MATHEMATICS FOR TTCs

STUDENT'S BOOK

YEAR

3

OPTION:

SCIENCE AND MATHEMATICS EDUCATION (SME)

FOREWORD

Dear Student,

Rwanda Education Board (REB) is honored to present Year Three Mathematics book for Sciences and Mathematics Education (SME) Student Teachers. This book serves as a guide to competence-based teaching and learning to ensure consistency and coherence in the learning of the Mathematics content. The Rwandan educational philosophy is to ensure that you achieve full potential at every level of education which will prepare you to be well integrated in society and exploit employment opportunities.

The government of Rwanda emphasizes the importance of aligning teaching and learning materials with the syllabus to facilitate your learning process. Many factors influence what you learn, how well you learn and the competences you acquire. Those factors include the relevance of the specific content, the quality of teachers' pedagogical approaches, the assessment strategies and the instructional materials available. In this book, we paid special attention to the activities that facilitate the learning process in which you can develop your ideas and make new discoveries during concrete activities carried out individually or with peers.

In competence-based curriculum, learning is considered as a process of active building and developing knowledge and meanings by the learner where concepts are mainly introduced by an activity, situation or scenario that helps the learner to construct knowledge, develop skills and acquire positive attitudes and values.

For efficiency use of this textbook, your role is to:

- Work on given activities which lead to the development of skills;
- Share relevant information with other learners through presentations, discussions, group work and other active learning techniques such as role play, case studies, investigation and research in the library, on internet or outside;
- Participate and take responsibility for your own learning;
- Draw conclusions based on the findings from the learning activities.

To facilitate you in doing activities, the content of this book is self-explanatory so that you can easily use it yourself, acquire and assess your competences. The book is made of units as presented in the syllabus. Each unit has the following structure: the key unit competence is given and it is followed by the introductory activity before the development of mathematical concepts that are connected to real world problems or to other sciences.

The development of each concept has the following points:

- It starts by a learning activity: it is a hand on well set activity to be done by students in order to generate the concept to be learnt;
- Main elements of the content to be emphasized;
- Worked examples; and
- Application activities: those are activities to be done by the user to consolidate competences or to assess the achievement of objectives.

Even though the book has some worked examples, you will succeed on the application activities depending on your ways of reading, questioning, thinking and grappling ideas of calculus not by searching for similar-looking worked out examples.

Furthermore, to succeed in Mathematics, you are asked to keep trying; sometimes you will find concepts that need to be worked at before you completely understand. The only way to really grasp such a concept is to think about it and work related problems found in other reference books.

I wish to sincerely express my appreciation to the people who contributed towards the development of this book, particularly, REB staff, UR-CE Lecturers, secondary school teachers and TTC Tutors for their technical support. A word of gratitude goes to Head Teachers and TTCs principals who availed their staff for various activities.

Any comment or contribution would be welcome to the improvement of this text book for the next edition.

Dr. NDAYAMBAJE Irénée

Director General, REB

ACKNOWLEDGEMENT

I wish to express my appreciation to the people who played a major role in the development of this Mathematics book for Year Three student teachers in Sciences and Mathematics Education (SME). It would not have been successful without active participation of different education stakeholders.

I owe gratitude to different universities and schools in Rwanda that allowed their staff to work with REB in the in-house textbooks production initiative.

I wish to extend my sincere gratitude to lecturers, secondary school teachers and TTC tutors whose efforts during writing exercise of this book were very much valuable.

Finally, my word of gratitude goes to the Rwanda Education Board staffs who were involved in the whole process of in-house textbook Elaboration.

Joan MURUNGI

Head of CTLR Department

Table of Contents

UNIT 1

COMPLEX NUMBERS

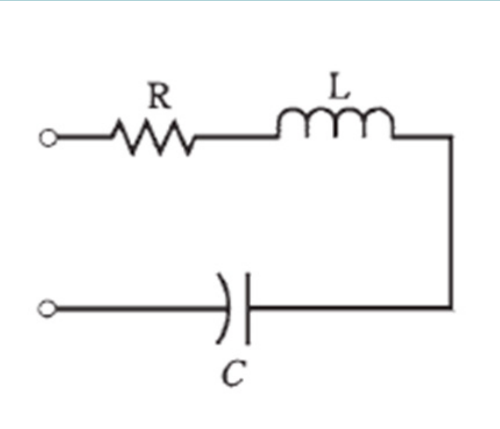
Key Unit competence:

Extend understanding on sets of numbers to complex numbers and solve equations in set of complex numbers.

1.0 Introductory activity

1) An alternating current is a current created by rotating a coil of wire through a magnetic field.

Consider an electric circuit consisting of a resistor, an inductor and a capacitor,



If the angular velocity of the wire is ω, the opposition to the current flow is given by the impedance $Z = R + \frac{1}{j\omega C} + j\omega L$, where j is a number such that $j^2 = -1$

Is j a real number? Explain.

2) Considering that $j^2 = -1$, solve the following equations:

i) $x^2 + 1 = 0$

ii) $x^2 + 4 = 0$

3) Can now any quadratic equation be solved considering that $j^2 = -1$?

1. 1 Concept of complex numbers and their algebraic form

1.1.1 Definitions

Activity 1.1.1

Using the formula of solving quadratic equations in the set of real numbers and considering that $\sqrt{-1} = i$, find the solution set of the following equation $x^2 + 16 = 0$.

What do you think about your answer? Is it an element of $\mathbb{R}$? Explain.

Content summary

To overcome the obstacle of unsolved equation in $\mathbb{R}$, Bombelli, Italian mathematician of the sixteenth century, created new numbers which were given the name of complex numbers.

The symbol "i" satisfying $i^2 = -1$ was therefore created. The equation $x^2 = -1$, which had nosolution in $\mathbb{R}$ gets two in the new set, because $x^2 = -1$ gives $x = i$ or $x = -i$ if we respect the properties of operations in $\mathbb{R}$.

Definition:

Given two real numbers a and b we define the complex number z as $z = a + ib$ with $i^2 = -1$. The number a is called the **real part of** z denoted by $\text{Re}(z)$ and the number b is called the imaginary part of z denoted by $\text{Im}(z)$; the set of complex numbers is noted by $\mathbb{C}$. Mathematically the set of complex numbers is defined as

$$\mathbb{C} = \left\{a + ib; \quad a, b \in \mathbb{R} \quad and\ i^2 = -1\right\}.$$

The complex number $z = a + ib$ is known as the algebraic form of a complex number z.

If b=0, then $z = a$ and z is a real number. Hence, any real number is a complex number

This gives $\mathbb{R} \subset \mathbb{C}$ to mean that the set of real numbers is a subset of the set of complex numbers.

If $a = 0$ and $b \neq 0$, then $z = ib$, and the number z is said to be pure imaginary. As in the previous classes, we can write that $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$.

Example

Find the real part and imaginary part of the following complex numbers and give your observations.

$$a)4+7i \quad b)5-3i \quad c)\frac{1}{2}-\frac{\sqrt{3}}{2}i \quad d)\pi i \quad e)\sqrt{2} \quad f)-\frac{4}{7} \quad g)-11.6.$$

Solution

Each of these numbers can be put in the form $a + ib$ where a and b are real numbers as detailed in the following table:

	Complex number	Real part	Imaginary part	Observations
a)	$4+7i$	4	7	A complex number
b)	$5-3i$	5	-3	A complex number
c)	$\frac{1}{2}-\frac{\sqrt{3}}{2}i$	$\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	A complex number
d)	πi	0	π	A complex number that is pure imaginary
e)	$\sqrt{2}$	$\sqrt{2}$	0	A complex number that is real number
f)	$-\frac{4}{7}$	$-\frac{4}{7}$	0	A complex number that is real number
g)	-11.6	-11.6	0	A complex number that is real number

Complex numbers are commonly used in electrical engineering, as well as in physics as it is developed in the last topic of this unit. To avoid the confusion between i representing the current and i for the imaginary number, physicists prefer to use j to represent the imaginary number. For example in the R-C (Resistor and Capacitor) current divider of the current from a generator, the capacitor has the impedance $Z_C = \frac{1}{j\omega C}$ where j is the imaginary unit.

1.1.2 Properties of the imaginary number "i"

Activity 1.1.2

Use the definition of a complex number, and the fact that $i^2 = -1$ to find the following

a) i^3 b) i^4 c) i^5 d) i^7 e) i^8.

Generalize the value of i^n for $n \in \mathbb{N}$

Content summary

From the activity 1.1.2, it is easy to find that

$$i^1 = i$$
$$i^2 = -1$$
$$i^3 = -i$$
$$i^4 = 1$$
$$i^{4n} = 1, \quad i^{4n+1} = i, i^{4n+2} = -1, i^{4n+3} = -i.$$

By agreement , $i^0 = 1$

Geometrically, we deduce that the imaginary unit, i , "cycles" through 4 different values each time we multiply as it is illustrated in Figure 1.1.

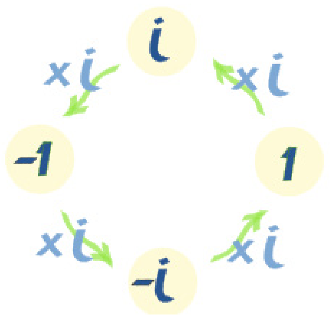


Figure 1. 1: Cycles of imaginary unit

From the figure 1.1, the following relations may be used:

$\forall n \in \mathbb{N}, i^{4n} = 1, \quad i^{4n+1} = i, i^{4n+2} = -1, i^{4n+3} = -i.$

Application activity 1.1

1. Observe the following complex numbers and identify the real part and imaginary part.

   $a)\ z = 4 + 2i \quad b)\ z = i \quad c)\ z = \sqrt{2} - i \quad d)\ z = -3.5$

2. Use the properties of the number i to find the value of the following:

   $a)\ i^{25} \quad b)\ i^{2310} \quad c)\ i^{71} \quad d)\ i^{51} \quad e)\ i^{28}$

1.2 Geometric representation of a complex number

Activity 1.2

1. Draw the Cartesian plane and plot the following points: $A(2,3), B(-3,5)$ and $C\left(\frac{1}{2}, 7\right)$.
2. Consider the complex number $z = -3 + 5i$ and plot the point $Z(-3,5)$ in plane xoy.

Discuss if all complex numbers of the form $z = a + bi$ can be plotted in plane xoy.

Content summary

The **complex plane** comprises two number lines that intersect in a right angle at the point (0,0). The horizontal number line (known as $x-axis$ in Cartesian plane) is the **real axis while** the vertical number line (the $y-axis$ in Cartesian plane) is the **imaginary axis**.

Every complex number $z = a + bi$ can be represented by a point $Z(a,b)$ in the complex plane.

The complex plane is also known as the Argand diagram. The new notation $Z(a,b)$ of the complex number $z = a + bi$ is the geometric form of z and the point $Z(a,b)$ is called the affix of $z=a+bi$. In the Cartesian plane, (a,b) is the coordinate of the extremity of the vector $\begin{pmatrix} a \\ b \end{pmatrix}$ from the origin (0,0).

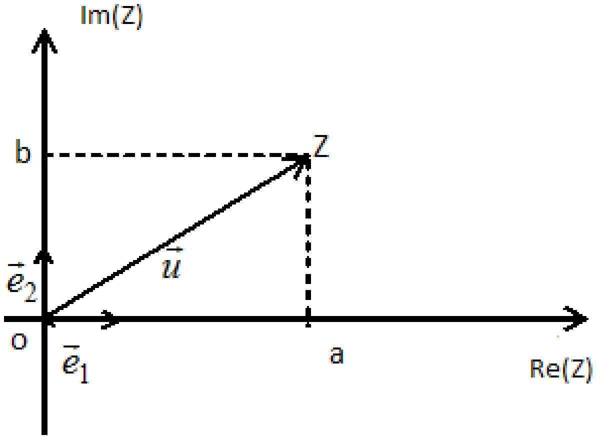


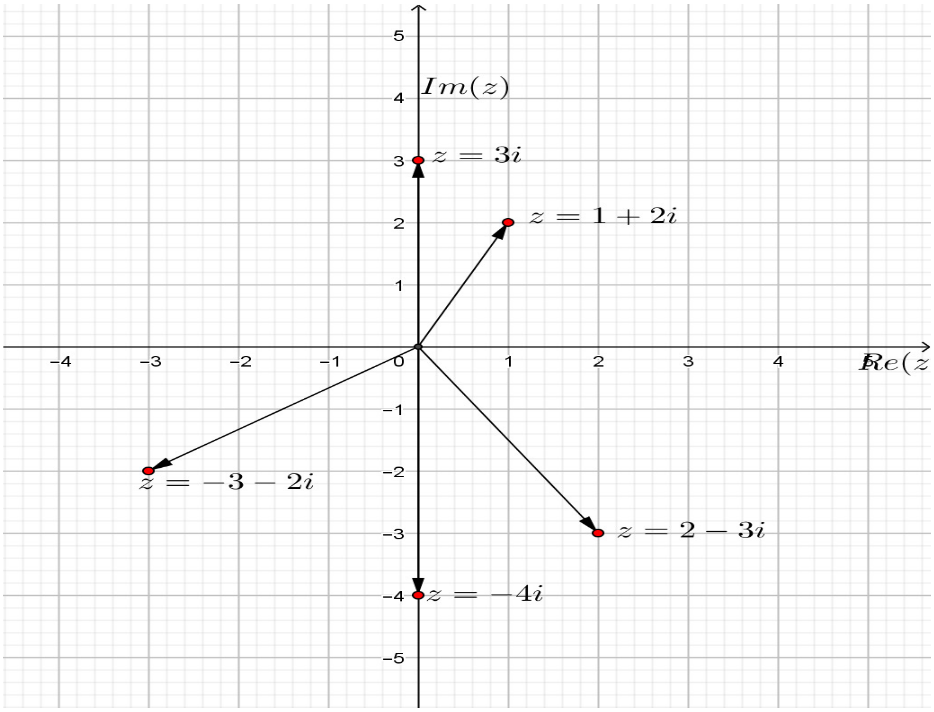
Figure 1. 2 The complex plane containing the complex number $z=a+bi$

