MATHEMATICS FOR TTCs

STUDENT'S BOOK

YEAR

1

OPTION:

Early Childhood and Lower Primary Education (ECLPE)

FOREWORD

Dear Student,

Rwanda Education Board (REB) is honored to present Year 1 Mathematics book for Early Childhood and Lower Primary Education (ECLPE) student teachers which serves as a guide to competence-based teaching and learning to ensure consistency and coherence in the learning of the Mathematics. The Rwandan educational philosophy is to ensure that you achieve full potential at every level of education which will prepare you to be well integrated in society and exploit employment opportunities.

The government of Rwanda emphasizes the importance of aligning teaching and learning materials with the syllabus to facilitate your learning process. Many factors influence what you learn, how well you learn and the competences you acquire. Those factors include the relevance of the specific content, the quality of teachers' pedagogical approaches, the assessment strategies and the instructional materials available. In this book, we paid special attention to the activities that facilitate the learning process in which you can develop your ideas and make new discoveries during concrete activities carried out individually or with peers.

In competence-based curriculum, learning is considered as a process of active building and developing knowledge and meanings by the learner where concepts are mainly introduced by an activity, situation or scenario that helps the learner to construct knowledge, develop skills and acquire positive attitudes and values.

For efficiency use of this textbook, your role is to:

- Work on given activities which lead to the development of skills;
- Share relevant information with other students through presentations, discussions, group work and other active learning techniques such as role play, case studies, investigation and research in the library, on internet or outside;
- Participate and take responsibility for your own learning;
- Draw conclusions based on the findings from the learning activities.

To facilitate you in doing activities, the content of this book is self explanatory so that you can easily use it by yourself, acquire and assess your competences. The book is made of units as presented in the syllabus. Each unit has the following structure: the key unit competence is given and it is followed by the introductory activity before the development of mathematical concepts that are connected to real world problems or to other sciences.

The development of each concept has the following points:

- It starts by a learning activity: a hand on well set activity to be done by students in order to generate the concept to be learnt;
- Main elements of the content to be emphasized;

- Worked examples;
- Application activities: to be done by the user to consolidate competences or to assess the achievement of objectives.

Even though the book has some worked examples, you will succeed on the application activities depending on your ways of reading, questioning, thinking and grappling ideas of calculus not by searching for similar-looking worked out examples.

Furthermore, to succeed in Mathematics, you are asked to keep trying; sometimes you will find concepts that need to be worked at before you completely understand. The only way to really grasp such a concept is to think about it and work related problems found in other reference books.

I wish to sincerely express my appreciation to the people who contributed towards the development of this book, particularly, REB staff, Secondary school teachers and TTC Tutors for their technical support. A word of gratitude goes to Head Teachers and TTCs principals who availed their staff for various activities.

Any comment or contribution would be welcome to the improvement of this book for the next edition.

Dr. NDAYAMBAJE Irénée

Director General, REB

ACKNOWLEDGEMENT

I wish to express my appreciation to the people who played a major role in the development of this Mathematics book for Year one student teachers in Early Childhood and Lower Primary Education (ECLPE). It would not have been successful without active participation of different education stakeholders.

I owe gratitude to different universities and schools in Rwanda that allowed their staff to work with REB in the in-house textbooks production initiative.

I wish to extend my sincere gratitude to secondary school teachers and TTC tutors whose efforts during writing exercise of this book were very much valuable.

Finally, my word of gratitude goes to the Rwanda Education Board staffs who were involved in the whole process of in-house textbook Elaboration.

Joan MURUNGI

Head of CTLR Department

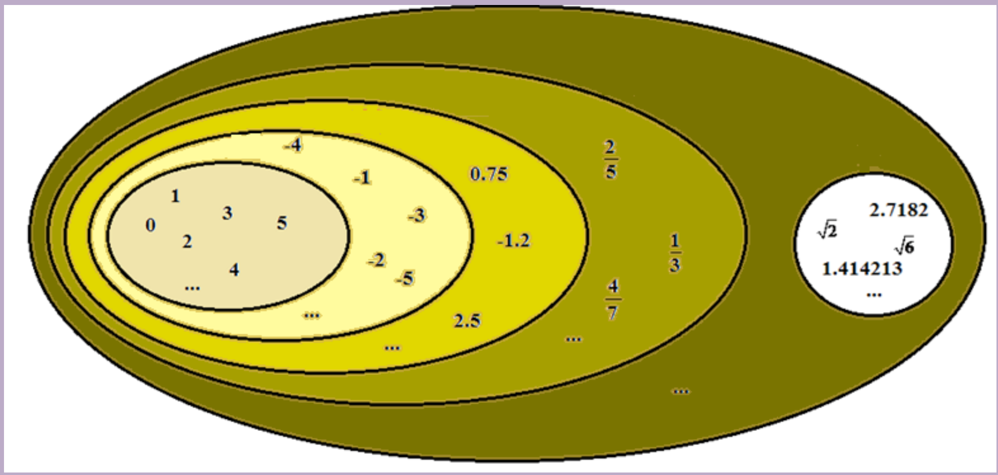
TABLE OF CONTENT

UNIT: 1 SET OF NUMBERS

Key Unit competence: Classify numbers into naturals, integers, rational and irrationals.

1.0. Introductory Activity 1

From the following diagram, discuss and work out the given tasks:



1. How many sets of numbers do you know? List them down.
2. Using a mathematical dictionary or the internet, define the sets of numbers you listed in (1).
3. Give an example of element for each set of numbers you listed.
4. Establish the relationship between the set of numbers that you listed.

1.1. Natural numbers

1.0.1. Definition

Activity 1.1.1

For any school or organization, they register their different assert for good management.

For example

1. The number of desks may be 152
2. The number of Mathematics textbooks may be 2000.
3. The number of classrooms is 15
4. The number of kitchen may be 1
5. The number of car may be 0
6. And so on

All of these numbers are elements of a set. Which set do you think, they can belong to?

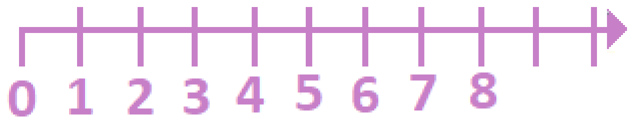
Can you give other example of elements of that set?

CONTENT SUMMARY

Usually, when counting, people begin by one, followed by two, then three and so on. The numbers we use in counting including zero, are called Natural numbers.

The set of natural numbers is denoted by $\mathbb{N}=\{0,1,2,3,4,\ldots\}$.

On a number line, natural numbers are represented as follows:



Application activity 1.1.1

1. Write down, first ten elements of natural numbers starting from zero.
2. Apart from recording assets, give two examples of where natural numbers can be used in daily life.

1.1.2. Sub sets of natural numbers

Activity 1.1.2

Carry out the following tasks.

(a) Use a dictionary or internet to define the terms: even, odd and prime numbers.

(b) You are given the set of natural numbers between 0 and 20,

(i) make a set of odd numbers.

(ii) make a set of even numbers.

(iii) make a set of prime numbers.

(iv) identify even numbers which are prime numbers?

(v) How many odd numbers are prime numbers?

(c) Represent the information from (b) in a Venn diagram.

CONTENT SUMMARY

There are several subsets of natural numbers:

(a)Even numbers

Even numbers are numbers which are divisible by 2 or numbers which are multiples of 2. Even numbers from 0 to 20 are 0, 2, 4, 6, 8, 10,12, 14, 16, 18 and 20.

The set of even numbers is $E=\{2,4,6,8,\ldots\}$ and it is a subset of natural numbers.

(b) **Odd numbers**

Odd numbers are numbers which leave a remainder of 1 when divided by 2. Odd numbers between 0 and 20 are 1,3,5,7,9,11,13,15,17 and 19. The set of odd numbers is $O=\{1,3,5,7,\ldots\}$ and it is a subset of natural numbers.

(c) **Prime numbers**

Prime number is a number that has only two divisors 1 and itself. Prime numbers between 0 and 20 are 2, 3, 5, 7, 11, 13, 17 and 19. The set of prime numbers is $P=\{2,3,5,7,11,13,19,\ldots\}$ and it is a subset of natural numbers.

Application activity 1.1.2

Given E = {1, 4, 8, 11, 16, 25, 49, 53, 75}, list the elements of the following subsets

(a) Even numbers

(b) Odd numbers

(c) Prime numbers

Represent the above information on a Venn diagram.

1.1.3 Operations and properties on natural numbers

Activity 1.1.3

1. From the given any three natural numbers a, b and c, investigate the following operations
   a) is $a+b$ always a natural number? Use example to justify your answer.
   b) is $a+b$ and $b+a$ always giving the same answer? Use example to justify your answer.
   c) is $a+(b+c)$ and $(a+b)+c$ always giving the same answer? Use example to justify your answer.
2. Given any three natural numbers a, b and c, investigate the following operations:
   $a-b$ and $b-a$, $a-(b-c)$ and $(a-b)-c$

   What do you notice? Is always the answer an element of natural numbers? Justify your answer.
3. Given any three natural numbers a, b and c, investigate the following operations : $a \times b$, $a \times b$ and $b \times a$, $a \times (b \times c)$ and $(a \times b) \times c$, $a \times (b+c)$ and $ab+ac$, $a \times (b-c)$ and $ab-ac$
4. Given any two natural numbers a, b different from zero, investigate the following operations $a \div b$, $a \div b$ and $b \div a$,

CONTENT SUMMARY

Addition

(i) Closure property: The sum of any two natural numbers is always a natural number. This is called 'Closure property of addition' of natural numbers. Thus, $\mathbb{N}$ is closed under addition. If ***a*** and ***b*** are any two natural numbers, then (a + b) is also a natural number.

Example: 2 + 4 = 6 is a natural number.

(ii) Commutative property: If a and b are any two natural numbers, then, a+b=b+a. Addition of two natural numbers is commutative.

Example: 2+ 4 = 6 and 4 + 2 = 6. Hence, 2 + 4 = 4 + 2

(iii) Associative property:

If a, b and c are any three natural numbers, then a + (b + c) = (a + b) + c. Addition of natural numbers is associative.

Example: 2 + (4 + 1) = 2 + (5) = 7 and (2 + 4) + 1 = (6) + 1 = 7. Hence, 2+(4+1)=(2+4)+1

Subtraction

(i) Closure property: The difference between any two natural numbers need not be a natural number. Hence $\mathbb{N}$ is not closed under subtraction.

Example: 2 - 5 = -3 is a not natural number.

(ii) Commutative property: If a and b are any two natural numbers, then

(a-b) ≠ (b-a). Subtraction of two natural numbers is not commutative.

Example: 5 - 2 = 3 and 2 - 5 = -3. Hence, 5 - 2 ≠ 2 – 5. Therefore, Commutative property is not true for subtraction.

(iii) Associative property: If a, b, c and d are any three natural numbers, then

a - (b - d) ≠ (a - b) – d. Subtraction of natural numbers is not associative.

Example: 2 - (4 - 1) = 2 - 3 = -1 and (2 - 4) - 1 = -2 - 1 = -3

Hence, 2 - (4 - 1) ≠ (2 - 4) – 1. Therefore, Associative property is not true for subtraction.

Multiplication

(i) Closure property: If a and b are any two natural numbers, then a x b = ab is also a natural number. The product of two natural numbers is always a natural number. Hence $\mathbb{N}$ is closed under multiplication.

Example: 5 x 2 = 10 is a natural number.

(ii) Commutative property: If a and b are any two natural numbers, then

a x b = b x a. Multiplication of natural numbers is commutative.

Example: 5 x 9 = 45 and 9 x 5 = 45. Hence, 5 x 9 = 9 x 5. Therefore, Commutative property is true for multiplication.

(iii) Associative property: If a, b and c are any three natural numbers,

then a x (b x c) = (a x b) x c. Multiplication of natural numbers is associative.

Example: 2 x (4 x 5) = 2 x 20 = 40 and (2 x 4) x 5 = 8 x 5 = 40 , Hence,

2 x (4 x 5) = (2 x 4) x 5. Therefore, Associative property is true for multiplication.

(iv) Multiplicative identity: a is any natural number, then a x 1 = 1 x a = a. The product of any natural number and 1 is the whole number itself. 'One' is the multiplicative identity for natural numbers.

Example: 5 x 1 = 1 x 5 = 5

Division

(i) Closure property: When we divide of a natural number by another natural number, the result does not need to be a natural number. Hence, ℕ is not closed under multiplication.

Example: When we divide the natural number 3 by another natural number 2, we get 1.5 which is not a natural number.

(ii) Commutative property: If a and b are two natural then a ÷ b ≠ b ÷ a. Division of natural numbers is not commutative.

Example: 2 ÷ 1 = 2 and 1 ÷ 2 = 1.5. Hence, 2 ÷ 1 ≠ 1 ÷ 2

Therefore, Commutative property is not true for division.

(iii) Associative property: If a, b and c are any three natural numbers,

then a ÷ (b ÷ c) ≠ (a ÷ b) ÷ c. Division of natural numbers is not associative.

Example : 3 ÷ (4 ÷ 2) = 3 ÷ 2 = 1.5 and (3 ÷ 4) ÷ 2 = 0.75 ÷ 2 = 0.375

Hence, 3 ÷ (4 ÷ 2) ≠ (3 ÷ 4) ÷ 2. Therefore, Associative property is not true for division.

Distributive Property

(i) Distributive property of multiplication over addition :

If a, b and c are any three natural numbers, then a x (b + c) = ab + ac. Multiplication of natural numbers is distributive over addition.

Example : 2 x (3 + 4) = 2 x 3 + 2 x 4 = 6 + 8 = 14

2 x (3 + 4) = 2 x (7) = 14. Hence, 2 x (3 + 4) = 2 x3 + 2 x 4

Therefore, Multiplication is distributive over addition.

(ii) Distributive property of multiplication over subtraction:

If a, b and c are any three natural numbers, then a x (b - c) = ab – ac. Multiplication of natural numbers is distributive over subtraction.

Example: 2 x (4 - 1) = (2x4) - (2x1) = 8 - 2 = 6

2 x (4 - 1) = 2 x (3) = 6. Hence, 2 x (4 - 1) = (2x4) - (2x1)

Therefore, multiplication is distributive over subtraction.

Application activity 1.1.3

Using your own choice of natural numbers, carry out the following operations and deduce the properties:

1) $a+b$, $a+b=b+a$, $a+(b+c)=(a+b)+c$

2) $a\times b$, $a\times b=b\times a$, $a\times(b\times c)=(a\times b)\times c$,

3) $a\times(b+c)=ab+ac$

1.2 Integers

1.2.1 Definition

Activity 7.2.1

Carry out the following activities

1. Using a Maths dictionary, define what an integer is.

2. What integer would you give to each of the following situation?

   (a) A fish which is 50 m below the water level.

   (b) Temperature of the room which is 42°C.

   (c) A boy who is 2 m below the ground level in a hole.

   (d) A bird which is 3 m high on a tree.

CONTENT SUMMARY

Integers are whole numbers which have either negative or positive sign and include zero. The set of integers is represented by $\mathbb{Z}$. Iintegers can be negative {-1,-2,-3,-4,-5,... }, positive {1, 2, 3, 4, 5, ... }, or zero {0}

The set of integers is represented using Carly brackets as follow :

{ ..., -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5, ... }

Example 1:

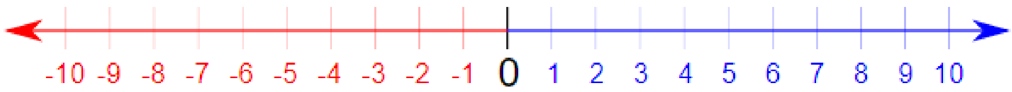
Taking 4 steps forward can be considered as +4 (positive 4). This means it is 4 steps ahead of the starting point.

Taking 4 steps backward is considered as -4 (negative 4). This means it is 4 steps behind the starting point.

Example 2:

When measuring temperature, 5 degrees above zero degree Celsius is considered as $+5^0C$. This means it is 5^0C hotter than 0^0C ; 5 degrees below zero degree Celsius is considered as -5^0C. This means it is 5^0C colder than 0^0C.

Integers can be represented on a number line as shown below



When you move towards the left of the number line, numbers become smaller and smaller . This means -5 is smaller than -2 and -100 is less than -1. We can represent this as -5<-2 and -100<-1.

Application activity 1.2.1

John borrowed $3 to pay for his lunch.

Alex borrowed $5 to pay for her lunch.

Virginia had enough money for lunch and has $3 left over.

Place these people on the number line to find who is poorest and who is richest given that they will pay the same amount.

1.2.2 Sub sets of integers

Activity 7.2.1

Discuss the following:

From integers between 12 and +20, form small sets which contain;

a) odd numbers

b) even numbers

c) factors of 6

d) multiples of 3.

CONTENT SUMMARY

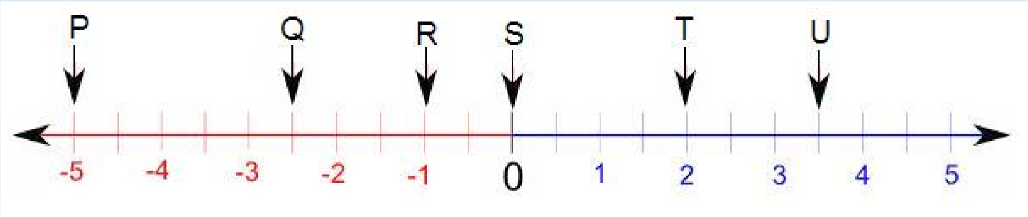
Integers have several subsets such as set of natural numbers, set of even numbers, set of odd numbers, set of prime numbers, set of negative numbers and so on.

Some of **special subset** of integers are the following:

- The set of **non-negative integers** denoted $\mathbb{Z}_0^+$ and $\mathbb{Z}_0^+ = \{0, 1, 2, 3, \ldots\}$; this set is also called a set of **whole integers**.
- The set of **positive integers denoted** as $\mathbb{Z}^+ = \{1, 2, 3, \ldots\}$
- The set of **negative integers** denoted as $\mathbb{Z}^- = \{\cdots, -4, -3, -2, -1\}$

Application activity 1.2.2

From the following figure, the points P, Q, R, S, T and U are located.



Among them,

- which are negative integers?
- Which are positive integers?
- Which are neither positive nor negative integers?
- Which are not integers?

1.2.3 Operations and properties on integers

Activity 1.2.3

1. Work out the following on a number line:

   (a) (+3) + (+2)   (b) –(5) + - (3)   (c) (+4) + (-3)

   (d) Which side of the number line did you move when adding a negative number to a positive number?

   (e) Which side of the number line did you move when adding the given numbers which are all negative?

2. Work out the following and show your solutions on a number line.

   (a) (-4)-(+3)   (b) (+5) – (+3)   (c) (–6) – (-6)

   (d) On your number line, which direction do you move when subtracting two negative numbers?

   (e) In case you have two positive numbers that you are finding the difference, which side of the number line would you move?

3. Work out the following:

   (a) (+5) × (-6)   (b) (+5) × (+6)   (c) (-5) × (+6)   (d) (-5) × (-6)

   (e) Did you obtain the same results in all the four tasks?

4. Work out the following and show your solutions on a number line.

   (a) (-4) ÷(+4)   (b) (+4) ÷ (+4)   (c) (–4) ÷ (-4)   (d) (+4) ÷ (-4)

   (e) Did you obtain the same results in all the four questions above?

CONTENT SUMMARY

The table below shows how addition, subtraction, multiplication and division of integers are performed.

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ADDITION

Addition of integers is best understood by illustrating the movements on the number line.

Example

(-3)+ (+2) = -1

• **Same sign:** Add the numbers and copy the sign

• **Different signs:** Subtract the numbers and copy the sign of larger number.

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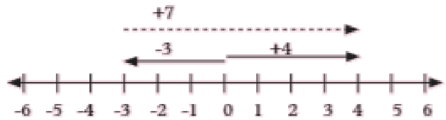
SUBTRACTION

Subtraction of integers is best understood by illustrating the movements on the number line.

Example

(+4)-(-3)=+7



1. Change the sign of subtrahend

2. Use the addition rule for integers

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MULTIPLICATION

• Same sign: Product is positive

• Different signs: Product is negative

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DIVISION

• Same sign: Quotient is positive

• Different signs: Quotient is negative

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Application activity 1.2.3

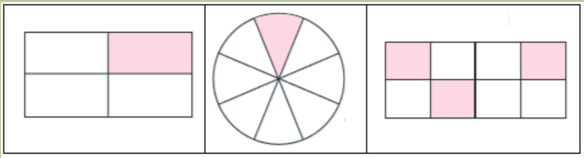
A common example of negative integer usage is the thermometer. Thermometers are similar to number lines, but vertical. They have positive integers above zero and negative integers below zero. Commonly, people recognize a temperature of -25°C as cold. People use this number system to measure and represent the temperature of the air. Also, if it is -23°C outside, and the temperature drops 3 degrees, what is temperature now? If we picture the thermometer, we know that as the temperature drops, we look downwards on the thermometer.

1.3 Rational and irrational numbers

1.3.1 Definition of rational numbers

Activity 1.3.1

Observe the following figures and answer the related questions



- Write the fractions of the shaded and unshaded parts
- Fractions are elements of which set?
- Given integers a and b, with b a non-zero integer, is $\frac{a}{b}$ always a fraction? Support your answer with examples

CONTENT SUMMARY

From any two integers a and b, we deduce fractions expressed in the form $\frac{a}{b}$, where b is a non-zero integer. **A rational number is a number that can be expressed as a fraction where** both the numerator and the denominator in the fraction are integers; the denominator in a rational number cannot be zero. As fractions are rational numbers, thus set of fractions is known as a set of rational numbers.

Application activity 1.3.1

From research on internet or using reference books, identify different type of fractions and give example for each type.

1.3.2 Sub sets of rational numbers

Activity 1.3.2

Knowing that a rational number is a number that can be in the form of $\frac{p}{q}$, where p and q are integers and q is not zero. Tell whether the given statements are true or false. Explain your choice.

1. All integers are rational numbers.
2. No rational numbers are whole numbers.
3. All rational numbers are integers.
4. All whole numbers are rational numbers.

CONTENT SUMMARY

From the concept of subset and definition of a set of rational numbers we can establish some subsets of rational numbers; among them we have:

- Integers and its subsets: $\mathbb{Z}, \mathbb{Z}_0^+, \mathbb{Z}^-, \mathbb{Z}^+, ...$
- Natural numbers and its subsets: $\mathbb{N}, \mathbb{N}^+$, prime numbers, odd numbers, even numbers,...
- Counting numbers and its subsets: $\mathbb{N}^+$, square numbers, prime numbers, odd numbers, even numbers ...

Application activity 1.3.2

Using Venn diagram, establish the relationship between the following sets: Natural numbers; Integers and rational numbers.

1.3.3 Operations and properties on rational numbers

Activity 1.3.3

1. From your own choice of three rational numbers a, b and c,

Investigate the following operations

a) is $a+b$ always a rational number? Use example to justify your answer.

b) is $a+b$ and $b+a$ always giving the same answer? Use example to justify your answer.

c) is $a+(b+c)$ and $(a+b)+c$ always giving the same answer? Use example to justify your answer.

2. Given any three rational numbers a, b and c, investigate the following operations : $a-b$, $a-b$ and $b-a$, $a-(b-c)$ and $(a-b)-c$

   What do you notice? Is always the answer an element of rational numbers? Justify your answer.

3. Given any three rational numbers a, b and c, investigate the following operations : $a \times b$, $a \times b$ and $b \times a$, $a \times (b \times c)$ and $(a \times b) \times c$

4. Given any two rational numbers a, b and c, investigate the following operations : $a \div b$, $a \div b$ and $b \div a$, $a \div (b \div c)$ and $(a \div b) \div c$

CONTENT SUMMARY

The following table shows how addition, subtraction, multiplication and division of rational numbers are performed

Operation	Calculations	Properties
Addition of Rational Numbers	For any two rational numbers a and b, $a + b$ is also a rational number.	Closure property under addition
	For any two rational numbers a and b, $a + b = b + a$. Two rational numbers can be added in any order and the result remain the same	Commutative property under addition
	For any three rational numbers a, b and c. $(a + b) + c = a + (b + c)$. Rational numbers can be added regardless of how they are grouped and the result remain the same.	Addition is associative for rational numbers.
Subtraction of Rational Numbers	For any two rational numbers a and b, $a - b$ is also a rational number.	Closure property under subtraction of rational numbers
	For any two rational numbers a and b, $a - b \neq b - a$.	Subtraction is not commutative for rational numbers.
	For any three rational numbers a, b and c. $(a - b) - c \neq a - (b - c)$.	Subtraction is not associative for rational numbers
Multiplication of Rational Numbers	For any two rational numbers a and b, $a \times b$ is also a rational number.	Closure Property under multiplication
	For any two rational numbers a and b, $a \times b = b \times a$ Two rational numbers can be multiplied in any order and the result remain the same	Multiplication is commutative for rational numbers.
	For any three rational numbers a, b and c. $(a \times b) \times c = a \times (b \times c$ Rational numbers can be multiplied regardless of how they are grouped and the result remain the same.	Multiplication is associative for rational numbers.
	For any three numbers a, b and c. $a \times (b + c) = (a \times b) + (a \times c)$	Distributive property states that for any three numbers a, b and c we have $a \times (b + c) = (a \times b) + (a \times c)$

Division of Rational Numbers	For any two rational numbers *a* and *b with b different from zero, a* ÷ *b* is also a rational number. But we know that any rational number *a*, *a* ÷ 0 is not defined.	Closure Property under division for non-zero rational numbers.
	For any two rational numbers *a* and *b*, $a \div b \neq b \div a$. The expressions on both the sides are not equal	Division is not commutative for rational numbers.
	For any three rational numbers *a* , *b* and *c*, $a \div (b \div c) \neq (a \div b) \div c$. The expressions on both the sides are not equal	Division is not associative for rational numbers

Application activity 1.3.3

Read carefully the following text and make a research to find out two more examples where fractions or rational numbers are used in real life and then present your findings in written and oral forms.

"Imagine you are shopping with your $100 in birthday money. You really want a few items you have had your eye on for a while, but they are all very expensive. You are waiting for the items to go on sale, and when they do, you rush down to the store. Instead of being marked with a new price, though, the store has a large sign that reads: All items are currently 75% off. This sounds like great news, but without doing some Math, there is no way to know if you have enough money. Knowing that 75% is ¾ off the cost of each item is the best way to get started".

1.3.4 Definition of irrational numbers

Activity 1.3.4

1. Express this recurring decimal $0.66666\ldots$ as a fraction and explain how you get the answer.

2. By using calculator, carry out the following;

   (a) $\sqrt{2}$ (b) $\sqrt[3]{5}$

Is it possible to express those numbers as fractions? Discuss your answer.

Can you indicate the set for which the given number belong?

CONTENT SUMMARY

Some decimals are expressed as a rational $\frac{p}{q}$. For example, $0.13 = \frac{13}{100}, 0.25 = \frac{1}{4}$. The recurring decimals such as $0.3333\ldots = \frac{1}{3}$ fall under rational numbers. From Activity 1.3.4, you notice that, there are some decimals which do not recur. Their values keep changing and they go on without an end. For example, $\sqrt{3} = 1.7320508\ldots$, $\sqrt[3]{4} = 1.587401\ldots$

Similarly, there are some numbers which do not have exact values neither can they be expressed as fractions, for example π etc. These numbers are called **irrational numbers**.

An irrational number can be written as a decimal, but not as a fraction.

An irrational number has endless non-repeating digits to the right of the decimal point

Although irrational numbers are not often used in daily life, they do exist on the number line. In fact, between 0 and 1 on the number line, there are an infinite number of irrational numbers!

Application activity 1.3.4

Using internet or reference books, from your own choice of appropriate numbers, verify and discuss if the given statement is always true, sometimes true or never true.

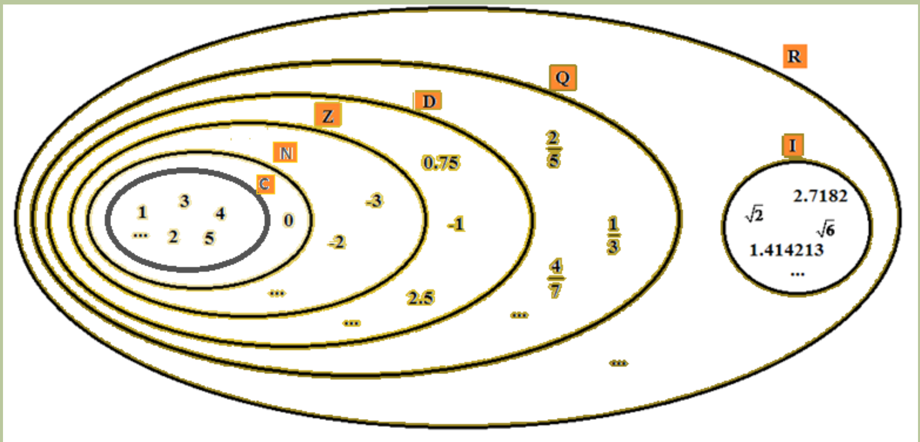
1. The sum of rational number and irrational number is irrational.
2. The product of rational number and irrational number is irrational.
3. The sum of two irrational numbers is irrational.
4. The product of two irrational numbers is irrational.
5. Between two rational numbers, there is an irrational number.
6. If you divide an irrational number by another, the result is always an irrational number.

1.4 Real numbers

1.4.1 Definition

Activity 1.4.1

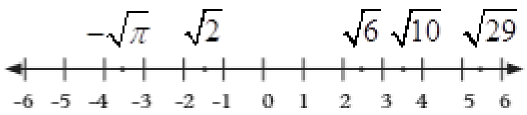
From the following Venn diagram, representing the set of counting numbers, natural numbers, integers, decimals, rational numbers and irrational numbers, verify and discuss if the given statement is whether true or false.



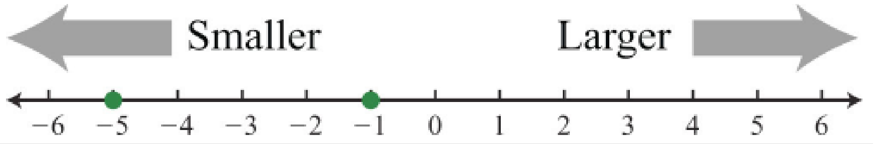
1. The set of counting numbers is a subset of natural numbers.
2. The intersection of set of integers and counting numbers is the set of natural numbers.
3. The intersection of set of integers and natural numbers is the set of counting numbers.
4. The union of set of natural numbers and counting numbers is the set of natural numbers.
5. The intersection of set of rational numbers and irrational numbers is the set of irrational numbers.
6. The union of set of rational numbers and irrational numbers is a set of irrational numbers.

CONTENT SUMMARY

The set of rational numbers and the set of irrational numbers combined together, form the set of real numbers. The set of real numbers is denoted by $\mathbb{R}$. Real numbers are represented on a number line as infinite points or they are set of decimal numbers found on a number line. This is illustrated on the number line below



The real numbers are ordered. We say is a ***less than*** b and write $a<b$ if $b-a$ is positive. **Geometrically** this means that a lies to the left of b on the number line. Equivalently, we say b is ***greater than*** a and write $b>a$. The symbol $a \le b$ $(\text{or } b \ge a)$ means that either $a<b$ or $a=b$ and is read a ***less than or equal to*** b. In fact, when comparing real numbers on a number line, the larger number will always lie to the right of the smaller one. It is clear that 5 is greater than 2, but it may not be so clear to see that –1 is greater than –5 until we graph each number on a number line.



Application activity 1.4.1

Some examples of applications of real numbers in our daily life are identified below. From research activity; find out at least 3 examples of other applications of real numbers in our daily life.

Real numbers help us to count and to measure out quantities of different items. For instance, in catering you may have to ask the client how many sandwiches they need for the event. Certainly, those working in accounts and other financial related jobs may use real numbers mostly. Even when relaxing at the end of the day in front of the television flicking from one channel to the next you are using real numbers.

1.4.2 Subsets of real numbers and intervals

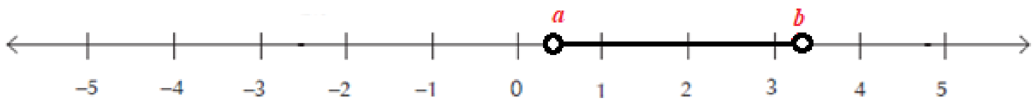
Activity 1.4.2

Carry out research on sets of real numbers to determine its subsets and present your finds using a number line.

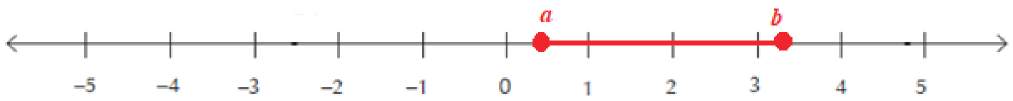
CONTENT SUMMARY

The main subsets of real numbers are sets of irrational numbers, rational numbers and its subsets. Certain subsets of real numbers are called **intervals**, they occur frequently in calculus and correspond geometrically to line segments. For example, if $a<b$, the **open interval** from a to b consists of all numbers between a and b and is denoted by the symbol (a,b) or $]a,b[$ and described as $(a,b)=\{x : a<x<b\}$.

For this case, the end points of the interval namely a and b are excluded. This is indicated by the **round brackets** $(\ )$ and by the **open dots** as illustrated in the following figure



The **closed interval** from a to b are is the set described as $[a,b]=\{x: a\le x\le b\}$. Here the endpoints of the interval are included. This is indicated by **square brackets** $[\ ]$ and by the **solid dots** as illustrated in the following figure.



