$$\Rightarrow \cos\frac{1}{2}A = \pm\sqrt{\frac{1-\frac{7}{25}}{2}} \qquad \Rightarrow \qquad \cos\frac{1}{2}A = \pm\sqrt{\frac{18}{50}}$$

Thus, $\cos\frac{1}{2}A = \pm\sqrt{\frac{9}{25}} = \pm\frac{3}{5}$

- $\tan\frac{1}{2}A = \pm\sqrt{\frac{1-\cos A}{1+\cos A}}$

$$\Rightarrow \tan\frac{1}{2}A = \pm\sqrt{\frac{1+\frac{7}{25}}{1-\frac{7}{25}}} \Rightarrow \tan\frac{1}{2}A = \pm\sqrt{\frac{32}{18}}$$

Finally, $\cos\frac{1}{2}A = \pm\sqrt{\frac{16}{9}} = \pm\frac{4}{3}$

APPLICATION ACTIVITY 3.7.3

1. If $\cos A = -\frac{1}{3}$, find the values of $\sin\frac{1}{2}A$, $\cos\frac{1}{2}A$
2. If $\tan 2A = \frac{7}{24}$, $0 < A < \frac{\pi}{4}$, find the value of $\tan A$
3. Find the value of $\sin 22\frac{1}{2}^0$,$\cos 22\frac{1}{2}^0$ and $\tan 22\frac{1}{2}^0$

3.7.4 Simpson's formulas

3.7.4.1 Product into Sum formulae

ACTIVITY 3.7.4.1

From addition and subtraction formulae, evaluate

1. $\sin(x+y)+\sin(x-y)$
2. $\sin(x+y)-\sin(x-y)$
3. $\cos(x+y)+\cos(x-y)$
4. $\cos(x+y)-\cos(x-y)$

CONTENT SUMMARY

From the activity above, derive the sum or difference formulae as follows:

$\sin(x+y) = \sin x\cos y + \sin y\cos x \quad (1)$

$\sin(x-y) = \sin x\cos y - \sin y\cos x \quad (2)$

$\cos(x+y) = cox\cos y - \sin x\sin y \quad (3)$

$\cos(x-y) = \cos x\cos y + \sin x\sin y \quad (4)$

By adding terms of the equality (1) and (2) then (3) and (4) respectively we get:

$$\sin(x+y)+\sin(x-y)=2\sin x\cos y \quad \text{(i)} \quad \Rightarrow \sin x\cos y=\frac{1}{2}[\sin(x+y)+\sin(x-y)]$$

$$\cos(x+y)+\cos(x-y)=2\cos x\cos y \quad (ii) \quad \Rightarrow \cos x\cos y=\frac{1}{2}[\cos(x+y)+\cos(x-y)]$$

By subtracting terms of the equality (2) from (1) then (3) and (4) respectively we get:

$$\sin(x-y)-\sin(x+y)=2\cos x\sin y \quad (iii) \quad \Rightarrow \cos x\sin y=\frac{1}{2}[\sin(x-y)-\sin(x+y)]$$

$$\cos(x+y)-\cos(x-y)=-2\sin x\sin y \ (iv) \quad \Rightarrow \sin x\sin y=-\frac{1}{2}[\cos(x+y)-\cos(x-y)]$$

Product to Sum or difference formulas:

1) $\sin x\sin y=\frac{1}{2}\left[\cos\left(x-y\right)-\cos\left(x+y\right)\right]$ or $\frac{1}{2}[\cos(\mathrm{x+y})-\cos(\mathrm{x-y})]$

2) $\cos x\cos y=\frac{1}{2}\left[\cos\left(x-y\right)+\cos\left(x+y\right)\right]$

3) $\sin x\cos y=\frac{1}{2}\left[\sin\left(x+y\right)+\sin\left(x-y\right)\right]$

4) $\cos x\sin y=\frac{1}{2}[\sin(\mathrm{x+y})-\sin(\mathrm{x-y})]$

Example 1:

Transform the following product into the sum $\sin 3x\cos 4x$

Solution

$$\sin 3x\cos 4x=\frac{1}{2}\left[\sin\left(3x+4x\right)+\sin\left(3x-4x\right)\right]$$

$$=\frac{1}{2}\left[\sin 7x+\sin\left(-x\right)\right]$$

$$=\frac{1}{2}[\sin 7x-\sin x]$$

Example 2:

Change the following product into a sum or difference: $\sin 9x \sin 11x$

Solution

$$\sin 9x \sin 11x = \frac{1}{2}(2\sin 9x \sin 11x)$$

$$= -\frac{1}{2}\left[\cos(9x+11x) - \cos(9x-11x)\right]$$

$$= -\frac{1}{2}\left[\cos 20x - \cos(-2x)\right]$$

$$= -\frac{1}{2}(\cos 20x - \cos 2x)$$

$$= \frac{1}{2}(\cos 2x - \cos 20x)$$

APPLICATION ACTIVITY 3.7.4.1

Transform in sum

a) $\sin x \cos 3x$ b) $\cos 12x \sin 9x$ c) $\sin 9x \sin 11x$

d) $2\cos 5x \sin 3x$ e) $\cos\frac{5x}{2}\cos\frac{3x}{2}$

3.7.4.2 Sum into product formulae

ACTIVITY 3.7.4.2

Given the sum and difference formula

1) $\cos(x+y) = \cos x \cos y - \sin x \sin y$

2) $\cos(x-y) = \cos x \cos y + \sin x \sin y$

3) $\sin(x+y) = \sin x \cos y + \cos x \sin y$

4) $\sin(x-y) = \sin x \cos y - \cos x \sin y$

$$5)\ \tan(x+y)=\frac{\tan x+\tan y}{1-\tan x\,.\,\tan y}$$

$$6)\ \tan(x-y)=\frac{\tan x-\tan y}{1+\tan x\,.\,\tan y}$$

$$7)\ \cot(x+y)=\frac{\cot x\,.\,\cot y-1}{\cot x+\cot y}$$

$$8)\ \cot(x-y)=\frac{\cot x\,.\,\cot y+1}{\cot x-\cot y}$$

Using the relations $x+y=p$ and $x-y=q$, and considering that ,

$$\begin{cases}x+y=p\\x-y=q\end{cases}\Rightarrow\begin{cases}x=\dfrac{p+q}{2}\\[2mm]y=\dfrac{p-q}{2}\end{cases}$$

Rewrite each of the formula above by transforming the sum into the product in function of p and q.

CONTENT SUMMARY

To express the sum into the product, formulas can be derived from the product-to-sum identities. For example, with simple substitutions, we can derive the sum-to-product identity for sine.

Let $x+y=p$ and $x-y=q$,

$$\begin{cases}x+y=p\\x-y=q\end{cases}\Rightarrow\begin{cases}x=\dfrac{p+q}{2}\\[2mm]y=\dfrac{p-q}{2}\end{cases}$$

by replacing $x+y=p$ and $x-y=q$, at the same time $x=\dfrac{p+q}{2}$ and $y=\dfrac{p-q}{2}$, to get:

Sum or difference to product formulas

a) $\sin p+\sin q=2\sin\left(\frac{p+q}{2}\right)\cos\left(\frac{p-q}{2}\right)$

b) $\sin p-\sin q=2\sin\left(\frac{p-q}{2}\right)\cos\left(\frac{p+q}{2}\right)$

c) $\cos p-\cos q=-2\sin\left(\frac{p+q}{2}\right)\sin\left(\frac{p-q}{2}\right)$

d) $\cos p+\cos q=2\cos\left(\frac{p+q}{2}\right)\cos\left(\frac{p-q}{2}\right)$

Example 1:

Transform $\cos 2x-\cos 4x$ in product

Solution

$$\cos 2x-\cos 4x=-2\left(\sin\frac{2x+4x}{2}\sin\frac{2x-4x}{2}\right)$$

Or $\cos 2x-\cos 4x=-2\sin 3x\sin(-x)$

So, $\cos 2x-\cos 4x=2\sin 3x\sin x$

Example 2:

Write $\sin 7x+\sin 5x$ as a product

Solution

$$\sin 7x+\sin 5x=2\sin\left(\frac{7x+5x}{2}\right)\cos\left(\frac{7x-5x}{2}\right)$$
$$=2\sin 6x\cos x$$

APPLICATION ACTIVITY 3.7.4.2

Transform in product

1. $\cos x + \cos 7x$
2. $\sin 4x - \sin 9x$
3. $\sin 3x + \sin x$
4. $\cos 2x - \cos 4x$

3.8. TRIGONOMETRIC EQUATIONS

The solution of equations reducible to the form

$$\sin(x+\alpha) = k \text{ , } \cos(x+\alpha) = k \text{ , } \tan(x+\alpha) = b \;\; for \;\; |k| \le 1 \;\text{ and } b \in \mathbb{R} \text{ ,}$$

ACTIVITY 3.8

1. Find at least three angles whose sine is $\frac{1}{2}$
2. Find at least three angles whose cosine is $\frac{\sqrt{2}}{2}$
3. Find at least three angles whose tangent is $\frac{\sqrt{3}}{3}$

CONTENT SUMMARY

The solutions of a trigonometric equation for which $0 \le x \le 2\pi$ are called **principal solutions** while the expression (involving integer k) of solution containing all values of the unknown angle is called the **general solution** of the trigonometric equation. When the interval of solution is not given, you are required to find general solution.

When solving trigonometric equations, the following identities are helpful:

$\sin\alpha = \sin(\alpha + 2k\pi),\ k \in \mathbb{Z}$ $\qquad$ $\sin\alpha = \sin(\pi - \alpha)$

$\cos\alpha = \cos(\alpha + 2k\pi),\ k \in \mathbb{Z}$ $\qquad$ $\cos\alpha = \cos(-\alpha)$

$\tan\alpha = \tan(\alpha + k\pi),\ k \in \mathbb{Z}$ $\qquad$ $\tan\alpha = \tan(\alpha + \pi)$

Example 1:

Find the principal solutions of the following equation $\sin x = \frac{1}{\sqrt{2}}$

Solution

$\sin x = \frac{1}{\sqrt{2}}$ is positive $\Rightarrow x$ lies in the 1^{st} or 2^{nd} quadrant.

Here $\sin x = \frac{1}{\sqrt{2}} = \sin\frac{\pi}{4}$ or $\sin\left(\pi - \frac{\pi}{4}\right)$

$$\Rightarrow x = \frac{\pi}{4}\ \ or\ \ x = \frac{3\pi}{4}$$

Thus $x = \frac{\pi}{4}$ or $x = \frac{3\pi}{4}$.

Example 2:

Solve in the set of real numbers $\cos 2x = -\frac{\sqrt{3}}{2}$

Solution

$\cos 2x = -\frac{\sqrt{3}}{2}$ is negative $\Rightarrow 2x$ lies in the 2^{nd} or 3^{rd} quadrant.

Here $\cos 2x = -\cos\frac{\pi}{6} = \cos\left(\pi - \frac{\pi}{6}\right)\ or\ \cos\left(\pi + \frac{\pi}{6}\right)$

$$\Rightarrow 2x = \frac{5\pi}{6} + 2k\pi\ \ or\ \ \frac{7\pi}{6} + 2k\pi, k \in \mathbb{Z}$$

$$x = \frac{5\pi}{12} + k\pi\ \text{,or}\ x = \frac{7\pi}{12} + k\pi, \text{k} \in \mathbb{Z}$$

Example 3:

Solve $\sin\frac{x}{3}=-\frac{\sqrt{3}}{2}$ for $x\in[0,2\pi]$

Solution

$$\sin\frac{x}{3}=-\frac{\sqrt{3}}{2} \text{ for } x\in[0,2\pi]$$

$$\frac{x}{3}=\begin{cases}-\frac{\pi}{3}+2k\pi \\ \pi-\left(-\frac{\pi}{3}\right)+2k\pi\end{cases}$$

$$\frac{x}{3}=\begin{cases}-\frac{\pi}{3}+2k\pi \\ \frac{4\pi}{3}+2k\pi\end{cases} \Rightarrow x=\begin{cases}-\pi+6k\pi \\ 4\pi+6k\pi\end{cases}$$

Since we are given the condition $x\in[0,2\pi]$, we need to substitute k with some integers

(..., -2, -1, 0, 1, 2 ...). However, doing this, no value of x can be found in the given interval.

Thus there is no solution.

APPLICATION ACTIVITY 3.8

Solve:

1. $\cos 8x = -\frac{1}{2}$
2. $\sin 3x \cos 7x = 0$ for $0^0 < \theta \leq 180^0$
3. $6\cos^2 \theta + \sin\theta - 5 = 0$ for $0^0 < \theta \leq 360^0$
4. $\frac{\tan 47^0 - \tan\theta}{1 + \tan 47^0 \tan\theta} = \frac{3}{2}$
5. $\sin 3x + \sin x = 0$
6. $\cos 5x + \cos 3x = 0$
7. $\sin 7x - \sin x = \sin 3x$

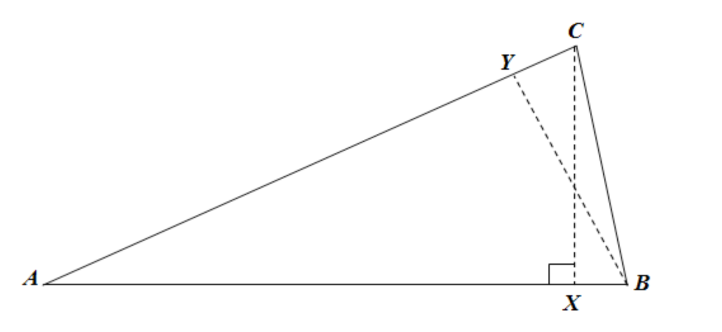
3.9 Triangles and Applications

3.9.1 Solving problems on a Triangle

a) SINE LAW

ACTIVITY 3.9.1.1

Consider the following triangle



CX is perpendicular to side AB and BY is perpendicular to side AC. Let AB=c, AC=b, BC=a, CX=h and BY=k

CX is perpendicular to side AB and BY is perpendicular to side AC. Let AB=c, AC=b, BC=a, CX=h and BY=k

1. In triangle BCX find $\sin B$. In triangle AXC find $\sin A$. Deduce the relationship between side a and side c.

2. In triangle ABY find $\sin A$. In triangle BCY find $\sin C$. Deduce the relationship between side b and side a

3. Deduce relationship between three sides.

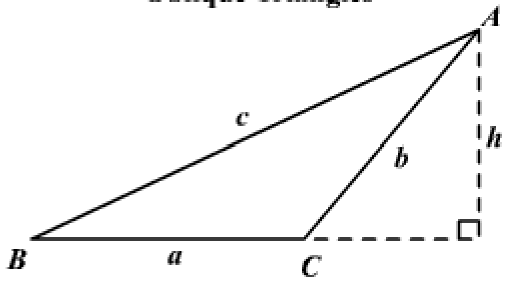
CONTENT SUMMARY

From geometry, a triangle has three sides (S), three angles (A) to determine triangle ,you need only three of those parts using the following acronyms: AAS, ASA, SAS, and SSS. The other two acronyms represent match-ups that don't quite work: AAA determines similarity only, while SSA does not even determine similarity. With trigonometry, we can find the other parts of the triangle once congruence is established.

The Law of Sines is the relationship between the sides and angles of non-right (oblique) triangles Simply, it states that the ratio of the length of a side of a triangle to the sine of the angle opposite that side is the same for all sides and angles in a given triangle.

Given the triangle ABC

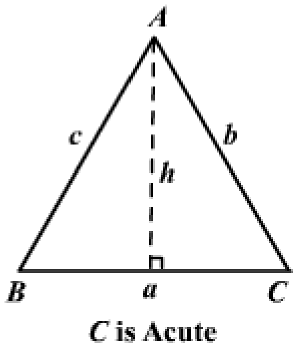
Oblique Triangles



C is Obtuse

In triangle ABC is an oblique triangle with sides $a, b \; and \; c$, then

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$



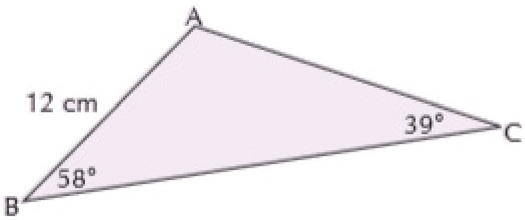
C is Acute

To use the Law of Sines you need to know either two angles and one side of the triangle (AAS or ASA) or two sides and an angle opposite one of them (SSA).

Notice that for the first two cases we use the same parts that we used to prove congruence of triangles in geometry but in the last case we could not prove congruent triangles given these parts. This is because the remaining pnieces could have been different sizes.

Example 1

Find the length of AC in the following triangle



Solution

The acronym that we have for this case is AAS (two angles and one side).

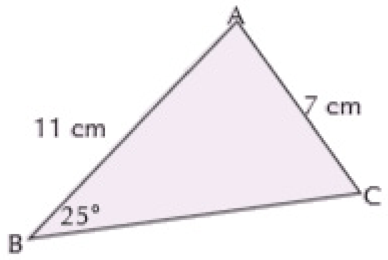
Let $AC = b$, using the sine rule, we get

$$\frac{\sin 58^0}{b} = \frac{\sin 39^0}{12} \ \textbf{or} \ b = \frac{12 \sin 58^0}{\sin 39^0} = 16.17.$$

Therefore, $AC = 16.17\,cm$.

Example 2:

Find the measure of angle C in triangle ABC if AC is 7 cm, AB is 11 cm and angle B measures 25°. Given that:



Solution

For this case, the acronym SAS (two sides and one angle).

From the sine rule, we have

$$\frac{\sin C}{c}=\frac{\sin B}{b}\Leftrightarrow\frac{\sin C}{11}=\frac{\sin 25^0}{7}\Leftrightarrow \sin C=\frac{11\sin 25^0}{7}$$

or $C=41.6^0$ *or* $180^0-41.6^0=138.4^0$.

Thus, $C=41.6^0$ or 138.4^0 as it can be acute or obtuse.

Notice:

While two angles and a side of a triangle are always sufficient to determine its size and shape, the same can not be said for two sides and an angle. Perhaps unexpectedly, it depends on where that angle is. If the angle is included between the two sides (the SAS case), then the triangle is uniquely determined up to congruence. If the angle is opposite one of the sides (the SSA case), then there might be one, two, or zero triangles determined.

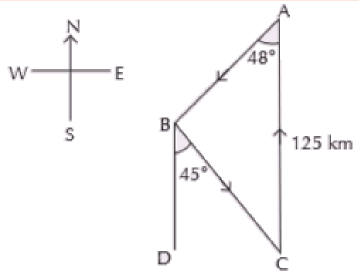
APPLICATION ACTIVITY 3.9.1.1

1. A tower of 30 m of height stands on top of a hill. From a point some distance from the base of the hill, the angle of elevation to the top of the tower is 33°. From the same point the angle of elevation to the bottom of the tower is 32°. Find the height of the hill.