Examples

1) Plot in the same Argand diagram the following complex numbers

$z_1=1+2i,\ z_2=2-3i,\ z_3=-3-2i,\ z_4=3i$ and $z_5=-4i$

Solution



Application activity 1.2

Represent in the complex plane the following numbers:

$a)\ z=-1+i \quad b)\ z=i \quad c)\ z=-4-i \quad d)\ z=-3.5+1.2i$

1.3 Operations on complex numbers

1.3.1 Addition and subtraction in the set of complex numbers

Activity 1.3.1

a) Using the Cartesian plane, plot the point $A(1,2)$ and $B(-2,4)$; deduce the coordinate of the vector $\overrightarrow{OA}+\overrightarrow{OB}$.

b) Basing on the answer found in a), deduce the affix of the complex number z_1+z_2 if $z_1=1+2i$ and $z_2=-2+4i$.

c) Check your answer using algebraic method/technique.

d) Express your answer in words.

Content summary

Complex numbers can be manipulated just like real numbers but using the property $i^2=-1$ whenever appropriate. Many of the definitions and rules for doing this are simply common sense, and here we just summarise the main definitions.

Equality of complex numbers: $a+bi=c+di$ if and only if $a=c$ and $b=d$.

To perform addition and subtraction of complex numbers we add (or subtract) real parts, and we add (or subtract) imaginary parts:

The sum of two complex numbers $z_1=a+bi \text{ and } z_2=c+di$ is the complex number whose real part is the sum of real parts of given complex numbers and the imaginary part is the sum of their imaginary parts. This means

$z_1+z_2=\left[\operatorname{Re}(z_1)+\operatorname{Re}(z_2)\right]+i\left[\operatorname{Im}(z_1)+\operatorname{Im}(z_2)\right]$ or $z_1+z_2=(a+c)+(b+d)i.$

The difference of $z_2=c+di$ from $z_1=a+bi$ is $z_1-z_2=(a-c)+(b-d)i.$

Examples

1) Determine $z_1 + z_2$ and $z_1 - z_2$ given that

a) $z_1 = 5 + 6i$ and $z_2 = 3 + 7i$ b) $z_1 = 2 + 4i$ and $z_2 = 3 - 6i$

Solution

a) $z_1 + z_2 = (5 + 6i) + (3 + 7i) = (5 + 3) + (6 + 7)i = 8 + 13i$

$z_1 - z_2 = (5 + 6i) - (3 + 7i) = (5 - 3) + (6 - 7)i = 2 - i$

b) $z_1 + z_2 = (2 + 4i) + (3 - 6i) = (2 + 3) + (4 - 6)i = 5 - 2i$

$z_1 - z_2 = (2 + 4i) - (3 - 6i) = (2 - 3) + (4 + 6)i = -1 + 10i$

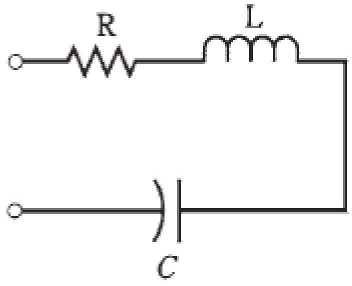
2) Impedances in series RLC circuits as a sum of complex numbers

Given two impedances, $Z_1 = R_1 + jX_1$ and $Z_2 = R_2 + jX_2$, the net impedance Z of these

in series is their vector sum given by $Z = (R_1 + R_2) + j(X_1 + X_2)$.

This means that the resistance and reactance components add separately. The reactances X_1 and X_2 might both be inductive (positive); they might both be capacitive (negative); or one might be inductive and the other capacitive.

When a coil, capacitor, and resistor are connected in series (see the figure below), the resistance R can be thought of as all belonging to the coil, when you use the above formulae.



A series RLC circuit

Then you have two vectors to sum up, when finding the impedance of a series RLC circuit:

$Z=(R+jX_L)+(0+jX_C)=R+j(X_L+X_C)$

Example

A resistor, coil, and capacitor are connected in series with $R=45\Omega$, $X_L=22\Omega$ and $X_C=-30\Omega$. What is the net impedance Z?

Solution

Consider the resistor to be part of the coil (inductor), obtaining two complex vectors, $45+22j$ and $0-30j$. Adding these gives the resistance component of $(45+0)\Omega=45\Omega$, and the reactive component of $(22j-30j)\Omega=-8j\Omega$. Therefore, the net impedance is $Z=(45-8j)\Omega$.

Application activity 1.3.1

Represent graphically the following complex numbers, and then deduce the numerical answers from the diagrams.

$a)(5-i)+(2-7i)$

$b)(6+3i)-(10+8i)$

$c)(4+2i)-(4-2i)+(4-0.6i)$

$d)(8-i)-(2-i)$

1.3.2 Conjugate of a complex number

Activity 1.3.2

In the complex plane,

1.Plot the affix of complex number$z=2+5i$

2. Find the image $P'$of the point Paffix of z by the reflection across the real axis. What is the coordinate ofP'?

Write the cwrit Write the complex number z' associated to $P'$and discuss the relationship between z and z' of P'?

Write algebraically the complex number z' associated to $P'$and discuss the relationship between z and z'.

Content summary

Every complex number $z = a + bi$ has a corresponding complex number $\bar{z}$ called conjugate of z such that $\bar{z} = a - bi$ and affix of $\bar{z}$ is the "reflection" of affix of z about the real axis as illustrated in Figure 1.3.

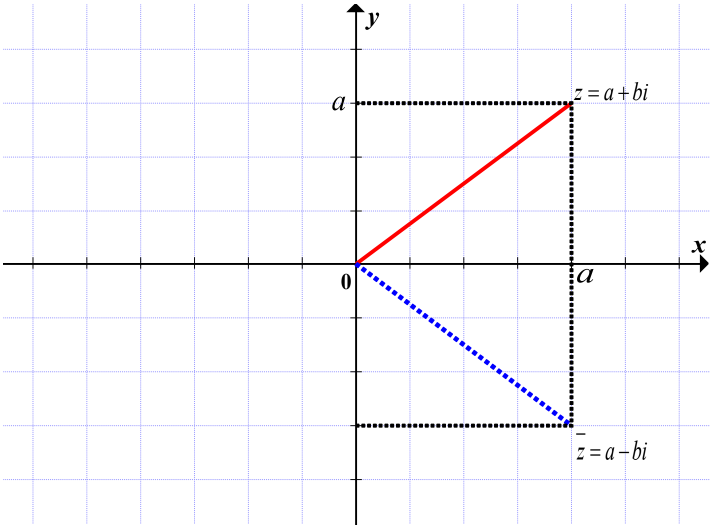


Figure 1. 3 Reflection of affix about the real axis

If $z = a + bi$, then $z + \bar{z} = 2a$ *and* $z - \bar{z} = 2bi$ which gives $\mathrm{Re}(z) = \frac{z + \bar{z}}{2}$ and $\mathrm{Im}(z) = \frac{z - \bar{z}}{2i}$.

Example

For each of the following complex numbers, find their conjugate

a) $2 + 4i$ b) $3 - 6i$ c) $2i - 4$

Solution

a) $\overline{2 + 4i} = 2 - 4i$

b) $\overline{3 - 6i} = 3 + 6i$

c) $\overline{2i - 4} = -2i - 4$

Application activity 1.3.1

a. Represent in Argand diagram, the following complex numbers z and find $\overline{z}$ and $\overline{\overline{z}}$.

1) $z = 2 + 4i$ 2) $z = 2 - 4i$ 3) $z = -2 + 4i$ 4) $z = -2 - 4i$

b. Establish a relationship between z and $\overline{\overline{z}}$.

1.3.3 Multiplication and powers of complex number

Activity 1.3.3

Given two complex numbers $z_1 = 4 - 7i$ and $z_2 = 5 + 3i$

Apply the rules of product calculation in $\mathbb{R}$ and the agreement $i^2 = -1$

to find the product $z_1 \cdot z_2$ and the powers z_1^2 and z_2^2.

Name and discuss the properties used while calculating the above product.

Express in your own words the property used and the answer you found.

Content summary

Let $z_1 = a + bi$ and $z_2 = c + di$ be complex numbers. We define multiplication and powers as follows:

a) $z_1 \cdot z_2 = (a+bi)(c+di) = ac - bd + i(ad + bc)$

b) $z_1^{\ 2} = z_1 \cdot z_1 = (a+bi)(a+bi) = (a+bi)^2 = a^2 - b^2 + 2abi$

c) $z_1^n = \underbrace{z_1 \ldots z_1}_{n\ times} = \underbrace{(a+bi)\ldots(a+bi)}_{n\ times} = (a+bi)^n$

We use the distributive property of multiplication over addition to find the above products.

Example

Find the product/power of the following complex numbers

a) $(3-2i)(5+4i)$ $\qquad$ $b)(4-3i)^2$ $\qquad$ $c)(1+i)^4$

Solution

$$a)(3-2i)(5+4i)=\left[(3\times5)-(-2)\times4\right]+i\left[(-2)\times5)+3\times4\right]=23+2i$$

$$b)(4-3i)^2=16-24i-9=7-24i$$

$$c)(1+i)^4=1+4i+6i^2+4i^3+i^4=1+4i-6-4i+1=-4$$

Application activity 1.3.3

Perform the following operations to find z

a)$z=i(3-7i)(2-i)$ $\qquad$ b) $z=(1+\mathrm{i})^2-3(2-\mathrm{i})^3$

1.3.4 Division in the set of complex numbers

Activity 1.3.4

Consider the complex number $z=\dfrac{2+3i}{5+i}$, apply the rules of rationalizing the denominator

in $\mathbb{R}$ and the agreement $i^2=-1$ to transform the denominator into real part without changing the value of z. Deduce the quotient of $2+3i$ by $5+i$.

Give the general rule of division in complex numbers. Express your answers in words.

Content summary

Given two complex numbers $z_1=a+bi$ and $z_2=c+di$, the quotient $\dfrac{z_1}{z_2}$ is defined by:

$$\frac{z_1}{z_2}=\frac{(a+bi)(c-di)}{(c+di)(c-di)}=\frac{(ac+bd)+(bc-ad)i}{c^2+d^2}$$

Example

Compute the following quotients

$$a)\frac{1+2i}{3+4i} \qquad b)\frac{1}{a+ib}$$

Solution

$$a)\ z=\frac{1+2i}{3+4i}=\frac{(1+2i)(3-4i)}{(3+4i)(3-4i)}=\frac{11+2i}{25}$$

$$b)\ z=\frac{1}{a+ib}=\frac{a-ib}{(a+ib)(a-ib)}=\frac{a-ib}{a^2+b^2}=\frac{a}{a^2+b^2}-i\frac{b}{a^2+b^2}$$

Application activity 1.3.4

1. Find the value z in algebraic form

$$a)\ z=\frac{1}{(2+i)(1-2i)}$$

$$b)\ z=\left(\frac{\sqrt{3}-i}{\sqrt{3}+i}\right)^4\left(\frac{1+i}{1-i}\right)^5$$

2. Determine the real numbers x and y given that:

$$a)\ x+4y+xyi=12-16i$$

$$b)\ x-7y+8i=6y+(6y-100)i$$

$$c)\ \frac{1}{x+iy}+\frac{1}{1+2i}=1\ (x\neq 0\ and\ y\neq 0)$$

Given that $T=\dfrac{x-iy}{x+iy}$ where $x,y\in\mathbb{R}$, show that $\dfrac{1+T^2}{2T}=\dfrac{x^2-y^2}{x^2+y^2}$

1.3.5 Modulus of a complex number

1.3.5.1 Definition and Interpretation of modulus

Activity 1.3.5.1

Let $z = -2+3i$ be a complex number.

a) Plot z in the complex plane

b) Determine the distance between the origin $(0,0)$ to the point $(-2,3)$ affix of z.

Express the formulae used in words.