Application activity 1.4.2

Write down the truth table for

1. Three propositions *p*, *q* and *r*
2. Four propositions *p*, *q*, *r and* ***s***

Note that, in writing intervals, it is also possible to include only one endpoint in an interval. Included point is geometrically represented by the solid dot. From this notice, complete the following table.

Notation	Set description	Geometrical representation
(a,b)	$\{x: a<x<b\}$	
$[a,b]$	$\{x: a\le x\le b\}$	
$[a,b)$		
$(a,b]$		
$]a,+\infty[$		
$[a,+\infty[$		
$]-\infty,b[$		

$]-\infty, b]$		
$]-\infty, +\infty[$		

1.4.3 Operations and properties on real numbers

Activity 1.4.3

1. From your own choice of three real numbers a, b and c, investigate the following operations

   a) is $a+b$ always a real number? Use example to justify your answer.

   b) is $a+b$ and $b+a$ always giving the same answer? Use example to justify your answer.

   c) is $a+(b+c)$ and $(a+b)+c$ always giving the same answer? Justify your answer by example.

2. Given any three real numbers a, b and c, investigate the following operations : $a-b$, $a-b$ and $b-a$, $a-(b-c)$ and $(a-b)-c$

   What do you notice? Is always the answer an element of real numbers? Justify your answer.

3. Given any three real numbers a, b and c, investigate the following operations : $a \times b$, $a \times b$ and $b \times a$, $a \times (b \times c)$ and $(a \times b) \times c$

4. Given any two real numbers a, b and c, investigate the following operations : $a \div b$, $a \div b$ and $b \div a$, $a \div (b \div c)$ and $(a \div b) \div c$

CONTENT SUMMARY

The following table shows how addition, subtraction, multiplication and division of real numbers are performed

Operation	Calculations	Properties
Addition of real Numbers	For any two real numbers a and b, $a + b$ is also a real number.	Closure property under addition
	For any two real numbers a and b, $a + b = b + a$. Two real numbers can be added in any order and the result remain the same	Commutative property under addition
	For any three real numbers a, b and c. $(a + b) + c = a + (b + c)$. Real numbers can be added regardless of how they are grouped and the result remain the same.	Addition is associative for real numbers.
Subtraction of Real Numbers	For any two real numbers a and b, $a - b$ is also a real number.	Closure property under subtraction of real numbers
	For any two real numbers a and b, $a - b \neq b - a$.	Subtraction is not commutative for real numbers.
	For any three real numbers a, b and c. $(a - b) - c \neq a - (b - c)$.	Subtraction is not associative for real numbers
Multiplication of real Numbers	For any two real numbers a and b, $a \times b$ is also a real number.	Closure Property under multiplication
	For any two real numbers a and b, $a \times b = b \times a$ Two real numbers can be multiplied in any order and the result remain the same	Multiplication is commutative for real numbers.
	For any three real numbers a, b and c. $(a x b) x c = a x (b x c$ Real numbers can be multiplied regardless of how they are grouped and the result remain the same.	Multiplication is associative for real numbers.
	For any three numbers a, b and c. $a \times (b + c) = (a \times b) + (a \times c)$	Distributive property states that for any three numbers a, b and c we have $a \times (b + c) = (a \times b) + (a \times c)$

Division of Real Numbers	For any two real numbers *a* and *b with b different from zero,* $a \div b \neq b \div a$ is also a real number. But we know that any real number *a*, $a \div 0$ is not defined.	Closure Property under division for real numbers different from zero.
	For any two real numbers *a* and *b*, $a \div b \neq b \div a$. The expressions on both sides are not equal	Division is not commutative for real numbers.
	For any three real numbers *a* , *b* and *c*, $a \div (b \div c) \neq (a \div b) \div c$. The expressions on both the sides are not equal	Division is not associative for real numbers

Application activity 1.4.3

1. Discuss whether Closure property under division for real numbers is satisfied.
2. Ngoma District wants to sell two fields on the same price. One has width of $60\,m$ out $160\,m$ of length. Another one has the width of $100\,m$ as it is its length. Among these fields, what is biggest? Interpret your result.

1.4. END UNIT ASSESSMENT 1

1. List three rational numbers between 0 and 1
2. Identify the sets to which each of the following numbers belong by ticking (v) in the appropriate boxes (cells).

	Number	Counting	Natural	Integers	Rational	Irrational	Real
1	-1						
2	$\frac{1}{7}$						
3	0						
4	$-\sqrt{7}$						
5	$\sqrt{0.03}$						

6	$\frac{\sqrt{4}}{2}$						
7	$\frac{1}{0}$						
8	$\frac{\pi}{2}$						
9	$\frac{\sqrt{27}}{\sqrt{3}}$						
10	$\sqrt[3]{-27}$						

Represent (approximately) by a point on a number line each of the real numbers in Exercise 2.

UNIT: 2 SET THEORY

Key Unit competence: **Solve problems that involve Sets operations, using Venn diagram.**

2.0. Introductory Activity 2

At a TTC school of 500 students-teachers, there are 125 students enrolled in Mathematics club, 257 students who play sports and 52 students that are enrolled in Mathematics club and play sports.

If M stands for the set of students in Mathematics club, S stands for the set of students in Sports and U stands for all students at the school or universal set,

1. Complete the following table

Symbol	Description	Value for this problem
$n(M)$	The number of elements in set M	
$n(S)$	The number of elements in set S	
$n(M \cap S)$	The number of elements in the intersection of sets M and S (all the elements that are in both sets-the overlap)	
$n(M \cup S)$	The number of elements in the union of sets M and S (all the elements that are in one or both of sets)	

2. Create a Venn diagram to illustrate this information.

2.1 Sets and Venn diagrams

Activity 2.1

Given set $A = \{ 1, 3, 5, 7, 9 \}$ and $B = \{ 1, 2, 3, 4, 5 \}$,

- Identify common elements for both sets.
- Determine elements of set A which are not in set B
- Determine elements of set B which are not in set A
- Determine all elements in set A and set B
- Represent elements of set A in a Venn diagram
- Represent elements of set B in a Venn diagram
- Represent elements of sets A and B using one Venn diagram

CONTENT SUMMARY

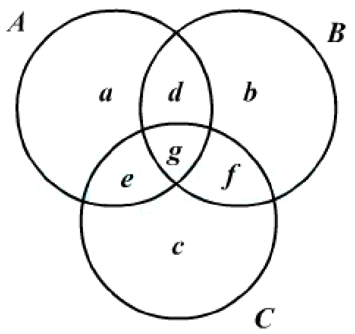
A well-defined collection of objects is called a set. Each member of a set is called an element. All elements of a set follow a certain rule and share a common property amongst them.

A set that contains all the elements and sets in a given scenario is called a Universal Set (U).

Venn Diagrams consist of closed shapes, generally circles, which represent sets. The capital letter outside the circle denotes the name of the set while the small letters inside the circle denote the elements of the set.

The various operations of sets are represented by partial or complete overlap of these closed figures. Regions of overlap represent elements that are shared by sets.

In practice, sets are generally represented by circles. The universal set is represented by a rectangle that encloses all other sets.



The given figure is a representation of a Venn diagram. Here each of the circles A, B and C represents a set of elements.

- Set *A* has the elements *a*, *d*, *e* and *g*.
- Set *B* has the elements *b*, *d*, *g* and *f*.
- Set *C* has the elements *e*, *g*, *f* and *c*.
- Both *A* and *B* have the elements *d* and *g*.
- Both *B* and *C* have the elements *g* and *f*.
- Both *C* and *A* have the elements *e* and *g*.
- *A*, *B* and *C* all have the element *g*.

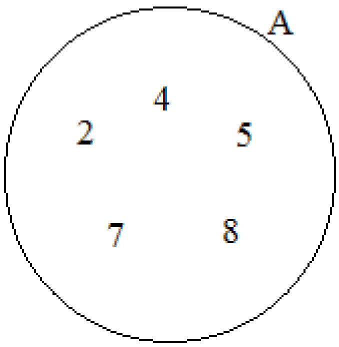
The circular pattern used to represent a set and its elements is called a **Venn diagram**.

Venn diagrams are an efficient way of representing and analysing sets and performing set operations. As such, the usage of Venn diagrams is just the elaboration of a solving technique. Problems that are solved using Venn diagrams are essentially problems based on sets and set operations.

Example

1. Given set A= {2, 4, 5, 7, 8}, represent set A on a Venn diagram.

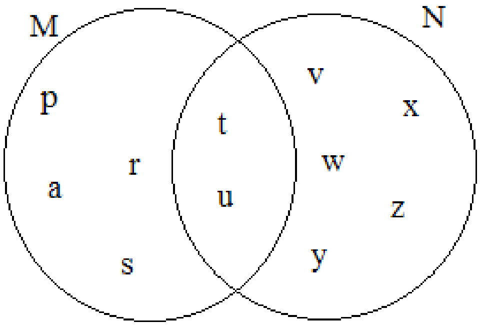
Solution:



First, express the data in terms of set notation and then fill the data in the Venn diagram for easy solution.

When drawing Venn diagrams, some important facts like "**intersection**", "**union**" and "**complement**" should be well considered and represented.

Example: Consider the Venn diagram below.



List the elements of set M and N.

Solution

$M = \{a, \text{p}, \text{r}, \text{s}, \text{t}, \text{u}\}$

$N = \{t, u, v, w, \text{x}, \text{y}, \text{z}\}$

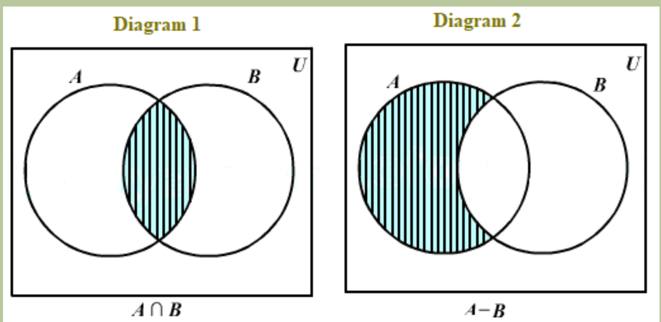
Application activity 2.1

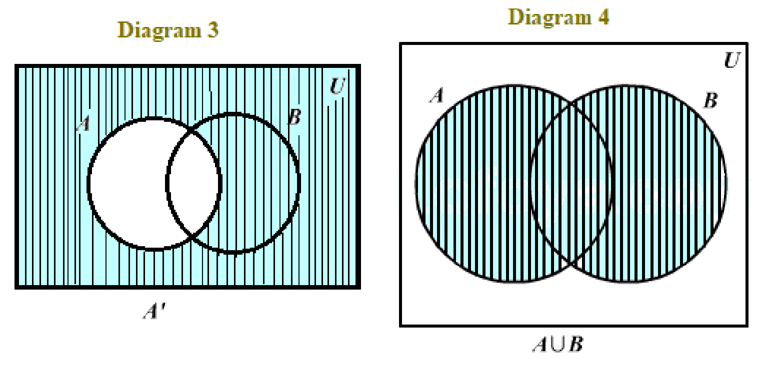
Consider these two sets $A = \{2, 4, 6, 8, 10\}$ *and* $B = \{2, 3, 5, 7\}$. Represent them in a Venn diagram

2.2 Operations of Sets

Activity 2.2

Consider a class of students that form the universal set. Set A is the set of all students who were present in the English class, while Set B is the set of all the students who were present in the History class. It is obvious that there were students who were present in both classes as well as those who were not present in either of the two classes. The shaded part shows the elements which are considered in the diagram.





Observe the diagrams and identify which one to represent the following:

- All students who were absent in the English class
- All students who were present in at least one of the two classes.
- All the students who were present for both English as well as History classes.
- All the students who have attended only the English class and not the History class
- All the students who have attended just the History class and not the English class.

CONTENT SUMMARY

From the activity, we realize that two or more sets can be represented using one Venn diagram and from the representations, different sets can be determined. To determine those sets, one may perform different operations on sets such as:

- Intersection of sets
- Union of sets
- Universal sets
- Simple difference of sets
- Symmetric difference of sets
- Complement of sets

a. The intersection of sets

The common elements which appear in two or more sets form the intersection of sets. The symbol used to denote the intersection of sets is $\cap$.

The intersection of sets A and B is denoted by $A \cap B$ and consist of those elements which belong to A and B that is $A \cap B = \{x | x \in A \text{ and } x \in B\}$

Example:

Given that set A = $\{\text{the first 5 letters of the alphabet}\}$ and set B = $\{\text{all the vowels}\}$;

(a) List the elements of each set.

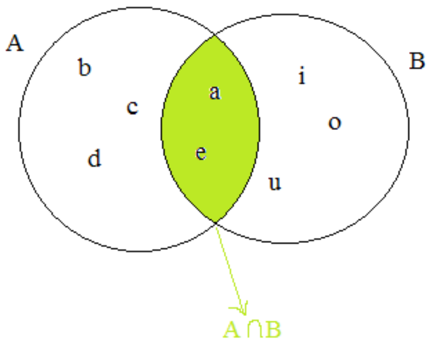
(b) Find $A \cap B$

(c) Draw a Venn diagram to represent set $A \cap B$

Solution:

A = $\{a, b, c, d, e\}$
B = $\{a, e, i, o, u\}$
A $\cap B = \{a, b, c, d, e\} \cap \{a, e, i, o, u\} = \{a, e\}$

The Venn diagram is as shown below.



b. The union of sets

Elements of two or more sets can be put together to form a set. The set formed is known as **the union of sets**. The symbol for the union of sets is $\cup$.

The union of two set A and B, is denoted by $A \cup B$ and consists of all the elements which are members of either A or B or both A and B that is $A \cup B = \{x | x \in A \text{ or } x \in B\}$

Example

Given the following sets $A = \{a, b, c, d, e, f\}$ and $B = \{a, b, c, h, i\}$

a. Find the number of elements of the following sets: A, B, $A \cup B$:

$(i)\ n(A)$

$(ii)\ n(B)$

$(iii)\ n(A \cup B)$

b. Draw Venn diagrams to represent $A \cup B$

Solution

a.

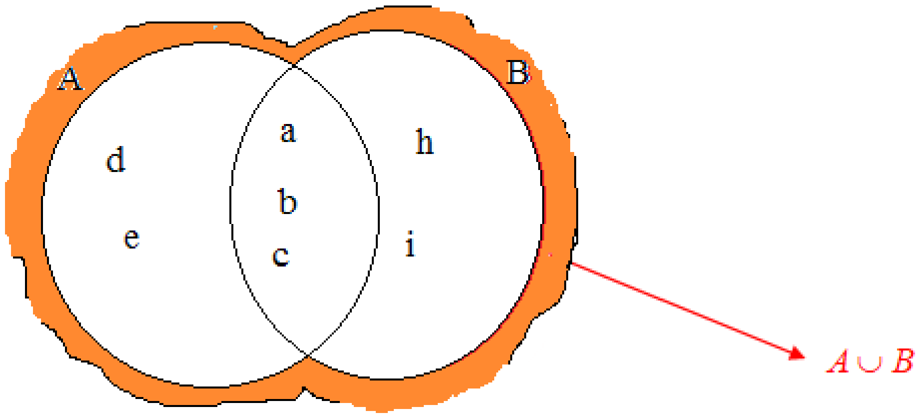
$(i)\ n(A) = 6$

$(ii)\ n(B) = 5$

$(iii)\ A \cup B = \{a,\ b,\ c,\ d,\ e,\ f\} \cup \{a,\ b,\ c,\ h,\ i\} = \{a,\ b,\ c,\ d,\ e,\ f,\ h,\ i\}$

$\Rightarrow n\ (A \cup B) = 8$

b.



a) Universal set

A set that contains all the subsets under consideration is known as a universal set. A Universal set is denoted by the symbol U .

Example

Consider a school in which one can find various categories of people such as pupils, teachers and other workers or staff.

1. Use S to represent the set of people in a school.
2. Write all the subsets of S.

Solution

1. We can present the set of all people in the school with sets as follows:

Set $S = \{pupils,\ teachers,\ workers\}$.

Thus, set S contains all the subsets of the various categories of people in the school.

Let us use sets P, T and W to represent the subsets of set S.

$Set\ P = \{pupils\}$
$Set\ T = \{teachers\}$
$Set\ W = \{other\ workers\}$.
$Therefore\ P \subset S,\ T \subset S\ and\ W \subset S.$

2. Thus, the set S is a universal set and is represented as:
$S = \{pupils\ ,\ teachers,\ workers\}$.

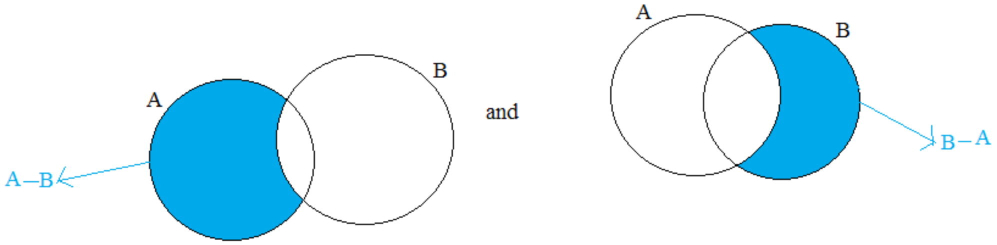
b. Difference and Symmetric difference of sets

i. Difference of sets

Difference between sets A and B written as $A - B$ or $A \setminus B$ is the set of the elements of set A which are not in set B. It means that $A - B = \{x | x \in A, x \notin B\}$

Likewise $B - A\ or\ B \setminus A$ is difference between sets B and A. This is the set of elements that are in set B and not in set A. It means that $B - A = \{x | x \in B, x \notin A\}$

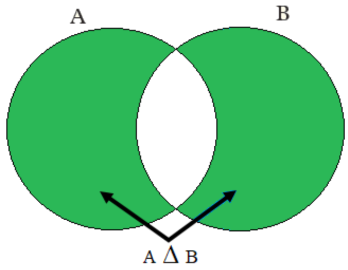
A–B and B–A can be shown using a Venn diagram as follows:



ii. Symmetric difference of sets

The union of sets $A - B\ and\ B - A$ is known as **the symmetric difference between sets A and B**. It is written in symbols as $A \Delta B$ to mean $A \Delta B = (A - B) \cup (B - A)$

$A \Delta B$ can be shown using a Venn diagram as follows



Example:

Given that $A = \{3, 4, 5, 6, 7, 8\}$ and $B = \{2, 4, 8, 12\}$, find:

$(a)\ A - B$
$(b)\ B - A$
$(c)\ A \Delta B$

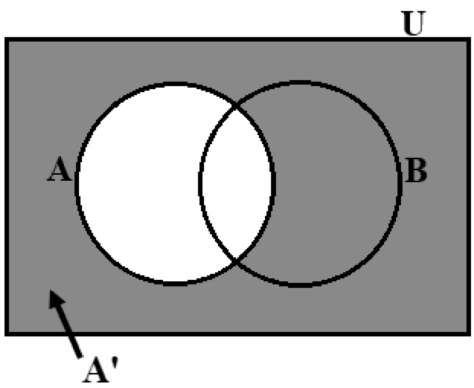
Solution

$(a)\ A - B = \{3, 5, 6, 7\}$
$(b)\ B - A = \{2, 12\}$
$(c)\ A\Delta B = (A-B)\cup(B-A = \{2,3,5,6,7,12\}$

c) The complement of set

Complement of a set is the set of all elements in the universal set that are not members of a given set. The symbol for the universal set is U.

The complement of A is denoted by A' and consist of all those elements in the universe which do not belong to A that is $A' = \{x | x \in U, x \notin A\}$



Note that $A - B = A \cap B'$ where B' is the complement of B

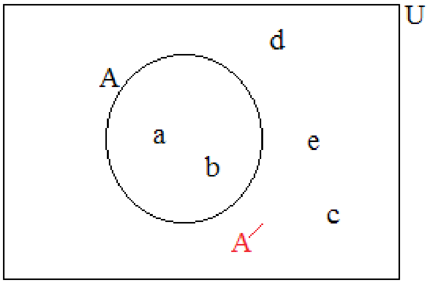
Example

Given that $U = \{a, b, c, d, e\}$ and $A = \{a, b\}$, find A' where A' is the complement of A.

Solution

$U = \{a, b, c, d, e\}, A = \{a, b\} \Rightarrow A' = \{c, d, e\}$.

This can be represented on a Venn diagram as shown by the diagram.



Application activity 2.2

1. Consider the sets

$A = \{1, 2, 3, 5, 6, 8\}, B = \{2, 4, 6, 8\}$ *and* $C = \{1, 3, 5, 7\}$.

Find and draw the Venn diagrams for:

(a) $B \cap C$ (b) $A \cap C$
(c) $A \cap B$ (d) $B \cup C$
(e) $A \cup B$.

2. Given $U = \{\text{letters of the word elephatiasis}\}$, Set $A = \{\text{all vowels}\}$, Set $B = \{\text{first five letters of the English alphabet}\}$ find:

a) A
b) B
c) $A - B$
d) $B - A$
e) A Δ B

3. If $U = \{a, e, i, o, u, c, d\}$, $X = \{a, b, e\}$ *and* $Y = \{c, d, e\}$, find:

(a) $(X \cap Y)'$
(b) $(X \cup Y)'$

2.3 Analysis, interpretation and presentation of a problem using Venn diagram

Activity 2.3

1. A survey was carried out in a shop to find the number of customers who bought bread or milk or both or neither. Out of a total of 79 customers for the day, 52 bought milk, 32 bought bread and 15 bought neither.

   a. Without using a Venn diagram, find the number of customers who:

      i. bought bread and milk

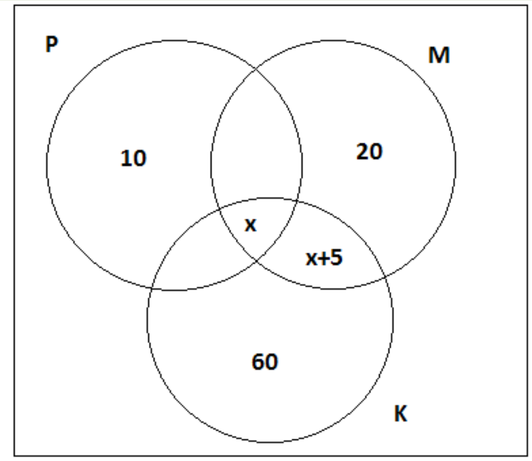
      ii. bought bread only

      iii. bought milk only

   b. With the aid of a Venn diagram, work out questions (i), (ii) and (iii) in (a) above.

   c. Which of the methods in (a) and (b) above is easier to work with? Give reasons for your answer.

2. The Venn diagram below shows the number of senior one students in a school who like Mathematics (M), Physics (P) and Kinyarwanda (K). Some like more than one subject in total 55 students like Mathematics.



   a. How many students like the three subjects?

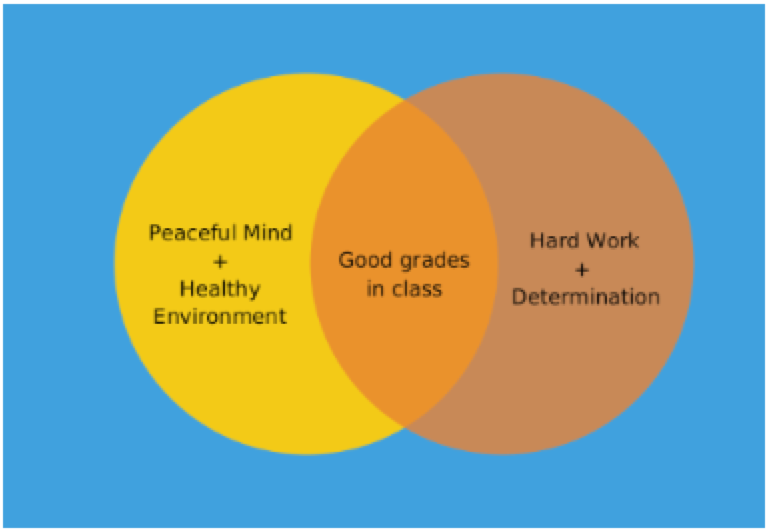
   b. Find the total number of senior one students in the school.

   c. How many students like Physics and Kinyarwanda only?

CONTENT SUMMARY

1. Venn diagrams are great for comparing things in a visual manner and to quickly identify overlaps. They are diagrams containing circles that show the logical relations between a collection of sets or groups.

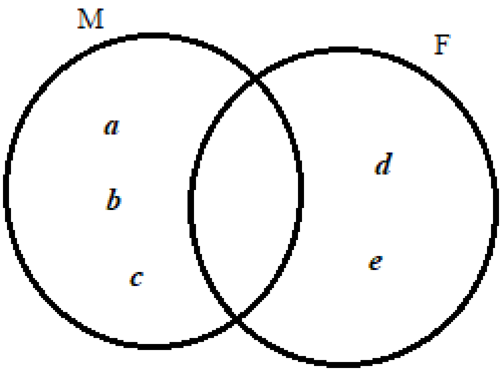
Understanding a student's interaction in class



Venn diagrams are used in many areas of life where people need to categorize or group items, as well as compare and contrast different items. Although Venn diagrams are primarily a thinking tool, they can also be used for assessment. However, students must already be familiar with them.

Example: In a room, there are 5 people a, b, c, d, e. Out of them, a, b and c are Males while d and e are Females. Also, a and e study science while b, c and d study English.

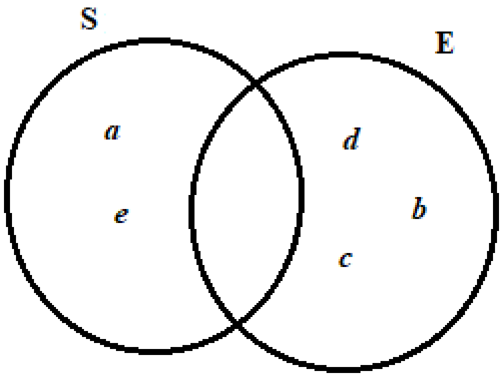
The set of males is M = {a, b, c} and the set of females is F = {d, e}



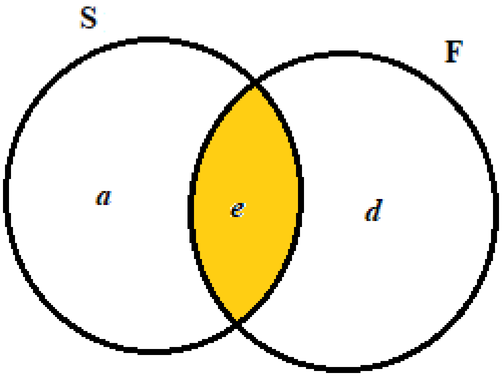
If we consider sets M and F, there is no common element between them. Hence, M ∩ F = ϕ

Such sets which have no elements in common are called disjoint sets.

- The set of science students is S = {a, e} and the set of English students is
  E = {b, c, d}

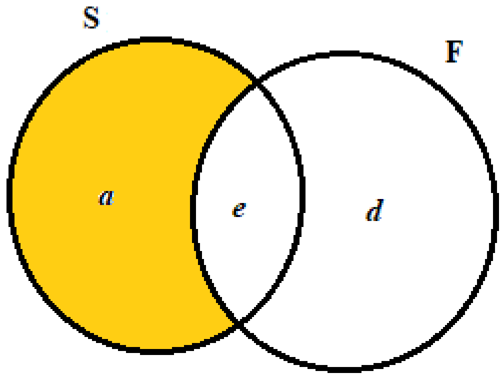


- If we wish to find out all female students who have taken science, we need to find out what is common in set F and set S. This is called an intersection of set F and set S and is denoted by F ∩ S. Here, F ∩ S = {e}



Thus, an intersection of two sets is formed by the elements which are common to both sets.

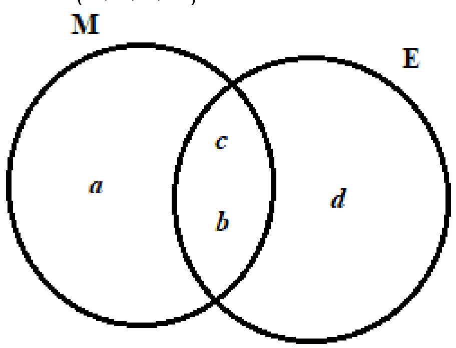
- If we wish to find out those females who have not taken science. Here, we have to check the set F and remove all elements of set S present in this set. This is called the difference between two sets. S - F = {a}



Thus, difference of set A and set B is defined as the set of all elements present in A but not in B.

A - B = {xΩ | x (A and x ∉ B}

- If we wish to represent a set containing" either males or English students or both". This would mean taking all the elements from set M and set E together into one set. This is called the union of set M and set E and is denoted by $M \cup E$. Thus, $M \cup E = \{a,b,c,d\}$



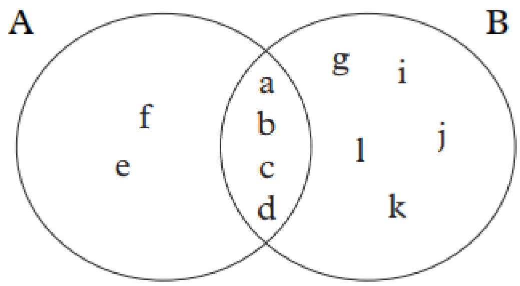
Note:

Though b and c exist in both sets E and M , they are written only once while writing the union. This is because no element is ever written twice while writing a set.

2. A Venn diagram plays a very important role in analysing the set problem and helps in solving the problem very easily. To well perform the task of solving problems using Venn diagram, we first express the data in terms of set notations and then fill the data in the Venn diagram for easy solution. Some important facts like "intersection", "union" and "complement" should be well considered and represented when drawing Venn diagrams.

Example: Consider two intersecting sets A and B such that $A = \{a,\ b,\ c,\ d,\ e,\ f\}$ and $B = \{a,\ b,\ c,\ d,\ g,\ i,\ j,\ k,\ l\}$.

We represent the two sets in a set diagram as follow.



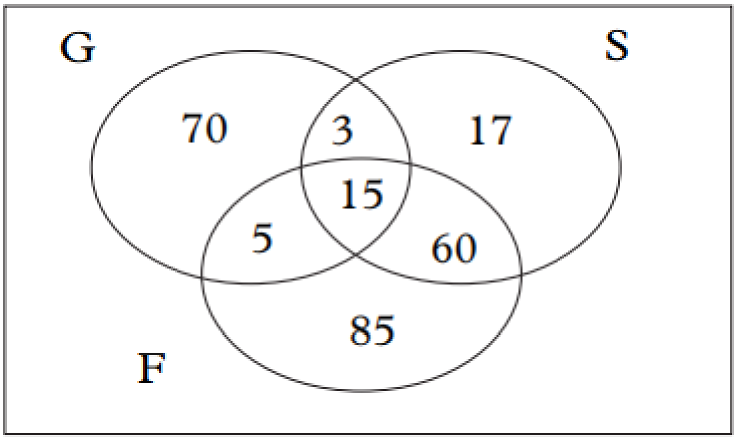
The number of elements in the union of sets A and B is given by:

$n\ (A \cup B)\ =\ n(A)\ +\ n(B)\ -\ n(A\ \cap\ B)$. In the Venn diagram above,

$n(A) = 6,\ n(B)$=9 and $n(A\ \cap\ B) = 4b \Rightarrow n(A \cup B) = 6 + 9 - 4 = 11$

A Venn diagram makes the problem easier because we can represent the data extracted in each region and then calculate the values required.

Example: Consider the Venn diagram showing the numbers of students who take the foreign languages German (G), Spanish(S) and French(F) in a college.



The total number of students taking languages is given by the union of the three sets as shown by the following formula.

$$n(G \cup S \cup F) = n(G) + n(S) + n(F) - \{n(G \cap S) + n(G \cap F) + (S \cap F) + n(G \cap S \cap F)\}$$

From the Venn diagram above, it is clear that:

$$n(G \cup S \cup F) = 93 + 95 + 165 - (18 + 20 + 20 + 75) + 15 = 353 - 113 + 15 = 255$$

Application activity 2.3

Let U be the universal set containing all the natural numbers between 0 and 11. Hence, $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$

Let P be the set containing all prime numbers between 0 and 11.

Thus, P= { 2,3,5, 7}

Let E be the set containing all the even numbers between 0 and 11. Hence, E={ 2, 4, 6, 8, 10}.

a. Find $P \cup E$ and $P \cap E$

b. Represent the above situations using a Venn diagram

2.4. Modelling and solving problems involve Set operations using Venn diagram

Activity 2.4

A survey was carried out in Kigali. 50 people were asked about their preferred hotel for taking lunch among Hilltop, Serena and Lemigo hotels. It was found out that 15 people ate at Hilltop, 30 people ate at Serena, 19 people ate at Lemigo, 8 people ate at Hilltop and Serena, 12 people ate at Hilltop and Lemigo, 7 people ate at Serena and Lemigo. 5 people ate at Hilltop, Serena, and Lemigo.

a. Model the problem using variables and represent the information on a Venn diagram.

b. How many people ate at Hilltop?

c. How many ate at Hilltop and Serena but not at Lemigo?

d. How many people did not eat from any of these three hotels?

CONTENT SUMMARY

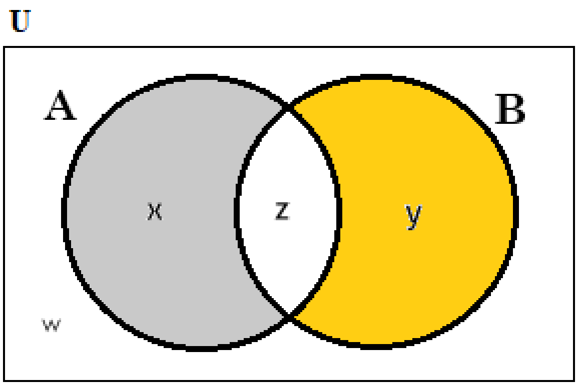
Pictorial representations of sets represented by closed figures are called set diagrams or Venn diagrams and they are used to illustrate various operations like union and intersection.