2. An aeroplane is 1200 m directly above one end of a field. The angle of depression of the other end of the field from the aeroplane is 64°. How long is the field?

3. Mutesi is standing on the bank of a river and observes that the angle subtended by a tree on the opposite bank is 60°. When she retreats 40 m from the bank, she finds the angle to be 30°. Find the height of the tree and the breadth of the river.

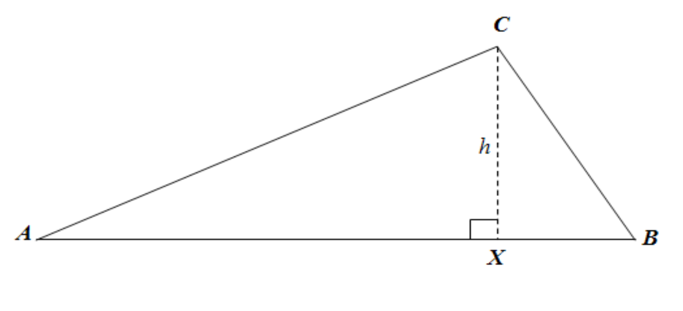
4. The course for a boat race starts at point A and proceeds in the direction S 48° W to point B. Then in the direction S 45° E to point C, and finally back to A, as illustrated in the Figure. Point C lies 125 km directly south of point A. Approximate the total distance of the race course.



b) COSINE LAW

ACTIVITY 3.9.1.2

Consider the following triangle



CX is perpendicular to side AB. Let AB=c, AC=b, BC=a, CX=h

1. In triangle AXC find $\cos A$
2. In triangle AXC use Pythagoras' theorem to find h^2
3. In triangle BCX use Pythagoras' theorem to find h^2
4. Combine results obtained in 1., 2., and 3. and give conclusion.

CONTENT SUMMARY

The Law of Cosines is an important extension of the Pythagorean theorem, with many applications. It is often called the "generalized Pythagorean theorem" because it contains that classic theorem. While the Law of Sines is the tool, we use to solve triangles in the AAS and ASA cases, the Law of Cosines is the required tool for SAS and SSS.

Let ABC be any triangle with sides and angles labeled in the usual way, then

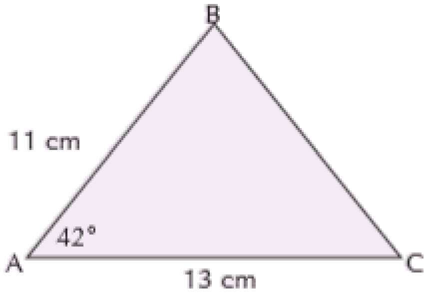
$$a^2 = b^2 + c^2 - 2bc\cos A$$

$$b^2 = a^2 + c^2 - 2ac\cos B$$

$$c^2 = a^2 + b^2 - 2ab\cos C$$

Example 1:

Find the length of BC from the following triangle.



Solution

Let a be the length of side BC, from cosine law $a^2 = b^2 + c^2 - bc\cos A$,we have,

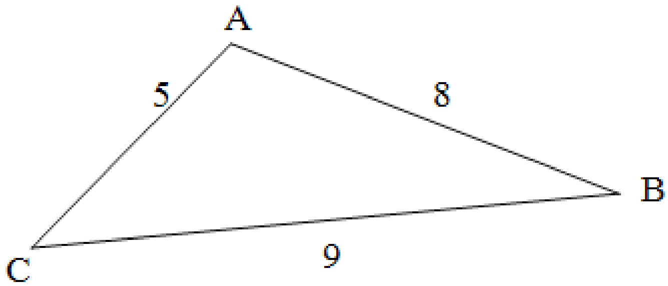
$$a^2 = (13)^2 + (11)^2 - 2(13)(11)\cos 42^0$$

$$a^2 = 169 + 121 - 286\cos 42^0 \Leftrightarrow a = \sqrt{290 - 286\cos 42^0} = 8.8.$$

Finally, the length of side BC is $8.8cm$.

Example 2:

Find the angle C



Solution

Let C be the required angle the given triangle;

From cosine law $c^2 = a^2 + b^2 - 2ab\cos C$, we have,

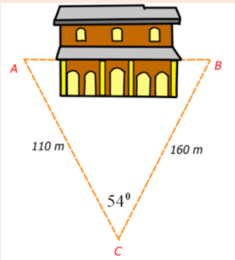
$$8^2 = (9)^2 + (5)^2 - 2(9)(5)\cos C \Leftrightarrow 64 = 81 + 25 - 90\cos C$$

Or $\cos C = \frac{42}{90}$, thus $C = 62.2^0$.

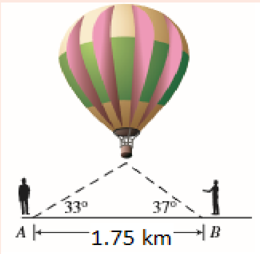
The required angle is $C = 62.2^0$.

APPLICATION ACTIVITY 3.9.1.2

1. Find the length of the side BC of triangle ABC in which $AB = 7cm, AC = 9cm$ and $\angle BAC = 71^\circ$
2. Find the length of the side AB of triangle ABC in which $BC = 15.3cm, AC = 9.4cm$ and $\angle ACB = 121^\circ$ *
3. Angelius wants to find the distance between two points A and B on opposite sides of a building. He locates a point C that is 110 m from A and 160 m from B, as illustrated in the figure. If the angle at C is 54^0, find distance AB.



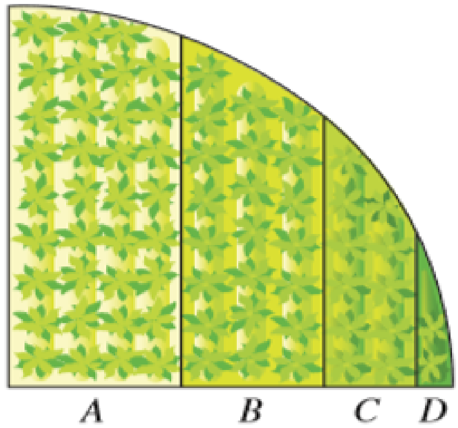
4. A hot-air balloon is seen over Tucson, Arizona, simultaneously by two observers at points A and B that are 1.75 km apart on level ground and in line with the balloon. The angles of elevation are as shown on the figure. How high above ground is the balloon?



3.9.2 Applications

ACTIVITY 3.9.2

1. Bwenge's garden is in the shape of a quarter-circle with radius 10 *m*. He wishes to plant his garden in four parallel strips, as shown in the figure 8.19, so that the four arcs along the circular edge of the garden are all of equal length. After measuring four equal arcs, he carefully measures the widths of the four strips and records his data in the table shown at the right below.



Strip	Width
A	3.827 *m*
B	3.344 *m*
C	2.068 *m*
D	0.761 *m*

Figure: Garden's design

Van Cool sees Bwenge's data and realizes that he could have saved himself some work by figuring out the strip widths by trigonometry. By checking his data with a calculator, she is able to correct two measurement errors he has made. Find Bwenge's two errors and correct them.

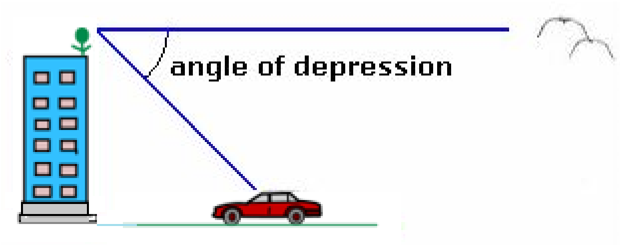
CONTENT SUMMARY

A triangle has six "parts", three angles and three sides, but you do not need to know all six parts to determine a triangle up to congruence. In fact, three parts are usually sufficient. The trigonometric functions take this observation a step further by giving us the means for actually finding the rest of the parts once we have enough parts to establish congruence. We will learn about solving general triangles in this unit, but we can already do some right triangle solving just by using the trigonometric ratios.

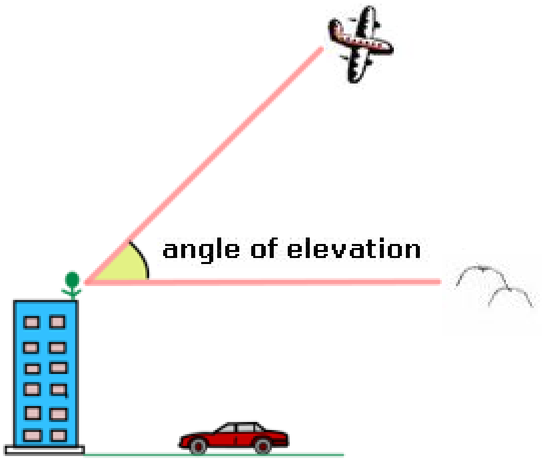
a) ANGLES OF ELEVATION AND ANGLE OF DEPRESSION

You can use right triangles to find distances, if you know an angle of elevation or an angle of depression. The figure below shows each of these kinds of angles.

Suppose that an observer is standing at the top of a building and looking straight ahead at the birds (horizontal line). The observer must lower his/her eyes to see the car parked (slanting line). The angle formed between the two lines is called the angle of depression.

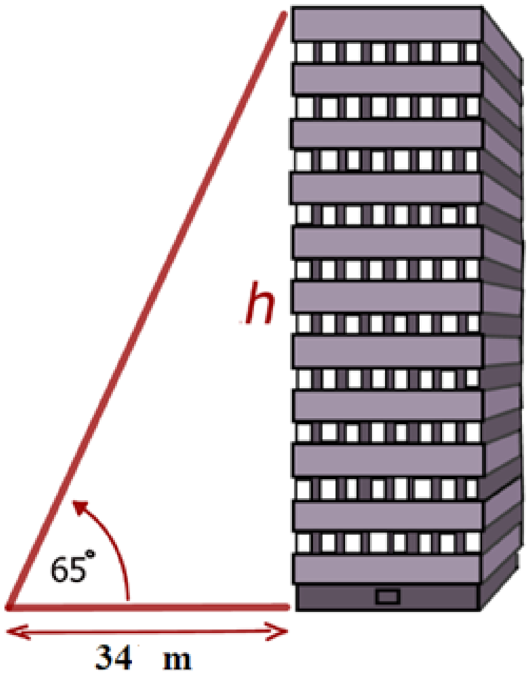


Suppose that an observer is standing at the top of a building and looking straight ahead at the birds (horizontal line). The observer must raise his/her eyes to see the airplane (slanting line). The angle formed between the two lines is called the angle of elevation.



Example 1: FINDING THE HEIGHT OF A BUILDING

From a point 34 m away from the base of the Peachtree Center Plaza in Atlanta, Georgia, the angle of elevation to the top of the building is 65°. Find the height h of the building.



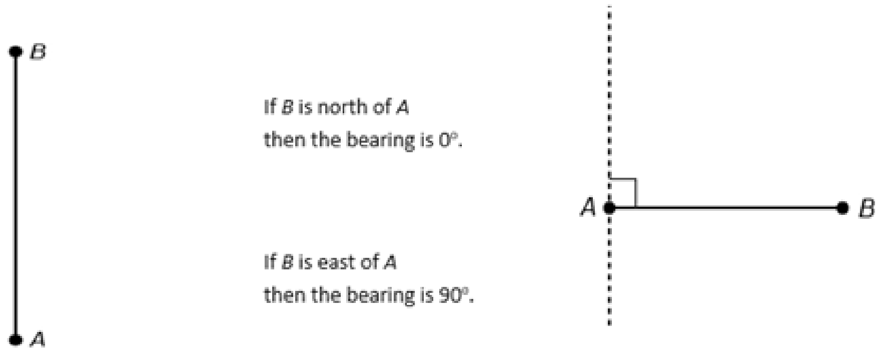
Solution

We need a ratio that will relate an angle to its hypotenuse and adjacent sides. The cosine function is the appropriate choice.

$$\tan 65^0 = \frac{h}{34} \text{ Or } h = 34 \tan 65^0 \simeq 72.9m.$$

b) BEARINGS AND AIR NAVIGATION

We say that point B has a bearing of α degrees from point A if the line connecting A to B makes an angle of α with a vertical line drawn through A, the angle being measured clockwise.



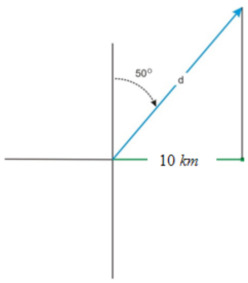
Similarly, if B is south of A then the bearing is 180o, and if B is west of A then the bearing is 270o. The bearing can be any number between 0 and 360, because there are 360 degrees in a circle. We can also use right triangles to find distances using angles given as bearings.

In navigation, a bearing is the direction from one object to another. Further, angles in navigation and surveying may also be given in terms of north, east, south, and west. For example, $N70^\circ E$ refers to an angle from the north, towards the east, while $N70^\circ W$ refers to an angle from the north, towards the west. $N70^\circ W$ would result in an angle in the second quadrant.

Example 2:

A boat travels on a $N50^\circ E$ course. The ship travels until it is due north of a port which is 10 kilometres due east of the port from which the ship originated. How far did the boat travel?

Solution



The angle between *d* and 10 km is the complement of 50° which is 40°.

$$\cos 40^\circ = \frac{adjacent}{hypothenuse} = \frac{10}{d} \Leftrightarrow \cos 40^\circ = \frac{10}{d}$$

$$\Leftrightarrow d \cos 40^\circ = 10$$

$$d = \frac{10}{\cos 40^0} \approx 13.05\ km$$

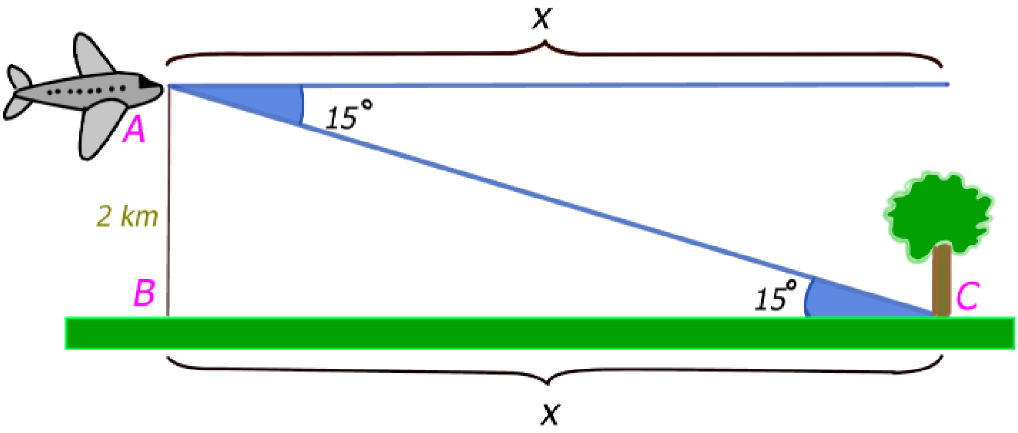
c) AIR NAVIGATION

Example 3:

An airplane is flying at a height of 2 kilometres above the level ground. The angle of depression from the plane to the foot of a tree is 15°.

Find the distance that the air plane must fly to be directly above the tree.

Solution



Let x be the distance the airplane must fly to be directly above the tree. The level ground and the horizontal are parallel, so the alternate interior angles are equal in measure.

$$\tan 15^\circ = \frac{2}{x}$$

$$x = \frac{2}{\tan 15^\circ}$$

$$\approx 7.46$$

So, the airplane must fly about 7.46 kilometres to be directly above the tree.

d) INCLINED PLANE

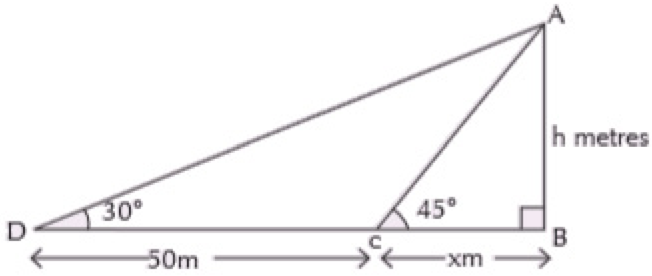
An **inclined plane**, also known as a **ramp**, is a flat supporting surface tilted at an angle, with one end higher than the other, used as an aid for raising or lowering a load.

Example 4:

A person standing on the river bank observes that the angle subtended by a tree on the opposite bank is 450. When he retreats 50 metres from the bank, he finds the angle to be 300. Find the breadth of the river and the height of the tree.

Solution

Let $AB = h$ metres be the height of the tree and $CB = x$ metres be the breadth of the river $\angle BCA = 45^0$. Consider C as the first position of the person and D as the second position of the person after he retreats.



From the above figure,

In triangle ABC, $\angle ABC = \angle ABD = 90^0$, $\angle BCA = 45^0$

$$\tan 45^0 = \frac{AB}{BC} = \frac{h}{x} \Leftrightarrow 1 = \frac{h}{x} \, or \; x = h.$$

In triangle $ABD, \angle BDA = 30^0$, $\tan 30^0 = \frac{AB}{BD} = \frac{h}{x+50}$.

As $x = h$, $\tan 30^0 = \frac{h}{x+50} \Leftrightarrow \frac{1}{\sqrt{3}} = \frac{h}{h+50} \Leftrightarrow h + 50 = h\sqrt{3} \Leftrightarrow h = \frac{50}{\sqrt{3}-1} = 40.98$.

Therefore, the height of the tree is the same as the breadth of the river that is $40.98\,m$

APPLICATION ACTIVITY 3.9.2

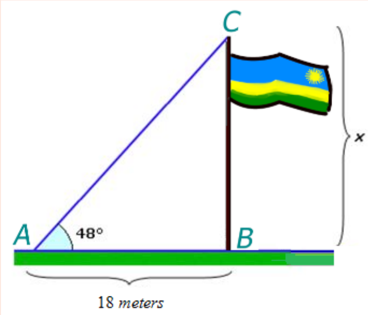
1. A tree is located on an incline of a hill. The tree is broken and the tip of the tree touches the hill farther down the hill and forms an angle of 30° with the hill. The broken part of the tree and the original tree form an angle of 50° at the break. The original part of the tree is 3 m tall. How tall was the tree before it broke?

2. Two ships are located 200 m and 300 m respectively from a lighthouse. If the angle formed by their paths to the lighthouse is 96°. What is the distance between the two ships?