Content summary

The distance from origin to the point (a,b) which is the affix of the complex number $z = a+bi$ is called the modulus or magnitude or absolute value of z and is denoted by $|z|$ or $|a+bi|: r = |z| = \sqrt{a^2+b^2}$ as illustrated on figure 1.4.

Notice that the modulus of a complex number is always a real number and in fact it will never be negative since square roots always return a positive number or zero depending on what is under the radical.

In addition, if z is a real number (i.e. $z = a+0i$) then, $|z| = \sqrt{a^2+0} = |a|$ (the absolute value of a).

We can compute the modulus of a complex number using its real and imaginary parts such that if $\operatorname{Re}(z) = a$ and $Im(z) = b$ the modulus of z is

$$|z| = \sqrt{z.\bar{z}} = \sqrt{[\operatorname{Re}(z)]^2 + [\operatorname{Im}(z)]^2} = \sqrt{a^2+b^2}$$

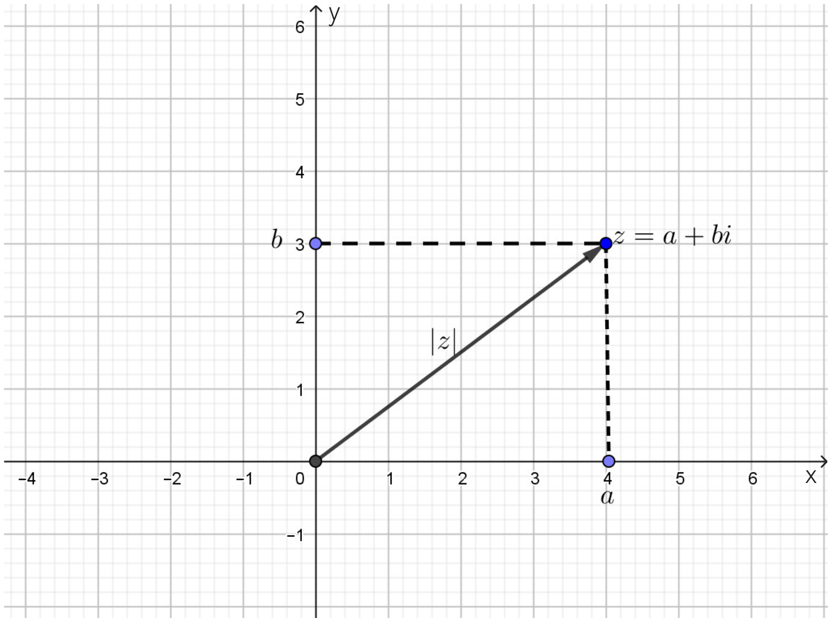


Figure 1. 4 Graph representing the modulus of $z = a + bi$

Properties of modulus

For complex numbers z and z', and for any integer n, we have:

$i) |z|^2 = z \cdot \bar{z}$

$ii) |z \cdot z'| = |z| \cdot |z'|$

$iii) \left|\frac{1}{z}\right| = \frac{1}{|z|} (z \neq 0)$

$iv) |z^n| = |z|^n (z \neq 0);$

$v) \left|\frac{z}{z'}\right| = \frac{|z|}{|z'|} (z' \neq 0)$

$vi) |z + z'| \leq |z| + |z'| \; (triangular\ inequality)$

Example

Calculate the modulus of the following complex numbers

a) $4-3i$ b) $1+i\sqrt{3}$ c) $-5i$ d) $\dfrac{5}{1+i\sqrt{3}}$

Solution

a) $|4-3i| = \sqrt{16+9} = 5$.

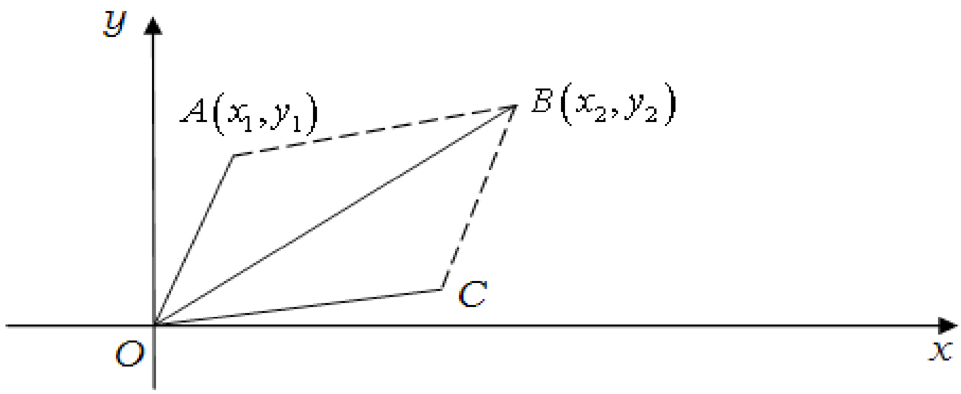
b) $\left|1+i\sqrt{3}\right| = \sqrt{1+3} = 2$.

c) $|-5i| = \sqrt{25} = 5$.

d) $\left|\dfrac{5}{1+i\sqrt{3}}\right| = \dfrac{|5|}{\left|1+i\sqrt{3}\right|} = \dfrac{5}{2}$.

Interpretation of $|z_B - z_A|$

Consider the complex number $z = z_B - z_A$, where $z_A = x_1 + iy_1$ and $z_B = x_2 + iy_2$. The points A and B represent z_A and z_B respectively.



Then $z = (x_2 - x_1) + i(y_2 - y_1)$ and is represented by the point C. This makes $OABC$ a parallelogram.

From this it follows that $|z_B - z_A| = \overline{OC} = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2}$. That is to say $|z_B - z_A|$ is the length AB in the Argand diagram.

If the complex number z_A is represented by the point A, and the complex number z_B is represented by the point B, then $|z_B - z_A| = \overline{AB}$,

Example

Let A and B be the points with affixes $z_A = -1 - i,\ z_B = 2 + 2i$. Find $\overline{AB}$

Solution

$$\begin{aligned}\overline{AB} = |z_B - z_A| &= |3+3i| \\ &= \sqrt{9+9} \\ &= 3\sqrt{2}\end{aligned}$$

Application activity 1.3.5.1

1. Determine the modulus of the following complex numbers:

   $z_1 = 2 - 3i$, $z_2 = 3 + 4i$, $z_3 = 6 + 4i$, and $z_4 = 15 - 8i$ then deduce the modulus of $z = \dfrac{(2-3i)(3+4i)}{(6+4i)(15-8i)}$

2. If $z_1 = 1 - i$, $z_2 = -2 + 4i$, $z_3 = 3 - 2i$, calculate :

   a) $|2z_2 - 3z_1|$;

   b) $|z_1.\overline{z}_2 + \overline{z}_1 . z_2|$;

   c) $\left|\dfrac{z_1 + z_2 + 1}{z_1 - z_2 + 1}\right|$,

   d) $|z_1^2 + z_2^2|^2 + |z_3^2 - z_2^2|^2$

1.3.5.2 Loci related to the distances on Argand diagram

Activity 1.3.5.2

Sketch the set of points determined by the condition

$|z - 1 + 3i| = 2$

Hint: Replace z by $x + yi$ and perform other operations

Content summary

A locus is a path traced out by a point subjected to certain restrictions. Paths can be traced out by points representing variable complex numbers on an Argand diagram just as they can in other coordinate systems.

Consider the simplest case first, when the point P represents the complex number z such that $|z| = R$. This means that the distance of OP from the origin O is constant and so P will trace out a circle.

> $|z| = R$ represents a circle with centre at origin and radius R.

If instead $|z - z_1| = R$, where z_1 is a fixed complex number represented by point A on Argand diagram, then

> $|z - z_1| = R$ represents a circle with centre z_1 and radius R.

Note that if $|z - z_1| \le R$, then the point P representing z cannot only lie on the circumference of the circle, but also anywhere inside the circle. The locus of P is therefore the region on and within the circle with centre A and radius R.

Now consider the locus of a point P represented by the complex number z subjected to the conditions $|z - z_1| = |z - z_2|$, where z_1 and z_2 are fixed complex numbers represented by the points A and B on an Argand diagram. Then

> $|z - z_1| = |z - z_2|$ represents a straight line- the perpendicular bisector (mediator) of the segment joining the points z_1 and z_2.

Note also that if $|z - z_1| \le |z - z_2|$ the locus of z is not only the perpendicular bisector of AB, but also the whole half line, in which A lies, bounded by this bisector.

Example

If $\left|\dfrac{z+2}{z}\right| = 2$ and P represents z in the Argand plane, show that P lies on a circle and find the centre and radius of this circle.

Solution

Let $z = x + iy$ where $x, y \in \mathbb{R}$

Then $\left|\frac{z+2}{z}\right|=2$

$\Rightarrow |z+2|=2|z|$

$\Rightarrow |x+iy+2|=2|x+iy|$

$\Rightarrow |x+2+iy|=2|x+iy|$

$\Rightarrow \sqrt{(x+2)^2+y^2}=2\sqrt{x^2+y^2}$

$\Rightarrow (x+2)^2+y^2=4x^2+4y^2 \qquad [squaring\ \ both\ \ sides]$

$\Rightarrow x^2+4x+4+y^2=4x^2+4y^2$

$\Rightarrow -3x^2-3y^2+4x=-4$

$\Rightarrow 3x^2+3y^2-4x=4$ which is the equation of a circle with centre at $\left(\frac{2}{3},0\right)$ and with radius of length $\frac{4}{3}$

Example:

Determine, in complex plane, the locus M of affix z such that $|z-2i|=|z+2|$.

Solution

Let $z=x+yi$, we have

$|x+yi-2i|=|x+yi+2|$

$\Rightarrow |x+i(y-2)|=|x+2+yi|$

$\Rightarrow \sqrt{x^2+(y-2)^2}=\sqrt{(x+2)^2+y^2}$

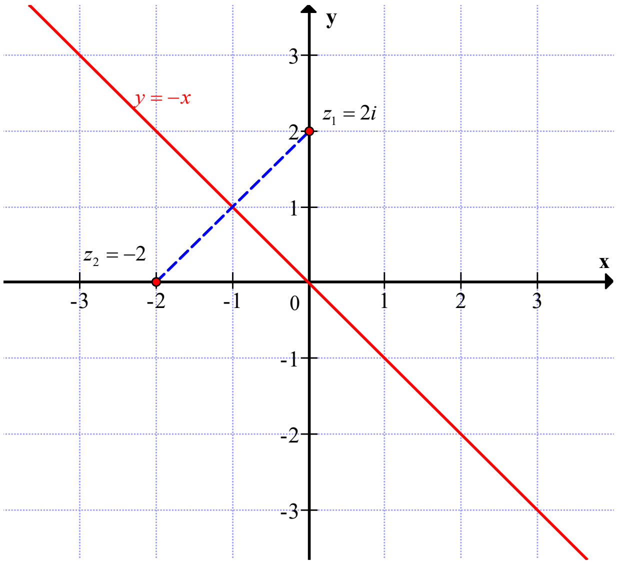
$\Rightarrow x^2+(y-2)^2=(x+2)^2+y^2$

$\Rightarrow x^2+y^2-4y+4=x^2+4x+4+y^2$

$\Rightarrow -4y=4x$

$\Rightarrow y=-x$

This is a straight line, perpendicular bisector of the segment joining the points $z_1 = 2i$ and $z_2 = -2$. See the following figure.



Application activity 1.3.5.1

1) If $\left|\frac{2z+1}{z}\right| = 1$ and P represents z in the Argand plane, show that P lies on a circle and find the centre and radius of this circle.

2) Determine, in complex plane, the set of points M of affix z such that

a) $|z| = 2$ b) $|z| < 2$ c) $|z| > 2$ d) $|z+1| = 1$ e) $|z+1| = |z-1|$

f) $|z - 1 + 3i| = 2$

1.3.6 Square root of a complex number

Activity 1.3.6

1.Consider the polynomials $P(x)$ and $Q(x)$ in one real variable defined as $P(x) = 2x^3 + 3x^2 - c$ and $Q(x) = ax^4 + bx^3 - dx^2 + fx + g$. Is it possible to have $P(x) = Q(x)$?

2. Given that $(a+bi)^2 = 6 - 4i$, where a and b are real numbers, discuss and determine the values of a and b. Using the values of a and b, calculate the square root of $z = 6 - 4i$. Express your answer in words.

Content summary

A complex number $z = x + yi$ is a square root of a complex number $Z = a + bi$ if $z^2 = Z$. This means that $(x + yi)^2 = a + bi$ equivalently $x^2 + 2xyi - y^2 = a + bi$.

Then $\begin{cases} x^2 - y^2 = a \\ 2xy = b \end{cases}$

Therefore $\begin{cases} x = \pm\sqrt{\frac{1}{2}\left(a + \sqrt{a^2 + b^2}\right)} \\ y = \pm\sqrt{\frac{1}{2}\left(\sqrt{a^2 + b^2} - a\right)} \end{cases}$

The sign cannot be taken arbitrary because the product xy has the sign of b.