A Mathematician John Venn introduced the concept of representing the sets pictorially by means of closed geometrical figures called Venn diagrams. In Venn diagrams, the Universal Set U is represented by a rectangle and all other sets under consideration by circles within the rectangle. Venn diagrams are useful in solving simple logical problems.

A Venn Diagram is an illustration that shows logical relationships between two or more sets (grouping items). Venn diagram uses circles (both overlapping and non-overlapping) or other shapes. Commonly, Venn diagrams show how given items are similar and different.

Despite Venn diagram with 2 or 3 circles are the most common type, there are also many diagrams with a larger number of circles $(5,6,7,8,10\ldots)$ theoretically, they can have unlimited circles.

Venn Diagram in case of two elements



Where;

x is the number of elements that belong to set A only

y is the number of elements that belong to set B only

z is the number of elements that belong to set A and B both (AÇB)
w is the number of elements that belong to none of the sets A or B
From the above figure, it is clear that

$n(A) = x + z$

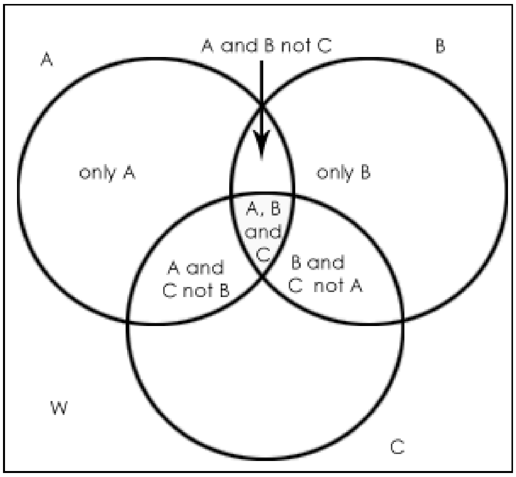
$n(B) = y + z$

$n(A \cap B) = z$

$n(A \cup B) = x + y + z$

Total number of elements is $x + y + z + w$

Venn Diagram in case of three elements

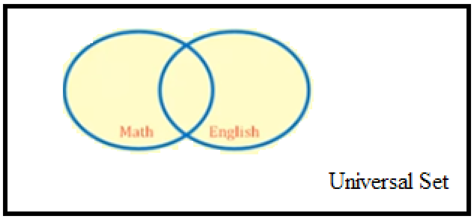


Where *w* is the number of elements that belong to none of the sets A, B or C

Tip: Always start filling values in the Venn diagram from the innermost value or intersection part.

Example:

150 TTC student-teachers were interviewed. 85 were were registered for a Math class, 70 were were registered for an English class while 50 were registered for both Math and English.



Model this problem using variables and Venn diagram and find out the following:

- How many student-teachers signed up only for Math class?
- How many student-teachers signed up only for English class?
- How many student-teachers signed up for Math or English?
- How many student-teachers signed up for neither Math or English?

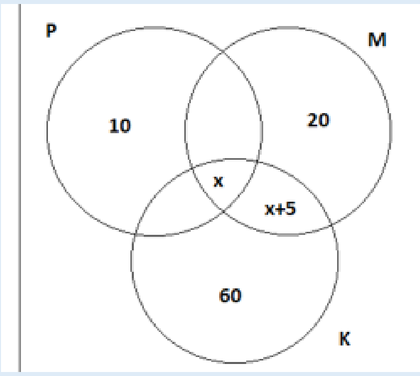
Solution

- Let x be the number of student- teachers who signed up for both Math and English.
- The number of student- teachers who signed up only for Math class is 85 – x. Knowing that x = 50, student- teachers who signed up only for Math is 35.
- The number of student- teachers who signed up only for English class is 70 – x. Knowing that x = 50, student- teachers who signed up only for English is 20.
- The number of student- teachers who signed up for Math or English is given by the total number of all student teachers in both sets. This is 35 +50 + 20 = 105
- The number of student- teachers who signed up for neither Math or English is given by the total number of all TTC student- teachers who were interviewed minus the total number of all student teachers in both sets.

  This is 150 - 105 = 45

Application activity 2.4

The Venn diagram below shows the number of year two student-teachers in SME who like Mathematics (M), Physics (P) and Kinyarwanda (K). Some like more than one subject in total 55 students like Mathematics.



a. How many student- teachers who like the three subjects?

b. Find the total number of year two student- teachers in SME.

c. How many student-teachers who like Physics and Kinyarwanda only?

2.5. END UNIT ASSESSMENT 2

1. In a class, 15 students play Volleyball, 11play Basketball, 6 play both games and everyone plays at least one of the games. Find the total number of students in the class.

2. Out of 17 teachers in a school, 10 teach Economics and 9 teach Mathematics. The number of teachers who teach both subjects is twice that of those who teach none of the subjects. With the aid of a Venn diagram, find the number that teach:

   (a) Both subjects

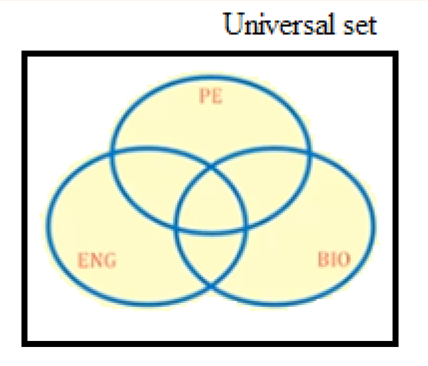
   (b) None of the subject

   (c) Only one subject

3. In a class the students are required to take part in at least two sports chosen from football, gymnastics and tennis. 9 students play football and gymnastics; 19 play football and tennis; 6 play all the three sports. If there were 30 students in the class, draw a Venn diagram to show this information. With the help of a Venn diagram, find out how many students did not participate in any of the sports.

4. At a certain school, 100 students were interviewed about the subject they like. 28 students took Physical Education (PE), 31 took Biology (BIO), 42 took English (ENG), 9 took PE and BIO, 10 took PE and ENG, 6 took BIO and ENG, while 4 students took all three subjects.

   - Represent the situation in the following Venn diagram.



   - Model the problem using variables and find out the following:

     a. How many students took none of the three subjects?

     b. How many students took PE, but not BIO or ENG?

     c. How many students took BIO and PE but not ENG?

UNIT: 3

PROBLEMS ON RATIOS AND PROPORTIONS

Key Unit competence: Apply ratios, proportions and multiplier proportion change to solve real life related problems

3.0. Introductory Activity 3

In daily life people compare quantities, share proportionally things or objects for different reasons. Do you ever wonder why ratio and proportion are necessary in daily life situations?

For example, look at your classmates:

a. Find out the number of boys, then the number of girls. Can you express the number of boys or girls in terms of fraction or ratio?

b. Can you equally share a certain number of Mathematics textbooks to different groups in your classroom and then figure out the ratio of Mathematics textbooks per learner?

c. Give examples where the concept of ratio and proportion are used in daily life situations

3.1 Equal and unequal share, Ratio, Direct and indirect proportions.

3.1.1 Equal and unequal share, Ratio and proportion

Activity 3.1.1

1. Suppose that two brothers from your village received 7 000 Frw from their child who lives in the city. The condition to share this amount of money is that for every 2 Frw that the young brother gets, the other one gets 3 Frw. The two brothers have come to you for help after they disagreed on how to share such money.

(i) In what ratio would you share the money between them?

(ii) Tell your partner how you would share the money and how much each would get.

2. Read carefully the following word problem and express the given data into fraction or ratio.

  a. John and Lucy partnered to save money for x time and later buy a taxi. For every 800 Frw that John saved, Lucy saved 120 Frw. In what fraction or ratio were their contributions? What other simplest fraction or ratio is same as this?

  b. Jane and David sold milk to a vendor in the morning. Jane sold 4 500 ml while David sold 7.5 litres. In what fraction or ratio are their milk sales?

CONTENT SUMMARY

- **Equal and unequal share**

There are many cases in real life where people or organizations need to share items or resources equally or unequally. For example, a father may want to share 24 acre of land among his two sons. One of them who is disabled gets double of what the other son gets. In such case, the land is unequally shared.

Assume that the whole land is first subdivided into equal parts (3 parts). The disabled son gets 2 parts out of 3 parts of the whole.

i.e $\frac{2}{3}$ *of* 24 acres $= \frac{2}{3} \times 24$ acres $= 16$ acres

The other son gets 1 part of 3 parts of the whole.

i.e. $\frac{1}{3}$ *of* 24 acres $= \frac{1}{3} \times 24$ acres $= 8$ acres. It is clearly observed that the two proportions add up to the whole $16 + 8 = 24$ acres.

Also, in every day situations, people should solve problems involving unequal sharing using knowledge of fractions.

Examples:

1. A book has 120 pages divided into 4 equal units. Aloys has read $\frac{3}{4}$ of the book units, and Angelius has read 2 units. How much more has Aloys read?

Solution

To find the number of units read by Aloys , we calculate a fraction of a whole number. This is $\frac{3}{4}$ of 4 units which is 3units. It is clear that Aloys read 1 more unit than Angelius.

2. A book has 120 pages diveded into 8 equal units. Aloys has read $\frac{3}{8}$ of the book, and Angelius has read 45 pages. Explain why Aloys and Angelius have read the same amount of the book units?

Solution

- Through calculations, 120 pages divided into 8 equal units means that every unit has 15 pages
- $\frac{3}{8}$ of 8 equal units is equivalent to 3 units of 45 pages
- $\frac{3}{8}$ of the book is equivalent to 45 pages . it is clear that Aloys and Angelius have read both 45 pages of the book

Ratios:

The mathematical term '**ratio**' defines the relationship between two numbers of the same kind. The relationship between these numbers is expressed in the form "**a to b**" or more commonly in the form $a : b$

A ratio is used to represent how much of one object or value there is in relation to another object or value.

Example: If there are 10 apples and 5 oranges in a bowl, then the ratio of apples to oranges would be 10 to 5 or 10: 5. This is equivalent to 2:1. In contrast, the ratio of oranges to apples would be 1:2.

Ratios occur in many situations such as in business where people compare profit to loss, in sports where compare wins to losses etc.

A comparison of two or more numbers is a **ratio.** You can write a 2-term ratio in fractional form when the second term is non-zero. Thus $a:b=\frac{a}{b}$, *where* $b \neq 0$ and we read *a is to b*.

To solve problems about ratios you need to know the following:

- The order of the terms of ratio is very important. If the order of the terms is changed, then the meaning of the ratio also changes.
- A ratio is in the lowest terms or simplest form if the terms of the ratio have no integral common factor. For example to write $4:8:12$ in lowest terms, you divide each term by the greatest common factor of the terms.

  Therefore, $4:8:12=1:2:3$ *it means that* $\frac{4}{4}:\frac{8}{4}:\frac{12}{4}$
- When you equate two ratios you are writing a **proportion.** A proportion is a relationship between four numbers or quantities in which the ratio of the first pair equals the ratio of the second pair, and is written as $a:b = c:d$ and is read as "**a is to b while c is to d**"

  Therefore $a:b=c:d$ *and* $4:8:12=1:2:3$ are proportions
- The proportion $a:b=c:d$ is written as $\frac{a}{b}=\frac{c}{d}$ where *a and d* are known as **extremes** of the proportion, while *b and c* are known as **means**
- The product of the extremes equals the product of the means.

  If $\frac{a}{b}=\frac{c}{d}$, $b \neq 0$ *and* $d \neq 0$, $a \times d = b \times c$

Example:

(a) Write $(x^2-y^2):(ax+ay)$ in lowest terms;

(b) Find the mean proportional between 3 *and* 75

Solution:

(a) Factor the terms of the ratio: $(x^2-y^2):(ax+ay)=(x-y)(x+y)=a(x+y)$

$$\Rightarrow \frac{(x-y)(x+y)}{x+y} : \frac{a(x+y)}{(x+y)}$$

$$(x-y):a \quad where \quad (x+y) \neq 0$$

(b) let x represent the mean proportion between 3 and 75, then $\frac{3}{x} = \frac{x}{75} \Rightarrow x^2 = 225$

$\therefore x = \pm 15$

So the mean proportional between 3 *and* 75 *are* $+15$ *or* -15

Note :

To share a quantity into two parts in the ratio $a:b$, the quantity is split into $a+b$ equal parts. The required parts become $\frac{a}{a+b}$ and $\frac{b}{a+b}$ of the quantity

Example

Share 38 400 Frw between Linda and Jean in the ratio 5:7 respectively.

Solution

38 400 Frw is to be shared in the ratio 5:7. It is split into 12 equal parts i.e 5 + 7 = 12 equal parts.

The amount of money Linda receives is $\frac{5}{12} \times 38\ 400\ Frw = 16\ 000\ Frw$

The amount of money Jean receives is $\frac{7}{12} \times 38\ 400\ Frw = 22\ 400\ Frw$

Application activity 3.1.1

Ingabire, Mugenzi and Gahima have jointly invested in buying and selling of shares in the Rwanda stock exchange market. In one sale as they invested different amount of money, they realised a gain of 1 080 000 Frw and intend to uniquely share it in the ratio $2:3:4$ respectively. How much did Mugenzi get?

3.1.2 Direct and indirect proportions

Activity 3.1.2

1. Observe the table below that represents the relationship between the number of pens and their costs

No of pens	1	2	3	4	5
Cost (FRW)	120	240	360	480	600

(i) Draw the graph of the number of pens (N) against cost (C).

(ii) Describe the graph you have drawn in (i) above.

2. Consider the relationship between the speed and time taken by a car to cover a fixed distance of 320 km.

Speed (km/h)	20	40	80	160
Time (h)	16	8	4	2

Take 20 km/h to be the original speed.

(i) What do you notice when the speed is doubled?
(ii) Plot the graph of speed against time.
(iii) Describe the graph you drew to your classmates.

CONTENT SUMMARY

- **Direct proportion**

The mathematical term '**proportional**' describes two quantities which always have the same relative size or 'ratio'.

Example: An object weighs 2kg on the 1st day. If weight is proportional to age, then the object will weigh 4kg on the 2nd day, 6kg on the 3rd day, 8kg on the 4th day, 10kg on the 5th day and so on.

There are many phenomena in real life that involve a squared relationship.

For example:

- The area (A) of a circle varies directly with the square of the radius r
- The value (V) of a diamond varies directly with the square of its mass (M) make the following comparison:

Direct variation	Direct square variation
$y \propto x$ or $y \sim x$	$y \propto x^2$ or $y \sim x^2$
$y=kx$, where k is constant	$y = kx^2$, where k is constant
y varies directly with x	y varies directly with the square of x
y is directly proportional to x	y is directly proportional to the square of x
$\frac{y_1}{x_1} = \frac{y_2}{x_2} = k$	$\frac{y_1}{x^2_1} = \frac{y_2}{x^2_2} = k$

Example:

The volume of water rushing from a hose on a unit of time varies directly with the square of the diameter of that hose (if water pressure is constant). From a fire hose, with a diameter of 8cm, 400kl of water were obtained. For the same amount of time, how much water could be obtained from a garden hose 2cm in diameter?

Solution: let V be the volume of water in kl, obtained in a unit of time. Let d be the diameter of that hose in cm; then $V = kd^2 \Leftrightarrow 400 = 8^2 k \Leftrightarrow k = \frac{400}{64} = \frac{25}{4}$

Thus, the variation equation is given by $V = \frac{25}{4} d^2$

As d=2cm; the volume $V = \frac{25}{4}(2)^2 \Leftrightarrow V = 25Kl$

Thus, the volume of water obtained from the garden hose is $25\ Kl$

- **Indirect proportion**

Consider this table:

Speed (v) in km per hour	Time (t) in hours
16	1
8	2
4	4
1	16

From this table we see that:

- If the speed v is increased, the time taken to travel 16km is decreased.
- If the speed is decreased, the time taken to travel 16km is increased

The formula, $vt = 16$ can be written as $v = \frac{16}{t}$ and you can write $v \propto \frac{1}{t}$

This is read as: "v **varies inversely with time** or v **varies indirectly proportional with t"**

In general, an inverse proportional is given as:

$xy = k \ or \ y = \frac{k}{x}$ where k is the constant of variation, $k \neq 0$

If (x_1, y_1) *and* (x_2, y_2) satisfy the inverse variation, then you may obtain a proportion as follows: $x_1 y_1 = k \quad and \quad x_2 y_2 = k$

Thus, $x_1 y_1 = x_2 y_2 \quad or \quad \frac{x_1}{x_2} = \frac{y_2}{y_1}$

Example: if s varies inversely or indirectly proportional with t and s=30 when t=25, then find the value of s when t is 150

Solution: write $s = \frac{k}{t}$ where k is constant

When

$s = 30, t = 25$

$s = \frac{k}{25} \Leftrightarrow 30 = \frac{k}{25} \Leftrightarrow k = 750$

The variation equation is

$s = \frac{750}{t}, t = 150; \quad s = \frac{750}{150} = 5$

Therefore, $s = 5$ when $t = 150$

Application activity 3.1.2

1. If b is directly proportional to c^2, then find the constant of variation if $b = 72$ when $c = 12$

2. Calculate the constant of variation for the following inverse variation:

   (a) m varies inversely with n where m=25 when n=2

   (b) p is inversely proportional to q where q=16 when p=4

3.2 Calculation of proportional and compound proportional change

Activity 3.2

1. Discuss with your classmate what you understand by the word multiplier.
   - Consider a shirt that is sold at a 20% discount.
   - What is the percentage of the selling price?
   - Convert this percentage you have gotten into fraction. What do you notice?
2. Consider a shirt with a marked price of 500 Frw. After negotiating with the customer, the shirt is sold at a 10% lower. Discuss with your classmate the change in price and the new price (selling price) of the shirt in Frw.
3. Consider that 3 people working at the same rate can cultivate 2 acres of land in 3 days. What do you think will happen if the working days are increased to five and people are still working at the same rate? Discuss.

CONTENT SUMMARY

Proportional change using multiplier

- Consider the price of a book being reduced by 15%, the percentage of the selling price is $100\% - 15\% = 85\%$
- 85% converted to decimal gives $= \frac{85}{100} = 0.85$
- We say that 0.85 is **the multiplier** of the price of the book.

Examples:

1. What is the multiplier for 15% increase?

Solution

A 15% increase means the final percentage for the quantity will be

$100\% + 15\% = 115\%$

115% as a decimal $= \frac{115}{100} = 1.15$

So 1.15 is the multiplier.

2. Increase 200 kg by 8%

Solution

8% means the overall percentage will be 100% + 8% = 108%

108% in decimals $= \frac{108}{100} = 1.08$

Multiplier $= 1.08$

New value = 1.08 × 200 kg = 216 kg

Note that the proportion of a given quantity can be reduced or increased:

i) **A decreasing multiplier** is a factor that reduces the proportion of a given quantity. To calculate the new price, we proceed as

$$New\ price = initial\ price \times multiplier, \text{ where,}$$

$multiplier = \frac{(100-x)}{100}$ and x is the percentage decrease on the cost price.

ii) **An increasing multiplier** is a factor that increases the proportion of a given quantity.

To calculate the new price, we proceed as

$New\ price = initial\ price \times multiplier$, where $multiplier = \frac{(100+x)}{100}$ and x is the percentage increase on the cost price.

Example

1. For purposes of sales promotion, the price of a book has been reduced by 20% to 3 600 Frw. What was the price before the reduction?

Solution

We are required to use the reverse process of decreasing the price by the given percentage.

Let the old price by y

Then $the\ new\ price = (100 - 20)\%\ of\ y$

$$= 80\%\ of\ y,\ 80\%\ of\ y = 3\ 600\ Frw$$

$$\frac{80}{100}y = 3\ 600\ Frw \Leftrightarrow y = \frac{3600 \times 100}{80} \Leftrightarrow y = 4500 Frw$$

2. A farmer gets 80 litres of milk from his cow. The amount of milk from the cow reduced by 5% after illness. What is the new amount of milk produced by the cow?

Solution

Initial amount = 80 *litres*

Percentage decrease = 5%

Amount decreased = 5% *of* 80 *litres*

$\frac{5}{100} \times 80 = 4$ *litres* . New amount = $(80 - 4)$ *litres* $= 76$ *litres*

Compound proportional change or continued proportions.

Sometimes, a quantity may be proportional to two or more other quantities. In such case, the quantities are said to be in **compound proportion** or continued proportions.

The mathematical term 'iteration' means to repeat an operation. Iterations involve inputting a value into a function, receiving an output value, and then using this output value as the input value for the function.

Example: To iterate the function 2x + 3, first you place any value of x into the function:

So if x = 1, then 2x + 3 = 5

Now you must take this output value of 5 and put it into the original function.

Therefore, when x = 5 , then 2x + 3 = 13

By continuing these iterations, you will be presented with an infinite sequence of numbers: 5, 13, 29, 61, ...

Repeated proportional change is an extremely useful mathematical process because it can be used to calculate real world financial problems such as compound interest.

'**Compound interest**' refers to the interest added to a deposit or loan. The added interest will also earn interest as time passes. To calculate the compound interest of a loan, we may use repeated proportional change.

Example

(a) £500 is borrowed for 6 years at 5 % compound interest. Calculate the amount of compound interest which will be paid

Solution

(a) From the question, we know that 5% compound interest is added each year. This means that there will be 105% of the original amount borrowed at the end of the first year or 1.05.

With 1.05 as your multiplier, you can calculate the total amount of money borrowed after 6 years. The money was borrowed for 6 years, so you must raise 1.05 to the power of 6. Therefore:

Total amount of money borrowed is 500 x $(1.05)^6$ = 670.047= £670

The question has asked you to calculate the amount of compound interest. To do so, you must subtract the original amount borrowed (£500) from the value you have just generated:

670 - 500 = 170. As a result, the amount of compound interest which will be paid is £170

To calculate compound proportionality problems, one can use the unitary method or compound rule of three

Example: 9 men working in a factory produce a certain number of pans in 6 working days. How long will it take 12 men to produce the same number of pans if they work at the same rate?

Solution

- 9 men work for 6 days
- 1 man works for 9 × 6 days = 54 complete working days
- 12 men will work for $\frac{9\times 6}{12}=4\frac{1}{2}$ working days.

The proportion involving two or more quantities is called Compound Proportion.

The following table is showing different quantities. Let us investigate different rules to solve compound proportions problem.

Quantity 1	Quantity 2	Quantity 3
A	B	C
D	E	x

CASE1

If quantity 1 and quantity 2 are directly related and quantity 2 and quantity 3 are also directly related, then we use the following rule:

$$\frac{a\times b}{c}=\frac{d\times e}{x}$$

CASE 2

If quantity 1 and quantity 2 are directly related and quantity 2 and quantity 3 are inversely related, then we use the following rule:

$$\frac{b\times c}{a}=\frac{e\times x}{d}$$

CASE3

If quantity 1 and quantity 2 are inversely related and quantity 2 and quantity 3 are directly related, then we use the following rule:

$$\frac{a\times b}{c}=\frac{d\times e}{x}$$

CASE 4

If quantity 1 and quantity 2 are inversely related and quantity 2 and quantity 3 are also inversely related, then we use the following rule: $a\times b\times c=d\times e\times x$

Example1:

195 men working 10 hours a day can finish a job in 20 days. How many men are employed to finish the job in 15 days if they work 13 hours a day?

Solution:

Let x be the number of men required

Days	Hours	Men
20	10	195
15	13	x

Analysing the problem, one can find that the number of days and hours required for one person to finish the same job are many than the one used by 195 men.

$$1day\rightarrow 10hours\rightarrow 195\times 20$$

$$1day\rightarrow 1hour\rightarrow 195\times 20\times 10$$

$$15days\rightarrow 1hour\rightarrow \frac{195\times 20\times 10}{15}$$

$$15days\rightarrow 13hours\rightarrow \frac{195\times 20\times 10}{15\times 13}=200$$

or

$$a\times b\times c=d\times e\times x$$

$$20\times 10\times 195=15\times 13\times x$$

$$39000=195x\Leftrightarrow x=\frac{39000}{195}=200$$

Application activity 3.2

1. What is the multiplier of 45% decrease?
2. Deborah's salary last year was 15 000 Frw. This year it was increased by 20%. What is her salary this year?
3. In 2004 a company processed 800 tonnes of maize. In 2005, the company decreased production by 30 %. How many tonnes did the company process in 2005?
4. Four men working at the same rate can dig a piece of land in ten days. How long would it take five men to do the same job?

3.3 Problems involving direct and indirect proportions

Activity 3.3

Read carefully the given problems and discuss how can you solve problems involving direct and indirect proportions.

1. F is directly proportional to x. When F is 6, x is 4. Find the value of F when x is 5.
2. A is directly proportional to the square of B. When A is 10, B is 2. Find the value of A when B is 3.
3. A is inversely proportional to B. When A is 10, B is 2. Find the value of A when B is 8
4. F is inversely proportional to the square of x. When A is 20, B is 3. Find the value of F when x is 5.

CONTENT SUMMARY

Two values x and y are **directly proportional** to each other when the ratio x : y or $x \propto y$ is a constant (i.e. always remains the same). This would mean that x and y will either increase together or decrease together by an amount that would not change the ratio.

Knowing that the ratio does not change allows you to form an equation to find the value of an unknown variable.

Example:

If two pencils cost $1.50, how many pencils can you buy with $9.00?

Solution:

The number of pencils (p) is directly proportional to the cost.

$$\frac{2}{1.5}=\frac{p}{9}\Rightarrow 1.5p=18\Rightarrow p=12 \text{ pencils}$$

Two values x and y are **inversely proportional** to each other when their product xy is a constant (always remains the same). This means that when x increases y will decrease, and vice versa, by an amount such that xy remains the same.

Knowing that the product does not change also allows you to form an equation to find the value of an unknown variable

Example:

It takes 4 men 6 hours to repair a road. How long will it take 8 men to do the job if they work at the same rate?

Solution:

The number of men is inversely proportional to the time taken to do the job. Let t be the time taken for the 8 men to finish the job

$$4\times 6=8\times t\Rightarrow 24=8t\Rightarrow t=3\, hours$$

Usually, you will be able to decide from the question whether the values are directly proportional or inversely proportional.

Problem solving plan

1. Understand the problem: Think what information is given and what information is required
2. Decide on a strategy: List the strategies with which you think the solution can be found
3. Apply the strategy: Find the solution using the strategy you have chosen
4. Look back:
    - Have you verified your solution?
    - Are there other solutions?
    - Can you solve a simpler problem?
    - Have you answered the question as it was initially stated?

Example

1. The voltage V(volts) across an electrical circuit is directly proportional to the current I (Amperes) flowing through the circuit. When I=1.2 A, V=78V

   a) Express V in terms of I

   b) Find V when I=2A

   c) Find I when V=162.5V

Solution

I=1.2 A, V=78V

$V \propto I$

$V = kI$

$78 = (1.2)k \Rightarrow k = \frac{78}{1.2} = 65$

$V = 65I$

a) $V = 65 \times 2 = 130V$

b) $162.5 = 65I \Rightarrow I = \frac{162.5}{65} = 2.5\,\text{A}$

Application activity 3.3

1. The area $A(cm^2)$ of a square is directly proportional to the square of its perimeter $P(cm)$. When $P = 8$, $A = 4$ .Find a formula for A in terms of P and P in terms of A

2. When a fixed volume of water is poured into a cylindrical jar, the depth $D(cm)$ of the water is indirectly proportional to the cross section area $A(cm^2)$ of that cylindrical jar. When A=14, D=120

   a) Find the formula for A in term of D

   b) Find A when D=150

   c) Find D when A=60

3.4. END UNIT ASSESSMENT 3

1. The force of attraction $F(Newtons)$ between two sphere is indirectly proportional to the square of the distance $D(metres)$ between the centres of spheres. When $D=2$, $F=0.006$

   a) Express F in terms of D

   b) Find F when $D=2.5$

   c) Find D when F=0.001 and give your answer correct to 3 significant figures

2. Find the missing(unknown) terms of each proportion:

   a) $2:5=4:x$

   b) $8:4=20:4x$

   c) $4:3x=45:63$

3. Express each of the following in lowest terms:

   a) $-16xy:32\text{y}$

   b) $(8k-2k):2k$

UNIT: 4

PROPOSITIONAL AND PREDICATE LOGIC

Key Unit competence: Use Mathematical logic as a tool of reasoning and argumentation in daily situation

4.0. Introductory Activity

1. Discuss whether the following logical arguments are valid or not valid. Justify your answer
   - If you give a child an orange and another child an orange. Children got an orange.
   - Kigali is in Rwanda and $2^n > 0,\ n \in \mathbb{Z}$
   - If you do not attend class, then either you read a book or you will not pass the exam.
   - All dogs are meat eaters, Cat is a meat eater, therefore, cat is a dog.
   - Some investors are wealthy. All wealthy people are happy, therefore, some investors are happy.
2. Complete the argument by adding a conclusion that makes the argument valid.
   - Some rules are unfair. All unfair rules should be eliminated, therefore, . . .
   - No team which plays in X- stadium has ever won the Super cup. Some teams that wear red uniforms have won the Super cup, therefore, . . .

4.1 Definition

4.1.1 Simple statement and compound statements

Activity 4.1.1

From the following expression, give your answer by true or false

1. Every integer larger than 1 is positive
2. Kampala is in Rwanda
3. How old are you?
4. Every liquid is water.
5. Write down the names of Rwandan president.
6. $1-x^2=0\cdot$
7. Rwanda is African country or Rwanda is a member of Commonwealth.

CONTENT SUMMARY

A sentence which is either true or false but not both simultaneously is named **statement or proposition**. In the context of logic, a proposition or a statement is the sentence in the grammatical sense conveying a situation which is neither imperative, interrogative nor exclamatory.

The expressions 1, 2, 4 and 7 are statements: the 2nd and 4th are false while 1st and 7th are true.

- The expression "How old are you?" is not a proposition since you cannot reply by true nor false (grammatically this sentence is interrogative).
- The equality "$1-x^2=0$" is not a statement because for some values of x the equality is true, whereas for others is false.
- The expression "Write down the names of Rwandan president" is not a proposition as the answer will be given by neither true nor false.

A statement that cannot be broken into two or more sentences is called **simple statement**. Combining two or more simple statements we form a **compound statement.**

In activity 4.1.1, the 1st, 2nd and 4th expressions are simple statements while the 7th is a compound statement.

In this unit, statements will be denoted by small letters such as ***p*, *q*, *r*, ...**

The logical statements are required to have a definite truth-value, or, to be either true or false, but never both, and to always have the same truth value. The two truth values of proposition are **true** and **false** and are denoted by the symbols **T** and **F** respectively. Occasionally they are also denoted by the symbols **1** and **0** respectively.

Application activity 4.1.1

1. Find out which of the following sentences are statements and which are not. Justify your answer.
   a) Uganda is a member of East African Community.
   b) The sun is shining.
   c) Come to class!
   d) The sum of two prime numbers is even.
   e) It is not true that China is in Europe.
   f) May God bless you!
2. Write down the truth value (T or F) of the following statements
   a) Paris is in Italy.
   b) 13 is a prime number.
   c) Kigeri IV Rwabugiri was the king of the Kingdom of Rwanda
   d) Lesotho is a state of South Africa.
   e) Tanzania is in east of Rwanda and is in SADC (Southern African Development Community)

4.1.2 Truth tables

Activity 4.1.2

Suppose we are given two simple statements, named p and q to get a compound statement $C(p,q)$.

a) What are the possibilities for the truth-values of p and of q?

b) Using a table,
   i) How many possibilities are there, for their pairs of truth-values?
   ii) How many possibilities are there, for the triples of truth-values of three statements p, q and r?

The way we will define compound statements is to list all the possible combinations of the truth-values (abbreviated T and F) of the simple statements (that are being combined into a compound statement) in a table, called a **truth table**. The name of each statement is at the top of a column of the table. The truth values used to define the truth-value of the compound statement appear in the last column.

If the compound statement contains n distinct simple statements, we will consider 2^n possible combinations of truth values in order to obtain the truth table.

Example 1

For one proposition p ($2^1 = 2$ possible combinations), the truth table is

p
T
F

Example 2

For two propositions p and q ($2^2 = 4$ possible combinations), the truth table:

p	q	$C(p,q)$
T	T	TT
T	F	TF
F	T	FT
F	F	FF

The truth values incorporated in 3rd column will then be replaced by the defined truth-value of the statement signified by compounded statement $C(p,q)$ as a statement must be true or false.

Application activity 4.1.2

Write down the truth table for three propositions p, q and r

4.2 Logical connectives

4.2.1 Negation

Activity 4.2.1

Let *p*, *q*, *r*, *s* be the propositions

1. It is raining
2. Uganda is African country
3. London is in France
4. Kagabo reads Newsweek

Give a verbal sentence which describes the opposite proposition

CONTENT SUMMARY

The negation of a statement by introducing the word "not" denoted by prefixing the statement has opposite truth value from the statement. It is denoted by $\neg p$ or $\overline{p}$ or $\sim p$.

From this definition, it follows that the negation of a true statement is false while the negation of false statement is true; simply

if p is true, then $\neg p$ is false and if p is false, then $\neg p$ true.

Example 1: Let *p:* " Haruna is a good player", *then* $\neg p$: " Haruna is not a good player".