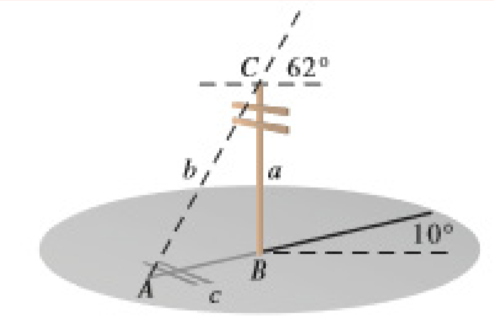
3. From a tower of 32 m of height, a car is observed at an angle of depression of 55°. Find how far the car is from the tower.

4. A town B is 13 km south and 18 km west of a town A. Find the bearing and distance of B from A.

5. The angle of elevation of the top of a pole measures 48° from a point on the ground 18 meters away from its base. Find the height of the flagpole.



6. A road slopes 10 above the horizontal, and a vertical telephone pole stands beside the road. The angle of elevation of the Sun is 62, and the pole casts a 14.5 metre shadow downhill along the road. Find the height of the telephone pole.



3.10 END UNIT ASSESSMENT

1. Verify the following identities

a) $\frac{\sin a}{1-\cos a}=\frac{1+\cos a}{\sin a}$

b) $\sec^2 a+\csc^2 a=(\sec^2 a)\csc^2 a$

c) $\sec^4 a-\tan^4 a=\sec^2 a+\tan^2 a$

d) $\sqrt{\frac{1-\cos a}{1+\cos a}}=\csc a-\cot a$

2. If $\sin\theta=0.954$ and $\cos\theta=0.3$, find the value of $\tan\theta$.

3. $\sin A=\frac{3}{5}$ and A is obtuse, find the values of $\cos A$ and $\tan A$

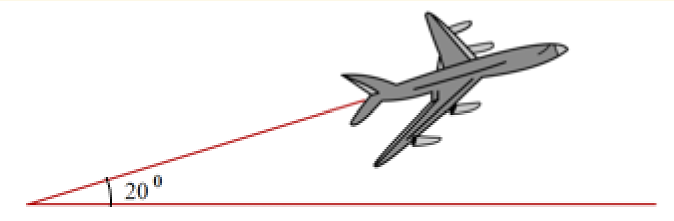
4. From a ship the angle of elevation of a point A at the top of a cliff is 12°. After the ship has sailed 400 m directly towards the foot of the cliff, the angle of elevation of A is 45°. Find the height of the cliff.

5. Give another angle which has the same sine as

a) 40°

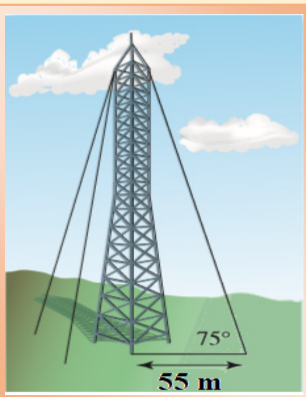
b) 300°

6. An aeroplane takes off at a constant angle of 20°.



By the time it has flown 1000 m, what is its altitude? Give your answer correct to the nearest metre.

7.



A guy wire from the top of the transmission tower at KARONGI forms a 75° angle with the ground at a 55-meters distance from the base of the tower. How tall is the tower?

UNIT 4

POLYNOMIAL FUNCTIONS

Key Unit competence: Use concepts and definitions of functions to determine the domain of polynomial functions, solve related problems and represent them graphically in simple cases (plotting linear and quadratic functions).

4.0 INTRODUCTORY ACTIVITY

1. Consider the following sentences:
   i. the **function** of the heart is to pump blood
   ii. Last Saturday, my sister got married; many people attended the **function.**
   iii. The area of a square is **function** of the length of its side.
   iv. Explain what is meant by the word **"Function"** in each of the three sentences above.
2. Which of the following illustrates the idea of a function as "**a quantity whose value depends on the value of another quantity**":
   i. $y = \frac{4x-4}{(x-1)^2}$
   ii. $A = \pi r^2$
   iii. $s = \sqrt{A}$
3. Any function involves at least two variables. Suggest, in each case in point (2) above, what is the "**independent variable**" and what is the "**dependent variable**"
4. Describe the similarity and the difference between the functions in part (2) above; make a proposal for the name of each type of function described in part (2) above.

5. Classify the following functions as "**polynomial**", "**rational**" or "**irrational**"

$$f(x)=\sqrt{\frac{x^2+1}{x-2}} \quad ; \quad f(x)=\frac{x+1}{x-5} \quad ; \quad f(x)=\sqrt{x^2-1} \quad ; \quad f(x)=2x-7 \quad ;$$

$$f(x)=\frac{x^3+2x-4}{5x}$$

6. If we agree that the set of all possible values the independent variable can assume is called the " **Domain**" of the function and the set of all possible values, the dependent value can assume is called the "**Range**" of the function, determine the range and the domain of each of the functions in part (2) above.

7. For each of the following functions $f(x)=\frac{1}{x}$; $g(x)=\sqrt{x}$; $h(x)=x^2$, write down the set of real numbers that are not in the:

   i. domain of the function

   ii. the range of the function

4.1. Generalities on numerical functions

4.1.1. Types of functions

ACTIVITY 4.1.1

Differentiate rational from irrational numbers. Guess which of the following functions is a polynomial, rational or irrational function

1. $f(x)=(x+1)^2$
2. $h(x)=\frac{x^3+2x+1}{x-4}$
3. $f(x)=\sqrt{x^2+x-2}$

CONTENT SUMMARY

Definition: a function is a relationship between the members of two sets(not necessarily distinct), such that to each value of the independent variable there corresponds at most one value of the dependent variable.

a) Polynomial

A function that is expressible as the sum of finitely many monomials in x is called **polynomial in *x***.

Example: x^3+4x+7; $17-\frac{2}{3}x$; y and x^5 are polynomials. Also $(x-2)^3$ is a polynomial in x because it is expressible as a sum of monomials.

In general, f is a polynomial in x if it is expressible in the form $f(x)=a_0+a_1x+a_2x^2+......+a_nx^n$ where n is a positive integer, and $a_0, a_1, ..., a_n$ are real constants. The greatest positive integer *n, for which* $a_n\neq 0$, *is called the* ***degree*** *of the polynomial, and, in this case,* a_n *is called the* ***leading coefficient*** *of the polynomial.*

A polynomial is called **linear** if it is of degree 1,that is, if it can be expressed in the form a_0+a_1x ,where $a_1\neq 0$

The polynomial is said to be **quadratic** if it is of degree 2, which means it can be expressed in the form $a_0+a_1x+a_2x^2$, $a_2\neq 0$,

The polynomial is **cubic,** or of degree 3**,** if it is possible to re write it as $a_0+a_1x+a_2x^2+a_3x^3$, $a_3\neq 0$,

An **n^th^ degree polynomial** has the form $a_0+a_1x^1+a_2x^2+a_3x^n+...+a_nx^n; a_n\neq 0$

b) Rational function

A function that is expressible as ratio of two polynomials is called **rational function.** It has the form *f(x)=*$\frac{a_0+a_1x+a_2x^2+......+a_nx^n}{b_0+b_1x+b_2x^2+......+b_mx^m}$.

Example:

$f(x)=\frac{x^2+4}{x-1},\ g(x)=\frac{1}{3x-5}$ are rational functions

c) Irrational function

A function that can be expressed as $\sqrt[n]{f(x)}$, where $f(x)$ contains the variable *x*.

Example:

$f(x)=\frac{\sqrt{x^2+4}}{\sqrt[3]{x-1}},\ g(x)=\sqrt{\frac{x}{x-5}}$ are irrational functions

APPLICATION ACTIVITY 4.1.1

Observe the given functions and categorize them into polynomial, rational or irrational functions.

$f(x)=x^3+2x^2-2$ $\quad g(x)=\frac{x^3+2x^2-2}{x-5}$ $\quad h(x)=\sqrt{x^3+2x^2-2}$

4.1.2. Injective, surjective and bijective functions

ACTIVITY 4.1.2

a. Consider the function f defined by $f:\mathbb{R}\to\mathbb{R}$

$$x\mapsto f(x)=x^2$$

i. Is there any real number missing an image?

ii. State whether *f* is a mapping or not

iii. Is there any real number that is image of more than one real number? If yes, give an example. If not, explain.

iv. State whether *f* is *one to one* or not.

v. Is there any real number that is not image under function *f?* If yes, give an example. If not, explain.

vi. State whether *f* is *onto* or not.

b. Consider now the function *f* defined by $f:\mathbb{R}\to\mathbb{R}^{+}$

$$x\mapsto f(x)=x^{2}$$

Repeat questions iii, iv, v and vi

c. Finally, consider the function *f* defined by $f:\mathbb{R}^{+}\to\mathbb{R}^{+}$

$$x\mapsto f(x)=x^{2}$$

Repeat questions iii, iv, v and vi

d. Consider the function f defined by $f:\mathbb{R}\to\mathbb{R}$

$$x\mapsto f(x)=-x^{2}+4x$$

Determine the greatest subsets *A* and *B* of $\mathbb{R}$ such that function by $f:A\to B$

$x\mapsto f(x)=-x^{2}+4x$ is bijective (that is, both one to one and onto)

CONTENT SUMMARY

Given sets A and B, a **function** from A to B is a correspondence, or a rule that associates to any element of A either one image in B, or no image in B

A function such that no element of A is missing an image is called a **mapping**, thus, under a mapping any element of A has exactly one image in B (not less than one, and not more than one)

A mapping such that any element of B is image of either one element of A, or of no element of A, is called a **one- to- one mapping**, or an **injective mapping** or simply an **injection;** under a one-to-one mapping no two elements of A share the common image in B.

Mathematically, $(\forall x_1\in A)(\forall x_2\in A); f(x_2)=f(x_1)\Rightarrow x_2=x_1$

A mapping such that any element of B is image of at least one element of A (image of one element of A, or image of more than one element of A), is called an **onto mapping**, or a **surjective mapping** or simply a **surjection**

Mathematically, $(\forall y \in B)(\exists x \in A); \mathrm{f(x)} = y$

A mapping both one-to-one and onto is said to be a bijective mapping , or simply ,a bijection

In particular, linear function $f : \mathbb{R} \to \mathbb{R}$

$x \mapsto f(x) = a\mathrm{x} + \mathrm{b}$, where $a \neq 0$, is bijective, there is no restrictions on the variables(independent or dependent)

Quadratic function $f : \mathbb{R} \to \mathbb{R}$

$x \mapsto f(x) = a\mathrm{x}^2 + bx + c$, where $a \neq 0$, is not bijective, since some real numbers share images, or some real numbers are not images under function *f*.

But the restrictions

$$f : [\frac{-b}{2a}, +\infty[\to [-\frac{\Delta}{4a}, +\infty[$$

$$x \mapsto f(x) = a\mathrm{x}^2 + bx + c, \quad a > 0;$$

$$f :]-\infty, -\frac{b}{2a}] \to [-\frac{\Delta}{4a}, +\infty[$$

$$x \mapsto f(x) = a\mathrm{x}^2 + bx + c, \ a > 0;$$

$$f : [\frac{-b}{2a}, +\infty[\to]-\infty, -\frac{\Delta}{4a}]$$

$x \mapsto f(x) = a\mathrm{x}^2 + bx + c, a < 0$ and

$$f :]-\infty, -\frac{b}{2a}] \to]-\infty, \frac{\Delta}{4a}]$$

$x \mapsto f(x) = a\mathrm{x}^2 + bx + c, \ a < 0$ are bijective

The homographic function $f:\mathbb{R}\to\mathbb{R}$

$$x \mapsto f(x)=\frac{ax+b}{cx+d}$$ is not bijective, since it is not a mapping($x=-\frac{d}{c}$ has no image under function *f*, or $y=\frac{a}{c}$ is not image under function *f*)

But the restriction $f:\mathbb{R}-\{-\frac{d}{c}\}\to\mathbb{R}-\{\frac{a}{c}\}$

$$x \mapsto f(x)=\frac{ax+b}{cx+d}$$ is bijective

Example1

Consider the set of pigeons and the set of pigeonholes on the diagram below to answer the questions:-

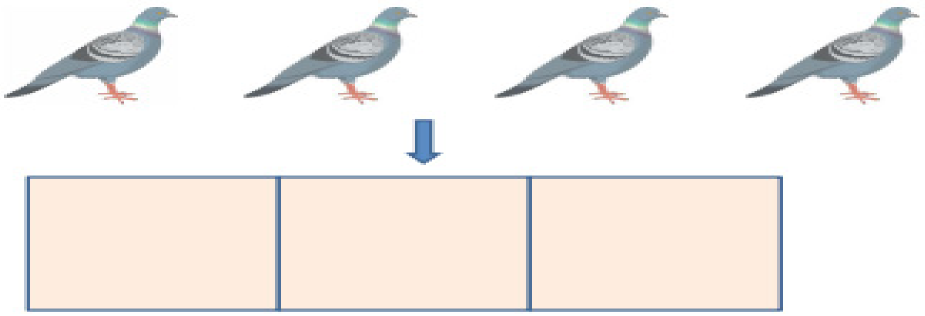
Determine whether it can be established or not between the two sets:

a. A mapping,

b. A one-to-one mapping,

c. An onto mapping,

d. A bijective mapping:



Solution:

Let the pigeons be numbered *a, b, c,* d and the pigeonholes be numbered *1,2,3.*

a) It is possible to establish a mapping between the two sets. For example, $\{(a,1);(b,2);(c,3);(d,3)\}$.This function is a mapping since each pigeon is accommodated in exactly one pigeonhole, though pigeons c and d are in the same pigeonhole.

b) It is not possible to establish a one-to-one mapping, since sharing images is not allowed. A function from one finite set to a smaller finite set cannot be one-to-one: there must be at least two elements that have the same image

c) The example given in part (a) illustrates a mapping that is onto: no pigeonhole is empty.

d) It is impossible to define a bijection, since it is already impossible to establish a one-to-one mapping

Example 2

Determine whether function $f:\mathbb{R}\to\mathbb{R}$

$$x \mapsto f(x) = 3x - 5 \text{ is (or is not)}$$

a) One-to-one

b) Onto

c) Bijective.

Solution:

a) Let x_1 and x_2 be real numbers such that $f(x_1) = f(x_2)$.Then $3x_1 - 5 = 3x_2 - 5$

This is equivalent, successively to $3x_1 = 3x_2$ (by adding 5 on both sides);

$$x_1 = x_2 \text{ (Dividing both sides by 3)}$$

Since the equality $f(x_1) = f(x_2)$ implies $x_1 = x_2$, the function is one-to-one.

b) Suppose y a real number. Let us look for real number x, if possible, such that $f(x) = y$.

Then $3x - 5 = y$.It follows that $x = \frac{y+5}{3}$; such x exists for any value of y;

$$f(\frac{y+5}{3}) = 3\left(\frac{y+5}{3}\right) - 5 = y$$

Therefore, function f *is onto.*

c) Since, from points (a) and (b), f is one-to-one and onto, function f is bijective.

Example3

Show that function f defined by $f:\mathbb{R}\to\mathbb{R}$

$x\mapsto f(x)=x^2+2x-3$ is neither one-to-one, nor onto

Solution

$f(-2)=(-2)^2+2(-2)-3=-3$ and $f(0)=(0)^2+2(0)-3=-3$

Since $f(-2)=f(0)$ and $-2\neq 0$, the function is not one-to-one.

On the other side, there is no x such that $f(x)=-5$;

in fact, $f(x)=-5 \Leftrightarrow x^2+2x-3=-5$

$$\Leftrightarrow x^2+2x+2=0$$

No such x since $\Delta=2^2-4(1)(2)=-4<0$

Therefore, the function is not onto

Example 4

Consider the function f defined by $f:\mathbb{R}\to\mathbb{R}$

$$x\mapsto f(x)=-x^2+4x$$

Determine the greatest subsets A and B of $\mathbb{R}$ such that function

$$f:A\to B$$

$$x\mapsto f(x)=-x^2+4x \text{ is bijective}$$

Solution:

The maximum value of the function occurs for $x=2$ and $f(2)=4$. Therefore,

$A=[2,+\infty[$ and $B=]-\infty,4]$, or $A=]-\infty,2]$ and $B=]-\infty,4]$

Example 5

Function $f:\mathbb{R}-\{a\}\to\mathbb{R}-\{b\}$

$$x\mapsto f(x)=\frac{2x-5}{3-x} \text{ is bijective}$$

a) Find the values of a and b

b) Show that f is one-to-one

c) Find the real number whose image is 2

Solution:

a) f is bijective if $a = 3$ and $b = -2$

b) Let $x_1 \neq 3$ and $x_2 \neq 3$ be such that $f(x_1) = f(x_2)$, that is $\frac{2x_1 - 5}{3 - x_1} = \frac{2x_2 - 5}{3 - x_2}$

Then $6x_1 - 2x_1x_2 - 15 + 5x_2 = 6x_2 - 2x_1x_2 - 15 + 5x_1$,

which is equivalent to $6(x_1 - x_2) - 5(x_1 - x_2) = 0 \Leftrightarrow x_1 - x_2 = 0 \Leftrightarrow x_1 = x_2$

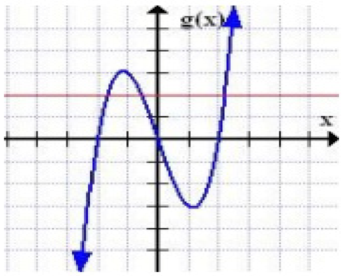
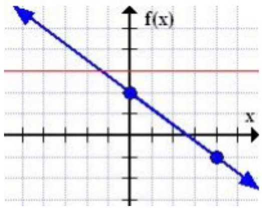
Therefore, f is one- to- one

c) Let x be the number. Then $f(x) = 2 \Leftrightarrow \frac{2x-5}{3-x} = 2$

Solving this equation, we get $x = 2$

Horizontal Line Test

Horizontal Line Test states that a function is a one to one(injective) function if there is no horizontal line that intersects the graph of the function at more than one point.

Graph representation	Interpretation	Conclusion
	You can see that for this graph, there are horizontal lines that intersect the graph more than once.	Not injective
	You can see that for this graph any horizontal line intersects the graph only once.	Injective function

APPLICATION ACTIVITY 4.1.2

1. Let A and B be two none-empty sets where $A = \{1, 2, 3, 4\}$ and $B = \{a, b, c\}$ Consider each of the following relations:

$T = \{(1,a),(2,b),(2,c),(3,c),(4,b)\}$

$U = \{(1,a),(2,b),(4,b)\}$

$V = \{(1,a),(2,b),(3,c),(4,b)\}$

Which of these relations $(T, U \; and \; V)$ qualify as functions?

2. i) Although the relation V in Question 1 above is a function, it is not a one-to-one (or injective) function. Why?

   ii) Is the relation V above defined an onto (surjective) function? Why?