Thus,

- if $b > 0$ you take the same sign.
- if $b < 0$ you interchange the signs.
- In each case, a complex number has two square roots.

Example

Find the square root of $8 - 6i$ b) $2i$ c) -2 d) $-3 - 4i$

Solution

a) Let $z^2 = (x + yi)^2 = 8 - 6i$

Develop the power: $(x^2 - y^2) + 2xyi = 8 - 6i$

Equate real parts and imaginary parts

$$x^2 - y^2 = 8 \quad (1)$$
$$2xy = -6 \quad (2)$$

Now, consider the modulus: $|z|^2 = |z^2|$

$$x^2 + y^2 = \sqrt{(8^2 + 6^2)} = 10 \qquad (3)$$

Solving (1) and (3), we get $x^2 = 9$ and $y^2 = 1$ leading to $x = \pm 3$ and $y = \pm 1$.

From (2), x and y are of opposite signs, thus: $x = 3$ and $y = -1$ or $x = 3$ and $y = -1$

Finally: $z_1 = 3 - i$ and $z_2 = -3 + i$

b) $\begin{cases} a=0 \\ b=2>0 \end{cases}$, thus $\begin{cases} x=\pm\sqrt{\frac{2}{2}}=\pm 1 \\ y=\pm\sqrt{\frac{2}{2}}=\pm 1 \end{cases}$

Since b is positive, we take the same sign. This yields to $\sqrt{-2i}=1+i$ or $\sqrt{-2i}=-1-i$.

c) From $z=-2,\ a=\mathbb{R}_e(z)=-2<0, b=\text{Im}(z)=0, x=0$ and $y=\pm\sqrt{2}$.

Thus, $\Rightarrow \sqrt{-2}=\pm i\sqrt{2}$

d) Let $z^2=(x+iy)^2=-3-4i$

By expending the power, we get

$(x^2-y^2)+2xyi=-3-4i$

Equating real parts and imaginary parts yields

$x^2-y^2=-3 \qquad (1)$

$2xy=-4 \qquad (2)$

Now, consider the modulus: $|z|^2 = |z^2|$

$x^2+y^2=\sqrt{9+16}=5 \qquad (3)$

Solving (1) and (3), we get

$x^2=1$ and $y^2=4$ leading to $x=\pm 1$ and $y=\pm 2$.

From (2), x and y are of opposite signs, thus: $x=1$ and $y=-2$ or $x=-1$ and $y=2$

Finally: $z_1=1-2i$ and $z_2=-1+2i$

We can write $\sqrt{-3-4i}=\pm(1-2i)$.

Application activity 1.3.6

Find the square roots of the following complex numbers.

a) $z=-3+4i$ b) $z=-2i$ c) $z=2-2i\sqrt{3}$

1.4 Equations in the set of complex numbers

1.4.1 Simple linear equations of the form $Az+B=0$

Activity 1.4.1

Given the complex number z such that $4z+5i=12-i$, discuss how to find the value of z. Express your answer in words.

Content summary

To find the solution set of the equation $Az+B=0$ (where A and B are two complex numbers, A different from zero) follows the same process involved in solving equation of the form $ax+b=0$ in the set of real numbers.

Therefore $Az+B=0 \Rightarrow z=-\frac{B}{A}$. The remaining task is to express z in the form of $x+yi$.

Example

Solve each of the following equations in the set of complex numbers

a) $(1-i)z=2+i$ b) $iz+(2-10i)z=3z+2i$

Solution

a) $(1-i)z=2+i \Leftrightarrow z=\frac{2+i}{1-i}=\frac{(2+i)(1+i)}{(1-i)(1+i)}=\frac{1+3i}{2}=\frac{1}{2}+\frac{3}{2}i$

b) $iz+(2-10i)z=3z+2i \Leftrightarrow iz+(2-10i)z-3z=2i \Leftrightarrow (i+2-10i-3)z=2i \Leftrightarrow z=-\frac{9}{41}-\frac{i}{41}$

Solve the following equation and the system of equations:

1) $(1+3i)\text{z}=2i+4i$

2) $\begin{cases} 7z+(8-2i)w=4-9i \\ (1+i)+(2-i)w=2+7i \end{cases}$

1.4.2 Quadratic equations of the form $Az^2 + Bz + C = 0$

Activity 1.4.2

Given the quadratic equation $z^2 - (1+i) = 0$, you can write it as $z^2 = 1+i$. Calculate the square root of $1+i$ to get the value of z and discuss how to solve equations of the form $Az^2 + C = 0$ where A and C are complex numbers and A is different from zero. Express in words the formula used.

Content summary

Solving simple quadratic equations in the set of complex numbers recalls the procedure of how to solve the quadratic equations in the set of real numbers considering that $\sqrt{-1} = i$.

There fore, $Az^2 + C = 0 \Leftrightarrow z^2 = \frac{-c}{A} \Leftrightarrow z = \sqrt{\frac{-c}{A}}$

Let's now discuss the general case where the coefficient B is not zero. We are already familiar with finding square root of a complex number, the process of solving the equation

$Az^2 + Bz + C = 0$ is the same as the process for solving quadratic equation in one real variable.

When solving equations of the form $Az^2 + Bz + C = 0, (A \neq 0)$;

take $z = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}$.

In particular, let a, b and c be real numbers $(a \neq 0)$, then the equation $az^2 + bz + c = 0$, has either two real roots, one repeated real root or two conjugate complex roots.

- If $\Delta > 0$, there are two distinct real roots:

  $z_1 = \frac{-b + \sqrt{\Delta}}{2a}$ and $z_2 = \frac{-b - \sqrt{\Delta}}{2a}$.

- If $\Delta = 0$, there is a double real root:

  $z_1 = z_2 = -\frac{b}{2a}$

– If $\Delta < 0$, there is no real roots. In this case there are two conjugate complex roots:

$$z_1 = \frac{-b + i\sqrt{-\Delta}}{2a} \text{ and } z_2 = \frac{-b - i\sqrt{-\Delta}}{2a}.$$

Where

$$\Delta = b^2 - 4ac$$

$$z_1 + z_2 = -\frac{b}{a},\ z_1 \cdot z_2 = \frac{c}{a}$$

Example

Solve the following equations a) $z^2 + 6z + 10 = 0$ b) $z^2 + \left(\sqrt{3} + i\right)z + 1 = 0$

Solution

a) $z^2 + 6z + 10 = 0$

$\Delta = 36 - 40 = -4$

Given that $i^2 = -1$, $\Delta = 4i^2$ and $\sqrt{\Delta} = 2i$ or $\sqrt{\Delta} = -2i$. The roots of the given equation are:

$$z_1 = \frac{-6 - 2i}{2} = -3 - i$$

$$z_2 = \frac{-6 + 2i}{2} = -3 + i$$

b) $z^2 + \left(\sqrt{3} + i\right)z + 1 = 0$

$$z = \frac{-\left(\sqrt{3} + i\right) \pm \sqrt{\left(\sqrt{3} + i\right)^2 - 4}}{2} = \frac{-\left(\sqrt{3} + i\right) \pm \sqrt{-2 + 2\sqrt{3}i}}{2}$$

Now we have to solve $w^2 = -2 + 2\sqrt{3}i$.

Put $w = x + iy$, you find $x = \pm 1$ and $y = \pm 3$.

Hence, $z = \frac{-\sqrt{3} - i \pm (1 + \sqrt{3}i)}{2}$, that gives: $z_1 = \frac{1 - \sqrt{3} + (1 + \sqrt{3})i}{2}$ and

$$z_1 = \frac{-1 - \sqrt{3} - (1 + \sqrt{3})i}{2}$$

Application activity 1.4.1

1) Solve the following equations

a) $z^2-(3+i)z+4+3i=0$ b) $z^2+9=0$

2) Prove algebraically that $(z-4)$ is a factor of $z^3-15z-4$ Based on your proof, discuss how to find the solution set of $z^3-15z-4=0$. List all solutions of the equations.

1. 5 Polar form of a complex number

1.5.1 Definition and argument of a complex number

Activity 1.5.1

Consider the vector $\overrightarrow{M}=\overrightarrow{e_1}+\overrightarrow{e_2}$, where $\overrightarrow{e_1}$ and $\overrightarrow{e_2}$ are unit vectors on axis, plot it in a Cartesian plane and show the position of the point M. Calculate $|\overrightarrow{M}|$, the modulus of $\overrightarrow{M}$ and write $\overrightarrow{M}$ in terms of $|\overrightarrow{M}|$ and the angle θ formed by $\overrightarrow{M}$ and $x-axis$. Express your answer in words.

Content summary

A complex number $z=a+bi$ plotted in the complex plane has the modulus $r=\sqrt{a^2+b^2}$.

Let θ be the angle defined by the vector $\vec{r}=a\vec{i}+b\vec{j}$ and the real axis as shown in figure 1.10.

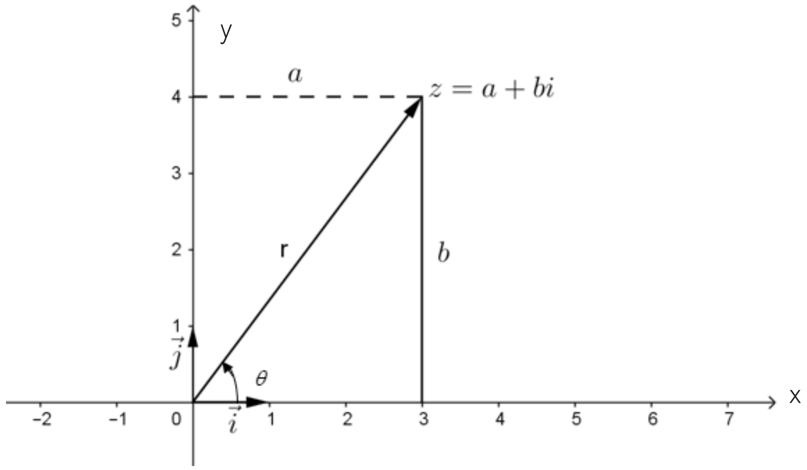


Figure 1.5 Argument of complex number $z=a+bi$

Using trigonometric ratios, we have

$$\begin{cases}\sin\theta = \dfrac{b}{r}\\ \cos\theta = \dfrac{a}{r}\end{cases} \Leftrightarrow \begin{cases} b = r\sin\theta \\ a = r\cos\theta\end{cases} \Rightarrow z = r(\cos\theta + i\sin\theta)$$

From affix of a complex number $z = a + bi$, there is a connection between its modulus and angle between the corresponding vector and positive $x-axis$ as illustrated in figure 1.5. This angle is called the argument of z and denoted by $\arg(z)$.

Hence, $z = a + ib = r(\cos\theta + i\sin\theta)$ with $r = |z| = \sqrt{a^2 + b^2}$ and

$\theta = \arg(z) = \arctan\dfrac{b}{a}$.

The expression $z = r(\cos\theta + i\sin\theta)$ is called polar form or trigonometric form of the complex number z.

The complex number $z = r(\cos\theta + i\sin\theta)$ can be written in brief as $z = rcis\theta$ or $r\angle\theta$.

Depending on the position of the affix of the complex number z, a and b can be of the same or different signs. Therefore it is very necessary to choose

$\theta = \arctan\dfrac{b}{a}$ carefully.

In summary,

$$\theta = arg(z) = \begin{cases} \arctan\dfrac{b}{a}, if\ a > 0 \\ \pi + \arctan\dfrac{b}{a}, if\ a < 0\, and\, b > 0 \\ -\pi + \arctan\dfrac{b}{a}, if\ a < 0\, and\, b < 0 \\ \dfrac{\pi}{2}, if\ a = 0\, and\, b > 0 \\ -\dfrac{\pi}{2}, if\ a = 0\, and\, b < 0 \\ Undefined, if\ a = 0\, and\, b = 0 \end{cases}$$

The value of $\theta = \arg(z)$ must always be expressed in radians. It can change by any multiple of 2π and still give the same trigonometric ratios. Hence, the arg function is sometimes considered as multivalued. Normally, it is advised to consider the argument in the interval $]-\pi,\pi]$ also called principal argument.

The argument of the complex number 0 is undefined.

Properties

1. Let z and z' be two non-zero complex numbers. We have $z=z'$ if and only if $|z|=|z'|$ and $\arg(z)=\arg(z')[2\pi]$ that is $\arg(z)=\arg(z')+2k\pi$, where $k\in\mathbb{Z}$ .This property is deduced from the definitions of modulus and an argument of a complex number.

2. The argument of z can be given by

$$\tan\theta = i\frac{z-\bar{z}}{z+\bar{z}}$$

Note that having a polar form of a complex number, you can get its corresponding algebraic form.