Example 2: Let p be a proposition. Construct the truth table of $\neg p$

Solution

p	$\neg p$
T	F
F	T

Application activity 4.2.1

1. Write the negation of each of the following statements.
   a) *Today is raining.*
   b) *Sky is blue*
   c) *My native country is Rwanda.*
   d) *Benimana is smart and healthy.*
2. Complete the following truth table

p	q	r	$\neg p$	$\neg q$	$\neg r$
T		T		F	
	T	F	F		
T	F	T			
T	F				T
	T	T	T		
F	T	F			
F				T	F
F	F	F			

4.2.2 Conjunction

Activity 4.2.2

Determine the truth value of each of the following statements

1. Paris is in France **and** $4+4=8$
2. $4+4=9$ **and** $5+8=11$
3. Paris is in England **and** $3+4=7$
4. Kigali is the capital of Burundi **and** $1+1=2$
5. The French revolution started in 1789 **and** ended in 1799
6. CO_2 is chemical formula of water **and** H_2O is chemical formula of carbon dioxide
7. m^2 is the unit of area and kg is one of the units of weight.

CONTENT SUMMARY

If two simple statements p and q are connected by the word **AND**, then the resulting compound statement p and q is called a conjunction of p and q and is written in symbolic form $p \wedge q$. It has the truth value **true** whenever both p and q have the truth value **true**; otherwise it has the truth value **false**.

Example 1

Let p be " It is raining today" and q be " There are fifteen chairs in this class room". Give a simple sentence which describes each of the following statements and construct its truth table

a) $p \wedge q$ b) $p \wedge \neg q$

Solution

a) From the given two simple statements, the resulting compound statement is "It is raining today and there are fifteen chairs in this class room".

The Truth table of related compound statement is as follows

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

b) From the given two simple statements, the resulting compound statement is "It is raining today and there are not fifteen chairs in this class room".

The Truth table of related compound statement is as follows

p	q	$\neg q$	$p \wedge \neg q$
T	T	F	F
T	F	T	T
F	T	F	F
F	F	T	F

Example 2

Let p and q be propositions. Construct the truth table for

a) $\neg p \wedge q$

b) $(\neg p \wedge q) \wedge \neg p$

Solution

p	q	$\neg p$	$\neg p \wedge q$	$(\neg p \wedge q) \wedge \neg p$
T	T	F	F	F
T	F	F	F	F
F	T	T	T	T
F	F	T	F	F

Application activity 4.2.2

If p stands for the statement " It is cold" and q stands for the statement " It is raining", then what does $\neg q \wedge \neg q$ *stands for?* Construct its truth table

4.2.3 Disjunction

Activity 4.2.3

Determine the truth value of each of the following statements

1. Paris is in France **or** $4+4=8$
2. The sun is a planet **or** the Jupiter is a star.
3. Paris is in Japan **or** $3+4=7$
4. The first president of United States of America is George Washington **or** was inaugurated in 1879.
5. Nairobi is the capital of Tanzania **or** $1+1=2$
6. The French revolution started in 1789 **or** ended in 1799

7. CO_2 is chemical formula of water **or** H_2O is chemical formula of carbon dioxide

8. m^2 is the unit of area **or** kg is one of the units of weight.

CONTENT SUMMARY

If two simple statements p and q are connected by the word **OR**, then the resulting compound statement p or q is called a **disjunction** of p and q and is written in symbolic form $p \vee q$. It has the truth value **false** only when p and q have truth value **false**, otherwise it has the truth value **true**.

Example 1

Let p be "Paris is in France" and q *be* " London is in England". Give a simple verbal sentence which describes each of the following statements and construct its truth table

a) $p \vee q$ b) $\neg p \vee q$

Solution

a) From the given two simple statements, the resulting compound statement is "Paris is in France or London is in England ".

The Truth table of related compound statement is as follows

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

b) From the given two simple statements, the resulting compound statement is "Paris is not in France or London is in England ".

The Truth table of related compound statement is as follows

p	q	$\neg p$	$\neg p \vee q$
T	T	F	T
T	F	F	F
F	T	T	T
F	F	T	T

Example 2: Construct the truth tables for $\neg(p \vee q)$ and $\neg p \wedge \neg q$ and compare them.

Solution: First, we make a table that displays all the possible combinations of truth-values for p and q, then we add two columns and put $p \vee q$ and $\neg(p \vee q)$ into its top cell and then start calculations.

p	q	$p \vee q$	$\neg(p \vee q)$
T	T	T	F
T	F	T	F
F	T	T	F
F	F	F	T

Application activity 4.2.3

1. Translate each of the following compound statements into symbolic form
   i) *Bwenge reads News Paper or Mathematics book.*
   ii) *Rwema is a student-teacher or not a book seller.*

2. Suppose that p is a false statement, and q is a true statement.
   a) *What is the truth-value of the compound statement* $\neg p \vee q$ *?*
   b) *What is the truth-value of the compound statement* $p \vee \neg q$ *?*
   c) *What is the truth-value of the compound statement* $p \vee q$ *? What is the truth-value of the compound statement* $\neg p \vee \neg q$ *?*

3. Let p and q be two propositions. Construct the truth table of
   a) $p \vee q$ b) $p \vee \neg q$ c) $p \wedge (p \vee \neg q)$

4.2.4 Conditional statement

Activity 4.2.4

Rephrase the following statements without changing the meaning

1. If you buy me a pen, I will go to school
2. If the earth is flat, then mars is flat
3. If you are tall, then you will be a member of our volleyball team
4. If you do not buy these shoes, then I will not go with you
5. If you do not pay school fees, then you will not get you school report

CONTENT SUMMARY

There are statements of the type if "$If\ p\ then\ q$". Such statements are called **Conditional statements** and are denoted by $p \rightarrow q$ or $p \Rightarrow q$ read as $p\ implies\ q$.

The conditional $p \Rightarrow q$ can also be read:

- If p, then q.
- q follows from p.
- q if p.
- p only if q.
- p is sufficient for q.
- q is necessary for p.

The statement $p \Rightarrow q$ has the truth value True in all case except when p is true while q is false.

Example 1

Rephrase the sentence "*If it is Sunday, you go to church*", then construct its related truth table.

Solution

Here are various ways of rephrasing the sentence:

- "You go to church if it is Sunday."

- "It is Sunday only if you go to church."
- "It can't be Sunday unless you go to church."

Let denote the given statements as:

p: It is Sunday   *q*: you go to church.

The statement ***If it is Sunday, you go to church*** is symbolized as follow: $p \Rightarrow q$.

Construction of its related truth table

p	q	$p \Rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

Example 2

Construct the truth table of the following statement:

"If either John takes Calculus or Betty takes Sociology then Peter will take English."

Solution

Let denote the statements as:

p: John takes Calculus,   *q*: Betty takes Sociology and *r*: Peter takes English.

The given statement can be symbolized as follow: $(p \vee q) \Rightarrow r$

Construct the truth table for $(p \vee q) \Rightarrow r$

p	q	r	$p \vee q$	$(p \vee q) \Rightarrow r$
T	T	T	T	T
T	T	F	T	F
T	F	T	T	T
T	F	F	T	F
F	T	T	T	T
F	T	F	T	F
F	F	T	F	T
F	F	F	F	T

Application activity 4.2.4

1. Using the statements p: Mico is fat and p: Mico is happy

   Assuming that "not fat" is thin, write the following statements in symbolic form

   a) If Mico is fat then she is happy

   b) Mico is unhappy implies that Mico is thin

2. Write the following statements in symbolic form and their truth table

   a) If n is prime, then n is odd or n is 2.

   b) If x is nonnegative, then x is positive or x is 0.

   c) If Tom is Ann's father, then Jim is her uncle and Sue is her aunt.

4.2.5 Bi-conditional statements

Activity 4.2.5

1. Let p be the statement: "Abijuru is intelligent"

   and q be the statement: "Abijuru is hard working"

   a) Give the truth value of $p \Rightarrow q$

   b) Give the truth value of $q \Rightarrow p$

   c) Give the truth value of $(p \Rightarrow q) \wedge (q \Rightarrow p)$

2. Let r be the statement: "$7 = 7$" and s be the statement: "$5 = 3$"

   a) Give the truth value of $r \Rightarrow s$

   b) Give the truth value of $s \Rightarrow r$ $s \rightarrow r$

   c) Give the truth value of $(r \Rightarrow s) \wedge (s \Rightarrow r)$.

CONTENT SUMMARY

Generally $p \Rightarrow q$ is not the same as $q \Rightarrow p$. It may happen, however, that both $p \Rightarrow q$ and $q \Rightarrow p$ are true or are false. The statement $p \Leftrightarrow q$ is defined to be the statement $(p \Rightarrow q) \wedge (q \Rightarrow p)$. For this reason, the double headed arrow $p \Leftrightarrow q$ is called the **bi-conditional**.

The **bi-conditional** $p \Leftrightarrow q$, which we read "*p* if and only if *q*" or "*p* is equivalent to *q*" is **true** if both *p* and *q* have the same truth values and **false** if *p* and *q* have opposite truth values.

Example 1

Let denote the statements as:

p: the number is divisible by 3

q: the sum of the digits forming the number is divisible by 3.

The compound statement "*The number is divisible by 3 if and only if the sum of the digits forming the number is divisible by three*"; means that If the sum of the digits forming the number is divisible by 3, then the number is divisible by 3 and if the number is divisible by 3, then the sum of the digits forming the number is divisible by 3. Symbolically $p \Leftrightarrow q$ means $(p \Rightarrow q) \wedge (q \Rightarrow p)$.

Example 2

Construct the truth table for $p \Leftrightarrow q$

Solution

p	q	$p \Leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

Application activity 4.2.5

1. Suppose that r is a false statement, and s is a true statement.

   a) What is the truth-value of the compound statement $(\neg r) \Leftrightarrow s$?

   b) What is the truth-value of the compound statement $r \Leftrightarrow (\neg s)$?

   c) What is the truth-value of the compound statement $r \Leftrightarrow s$?

2. Construct the truth table for

a) $p \leftrightarrow q$ and $(p \rightarrow q) \wedge (q \rightarrow p)$

b) $p \leftrightarrow q$ and $(\neg p \vee q) \wedge (\neg q \vee p)$

c) $\neg(p \leftrightarrow q)$ and $(p \vee q) \wedge \neg(p \wedge q)$

4.2.6 Tautologies and Contradictions

Activity 4.2.6

Let p and q be two propositions. Construct the truth table of

1. $p \vee \neg p$
2. $p \wedge (\neg p)$
3. $\neg p \wedge (p \wedge q)$
4. $\neg(p \wedge q) \vee (p \vee q)$

CONTENT SUMMARY

A tautology is a compound statement that is always **true** regardless of the truth values of the individual statements substituted for its statement variables.

Example

- The statement "**The main idea behind data compression is to compress data**" is a tautology since it is always true.
- The statement **"I will either get paid or not get paid"** is a tautology since it is always true

If you are given a statement and want to determine if it is a tautology, then all you need to do is construct a truth table for the statement and look at the truth values in the final column. If all of the values are T (for true), then the statement is a tautology.

The statement **"I will either get paid or not get paid"** is a tautology since it is always true. We can use p to represent the statement "**I will get paid**" and not p (written $\neg p$) to represent "**I will not get paid**."

- p : I will get paid
- $\neg p$: I will not get paid
- So, $p \vee (\neg p)$: I will either get paid or not get paid
- A truth table for the statement would look like:

p	$\neg p$	$p \vee (\neg p)$
T	F	T
F	T	T

Looking at the final column in the truth table, one can see that all the truth values are T (for true). Whenever all of the truth values in the final column are true, the statement is a tautology. So, our statement 'I will either get paid or not get paid' is always a true statement, a tautology.

A contradiction is a compound statement that is always **false** regardless of the truth values of the individual statements substituted for its statement variables.

Example

- The statement "**I don't believe in reincarnation, but I did in my past life**", is always false.
- The statement $p \wedge (\neg p)$ is always false, because p and $\neg p$ cannot both be true.

Application activity 4.2.6

From the following compound statements, indicate which is tautology, contradiction or neither.

1. $p \wedge \neg(p \wedge q)$
2. $\neg q \wedge (q \wedge r)$

4.3 Quantifiers and their negations: Existential and Universal quantifiers

4.3.1 Predicates

Activity 4.3.1

State whether the following sentences are true, false or neither. Justify your answer

1. That guy is sick
2. He is the president of America
3. A cuboid has 6 faces, 12 edges and 8 vertices.
4. $x+y=5$
5. x^2 is never equal to $3x+5$
6. $\frac{x}{7}$ is an integer
7. If we multiply both the numerator and denominator of a fraction by a given number, then the new fraction is equivalent to the original fraction
8. Except for 1, no cube number is also a square number.

CONTENT SUMMARY

A declarative sentence is an open statement or a predicate if:

- It contains one or more variables, and
- It is not a statement, but It becomes a statement when the variables in it are replaced by certain allowable choices.

A predicate can be denoted by a small letter followed by its variables.

A predicate requiring m variables is called an ***m*-place predicate.**

An m-place predicate can be noted as $p(x_1, x_2, x_3, \ldots, x_m)$

Example 1

Consider the following statements:

- Erick is taller than Bill

- Canada is to the north of United States

The predicates "**is taller than**" and "**is to the north of**" are 2-place predicates because names of two objects are needed to complete a statement involving these predicates.

If ***g*** symbolizes "is taller than", ***e*** denotes "Erick" and ***b*** denotes "Bill".

n symbolizes "is to the north", ***c*** denotes "Canada" and ***s*** denotes "United States".

The above 2 statements can be written as ***g(e,b)*** and ***n(c,s)*** which are different by ***g(b,e)*** and ***n(s,c).***

The order in which the names appear in the statement as well as in the predicate is important.

When a predicate is described, it usually happens that the set of possible values for the variables is specified. This set is called the **domain of interpretation or the universe of discourse,** or often simply the **domain**. It is commonly denoted by **D.**

This set is never empty; that is, there is always at least one element in any domain of interpretation.

If $p(x)$ is a predicate and x has domain U, the **truth set** of $p(x)$ is the set of all elements t of U such that $p(t)$ is true.

Example 2

Let $p(x_1, x_2, x_3, x_4)$ be a predicate $x_1 + x_2 \leq x_3 + x_4$ and let the domain of interpretation be $D = \{0,1,2\}$. Then, the predicate may become true or false depending on how we choose $x_1, x_2, x_3,$ *and* x_4.

For example, choosing all of them to be 1 changes the predicate into true statement, while a careful choice for the values of the variables can change the predicate into a false statement.

Example 3

Assume a predicate $p(x)$ that represents statement: "x is a prime number" from domain $D = \{2,3,4,5,6,7\}$

Truth values for different x:

$p(2)=T$
$p(3)=T$
$p(4)=F$
$p(5)=T$
$p(6)=F$
$p(7)=T$

All statements $p(2), p(3), p(4), p(5), p(6), p(7)$ are propositions. But $p(x)$ with variable x is not a proposition.

The truth set of this predicate is $\{2,3,5,7\}$

Application activity 4.3.1

1. Let $p(x)$ be the predicate: "x is even number" from domain $U=\{1,2,3,4,5,6,7,8,9,10\}$ Give the truth set of $p(x)$

2. Assume a predicate $p(x,y)$ that represents the statement: $x+y=8$. Give the truth value of a) $p(3,5)$ b) $p(1,5)$

4.3.2 Quantifiers

Activity 4.3.2

State whether the following statements are true, false or neither. Write down a counter example where the statement is false

1. $\frac{1}{x}$ is always less than x, for any real number x
2. All human beings are mortal
3. All birds can fly
4. The square of an even number is also an even number.
5. All months has at least 30 days
6. There exist a unique natural number x such that $x+1>x$

CONTENT SUMMARY

Given a predicate *p* and a domain *D*, how often does the predicate become true?

Certain words, symbols or groups of words that help to answer such a question are called **quantifiers.**

They help us to decide the frequencies with which a predicate becomes true, whether it is satisfied by no element of a domain, or one element, or some elements, or all elements.

The most important quantifiers are:

The existential quantifiers $\exists$ ("**there exist**"), and

The universal quantifier $\forall$ ("**for all**")

The appearance of each quantifier is signed to evoke its meaning. The symbol $\exists$ is similar to a back-to-front E (the first letter of the word **EXIST**), while $\forall$ is similar to an upside-down A (the first letter of the word **ALL**).

Existential quantifier

The statement $(\exists x)\, p(x)$ is read as "there exists an *x* such that $p(x)$ is true".

This means that $p(x)$ is sometimes true. That is, at least one value *a* in *D* for which $p(a)$ is true.

We also write $\exists x, p(x)$ or $\exists x \in D, p(x)$. Where the second version "there exit *x* in *D* such that $p(x)$ is true".

Universal quantifier

The statement $(\forall x)\, p(x)$ is read "for all *x* $p(x)$ is true." This means that $p(x)$ is always true; that is no matter what value *a* from domain *D* is chosen for *x*, the resulting statement $p(a)$ is true.

We also write $\forall x, p(x)$ or $\forall x \in D, p(x)$. Where the second version is read as "for all *x* in *D*, $p(x)$ is true".

In general, when are quantified statements true/false?

Truth Values of Quantifiers:

Statement	True When	False When
∀xP(x)	P(x) is true for every x.	There is an x for which P(x) is false.
∃xP(x)	There is an x for which P(x) is true.	P(x) is false for every x.

Example 1

Let $p(x)$ *means* $x^2 \geq x$. *Let* $D_1 = \mathbb{N}$ *and* $D_2 = \mathbb{R}$ *be two domains of interpretation.*

The statement $(\forall x)\, p(x)$ is true in D_1 but false in D_2, since for example $\frac{1}{2} \in D_2$, *and* $\left(\frac{1}{2}\right)^2 = \frac{1}{4} < \frac{1}{2}$.

Let $q(x)$ *means* $x^2 = -x$. *Let* $D_1 = \mathbb{N}_0$ *and* $D_2 = \mathbb{Z}$ *be two domains of interpretation.*

The statement $(\exists x)\, p(x)$ is false in D_1 but true in D_2, since for example $0 \in D_2$, *and* $(0)^2 = -0$.

Example 2

Translate the following into symbolic form: "Everybody likes him"

Solution

$\forall x, p(x)$ which means $p(x)$: x likes him

Example 3

Consider the predicates: $r(x)$: $x - 7 = 2$ *and* $s(x)$: $x > 9$. *If the universe of discourse is the real numbers, give the truth value of propositions:*

$(\exists x)\, s(x) \wedge \neg(\forall x)\, r(x)$

Solution

True. There exist real numbers that are greater than 9, and not all real numbers are equal to 9.

Application activity 4.3.2

1. Translate the following into symbolic form:
   a) Somebody cried out for help and called the police
   b) Nobody can ignore her
2. Consider the following predicates:

$p(x, y): x > y \qquad q(x, y): x \leq y \qquad r(x): x-7 = 2 \qquad s(x): x > 9$

If the universe of discourse is the real numbers, give the truth value of each of the following propositions: a) $(\exists x)r(x)$ b) $(\forall y)[\neg s(y)]$

4.3.3 Negation of quantifiers

Activity 4.3.3

Negate the following statements

1. All grapefruit are pink.
2. Some celebrities are modest.
3. No one weighs more than two thousand pounds.
4. Some people are more than ten feet tall.
5. All snakes are poisonous.
6. Some whales can stay under water for two days without surfacing for air.
7. All birds can fly.

CONTENT SUMMARY

Considering the statement: "All dogs have tails" and negating it we get the statement: "Not all dogs have tails" which means that: "There is at least one dog that does not have a tail."

Let us express these ideas symbolically:

Let $p(x)$ mean "dog x has a tail". Then all dogs have tails is represented by $\forall x, p(x)$

and dog x doesn't have a tail is $\neg p(x)$. So not all dogs have tails is $\neg[\forall x, p(x)]$. Abbreviating there is at least one dog that doesn't have a tail or there exists a tailed dog, we can say that there exists a tailless dog is $\exists x, p(x)$.

Since the last two statements have, in words, the same meaning; we can declare the predicate formulae representing them to be logically equivalent: $\neg[\forall x, p(x)] \equiv \exists x[\neg p(x)]$.

By generalizing the argument that we have used to produce two results, it can be shown that the logical equivalence holds true no matter what predicate is represented by $p(x)$. In fact, this is one of the generalized of the De Morgan's laws. For predicate logic, the other is similar: $\neg[\exists x, p(x)] \equiv \forall x[\neg p(x)]$ that can be interpreted as "there are no dogs with tails" has the same meaning as "every dog is tailless."

Example 1

Let $p(x)$ mean "country x has a president." Interpret the following statements. Establish the equivalence of their negation.

a) $\forall x, p(x)$ b) $\exists x, p(x)$

Solution

Since $p(x)$ mean "country x has a president" , thus

a) $\forall x, p(x)$ means "all countries have president"

b) $\exists x, p(x)$ means "there is a country with a president."

Negating the given statements, we get

a) $\neg[\forall x, p(x)]$ that is interpreted as "it is not true that all counties have presidents."

This sentence can be express as "there is a country with no president" symbolically $\exists x[\neg p(x)]$.

Thus, $\neg[\forall x, p(x)] \equiv \exists x[\neg p(x)]$

b) $\neg[\exists x, p(x)]$ means "it is not true that there is a country with a president."

This sentence is equivalent to" all countries have no president" symbolically

$\forall x[\neg p(x)]$.

Hence $\neg[\exists x, p(x)] \equiv \forall x[\neg p(x)]$.

These laws can be summarized as follow:

"Negating a universally quantified formula changes it into an existentially quantified formula and vice-versa with the part of the formula after the quantifier becoming negated."

Application activity 4.3.3

Negate each of the following statements and write the answer in symbolic form

1. Some students are math majors
2. Every real number is positive, negative or zero
3. Every good boy does fine
4. There is a broken desk in our classroom
5. Lockers must be turned in by the last day of class
6. Haste makes waste

4.4 END UNIT ASSESSMENT 4

1. Which of the following sentences are propositions?
    a) Pretoria is the capital city of South Africa
    b) Is this concept important?
    c) 3+4=5
    d) If x is a real number, then $x^2 < 0$
    e) Wow, what a day!
2. Find the negation of the proposition "Today is Monday"

3. Find the conjunction of the propositions p and q, where p is the proposition "Today is Sunday" and q is the proposition "The moon is made of cheese".

4. Find the disjunction of the propositions p and q, where p is the proposition "Today is Sunday" and q: "The moon is made of cheese".

5. Show $[p \wedge (p \rightarrow q)] \rightarrow q$ is a tautology.

6. Find the truth value of the bi-conditional "The moon is made of cheese if and only if 1=2".

7. Construct the truth table for the proposition $(\neg p \rightarrow q) \wedge r$

8. Show, by constructing its truth table, that $\neg(p \vee q) \Leftrightarrow \neg p \wedge \neg q$ is a tautology.

9. Consider the predicate $p(x, y)$: "$y = x + 3$". What are the truth values of the propositions $p(1,2)$ and $p(0,3)$?

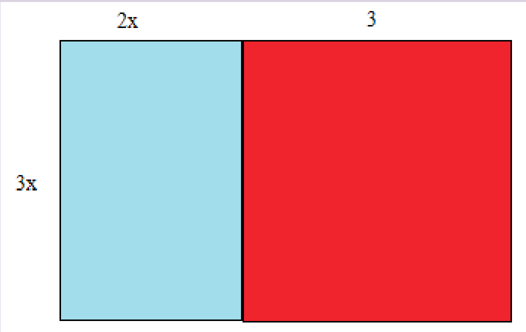
10. Express the statement "Every student in this class has seen a computer" as a universal quantification.

UNIT: 5 OPERATION ON POLYNOMIALS

Key Unit competence: **Perform operations, on polynomials and solve related problems.**

5.0. Introductory Activity 5

1. Research on the definition of polynomial, discuss with your colleague to come up with a concise definition and make a presentation to the whole class.
2. The length of a rectangle is represented by $x^2 + 3x + 2$, and the width is represented by $4x$.
   i. Express the perimeter of the rectangle as a polynomial.
   ii. Express the area of the rectangle as a polynomial.
3. Observe this shape.



Express the area of the full rectangle as a polynomial.

5.1 Defining and comparing polynomials

Activity 5.1

Consider the following algebraic expression

(a) $2x$

(b) $2x + 3$

(c) $x^2 + 2x - 3$

(d) $x^3 + 2x^2 + 3x - 1$

(e) $x^4 + x^3 - 2x^2 + 3x - 2$

Answer the following questions based on the expressions above.

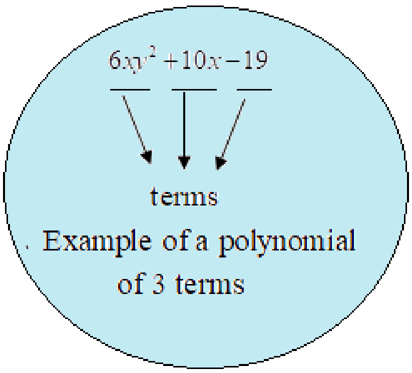
1. How many terms does each of the expression have?
2. State the highest power of x in each expression.
3. Using a dictionary or internet, find out the classification of polynomials based on the number of terms.

CONTENT SUMMARY

Polynomial comes from **poly**- meaning "**many**" and -**nomial** meaning "**term**".

A polynomial looks like this:

$6xy^2 + 10x - 19$ terms



A polynomial can have:

- **constants** (like **6**, **10**, or -19)
- **variables** (like ***x*** , ***y, z,…etc***)
- **exponents** (like the 2 in y^2), but only **0, 1, 2, 3,** ... etc

That can be combined using operations such as **addition, subtraction, multiplication and division.**

A polynomial can never be divided by a variable such as $\frac{7}{x}$

Let $a_0, a_1, a_2, ..., a_{n-1}, a_n$ be real numbers. Then $a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$ is a polynomial in x (with real coefficient). When $a_0, a_1, a_2, ..., a_{n-1}, a_n$ are not all zero, it can be assumed that $a_n \neq 0$ and the polynomial has degree n

Example:

$7x^2 + 4x + 4$ is a polynomial while y^{-3} is not a polynomial because it has a negative exponent (-3)

The classification of polynomials is based on the number of terms. The following table identifies the types of polynomial.

Number of terms	Type of polynomial	Example
One term	Monomial	$2x$
Two terms	Binomial	$5x - 1$
Three terms	Trinomial	$3a + 7b + c$
....	....	....
n terms	Polynomial	$a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$ where $a_n \neq 0$ and the degree $n \geq 0$

The **degree** of a polynomial with only one variable is the **highest exponent** of that variable. For example, in $4x^3 - x + 3$ the degree is **3** which is the highest exponent of *x*.

Note that: A polynomial containing two or more variables are said to be **homogenous** if every term is of the same degree.

For example, $xy^2 + x^2y + 3x^3$ *and* $3x + 2y - 4z$ are homogenous polynomials of degree 3 and 1 respectively.

Application activity 5.1

1. Identify each of the following expression as a monomial, binomial, trinomial or polynomial

$(a)\ 5x^3 + x^2 - 3x - 4$

$(b)\ 3ab^2c - 6b$

$(c)\ 4x^2y^2$

$(d)\ 4 - 6ab^2$

$(e)\ x^2 + y^3$

2. Which of the following are homogenous? State the degree of those that are homogenous.

$(a)\ zx + xy$

$(b)\ a^2b^2 + 2a + 2b$

$(c)\ x^3 + y^3 - z^3$

$(d)\ x^2 - 3xy - 40y^2$

5.2. Operations on polynomials

Activity 5.1

1.Consider the expressions;

$(i)\ 3x^3 - 13x^2 + 4x - 2$

$(ii)\ x^2 + 3x + 4$

$(iii)\ 5x^2 - 3x + 3$

(a) Identify the terms that are similar in (i), (ii) and (iii) and group them together.

(b) Combine the sets of terms in (a) above.

(c) Simplifying each group to a single term.

(d) Now add all the expressions together

$(3x^3 - 13x^2 + 4x - 2) + (x^2 + 3x + 4) + (5x^2 - 3x + 3)$

2. Consider the polynomial expressions

$(a)\, x^2 + y + 1$

$(b)\ 3x^2 + 2y - 3$

Given $x = 2$ *and* $y = 3$, solve the expressions. Then Compare your findings with other classmates.

CONTENT SUMMARY

a) Addition, subtraction and substitution of polynomials

When simplifying algebraic expressions by adding or subtracting, first collect or group the like terms together. Simplification is usually easier if the positive like terms are separated from the negative ones.

Example

Simplify the expression

$2x - 4y + 5x - 3y$

Solution

To simplify the expression, you first collect like terms together for example,

$2x - 4y + 5x - 3y$

$= 2x + 5x - 4y - 3y$ $(like\ term\ together)$

$= 7x - 7y$ $(combine\ like\ terms)$

$7x - 7y$ (this expression cannot be simplified further because $7x$ *and* $-7y$ are unlike terms)

Polynomials can be evaluated numerically if some numerical values are attached or attributed to the variable or variables. We use the method of substitution in the given expression and then simplify.

Substitution involves replacing variables, in an algebraic expression, with specific values. The expression may then be evaluated.

Example

If, find $x = 3,\ y = -2\ and\ z = 5$ the value of

$(a)\ xy + z^2$

$(b)\ (x + y)(3x - 4z)$

Solution

$$
\begin{aligned}
(a)\ xy + z^2 &= 3 \times (-2) + 25 \\
&= -6 + 25 \\
&= 19
\end{aligned}
$$

$$
\begin{aligned}
(b)\ (x + y)(3x - 4z) &= (3 + (-2))(3 \times 3 (- 4) \times 5) \\
&= (1)(9 - 20) \\
&= 1 \times (-11) \\
&= -11
\end{aligned}
$$

b) Multiplication of polynomials

The multiplication of polynomials has to follow these rules, therefore:

1. If there is a positive (plus) sign just before a bracket, the sign of each term inside the bracket is unchanged when the bracket is removed (i.e. when the expression is expanded).
2. If there is a negative (minus) sign just before a bracket, the sign of each term inside the bracket must be changed to the opposite sign when the bracket is removed. Removing the bracket is like multiplying each term by -1

In general, when expression in a bracket is multiplied by a number in order to remove the brackets, every term inside the bracket must be multiplied by that number.

Thus,

$$
\begin{aligned}
a(x+y) &= a \times x + a \times y = ax + ay \\
and\ a(x-y) &= a \times x - a \times y = ax - ay
\end{aligned}
$$

Example

1. Remove the brackets and simplify:

$(a)\ 7g + (3g - 4h) - (2g - 9h)$

$(b)\ (6x - y + 3z) - (2x + 5y - 4z)$

Solution

$$(a)\ 7g + (3g - 4h) - (2g - 9h) = 7g + 3g - 4h - 2g + 9h$$

$$= 7g + 3g - 2g + 9h - 4h$$

$$= 8g + 5h$$

$$(b)\ (6x - y + 3z) - (2x + 5y - 4z) = 6x - y + 3z - 2x - 5y + 4z$$

$$= 6x - 2x - y - 5y + 3z + 4z$$

$$= 4x - 6y + 7z$$

2. Remove the brackets and simplify:

$$2(3x - y) + 4(x + 2y) - 3(2x - 3y)$$

Solution

$$2(3x - y) + 4(x + 2y) - 3(2x - 3y)$$

$$= 2 \times 3x - 2 \times y + 4 \times x + 4 \times 2y - 3 \times 2x + 3 \times 3y$$

$$= 6x - 2y + 4x + 8y - 6x + 9y$$

$$= 6x + 4x - 6x + 8y + 9y - 2y$$

$$= 4x + 15y$$

c) Division of polynomials

For the division of one polynomial by another to be possible, the degree of the dividend (Numerator) must be greater or equal than that of the divisor (denominator).

Remember:

Like in division of numbers, not all polynomials divide exactly, some will have remainders.

Suppose the polynomial to be divided is denoted by $f(x)$, and the divisor by the polynomial $g(x)$, we can denote the result of division as follow:

$$\frac{f(x)}{g(x)} = Q + \frac{R}{g(x)}$$

Hence $f(x) = Q.g(x) + R$, where Q is the quotient and R is the remainder, the division process terminates as soon as the degree of R is less than the degree of division $g(x)$.

In division, order of the terms is important:

i. Both the dividend and the divisor must be written in descending powers of the variable.

ii. If a term is missing, a zero term must be inserted in its place.

Example

Given that $f(x) = x^2 + 7x + 12$ and $g(x) = x + 4$, divide $f(x)$ *by* $g(x)$.

Solution

Using skills of long division of numbers we write $(x^2 + 7x + 12) \div (x + 4) = x + 3$ in the form

$$\begin{array}{r|l} & \quad\quad\quad x+3 \\ \hline x+4 & x^2 + 7x + 12 \\ & -(x^2 + 4x) \quad \downarrow \\ \hline & \quad\quad 3x + 12 \\ & \quad -(3x + 12) \\ \hline & \quad\quad\quad 0 \end{array}$$

i. Divide x^2 *by* x to obtain x.

ii. Multiply $x(x + 4)$ and subtract then bring down the next term.

iii. Divide $3x$ *by* x to obtain and multiply by $x + 4$.

iv. Subtract.

Thus $(x^2 + 7x + 12) \div (x + 4)$

The division is exact, the quotient is $x + 3$ and there is no remainder.

Application activity 5.2

1. Given that $x = -3$ *and* $y = 2$, find the value of the expressions below:
   $(a)\ 3x - 6y - 4x + 8y$
   $(b)\ 6x^2 - 9x + 6x - 8$
2. Expand and simplify the following
   $(a)\ (x+y)(x-y-2)$
   $(b)\ (3x-2)(2x^2-2x+1)$
3. Divide $-20 + 6x^3 - 4x^2$ *by* $2x - 4$ and state the quotient and the remainder.

5.3 Factorization of polynomials

Activity 5.3

Consider the following expressions:

$(a)\ 2a + 2b$

$(b)\ 3r + 6r^2$

$(c)\ xy + axy$

$(d)\ 9x^2y + 15xy^2$

For each expression above, identify the common factors for both terms and rewrite the expression in factor form. Compare your results with those of other classmates.