   (iii) Does the function f, defined by the relation V, have an inverse?

3. a) Find a function $f : \mathbb{N} \to \mathbb{N}$ that is injective, but not surjective.

   b) Find a function $f : \mathbb{N} \to \mathbb{N}$ that is surjective, but not injective.

4. Discuss whether the following functions are bijective: $f(x) = 3^x$ and $g(x) = x^3$ if no, which condition fails?

5. Let W and Z be sets; $W \times Z$ the Cartesian product of W and Z ;

C be set the set of all correspondences (relations) from W to Z ;

F be set the set of all functions from W to Z ;

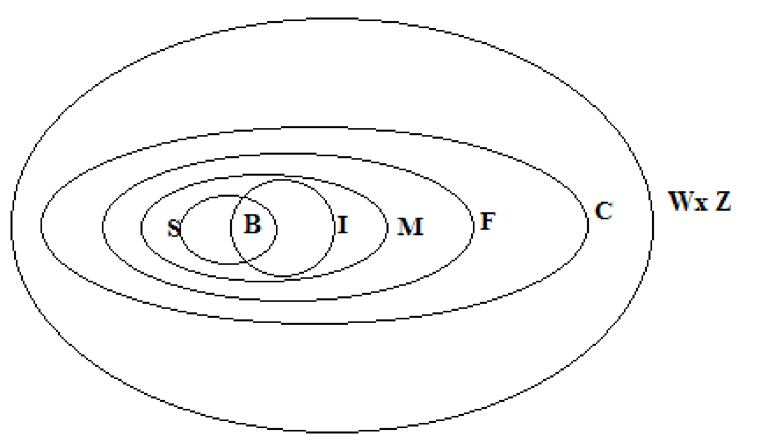
M be set the set of all mappings from W to Z ;

I be set the set of all one to one mappings from W to Z ;

S be set the set of all onto mappings from W to Z ;

B be set the set of all bijective mappings from W to Z ;

Then we have the following sequence of inclusion of sets



Using examples, explain in your own words the relationship amongst these sets, and decide on the following inclusion: $S \subset M \subset F \subset C \subset (W \times Z)$.

4.1.3 Existence condition for a given function

ACTIVITY 4.1.3

Consider the following functions and calculate their numerical value at the given value(s) of x. Discuss your findings

a) $f(x) = \frac{1}{x}$ at $x = 0$

b) $f(x) = \frac{2+x}{x-3}$ at $x = 1,\ x = 2,\ x = 3$

c) $f(x) = \frac{2(x-1)}{x-1}$ at $x = 0,\ x = 1$

d) $f(x) = \frac{x+1}{x^2-1}$ at $x = -1, x = 1$ *and* $x = 2$

e) $f(x) = \frac{2(x-1)}{x^2+1}$ at $x = -1, x = 1$ *and* $x = 2$

CONTENT SUMMARY

From activity 4.1.3 calculating the numerical values of the given functions at the given value(s) of x leads to the all values for which the function is defined known as the existence condition for a given function.

From activity 4.1.3 one can find the following:

a) $f(x)=\frac{1}{x}$ at $x=0$ if $x=0$ you would be dividing by 0. So $x\neq 0$

b) $f(x)=\frac{2+x}{x-3}$ at $x=1,\ x=2,\ x=3$, if $x=3$ you would be dividing by 0. So $x\neq 3$

c) $f(x)=\frac{2(x-1)}{x-1}$ at $x=0,\ x=1$, if $x=1$ you would be dividing by 0. Therefore, $x\neq 1$

d) $f(x)=\frac{x+1}{x^2-1}$ at $x=\text{-}1, x=1\ and\ x=2$, Both $x=1$ and $x=-1$ would make the denominator 0. Again, this function can be simplified to $f(x)=\frac{1}{x-1}$, but when $x=1$ or $x=-1$ the *original* function would include division by 0. Therefore, $x\neq 1$ and $x\neq -1$

$f(x)=\frac{2(x-1)}{x^2+1}$ at $x=\text{-}1, x=1\ and\ x=2$. This is an example with *no* domain restrictions, even though there is a variable in the denominator. Since $x^2\geq 0, x^2+1$ can never be 0. The least it can be is 1. A radical function is a function that contains the independent variable under the square root sign. For example, $f(x)=\sqrt{7-x}$ is a radical function

An irrational function is a function that contains the independent variable in the radicand; the index may be any positive integer ≥ 2.Thus all radical functions are irrational functions, but the converse is not true.

The following table gives examples of domain restrictions for irrational functions.

Function	Restriction
$f(x)=\sqrt{x}$	$x \geq 0$
$f(x)=\sqrt{x+10}$	$x+10 \geq 0$
$f(x)=\sqrt{-x}$	$-x \geq 0; x \leq 0$
$f(x)=\sqrt{x^2-1}$	$x^2-1 \geq 0$
$f(x)=\sqrt{x^2+1}$	$x^2+1 \geq 0$ (always true for all real values of x), therefore, there is no restriction for this function.

Examples

For each of the following functions determine the restrictions on the independent variable $f(x)=\sqrt{x^2-1}+\frac{1}{x}$

1. $f(x)=\tan x$
2. Let $f(x)=x+2$ and $g(x)=\sqrt{x+1}$, hence find the restrictions for the function $h(x)=\frac{g(x)}{f(x)}$.
3. $k(x)=\frac{3x^2}{5x-1}$

Solution:

1. $x^2-1 \geq 0$ and $x \neq 0$. This is equivalent to $x \leq -1$ or $x \geq 1$
2. $\cos x \neq 0$, that is $x \neq \pm\frac{\pi}{2}+2k\pi; k \in \mathbb{Z}$
3. For $f(x)$: no restriction

   For $g(x): x+1 \geq 0$; that is $x \geq -1$

For $\frac{f(x)}{g(x)}$: $x+1\geq 0$ and $x+2\neq 0$, then $x\geq -1$

4. $5x-1\neq 0; x\neq \frac{1}{5}$

APPLICATION ACTIVITY 4.1.3

Find out the condition (s) or the restriction(s) for the existence of the image under the following functions:

i) $h(x)=\frac{x^2-3}{x-4x^3}$

ii) $f(x)=\frac{9}{\sqrt{4-x^2}}$

iii) $f(x)=\frac{\sqrt[3]{x^3+x}}{5x}$

iv) $f(x)=\frac{3x}{x^2+4}$

4.2. Domain of definition and range of a numerical function.

4.2.1. Domain and range of polynomial functions

ACTIVITY 4.2.1

1) Given the function $f(x)=x-4$. Plot the graph of $f(x)$ and discuss whether it continues endlessly or not.

2) Consider the function $f(x)=\frac{1}{2}(x-2)^2-4$ .complete the following table and use it to plot the graph of $f(x)$. Can you predict all values of x which can make $f(x)$ not defined.

x	-5	-4	-3	-2	-1	0	1	2	3	4
y										

3) Consider the function $f(x)=x^3-3x^2+3x-2$ .complete the following table and use it to plot the graph of $f(x)$. Can you predict all values of x which can make $f(x)$ defined.

x	-8	-4	-2	0	1	2	3	4	5
y									

4) For which value(s) of x is the function

$f(x)=a_nx^n+a_{n-1}x^{n-1}+\ldots+a_2x^2+a_1x+a_0$ defined ?

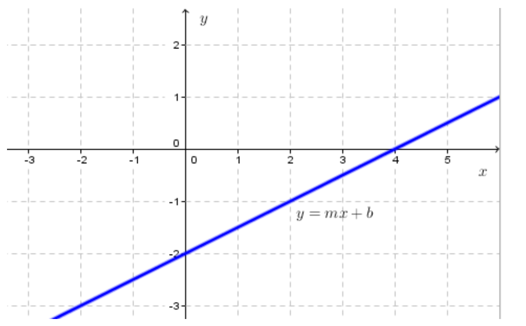
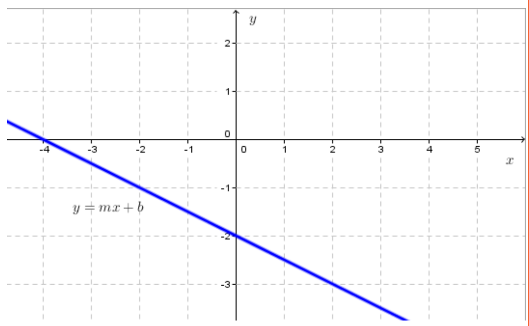
$a_n\ ,a_{n-1},\ldots,a_2,a_1,a_0$ are real coefficients and $n=0,1,2,3,4,\ldots$

CONTENT SUMMARY

The domain of any **linear functions** is the set of all real numbers ,that is

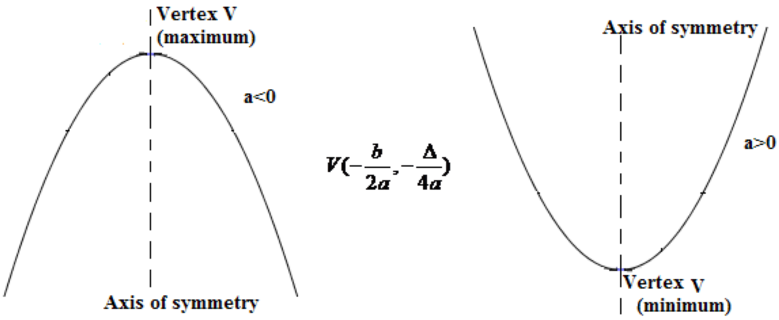
Domf $\mathbb{R}$ or Domf= $]-\infty,+\infty[$ Similarly, the range of a linear function f, denoted Imf, is the set of all real numbers, that is $\operatorname{Im} f=]-\infty,+\infty[$

Depending on the sign of m *in the equation* $y=mx+b$, the trend of the graph is as follows:

$m>0$	$m<0$
	

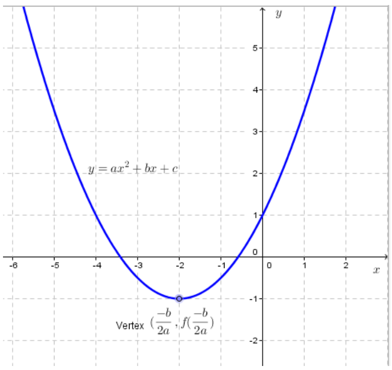
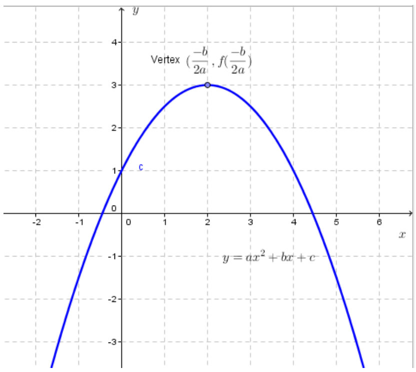
From the graphs, one can observe that each value of x has its corresponding y value.

For quadratic functions $y = ax^2 + bx + c$, the main features are summarized on the graph below:

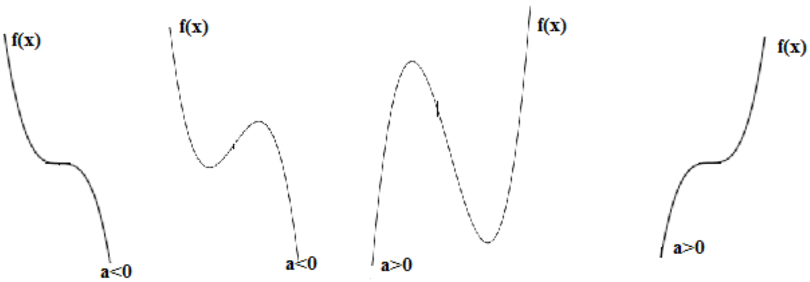


If $a > 0$, then the range of function $y = ax^2 + bx + c$ is $[-\frac{\Delta}{4a}, +\infty[$

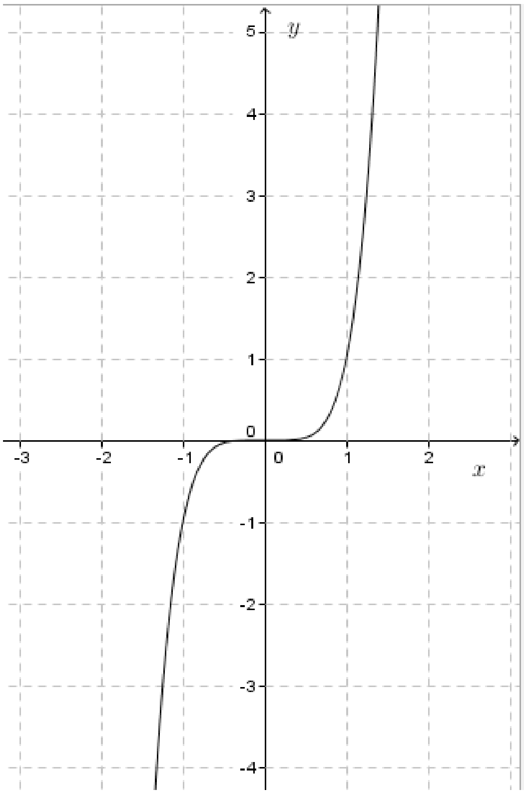
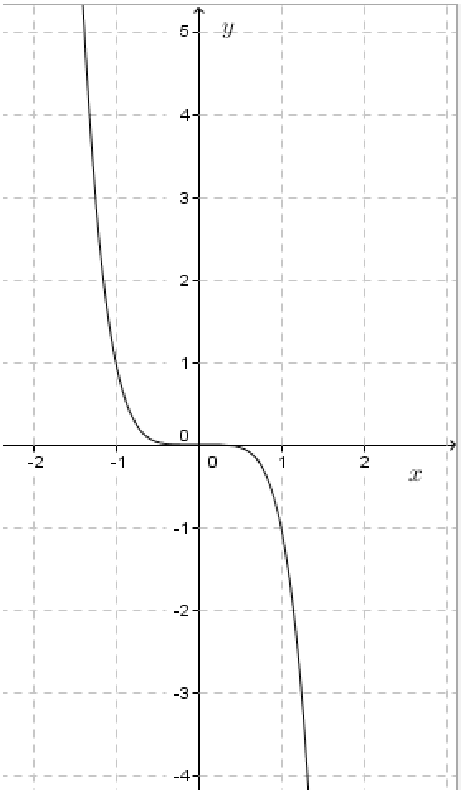
If $a < 0$, then the range of function $y = ax^2 + bx + c$ is $]-\infty, -\frac{\Delta}{4a}]$

$a > 0$, minimum	$a < 0$, maximum
	

For cubic functions $f(x) = ax^3 + bx^2 + cx + d; a \neq 0$, the trends of the graphs are as shown below:



In each case, the domain is $]-\infty,+\infty[$ and the range is $]-\infty,+\infty[$

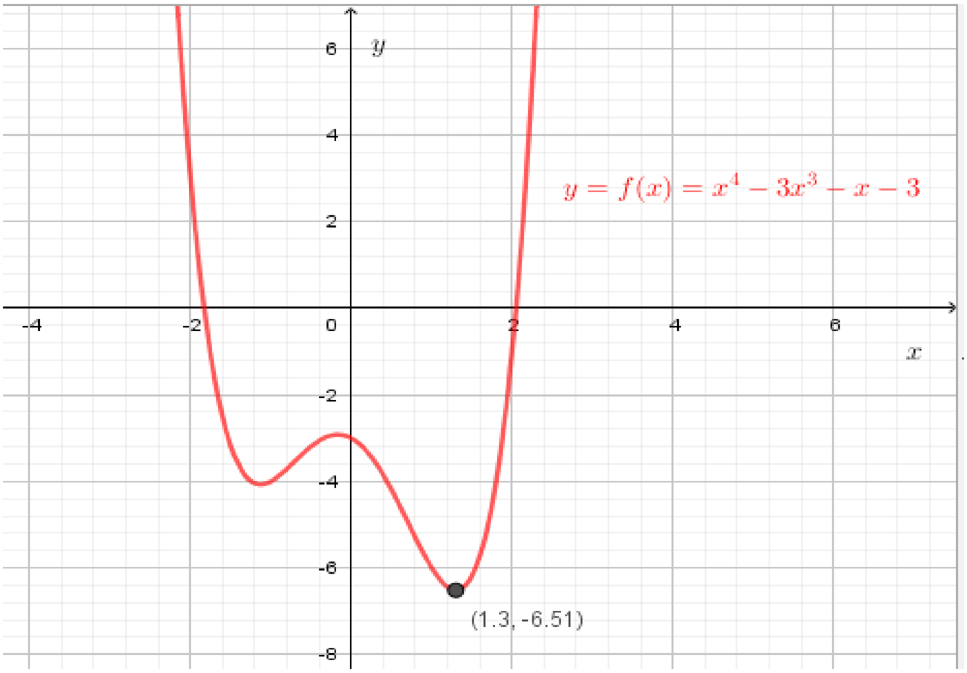
$a>0$	$a<0$
	

$D=]-\infty,\ \infty[\quad R=]-\infty,\ \infty[$.

If for polynomials of odd degrees the range is the set of all real numbers, it is not the case for polynomials of even degree, greater or equal to 4.The determination of the range is not easy unless the function is given by its graph; in this case, find by inspection, on the y-axis, the set of all points such that the horizontal lines through those points cut the graph.

Example1

Determine the domain and range of $f(x)=x^4-3x^3-x-3$ shown on the graph below:



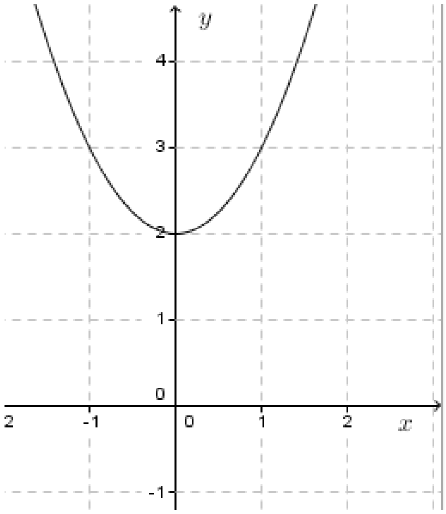
Solution:

$domf =]-\infty, +\infty[$ and $\operatorname{Im} f = [-6.51, +\infty[$

Example 2: Find the range for the function $f(x) = x^2 + 2$

Solution:

Let $f: \mathbb{R} \to \mathbb{R}$ be the function given by $f(x) = x^2 + 2$. Then the range is $[2, +\infty[$ as shown on the graph below:



APPLICATION ACTIVITY 4.2.1

Find the domain and range of the following functions.

a) $f(x) = -x^2 + 1$

b) $f(x) = 2x^3 - x + 1$

c) $f(x) = 3x + 1$

4.2.2. Domain and range of rational functions

ACTIVITY 4.2.2

For which value(s) the following functions are not defined

a) $f(x) = \frac{1}{x}$

b) $f(x) = \frac{x}{(x-1)(x+3)}$

CONTENT SUMMARY

Given that $f(x) = \frac{g(x)}{h(x)}$ where $g(x)$ and $h(x)$ are polynomials, then the domain of definition is the set of real numbers excluding all values where the denominator is zero. That is $Domf = \{x \in \mathbb{R} : h(x) \neq 0\}$

From activity 4.2.2, one can determine the intervals for which the given functions are valid.