Example

a) Write the complex numbers in the polar form

i) $z=1+i$ ii) $z=\frac{1}{2}-i\frac{\sqrt{3}}{2}$

b) Convert the following complex numbers in algebraic form

i) $\text{cis}\left(-\frac{\pi}{3}\right)$ ii) $2\,\text{cis}\left(-\frac{\pi}{2}\right)$

Solution

a)

i) $z=1+i$, $|z|=\sqrt{2}$,

Let θ be $Arg(z)$, thus $\begin{cases}\cos\theta=\frac{1}{\sqrt{2}}=\frac{\sqrt{2}}{2}\\ \sin\theta=\frac{1}{\sqrt{2}}=\frac{\sqrt{2}}{2}\end{cases}$ which gives that θ lies in 1st quadrant and $Arg(z)=\theta=\frac{\pi}{4}$.

Hence, $z=\sqrt{2}\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)$

ii) $z=\frac{1}{2}-i\frac{\sqrt{3}}{2}\Rightarrow |z|=1, Arg(z)=-\frac{\pi}{3}$. Thus

$$z=\frac{1}{2}-i\frac{\sqrt{3}}{2}=\cos\left(-\frac{\pi}{3}\right)+i\sin\left(-\frac{\pi}{3}\right)$$

b) i) $cis\left(-\frac{\pi}{3}\right)=\cos\left(-\frac{\pi}{3}\right)+i\sin\left(-\frac{\pi}{3}\right)=\frac{1}{2}-i\frac{\sqrt{3}}{2}$

ii) $2cis\left(-\frac{\pi}{2}\right)=2\left(\cos\left(-\frac{\pi}{2}\right)+i\sin\left(-\frac{\pi}{2}\right)\right)=2(0-i)=-2i$.

Application activity 1.4.1

1) Find the principal argument of the following complex numbers

a) $-2i$ b) $1-i$ c) $1-i\sqrt{3}$

d) $-1+i\sqrt{3}$ e) $-\sqrt{3}-i$

2) Write the following complex numbers in the polar form

a) $z=-1+i$ b) $z=-\frac{1}{2}+i\frac{\sqrt{3}}{2}$ c) $z=-2\left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)$ d) $z=2$

e) $z=-i$

3) Convert the following complex numbers in algebraic form

a) $5\,\text{cis}\,270^0$ b) $4\,\text{cis}\,300^0$ c) $\sqrt{2}\,\text{cis}\left(-\frac{\pi}{4}\right)$ d) $3\,\text{cis}\left(\frac{\pi}{2}\right)$

1.5.2 Operations on complex numbers in polar form

1.5.2.1 Multiplication and division of complex numbers

Activity 1.5.1

Given two complex numbers $z_1 = \sqrt{3} - i$ and $z_2 = 1 + i$,

a) Write z_1 and z_2 in polar form.

b) Determine the product $z_1.z_2$ in algebraic form and convert it in polar form

c) Deduce from (b) the argument and the modulus of $z_1.z_2$

d) Compare the argument of the product $z_1.z_2$ and those for z_1 and z_2, then establish any relationship among them. Express your answer in words

Content summary

For two complex numbers $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$, their product is given by $z_1 z_2 = r_1 r_2\left[\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)\right]$ provided that 2π may be added to or substracted from $\theta_1+\theta_2$ if $\theta_1+\theta_2$ is outside the permitted range of the principal argument.

Similarly, division is given by:

$\frac{z_1}{z_2} = \frac{r_1}{r_2}\left[\cos(\theta_1-\theta_2) + i\sin(\theta_1-\theta_2)\right]$ provided that 2π may be added to or substracted from, $\theta_1-\theta_2$ if $\theta_1-\theta_2$ is outside the permitted range of the principal argument.

Therefore, for non-zero complex numbers z, z' and any integer n, we have:

i) $\arg(z.z') = \arg(z) + \arg(z')[2\pi]$

ii) $\arg\left(\frac{1}{z}\right) = -\arg(z)[2\pi]$

iii) $\arg(z^n) = n\arg(z)[2\pi]$

iv) $\arg\left(\frac{z}{z'}\right) = \arg(z) - \arg(z')[2\pi]$

Example

Compute using polar form

a) $z=\left(\sqrt{3}-i\right)(1+i)$ b) z' = $\dfrac{\sqrt{3}+i}{1+i}$

Solution

a) Let $z_1=\sqrt{3}-i$, thus $|z_1|=\sqrt{3+1}=2$ and $z_1=2\left(\dfrac{\sqrt{3}}{2}-\dfrac{1}{2}i\right)$.

Let $\alpha_1=\arg(z_1)$, thus

$$\left.\begin{array}{l}\cos\alpha_1=\dfrac{\sqrt{3}}{2}\\ \sin\alpha_1=-\dfrac{1}{2}\end{array}\right\}\Leftrightarrow \alpha_1=-\frac{\pi}{6}[2\pi]=arg(z_1).$$

$z_2=1+i\Rightarrow|z_2|=\sqrt{1+1}=\sqrt{2}$, and then

$$z_2=\sqrt{2}\left(\frac{\sqrt{2}}{2}+i\frac{\sqrt{3}}{2}\right)$$

Let $\alpha_2=\arg(z_2)$, thus

$$\left.\begin{array}{l}\cos\alpha_2=\dfrac{\sqrt{2}}{2}\\ \sin\alpha_2=\dfrac{\sqrt{2}}{2}\end{array}\right\}\Leftrightarrow \alpha_2=\frac{\pi}{4}[2\pi]=\arg(z_2).$$

Then $\arg(z)=-\dfrac{\pi}{6}+\dfrac{\pi}{4}[2\pi]=\dfrac{\pi}{12}[2\pi]$.

Therefore:

$$z_1.z_2=2\sqrt{2}\left(\cos\frac{\pi}{12}+i\sin\frac{\pi}{12}\right)$$

b) Let $z_3=\sqrt{3}+i$, thus $|z_3|=\sqrt{3+1}=2$ and $z_3=2\left(\dfrac{\sqrt{3}}{2}+\dfrac{1}{2}i\right)$.

Let $\alpha_3=\arg(z_3)$

$$\left.\begin{array}{l}\cos\alpha_3=\dfrac{\sqrt{3}}{2}\\ \sin\alpha_3=\dfrac{1}{2}\end{array}\right\}\Rightarrow \alpha3 = \arg(z_3)=\frac{\pi}{6}\ [2\pi].$$

$$\frac{z_3}{z_2}=\frac{2\left(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right)}{\sqrt{2}\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)}=\frac{\sqrt{2}\left(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right)}{\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)}=\sqrt{2}\left(\cos\frac{\pi}{12}-i\sin\frac{\pi}{12}\right).$$

Application activity 1.5.2.1

Given 3 complex numbers z, y and w such that

$z=1+i,\ w=-\sqrt{3}+i\quad and\quad y=-3+i\sqrt{3}$

Convert them in polar form then compute $z.w,\ z.y,\ \frac{w}{y},\frac{y}{z}$

1.5.2.2 Powers of a complex number in polar form

Activity 1.5.2.2

Given complex number $z=r(\cos\theta+i\sin\theta)$

1) Find the expression for $z^2=z\cdot z$
2) Find the expression for $z^3=z^2\cdot z$
3) Using results from 1 to 2, guess the expression for z^n
4) Express your answers in words.

Content summary

Power of a complex number z is given by

$$z^n=\left(r(\cos\theta+i\sin\theta)\right)^n=r^n(\cos n\theta+i\sin n\theta);\ n\in\mathbb{Z}_0$$

In particular, if $r=1$, we have the equality known as de Moivre's theorem

$(\cos\theta + i\sin\theta)^n = (\cos n\theta + i\sin n\theta)$

that is valid for any rational number n .

Example

Consider the complex number $z = -1 + i$; determine z^{20}

Solution

$z = -1 + i \Rightarrow |z| = \sqrt{2}$ and $\arg(z) = -\frac{3\pi}{4}[2\pi]$ and $Arg(z) = -\frac{\pi}{4}$ leading to

$z = \sqrt{2}\left[\cos\left(-\frac{3\pi}{4}\right) + i\sin\left(-\frac{3\pi}{4}\right)\right]$

Hence,

$$z^{20} = \sqrt{2^{20}}\left[\cos\left(-\frac{3\pi}{4}\right) + i\sin\left(-\frac{3\pi}{4}\right)\right]^{20} = 2^{10}\left[\cos(-15\pi) + i\sin(-15\pi)\right] = -2^{10}$$

Application activity 1.5.2.2

1) Simplify the following complex numbers using De Moivre's theorem

a) $(\cos 3\pi + i\sin 3\pi)^9$ b) $\left(\cos\frac{5\pi}{2} + i\sin\frac{5\pi}{2}\right)^{\frac{2}{5}}$ c) $\left(\cos\frac{\pi}{5} + i\sin\frac{\pi}{5}\right)^5$

d) $(\cos 45^0 + i\sin 45^0)^2$ e) $(\cos 60^0 + i\sin 60^0)^3$

2) Using De Moivre's theorem, perform the following powers

$a)\left(1 + i\sqrt{3}\right)^6$ $b)\left(-\sqrt{3} + i\right)^{10}$ $c)(1 - i)^7$

1.5.2.3. Nthroots of a complex number

Activity 1.5.2.3

Given $z=4$

1) Express z in polar form

2) Let $z_k = r'\text{cis}\,\theta'$ for $k=0,1,2,3$ be the four 4^{th} roots of z. Using result in 1) and the expression $(z_k)^4 = z$, find all four 4^{th} roots of z in polar form.

Content summary

From this activity,

If $(z_k)^n = z$ for $z = r cis\theta$, then

$$z_k = \sqrt[n]{r}\; cis\left(\frac{\theta + 2k\pi}{n}\right) \qquad k = 0,1,2,3,\ldots\ldots,n-1$$

Here $\sqrt[n]{r}$ is the usual (positive) n^{th} root of the positive real number r.

Example

Determine the 4^{th} roots of -4

Solution

$|-4| = 4$

$\arg(-4) = \pi + \arctan 0 = \pi$ and

$$z_k = \sqrt[4]{4}\left(cis\frac{\pi + 2k\pi}{4}\right);\; k = 0,1,2,3$$

$$z_0 = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) \qquad z_1 = \sqrt{2}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)$$

$$= \sqrt{2}\left(\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right) \qquad = \sqrt{2}\left(-\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right)$$

$$= 1 + i \qquad = -1 + i$$

$$z_2=\sqrt{2}\left(\cos\frac{5\pi}{4}+i\sin\frac{5\pi}{4}\right) \quad z_3=\sqrt{2}\left(\cos\frac{7\pi}{4}+i\sin\frac{7\pi}{4}\right)$$

$$=\sqrt{2}\left(-\frac{\sqrt{2}}{2}-i\frac{\sqrt{2}}{2}\right) \quad =\sqrt{2}\left(\frac{\sqrt{2}}{2}-i\frac{\sqrt{2}}{2}\right)$$

$$=-1-i \quad =1-i$$

Then the 4th roots of -4 are $1+i, -1+i, -1-i,$ and $1-i$

Special case: nth roots of unity

Here $z=1$ and $|z|=1$, $\arg(z)=0$

Then

$$z_k=\sqrt[n]{1}\ \text{cis}\frac{0+2k\pi}{n}=\text{cis}\frac{2k\pi}{n}$$

And then, the nth roots of unity are given by

$$z_k=\text{cis}\frac{2k\pi}{n};\ k=0,1,2,3,\ldots..,n-1$$

This shows that the first root among the nth roots of unity is always 1.

Notice

1. The *nth* roots of unity can be used to find the *nth* roots of any complex number if one of these roots is known.

   If one of the *nth* roots of a complex number z is known, the other roots are found by multiplying that root with *nth* roots of unity.

2. The sum of *nth* roots of unity is zero.

Example

Find cubic roots of unity

Solution

$z=1, n=3$

$$z_k=cis\frac{2k\pi}{3}; \quad k=0,1,2$$

$$z_0 = \operatorname{cis} 0$$
$$= 1$$

$$z_1 = \operatorname{cis}\frac{2\pi}{3}$$
$$= -\frac{1}{2} + i\frac{\sqrt{3}}{2}$$

$$z_2 = \operatorname{cis}\frac{4\pi}{3}$$
$$= -\frac{1}{2} - i\frac{\sqrt{3}}{2}$$

Example

Using cubic roots of unity, find the cubic root of -27, known that -3 is one of the roots.

Solution

We have one of the cubic roots of -27, which is -3.

We have seen that the cubic roots of unity are

$$\left.\begin{aligned} z_0 &= 1 \\ z_1 &= -\frac{1}{2} + i\frac{\sqrt{3}}{2} \\ z_2 &= -\frac{1}{2} - i\frac{\sqrt{3}}{2} \end{aligned}\right\} \Rightarrow \sum_{k=0}^{2} z_k = 0$$

Then, cubic roots of -27 are

$$z_0 = 1 \times (-3)$$
$$= -3$$

$$z_1 = \left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) \times (-3)$$
$$= \frac{3}{2} - i\frac{3\sqrt{3}}{2}$$

$$z_2 = \left(-\frac{1}{2} - i\frac{\sqrt{3}}{2}\right) \times (-3)$$

$$= \frac{3}{2} + i\frac{3\sqrt{3}}{2}$$

Example

Using 5^{th} roots of unity, find the exact value of $\cos\frac{2\pi}{5}$.