CONTENT SUMMARY

In arithmetic, you are familiar with factorization of integers into **prime factors**.

For example, $15 = 3 \times 5$.

15 is called the **multiple**, while 3 and 5 are called its **divisors or factors**.

The process of writing 15 as product of 3 and 5 is called **factorization.**

Factors 3 and 5 cannot be further reduced into other factors.

Like factorization of integers in arithmetic, one can make factorization of polynomials into other irreducible polynomials in algebra.

For example, $x^2 + 2x$ is a polynomial. It can be factorized into x *and* $(x + 2)$.

$x^2 + 2x = x(x + 2)$.

So, x *and* $x + 2$ are two factors of $x^2 + 2x$. While x is a monomial factor, $x + 2$ is a binomial factor.

Application activity 5.2

Factorise each of the following expressions:

(a) $2ab + 4c$
(b) $-3b^2 - 9b$
(c) $3x^3 + 6x^2 - 9x$

5.4 Expansion of polynomials

Activity 5.4

Apply distributive properties to perform the following

(i) $(x+4)(x-2)$
(ii) $(x+4)(x+4)$
(iii) $(x-1)(x-1)$

a) How many terms does each result have?

b) Find out the common characteristics for the all above expressions. What is the highest and lowest exponent for the variable x in all expressions?

CONTENT SUMMARY

Given the expression; $(x+3)(x+2)$. Expansion of the expression takes the steps below:

$$(x+3)(x+2) = x(x+2) + 3(x+2) = x^2 + 2x + 3x + 6$$

$$= x^2 + 5x + 6 \text{ (since } 2x \text{ and } 3x \text{ are like terms).}$$

This means that $(x+3)$ *and* $(x+2)$ are factors of $x^2 + 5x + 6$.

$\Rightarrow x^2 + 5x + 6 = (x+3)(x+2)$ (in factor form).

Note: In $x^2 + 5x + 6$,

a) The coefficient of the highest degree of this trinomial is 1,

b) The coefficient of the linear term is 5, the sum of the constant terms in the binomial factors, and

c) The constant term is 6, the product of the constant terms in the binomial factors.

Generally,

In a simple expression like $ax^2 + bx + c$, where $a = 1$, the factors are always of the form $(x+m)(x+n)$, where m and n are constants. The expression $ax^2 + bx + c$ is factorizable only if there exists two integers m and n such that $m \times n = c$ (product of the factors) and $m + n = b$ (sum of the factors).

To factorise a trinomial whose form: $ax^2 + bx + c$, where $a = 1$, follow the steps below.

1. List all the possible pairs of integers whose product equals the constant term.
2. Identify the only pair whose sum equals the coefficient of the linear term.
3. Rewrite the given expression with the linear term split as per the factors in 2 above.
4. Factorise your new expression by grouping, i.e. taking two terms at a time.
5. Check that the factors are correct by expanding and simplifying.

Example

Factorise $x^2 + 8x + 12$.

Solution

In this example, $a = 1$, $b = 8$ *and* $c = 12$.

1. List all the pairs of integers whose product is 12. These are:

$1 \times 12 \quad 3 \times 4 \quad 2 \times 6$

$1 \times -12 \quad -3 \times -4 \quad -2 \times -6$

2. Identify the pair of numbers whose sum is 8. The numbers are 6 and 2.

3. Rewrite the expression with the middle term split.

$$x^2 + 8x + 12 = x^2 + 2x + 6x + 12$$

Factorize $x^2 + 2x + 6x + 12$ by grouping. $x^2 + 2x + 6x + 12$ *has* 4 terms which we can group in twos so that first and second terms make one group and third and fourth terms make another group.

i.e $x^2 + 2x + 6x + 12$ In each group, factor out the common factor.

Thus,

$x^2 + 2x + 6x + 12 = x(x + 2) + 6(x + 2)$ We now have two terms, i.e. $x(x + 2)$ and $6(x + 2)$, whose common factor is $(x + 2)$

$\therefore x^2 + 8x + 12 = (x + 2)(x + 6)$ (Factor out the common factor $(x + 2)$)

Check that $(x + 2)(x + 6) = x^2 + 8x + 12$.

Note: Since all the terms in the example are positive, the negative pairs of factors of 12 could have been omitted altogether.

Note that:

- If the third term in the split form of the expression is negative, we factor out the negative common factor.

Example:

$y^2 + 2y - 35 = y^2 + 7y - 5y - 35$ (the third term is negative)

$= y(y + 7) - 5(y + 7)$ (we factor out -5)

$= (y + 7)(y - 5)$.

- The order in which we write mx *and* nx in the split form of the expression does not change the answer.

Consider again the expressions $(x + 2)^2$ *and* $(x - 3)^2$, expand and simplify them

Each binomial expansion has three terms

The first term is the square of the first term of the binomial

The third term is the square of the second term of the given binomial

The middle term is twice the product of the two terms of the binomial

$$i.e.\ (x+2)^2 = (x)^2 + 2(2 \times x) + (2)^2$$
$$= x^2 + 4x + 4$$

$$(x-3)^2 = (x)^2 + 2(x \times -3) + (-3)^2$$
$$= x^2 - 6x + 9$$

Just like we have square numbers in arithmetic, we also have square trinomials in algebra.

Remember $(a+b)^2 = (a+b)(a+b) = a^2 + 2ab + b^2$

In this case $a^2 + 2ab + b^2$ is a perfect square because it has two identical factors.

Remarks

If a trinomial is a perfect square,

1. The first term must be a perfect square.
2. The last term must be a perfect square.
3. The middle term must be twice the product of numbers that were squared to give the first and last terms.

Example

Show that the following expressions are perfect squares and give the factor of each.

(a) $9x^2 + 12x + 4$

(b) $9x^2 - 30x + 25$

Solution

(a) $9x^2 + 12x + 4$

Condition (1): first term $9x^2 = (3x)^2$

Condition (2): last term $4 = (2)^2$

Condition (3): middle term $12x = 2(3x)(2)$

$\therefore 9x^2 + 12x + 4$

$= (3x)^2 + 2(2)(3x) + 2^2$

$= (3x + 2)^2$

(b) $9x^2 - 30x + 25$

First term $9x^2 = (3x)^2$

Last term $25 = (-5)^2$

Middle term $-30x = 2(-5)(3x)$

$\therefore 9x^2 - 30x + 25$ is a perfect square which factorizes to $(3x - 5)^2$.

Note: In $9x^2 - 30x + 25$, middle term of the expression is negative, hence the constant term in the binomial factor must be negative.

Application activity 5.4

1. Factorize the following expressions:

(**i**) $x(x+1) + 3(x+1)$
(**ii**) $3(2x+1) - x(2x+1)$
(**iii**) $4a(2a-3) - 3(2a-3)$
(**iv**) $4b(b+6) - (b+6)$
(**v**) $3y(4-y) + 6(4-y)$

2. Show that the following are perfect squares. Hence state their factors

(**i**) $x^2 + 8x + 16$
(**ii**) $x^2 + 12x + 36$
(**iii**) $x^2 - 14x + 49$

5.5 END UNIT ASSESSMENT 5

1. Classify the following expression into monomial, binomial or trinomial

   $(a)\ a^2b^3 - 25$

   $(b)\ -yx^2z^3$

   $(c)\ 2a + 3a^2c - b$

   $(d)\ -5x^2 + 6x + 3$

2. Which of the following are homogenous? State the degree of those that are homogenous

   $(a)\ 6x - 5y + 6z$

   $(b)\ ab + ac + bc$

   $(c)\ x^3 + y^3 + z^3 + 3a^2c + 3ac^2$

   $(d)\ 2a^2 - 7ab - 30y^2$

   $(e)\ 5x^3 + 6x^2y - 7xy^2 + 6y^3$

3. Expand and simplify the following

   $(a)\ (2x + 3 - y)^2$

   $(b)\ (a - 2b)(3a^2 - 2ab + b^2)$

   $(c)\ -x(2x - 3x - 1)$

   $(d)\ (x - y - 2)(2x + 3 - y)$

4. Factorize:

   a) $ax + ay$

   b) $3x + 3z$

   c) $21xy - 6x^2$

   d) $6x^2 + 15xy$.

   e)$9x^2 - 45y^2x^3$

   f) $4x + 14x^2$

5. Show that the following are perfect squares. Hence state their factors.

   (i) $9p^2 + 24pq + 16q^2$

   (ii) $4x^2 + 12x + 9$

UNIT: 6 LINEAR AND QUADRATIC EQUATION AND INEQUALITIES

Key Unit competence: Solve algebraically or graphically linear, quadratic equations or inequalities

6.0. Introductory Activity 6

An equation is a statement in which the values of two mathematical expressions are equal while an inequality is a statement in which the values of two mathematical expressions are not equal.

1. Make a research on internet or using Math reference books and make a short presentation on the following:
   a. Linear equation in one unknown
   b. Inequality in one unknown
   c. Quadratic equation
   d. Find out two examples where linear and quadratic equations and inequalities are used in real life.
2. Discuss and make a presentation on how to find the value of x such that the following statements are true

   $x+1=5$ $\quad$ $(x+1)(x-1)<0$ $\quad$ $x^2-1=0$

6.1 Linear and quadratic equations

6.1.1. linear equations

Activity 6.1.1

The following mathematical statements are always true for only one value of x. For each statement, find out the real value of x

1) $x+1=5$ $\quad$ 2) $2x-4=0$

3) $2x+1=-5$ $\quad$ 4) $x-4=10$

CONTENT SUMMARY

An equation is a statement in which the values of two mathematical expressions are equal. Consider the statement $ax+b=0$, where $a,b\in\mathbb{R}$ and $a\neq 0$. This statement is true when $x=-\frac{b}{a}$ (the solution or the root of the equation $ax+b=0$). Thus, to find a solution to the given equation is to find the value that satisfy that equation.

In general, linear equation in one unknown is of the form $ax+b=0$, with a and b constant and $a\neq 0$

Examples: Solve in set of real numbers

a) $x+6=14$

b) $4x+5=20+x$

c) $x=14-x$

Solutions

a)	b)	c)
$x+6=14$ $\Leftrightarrow x=14-6$ $\Rightarrow x=8$ $S=\{8\}$	$4x+5=20+x$ $\Leftrightarrow 4x-x=20-5$ $\Leftrightarrow 3x=15$ $\Leftrightarrow x=\frac{15}{3}$ $\Rightarrow x=5$ $S=\{5\}$	$x=14-x$ $\Leftrightarrow 2x=14$ $\Rightarrow x=7$ $S=\{7\}$

Application activity 6.1.1

Solve the following linear equations in the set of real numbers

1) $x+5=9$

2) $6x+5=5$

3) $x+5=9x+1$

4) $-6x-5=9$

5) $6x-51=9$

6.1.2. Quadratic equations

Activity 6.1.2

1. The following mathematical statements are always true for only some values of x. Find out the real values of x in the following statement $x^2+2x-24=0$
2. Suppose that you own a plot of land and want to build a fence there. Let the area of the plot be 900 m^2. You want your fence to have a length equal to twice its width. Let X be the width, then the length is 2X. Knowing that the Length times width is equal to the area, how will you optimise the area of land you have? Do you think the statement $2x^2=900$ can help you to find out the length and width of your fence? Explain your answer.

CONTENT SUMMARY

Equations of the type $ax^2+bx+c=0$ $(a \neq 0)$ are called quadratic equations.

There are three main ways of solving such equations:

a) By factorizing or finding square roots

b) By using the formula $x=\dfrac{-b \pm \sqrt{b^2-4ac}}{2a}$

c) By completing the square

1. Quadratic equations by factorizing or finding square roots

The method of solving quadratic equations by factorization should only be used if is readily factorized by inspection.

To factorize a quadratic equation, one can use the sum and product of its roots.

Let x and y be two real numbers such that $x+y=s$ and $xy=p$.

Here $y=s-x$ and $x(s-x)=p$. Or $sx-x^2=p$ or $x^2-sx+p=0$. This equation is said to be quadratic equation and s, p are the sum and product of the two roots respectively.

Quadratic equation or equation of second degree has the form $ax^2+bx+c=0$,

where the sum of two roots is $s=-\dfrac{b}{a}$ and their product is $p=\dfrac{c}{a}$.

Example 1: *Solve in* $\mathbb{R}: x^2+2x-24=0$	**Example 2:** *Solve in* $\mathbb{R}: 5x^2+7x-6=0$
Solution $x^2+2x-24=0 \Leftrightarrow (x+6)(x-4)=0$ So, either $x+6=0$ or $x-4=0$ giving $x=-6$ or $x=4$.	Solution $5x^2+7x-6=0$ $\Leftrightarrow 5x^2-3x+10x-6=0$ $\Leftrightarrow x(5x-3)+2(5x-3)=0$ $\Leftrightarrow (5x-3)(x+2)=0$ So, either $5x-3=0$ or $x+2=0$ giving $x=\frac{3}{5}$ or $x=-2$.

Example 3:: *Solve in* $\mathbb{R}: x^2-7x+5=-5$

Solution

$x^2-7x+5=-5$

$\Leftrightarrow x^2-7x+10=0$

As we saw it, in this equation the sum of two roots is -7 and the product is 10. To find those roots we can think about two numbers such that their sum is -7 and their product is 10. Those numbers are -2 and -5. Thus $S=\{2,5\}$

2. Quadratic equations by the formula

To solve this equation, first we find the discriminant (delta): $\Delta=b^2-4ac$

In fact,

$ax^2+bx+c=0$

$\Leftrightarrow ax^2+bx=-c$

$\Leftrightarrow a\left(x^2+\frac{b}{a}x\right)=-c \; as \; a\neq 0$

$\Leftrightarrow x^2+\frac{b}{a}x=-\frac{c}{a} \; as \; a\neq 0$ (making the coefficient of x^2 one)

$$\Leftrightarrow \left(x+\frac{b}{2a}\right)^2=-\frac{c}{a}+\frac{b^2}{4a^2} \leftarrow \frac{b^2}{4a^2} \textit{ is the square of half the coefficient of } x, \left(\frac{b}{a}\right),$$

$in\, x^2+\frac{b}{a}x$

$$\Leftrightarrow \left(x+\frac{b}{2a}\right)^2=\frac{b^2-4ac}{4a^2}$$

$$\Leftrightarrow x+\frac{b}{2a}=\pm\sqrt{\frac{b^2-4ac}{4a^2}}$$

$$\Leftrightarrow x+\frac{b}{2a}=\pm\frac{\sqrt{b^2-4ac}}{2a}, if\ a>0$$

$$or\ x+\frac{b}{2a}=\mp\frac{\sqrt{b^2-4ac}}{2a}, if\ a<0$$

Simply,

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}.$$

Let $\Delta=b^2-4ac$, There are three cases:

- If $\Delta>0$, there are two distinct real roots:
- $x_1=\frac{-b+\sqrt{\Delta}}{2a}$ and $x_2=\frac{-b-\sqrt{\Delta}}{2a}$
- If $\Delta=0$, there is one repeated real root (one double root):
- $x_1=x_2=\frac{-b}{2a}$
- If $\Delta<0$, there is no real root.

Example 1: *Solve in* $\mathbb{R}$ *:* $x^2+2x+1=0$ **Solution** $x^2+2x+1=0$ $\Delta=2^2-4(1)(1)=0$ $x_1=x_2=\frac{-2}{2}=-1$ $S=\{-1,-1\}$	Example 2: *Solve in* $\mathbb{R}$ *:* $x^2-7x+5=-5$ **Solution** $x^2-7x+5=-5$ $\Leftrightarrow x^2-7x+10=0$ $\Delta=(-7)^2-4(1)(10)=9$ $x_1=\frac{-(-7)+\sqrt{9}}{2}=5,\ x_1=\frac{-(-7)-\sqrt{9}}{2}=2$ $S=\{2,5\}$
Example 3: *Solve in* $\mathbb{R}$ *:* $2x^2+3x+4=0$ **Solution** $2x^2+3x+4=0$ $\Delta=3^2-4(2)(4)=-23$ Since $\Delta<0$, there is no real root. Then, $S=\varnothing$	*Example 4: For what value of k will the equation* $x^2+2x+k=0$ *have one double roots? Find that root.* **Solution** For one double root $\Delta=0$. $\Delta=4-4k$ $4-4k=0\Rightarrow k=1$ Thus, the value of k is 1. That root is $x=-\frac{2}{2}=-1$.

3. Quadratic equations by completing the square

Before solving quadratic equations by completing the square, let's look at some examples of expanding a binomial by squaring it.

- $(x+3)^2=x^2+6x+9$.
- $(x-5)^2=x^2-10x+25$

Notice that the constant term (k^2) of the trinomial is the square of half of the coefficient of trinomial's x-term .Thus, to make the expression x^2+kx a perfect square, you must add $\left(\frac{1}{2}k\right)^2$ to the expression.

When completing the square to solve quadratic equation, remember that you must preserve the equality. When you add a constant to one side of the equation, be sure to add the same constant to the other side of equation.

Example 1: Solve $x^2-4x+1=0$ by completing the square	**Example 2:** Solve $4x^2+2x-5=0$ by completing the square
Solution	**Solution**
$x^2-4x+1=0$ Rewrite original equation	$4x^2+2x-5=0$ Rewrite original equation
$x^2-4x=1$ Subtract 1 from both sides.	$4x^2+2x=5$ Add 5 to both sides.
$x^2-4x+(-2)^2=1+(-2)^2$ Add $(-2)^2=4$ to both sides.	$x^2+\frac{1}{2}x=\frac{5}{4}$ Divide both sides by 4.
$(x-2)^2=5$ Binomial squared.	$x^2+\frac{1}{2}x+\left(\frac{1}{4}\right)^2=\frac{5}{4}+\frac{1}{16}$ Add $\left(\frac{1}{4}\right)^2=\frac{1}{16}$ to both sides.
$x-2=\pm\sqrt{5}$ Take square roots.	$\left(x+\frac{1}{4}\right)^2=\frac{21}{16}$ Binomial squared.
$x=2\pm\sqrt{5}$ Solve for x.	$x+\frac{1}{4}=\pm\frac{\sqrt{21}}{2}$ Take square roots.
The equation has two solutions: $x=2+\sqrt{5}$ ***and*** $x=2-\sqrt{5}$	$x=-\frac{1}{4}\pm\frac{\sqrt{21}}{2}$ Solve for x.
	The equation has two solutions: $x=-\frac{1}{4}+\frac{\sqrt{21}}{2}$ and $x=-\frac{1}{4}-\frac{\sqrt{21}}{2}$

Application activity 6.1.2

1. Solve in set of real numbers the following equation $x^2+6x+8=0$ by factorization
2. Solve in set of real numbers the following equation $x^2-12x+11=0$ using the quadratic formula
3. Solve in set of real numbers the following equation $x^2+5x-24=0$ by completing the square

6.2 Equations reducible to quadratic

Activity 6.2

Let $u=x^2$ in each of the following equations of degree 4 and rewrite the given equation by making it an equation of degree 2.

1. $x^4-2x^2+4=0$
2. $6x^4+5x^2+1=0$

CONTENT SUMMARY

Biquadratic equations are the equations that can be reducible to quadratic equations and they are on the form $ax^4+bx^2+c=0$

To solve a biquadratic equations $ax^4+bx^2+c=0$, change $x^2=y$. This generates a quadratic equation $ay^2+by+c=0$ with the unknown y.

For every positive value of $\boldsymbol{y}$ there are **two values of** $\boldsymbol{x}$, find: $x=\pm\sqrt{y}$

Example: *Solve* $x^4-7x^2+12=0$

Solution

Here $x^4=\left(x^2\right)^2$

Let $y=x^2$, then $y^2=x^4$

Now, the given equation becomes $y^2-7y+12=0$

$y^2-7y+12=(y-3)(y-4)=0 \Rightarrow y=3$ or $y=4$

But $y=x^2$

$y = 3: \quad 3 = x^2 \Rightarrow x = \pm\sqrt{3}$

$y = 4: \quad 4 = x^2 \Rightarrow x = \pm 2$

So, we have four solutions to the original equation, $x = \pm 2$ *and* $\pm\sqrt{3}$.

The basic process is to check if the equation is reducible to quadratic in form and then make a quick substitution to turn it into a quadratic equation. In most cases to make the check that it's reducible to quadratic in form all that we really need to do is to check if one of the exponents is twice the other.

Application activity 6.2

Solve the following biquadratic equations in the set of real numbers

1. $x^4 - 13x^2 + 36 = 0$
2. $x^4 - 10x^2 + 9 = 0$

6.3.1 Linear and quadratic inequalities

Activity 6.3.1

Find at least 5 value(s) of x such that the following statements are true

1) $x < 5$
2) $x > 0$
3) $2x \le 0$
4) $-2 < x < 4$

CONTENT SUMMARY

The statement $x + 3 = 10$ is true only when $x = 7$. If x is replaced by another number (for example 5), the statement is false. To be true we may say that 5 + 3 is less than 10 or in symbol $5 + 3 < 10$. In this case we no longer have equality but **inequality**.

Suppose that we have the inequality $x + 3 < 10$, in this case we have an inequality with one unknown. Here the real value of x to satisfy this inequality is not unique. For example, 1 is a solution but 3 is also a solution. In general, all real numbers less than 7 are solutions. In this case we will have many solutions combined in an interval.

Now, the solution set of $x + 3 < 10$ is an open interval containing all real numbers less than 7 whereby 7 is excluded. Solving the inequality $x + 3 < 10$, one can easily find that the values of x are less than 7. Mathematically is written as follow: $S =]-\infty, 7[$

In general, inequalities in one unknown are of the forms:
$ax + b > 0, \quad ax + b < 0, \quad ax + b \ge 0, \quad ax + b \le 0$, with a and b constant and $a \ne 0$

Recall that

- When the same real number is added or subtracted from each side of inequality the direction of inequality is not **changed**.
- The direction of the inequality is not **changed** if both sides are multiplied or divided by the same **positive real number** and is **reversed** if both sides are multiplied or divided by the **same negative real** number.

Examples: *Algebrically solve the following inequalities in the set of real numbers*

Example 1:	**Example 2:**	**Example 3:**
$-2x+5\leq 0$ **Solution** $-2x+5\leq 0$ $\Leftrightarrow -2x\leq -5$ $\Leftrightarrow x\geq \frac{5}{2}$ $S=\left[\frac{5}{2},+\infty\right[$	$x-4>0$ **Solution** $x-4>0$ $\Leftrightarrow x>4$ $S=\left]4,+\infty\right[$	$x>2x-4$ **Solution** $x>2x-4$ $\Leftrightarrow -x>-4$ $\Leftrightarrow x<4$ $S=\left]-\infty,4\right[$

Example 5:

$2x+5\leq 2x+4$

Solution

$2x+5\leq 2x+4 \quad 0x\leq -1$

Since any real number times zero is zero and zero is not less or equal to -1 then the solution set is the empty set. $S=\varnothing$

Example 4:

$2(x+5)>2x-8$

Solution

$2(x+5)>2x-8$

$\Leftrightarrow 2x+10>2x-8$

$\Leftrightarrow 0x>-18$

Since any real number times zero is zero and zero is greater than -18, then the solution set is the set of real numbers. $S=\mathbb{R}=\left]-\infty,+\infty\right[$

Application activity 6.3.1

Solve the following inequalities

1) $x+6<15$
2) $2x-4<16$
3) $5x\leq 25$
4) $3x-5>21$
5) $-2x+5<0$

6.3.2. Inequalities products / quotients

Activity 6.3.2

Find at least 5 value(s) of x such that the following statements are true

1) $(x+1)(x-1)<0$

2) $\frac{x+2}{x-1}\leq 0$

CONTENT SUMMARY

- To solve the inequality of the form $(ax+b)(cx+d)<0$ or $(ax+b)(cx+d)\leq 0$ one can find out all real numbers that make the left hand side negative.
- To solve the inequality of the form $\frac{ax+b}{cx+d}<0$ or $\frac{ax+b}{cx+d}\leq 0$, one can find out all real numbers that make the left hand side negative.
- To solve the inequality of the form $(ax+b)(cx+d)>0$ or $(ax+b)(cx+d)\geq 0$, one can find out all real numbers that make the left hand side positive.
- To solve the inequality of the form $\frac{ax+b}{cx+d}>0$ or $\frac{ax+b}{cx+d}\geq 0$ one can find out all real numbers that make the left hand side negative.

The following steps are essential:

a) First we solve for $(ax+b)(cx+d)=0$

b) We construct the table of signs, by finding the sign of each factor and then the sign of the product or quotient.

For the quotient the value that makes the denominator to be zero is always excluded in the solution. For that value we use the symbol || in the row of quotient sign.

c) Write the interval considering the given inequality sign.

Examples: *Solve inequalities in the set of real numbers*

1. $(3x+7)(x-2)<0$

Start by solving $(3x+7)(x-2)=0$

$3x+7=0$ or $x-2=0$

$\Leftrightarrow x=-\frac{7}{3}$ or $\Leftrightarrow x=2$

The next is to find the sign table.

x	$-\infty$		$-\frac{7}{3}$		2		$+\infty$
$3x+7$		-	0	+		+	
$x-2$		-		-	0	+	
$(3x+7)(x-2)$		+	0	-	0	+	

Since the inequality is $(3x+7)(x-2)<0$ we will take the interval where the product is negative. Thus, $S=\left]-\frac{7}{3},2\right[$

2. $(x-3)(-2x+4)(x+1)\geq 0$

$x-3=0\Rightarrow x=3,\quad -2x+4=0\Rightarrow x=2,\ x+1=0\Rightarrow x=-1$

x	$-\infty$		-1		2		3		$+\infty$
$x-3$		-		-		-	0	+	
$-2x+4$		+		+	0	-		-	
$x+1$		-	0	+		+		+	
$(x-3)(2x-4)(x+1)$		+	0	-	0	+	0	-	

$S=\left]-\infty,-1\right]\cup\left[2,3\right]$

3. $\dfrac{x+4}{2x-1} \geq 0$

$x+4=0 \Rightarrow x=-4$

$2x-1=0 \Rightarrow x=\dfrac{1}{2}$

x	$-\infty$		-4		$\frac{1}{2}$		$+\infty$
$x+4$		-	0	+		+	
$2x-1$		-		-	0	+	
$\dfrac{x+4}{2x-1}$		+	0	-	||	+	

$S =]-\infty, -4] \cup \left]\dfrac{1}{2}, +\infty\right[$

4. $\dfrac{3x+6}{-x-1} > 0$

$3x+6=0 \Rightarrow x=-2$

$-x-1=0 \Rightarrow x=-1$

x	$-\infty$		-2		-1		$+\infty$
$3x+6$		-	0	+		+	
$-x-1$		+		+	0	-	
$\dfrac{3x+6}{-x-1}$		-	0	+	||	-	

$S =]-2, -1[$

Application activity 6.3.2

Solve the following inequalities

1. $(x-3)(x+3) > 0$

2. $\dfrac{2x-6}{x+2} \geq 0$

6.3.3. Quadratic inequalities

Activity 6.3.3

Use at least 5 values of x to find out the range where

1. $(x-2)(x+1)$ is positive
2. $(x-3)^2$ is negative
3. $(x-1)(x-2)$ is negative

CONTENT SUMMARY

We saw how to solve the inequality like $(ax+b)(cx+d)>0$. If we find the product of the left hand side, the result will be a quadratic expression of the form ax^2+bx+c. Then to solve the inequality of the form $ax^2+bx+c>0$, we need to put the expression ax^2+bx+c in factor form and use the method used to solve inequality product.

If the expression to be transformed in factor form has no factor form, we find its sign by replacing the unknown by any chosen real number. We may find that the expression is always positive or always negative. If the expression to be transformed in factor form has a repeated root, it is zero at that root and positive elsewhere.

Examples: *Solve the following quadratic inequalities*

$x^2-2x+1\le 0$

Solution

$x^2-2x+1=0$

$x_1=x_2=1$ and $x^2-2x+1=(x-1)(x-1)$

The expression x^2-2x+1 is zero for $x=1$ otherwise it is positive since

$x^2-2x+1=(x-1)(x-1)=(x-1)^2$

The solution is only $x=1$ since we are given $x^2-2x+1\le 0$.

$x^2-5x+6\geq 0$

Solution

$x^2-5x+6=0$, either $x=2$ or $x=3$ and then $x^2-5x+6=(x-2)(x-3)$

x	$-\infty$		2		3		$+\infty$
$x-2$		-	0	+		+	
$x-3$		-		-	0	+	
x^2-5x+6		+	0	-	0	+	

$S=]-\infty,2]\cup[3,+\infty[$

$3x^2+x-14<0$

Solution

$3x^2+x-14=0$, $x_1=-\frac{7}{3}$ or $x_2=2$,

$3x^2+x-14=3\left(x+\frac{7}{3}\right)(x-2)=(3x+7)(x-2)$

x	$-\infty$		$-\frac{7}{3}$		2		$+\infty$
$3x+7$		-	0	+		+	
$x-2$		-		-	0	+	
$3x^2+x-14$		+	0	-	0	+	

Thus, $S=\left]-\frac{7}{3},2\right[$

$x^2-4x+4\geq 0$

Solution

$x^2-4x+4=0$.

$x_1=x_2=2$ and $x^2-4x+4=(x-2)(x-2)$ or $x^2-4x+4=(x-2)^2$

The expression x^2-4x+4 is zero for $x=2$ and positive elsewhere.

Then, $S=\mathbb{R}$

$2x^2+2x\leq 4x-10$

Solution

$2x^2+2x\leq 4x-10$

$\Leftrightarrow 2x^2+2x-4x+10\leq 0$

$\Leftrightarrow 2x^2-2x+10\leq 0$

$\Leftrightarrow x^2-x+5\leq 0$

$x^2-x+5=0$

$\Delta=1-20=-19$

The expression x^2-x+5 cannot be factorized. Let $x=0$, $0^2-0+5=5>0$. Then the expression x^2-x+5 is always positive and the solution set is an empty set.

$-2x^2+2x-10<0$

Solution

$-2x^2+2x-10<0$

$\Leftrightarrow x^2-x+5>0$

$x^2-x+5=0$

$\Delta=1-20=-19$

The expression x^2-x+5 cannot be factorized. Let $x=0$, $0^2-0+5=5>0$. Then the expression x^2-x+5 is always positive and the solution set is the set of real numbers.

Application activity 6.3.3

Solve in the set of real numbers

1. $x^2 - 10x - 20 > 0$

2. $6x^2 - 5x + 1 \leq 0$

6.4 Solving word problems involving linear or quadratic equations

Activity 6.4

1. Make a research on internet and by means of examples prepare a short presentation on how linear equations and quadratic equations can be used in daily life.

2. Read carefully the following word problems and model or rewrite them using a mathematical statement.

   - Six less than two times a number is equal to nine
   - Jane paid 22100 Frw for shoes and clothes. She paid 2100 Frw more for clothes than she did for shoes. How much did Jane pay for shoes?

3. A manufacturer develops a formula to determine the demand for its product depending on the price in Rwandan franc. The formula is , where is the price per unit, and is the number of units in demand. At what price will the demand drop to 1000 units?

CONTENT SUMMARY

It is not always easy to tell what kind of equation a word problem involves, until you start translating it to math symbols.

Words like "together," "altogether," or "combined" often indicate that the problem involves addition. The word "left," as in "he had xx amount left,» often indicates subtraction.

The following table explains the keywords used when writing equations from verbal models.

Addition	Subtraction	Multiplication	Division
Sum	difference	product	quotient
Plus	less than	times	divided
And	subtract		split up
Altogether	left		
Combined	decreased		
more than			
Increased			

Example 1:

For every subscription she sells, Joan earns a commission of $125. If she earned $1750 last month in commissions, how many subscriptions did she sell?

Solution

Here the unknown is x and it is the number of subscriptions sold. The equation is $125x = 1750$

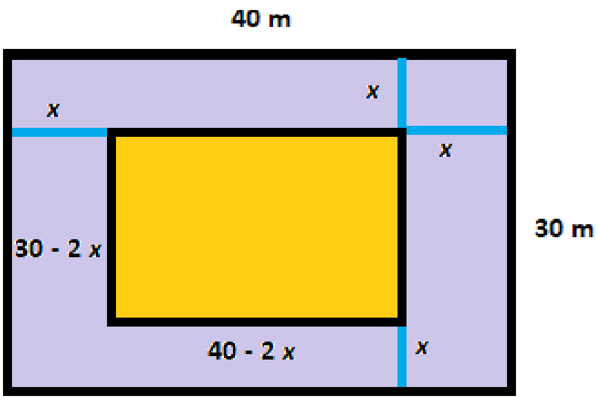
This can be solved by dividing both sides by 125.

$$x = \frac{1750}{125} \Rightarrow x = 14$$

Example 2

A rectangular building is to be placed on a plot that measures 30 m by 40 m. The building must be placed in the plot so that the width of the lawn is the same on all four sides of the building. Local restrictions state that the building cannot occupy any more than 50% of the property. What are the dimensions of the largest building that can be built on the property?