Example1

Find the domain of each of the functions:

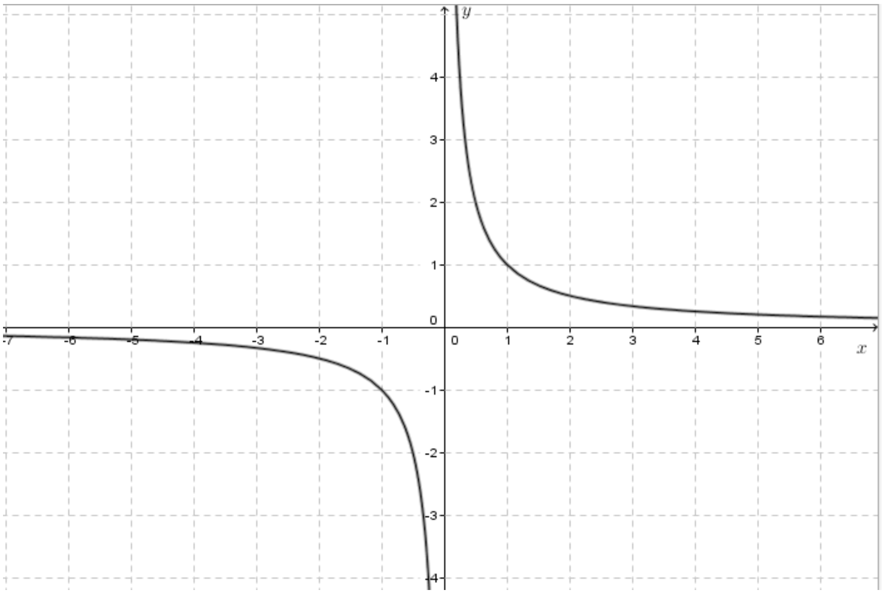
a. $f(x) = \frac{1}{x}$

b. $f(x) = \frac{x}{(x-1)(x+3)}$

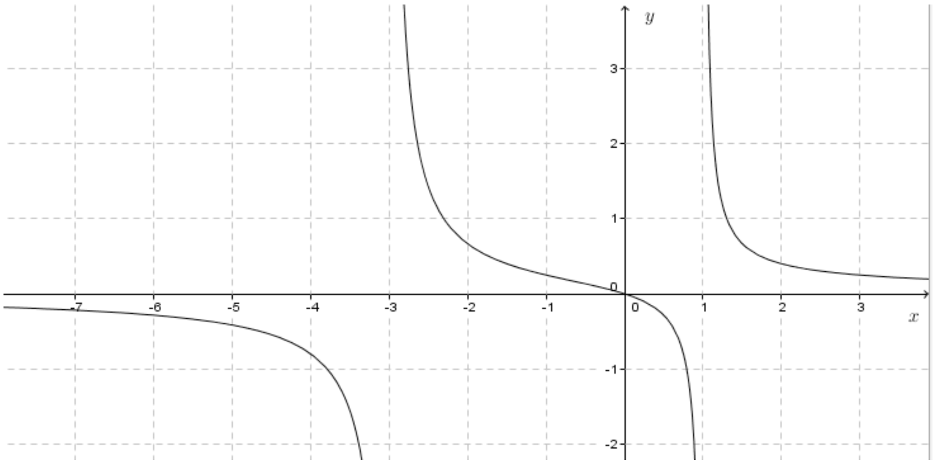
c. $f(x)=\dfrac{x+1}{3x+6}$

Solution:

a) The denominator should be different from zero ($x\neq 0$), domain of definition is $\{x\in\mathbb{R}:x\neq 0\}$ or $\mathbb{R}^{*} or\ \mathbb{R}^{+}$. The domain can be written as an interval as follows: $]-\infty,0[\cup]0,+\infty[$. Observing the graph of the function $f(x)=\dfrac{1}{x}$, one can early realize that the function has no value only if $x=0$



b) $f(x)=\dfrac{x}{(x-1)(x+3)}$. The denominator should be different from zero, $(x-1)(x+3)\neq 0$. domain of definition is $\{x\in\mathbb{R}:x\neq 1 \text{ and } x\neq -3\}$ or simply $]-\infty,-3[\cup]-3,1[\cup]1,+\infty[$. Observing the graph of the function $f(x)=\dfrac{x}{(x-1)(x+3)}$, one can early realize that the function has no value only if $x=1\ and\ x=-3$



c) Condition: $3x+6\neq 0$

$3x+6=0 \Rightarrow x=-2$

Then, $Domf=\mathbb{R}\setminus\{-2\}$ or $Domf=\left]-\infty-2\right[\cup\left]-2,+\infty\right[$

Example 2: Find the range for the function $f(x)=\dfrac{1}{x-2}$

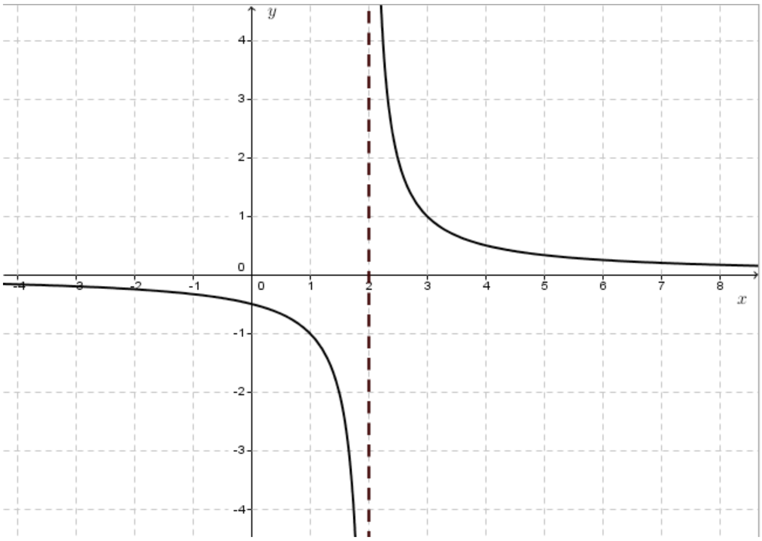
Solution

(1) Put $y=f(x)=\dfrac{1}{x-2}$

(2) Solve for x, $y=\dfrac{1}{x-2}\Leftrightarrow x=\dfrac{1}{y}+2$. Note that x can be solved if and only if $y\neq 0$.

The range of $f(x)$ is $\{y\in\mathbb{R}: y\neq 0\}=\mathbb{R}\setminus\{0\}$.

Alternatively, one can see on the graph that the range of $f(x)$ is $\mathbb{R}\setminus\{0\}$.



Example 3: Find the range for the function $f(x)=\dfrac{2x+1}{x^2+2}$

Solution

(1) Put $y=f(x)=\dfrac{2x+1}{x^2+1}$

(2) Solve for x, $y=\dfrac{2x+1}{x^2+1}\Leftrightarrow yx^2+y=2x+1$.

$$yx^2+y=2x+1\Leftrightarrow yx^2-2x+(y-1)=0$$

$$x=\frac{2\pm\sqrt{4-4y(y-1)}}{2y} \text{ if } y\neq 0, \quad x=-\frac{1}{2} \text{ if } y=0$$

$$=\frac{1\pm\sqrt{1-y^2+y}}{y} \text{ if } y\neq 0$$

Comparing the two, we see that x exists in the set of real numbers if and only if $1-y^2+y\geq 0$, that is $y^2-y-1\leq 0$

The range of $f(x)$ is $\{y\in\mathbb{R}: y^2-y-1\leq 0\}$. Solving the inequality $y^2-y-1\leq 0$ we get $y=\dfrac{1\pm\sqrt{5}}{2}$. Then studying the sign of the quadratic expression,

$$y^2 - y - 1 = \left(y - \frac{1-\sqrt{5}}{2}\right)\left(y - \frac{1+\sqrt{5}}{2}\right)$$

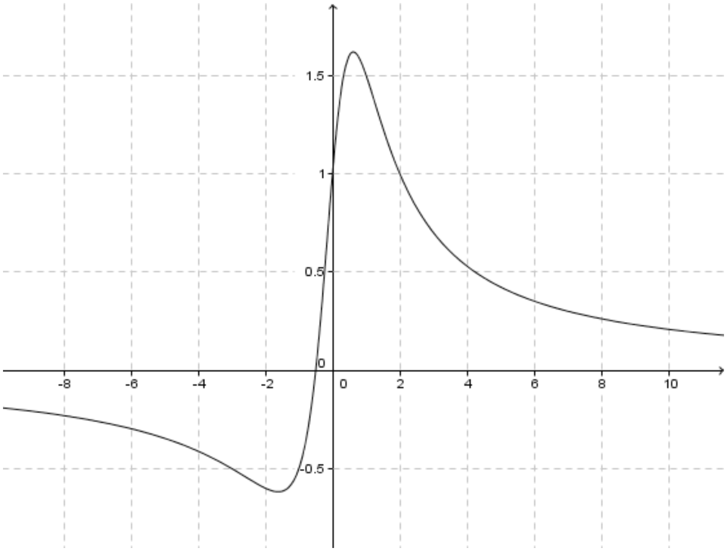
	$y < \frac{1-\sqrt{5}}{2}$	$y = \frac{1-\sqrt{5}}{2}$	$\frac{1-\sqrt{5}}{2} < y < \frac{1+\sqrt{5}}{2}$	$y = \frac{1+\sqrt{5}}{2}$	$y > \frac{1+\sqrt{5}}{2}$
$y - \frac{1-\sqrt{5}}{2}$	— —–	0	+ + + +	+ + ++	+ + + +
$y - \frac{1+\sqrt{5}}{2}$	———	——-	————-	0	+ + + +
$y^2 - y - 1$	+ + + + +	0	—–——	0	+ + +

From the table , we see that the range of the function $f(x) = \frac{2x+1}{x^2+2}$ is:

$$R(f) = \left\{ y \in \mathbb{R} : \frac{1-\sqrt{5}}{2} \le y \le \frac{1+\sqrt{5}}{2} \right\} = \left[\frac{1-\sqrt{5}}{2}, \frac{1+\sqrt{5}}{2} \right]$$

One can see on the graph that the range of $f(x)$ is

$$\left[\frac{1-\sqrt{5}}{2}, \frac{1+\sqrt{5}}{2} \right] \simeq [-0.618034; 1.61803]$$



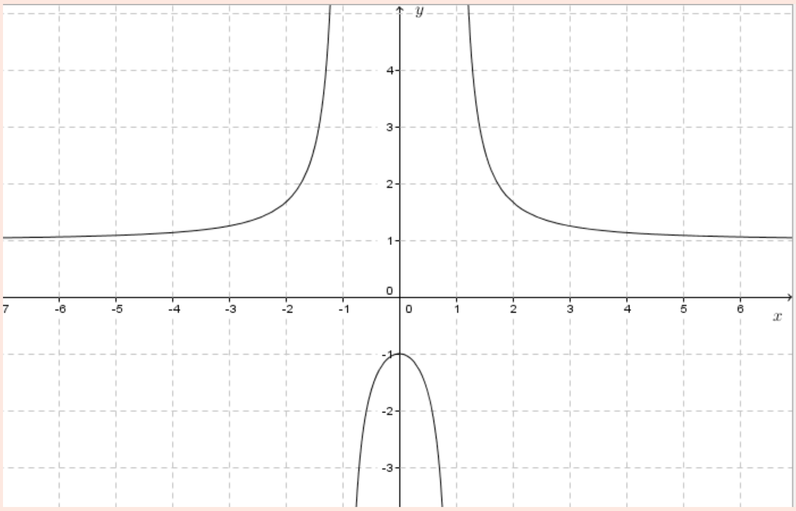
APPLICATION ACTIVITY 4.2.2

1. Find the domain and range of the following functions

a) $f(x)=\dfrac{\sqrt{x+2}}{x^2-9}$ b) $f(x)=\dfrac{2x+1}{6x^2-x-2}$ c) $f(x)=\dfrac{-6}{25x^2-4}$

d) $f(x)=\dfrac{2x-9}{x^3+2x^2-8x}$ e) $f(x)=\dfrac{5}{x-2}$

2. Given the function $f(x)=\dfrac{x^2+1}{x^2-1}$, observe its graph and write down x-values where the function is not defined.



4.2.3. Domain and range of irrational functions

ACTIVITY 4.2.3

For each of the following functions, give the range of values of the variable x for which the function is not defined

1. $f(x)=\sqrt{2x+1}$ 2. $f(x)=\sqrt[3]{x^2+x-2}$

3. $g(x)=\sqrt{\dfrac{x-2}{x+1}}$

CONTENT SUMMARY

Given that $f(x)=\sqrt[n]{g(x)}$ where $g(x)$ is a polynomial, there are two cases

a) If n is odd number, then the domain is the set of real numbers. That is $domf=\mathbb{R}$

b) If n is even number, then the domain is the set of all values of x such $g(x)$ is positive or zero. That is $domf=\{x\in\mathbb{R}: g(x)\geq 0\}$

Example 1: Given the function $g(x)=\sqrt{x^2-1}$, determine the domain.

Solution

Condition of existence: $x^2\text{-}1\geq 0$ this implies that we need to determine the interval where $x^2\text{-}1$ is positive.

The corresponding equation is $x^2\text{-}1=0$, solving for the variable, we obtain:

$$x^2\text{-}1=0 \Leftrightarrow (x-1)(x+1)=0$$

$$x-1=0 \Rightarrow x=1$$

$$x+1=0 \Rightarrow x=-1$$

x	$-\infty$ -1 1 $+\infty$
$x+1$	- - - 0 + + + + + + +
$1-x$	- - - - - - - 0 + + + + + + + +
x^2-1	+ + + 0 - - - - 0 + + + + + + +

Therefore $Domf=]-\infty,-1]\cup[1,+\infty[$

Example2: Find the domain of the function $h(x)=\dfrac{\sqrt{1-x^2}}{x}$

Solution

Conditions of existence $1-x^2\geq 0$ and $x\neq 0$

- For the corresponding equation is $1-x^2=0$ and solving for the variable,

we get: $(1-x)(1+x)=0\Rightarrow\begin{cases}1-x=0 \text{ or } x=1\\ 1+x=0 \text{ or } x=-1\end{cases}$

- For $x \neq 0$ all real numbers are accepted except from zero. Combining the two conditions we get:

x	$-\infty$ -1 0 1 $+\infty$
$1+x$	- - - - 0 + + + + + + + + + + +
$1-x$	+ + + + + + + + 0 - - - - -
$1-x^2$	- - - 0 + + + + + + 0 - - - - -
x	- - - - - - - 0 + + + + + + + + +
$f(x)=\frac{\sqrt{1-x^2}}{x}$	undefined 0 - - || + + + + + 0 undefined

$-1 \leq x \leq 1$ and $x \neq 0$. Therefore, $domf = [-1,0[\cup]0,1]$

Example 3: Find domain of definition of $f(x) = \sqrt[3]{x+1}$

Solution

Since the index in radical sign is odd number, then $Domf = \mathbb{R}$

Example 4: Find the domain of definition of $g(x)$ if $g(x) = \sqrt[4]{x^2+1}$

Solution

Condition: $x^2+1 \geq 0$

Clearly x^2+1 is always positive. Thus $Domg = \mathbb{R}$

Example 5: Find domain of $f(x) = \frac{x}{\sqrt{x^3-4x^2+x+6}}$

Solution

Conditions: $x^3-4x^2+x+6 \geq 0$ and $x^3-4x^2+x+6 \neq 0$.The two conditions are combined in one: $x^3-4x^2+x+6 \geq 0$ and $x^3-4x^2+x+6 \neq 0$

$x^3 - 4x^2 + x + 6 = (x+1)(x-2)(x-3)$

x	$-\infty$		-1		2		3		$+\infty$
$x+1$		-	0	+	+	+	+	+	
$x-2$		-	-	-	0	+	+	+	
$x-3$		-	-	-	-	-	0	+	
x^3-4x^2+x+6		-	0	+	0	-	0	+	

Then, $Domf =]-1,2[\cup]3,+\infty[$

Example 6: Find the range for the function $f(x) = \sqrt{1+5x}$

Solution

$1+5x \geq 0$ (Restrictions on x);

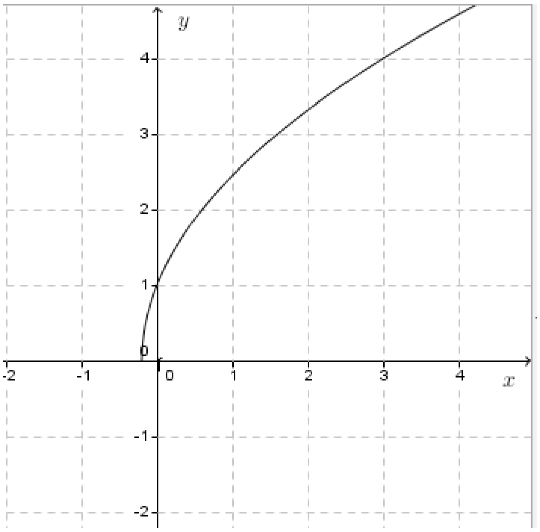
$\sqrt{1+5x} \geq 0$ (Taking the square roots);

But $f(x) = \sqrt{1+5x}$;

Therefore, $f(x) \geq 0$

The range of $f(x)$ is $\operatorname{Im} f = [0,+\infty[$.

The graph below illustrates the range:



APPLICATION ACTIVITY 4.2.3

1. Find the domain of definition for each of the following functions

$f(x)=\sqrt{4x-8}$	$g(x)=\sqrt{x^2+5x-6}$	$h(x)=\dfrac{x^3+2x^2-2}{\sqrt[3]{x+4}}$	$f(x)=\dfrac{x-2}{\sqrt[4]{x^2-25}}$
$f(x)=\sqrt{\dfrac{(x-1)^2}{x+4}}$	$h(x)=\sqrt{\dfrac{(x-1)(x+3)}{8-2x}}$	$f(x)=\dfrac{x-1}{\sqrt{2-x}}$	$f(\text{x})=\sqrt{4-x^2}$

Find the range of each of the following functions

$a)\, f(x)=\sqrt{1-x}$

$b)\, f(x)=\sqrt{-x-3}$

4.3 Composition of Functions

ACTIVITY 4.3

Given two functions $f(x)=2x^2+2x-5$ and $g(x)=3x+2$

a) Substitute the variable in $f(x)$ by the value of $g(x)$. Is the obtained function a function of x?