Solution

The 5^{th} roots of unity are given by $z_k = cis\frac{2k\pi}{5}$, $k = 0,1,2,3,4.$

$$z_0 = 1$$

$$z_1 = \text{cis}\frac{2\pi}{5}$$

$$= \cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5}$$

$$z_2 = \text{cis}\frac{4\pi}{5}$$

$$= \cos\frac{4\pi}{5} + i\sin\frac{4\pi}{5}$$

$$z_3 = \text{cis}\frac{6\pi}{5}$$

$$= \cos\frac{6\pi}{5} + i\sin\frac{6\pi}{5}$$

$$z_4 = \text{cis}\frac{8\pi}{5}$$

$$= \cos\frac{8\pi}{5} + i\sin\frac{8\pi}{5}$$

The sum of these roots must be zero, then

$$1 + \cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5} + \cos\frac{4\pi}{5} + i\sin\frac{4\pi}{5} + \cos\frac{6\pi}{5} + i\sin\frac{6\pi}{5} + \cos\frac{8\pi}{5} + i\sin\frac{8\pi}{5} = 0$$

$$1 + \cos\frac{2\pi}{5} + \cos\frac{4\pi}{5} + \cos\frac{6\pi}{5} + \cos\frac{8\pi}{5} + i\left(\sin\frac{2\pi}{5} + \sin\frac{4\pi}{5} + \sin\frac{6\pi}{5} + \sin\frac{8\pi}{5}\right) = 0$$

Taking only the real parts, we have

$$1+\cos\frac{2\pi}{5}+\cos\frac{4\pi}{5}+\cos\frac{6\pi}{5}+\cos\frac{8\pi}{5}=0 \quad (1)$$

We know that $\cos\alpha=\cos(2\pi-\alpha)$, then

$$\cos\frac{6\pi}{5}=\cos\left(2\pi-\frac{6\pi}{5}\right)$$
$$=\cos\left(\frac{10\pi-6\pi}{5}\right)$$
$$=\cos\frac{4\pi}{5}$$

$$\cos\frac{8\pi}{5}=\cos\left(2\pi-\frac{8\pi}{5}\right)$$
$$=\cos\left(\frac{10\pi-8\pi}{5}\right)$$
$$=\cos\frac{2\pi}{5}$$

(1) becomes, $1+\cos\frac{2\pi}{5}+\cos\frac{4\pi}{5}+\cos\frac{4\pi}{5}+\cos\frac{2\pi}{5}=0$ and we have

$$1+2\cos\frac{2\pi}{5}+2\cos\frac{4\pi}{5}=0$$

$$\Leftrightarrow 1+2\cos\frac{2\pi}{5}+2\cos 2\left(\frac{2\pi}{5}\right)=0$$

$$\Leftrightarrow 1+2\cos\frac{2\pi}{5}+2\left(2\cos^2\frac{2\pi}{5}-1\right)=0, \qquad as\ \ \cos 2\alpha=2\cos^2\alpha-1$$

$$\Leftrightarrow 1+2\cos\frac{2\pi}{5}+4\cos^2\frac{2\pi}{5}-2=0$$

$$\Leftrightarrow 4\cos^2\frac{2\pi}{5}+2\cos\frac{2\pi}{5}-1=0$$

Let $t=\cos\frac{2\pi}{5}$, we have

$$4t^2+2t-1=0$$

$\Delta = 4 + 16 = 20$

$$t_1 = \frac{-2+\sqrt{20}}{8} = \frac{-2+2\sqrt{5}}{8} = \frac{\sqrt{5}-1}{4} \qquad t_2 = \frac{-2-\sqrt{20}}{8} = \frac{-2-2\sqrt{5}}{8} = \frac{-\sqrt{5}-1}{4}$$

The angle $\frac{2\pi}{5}$ is an angle of the first quadrant. Then the cosine of this angle must be positive. Thus, $t = \frac{-\sqrt{5}-1}{4}$ is to be rejected.

Hence, the exact value of $\cos\frac{2\pi}{5}$ is $\frac{\sqrt{5}-1}{4}$.

Graphical representation of n^{th} roots of a complex number

The n nth roots of a complex number are equally spaced around the circumference of a circle of centre o in the complex plane.

If the complex number for which we are computing the n^{th} roots is $z = rcis\theta$, the radius of the circle will be $R = \sqrt[n]{r}$ and the first root z_0 corresponding to $k = 0$ will be at an amplitude of $\varphi = \frac{\theta}{n}$. This root will be followed by the $n-1$ remaining roots at equal distances apart, consecutively.

The angular amplitude between each root is $\beta = \frac{2\pi}{n}$.

Now, if $z = 1$ the radius of the circle is 1.

Thus, the n^{th} roots of unity are equally spaced around the circumference of a unit circle (circle of centre o and radius 1) in the complex plane.

Example

Represent graphically the 4^{th} roots of $z = 8\left(1 - i\sqrt{3}\right)$

Solution

$|z| = 8\sqrt{4} = 16$

$$\arg z = \arctan \frac{-8\sqrt{3}}{8}$$
$$= \arctan\left(-\sqrt{3}\right)$$
$$= -\frac{\pi}{3}$$

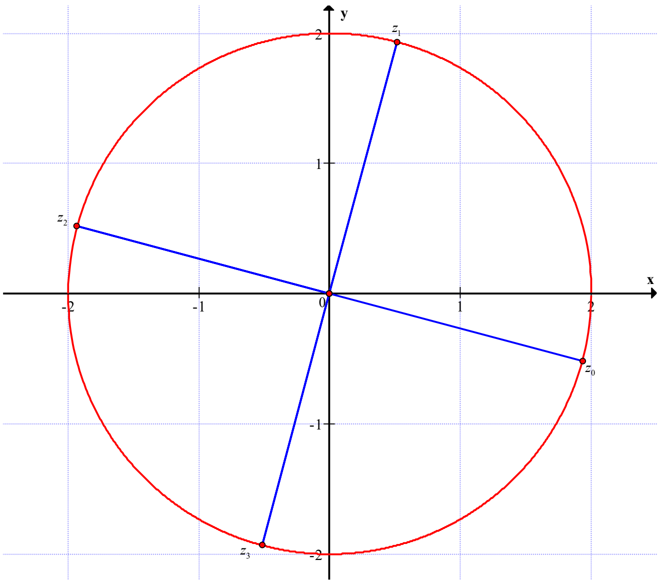
The roots are given by

$$z_k = \sqrt[4]{16}\ \text{cis}\left(\frac{-\frac{\pi}{3} + 2k\pi}{4}\right)$$
$$= 2\,\text{cis}\left(\frac{-\pi + 6k\pi}{12}\right) \quad k = 0, 1, 2, 3$$

This is

$$z_0 = 2\,\text{cis}\left(-\frac{\pi}{12}\right) \quad z_2 = 2\,\text{cis}\left(\frac{11\pi}{12}\right)$$
$$z_1 = 2\,\text{cis}\left(\frac{5\pi}{12}\right) \quad z_3 = 2\,\text{cis}\left(\frac{17\pi}{12}\right)$$

The circle will have radius 2.



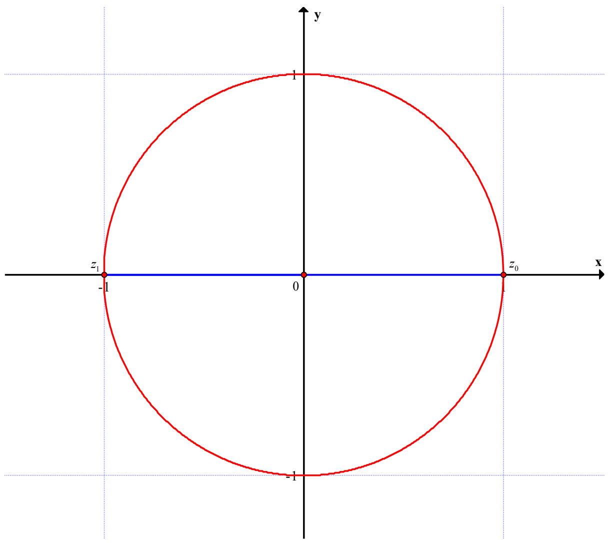
Example

Represent graphically the n^{th} roots of unity for $n=2$, $n=3$ and $n=4$.

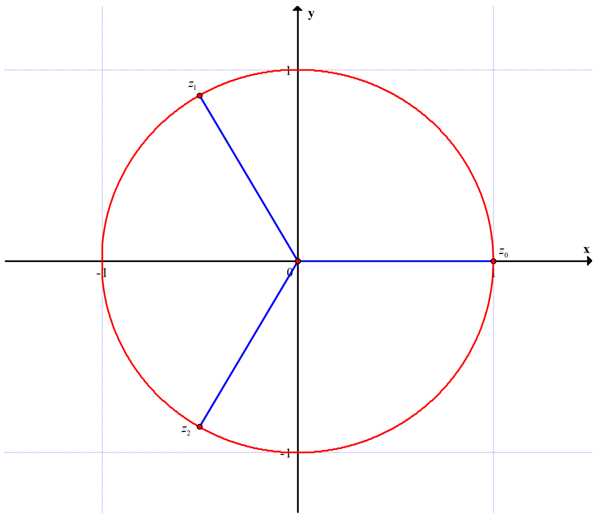
Solution

$$z_k = \text{cis}\left(\frac{2k\pi}{n}\right), \quad k = 0,1,2,3$$

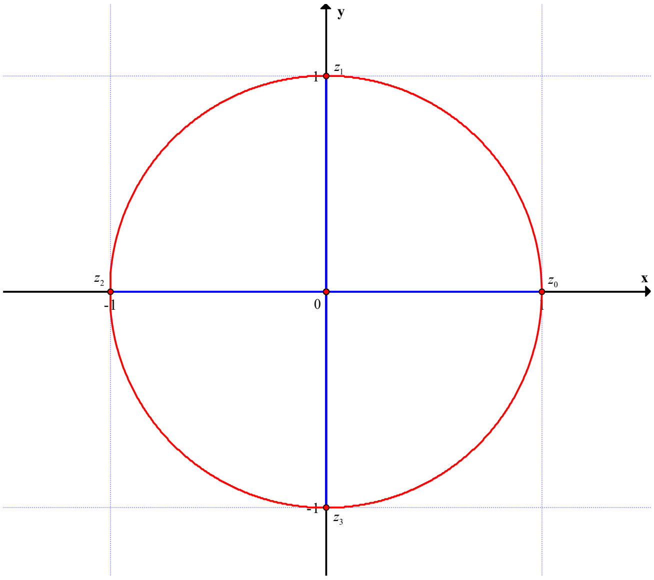
$$n=2: z_0 = 1, \quad z_1 = -1$$



$$n=3: z_0 = 1, \quad z_1 = -\frac{1}{2} + i\frac{\sqrt{3}}{2}, \quad z_2 = -\frac{1}{2} - i\frac{\sqrt{3}}{2}$$



$$n=4: z_0 = 1, \quad z_1 = i, \quad z_2 = -1, \quad z_3 = -i$$



We can see that the n^{th} roots of unity for $n > 2$ are the vertices of a regular polygon inscribed in a circle of centre 0 and radius 1.

Application activity 1.5.2.3

1) Solve the equation $z^4 = i$

2) Find the five fifth roots of 32

3) Find five fifth roots of 1

4) Find four fourth roots of $8 + 8i\sqrt{3}$

5) Using 5^{th} roots of unity, find the exact value of $\sin\dfrac{2\pi}{5}$.

6) On Argand diagram, represent

a) the three cube roots of -27

b) the four fourth roots of -4

c) the cube roots of $8i$

d) the fourth roots of -1

1.5.3 Construction of regular polygons

Activity 1.5.3

1) Find three 3^{rd} roots of unity

2) Represent the roots obtained in 1) on Argand diagram

3) Use a ruler to join the points obtained in 2).

4) What can you say about the figure obtained in 3)?

Content summary

A regular polygon is a polygon that is equiangular (all angles are equal in measure) and equilateral (all sides have the same length).

As illustrated on figure 1.6, we call apothem, the perpendicular distance from the centre (the interior point) to any side. We can draw a line segment from the centre to one of the vertices. The length of this segment is called the radius of the polygon.

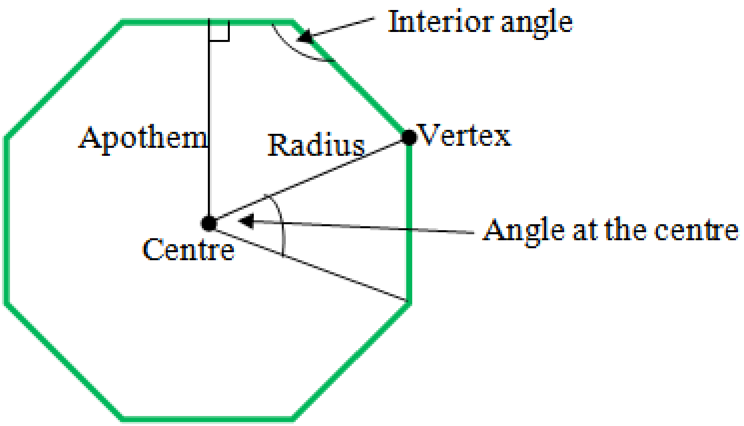


Figure 1.6: Regular polygon

Recall that the n^{th} roots of unity are given by $z_k = \text{cis}\,\frac{2k\pi}{n}, \quad k = 0,1,2,.....,n-1.$

The n^{th} roots of unity for $n \geq 3$ are the vertices of a regular polygon with n sides inscribed in a circle of centre o and radius 1. The vertices of a polygon are the points where its any two consecutive sides intersect. The angle at the centre is $\frac{2\pi}{n}$.