Solution



- Let x represent the width of the lawn
- Let $40-2x$ represent the length of the building
- Let $30-2x$ represent the width of the building
- The area of the plot is 1200m^2
- The maximum area of the building is 600m^2
- $(40-2x)(30-2x)=600 \Leftrightarrow 4x^2-140x+1200=600 \Leftrightarrow 4x^2-140x+600=0$
- Solving this equation, one can find that $x=5$ or $x=30$

Application activity 6.4

Read carefully the following word problems and solve the related equations

Assume that in a competitive market the demand schedule is $p=420-0.2q$ and the supply schedule is $p=60+0.4q$ ($p=price,\ q=quantity$). If the market is in equilibrium, then the equilibrium price and quantity will be where the demand and supply schedules intersect. As this will correspond to a point which is on both the demand schedule and the supply schedule the equilibrium values of p and q will be such that both equations hold. Find the equilibrium quantity and the equilibrium price by solving the following $420-0.2q=60+0.4q$

6.5 Solving and discussing parametric equations

Activity 6.5

Let λ take positive values and negative values on your choice , discuss about the parameter λ in the equation $\lambda x^2+(\lambda-1)x+2=0$

CONTENT SUMMARY

To discuss a parametric equation of the second degree with parameter λ, one can do the following:

1. Find the values of λ for which the given equation is of the second degree

2. Find the values of λ for which the equation has 0, 1 or 2 roots studying sign of Δ

3. Determine the roots using sign of product and sum in a table.

Examples

1. Discuss the parametric equation $\lambda x^2+(\lambda+1)x+\lambda=0$

Solution

$\lambda x^2+(\lambda+1)x+\lambda=0$

This equation is of the second degree if $\lambda\neq 0$

$$\Delta=(\lambda+1)^2-4\lambda^2$$
$$=\lambda^2+2\lambda+1-4\lambda^2$$
$$=-3\lambda^2+2\lambda+1$$
$$=(\lambda-1)(-3\lambda-1)$$

λ	$-\infty$		$-\frac{1}{3}$		1		$+\infty$
$-3\lambda-1$		+	0	-		-	
$\lambda-1$		-		-	0	+	
Δ		-	0	+	0	-	

Product: $p=\frac{c}{a}=\frac{\lambda}{\lambda}>0$

Sum: $s=-\frac{b}{a}=\frac{-\lambda-1}{\lambda}$

λ	$-\infty$		-1		0		$+\infty$
$-\lambda-1$		+	0	-		-	
λ		-		-	0	+	
s		-	0	+	||	-	

Summary table

λ	Δ	p	s	Conclusion
$\left]-\infty,-\frac{1}{3}\right[$	-	+	0	No real roots

$-\frac{1}{3}$	0	+	+	One positive double root equal to $\frac{s}{2}=1$
$\left]-\frac{1}{3},0\right[$	+	+	+	Two distinct positive roots
0	||	||	||	The equation is of the first degree, $x=0$
$]0,1[$	+	+	-	Two distinct negative roots
1	0	+	-	One negative double root equal to $\frac{s}{2}=-1$
$]1,+\infty[$	-	+	-	No real roots

2. Discuss the parametric equation $\lambda x^2+2(\lambda+2)x+2\lambda+7=0$

Solution

$\lambda x^2+2(\lambda+2)x+2\lambda+7=0$

This equation is of the second degree if $\lambda \neq 0$

$$\Delta=4(\lambda+2)^2-4\lambda(2\lambda+7)$$
$$=4\left(\lambda^2+4\lambda+4-2\lambda^2-7\lambda\right)$$
$$=4\left(-\lambda^2-3\lambda+4\right)$$
$$=4(-\lambda-4)(\lambda-1)$$

λ	$-\infty$		-4			1		$+\infty$
$-\lambda-4$		+	0	-	-			
$\lambda-1$		-		-		0	+	
Δ		-	0	+		0	-	

Product: $p=\frac{c}{a}=\frac{2\lambda+7}{\lambda}$

λ	$-\infty$		$-\frac{7}{2}$		0		$+\infty$
$2\lambda+7$		-	0	+		+	
λ		-		-	0	+	
p		+	0	-	$\|\|$	+	

Sum: $s=-\frac{b}{a}=\frac{-2\lambda-4}{\lambda}$

λ	$-\infty$		-2		0		$+\infty$
$-2\lambda-4$		+	0	-		-	
λ		-		-	0	+	
s		-	0	+	$\|\|$	-	

Summary table

λ	Δ	p	s	Conclusion
$]-\infty,-4[$	-	+	-	No real roots
-4	0	+	-	One negative double root equal to $\frac{s}{2}=-\frac{1}{2}$
$\left]-4,-\frac{7}{2}\right[$	+	+	-	Two distinct negative roots
$-\frac{7}{2}$	**+**	**0**	**-**	Two distinct roots: $x_1=0$ and $x_2=s=\frac{6}{7}$
$\left]-\frac{7}{2},-2\right[$	+	-	-	Two distinct roots with different sign
-2	+	-	0	Two opposite real roots: $x_1=-\frac{\sqrt{6}}{2}$, $x_2=\frac{\sqrt{6}}{2}$
$]-2,0[$	+	-	+	Two distinct roots with different sign

0	||	||	||	Equation of the first degree, $x=-\frac{7}{4}$
$]0,1[$	+	+	-	Two distinct negative roots
1	0	+	-	One negative double root equal to $\frac{s}{2}=-6$
$]1,+\infty[$	-	+	-	No real roots

Application activity 6.5

Discuss the following parametric equation: $(10-\lambda)x^2-6x+\lambda=0$

6.6. END UNIT ASSESSMENT 6

1. Solve the following equations and inequalities in set of real numbers

    a. $x+5=2x-8$

    b. $2x-8\geq 0$

    c. $(x+3)(x-2)<0$

    d. $x^2-10x+1=0$

    e. $6x^2-5x+1\leq 0$

    f. $x^4-3x^3+4x^2-3x+1=0$

2. Discuss the following parametric equation: $x^2-\lambda x+\lambda-4=0$

3. Suppose your cell phone plan is 3000 Frw per month plus 20 Frw per minute. Your bill is 7 000 Frw. Use the equation $3\,000+20x=7\,000$, to find out how many minutes are on your bill.

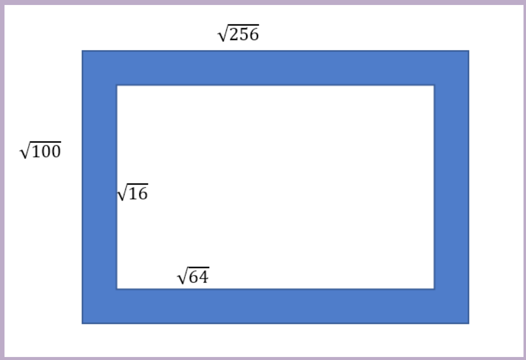
UNIT: 7

PROBLEMS ON POWERS, INDICES, RADICALS AND LOGALITHMS

Key Unit competence: Solve problems related to powers, indices, radical and common logarithms

7.0. Introductory Activity

1. The diagram below shows the design made on the table top. Discuss on how to find the area of painted region in cm^2 and calculate the area of the painted region.



2. Let $P = b^p$ *and* $Q = b^q$

   a) Using he numbers on your choice, discuss and show that:

   $1) P.Q = b^{p+q}$ $\qquad 2) \frac{P}{Q} = b^{p-q}$ $\qquad 3) P^n = b^{np}$ $\qquad 4) \sqrt[n]{P} = b^{\frac{p}{n}}$

   b) Evaluate each of the above expression for

   $n = 3, \quad b = 2, \quad p = 3 \quad and \quad q = 7$

7.1 Powers and radicals

7.1.1 Definition of powers/ indices and radicals

Activity 7.1.1

1. Knowing that $4 = 2\times 2 \text{ or } 2^2$, examine each expression and rewrite it in power form.

   a) $5\times5\times5\times5\times5\times5\times5$ b) $3\times3\times3\times3\times t\times t\times s\times s\times s\times s\times s$

   c) $(x+2)(x+2)(x+2)$

2. Express the number inside the radical symbol in power form. Observe the exponent of the number and the index of the radical sign and simplify the expression. What do you notice?

   a) $\sqrt{4}$ b) $\sqrt[3]{8}$ c) $\sqrt[4]{16}$ d) $\sqrt[5]{32}$ e) $\sqrt[6]{64}$ f) $\sqrt[7]{128}$

CONTENT SUMMARY

Powers/ indices or exponents

The power (or exponent) of a number says how many times to use the number in a multiplication. It is written as a small number to the right and above the base number.

Suppose that a is a real number. When the product $a\times a\times a\times a\times a\times a$ is written in simplest form as a^6, the number 6 is called **index/power/exponent.** So far, when the index is a **positive integer** n, then a^n means $a\times a\times a\times\cdots\times a$ n times and $a^n = \underbrace{a\times a\times a\times\ldots\times a}_{n-times}$ is the general form of powers.

Example1: **Write each expression in its simplest form of powers.**

1. $c^3.c^3.c^3.c^3.c^3$

2. $y^{-3}.y^{-3}.y^{-3}$

Solution

$$a)c^3.c^3.c^3.c^3.c^3 = c^{15}$$

$$b)y^{-3}.y^{-3}.y^{-3} = y^{-9}$$

Example2: Expand the expressions below

a. x^5 **b.** $(a-e)^3$

Solution

a) $x^5 = x \cdot x \cdot x \cdot x \cdot x$

b) $(a-e)^3 = (a-e)(a-e)(a-e)$

Radicals

In mathematics, a radical expression is defined as any expression containing a radical ($\sqrt{\ }$) symbol.

- The number under the root symbol is called radicand.
- The expression $\sqrt[n]{a}$ is read as «a radical n» or «the n th root of a"
- The expression $\sqrt[a]{b^c}$ is read as "ath root of b raised to the c power. The value of the index a is always positive.

In general

- If $n=2$ and a is a real number, then the symbol $\sqrt{a}$ denotes **the square root of** a .
- If $n=3$ and a is a real number, then the symbol $\sqrt[3]{a}$ denotes **the cubic root of** a **.**
- If $n=4$ and a is a real number ,then the symbol $\sqrt[4]{a}$ denotes **the 4th root of** a
- If n and a are positive real numbers, then the symbol $\sqrt[n]{a}$ denotes **the** n^{th} **roots of** a **.**
- **If *n*** is **an even** positive integer, then in the real number system we have the following:

  a) If $a>0$, then a has two n^{th} roots, one positive and one negative.

  b) If $a=0$, then a has one n^{th} root which is zero.

  c) If $a<0$, then a has no real n^{th} roots.

- **If n is an odd** positive integer other than 1, then in the real number system

  a. If $a > 0$, then a has one n^{th} root which is positive.

  b. If $a = 0$, then a has one n^{th} root which is zero.

  c. If $a < 0$, then a has one n^{th} roots which is negative.

Example 3: Rewrite in power form the number under radical symbol and simplify the radicals

$a)\sqrt{4}$ $\quad b)\sqrt[3]{125}$ $\quad \sqrt[4]{256}$

Solution

a) $\sqrt{4} = \sqrt{2^2} = 2$, where $4=2^2$

b) $\sqrt[3]{125} = \sqrt[3]{5^3} = 5$, where $125=5^3$

c) $\sqrt[4]{256} = \sqrt[4]{4^4} = 4$, where $256=4^4$

Application activity 7.1.1

1. Rewrite in power form the number under radical symbol and simplify each of the radicals

a) $\sqrt{700}$ b) $\sqrt[3]{24}$ c) $\sqrt[5]{-64}$ d) $\sqrt{72}$ e) $\sqrt{\frac{25}{144}}$ f) $\sqrt{\frac{5}{8}}$

2. A square window has an area of 22500square centimetres. What is the length of each side of the window?

7.1.2 Properties of indices and radicals

Activity 7.1.2

1. Work out:

   a) $10^2.10^3 =$

   b) $(2^5)^2 =$

   c) i) $\frac{5^4}{5^2} =$

   ii) $\frac{3^6}{3^8} =$

   iii) using numbers of your own choice, what will be the answer when you divide numbers or variables with the same base and the same powers? Is a^{m-m} always 1? Justify your answer.

2. Say whether the statement is true or false. Justify your answer.

a. $\sqrt{36.81} = \sqrt{36}.\sqrt{81}$

b. $\sqrt[3]{\frac{8}{27}} = \frac{\sqrt[3]{8}}{\sqrt[3]{27}}$

CONTENT SUMMARY

Properties of indices/powers/exponents.

- **Multiplying powers with the same base**

To multiply numbers or variables with the same base, add their powers.

$$a^m \cdot a^n = a^{m+n}$$; for positive integers m and n.

Example: Simplify each expression.

$a) 3^5 . 3^4$ $\quad b) h^2 . h^9$ $\quad c) a^3 . b^4$ $\quad d) -2y^3 . 3y^4$

Solution

a) $3^5.3^4 = 3.3.3.3.3.3.3.3.3 = 3^9$ *or* $3^5.3^4 = 3^{5+4} = 3^9$

b) $h^2.h^9 = h.h.h.h.h.h.h.h.h.h.h = h^{11}$ *or* $h^2.h^9 = h^{2+9} = h^{11}$

c) $a^3.b^4.a^2.a^5.b^8 = a^{3+2+5}.b^{4+8} = a^{10}.b^{12}$

d) $-2y^3.3y^4 = -2.\ 3.y^3.y^4 = -6y^{3+4} = -6y^7$

- **Finding a power of a power**

To find a power of a power, multiply those powers.

$(a^m)^n = a^{m.n}$; for positive integers m and n.

Example: Simplify each expression.

a) $(2^3)^4$ b) $p^5.(p^3)^2$

Solution

a) $(2^3)^4 = (2^3).(2^3).(2^3).(2^3) = 2^{3+3+3+3} = 2^{12}$ or $(2^3)^4 = 2^{3.4} = 2^{12}$

b) $p^5.(p^3)^2 = p^5.p^{3.2} = p^{5+6} = p^{11}$

- **Raising a product to a power**

For every nonzero number a and b and positive integer n then,

$(ab)^n = a^n.b^n$.

Example: Simplify each expression.

a) $(xyz)^4$ b) $(x^4)^2(3xy^4)^4$

Solution

a) $(xyz)^4 = (x^4).(y^4).(z^4) = x^4y^4z^4$

b) $(x^4)^2(3xy^4)^4 = (x^4)^2 3^4x^4(y^4)^4 = x^8.3^4.x^4.y^{16} = 81x^{12}y^{16}$

- **Dividing powers with the same bases**

To divide numbers or variables with the same base, subtract their powers.

$\frac{a^m}{a^n} = a^{m-n}$; for $a \neq 0$ and positive integers m and n.

Example: Simplify each expression.

a) $\frac{3^8}{3^5}$ b) $\frac{v^{37}u^5}{u^5v^{15}}$

Solution

a) $\frac{3^8}{3^5} = 3^{8-5} = 3^3 = 27$ b) $\frac{v^{37}u^5}{u^5v^{15}} = v^{37-15}.u^{5-5} = v^{22}.u^0 = v^{22}$

- **Raising a ratio to a power**

For every nonzero number a and b and positive integer n then, $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$

Example: Simplify each expression

a) $\left(\frac{a}{b}\right)^3$ b) $\left(\frac{2r}{t}\right)^n$

Solution

a) $\left(\frac{a}{b}\right)^3 = \frac{a^3}{b^3}$ b) $\left(\frac{2r}{t}\right)^n = \frac{2^n r^n}{t^n}$

- **Zero as a power**

If a power of a nonzero number or variable is zero, then $a^0 = 1$; for $a \neq 0$.

Example: Simplify each expression

a) 23^0 b) $(-2)^0$ c) $(1.02)^0$ d) $\left(\frac{2}{5}\right)^0$

Solution

a) $23^0 = 1$ b) $(-2)^0 = 1$ c) $(1.02)^0 = 1$ d) $\left(\frac{2}{5}\right)^0 = 1$

Negative powers

$a^{-n} = \frac{1}{a^n}$; for $a \neq 0$.

Example1: Simplify each expression

a) 4^{-3} b) $(-7)^{-1}$

Solution

a) $4^{-3}=\frac{1}{4^3}=\frac{1}{64}$ b) $(-7)^{-1}=\frac{1}{(-7)^1}=\frac{1}{-7}=-\frac{1}{7}$

Example2: Simplify and write each expression with positive exponents.

a) $\frac{a^3}{a^5}$ b) $\frac{r^3s}{s^5r}$

Solution

a) $\frac{a^3}{a^5}=a^{3-5}=a^{-2}=\frac{1}{a^2}$ b) $\frac{r^3s}{s^5r}=r^{3-1}s^{1-5}=r^2s^{-4}=\frac{r^2}{s^4}$

- **Fractional index/power property and its converse**

For every nonzero number a and positive integers m and n, the following identities are equivalent:

$$a^{\frac{m}{n}}=\sqrt[n]{a^m} \quad \text{and} \quad \sqrt[n]{a^m}=a^{\frac{m}{n}}$$

$$a^{\frac{1}{n}}=\sqrt[n]{a} \quad \text{and} \quad \sqrt[n]{a}=a^{\frac{1}{n}}$$

Example: Evaluate

a) $\sqrt[5]{32}$ b) $\sqrt[5]{p^5q^{10}}$

Solution

a) $\sqrt[5]{32}=\sqrt[5]{2^5}=2^{\frac{5}{5}}=2$

b) $\sqrt[5]{p^5q^{10}}=p^{\frac{5}{5}}q^{\frac{10}{5}}=pq^2$

Properties of n^{th} roots.

- **Multiplication property of radicals and its converse**

For every number $a\geq 0$ and $b\geq 0$ and positive integer n

$$\sqrt[n]{a.b}=\sqrt[n]{a}.\sqrt[n]{b} \quad \text{and} \quad \sqrt[n]{a}.\sqrt[n]{b}=\sqrt[n]{a.b}$$

Example: simplify

a) $\sqrt[3]{8x^{15}y^{12}}$ b) $\sqrt[6]{64p^{18}r^{15}}$

Solution

a) $\sqrt[3]{8x^{15}y^{12}} = \sqrt[3]{2^3}.\sqrt[3]{x^{15}}.\sqrt[3]{y^{12}} = 2^{\frac{3}{3}}.x^{\frac{15}{3}}.y^{\frac{12}{3}} = 2x^5y^4$

b) $\sqrt[6]{64p^{18}r^{15}} = \sqrt[6]{2^6}.\sqrt[6]{p^{18}}\sqrt[6]{r^{12}.r^3} = 2p^3r^2\sqrt{r}$

- **Division property of radicals and its converse**

For every number $a \geq 0$ and $b > 0$ and positive integer n

$$\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}} \quad \text{and} \quad \frac{\sqrt[n]{a}}{\sqrt[n]{b}} = \sqrt[n]{\frac{a}{b}}$$

Example: show that $\sqrt[4]{\frac{16w^8m^{12}}{81n^4}}$

Solution

$$\sqrt[4]{\frac{16w^8m^{12}}{81n^4}} = \sqrt[4]{\frac{(2w^2)^4(m^3)^4}{(3n)^4}} = \left[\frac{(2w^2)^4(m^3)^4}{(3n)^4}\right]^{\frac{1}{4}} = \frac{2w^2m^3}{3n}$$

- **A radical of a radical property its converse**

a) $\sqrt[n]{\sqrt[m]{a}} = \sqrt[nm]{a} = a^{\frac{1}{nm}}$

In fact,

$$\sqrt[n]{\sqrt[m]{a}} = \left(a^{\frac{1}{m}}\right)^{\frac{1}{n}} = a^{\frac{1}{m}\times\frac{1}{m}} = a^{\frac{1}{mn}} = \sqrt[mn]{a}$$

Example

Simplify : $\sqrt[3]{\sqrt{64}}$

Solution: $\sqrt[3]{\sqrt{64}} = \sqrt[6]{64} = \sqrt[6]{2^6} = 2$

- **But it must be carefully noted that**

For nonzero a and b: $\sqrt[n]{a\pm b}\neq\sqrt[n]{a}\pm\sqrt[n]{b}$

Application activity 7.1.2

1. Simplify each of the following expression

   a) $\sqrt[10]{a^{15}b^{10}}$

   b) $\sqrt{a^3b^5c^3}$

   c) $\sqrt{\dfrac{3s^3}{27s}}$

2. A student wrote that $-5^0=1$. What was the student's error?
3. Is -3^{-2} positive or negative? Justify.
4. The length of one side of a cube is $\left(3^2\times2^3\right)^2 m$. What is the volume of the cube?
5. Simplify $\left(u^3\right)^4$ and u^{3^4}. Are the expression equivalent?
6. a)Does have the same value as ? Justify your answer.
   b) Does ? Explain.
7. Which of or is twice the value of . Explain.

7.2 Operations on indices and radicals

Activity 7.2

1. Express the following to the simplest form and explain your working steps

   a) x^3x^2 b) $\dfrac{6xy^2}{3xy}$ c) $\dfrac{yx}{4xy}$

2.Simplify the following expressions and explain your working steps

1. $\sqrt{18}+\sqrt{2}$
2. $\sqrt{12}-3\sqrt{3}$
3. $\sqrt{2}\times\sqrt{3}$
4. $\dfrac{\sqrt{6}}{\sqrt{2}}$

CONTENT SUMMARY

Addition and subtraction of indices and radicals

When adding or subtracting the indices, we may need to combine like terms together to make one.

Example: carry out the following addition

$x^3 + x^3$

Solution

$x^3 + x^3 = 2x^3$

When adding or subtracting the radicals we may need to simplify if we have similar radicals. Similar radicals are the radicals with the same indices and same bases.

Example: carry out the following addition

$\sqrt{2} + \sqrt{8}$

Solution

$\sqrt{2} + \sqrt{8} = \sqrt{2} + \sqrt{2 \times 4} = \sqrt{2} + \sqrt{2} \times \sqrt{4} = \sqrt{2} + 2\sqrt{2} = 3\sqrt{2}$

Multiplication and division of indices and radicals

When multiplying or dividing the indices, we may need to apply the following rules or properties

a) $a^m \cdot a^n = a^{m+n}$

b) $\left(a^m\right)^n = a^{mn}$

c) $\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$

d) $\frac{1}{b^m} = b^{-m}$

e) $\frac{a^m}{a^n} = a^{m-n}$

f) $(ab)^m = a^m b^m$

These rules help to simplify some powers.

Example: simplify the following

$a)2^4.2^3.4$ $\quad b)a^4.b^3.\ a^5.\ b^8$ $\quad c)\dfrac{y^9}{y^2}$

Solution

a) $2^4 \cdot 2^3 \cdot 4 = 2^4 \cdot 2^3 \cdot 2^2 = 2^9 = 512$

b) $a^4 \cdot b^3 \cdot a^5 \cdot b^8 = a^4 \cdot a^5 \cdot b^3 \cdot b^8 = a^9 \cdot b^{11}$

$a^9 \cdot b^{11}$ cannot be simplified further because the bases are different.

c) $\dfrac{y^9}{y^2} = y^{9-2} = y^7$

When multiplying or dividing the **radicals**, we may need to apply the rules

$\forall n \in \mathbb{N} \setminus \{1\}, m \in \mathbb{R}$

a) $\sqrt[n]{a^m} = a^{\frac{m}{n}}$

b) $\sqrt[n]{ab} = \sqrt[n]{a} \cdot \sqrt[n]{b}$

c) $\sqrt[n]{\dfrac{a}{b}} = \dfrac{\sqrt[n]{a}}{\sqrt[n]{b}}$

d) $\sqrt[n]{\sqrt[m]{a}} = \sqrt[nm]{a} = a^{\frac{1}{nm}}$

Rationalizing the denominator.

Rationalizing the denominator is to convert a fraction with an irrational denominator to a fraction with rational denominator. To do this, if the denominator involves radicals we multiply both the numerator and denominator by the conjugate of the denominator.

The conjugate of $a \pm \sqrt{b}$ is $a \mp \sqrt{b}$.

The conjugate of is $\sqrt{a} \pm \sqrt{b}$ is $\sqrt{a} \mp \sqrt{b}$

Remember that $(a+b)(a-b) = a^2 - b^2$

Note: in case of one term, the squaring approach is used as follows:

- $\sqrt{a}.\sqrt{a} = a$
- $a\sqrt{b}.\sqrt{b} = ab$

Example: remove the radical on denominator

$a)\dfrac{1}{1+\sqrt{2}}$ $b)\dfrac{\sqrt{2}}{\sqrt{5}-\sqrt{3}}$ $c)\dfrac{\sqrt{3}+\sqrt{7}}{4\sqrt{2}}$

Solution

a) $\dfrac{1}{1+\sqrt{2}} = \dfrac{1-\sqrt{2}}{\left(1+\sqrt{2}\right)\left(1-\sqrt{2}\right)} = \dfrac{1-\sqrt{2}}{1-2} = \dfrac{1-\sqrt{2}}{-1} = \sqrt{2}+1$

b) $\dfrac{\sqrt{2}}{\sqrt{5}-\sqrt{3}} = \dfrac{\sqrt{2}\left(\sqrt{5}+\sqrt{3}\right)}{\left(\sqrt{5}-\sqrt{3}\right)\left(\sqrt{5}+\sqrt{3}\right)} = \dfrac{\sqrt{10}+\sqrt{6}}{5-3} = \dfrac{\sqrt{10}+\sqrt{6}}{2}$

c) $\dfrac{\sqrt{3}+\sqrt{7}}{4\sqrt{2}} = \dfrac{\left(\sqrt{3}+\sqrt{7}\right)\sqrt{2}}{\left(4\sqrt{2}\right)\sqrt{2}} = \dfrac{\sqrt{6}+\sqrt{14}}{8}$

Application activity 7.2

1. Simplify each of the following

   a. $\left(xy^3\right)^2 + 4x^2y^6$ b. $\dfrac{ab}{a^3} - \dfrac{a^3b^2}{a^5b}$

2. Evaluate the following:

   a. $\sqrt{20}+\sqrt{5}$ b. $5\sqrt{7}-\sqrt{28}$ c. $4\sqrt{3}-\sqrt{12}$ d. $\sqrt{18}\times\sqrt{8}$

   e. $\sqrt{45}+\sqrt{80}+\sqrt{180}$

3. Make the denominator of each of the following rational

   a) $\dfrac{1}{\sqrt{2}}$ b) $\dfrac{2-\sqrt{3}}{2\sqrt{5}}$ c) $\dfrac{2}{1-\sqrt{6}}$ d) $\dfrac{\sqrt{2}+\sqrt{3}}{\sqrt{3}+\sqrt{5}}$

7.3. Decimal logarithm

7.3.1 Definition

Activity 7.3.1

What is the real number for which 10 must be raised to obtain:

1) 1	2) 10
3) 100	4) 1000
5) 10000	6) 100000

CONTENT SUMMARY

The **decimal logarithm** of a positive real number is defined to be a real number for which 10 must be raised to obtain . It means that y=10^x and we write

$\forall x > 0, \quad y = \log x$

$\log x$ is the same as $\log_{10} x$ and is defined for all positive real numbers only, 10 is the base of this logarithm. In general notation we do not write this base for decimal logarithm.

In the notation $y = \log x$, x is said to be the **antilogarithm of y**.

Example : log (100) = ?

We are required to find the power to which 10 must be raised to obtain 100

So $\log(100) = \log(10^2) = 2$

$y = \log x$ means $10^y = x$

Be careful!

- $\log 2x + 1 \neq \log(2x+1)$
- $\log 2x + 1 = (\log 2x) + 1$

Since logarithms are defined using exponentials, any "log " has an equivalent "exponent" form, and vice-versa.

Example 1 : $\log 10^5 = 5$

Example 2 : $\log 0.01 = \log 10^{-2} = -2$

Co-logarithm

Co-logarithm, sometimes shortened to **colog** of a number is the logarithm of the reciprocal of that number, equal to the negative of the logarithm of the number itself,

$$\text{colog}\, x = \log\left(\frac{1}{x}\right) = -\log x$$

Example 3: colog 200 = –log 200 = –2.3010

Application activity 7.3.1

1. Change each exponential form below into logarithmic form

   a) $7^2 = 49$ b) $6^{-1} = \frac{1}{6}$ c) $10^0 = 1$ d) $\sqrt[3]{8} = 2$

2. Find co-logarithm of

   a) 100 b) 42 c)15

3. Change each logarithmic form below into exponential form

   a) $\log_3 81 = 4$ b) $\log_{10} 10 = 1$ c) $\log_5 \frac{1}{625} = -4$ d) $\log_9 27 = \frac{3}{2}$

4. Find the value of x, such that

   a) $x = \log_5 125$ b) $x = \log_{10} 0.001$ c) $x = \log_8 2$

   d) $x = \log_5 \frac{1}{25}$ e) $\log_x 243 = 5$ f) $\log_6 x = 2$

   g) $x = \log_{\frac{1}{2}} 32$

7.3.2 properties and operations on decimal logarithms

Activity 7.3.2

1. Calculate: a) log 100 b) log 1000
2. Calculate log(100)x(1000) .
3. Compare the results in 2 and the sum log 100 + log 1000 . Deduce log(*axb*) when *a and b* are positive real numbers

CONTENT SUMMARY

$\forall a,b \in]0,+\infty[$

- **Logarithm of the product of two or more positive numbers is the sum of logarithm of those numbers**

  a) $\log ab = \log a + \log b$

- **Logarithm of the quotient of two positive numbers is the difference of logarithm of those numbers**

  b) $\log \frac{a}{b} = \log a - \log b$

- **Logarithm of the power of a positive number is the product of the exponent with the logarithm of that number.**

  c) $\log a^n = n \log a$

- **Others properties**

$$\log \frac{1}{b} = -\log b$$

$$\log_a a = 1$$

$$\log_a 1 = 0$$

$$\log_a \sqrt[n]{a} = \frac{1}{n} \quad \text{and} \quad \log_a \sqrt[n]{b} = \frac{1}{n}\log_a b$$

$$\log_a \sqrt[n]{a^m} = \frac{m}{n} \quad \text{and} \quad \log_a \sqrt[n]{b^m} = \frac{m}{n}\log_a b$$

Example 1

Calculate in function of $\log a, \log b \; and \; \log c$

a) $\log a^2b^2$

b) $\log \frac{ab}{c}$

c) $\log \frac{ab}{\sqrt{c}}$

Solution

a) $\log a^2b^2 = \log(ab)^2$

$$= 2\log ab$$
$$= 2(\log a + \log b)$$

b) $\log \frac{ab}{c} = \log ab - \log c$
$$= \log a + \log b - \log c$$

c) $\log \frac{ab}{\sqrt{c}} = \log ab - \log \sqrt{c}$
$$= \log a + \log b - \log (c)^{\frac{1}{2}}$$
$$= \log a + \log b - \frac{1}{2}\log c$$

Example 2

Given that $\log 2 = 0.30$, $\log 3 = 0.48$ and $\log 5 = 0.7$. Calculate

a) $\log 6$

b) $\log 0.9$

Solution

a) $\log 6 = \log(2\times 3)$
$$= \log 2 + \log 3$$
$$= 0.30 + 0.48$$
$$= 0.78$$

b) $\log 0.9 = \log \frac{9}{10}$
$$= \log 9 - \log 10$$
$$= \log 3^2 - \log(2\times 5)$$
$$= 2\log 3 - \log 2 - \log 5$$
$$= 2(0.48) - 0.30 - 0.7$$
$$= -0.04$$

Change of base formula

If u ($u > 0$) and if a and b are positive real numbers different from 1, $\log_b u = \frac{\log_a u}{\log_a b}$

This means that if you have a logarithm in any other base, you can convert it in the decimal logarithm in the following way where $a = 10$:

$$\log_b u = \frac{\log_{10} u}{\log_{10} b} = \frac{\log u}{\log b}.$$

Example 3: $\log_2 5 = \frac{\log_{10} 5}{\log_{10} 2} \approx 2.322$

More generally, we apply power properties to solve equations.

Examples:

Solve the following equations:

$a) 8^{1-x} = 4^{2x+3}$

$b) a^{\frac{3}{2}} = 8$

Solution

$a) 8^{1-x} = 4^{2x+3}$

$\Rightarrow \left(2^3\right)^{1-x} = \left(2^2\right)^{2x+3} \Leftrightarrow 2^{3-3x} = 2^{4x+6}$

$\Rightarrow 3 - 3x = 4x + 6 \Leftrightarrow x = -\frac{3}{7}$

$b) a^{\frac{3}{2}} = 8$

$\Rightarrow a = \sqrt[3]{8^2} \Leftrightarrow a = \sqrt[3]{\left(2^3\right)^2} \Leftrightarrow a = \sqrt[3]{\left(2^2\right)^3}$

$\Rightarrow a = 2^2 = 4$

We may use logarithmic properties/ or laws to simplify and solve exponential equations.

Examples

1) If $y = 5x^7$, find a linear expression connecting $\log x$ and $\log y$.

Solution

Since $y = 5x^7$, then $\log y = \log 5x^7 = \log 5 + \log x^7 = \log 5 + 7\log x$

$$\log y = \log 5 + 7\log x$$

2) Write an expression equivalent to $\log y = 4 - 3\log x$.

Solution

Since $\log y = 4 - 3\log x$, knowing that $4 = 4\log 10 = \log 10^4$, then

$$\log y = \log 10^4 - \log x^3$$

$$\Rightarrow \log y = \log \frac{10^4}{x^3}$$

$$\Rightarrow \log y = \log \frac{10,000}{x^3}$$

3) Use logarithms to solve the equations below:

$a)5^x = 0.04$ $\quad b)5^x = 0.4$ $\quad c)2^{3x} = 3^{2x-1}$

Solution

a) Since $0.04 = \frac{1}{25} = 5^{-2}$ then $5^x = 5^{-2}$.

By introducing logarithm to both sides of equation $\log 5^x = \log 5^{-2}$ and using the power law $\log x^n = n\log x$, then

$$x\log 5 = -2\log 5$$

$$\Rightarrow x = \frac{-2\log 5}{\log 5} = -2 \qquad \therefore x = -2$$

b) Since $5^x = 0.4$, introduce logarithm to both sides of equation.

$$\log 5^x = \log 0.4$$

$$\Rightarrow x\log 5 = \log 0.4$$

$$\Rightarrow x = \frac{\log 0.4}{\log 5} = -0.569.$$

c) **Since** $2^{3x} = 3^{2x-1}$, then

$$\log 2^{3x} = \log 3^{2x-1}$$
$$\Rightarrow 3x\log 2 = (2x-1)\log 3$$
$$\Rightarrow 3x\log 2 = 2x\log 3 - \log 3$$
$$\Rightarrow x(3\log 2 - 2\log 3) = -\log 3$$
$$\Rightarrow x = \frac{-\log 3}{3\log 2 - 2\log 3}$$

Note: logarithmic properties are very important to solve some real life problems.

a. Logarithms are used to calculate the compound interest.