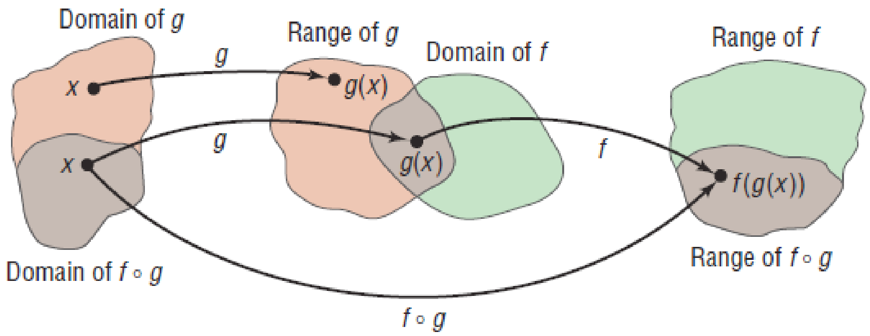
b) Substitute the variable in $g(x)$ by the value of $f(x)$. What do you notice?

c) Name the obtained functions by $h(x)$ and $j(x)$ respectively. Compare the two new functions and explain if they are the same or different.

CONTENT SUMMARY

Given two functions $f(x)$ and $g(x)$, one can "combine" them by substituting one function into the other as follow: $f[g(x)]$ or $g[f(x)]$.

Definition: Let $f(x)$ and $g(x)$ be two functions of x such that the codomain of $f(x)$ is a subset of the domain of $g(x)$. The *composition* of $f(x)$ with $g(x)$, denoted by $(fog)(x)$ is given by $(fog)(x) = f(g(x))$ Indicates that $f(x)$ is a function from A to B and $g(x)$ is a function from C to D where $B \subseteq C$. The function given by $(fog)(x) = f(g(x))$ is the composite of $f(x)$ and $g(x)$. The domain of $(fog)(x)$ is the set of all x in the domain of $g(x)$ such that $g(x)$ is in the domain of $f(x)$ as shown by the figure below.



Only the $x's$ in the domain of g for which $g(x)$ is in the domain of f, can be in the domain of $f \circ g$. The reason is that if $g(x)$ is not in the domain of f then $f[g(x)]$ is not defined. Because of this, the domain of $f \circ g$ is a subset of the domain of g; the range of $f \circ g$ is a subset of the range of f.

The domain of $(f \circ g)(x)$ is the set of all numbers x in the domain of g such that $g(x)$ is in the domain of f

Here, we note that $(f \circ g)(x) \neq (g \circ f)(x)$

Examples

1. *Let* $f(x) = x^2$ *and* $g(x) = 2x + 1$. *Find* $(fog)(x)$ *and* $(gof)(x)$.

Solution

By the definition of composition, we have,

$$(fog)(x) = f(g(x)) = f(2x+1) = (2x+1)^2 = 4x^2+4x+1$$

$$(gof)(x) = g(f(x)) = g(x^2) = 2x^2+1.$$

Remark If the range of $f(x)$ is not contained in the domain of $g(x)$, then we have to restrict $f(x)$ to a smaller set so that for every x in that set, $f(x)$ belongs to the domain of $g(x)$. The domain of $(gof)(x)$ is taken to be the following: $\textbf{dom } (gof)(x) = \{x \in \textbf{dom } (f(x)) : f(x) \in \textbf{dom } (g(x))\}.$

2. Let $f(x) = x+1$ and $g(x) = \sqrt{x}$ .find $(gof)(x)$

Solution

Note that the domain of $f(x)$ is $\mathbb{R}$ and the domain of $g(x)$ is $[0,+\infty[$. Thus the domain of $(gof)(x)$ is $\text{Dom}(gof) = \{x \in \mathbb{R}; x+1 \in [0,+\infty[\}$ $= \{x \in \mathbb{R}; x \geq -1\} = [-1,+\infty[$

APPLICATION ACTIVITY 4.3

Given the two functions $f(x) = x^2+1$ and $g(x) = x+1$. Find the following

a) $fog(x)$ b) $gof(x)$ c) $fog(1)$ d) $gof(1)$ e) $fog(x^2)$ e) $gof(\sqrt{x})$

4.4. Inverse Functions

ACTIVITY 4.4

Given the function $f(x)=4x+6$

a) Substitute $f(x)$ by y and solve for x

b) After solving for x, replace x on (a) by $f^{-1}(x)$ and y by x.

c) Find $f\left[f^{-1}(x)\right]$ and $f^{-1}\left[f(x)\right]$ and compare the results .What have you noticed?

CONTENT SUMMARY

Definition: Let $f(x)$ and $g(x)$ be two functions such that $f\left[g(x)\right]=x$ for each x in the domain of g and $g\left[f(x)\right]=x$ for each x in the domain of f.

Under these conditions, the function g is the inverse of f. The function g is denoted by $f^{-1}(x)$, which is read as "f - inverse". So $f\left(f^{-1}(x)\right)=x$ and $f^{-1}\left(f(x)\right)=x$

Note:

- If $f(x)$ is inverse of $g(x)$, then $g(x)$ is inverse of $f(x)$.
- The domain of f must be equal to the range of f^{-1}, and the range of f must be equal to the domain of f^{-1}.
- The notation $f^{-1}(x)\neq\dfrac{1}{f(x)}$

Example

Let the function $f(\mathrm{x})=\sqrt{2x-3}$ find the inverse function $f^{-1}(\mathrm{x})$.

Solution

$f(x)=\sqrt{2x-3}$ Rewrite the original function

$y=\sqrt{2x-3}$ Replace $f(x)$ with y

$x=\sqrt{2y-3}$ Interchange x and y

$x^2=2y-3$ Square each side to find y

$x^2+3=2y \Rightarrow y=\frac{x^2+3}{2}$

Therefore, $f^{-1}(x)=\frac{x^2+3}{2}$

APPLICATION ACTIVITY 4.4

1. The demand function for a commodity is $p=\frac{14.75}{1+0.01x}, x\geq 0$. Where p is the price per unit and x is the number of units sold.
   a) Find x as a function of p
   b) Find the number of units sold if the unit price is 10$.
2. For each of the following pair of functions $f(x)$ and g(x) show that they are inverse to each other.
   a) $f(x)=5x+1$ and $g(x)=\frac{x-1}{5}$
   b) $f(x)=9-x^2, x\geq 0$ and $g(x)=\sqrt{9-x}, x\leq 9$
   c) $f(x)=1-x^3$ and $g(x)=\sqrt[3]{1-x}$

4.5. Even functions and odd functions

4.5.1. Even functions

ACTIVITY 4.5.1

Consider the function $f(x)=x^2-1$

i. Complete the table below

x	.....	....	-3	-2	-1	0	1	2	3	.....	....
$y=x^2-1$											

ii. How are the images of any two opposite values of *x?* Express this mathematically,

iii. Plot each pair of obtained points (x,y) in the Cartesian plane and deduce the graph.

iv. From the graph of $f(x)=x^2-1$, what can you say about the line $x=0$

CONTENT SUMMARY

Let $f(x)$ be a numerical function whose domain is $Domf$.

$f(x)$ is said to be an **even function** if and only if:

i. $(\forall x \in Domf), -x \in Domf$

ii. $f(-x)=f(x)$, that is, any two opposite values of the independent variable have the same image under the function.

The graph of an even function is symmetrical about the *y*-axis.

Example1

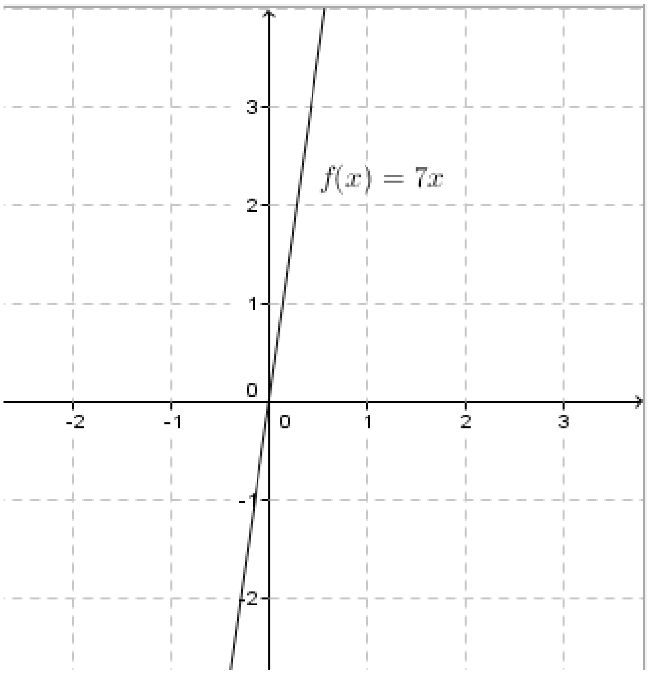
Determine whether the function $f(\mathrm{x})=7\,\mathrm{x}$ is even or not.

Solution

The domain of function *f* is the set of all real numbers. For any real number x, the opposite –x is also a real number and $f(-x)=-7x \neq 7x = f(x)$

Since $f(-x) \neq f(x)$, function f is not even.

Graphically,



The graph is not symmetrical about the y-axis

Example2

Determine whether the function $f(x) = 3x^2 - 4$ is even or not.

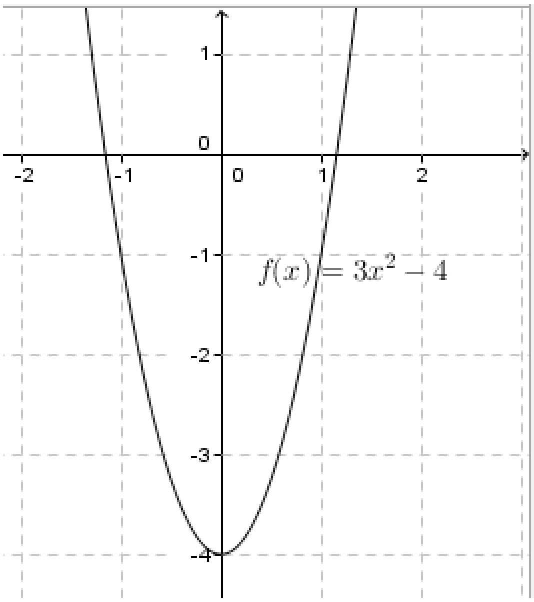
Solution

$f(-x) = 3(-x)^2 - 4 = 3x^2 - 4 = f(x)$

Since $f(-x) = f(x)$, function f is even.

Remember that $(-x)^n = \begin{cases} x^n, if \text{ n is even} \\ -x^n, if \text{ n is odd} \end{cases}$

The graph of the function is symmetrical about the y-axis as shown on the diagram below:



Example 3

Determine whether the function $g(x) = x^6 - x^4 + x^2 + 9$ is even or not.

Solution

$$g(-x) = (-x)^6 - (-x)^4 + (-x)^2 + 9$$
$$= x^6 - x^4 + x^2 + 9$$
$$= g(x)$$

Therefore, the function is even.

Example4

Determine whether the function $f(x) = \frac{3x+1}{x^2-25}$ is even or not.

Solution

$f(-x) = \frac{3(-x)+1}{(-x)^2-25} \Leftrightarrow f(-x) = \frac{-3x+1}{x^2-25}$. Therefore the function is not even since $f(-x) \neq f(x)$

Example5

Given functions $f(x) = \sqrt{x}$ and $g(x) = \sqrt{x-4}$, find the $f(x).g(x)$ and determine if the result is an even function or not.

Solution

Provided $x \geq 0$ and $x-4 \geq 0$,

$$f(x).g(x)=\sqrt{x}\ .\ \sqrt{x-4} \Leftrightarrow f(x).g(x)=\sqrt{x^2-4x}$$

$f.g(-x)=\sqrt{(-x)^2-4(-x)} \Leftrightarrow f.g(-x)=\sqrt{x^2+4x}$. Therefore the function is not even since $f(-x) \neq f(x)$

Notice that the conclusion could have been drawn from the fact that $x \geq 4$ does not imply $-x \geq 4$, thus function *f* is not even

APPLICATION ACTIVITY 4.5.1

Determine whether the following functions are even or not.

1. $f(x)=\dfrac{x^2+1}{x^4+3}$
2. $f(x)=x^{\frac{2}{3}}(x-4)$
3. $f(x)=x\sqrt{9-x}$

4.5.2. Odd function

ACTIVITY 4.5.2

Given the function $f(x)=x^3$

i. Complete the table below

x	-5	-4	-3	-2	-1	0	1	2	3	4	5
y											

ii. How are the images of any two opposite values of *x?* Express this mathematically,

iii. Plot each pair obtained in the Cartesian plane.

iv. How is the graph, with respect to the origin O?

v. Which name can you give to the origin $(0,0)$ with respect to the graph of $f(x)$?

CONTENT SUMMARY

Let $f(x)$ be a numerical function whose domain is $Domf$.

$f(x)$ is said to be an **odd function** if and only if:

i. $(\forall x \in Domf), -x \in Domf$

ii. $f(-x) = -f(x)$, that is, any two opposite values of the independent variable have opposite images under the function.

The graph of an even function is symmetrical about the *origin*.

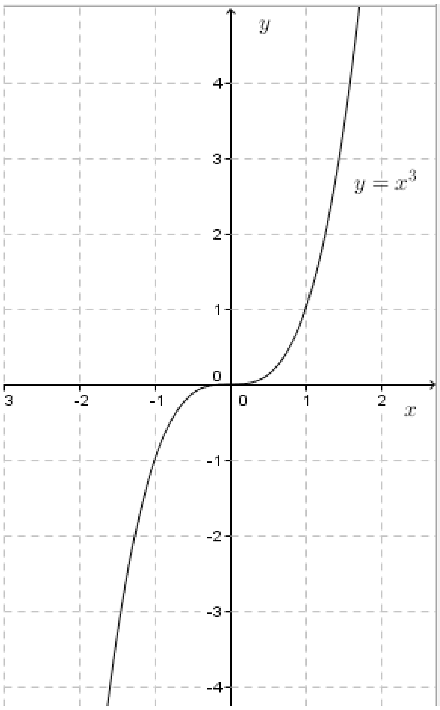
Note: Some functions are neither even nor odd

Example1 : Determine whether the function $f(x) = x^3$ is odd or not

Solution

$f(-x) = (-x)^3 \Leftrightarrow f(-x) = -x^3$ *and* $-f(x) = -x^3$. Therefore, $f(-x) = -f(x)$ and the function $f(x) = x^3$ is odd.

Graphically, the point $(0,0)$ is the center of symmetry for the graph of the function $f(x) = x^3$.



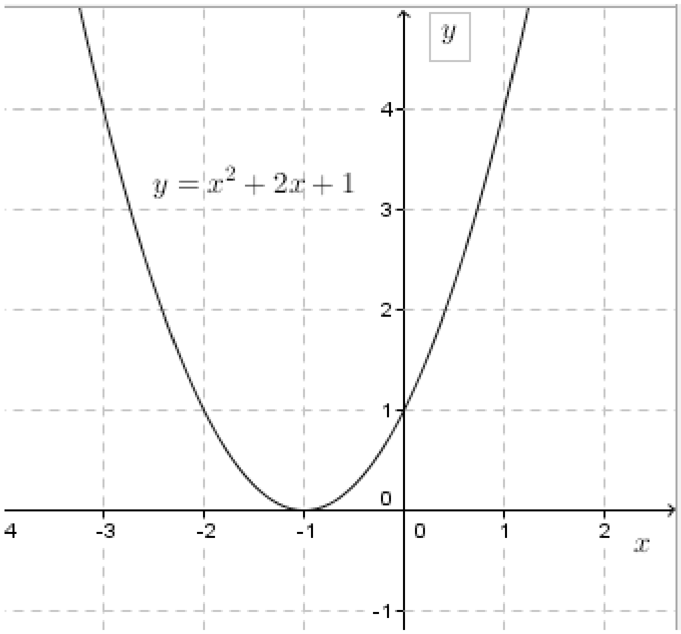
Example 2: Determine whether the function $f(x)=x^2+2x+1$ is odd ,even or neither

Solution

$f(-x)=(-x)^2+2(-x)+1 \Leftrightarrow f(-x)=x^2-2x+1 and -f(x)=-x^2-2x-1$.

$f(-x)\neq -f(x)$ and $f(-x)\neq f(x)$, therefore, the function $f(x)=x^2+2x+1$ is not odd neither even.

Graphically, point $(0,0)$ is not the center of symmetry for the graph of the function $f(x)$,and the line $x=0$ is not the axis of symmetry for the graph of function $f(x)=x^2+2x+1$.-



Example 3: Determine if the function $f(x)=\dfrac{x^3}{x^2-1}$ is even , odd ,or neither and deduce the symmetry of its graph.

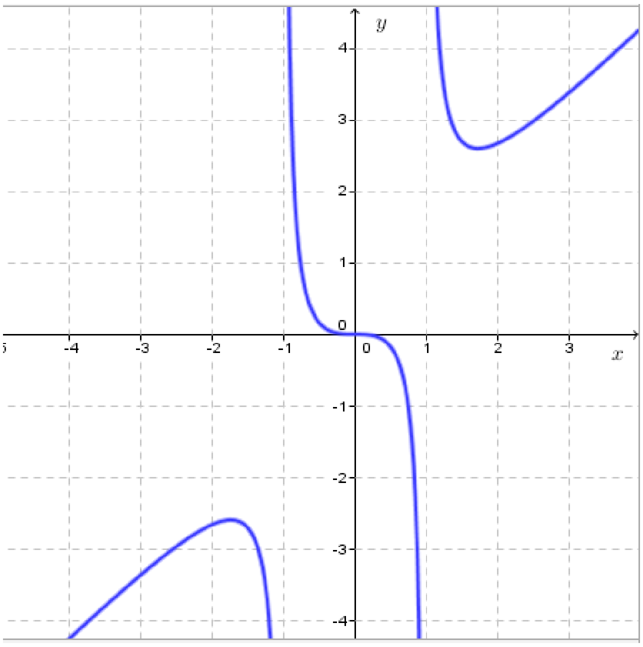
Solution

$$f(-x)=\frac{(-x)^3}{(-x)^2-1}=\frac{-x^3}{x^2-1}\neq f(x)$$

Therefore, the function is not even

But, $f(-x) = -f(x)$;it follows that f is an odd function.