To draw a regular polygon with n sides follows the following steps:

- Start by drawing a circle in Argand diagram, with the centre at the origin. The radius and the centre of this circle will be the radius and centre of the regular polygon.
- Around the circle, place the points with affixes

  $z_k = \text{cis}\frac{2k\pi}{n},\ k = 0,1,2,.....,n-1$. Those points are the vertices of the polygon.
- Using a ruler join the obtained points around the circle.
- The obtained figure is the needed regular polygon.

Example

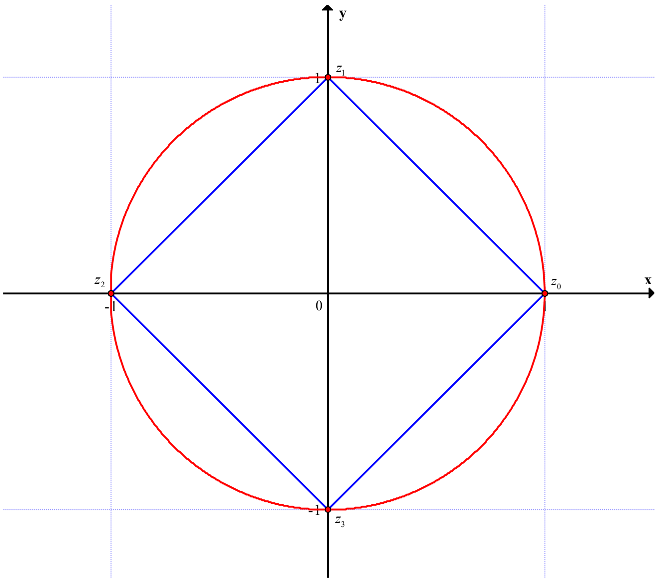
Construct, in Argand diagram, a square.

Solution

A square is a regular polygon with four sides.

We have four vertices: $z_k = \text{cis}\frac{2k\pi}{4} = \text{cis}\frac{k\pi}{2},\ k = 0,1,2,3$

$z_0 = \text{cis}\,0 = 1,\ z_1 = \text{cis}\frac{\pi}{2} = 1,\ z_2 = \text{cis}\,\pi = -1,\ z_3 = \text{cis}\frac{3\pi}{2} = -1$



Example

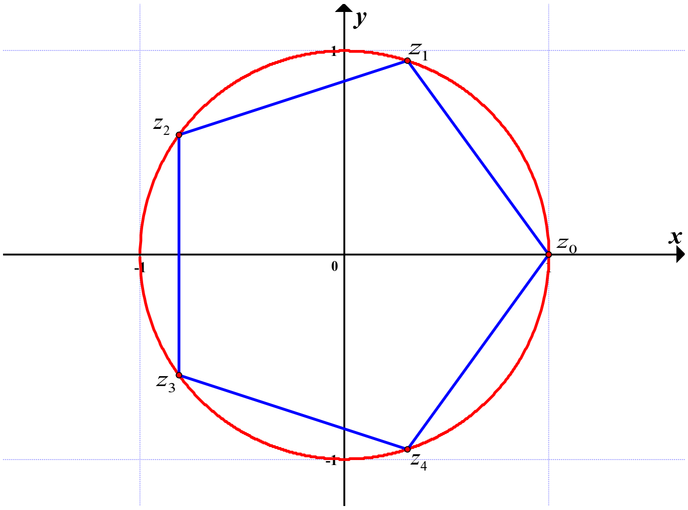
Construct, in Argand diagram, regular pentagon.

Solution

A regular pentagon has 5 sides.

We have five vertices: $z_k = \text{cis}\frac{2k\pi}{5},\ k = 0,1,2,3,4$

$$z_0 = \text{cis}\,0 = 1,\ \ z_1 = \text{cis}\frac{2\pi}{5},\ \ z_2 = \text{cis}\frac{4\pi}{5},\ \ z_3 = \text{cis}\frac{6\pi}{5},\ \ z_4 = \text{cis}\frac{8\pi}{5}$$



Application activity 1.5.3

In Argand diagram, construct the following polygons

1) A regular hexagon
2) A regular heptagon
3) A regular octagon
4) A regular nonagon

1.6 Exponential form of complex numbers

1.6.1 Definition of exponential form of a complex number

Activity 1.6.1

Given that $e^{i\theta}=\cos\theta+i\sin\theta$, express $U(t)=5e^{i\omega t}$ in an algebraic form of a complex number. The number ω is the angular frequency and t is the time the voltage $U(t)$ appears somewhere in the circuit.

Write this voltage $U(t)$ as a complex number in the polar form and deduce its modulus and argument when $t=90$ seconds.

Content summary

Every complex number z of modulus $|z|$ and argument θ, can be written as $z=|z|(\cos\theta+i\sin\theta)=|z|e^{i\theta}$.

The expression $z=r.e^{i\theta}$ where r and θ are the modulus and the argument of z respectively is called exponential form of the complex number z.

Properties

1. Properties of powers learnt are used for complex numbers expressed in exponential form

   $$e^{in\theta}=(\cos\theta+i\sin\theta)^n=\cos(n\theta)+i\sin(n\theta)$$

   This form is used by engineers who deal with altenating current to simplify calculation such that

   $$U(t)=U_0\,e^{i\omega t}=U_0(\cos\omega t+i\sin\omega t).$$

2. Given two complex numbers $z_1=r_1e^{i\theta_1}$ and $z_2=r_2e^{i\theta_2}$, the following identities are correct

   a) $z_1.z_2=r_1.r_2e^{i(\theta_1+\theta_2)}$

   b) $\dfrac{z_1}{z_2}=\dfrac{r_1}{r_2}.e^{i(\theta_1-\theta_2)}$

   c) $z_1^n=r_1^ne^{in\theta_1}$

Example

Express the complex numbers in exponential form

i) $-\sqrt{3}+i$ ii) -3 iii) $-2i$

Solution

i) Exponential form of a complex number whose modulus r and argument θ is $re^{i\theta}$.

Here $r=\left|-\sqrt{3}+i\right|=2$

Let θ be argument of complex number $-\sqrt{3}+i$;

We have

$$\begin{cases}\cos\theta=-\dfrac{\sqrt{3}}{2}\\ \sin\theta=\dfrac{1}{2}\end{cases} \Rightarrow \theta \text{ lies in } 2^{\text{nd}} \text{ quadrant and } \theta=\pi-\frac{\pi}{6}=\frac{5\pi}{6}.$$

Thus, $-\sqrt{3}+i=2e^{\frac{5\pi i}{6}}$.

ii) $z=-3$ $r=\left|-3\right|=3$

Let θ be argument of complex number -3;

Then,

$$\begin{cases}\cos\theta=-1\\ \sin\theta=0\end{cases} \Rightarrow \theta \text{ lies on negative } \textit{real axis} \text{ thus } \theta=\pi.$$

Thus, $-3=3e^{\pi i}$.

iii) $z=-2i$ $r=\left|-2i\right|=2, \theta=-\dfrac{\pi}{2}$; so $-2i=2e^{-i\frac{\pi}{2}}$.

Application activity 1.6.1

1) Plot the following complex on the Argand diagram and put them on exponential form

a) $1-i$ b) $2i$ c) $\frac{\sqrt{3}}{2}+\frac{1}{2}i$ d) $-\sqrt{3}-i$

2) Express the following numbers in the algebraic form

a) $e^{i\frac{\pi}{3}}$ b) $e^{-i\frac{\pi}{4}}$ c) $3e^{\frac{\pi i}{6}}$ d) $2e^{\frac{2\pi i}{3}}$ e) $2e^{-\pi}$

3) Express the following complex numbers in exponential form:

a) $z=-1+i\sqrt{3}$

b) $z=3+4i$

c) $z=2-2i$

d) $z=-3+i\sqrt{3}$

1.6.2 Euler's formulae

Activity 1.6.2

From De Moivre's theorem, consider expressions of $e^{i\theta}$ and $e^{-i\theta}$ in algebraic form. Discuss how to get the value of $\cos\theta$ and $\sin\theta$ in terms of $e^{i\theta}$ and $e^{-i\theta}$.

Content summary

The following formulae are correct for the argument θ given in radians and called

Euler's formulae:

$$\cos\theta=\frac{1}{2}\left(e^{i\theta}+e^{-i\theta}\right)$$

$$Sin\theta=\frac{1}{2i}\left(e^{i\theta}-e^{-i\theta}\right)$$

The Euler's formulae are used to write the product of trigonometric expressions in form of the sum of trigonometric expressions. This method is called linearization and is most used when integrating trigonometric functions.

Example

Use Euler's formula to show that

a) $2\sin x\cos y=\sin(x+y)+\sin(x-y)$ b) $\sin x\cos^2 x=\frac{1}{4}(\sin 3x+\sin x)$

Solution

Using Euler's formula for $\cos x$ and $\sin x$ we get

a) $2\sin x\cos y=2\left(\frac{e^{ix}-e^{-ix}}{2i}\right)\left(\frac{e^{iy}+e^{-iy}}{2}\right)$

$$=\frac{1}{2i}\left(e^{i(x+y)}+e^{i(x-y)}-e^{i(y-x)}-e^{i(-x-y)}\right)$$

$$=\frac{1}{2i}\left(e^{i(x+y)}-e^{-i(x+y)}+e^{i(x-y)}-e^{-i(x-y)}\right)$$

$$=\frac{1}{2i}\left(e^{i(x+y)}-e^{-i(x+y)}\right)+\frac{1}{2i}\left(e^{i(x-y)}-e^{-i(x-y)}\right)$$

$=\sin(x+y)+\sin(x-y)$ (As requested).

b) $\sin x\cos^2 x=\left(\frac{e^{ix}-e^{-ix}}{2i}\right)\left(\frac{e^{ix}+e^{-ix}}{2}\right)^2$

$$=\left(\frac{e^{ix}-e^{-ix}}{2i}\right)\left(\frac{e^{i2x}+2+e^{-i2x}}{4}\right)$$

$$=\frac{e^{i3x}+e^{-ix}+2e^{ix}-e^{ix}-e^{-i3x}-2e^{-ix}}{8i}$$

$$=\frac{1}{4}\frac{e^{i3x}-e^{-i3x}}{2i}+\frac{e^{ix}-e^{-ix}}{2i}$$

Therefore, $\sin x\cos^2 x=\frac{1}{4}(\sin 3x+\sin x)$

Application activity 1.6.2

Apply Euler's formula to linearize the following:

a) $\sin^2 x \cos x$

b) $\cos^2 x \cos y$

c) $\cos^3 x$

1.7 Application of complex numbers in Physics

1.7.1.Application of algebraic form of a complex number

Activity 1.7.1

Conduct a research from different books of the library or browse internet to discover the application of complex numbers in other subjects such as physics, applied mathematics, engineering, etc

Content summary

Complex numbers are applied in other subjects to express certain variables or to facilitate the calculation in complicated expressions. They are mostly used in physics ,in electrical engineering, electronics engineering, signal analysis, quantum mechanics, relativity, applied mathematics, fluid dynamics, electromagnetism, civil and mechanical engineering.

To avoid the confusion between i representing the current and i for the imaginary number, physicists prefer to use j to represent the imaginary number.

Example

The figure below shows a simple current divider made up of a capacitor and a resistor.

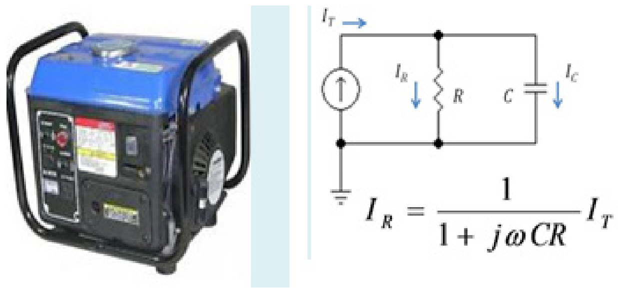


Figure 1. 7 A generator and the R-C current divider

The quantity $z_C = \dfrac{1}{j\omega C}$ is the impedance of the capacitor, where j is the imaginary unit,

Express I_R in terms of z_C.