Considering the case in which P is invested in a bank for n interest periods per year at the rate r expressed as a decimal. The accumulated amount A after the time t (number of years P is invested) is given by:

$A = P\left(1+\frac{r}{n}\right)^{nt}$ where n is the number of interest periods per year, r is the interest rate expressed as decimal, A is the amount after t -years.

Note: When the interest rate is compounded per year, $A = P(1+r)^n$ where r is expressed as a decimal for example $r = 9\% = 0.09$. When the interest rate is compounded monthly, $A = P\left(1+\frac{r}{12}\right)^{12.t}$ where t is the number of years and r expressed as a decimal.

Example 1:

How long will it take10,000Frw to double in an account earning 2% compounded quarterly?

Solution

For this problem, we'll use the compound Interest formula,

$F = P(1+i)^n$ where F is the final value, P the initial value of investment.

Since we want to know how long it will take, let t represent the time in years. The number of compounding periods is four times the time or $n = 4t$. The original amount is $P = 10,000$ and the future value is the double or F = 20,000. The interest rate per period is $i = \frac{0.02}{4} = 0.005$

When these values are substituted into the compound interest formula, we get the exponential equation

$$20000 = 10000(1.005)^{4t}$$

To solve this equation for t, isolate the exponential factor by dividing both sides by 10,000 to give $2 = (1.005)^{4t}$

Convert this exponential form to logarithm form and divide by 4

$$4t = \log_{1.005}(2)$$

$$t = \frac{\log_{1.005}(2)}{4}$$

To find an approximate value, use the Change of base formula to convert to a natural logarithm (or a common logarithm)

$t = \dfrac{\frac{\ln(2)}{\ln(1.005)}}{4} \approx 34.7\,years$; where $\log_e x = \ln x$. This is another special logarithm called natural logarithm $e \approx 2.71828$

It is interesting to note that the starting amount is irrelevant when doubling. If we started with P dollars and wanted to accumulate 2P at the same interest's rate and compounding periods, we would need to solve

$$2P = P(1.005)^{4t}.$$

This reduces to the same equation as above $2 = (1.005)^{4t}$ when both sides are divided by P. This means it takes about 37.4 years to double any amount of money at an interest rate of 2% compounded quarterly.

b. Logarithms are used to calculate the final value of an investment.

Consider an investment at compound interest where:

P is the initial sum invested, A is the final value of the investment, r is the interest rate per time period (as a decimal fraction) and n is the number of time periods.

The value of the investment at the end of each year will be $1+r$ times the sum invested at the start of the year.

Thus, for any investment,

- The value after one year $= P(1+r)$
- The value after 2 years $= P(1+r)(1+\mathrm{r}) = \mathrm{P}(1+\mathrm{r})^2$
- The value after 3 years $= P(1+r)(1+\mathrm{r})(1+r) = \mathrm{P}(1+r)^3$ *etc*

We can see that each value is multiplied by $(1+r)$ to the power of number of years that the sum is invested. Thus, after n years the initial sum A is multiplied by $(1+r)^n$.

The formula for the initial value A. Thus, after an investment of P money, for n time periods at interest rate r is therefore $A = P(1+r)^n$.

Examples:

1) If \$600 is invested for 3 years at 8% then the known values for the formula will be

$P = \$600; \quad n = 3; \quad r = 8\% = 0.08$. Thus the final sum will be

$$A = P(1+r)^n = 600(1.08)^3 = \$755.83$$

2) If \$4,000 is invested for 10 years at an interest rate of 11% per annum what will the final value of the investment be?

Solution:

$\mathrm{P} = \$4{,}000, \mathrm{n} = 10 \quad \mathrm{r} = 11\% = 0.11$

$F = A(1 + i)n = 4{,}000(1.11)10$

$$A = P(1+r)^n = 4000(1.11)^{10} = 11{,}253.68$$

The final value of the investment be $\$11{,}253.68$

c. Logarithms are used to time periods, initial amounts and interest rates

The formula for the final sum of an investment contains the four variables F, A, i and n.

So far we have only calculated F for given values of A, i and n. However, if the values of any three of the variables in this equation are given then one can usually calculate the fourth.

- ***Initial amount***

A formula to calculate A, when values for F, i and n are given, can be derived as follows.

Since the final sum formula is

$$A = P(1+r)^n,$$

then, dividing through by $(1+r)^n$ (we get the initial sum formula

$$P = \frac{A}{(1+r)^n} \quad or\ P = A(1+r)^{-n}$$

Example:

How much money needs to be invested now in order to accumulate a final sum of

$12,000 in 4 years' time at an annual rate of interest of 10%?

Solution:

Using the formula derived above, the initial amount is

$$P = A(1+r)^{-n} = 12,000(1.1)^{-4} = \$8,196.16$$

What we have actually done in the above example is find the sum of money that is equivalent to $12,000 in 4 years' time if interest rates are 10%. An investor would therefore be indifferent between (a) $8,196.16 now and (b) $12,000 in 4 years' time. The $8,196.16 is therefore known as the 'present value' (PV) of the $12,000 in 4 years' time. We shall come back to this concept in the next few sections when methods of appraising different types of investment project are explained.

- **Time period**

Calculating the time period is rather more tricky than the calculation of the initial amount.

From the final sum formula

$$A = P(1+r)^n \text{, then } \frac{A}{P} = (1+r)^n$$

If the values of A, P and r are given and one is trying to find n this means that one has to work out to what power (1 + r) has to be raised to equal $\frac{A}{P}$. One way of doing this is via logarithms.

Example

For how many years must $1,000 be invested at 10% in order to accumulate $1,600?

Solution

$P = \$1,000,\ \ A = \$1,600,\ \ \ r = 10\% = 0.1$

Substituting these values into the formula

$$\frac{A}{P} = (1+r)^n \text{ then, } \frac{1600}{1000} = (1+0.1)^n$$

We get $1.6 = (1.1)^n$

Since to find the nth power of a number its logarithm must be multiplied by n. Finding logs, this means that our equation becomes

$$\log 1.6 = n\log(1.1)$$

And $n = \frac{\log(1.6)}{\log(1.1)} = 4.93$. Given that 4,93 years is approximately 5years,
If investments must be made for whole years then the answer is 5 years.

This answer can be checked using the final sum formula

$$A = P(1+r)^n = 1000(1.1)^5 = 1,610.51 \approx 1600$$

If the $1,000 is invested for a full 5 years then it accumulates to just over $1,600, which checks out with the answer above.

A general formula to solve for n can be derived as follows from the final sum formula:

$$A = P(1+r)^n, \quad \frac{A}{P} = (1+r)^n \text{ and } \quad n = \frac{\log(A/P)}{\log(1+r)}$$

An alternative approach is to use the iterative method and plot different values on a spreadsheet. To find the value of n for which $1.6 = (1.1)^n$.

This entails setting up a formula to calculate the function $y = (1.1)^n$ and then computing it for different values of n until the answer 1.6 is reached. Although some students who find it difficult to use logarithms will prefer to use a spreadsheet, logarithms are used in the other examples in this section. Logarithms are needed to analyze other concepts related to investment and so you really need to understand how to use them.

Example:

1) How many years will $2,000 invested at 5% take to accumulate to $3,000?

Solution:

$P = 2{,}000$; $A = 3{,}000$; $r = 5\% = 0.05$

Using these given values in the time period formula derived above gives

$$n = \frac{\log(A/P)}{\log(1+r)} = \frac{\log 1.5}{\log 1.05} = 8.34$$

This money will need to be invested in 8.34 years.

2) How long will any sum of money take to double its value if it is invested at 12.5%?

Solution

Let the initial sum be A. Therefore, the final sum is

$$A = 2P \text{ and } r = 12.5\% = 0.125$$

Substituting these value for A and r into the final sum formula $A = P(1+r)^n$, we find $2A = A(1.125)^n$

Or $2 = (1.125)^n$ which gives $n = \dfrac{\log 2}{\log 1.125} = 5.9$

For any sum of money, it takes 5.9 years to double its value if it is invested at 12.5%.

- ***Interest rates***

A method of calculating the interest rate on an investment is explained in the following example.

If $4,000 invested for 10 years is projected to accumulate to $6,000, what interest rate is used to derive this forecast?

Solution

$P = 4{,}000$ $\quad$ $A = 6{,}000$ and $n = 10$

Substituting these values into the final sum formula

$A = P(1+r)^n$ gives $6000 = 4000(1+r)^{10}$

$1.5 = (1+r)^{10}$

$1 + r = \sqrt[10]{1.5}$

$r = 4.14\%$

A general formula for calculating the interest rate can be derived. Starting with the familiar final sum formula

$$A = P(1+r)^n \Leftrightarrow \frac{A}{P} = (1+r)^n \Leftrightarrow r = \left(\frac{A}{P}\right)^{1/n} - 1$$

Therefore, $r = \left(\dfrac{A}{P}\right)^{1/n} - 1$

Example:

1) At what interest rate will 3000Frw accumulate to 10,000Frw after 15 years?

Solution:

$$r = \left(\frac{A}{P}\right)^{1/n} - 1 = \left(\frac{10000}{3000}\right)^{1/15} - 1 = 0.083574 = 8.36\%$$

2) An initial investment of 50,000Frw increases to 56,711.25Frw after 2 years. What interest rate has been applied?

Solution

Given that P = 50,000 A = 56,711.25 n = 2 and $r=\sqrt[n]{\frac{A}{P}}-1$

We have $r=\sqrt[2]{\frac{56,711.25}{50,000}}-1=0.065=6.5\%$

d. Logarithms are used to determine ggrowth decay

Example:

The amount, $A(t)\,gram$ of radioactive material in a sample after t years is given by

$A(t)=80\left(2^{\frac{-t}{100}}\right)$.

a) Find the amount of material in the original sample.

b) Calculate the half-life of the material. [the half-life is the time taken for half of the original material].

c) Calculate the time taken for the material to decay to $1gram$.

Solution

a) The original amount of material present is $A(0)=80\left(2^{0}\right)=80gram.$

b) For the half –life, $A(\mathrm{t})=\frac{1}{2}(80gram)=40gram.$ Then ,

$$40=80\left(2^{-\frac{t}{100}}\right)\Rightarrow\frac{40}{80}=2^{-\frac{t}{100}}\Leftrightarrow\frac{1}{2}=2^{-\frac{t}{100}}\Rightarrow 2^{-1}=2^{-\frac{t}{100}}$$

$$\Rightarrow -1=-\frac{t}{100}\Leftrightarrow t=100$$

Therefore, the half-life is 100 years.

c) When $A(t)=1$, then

$$80\left(2^{-\frac{t}{100}}\right)=1 \Leftrightarrow 2^{-\frac{t}{100}}=\frac{1}{80} \Rightarrow -\frac{t}{100}\log 2=\log\frac{1}{80}$$

$$\Rightarrow \frac{t}{100}=\frac{-\log 80}{-\log 2} \Leftrightarrow t=\frac{100\log 80}{\log 2}=632$$

Therefore, it will takes 632 years for the material to decay 1*gram*.

Application activity 7.3.2

1. Without using calculator, compare the numbers ***a*** and ***b***.

a) $a=3\log 2$ and $b=\log 7$

b) $a=\log 2+\log 40$ and $b=4\log 2+\log 5$

c) $a=2\log 2$ and $b=\log 16-\log 3$

2. Given that $\log 2=0.30$, $\log 3=0.48$ and $\log 5=0.69$. Calculate

a) $\log 150$

b) $\log\frac{9}{2}$

c) $\log 0.2+\log 10$

3. Solve each of the following:

a) $2^{x-1}=16$ *b*) $5^{x-1}=0.008$ *c*) $3^{x}=\frac{1}{81}$

4. An initial investment of £50,000 increases to £56,711.25 after 2 years. What interest rate has been applied?

7.4. END UNIT ASSESSMENT 7

1. Simplify

a) $\sqrt{a}\times\sqrt{ab^3}\times\sqrt{bc^2}$ b) $\sqrt[3]{abc}\times\sqrt[3]{a^2b^2c^2}$

c) $\sqrt[3]{\frac{8}{27}}$ d) $\sqrt{\sqrt[4]{x^8}}$ e) $\sqrt{\frac{x^3y^4}{4x}}$

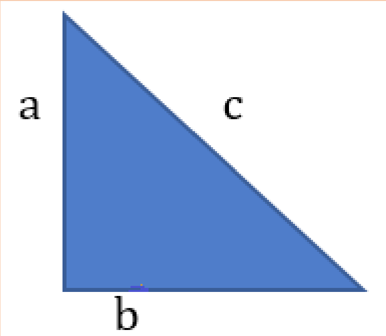
2. Expand the following logarithms in terms of their composites.

a) $\log_b\frac{xy}{z}$ b) $\log_b\frac{x}{yz}$ c) $\log_b p^2.\sqrt[3]{q}$ d) $\log_b\sqrt{\frac{p.q^3}{r^{\frac{1}{2}}s}}$

3. Express each expression below as simple logarithm

a) $\log_b x-2\log_b y+\log_b z$ b) $\log_b 2+\log_b \pi+\frac{1}{2}\log_b l-\frac{1}{2}\log_b g$

4. Use the right triangle below. Find the missing length(keep your answer in cm)



a) a=6cm, b=8cm

b) a=0.3dm, c=5cm

c) b=0.12m, c=13cm

5. Solve each of the following:

a) $9^{1-x}=1$ b) $4^x=\frac{1}{8}$ c) $9^x-10\left(3^x\right)+9=0$ d) $2^{4x+3}-33\left(2^{2x-1}\right)+1=0$

UNIT: 8

PARAMETERS OF CENTRAL TENDENCES AND DISPERSION

Key Unit competence: **Extend understanding, analysis and interpretation of data arising from problems and questions in daily life to the standard deviation**

8.0. Introductory Activity 8

1. During 6 consecutive days, a fruit-seller has recorded the number of fruits sold per type.

The table below shows the types and the number of sold fruits in one week.

Type of fruit	A (Banana)	B (Orange)	C (Pineple)	D (Avocado)	E (Mango)	F (apple)
Number of fruits sold	1100	962	1080	1200	884	900

Which type of fruits had the highest number of fruits sold?

b) Which type of fruits had the least number of fruits sold?

c) What was the total number of fruits sold that week?

d) Find out the average number of fruits sold per day.

2. During the welcome test of Mathematics out of 10 , 10 student-teachers of year one ECLPE scored the following marks: 3, 5,6,3,8,7,8,4,8 and 6.

a) Determine the mean mark of the class.

b) What is the mark that was obtained by many students?

c) Compare and discuss about the mean mark of the class and the mark for every student-teacher. What advice could you give to the Mathematics tutor?

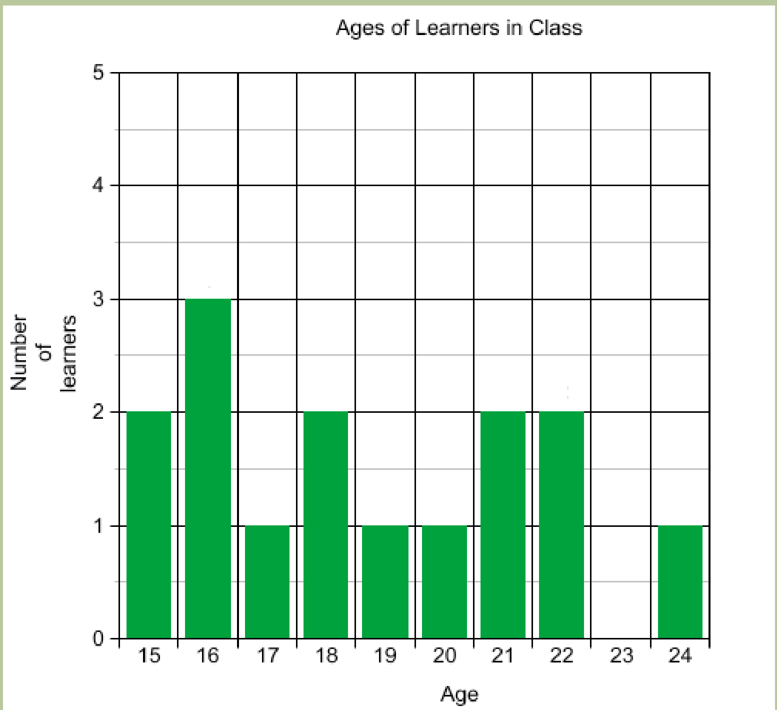
d) organize all marks in a table and try to present them in an X-Y Cartesian plane by making x-axis the number of marks and y-axis the number of student-teachers.

8.1 Collection and presentation of grouped and ungrouped data

8.1.1. Collection and presentation of ungrouped data

Activity 8.1.1

1. Observe the information provided by the graph and complete the table by corresponding the number of learners and their age



Age										
Number of learners										

2. Read carefully the given statement and categorize them into categorical or numerical data.

Amount of money earned last week	Opinions on environmental conservation
Arm span	School post code
Birthdate	State/Territory live in
Dominant hand reaction time	Travel method to school
Favourite sport	Travel time to school
Height	Year level
Hours slept per night	
Language mostly spoken at home	
Foot length	

Categorical	Numerical
1.	1.
2.	2.

CONTENT SUMMARY

Every day, people come across a wide variety of information in form of facts or categorical data, numerical data in form of tables.

For example, information related to profit/ loss of the school, attendance of students and tutors, used materials, school expenditure in term or year, student-teachers' results. These categorical or numerical data which is numerical or otherwise, collected with a definite purpose is called data.

Statistics is the branch of mathematics that deals with the collection, presentation, interpretation and analysis of data.

A sequence of observations, made on a set of objects included in the sample drawn from population, is known as statistical data.

Statistical data can be organized and presented in different forms such as raw tables, frequency distribution tables, graphs, etc.

Qualitative data

Qualitative data is a categorical measurement expressed not in terms of numbers, but rather by means of a natural language description. In statistics, it is often used interchangeably with "categorical" data. Categorical data represent characteristics such as a person's gender, marital status, hometown, or the types of movies they like. Categorical data can take on numerical values (such as "1" indicating male and "2" indicating female), but those numbers don't have mathematical meaning. It couldn't add them together, for example.

Example of qualitative data	Possible categories variable
– Marital status	– Single, married, divorced
– Gender	– Male, Female
– Pain level	– None, moderated, severe
– colour	– Red, black, green, yellow

Quantitative data

Quantitative data is a numerical measurement expressed not by means of a natural language description, but rather in terms of numbers. These data have meaning as a measurement, such as a person's height, weight, IQ, or blood pressure; or they're a count, such as the number of stock shares a person owns, or how many pages you can read of your favorite book before you fall asleep.

Numerical data can be further broken into two types: discrete and continuous.

- ***Discrete data*** represent items that can be counted; they take on possible values that can be listed out. The list of possible values may be fixed (also called *finite*); or it may go from 0, 1, 2, on to infinity (making it *countably infinite*). For example, the number of heads in 100 coin flips takes on values from 0 through 100 (finite case), but the number of flips needed to get 100 heads takes on values from 100 (the fastest scenario) up to infinity (if you never get to that 100th heads). Its possible values are listed as 100, 101, 102, 103, . . . (representing the countably infinite case).
- ***Continuous data*** represent measurements; their possible values cannot be counted and can only be described using intervals on the real number line. For example, the exact amount of gas purchased at the pump for cars with 20-gallon tanks would be continuous data from 0 gallons to 20 gallons, represented by the interval [0, 20], inclusive. You might pump 8.40 gallons, or 8.41, or 8.414863 gallons, or any possible number from 0 to 20. In this way, continuous data can be thought of as being unccountably infinite.

After the collection of data, tally is used to organize and present in a frequency distribution table. Tally means to count by grouping the number of times an item has occurred. When the data are arranged in this way we say that we have obtained the frequency distribution.

Raw data

Data which have been arranged in a systematic order are called raw data or ungrouped data.

For Example, the following are marks out of 20 for 12 student-teachers.

13	10	15	17	17	18
17	17	11	10	17	10

Frequency distribution

A frequency distribution is a table showing how often each value (or set of values) of the collected data occurs in a data set. A frequency table is used to summarize categorical or numerical data. Data presented in the form of a frequency distribution are called grouped data.

Example

The following data of marks, out of 20, obtained by 12 student-teachers can be presented in a frequency distribution table.

13	10	15	17	17	18
17	17	11	10	17	10

The set of outcomes is displayed in a frequency table, as illustrated below:

Marks	Tallies	Frequencies (fi)
10	||	2
11	|	1
13	|	1
15	||	2
17	~~||||~~	5
18	|	1
Total		12

Cumulative frequency

The cumulative frequency corresponding to a particular value is the sum of all frequencies up to the last value including the first value. Cumulative frequency can also defined as the sum of all previous frequencies up to the current point.

Example:

The set of data below shows marks obtained by student-teachers in Mathematics. Draw a cumulative table for the data.

11	15	18	15	10	16	11	10	17
13	17	11	17	16	17	15	13	16

Solution:

The cumulative frequency at a certain point is found by adding the frequency at the present point to the cumulative frequency of the previous point.

Marks	Frequencies (fi)	Cumulative frequencies (cufi)
10	2	2
11	3	$2+3=5$
13	2	$5+2=7$
15	3	$7+3=10$
16	3	$10+3=13$
17	4	$13+4=17$
18	1	$17+1=18$

STEM AND LEAF DISPLAYS

Is a plot where each data value is split into a leaf usually the last digit and a stem the other digit. The stem values are listed down, and the leaf values are listed next to them. This way the stem groups the scores and each leaf indicates a score within that group

Example: The mathematical competence scores of 10 student-teachers participating in mathematics competition are as follows:15,16,21,23,23,26,26,30,32,41. Construct a stem and leaf display for these data by using1, 2, 3, and 4 as your stems.

Solution

Stem	Leaf
1	5 6
2	1 3 3 6 6
3	0 2
4	1

This means that data are concentrated in twenties.

Example: The following are results obtained by student-teachers in French out of 50.

37, 33, 33, 32, 29, 28, 28, 23, 22, 22, 22, 21, 21, 21, 20,

20, 19, 19, 18, 18, 18, 18, 16, 15, 14, 14, 14, 12, 12, 9, 6

Use stem and leaf to display data

Solution:

Numbers 3, 2, 1, and 0, arranged as a stems to the left of the bars. The other numbers come in the leaf part.

3|2337

2|001112223889

1|2244456888899

0|69

From this, data are concentrated in ones

Application activity 8.1.1

1. At the beginning of the school year, a Mathematics test was administered in year 1 to 50 student-teachers to test their level of understanding. Their results out of 20 were recorded as follow:

16	10	11	11	17	12	13	16	15	15
10	9	9	10	11	9	17	10	16	8
18	10	17	9	11	10	14	12	9	8
17	18	15	13	10	10	18	9	10	18
9	10	10	11	11	9	16	11	11	9

Construct a frequency distribution table to help the tutor and the school administration to easily recognize the level of understanding for the Year 1 student-teachers.

2. Differentiate qualitative and quantitative data from the list below: Product rating, basketball team classification, number of student-teachers in the classroom, weight, age, Number of rooms in a house, number of tutors in school.

8.1.2. Collection and presentation of grouped data

Activity 8.1.2

The mass of 50 tomatoes (measured to the nearest g) were measured and recorded in the table below.

86	101	114	118	87	92	93	116	105
102	92	93	101	111	96	117	100	106
118	101	107	96	101	102	104	92	99
107	98	105	113	100	103	108	92	109
95	100	103	110	113	99	106	116	101
105	86	88	108	92				

Construct a frequency distribution table, using equal class intervals of width 5g and taking the lower class boundary of the first interval as 84.5g.

CONTENT SUMMARY

When the range of data is large, the data must be grouped into classes that are more than one unit in width. In this case a grouped frequency distribution is used. Data in this case are grouped in a frequency distribution using groups **or classes.**

- **Class limits:** The class limits are the lower and upper values of the class
- **Lower class limit:** Lower class limit represents the smallest data value that can be included in the class.
- **Upper class limit:** Upper class limit represents the largest data value that can be included in the class.
- $$\text{Class midpoint} = \frac{\text{Lower class limit} + \text{Upper class limit}}{2}$$
- **Class boundaries:** Class boundaries are the midpoints between the **upper class limit** of a class and **the lower class limit of the next class.** Therefore, each class has a lower and an upper class boundary.

Example: The following data represent the marks obtained by 40 students in Mathematics test. Organize the data in the frequency table; grouping the values into classes, starting from 41-50

54 83 67 71 80 65 70 73 45 60 72 82 79 78 65 54 67 64 54 76 45 63 49 52

60 70 81 67 45 58 69 53 65 43 55 68 49 61 75 52

Solution:

Classes	Class midpoint	Frequency
41-50	45.5	6
51-60	55.5	10
61-70	65.5	13
71-80	75.5	8
81-90	85.5	3

The classes: 41-50, 51-60, 71-80, 81-90

Lower class limits: 41, 51, 61, 71, and 81

Upper class limits: 50, 60, 70, 80, and 90

$$\text{Class midpoint for the first class} = \frac{41+50}{2} = 45.5$$

Example

Using the frequency table above, determine the class boundaries of the three classes.

Solution

For the first class, 41-50

The lower class boundary is the midpoint between **40 and 41,** that is 40.5

The upper class boundary is the midpoint between **50 and 51,** that is 50.5

For the second class, 51-60

The lower class boundary is the midpoint between **50 and 51,** that is 50.5

The upper class boundary is the midpoint between **60 and 61,** that is 60.5

For the third class, 61-70

The lower class boundary is the midpoint between **60 and 61,** that is 60.5

The upper class boundary is the midpoint between **70 and 71,** that is 70.5

Class width

Class width is the difference between the upper class boundary and lower class boundary

If the classes are presented in the form,, $[a,b[,[b,c[,[c,d[$

Class limit = class boundary

Example

Classes	Class midpoint	Frequency
[5,10[	7.5	2
[10,15[	12.5	6
[15,20[	17.5	4
[2,25[	22.5	3
[25,30[	27.5	4
[30,35[	32.5	1

For $[10,15[$ the lower class boundary is 10, the upper class boundary is 15.

Class width is 15-10 = 5

Application activity 8.1.2

Suppose a researcher wished to do a study on the distance (in kilometres) that the employees of a certain department store travelled to work each day. The researcher collected the data by asking each employee the approximate distance is the store from his or her home. Data are collected in original form called raw data as follow:

1	2	6	7	12	13	2	6	9	5
18	7	3	15	15	4	17	1	14	5
4	16	4	5	8	6	5	18	5	2
9	11	12	1	9	2	10	11	4	10
9	18	8	8	4	14	7	3	2	6

Since little information can be obtained from looking at raw data, help the researcher to make a frequency distribution table (use 1-3 km and 16-18 km as class limits) so that some general observations can easily be done by the researcher.

8.2 Measure of central tendencies: Mean, median and mode

Activity 8.2

1. Observe the following marks of 10 students in mathematics test: 5, 4, 10, 3, 3, 4, 7, 4, 6, 5 and complete the table below.

Marks (x)	Frequency (f)	f x
3		
4		
5		
6		
7		
10		
Total		

Calculate the mean, median, mode and range of this set

2. In three classes of Year three in TTC, during the quiz of mathematics out of 5 marks, 100 student-teachers obtain marks as shown in the table below:

Marks (x)	0	1	2	3	4
Frequency (f_i)	4	19	25	29	23

- What is the marks obtained by most of the students?
- You are asked to calculate the mean mark for the class. Explain how you should find it.

CONTENT SUMMARY

For the comparison of one frequency distribution with another, it requires to summarize it in such a manner that its essence is expressed in few figures only. For this purpose, we find most representative value of data. This representative value of the data is known as the measure of central tendency or averages.

a) Mean, mode , median and range of ungrouped data

Thus for any particular set of ungrouped data, it is possible to select some typical values or average to represent or describe the entire set such a descriptive value is referred to as a measure of central tendency or location such as mean, mode, median and range.

The median:

The data is arranged in order from the smallest to the largest, the middle number is then selected. This really the central number of the range and is called the median.

When total observation ($\sum fi = n$) is odd the median is given by $Me \to \left(\frac{n+1}{2}\right)^{th}$ or $Me = x_{\frac{n+1}{2}}$ and read the number which located on this position. On the other side when n is even, $Me = \frac{1}{2}\left[\left(\frac{n}{2}\right)^{th} + \left(\frac{n}{2}+1\right)^{th}\right]$ or $Median = \frac{x_{\frac{n}{2}} + x_{\frac{n}{2}+1}}{2}$ then the median is a half of the sum of number located on those two positions.

Examples

1. Calculate the median of the following numbers: 4, 5, 7,2, 1

Solution: Arrange data from lowest to highest number as 1, 2, 4, 5, 7

$$Me \to \left(\frac{n+1}{2}\right)^{th}$$

$$Me \to \left(\frac{5+1}{2}\right)^{th} = 3^{rd} \text{ Then, } Me = 4$$

2. Calculate the median of the following numbers: 4, 5, 7, 2, 1 and 8

Solution: arrange numbers as 1, 2, 4, 5, 7, 8. Total observation $(n) = 6$ since the total observation When is even then, $Me = \frac{1}{2}\left[\left(\frac{n}{2}\right)^{th} + \left(\frac{n}{2}+1\right)^{th}\right]$

$$Me = \frac{1}{2}\left[\left(\frac{6}{2}\right)^{th} + \left(\frac{6}{2}+1\right)^{th}\right] = \frac{1}{2}\left[(3)^{rd} + (4)^{th}\right]$$

$$Me = \frac{4+5}{2} = 4.5$$

The mean:

All the data is added up and the total divided by the number of items. This is called the mean and is equivalent to sharing out all data evenly. Mean is given by $\bar{x} = \frac{1}{n}\sum xfi$

The mode:

The mode is the number that appears the most often from the set of data

The range

In the case of ungrouped data, the range of a set of observations is the difference in values between the largest and the smallest observations in the set.

Example 1

Calculate the mean, median, mode and range of the following set of numbers: 3, 4, 4, 6, 8, 5, 4, and 8

Solution: $\bar{x} = \frac{1}{n}\sum xfi$

x	fi	xfi
3	1	3
4	3	12
5	1	5
6	1	6
8	2	16
	$\sum fi = n = 8$	$\sum xfi = 42$

The mean is given by

$\bar{x} = \frac{1}{n}\sum xfi$, $\bar{x} = \frac{42}{8} = 5.25$

- The median: Arrange data first 3, 4, 4, 4, 5, 6, 8,8.

  Then the median $= \frac{4+5}{2} = 4.5$
- The mode is 4
- The range $8 - 3 = 5$

Mean, mode, median and range of grouped data

For any particular set of grouped data, it is possible to select some typical values or average to represent or describe the entire set such a descriptive value is referred to as a measure of central tendency or location such as mean, mode, median and range.

Mean

Mean is given by $\bar{x} = \frac{1}{n}\sum xfi$, where $n = \sum f$: the sum of frequencies of data.

Mode

$$Mode = L_m + c\left(\frac{\Delta_1}{\Delta_1 + \Delta_2}\right)$$

Specifically, these notation symbols used in above formula;

L_m is lower boundary of modal class.

C is class width: the difference between upper and lower boundary of modal class ($C = U_m - L_m$)

f_m : frequency of modal class.

$\Delta_1 = f_m - f_b$

$\Delta_2 = f_m - f_a$

f_b is frequency followed by f_m and f_a is frequency follows f_m.

Median

$$Median = L_m + C\left(\frac{\frac{n}{2} - CF_b}{f_m}\right)$$

Specifically, these notation symbols used in above formula;

L_m this is lower boundary of modal class.

C this is class width: the difference between upper and lower boundary of modal class ($C = U_m - L_m$)

f_m : frequency of modal class

$n = \sum f$: the sum of frequencies of data.

CF_b : cumulative frequency followed by cumulative frequency of modal class

(cumulative frequency before modal class)

Range

In the case of grouped data the range is defined as the difference between the upper limit of the highest class and the lower limit of the smallest class.

Application activity 8.2

1. A group of student-teachers from SME were asked how many books they had read in previous year; the results are shown in the frequency table below. Calculate the mean, median and mode number of books read.

Number of books	0	1	2	3	4	5	6	7	8
frequency	5	5	6	9	11	7	4	2	1

2. During oral presentation of internship report for year three student-teachers the first 10 student-teachers scored the following marks out of 10:

   8, 7, 9, 10, 8, 9, 8, 6, 7 and 10

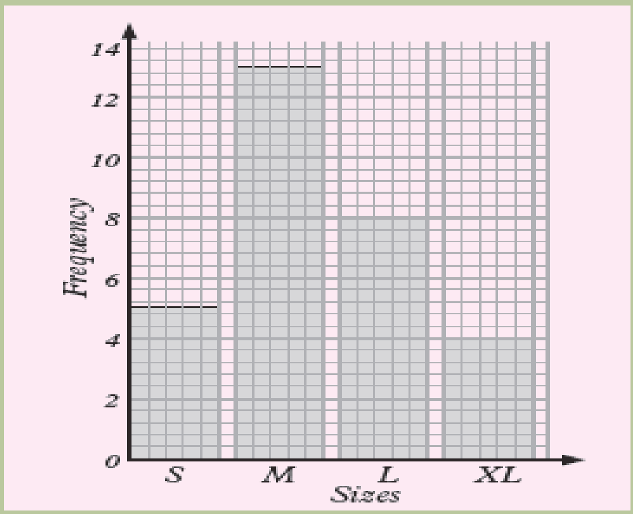
   a) Calculate the mean of the group

   b) Calculate the median and Mode

8.3 Graphical representation of grouped and ungrouped data

Activity 8.3

1. In a class of year 1, student-teachers are requested to provide their sizes in order to be given their sweaters. The graph below shows the sizes of sweaters for 30 students. Observe and interpret it by finding out the number of students with small, medium, large, extra-large. Guess the name of the graph.