The graph of $f(x) = \dfrac{x^3}{x^2 - 1}$ is shown below. It can be seen that point $(0,0)$ is the center of symmetry for the graph



APPLICATION ACTIVITY 4.5.2

1. Consider the function $f(x)$ defined from $\mathbb{R} \to \mathbb{R}$ by

   $f(x) = \dfrac{x^3}{9 - x^2}$

   i) Find the domain of definition of $f(x)$

   ii) Determine if $f(x)$ is even, or odd function and deduce the symmetry of the graph of $f(x)$

2. For which value(s) of x the function $f(x) = \sqrt{x^2 + 5x + 6}$ is valid? Determine if $f(x)$ is an even or odd function.

4.6 Graphs of linear and quadratic equations

ACTIVITY 4.6

1. Copy and complete the tables below.

x	-3	-2	-1	0	1	2	3
$y = 2x - 1$							

x	-3	-2	-1	0	1	2	3
$y = x^2 - 1$							

2. Use the coordinates from each table to plot the graphs on separate Cartesian planes.
3. What is your conclusion about the shapes of the graphs?

CONTENT SUMMARY

Linear functions

Definition of linear function

Any function of the form $f(x) = mx + b$, where m is not equal to 0 is called a linear function. The ***domain*** of this function is the set of all real numbers. The ***range*** of f is the set of all real numbers. The graph of f is a line with slope m and y intercept b.

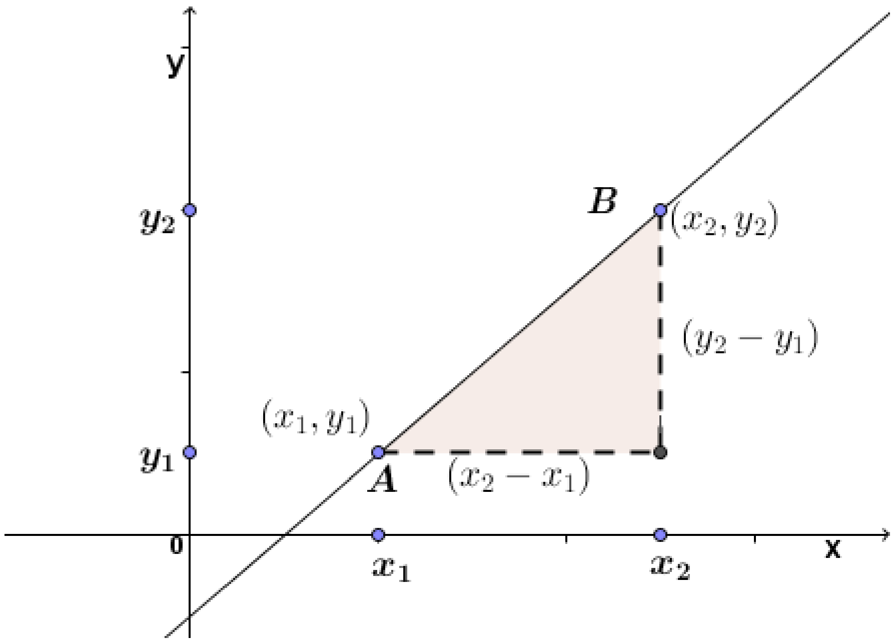
Note:

A function $f(x) = b$, where b is a constant real number is called a constant function. Its graph is a horizontal line at $y = b$.

Examples of a linear function are $y = x + 1, \quad y = 2x - 3, \quad y = -3x + 4, \ldots$

Graphs of linear functions.

The ordered pair (x, y) represents coordinates of any point on the Cartesian plane. Consider a line passing through the points $A(x_1, y_1)$ and $B(x_2, y_2)$.



From A to B, the change in the x-coordinate (horizontal change) is $x_2 - x_1$ and the change in the y-coordinate (vertical change) is $y_2 - y_1$.

By definition, gradient / slope is equal to $\frac{change\ in\ y-coordinates}{change\ in\ x-coordinates} = \frac{y_2 - y_1}{x_2 - x_1}$

In the Cartesian plane, the gradient of a line is the measure of its slope or inclination to the x-axis. It is defined as the ratio of the change in y-coordinate (vertical) to the change in the x-coordinate (horizontal).

When drawing a graph of a linear function, it is sufficient to plot only two points and these points may be chosen as the x and y intercepts of the graph. In practice, however, it is wise to plot three points. If the three points lie on the same line, the working is probably correct, if not you have a chance to check whether there could be an error in your calculation.

If we assign x any value, we can easily calculate the corresponding value of y.

Determine the x intercept, set $f(x) = 0$ and solve for x and then determine the y intercept, set x=0 to find $f(0)$.

Consider the equation $y = 2x + 3$.

- When $x = 0,\ y = 2\times 0 + 3 = 3$
- When $x = 1,\ y = 2\times 1 + 3 = 5$
- When $x = 2,\ y = 2\times 2 + 3 = 7$ and so on.

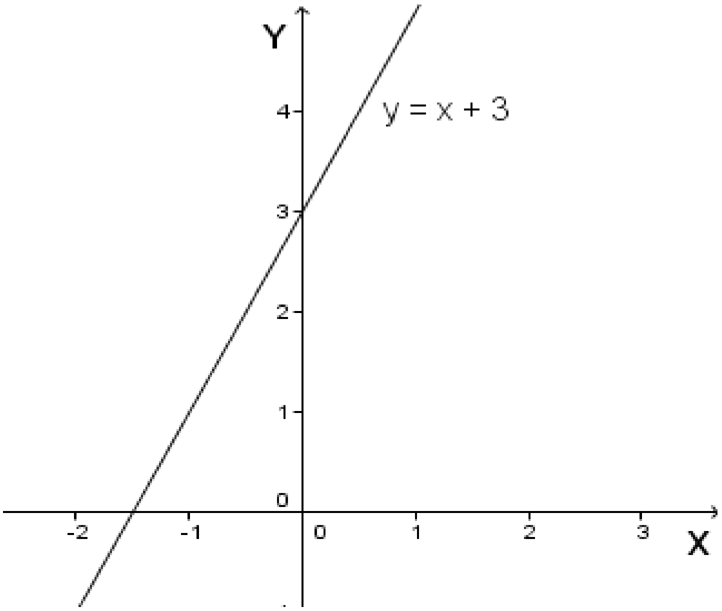
For convenience and ease while reading, the calculations are usually tabulated as

shown below in the table **of values for $\mathbf{y = 2x + 3}$**:

x	0	1	2	3	4
$2x$	0	2	4	6	8
$+3$	3	3	3	3	3
$y = 2x + 3$	3	5	7	9	11

From the table the coordinates (x, y) are $(0,3),\ (1,5),\ (2,7),\ (3,9),\ (4,11)$....

When drawing the graph, the dependent variable is marked on the vertical axis generally known as the y – axis. The independent variable is marked on the horizontal axis also known as the $OX - axis$



Quadratic function

Definition of quadratic function

A polynomial equation in which the highest power of the variable is 2 is called a quadratic function. The expression $y = ax^2 + bx + c$, where a, b and c are constants and a ≠ 0, is called a quadratic function of x or a function of the second degree (highest power of x is two).

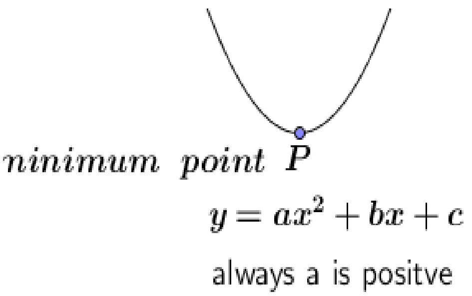
Table of values are used to determine the coordinates that are used to draw the graph of a quadratic function. To get the table of values, we need to have the domain (values of an independent variable) and then the domain is replaced in a given quadratic function to find range (values of dependent variables). The values obtained are useful for plotting the graph of a quadratic function. All quadratic function graphs are parabolic in nature.

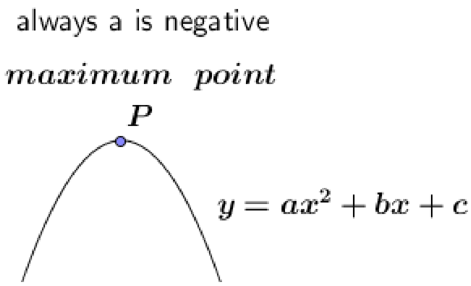
Any quadratic function has a graph which is symmetrical about a line which is parallel to the y-axis i.e. a line $x = h$ where h is constant value. This line is called **axis of symmetry**.

For any quadratic function $f(x) = ax^2 + bx + c$ whose axis of symmetry is the line $x = h$, the vertex is the point $(h, f(h))$.

The vertex of a quadratic function is the point where the function crosses its axis of symmetry.

If the coefficient of the x^2 term is positive, the vertex will be the lowest point on the graph, the point at the bottom of the U-shape. If the coefficient of the term x^2 is negative, the vertex will be the highest point on the graph, the point at the top of the ∩-shape. The shapes are as below.





There are two intercepts i.e. $x - intercept\ and\ y - intercept$.

$x-$intercept for any quadratic function is calculated by letting $y=0$

$y-$ intercept is calculated by letting $x=0$

Graph of quadratic function

The graph of a quadratic function can be sketched without table of values as long as the following are known.

- The vertex
- The x-intercepts
- The y-intercept

Example 1:

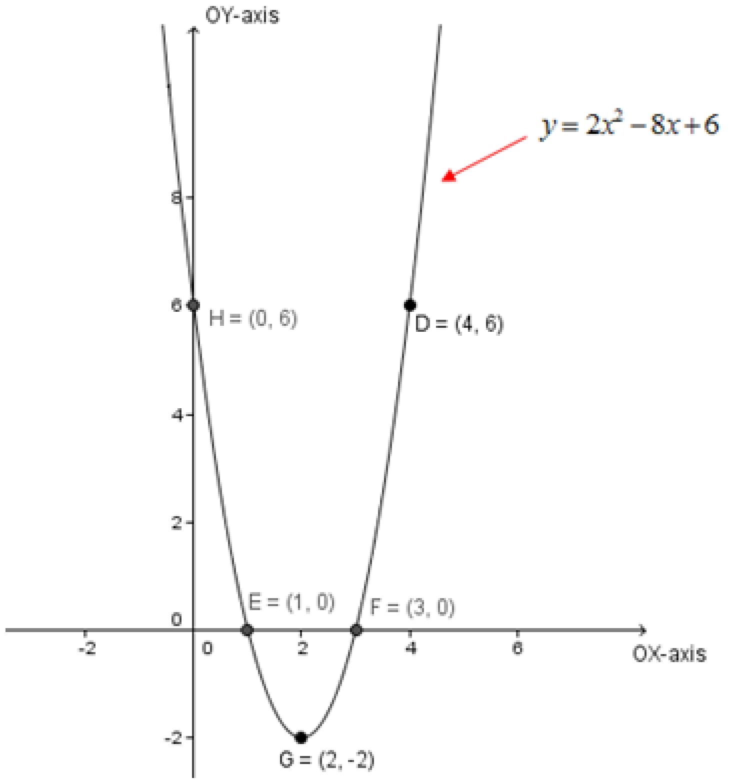
Find the vertex and axis of symmetry of the parabolic curve $y=2x^2-8x+6$

Solution

- The coefficients are a = 2, b = −8 and c = 6
- The x-coordinate of the vertex is $h=-\frac{b}{2a}=\frac{-(-8)}{2(2)}=\frac{8}{4}=2$
- The y-coordinate of the vertex is obtained by substituting the x-coordinate of the vertex to the quadratic function. We get $y=2(2)^2-8(2)+6=-2$
- The vertex is (2, −2) and the axis of symmetry is $x=2$.
- When x = 0, $y=2(0)^2-8(0)+6=6$.
- The y-intercept is (0, 6)

When $y=0$, $0=2x^2-8x+6$, we therefore solve the quadratic equation for the values of x and we find the x-intercepts are $(1,0)$ *or* $(3,0)$

The following is the graph of the function $y = 2x^2 - 8x + 6$ graph is as below.



APPLICATION ACTIVITY 4.6

1. Using the table of values sketch the graph of the following functions

    a. $y = -3x + 2$

    b. $y = x^2 - 3x + 2$

2. Without tables of values, state the vertices, intercepts with axes, axes of symmetry, and sketch the graphs.

3. (a) $y = 2x^2 + 5x - 1$

    (b) $y = 3x^2 + 8x - 6$

4.7 Application of functions in real life situation

ACTIVITY 4.7

1) Give three examples illustrating where you think functions can be used in daily life.

2) The cost C, in Rwandan francs, to produce x units of a product in a certain company is given by $C(x) = 80x + 150$ for $x \geq 0$.

   (i) Find and interpret C(10).

   (ii) How many products can be produced for 15,000 Frw?

   (iii) Explain the significance of the restriction on the domain, $x \geq 0$.

   (iv) Find and interpret C(0).

   (v) Find and interpret the slope of the graph of $y = C(x)$

CONTENT SUMMARY

Polynomials are used to describe curves of various types; people use them in the real world to graph curves. For example, roller coaster designers may use polynomials to describe the curves in their rides. Polynomials can be used to figure how much of a garden's surface area can be covered with a certain amount of soil.

The same method applies to many flat-surface projects, including driveway, sidewalk and patio construction. Polynomials can also be used to model different situations, like in the stock market to see how prices will vary over time.

Business people also use polynomials to model markets, as in to see how raising the price of a good will affect its sales. For people who work in industries that deal with physical phenomena or modelling situations for the future, polynomials come in handy every day. These include everyone from engineers to businessmen. For the rest of us, they are less apparent but we still probably use them to predict how changing one factor in our lives may affect another--without even realizing it.

Functions are important in calculating medicine, building structures (houses, businesses,…), vehicle design, designing games, to build computers (formulas that are used to plug to computer programs), knowing how much change you should receive when making a purchase, driving (amount of gas needed for travel).

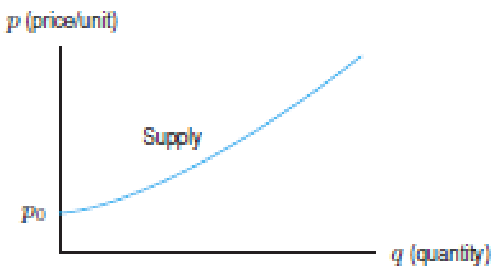
4.7.1 Price as function of quantity supplied

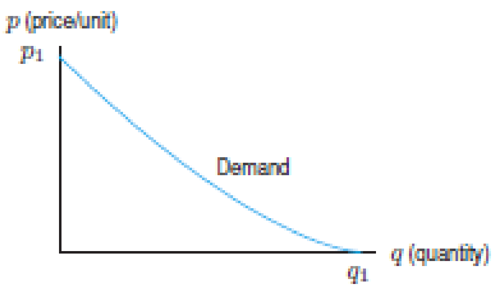
The quantity, q, of an item that is manufactured and sold depends on its price, p. As the price increases, manufacturers are usually willing to supply more of the product, whereas the quantity demanded by consumers falls.

The supply curve, for a given item, relates the quantity, q, of the item that manufacturers are willing to make per unit time to the price, p, for which the item can be sold.

The demand curve relates the quantity, q, of an item demanded by consumers per unit time to the price, p, of the item.

Economists often think of the quantities supplied and demanded Q as functions of price P. However, for historical reasons, the economists put price (the independent variable) on the vertical axis and quantity (the dependent variable) on the horizontal axis. (The reason for this state of affairs is that economists originally took price to be the dependent variable and put it on the vertical axis





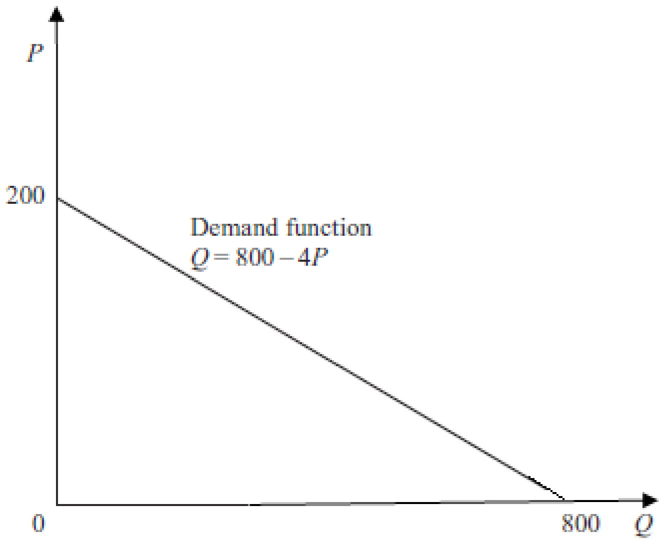
Theoretically, it does not matter which axis is used to measure which variable. However, one of the main reasons for using graphs is to make analysis clearer to understand. Therefore, if one always has to keep checking which axis measures which variable this defeats the objective of the exercise. Thus, even though it may upset some mathematical purists, the economists sometimes stick to the convention of measuring **quantity on the horizontal axis and price on the vertical axis**, even if price is the independent variable in a function.

This means that care has to be taken when performing certain operations on functions. If necessary, one can transform monotonic functions to obtain the inverse function (as already explained) if this helps the analysis.

Examples

a) The demand function $Q = 800 - 4P$ has the inverse function

$$P = \frac{800 - Q}{4} = 200 - 0.24Q$$



This figure shows that when the quantity Q is increasing, the price P reduces progressively. This can be caused by the fact that every consumer has sufficient quantity of goods and does not want to by any more.

b) Suppose that a firm faces a linear demand schedule and that 400 units of output Q are sold when price is \$40 and 500 units are sold when price is \$20. Once these two price and quantity combinations have been marked as points A and B, then the rest of the demand schedule can be drawn in. Use this data to determine the function that can help to predict quantities demanded at different prices and draw the corresponding graph.

Solution:

Accurate predictions of quantities demanded at different prices can be made if the information that is initially given is used to determine the algebraic format of the function.

A linear demand function must be in the form $P = a - bQ$, where a and b are parameters that we wish to determine the value of.

when P = 40 then Q = 400 and so 40 = a – 400b (1) when P = 20 then Q = 500 and so 20 = a – 500b (2)

Equations (1) and (2) are what is known as simultaneous linear equations.

$$\begin{cases} 40 = a - 400b \\ 20 = 1 - 500b \end{cases}$$

We can solve this by one of the methods we used above. We find $a = 120, b = 0.02$.

Our function can now be written as $P = 120 - 0.2Q$.