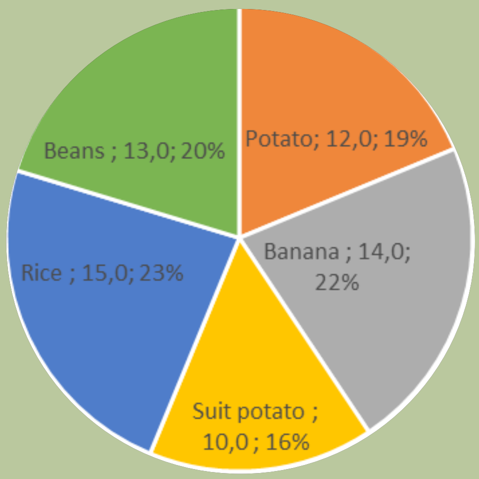
2. The mass of 50 tomatoes (measured to the nearest g) were measured and recorded in the table below.

Class boundaries (in g)	84.5-89.5	89.5-94.5	94.5-99.5	99.5-104.5	104.5-109.5	109.5-114.5	114.5-119.5
Frequency	4	7	6	13	10	5	5
Cumulative frequency							

a) Determine cumulative frequency

b) Try to draw a histogram, the frequency polygon and the cumulative frequency polygon to illustrate the data.

3. A pie chart shows the mass (in tonnes) of each type of food eaten in a certain month. Observe the pie chart and make a corresponding frequency distribution table by considering the number of tones for each type of food as frequency.



CONTENT SUMMARY

After the data have been organized into a frequency distribution, they can be presented in a graphical form. The purpose of graphs in statistics is to convey the data to the viewers in a pictorial form. It is easier for most people to comprehend the meaning of data presented graphically than data presented numerically in tables or frequency distributions.

The most commonly used graphs are: Bar graph, Pie chart, Histogram, Frequency polygon, Cumulative frequency graph or Ogive.

A bar graph

A bar chart or bar graph is a chart or graph that presents numerical data with rectangular bars with heights or lengths proportional to the values that they represent. The bars can be plotted vertically or horizontally. A vertical bar chart is sometimes called a line graph.

Pie chart

A pie chart is used to display a set of categorical data. It is a circle, which is divided into segments. Each segment represents a particular category. The area of each segment is proportional to the number of cases in that category.

$$\text{Angle for sector S} = \frac{\text{Frequency of S} \times 360^0}{\text{Total frequency}}$$

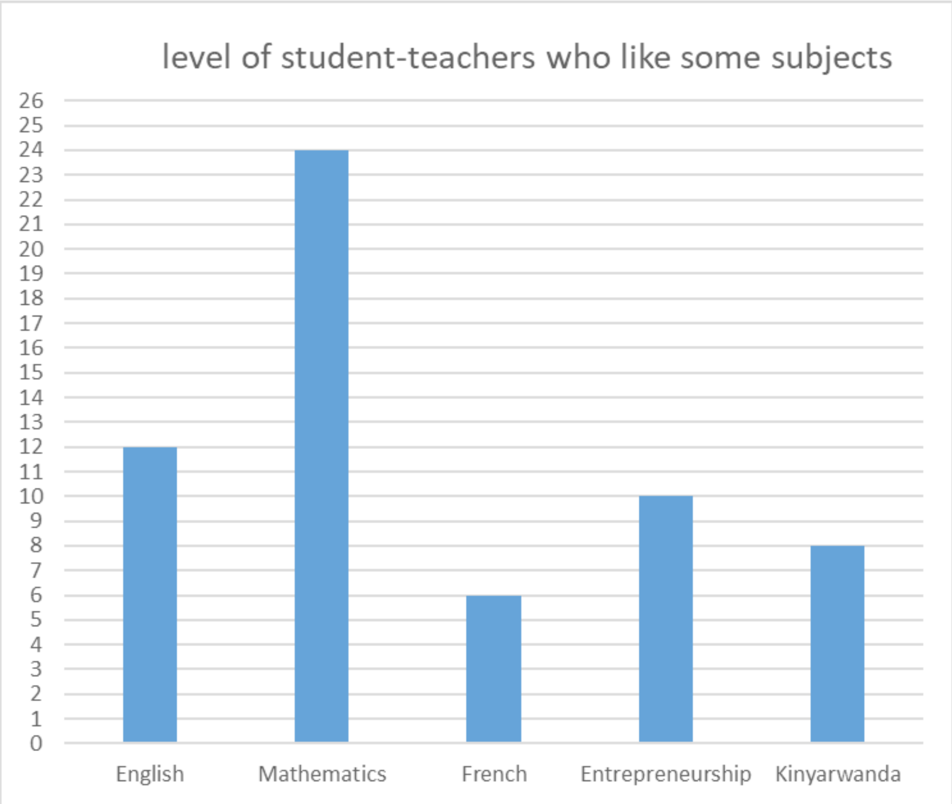
Example:

A TTC tutor conducted a survey to check the level of how student-teachers like some subjects (English, Mathematics, French, Entrepreneurship, Kinyarwanda) taught in SME option. The survey was done on 60 student-teachers and the table below summarizes the results.

Subject	Number of students
English	12
Mathematics	24
French	6
Entrepreneurship	10
Kinyarwanda	8

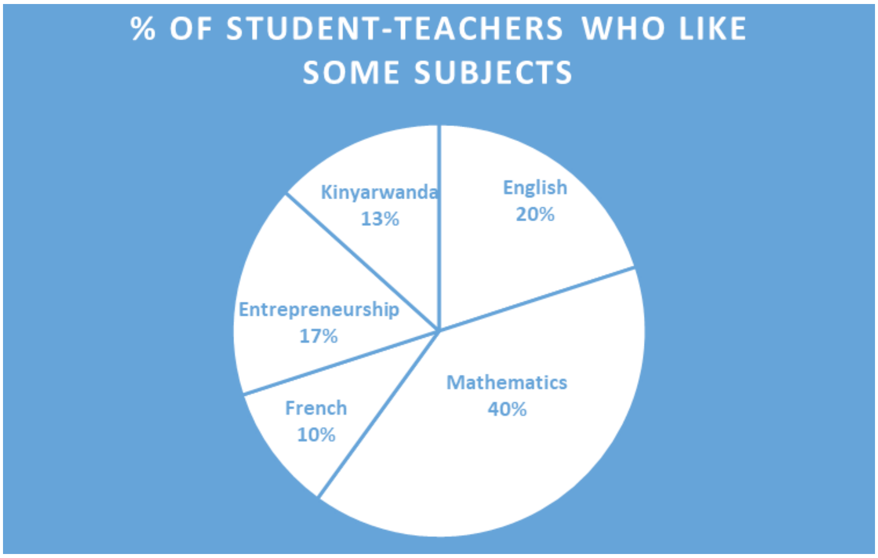
In order to clearly present his/ her findings, the tutor presented the data on a bar graph and then on a pie chart as follows:

1. **Bar graph**



2. A pie chart

Students who like Mathematics $= \frac{24x360^0}{60} = 144^0$ 144^0 represents 40% ($\frac{144^0 x100}{360^0} = 40\%$)	*Students who like Mathematics* $= \frac{12x360^0}{60} = 72^0$ 72^0 represents 20% ($\frac{72^0 x100}{360^0} = 20\%$)
Students who like French $= \frac{6x360^0}{60} = 36^0$ 36^0 represents 10% ($\frac{36^0 x100}{360^0} = 10\%$)	*Students who like Entrepreneurship* $= \frac{10x360^0}{60} = 60^0$ 60^0 represents 17% ($\frac{60^0 x100}{360^0} = 17\%$)
Students who like Kinyarwanda $= \frac{8x360^0}{60} = 48^0$ which is representing 13% ($\frac{8^0 x100}{360^0} = 13\%$)	



Histogram

Histogram is a statistical graph showing frequency of data. The horizontal axis details the class boundaries, and the vertical axis represents the frequency. Blocks are drawn such that their areas (rather than their height, as in a bar chart) are proportional to the frequencies within a class or across several class boundaries. There are no spaces between blocks.

Example: Draw a histogram for the frequency distribution given in the table below.

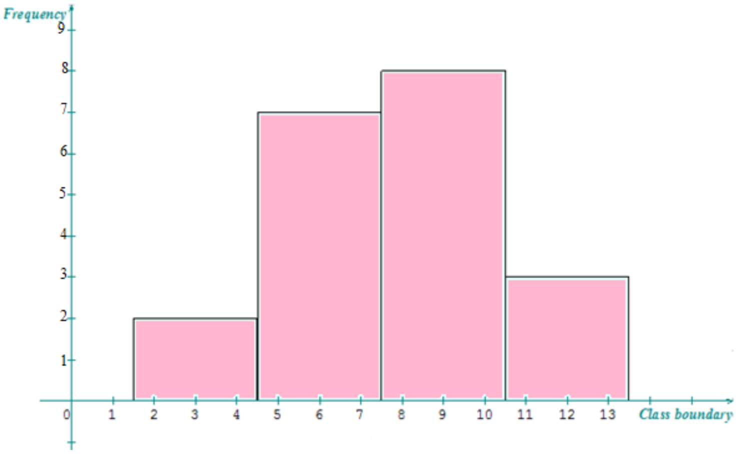
classes	Frequency
2-4	2
5 - 7	7
8-10	8
11-13	5

Solution:

Step 1. Find the class boundaries

classes	Frequency
$[1.5, 4.5[$	2
$[4.5, 7.5[$	7
$[7.5, 10.5[$	8
$[10.5, 13.5[$	3

Step 2: Draw Histogram (frequency against class boundaries)



Frequency polygon

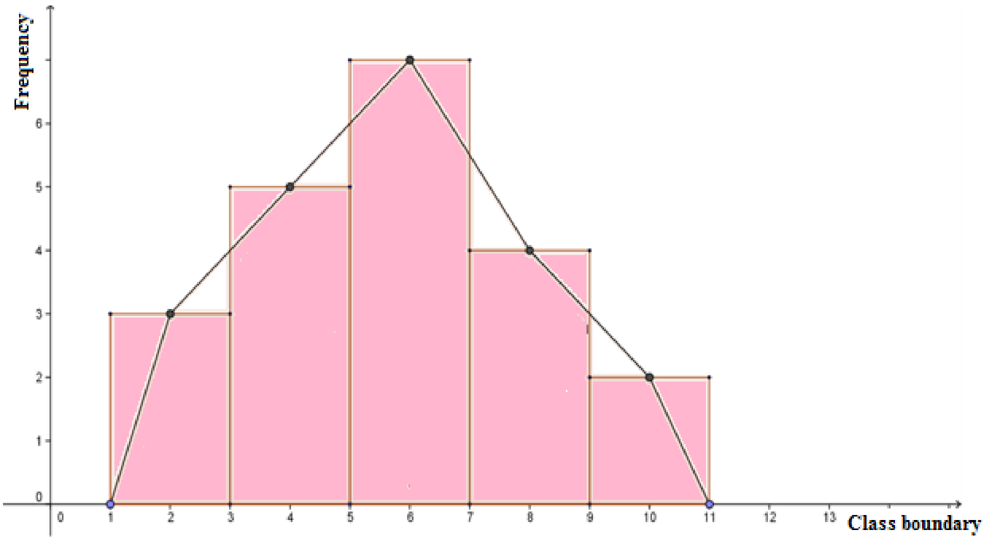
In a frequency polygon, a line is drawn by joining all the midpoints of the top of the bars of a histogram.

A frequency polygon gives the idea about the shape of the data distribution. The two end points of a frequency polygon always lie on the x-axis.

Example

Use the histogram above to draw a frequency polygon.

Solution

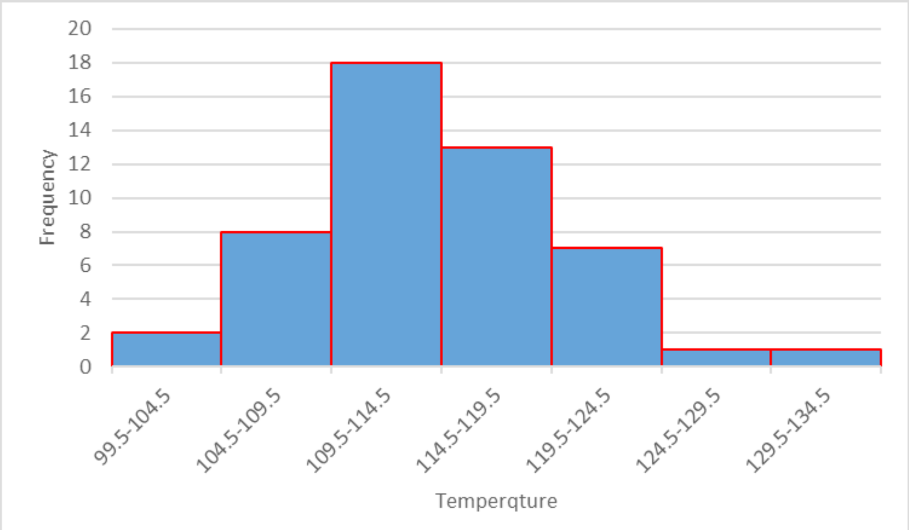


Example:

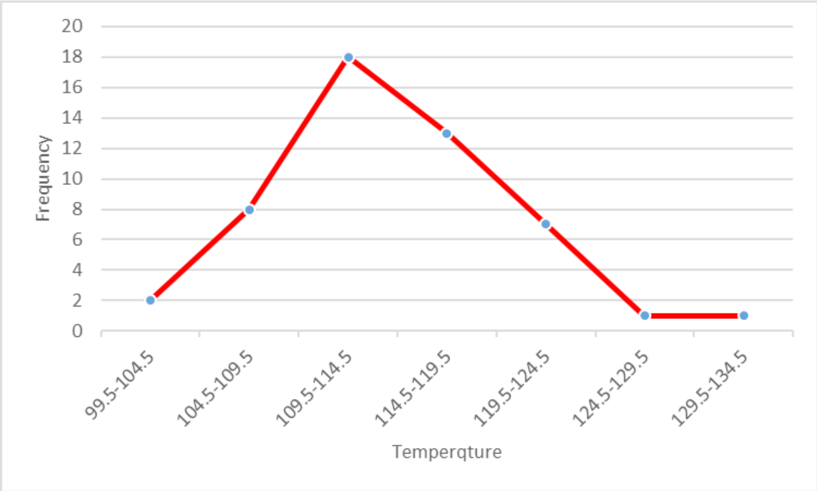
Construct a histogram, a frequency polygon, a cumulative frequency graph or ogive to represent the data shown below for the record high temperatures for each of the 50 states

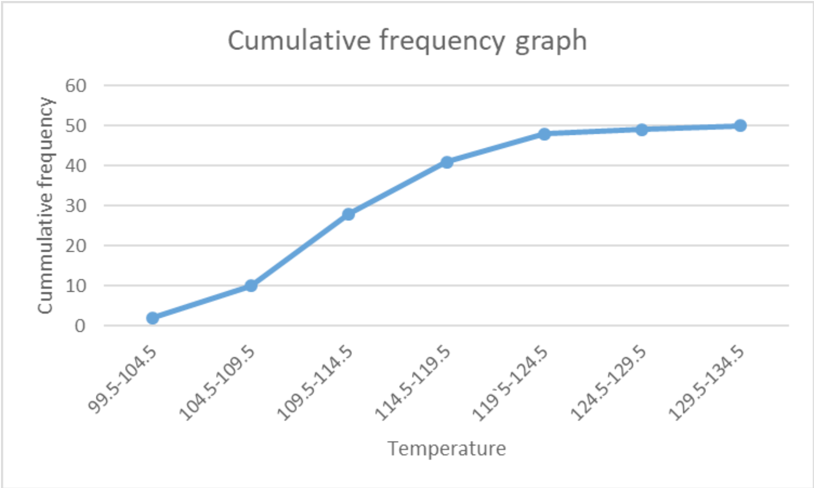
Class boundaries	Frequency
99.5-104.5	2
104.5-109.5	8
109.5-114.5	18
114.5-119.5	13
119.5-124.5	7
124.5-129.5	1
129.5-134.5	1

Step1: Draw Histogram



Step 2: Draw Frequency Polygon





Application activity 8.3

1. At the beginning of the school year, a Mathematics test was administered in year 1 for 50 student-teachers to test their levels in this subject.. Their results out of 20 were recorded as follow:

Marks	8	9	10	11	12	13	14	15	16	17	18
Frequency	2	9	11	8	2	2	1	3	4	4	4

Present the data using a bar graph to help the tutor and the school administration to easily recognize the level of understanding for the Year 1 student-teachers.

2. The cumulative frequency distribution table below shows distances (in km) covered by 20 runners during the week of competition.

Class boundaries	Frequency	Cumulative frequency
5.5-10.5	1	1
10.5-15.5	2	3
15.5-20.5	3	6
20`5-25.5	5	11
25.5-30.5	4	15
30.5-35.5	3	18
35.5-40.5	2	20

a) Construct a histogram

b) Construct a frequency polygon

c) Construct an ogive.

8.4 Measure of dispersion: Range, variance, Standard Deviation and coefficient of variation

Activity 8.4

Before starting the third term, Mathematics tutor calculated the mean mark of five student-teachers got in Mathematics test for the second term. He/she found that the mean mark is $\bar{x} = 16.875$. Use this mean to complete the table below:

x	f	$x-\bar{x}$	$(x-\bar{x})^2$	$f(x-\bar{x})^2$
12	4			
13	2			
15	1			
19	4			
21	5			
	$\sum f =$			$\sum f(x-\bar{x})^2 =$

CONTENT SUMMARY

The word dispersion has a technical meaning in statistics. The average measures the center of the data. It is one aspect of observations. Another feature of the observations is how the observations are spread about the center. The observation may be close to the center or they may be spread away from the center. If the observations are close to the center (usually the arithmetic mean or median), we say that dispersion, scatter or variation is small. If the observations are spread away from the center, we say that dispersion is large.

The study of dispersion is very important in statistical data. If in a certain factory there is consistence in the wages of workers, the workers will be satisfied. But if some workers have high wages and some have low wages, there will be unrest among the low paid workers and they might go on strikes and arrange demonstrations. If in a certain country some people are very poor and some are very high rich, we say there is economic disparity. It means that dispersion is large.

The extent or degree in which data tend to spread around an average is also called the dispersion or variation. Measures of dispersion help us in studying the extent to which observations are scattered around the average or central value. Such measures are helpful in comparing two or more sets of data with regard to their variability.

Properties of a good measure of dispersion

i. It should be simple to calculate and easy to understand

ii. It should be rigidly defined

iii. Its computation be based on all the observations

iv. It should be amenable to further algebraic treatment

Some measures of dispersion are Quartiles, variance, Range, standard deviation, coefficient of variation

$$Q_1 \to \frac{1}{4}(n+1)^{th} \text{ or } \quad Q_1 = x_{\frac{n+1}{4}} . \qquad Q_2 \to \frac{1}{2}(n+1)^{th} \text{ or } \quad Q_2 = x_{\frac{n+1}{2}} = Me$$

$$Q_3 \to \frac{3}{4}(n+1)^{th} \text{ or} Q_3 = x_{\frac{3(n+1)}{4}}$$

The inter-quartile range is given by the difference between third quartile and the first quartile.

Example 1

Find the first and the second quartiles of the data set: 1, 3, 4, 5, 5, 6, 9, 14, and 21

Solution:

$$Q_1 = \frac{1}{4}(n+1)^{th} = \frac{1}{4}(9+1)^{th} = (2.5)^{th},$$

$$\therefore Q_1 = 4$$

$$Q_2 = Me, \; Q_2 \to \frac{1}{2}(n+1)^{th} = \frac{1}{2}(9+1)^{th} = 5^{th}, \; Q_2 = 5$$

Example 2

In the series given below, calculate the first quartile, second quartile and interquartile range

2, 4, 6, 8, 10, 12, 14, 16, 18, 20, 22

Solution

$Q_1 \to \frac{1}{4}(n+1)^{th} = \left(\frac{11+1}{4}\right)^{th} = 3^{rd}$ Then the first quartile is $Q_1 = 6$

$Q_3 \to \frac{3}{4}(n+1)^{th} = \left(\frac{3}{4}(11+1)\right)^{th} = 9^{th}$ Then the third quartile is $Q_3 = 18$

Inter-quartile range= $Q_3 - Q_1 = 18 - 6 = 12$

Variance

Variance measures how far a set of numbers is spread out. A variance of zero indicates that all the values are identical. Variance is always non-negative: a small variance indicates that the data points tend to be very close to the mean and hence to each other, while a high variance indicates that the data points are very spread out around the mean and from each other.

The variance is denoted and defined by

$$\sigma^2 = \frac{\sum_{i=1}^{n}\left(x_i - \bar{x}\right)^2}{n}$$

Developing this formula we have

$$\sigma^2 = \frac{\sum_{i=1}^{n}\left(x_i^2 - 2x_i\bar{x} + \left(\bar{x}\right)^2\right)}{n}$$

$$= \frac{1}{n}\sum_{i=1}^{n} x_i^2 - \frac{1}{n}2\bar{x}\sum_{i=1}^{n} x_i + \frac{1}{n}\left(\bar{x}\right)^2\sum_{i=1}^{n} 1$$

$$= \frac{1}{n}\sum_{i=1}^{n} x_i^2 - 2\bar{x}\bar{x} + \left(\bar{x}\right)^2$$

$$= \frac{1}{n}\sum_{i=1}^{n} x_i^2 - \left(\bar{x}\right)^2$$

Thus, the variance is also defined by

$$\sigma^2 = \frac{1}{n}\sum_{i=1}^{n} x_i^2 - \left(\bar{x}\right)^2$$

Recall that the mean of the set of n values $x_1, x_2, x_3, \ldots, x_n$ is denoted and defined by

$$\bar{x} = \frac{x_1 + x_2 + x_3 + \ldots + x_n}{n}$$

$$= \sum_{i=1}^{n} \frac{x_i}{n}$$

$$= \frac{1}{n}\sum_{i=1}^{n} x_i$$

Example 1

Calculate the variance of the following distribution: ***9, 3, 8, 8, 9, 8, 9, 18***

Solution

$$\bar{x} = \frac{9+3+8+8+9+8+9+18}{8} = 9$$

$$\sigma^2 = \frac{(9-9)^2+(3-9)^2+(8-9)^2+(8-9)^2+(9-9)^2+(8-9)^2+(9-9)^2+(18-9)^2}{8} = 15$$

Example 2

Calculate the variance of the distribution of the following grouped data:

Class	[10,20[	[20,30[	[30,40[	[40,50[	[50,60[	[60,70[	[70,80[
Frequency	1	8	10	9	8	4	2

Solution

Class	x	f	xf	x^2	$x^2 f$
[10,20[	15	1	15	225	225
[20,30[	25	8	200	625	5000
[30,40[	35	10	350	1225	12250
[40,50[	45	9	405	2025	18225
[50,60[	55	8	440	3025	24200
[60,70[	65	4	260	4225	16900
[70,80[	75	2	150	5625	11250
		$\sum x = 42$	$\sum xf = 1820$	$\sum x_i^2 = 16975$	$\sum x^2 f = 88050$

$$\bar{x} = \frac{1820}{42} = 43.33$$

$$\sigma^2 = \frac{88050}{42} - (43.33)^2 = 218.94$$

STANDARD DEVIATION

The standard deviation has the same dimension as the data, and hence is comparable to deviations from the mean. We define the **standard deviation** to be the square root of the variance.

Thus, the standard deviation is denoted and defined by

$$\sigma = \sqrt{\frac{\sum_{i=1}^{n}\left(x_i - \bar{x}\right)^2}{n}} \text{ or } \sigma = \sqrt{\frac{1}{n}\sum_{i=1}^{n} x_i^2 - \left(\bar{x}\right)^2}$$

The above results are used for the grouped data where x_i is the mid-interval value for the i^{th} group.

The following results follow directly from the definitions of mean and standard deviation:

- When all the data values are multiplied by a constant *a*, the new mean and new standard deviation are equal to *a* times the original mean and standard deviation. That is, the mean of $ax_1, ax_2, ax_3, ..., ax_n$ is $a\left(\bar{x}\right)$ and the standard deviation is $a\sigma$.
- When a constant value, *b*,is added to all data values, then new mean is increased by *b*. However standard deviation does not change. That is, the mean of $ax_1, ax_2, ax_3, ..., ax_n$ is $\bar{x} + b$ and the standard deviation is σ.

Example 1

The six runners in a 200 meter race clocked times (in seconds) of 24.2, 23.7, 25.0, 23.7, 24.0,4.6

a) Find the mean and standard deviation of these times.

b) These readings were found to be 10% too low due to faulty timekeeping. Write down the new mean and standard deviation.

Solution

a) $\bar{x} = \frac{24.2 + 23.7 + 25.0 + 23.7 + 24.0 + 24.6}{6} = 24.2 \text{ seconds}$

$$\sigma = \sqrt{\frac{(24.2-24.2)^2 + (23.7-24.2)^2 + (25.0-24.2)^2 + (23.7-24.2)^2 + (24.0-24.2)^2 + (24.6-24.2)^2}{6}}$$

$= 0.473 \text{ seconds}$

b) We must divide each term 0.9 to find the correct time. The new mean is

$$\bar{x} = \frac{24.2}{0.9} = 26.9 \text{ sec}$$

The new standard deviation is $\sigma = \frac{0.4726}{0.9} = 0.525 \text{ sec}$

The method which uses the formula for the standard deviation is not necessarily the most efficient. Consider the following:

$$\text{variance} = \frac{\sum\left(x-\bar{x}\right)^2}{n}$$

$$= \frac{\sum\left(x^2 - 2x\bar{x} + \left(\bar{x}\right)^2\right)}{n}$$

$$= \frac{\sum x^2}{n} - \frac{\sum 2x\bar{x}}{n} + \frac{\sum\left(\bar{x}\right)^2}{n}$$

$$= \frac{\sum x^2}{n} - 2\bar{x}\frac{\sum x}{n} + \left(\bar{x}\right)^2\frac{\sum 1}{n} \quad \left(\text{since } \bar{x} \text{ is a constant}\right)$$

$$= \frac{1}{n}\sum x^2 - 2\left(\bar{x}\right)^2 + \left(\bar{x}\right)^2$$

$$= \frac{1}{n}\sum x^2 - \left(\bar{x}\right)^2$$

Example 2

The heights (in meters) of six children are 1.42, 1.35, 1.37, 1.50, 1.38 and 1.30. Calculate the mean height and the standard deviation of the heights.

Solution:

$$\text{Mean} = \frac{1}{6}\left(1.42 + 1.35 + 1.37 + 1.50 + 1.38 + 1.30\right) = 1.39 \quad m$$

$$\text{Variance} = \frac{1}{6}\left(1.42^2 + 1.35^2 + 1.37^2 + 1.50^2 + 1.38^2 + 1.30^2\right) - 1.39^2 = 0.00386 \ m^2$$

$$\text{Standard deviation} = \sqrt{0.00386 \ m^2} = 0.0621 \ m$$

Coefficient of variation

The coefficient of variation measures variability in relation to the mean (or average) and is used to compare the relative dispersion in one type of data with the relative dispersion in another type of data. It allows us to compare the dispersions of two different distributions if their means are positive. The greater dispersion corresponds to the value of the coefficient of greater variation.

The coefficient of variation is a calculation built on other calculations: the standard deviation and the mean as follows:

$$Cv = \frac{\sigma}{\bar{x}} \times 100$$

Where:

- σ is the standard deviation
- $\bar{x}$ Is the mean.

Example:

One data series has a mean of 140 and standard deviation 28.28. The second data series has a mean of 150 and standard deviation 24. Which of the two has a greater dispersion?

Solution:

$$Cv_1 = \frac{28.28}{140} \times 100 = 20.2\%$$

$$Cv_2 = \frac{24}{150} \times 100 = 16\%$$

The first data series has a higher dispersion.

Range

In the case of ungrouped data, the range of a set of observations is the difference in values between the largest and the smallest observations in the set. In the case of grouped data the range is defined as the difference between the upper limit of the highest class and the lower limit of the smallest class.

Example

Calculate the range of the following set of the data set: 1, 3, 4, 5, 5, 6, 9, 14 and 21

Solution: From the given series the lowest data is 1 and the highest data is 21

The Range $=$ highest value $-$ lowest value

Range $= 21 - 1 = 20$

Application activity 8.4

1. Out of 4 observations done by tutor of English, arranged in descending order, the 5^{th}, 7^{th}, 8^{th} and 10^{th} observations are respectively 89, 64, 60 and 49. Calculate the median of all the 4 observations.

2. In the following statistical series, calculate the standard deviation of the following set of data

   56,54,55,59,58,57,55

3. In the classroom of SME the first five student-teachers scored the following marks out of 10 in a quiz of Mathematics

   5, 6, 5, 2, 4, 7, 8, 9, 7, 5

   a) Calculate the mean, median and the modal mark

   b) Determine the quartiles and interquartile range

   c) Calculate the variance and the standard deviation

   d) Determine the coefficient of variation

8.5 Application of statistics in daily life

Activity 8.5

Using internet or reference books from the school library, make a research to provide in written form at least one example of where the following statistical terminologies are needed and used in real life situations:

- Frequency distribution
- Statistical graphs like Bar graph, Histogram, Frequency polygon Mean,
- Variance

Make a presentation of your findings to the whole class

CONTENT SUMMARY

Statistics is concerned with scientific method for collecting and presenting, organizing and summarizing and analyzing data as well as deriving valid conclusions and making reasonable decisions on the basis of this analysis

Today, statistics is widely employed in government, business and natural and social sciences. Statistical methods are applied in all field that involve decision making.

For example, in agriculture, statistics can help to ensure the amount of crops are grown this year in comparison to previous year or in comparison to required amount of crop for the country. It may also be helpful to analyze the quantity and size of grains grown due to use of different fertilizer.

In education, statistics can be used to analyze and decide on the money spend on girls education in comparison to boys education

Nowadays, graphs are used almost everywhere. In stock market, graphs are used to determine the profit margins of stock. There is always a graph showing how prices have changed over time, food price, budget forecasts, exchange rates…

Mean is used as one of comparing properties of statistics. It is defined as the average of all the clarifications. Mean helps Teachers to see the average marks of the students.

A large standard deviation indicates that the data points can spread far from the mean and a small standard deviation indicates that they are clustered closely around the mean. Standard deviation is often used to compare real-world data against a model to test the model.

In finance, standard deviation is often used as a measure of the risk associated with price-fluctuations of a given asset (stocks, bonds, property, etc.), or the risk of a portfolio of assets, Standard deviation provides a quantified estimate of the uncertainty of future returns.

Application activity 8.5

1. Using internet or reference books from the school library, make a research to provide in written form at least one example of where the following statistical terminologies are needed and used in real life situations:

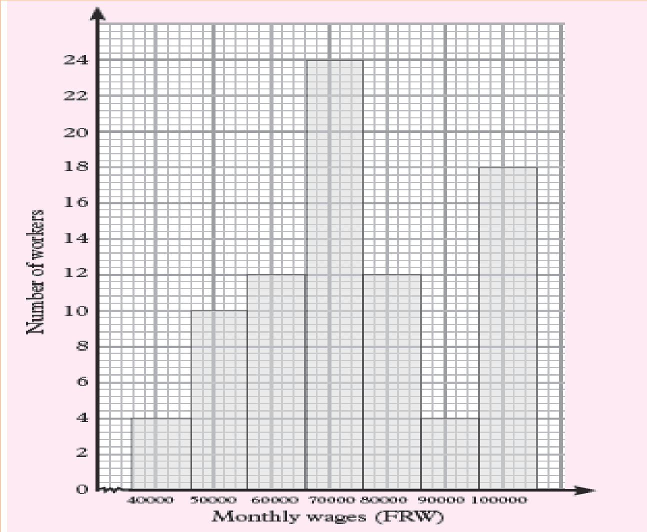
   - Frequency distribution
   - Statistical graphs like Pie chart, cumulative frequency polygon or ogive
   - Mode and median
   - Standard deviation

Make a presentation of your findings to the whole class

2. Collect data on 20 students' heights in your class and organize them in a frequency distribution. Calculate the mean of 20 students' heights, variance as well as standard deviation. What conclusion can you make based on the calculated mean, variance and standard deviation?

8.6. END UNIT ASSESSMENT 8

1. Use the graph below to answer the questions that follows



a) Use the graph to estimate the mode.

b) State the range of the distribution.

c) Draw a frequency distribution table from the graph

2. In test of mathematics, 10 student-teachers got the following marks:

   6, 7, 8, 5, 7, 6, 6, 9, 4, 8

   a) Calculate the mean, mode, quartiles and interquartile range

   b) Calculate the variance and standard deviation

   c) Calculate the coefficient of variation.

3. A survey taken in a restaurant shows the following number of cups of coffee consumed with each meal. Construct an ungrouped frequency distribution.

   0 2 2 1 1 2 3 5 3 2 2 2 1 0 1 2 4 4 0 1 0 1 4
   4 2 2 0 1 1 5

4. The amount of protein (in grams) for a variety of fast-food sandwiches is reported here. Construct a frequency distribution using six classes. Draw a histogram, frequency polygon, and ogive for the data, using relative frequencies. Describe the shape of the histogram.

23	30	20	27	44	26	35	20	29	29
25	15	18	27	19	22	12	26	34	15
27	35	26	43	35	14	24	12	23	31
40	35	38	57	22	42	24	21	27	33

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