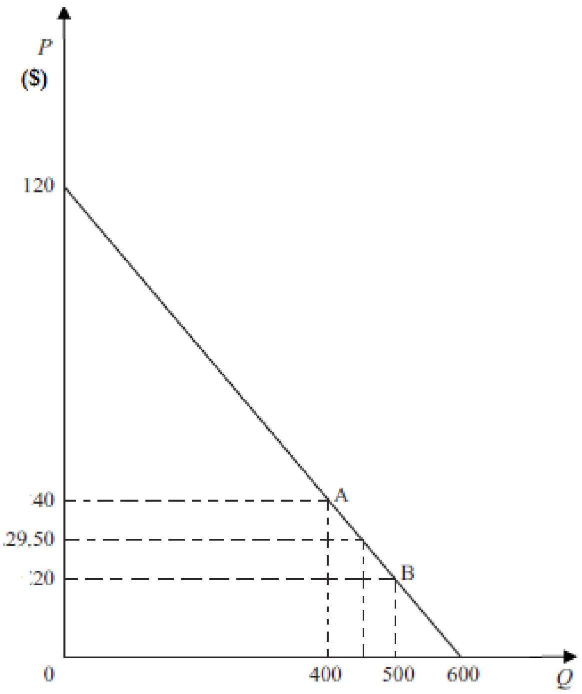
We can check that this is correct by substituting the original values of Q into the function.

If $Q = 400$ then $P = 120 - 0.2(400) = 120 - 80 = 40$

If $Q = 500$ then $P = 120 - 0.2(500) = 120 - 100 = 20$

These are the values of P originally specified and so we can are satisfied that the line that passes through points A and B is the linear function $P=120-0.2Q$.

The inverse of this function will be $Q = 600-5P$. Precise values of Q can now be derived for given values of P. For example, when P= £29.50 then $Q=600-5(29.50) = 452.5$.



4.7.2 Consumption as function of income

It is assumed that consumption C depends on income Y and that this relationship takes the form of the linear function $C = a + bY$.

Example:

When *the income* is \$600, the consumption observed is \$660. When *the income* is \$1,000, the consumption observed is \$900. Determine the equation "consumption function of income".

Solution:

To determine the required equation we can solve this system of equations

$$\begin{cases} 660 = a + 600b \\ 900 = a + 1000b \end{cases}$$

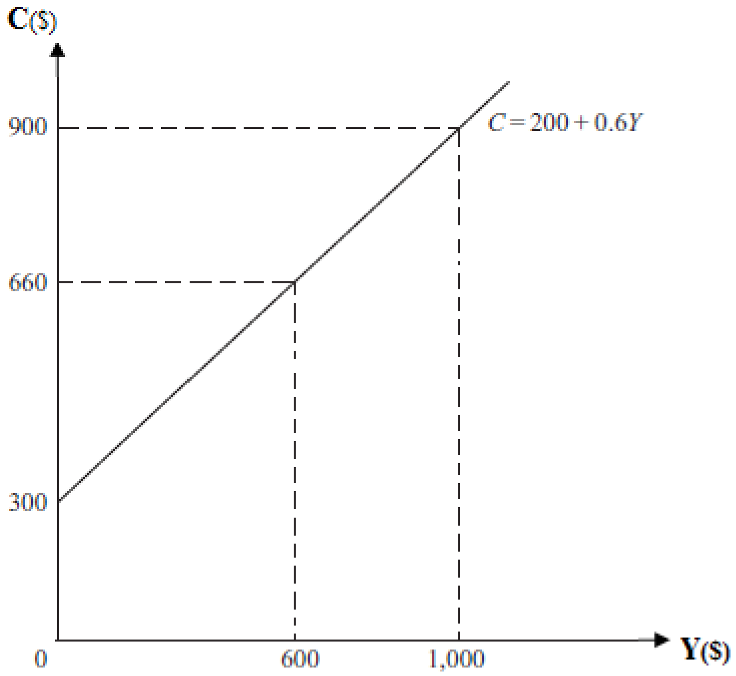
However, let us use another way as follows: We expect b to be positive, i.e. consumption increases with income, and so our function will slope upwards. As this is a linear function then equal changes in Y will cause the same changes in C.

A decrease in Y of \$400, from \$1,000 to \$600, causes C to fall by \$240, from \$900 to £660.

If Y is decreased by a further \$600 (i.e. to zero) then the corresponding fall in C will be 1.5 times the fall caused by an income decrease of \$400, since \$600=1.5×\$400.

Therefore the fall in C is 1.5 × \$240 = \$360. This means that the value of C when Y is zero is £660 − £360 = £300. Thus a = 300.

A rise in Y of \$400 causes C to rise by \$240. Therefore a rise in Y of \$1 will cause C to rise by $\$\frac{240}{400} = \$0.6$. Thus b = 0.6.
The function can therefore be specified as $C = 300 + 0.6Y$.



The graph shows that when the income increases, the consumption increases also.

4.7.3 Price as function of quantity demanded

The linear demand function is in the form $P = a - bQ$, where a and b are parameters, P is the price and Q is the quantity demanded.

Example:

Consider the function $P = 60 - 0.2Q$ where P is price and Q is quantity demanded. Assume that P and Q cannot take negative values, determine the slop of this function and sketch its graph.

Solution:

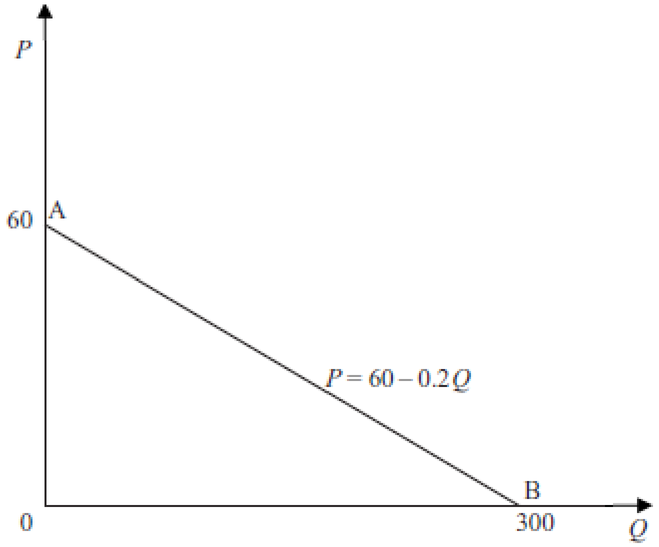
When $Q = 0$ then $P = 60$

When $P = 0$ then $0 = 60 - 0.2Q$

$0.2Q = 60$

$$Q = \frac{60}{0.2} = 300$$

Using this points: $(0,60)$ (0, 60) and $(300,0)$, we can find the graph as follows:



The slope of a function which slopes down from left to right is found by applying the formula

$$slope = (-1)\frac{height}{base}$$

to the relevant right-angled triangle. Thus, using the triangle 0BA, the slope of our function is

$$(-1)\frac{60}{300} = -0.2$$

This, of course, is the same as the coefficient of Q in the function $P = 60 - 0.2Q$.

Remember that in economics the usual convention is to measure P on the vertical axis of a graph. If you are given a function in the format $Q = f(P)$ then you would need to derive the inverse function to read off the slope.

Example:

What is the slope of the demand function $Q = 830 - 2.5P$ when P is measured on the vertical axis of a graph?

Solution:

If $Q = 830 - 2.5P$; then $2.5P = 830 - Q$

$P = 332 - 0.4Q$

Therefore the slope is the coefficient of Q, which is -0.4.

4.7.4 The cost function

The **cost function**, C(q), gives the total cost of producing a quantity q of some good. Costs of production can be separated into two parts: the *fixed costs,* which are incurred even if nothing is produced, and the *variable costs*, which depend on how many units are produced.

Example:

Let's consider a company that makes radios. The factory and machinery needed to begin production are fixed costs, which are incurred even if no radios are made. The costs of labor and raw materials are variable costs since these quantities depend on how many radios are made. The fixed costs for this company are \$24,000 and the variable costs are \$7 per radio.

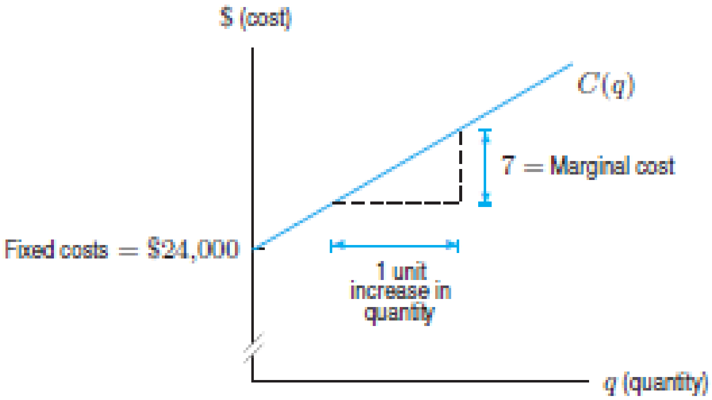
Then, Total costs for the company = Fixed costs + Variable costs

= 24,000 + 7.(Number of radios).

So, if q is the number of radios produced,

$$C(q) = 24,000 + 7q.$$

This is the equation of a line with slope 7 and vertical intercept 24,000.



If C(q) is a linear cost function,

- Fixed costs are represented by the vertical intercept.
- Marginal cost is represented by the slope.

4.7.5 The revenue function

The **revenue function**, R(q), gives the total revenue received by a firm from selling a quantity, q, of some good.

If the good sells for a price of p per unit, and the quantity sold is q,

then Revenue = Price · Quantity, so R = pq.

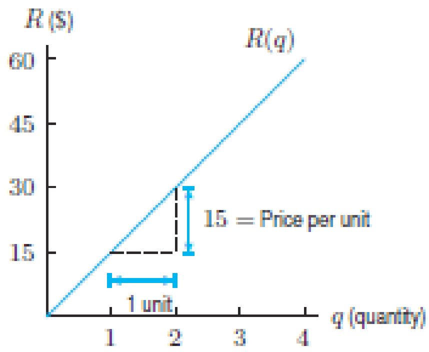
If the price does not depend on the quantity sold, so p is a constant, the graph of revenue as a function of q is a line through the origin, with slope equal to the price p.

Example:

1) If radios sell for $15 each, sketch the manufacturer's revenue function. Show the price of a radio on the graph.

Solution:

Since R(q) = pq = 15q, the revenue graph is a line through the origin with a slope of 15. See the figure. The price is the slope of the line.



2) Graph the cost function C(q) = 24,000 + 7q and the revenue function R(q) = 15q on the same axes. For what values of q does the company make money?

Solution:

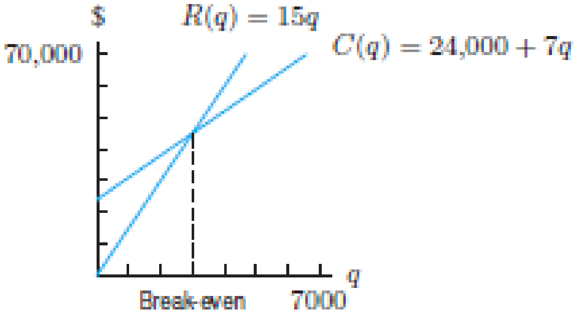
The company makes money whenever revenues are greater than costs, so we find the values of q for which the graph of R(q) lies above the graph of C(q). See Figure 1.45.

We find the point at which the graphs of R(q) and C(q) cross:

Revenue = Cost

$15q = 24{,}000 + 7q \quad \Rightarrow \quad 8q = 24{,}000 \quad \Rightarrow \quad q = 3000.$

The company makes a profit if it produces and sells more than 3000 radios. The company loses money if it produces and sells fewer than 3000 radios.



4.7.6 The profit function

Decisions are often made by considering the profit, usually written as π to distinguish it from the price, p .

We have: **Profit = Revenue – Cost.**

So, $\pi = R - C$

The *break-even point* for a company is the point where the profit is zero and revenue equals cost.

Example:

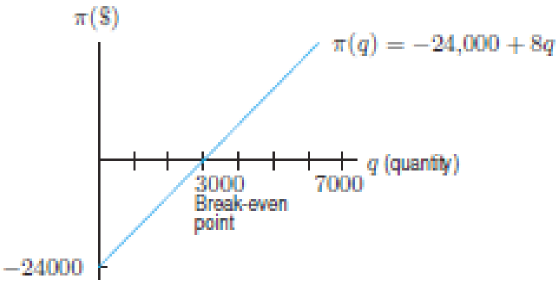
Find a formula for the profit function of the radio manufacturer. Graph it, marking the break-even point.

Solution:

Since R(q) = 15q and C(q) = 24,000 + 7q, we have

$$\pi(q) = R(q) - C(q)$$
$$= 15q - (24000 + 7q) = -2400 + 8q$$

Notice that the negative of the fixed costs is the vertical intercept and the break-even point is the horizontal intercept. See the figure;



4.7.7 The marginal cost, marginal revenue, and marginal profit

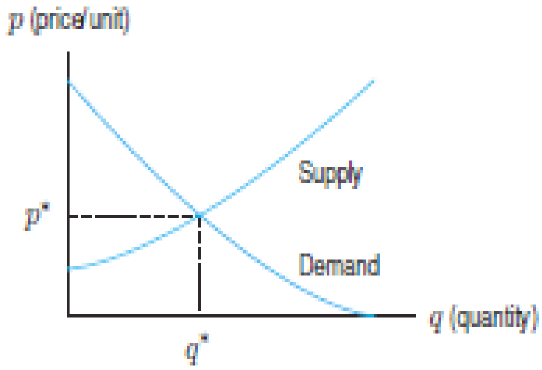
Just as we used the term marginal cost to mean the rate of change, or slope, of a linear cost function, we use the terms *marginal revenue* and *marginal profit* to mean the rate of change, or slope, of linear revenue and profit functions, respectively.

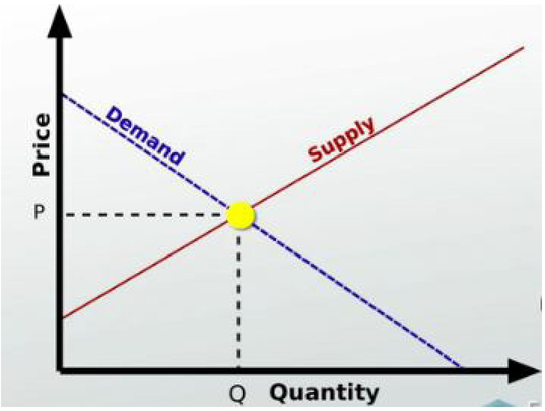
The term *marginal* is used because we are looking at how the cost, revenue, or profit change "at the margin," that is, by the addition of one more unit.

For **example**, for the radio manufacturer, the marginal cost is 7 dollars/item (the additional cost of producing one more item is $7), the marginal revenue is 15 dollars/item (the additional revenue from selling one more item is $15), and the marginal profit is 8 dollars/item (the additional profit from selling one more item is $8).

4.7.8 Equilibrium price and quantity

If we plot the supply and demand curves on the same axes, the graphs cross at the *equilibrium point*. The values $p*$ and q* at this point are called the *equilibrium price* and *equilibrium quantity*, respectively. It is assumed that the market naturally settles to this equilibrium point.





Example:

Find the equilibrium price and quantity if Quantity supplied equals 3p – 50 and Quantity demanded equals 100 – 2p.

Solution:

To find the equilibrium price and quantity, we find the point at which

Supply = Demand

3p – 50 = 100 – 2p

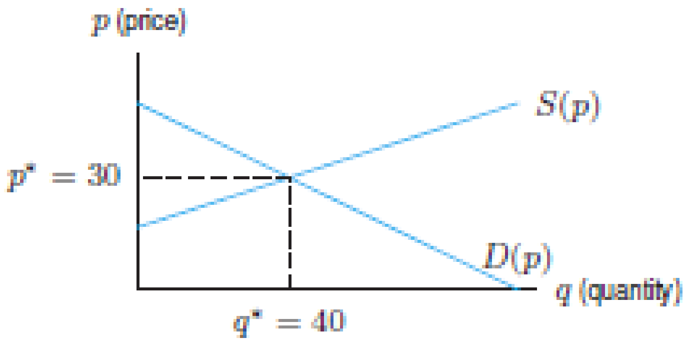
5p = 150

p = 30.

The equilibrium price is $30. To find the equilibrium quantity, we use either the demand curve or the supply curve. At a price of $30, the quantity produced is 100 – 2 (30) = 40 items.

The equilibrium quantity is 40 items.

In the figure, the demand and supply curves intersect at $p^* = 30$ and $q^* = 40$.



APPLICATION ACTIVITY 4.7

1. The local retailer has determined that the number x units of products sold in a week is related to the price p in dollars of each product. When the price was \$220, 20 products were sold in a week. When the products went on sale the following week, 40 products were sold at \$190 a piece

   a. Find a linear function which fits this data. Use the weekly sales x as the independent variable and the price p as the dependent variable.

   b. Find a suitable applied domain.

   c. Interpret the slope.

   d. If the retailer wants to sell 150 products next week, what should the price be?

   e. What would the weekly sales be if the price were set at \$150 per product?

2. Suppose $C(x) = x^2 - 10x + 27$ represents the costs, in hundreds, to produce x thousand pens. Find and interpret the average rate of change as production is increased from making 3000 to 5000 pens.

3. Make a research on internet or in reference books and find out three examples of where you think linear and quadratic functions can be used in daily life

4.8 END UNIT ASSESSMENT

1. Determine if the given functions are even, odd or neither and deduce the symmetry of their graphs.

   a) $f(x)=\frac{x^2+4}{x^3-x}$ b) $f(x)=3-2\sqrt{x+2}$

2. Determine if the relation h is a function which is injective, surjective or neither and explain why.

   $h=\{(-3,8),(-11,-9),(5,4),(6,-9)\}$

3. Given the function $h(x)=\sqrt{x^2-9}$ find

   i) $h(5)$

   ii) The value(s) of x if $h(x)=2$

   iii) The domain and range of h

4. Use the functions $f(x)=-2x+1$, $g(x)=\frac{x^2}{x-3}$ and $h(x)=\sqrt{5-x}$ to determine

   i) $h(-4)-f(-1)$

   ii) x if $g(x)=0$

   iii) The domain of $g(x)$ and $h(x)$

5. Find the domain of definition for each of the following functions

   a) $f(x)=x^3+2x^2-2$

   b) $g(x)=-2$

   c) $f(x)=(x+6)^2$

6. For which value(s) the following functions are not defined

a) $f(x) = x^3 + 2x + 1$

b) $g(x) = 9$

7. The total cost C for units produced by a company is given by $C(q) = 50000 + 8q$ where q is the number of units produced.

a) What does the number 50000 represent?

b) What does the number 8 represent?

c) Plot the graph of C and indicate the cost when $q = 5$.

d) Determine the real domain and the range of $C(q)$.

e) Is $C(q)$ an odd function?

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