Solution

Using the formula for I_R on the figure, dividing both the numerator and denominator by $j\omega C$, the current in the resistor is given by

$$I_R = \frac{1}{1 + j\omega CR} I_T \Leftrightarrow I_R = \frac{\dfrac{1}{j\omega C}}{R + \dfrac{1}{j\omega C}} I_T$$

Therefore, $I_R = \dfrac{z_C}{R + z_C} I_T$

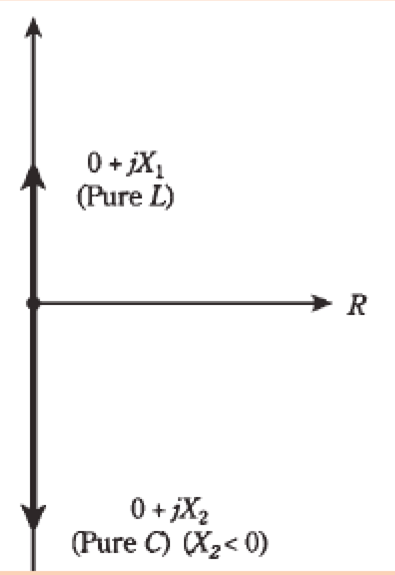
Application activity 1.6.2

In electricity when dealing with direct currents (DC), we encountered Ohm's law, which states that the resistance R is the ratio between voltage V and current I or $R = \dfrac{V}{I}$. With alternating currents (AC) both V and current I are expressed by complex numbers, so the resistance is now also complex. A complex resistance is called impedance and denoted by symbol Z. The building blocks of AC circuits are resistors (R, [Ω]), inductors (coils, L, [H=Henry]) and capacitors (C, [F=Farad]). Their respective impedances are $Z_R = R$, $Z_L = j\omega L$ and $Z_C = \dfrac{1}{j\omega C}$; precise the imaginary part of each of them?

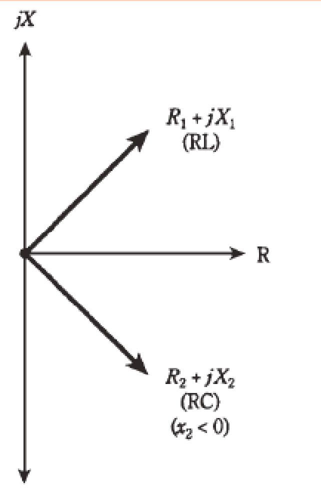
1.7.2. Application of geometric representation of a complex number

Activity 1.7.2

1. The figure below shows impedance vectors for pure inductance and pure capacitance. Represent the net impedance.



2. The figure below shows impedance vectors for resistance and reactance. Represent the net impedance



Find the net impedance or the vector sum(or resultant)

Content summary

In electrical engineering, the treatment of resistors, capacitors, and inductors can be unified by introducing imaginary, frequency-dependent resistances for the later two (capacitor and inductor) and combining all three in a single complex number called the impedance. Get familiar with the RC (Resistor-Capacitor) plane, just as with the RL (Resistor-Inductor) plane.

Each component (resistor, an inductor or a capacitor) has an impedance that can be represented as a vector in the RX plane. The vectors for resistors are constant regardless of the frequency.

Pure inductive reactances (X_L) and capacitive reactances (X_C) simply add together when coils and capacitors are in series. Thus, $X = X_L + X_C$.

In an alternating current series circuit containing a coil and capacitor, there is resistance, as well as reactance.

Whenever the resistance in a series circuit is significant, the impedance vectors no longer point straight up and straight down. Instead, they run off towards the "northeast" (for the inductive part of the circuit) and "southeast" (for the capacitive part).

Given two impedances, $Z_1 = R_1 + jX_1$ and $Z_2 = R_2 + jX_2$, the net impedance Z of these

in series is their vector sum, given by $Z = (R_1 + R_2) + j(X_1 + X_2)$.

In calculating the vector sum using the arithmetic method the resistance and reactance components add separately. The reactances X_1 and X_2 might both be inductive (positive); they might both be capacitive (negative); or one might be inductive and the other capacitive.

When a coil, capacitor, and resistor are connected in series (Figure 1.7), the resistance R can be thought of as all belonging to the coil, when you use the above formulae. (Thinking of it all as belonging to the capacitor will also work.)

Then you have two vectors to sum up, when finding the impedance of a series RLC circuit:

$$Z = (R + jX_L) + (0 + jX_C) = R + j(X_L + X_C)$$

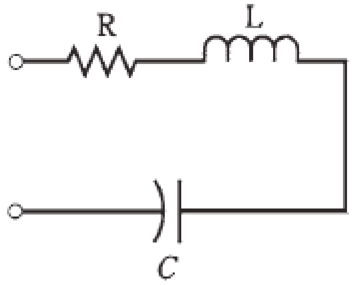


Figure 1. 6 A series RLC circuit

Example

A coil and capacitor are connected in series, with $jX_L = 30j$ and $jX_C = -110j$. What is the net reactance vector? Give comments on your answer.

Solution

Since $X = X_L + X_C$, the net reactance vector is $jX_L + jX_C = 30j - 110j = -80j$.

This is a capacitive reactance, because it is negative imaginary.

Application activity 1.7.2

1) A coil and capacitor are connected in series, with $jX_L = 200j$ and $jX_C = -150j$. What is the net reactance vector? Interpret your answer.

2) A resistor, coil, and capacitor are connected in series with $R = 560\Omega$, $X_L = 400\Omega$ and $X_C = -410\Omega$. What is the net impedance, Z?

1.7.3. Application of polar and exponential form of complex numbers

Activity 1.7.3

In an AC circuit, the voltage V can be represented in the Argand diagram by a vector of magnitude V_0, the vector making a variable angle ω with the real axis.

Express V in polar form and in exponential form.

Content summary

In electrical engineering, the treatment of resistors, capacitors and inductors can be unified by introducing imaginary, frequency-dependent resistances for capacitor, inductors and combining all three (resistors, capacitors, and inductors) in a single complex number called the impedance. This approach is called phasor calculus. As we have seen, the imaginary unit is denoted by j to avoid confusion with i which is generally in use to denote electric current. Since the voltage in an AC circuit is oscillating, it can be represented as

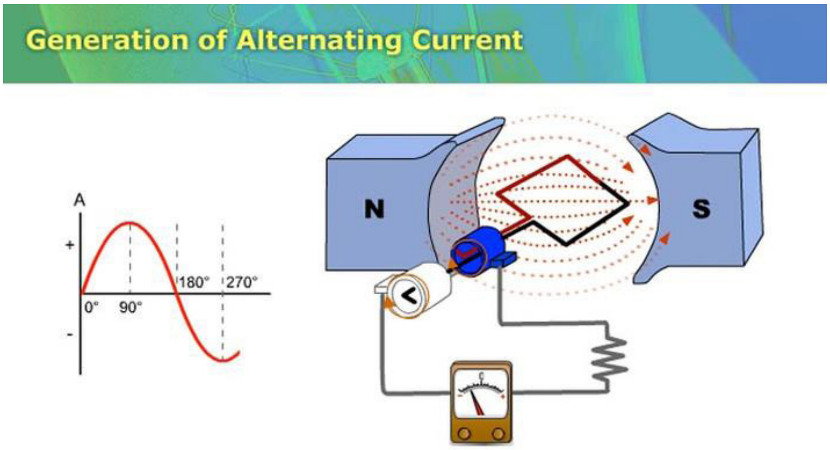
$$
\begin{aligned}
V &= V_0 e^{j\omega t} \\
&= V_0 \left(\cos \omega t + j \sin \omega t\right)
\end{aligned}
$$

Which denotes Impedance, for which V_o is peak value of impedance and $\omega = 2\pi f$ where f is the frequency of supply.

To obtain the measurable quantity, the real part is taken:

$\operatorname{Re}(V) = V_0 \cos wt$ and is called Resistance while $\operatorname{Im}(V) = V_o \sin \omega t$ denotes Reactance (inductive or capacitive).

An alternating current is a current created by rotating a coil of wire through a magnetic field.



If the angular velocity of the wire is ω, respective impedances are $Z_R = R,\ Z_L = j\omega L$ and $Z_C = \dfrac{1}{j\omega C}$; their moduli are the resistance R, the capacitive reactance is $|Z_C| = \dfrac{1}{\omega C}$ and the inductive reactance is given by

$|Z_L| = \omega L$.

and is called Resistance while imaginary part denotes Reactance (inductive or capacitive).

Thus $V_R + jV_L = V$ as $V_R = IR$, $V_L = IX_L$ where X_L is the inductive reactance $2\pi fL$ ohms and

$V = IZ$ (Z being the impedance) from which $R + jX_L = Z$.

Similarly, for the Resistance and Capacitance $(R - C)$ circuit shown in above figure(b), V_C lags I by 90∘ (i.e. I leads V_C by 90∘) and $V_R - jV_C = V$, from which $R - jX_C = Z$ (where X_C is the capacitive reactance $\frac{1}{2\pi fC}$ ohms).

Example1

Using "Ohm's law" for reactive $I = \frac{V_0}{Z}$,where $Z = R + \frac{1}{j\omega C} + j\omega L$,express Z in algebraic form and find the modulus of Z. Denote the argument of Z by Φ .Hence, express Z and I in exponential.

Solution

$$Z = R + \frac{1}{j\omega C} + j\omega L \Leftrightarrow Z = R - \frac{j}{\omega C} + j\omega L$$

$$\Leftrightarrow Z = R + j\left(\omega L - \frac{1}{\omega C}\right)$$: Algebraic form

Converting the impedance to exponential form:

The modulus of Z is $\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}$,

Therefore,

$$Z = R + j\left(\omega L - \frac{1}{\omega C}\right) = \sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}\, e^{j\phi}$$

with $\phi = \arctan\frac{\left(\omega L - \frac{1}{\omega C}\right)}{R} = \arctan\frac{\left(\omega^2 - \omega_0^2\right)}{\gamma\omega}$, $\omega_0 = \frac{1}{\sqrt{LC}}$ and $\gamma = \frac{R}{L}$.

Thus, the current is given by

$$I = \frac{V_0}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}} e^{j\phi} = \frac{V_0 \cdot \frac{\omega}{L}}{\sqrt{\gamma^2\omega^2 + \left(\omega^2 - \omega_0^2\right)^2}} e^{j\phi}$$

Example 2

Determine the resistance and series inductance (or capacitance) for each of the following impedances, assuming a frequency of 50 Hz:

a) $4 + j7\ \Omega$ b) $-j20\ \Omega$ c) $15cis\left(-60^0\right)\ \Omega$

Solution

a) Impedance, $Z = 4 + j7\ \Omega$ hence, Resistance is 4Ω and Reactance $7\ \Omega$.

Since the imaginary part is positive, the reactance is inductive, i.e. $X_L = 7\ \Omega$

Since $X_L = 2\pi fL$ then inductance, $L = \frac{X_L}{2\pi f} = \frac{7}{2\pi \times 50} = 0.0223H$ or $22.3mH$

b) Impedance, $Z = -j20\ \Omega$, i.e. $Z = 0 - j20\ \Omega$ hence Resistance is 0 and Reactance 20Ω.

Since the imaginary part is negative, the reactance is capacitive,

i.e. $X_C = 20\ \Omega$ and since $X_C = \frac{1}{2\pi fC}$ then:

capacitance, $C = \frac{1}{2\pi fX_C} = \frac{1}{2\pi(50)(20)} = \frac{10^6}{2\pi(50)(20)}\mu F = 159.2\mu F$

c) Impedance, $Z = 15cis\left(-60^0\right) = 15\left[\cos\left(-60^0\right) + j\sin\left(-60^0\right)\right] = 7.5 - 7.5j\sqrt{3}$

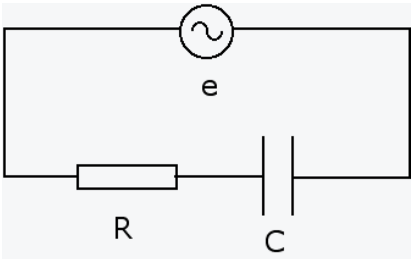
Hence resistance is 7.5Ω and capacitive reactance, $X_C = 7.5\sqrt{3} = 12.99\Omega$

Since $X_C = \frac{1}{2\pi fC}$ then capacitance,

$$C = \frac{1}{2\pi fX_C} = \frac{1}{2\pi(50)(20)} = \frac{10^6}{2\pi(50)(12.99)}\mu F = 245\mu F$$

Example 3

Consider the RC series of the alternating current circuits. The e.m.f (electromotive force) E that is supplied to the circuit is distributed between the resistor R and the capacitor C.



Given that the same current must flow in each element, the resistor and capacitor are in series such that the common current can often be taken to have the reference phase.

If the current is $I = \mathrm{I}_m\, e^{j\omega t}$, find the expression of the applied electromotive force E.

Hint: $V_C = \dfrac{1}{j\omega C} I$

Solution

In a series circuit, the potential differences are added up around the circuit:

$$E = V_R + V_C$$

$$= RI - \frac{j}{\omega C} I = \left(R - \frac{j}{wc} \right) I_m e^{j\omega t}$$

$$= \sqrt{R^2 + \frac{1}{(\omega C)^2}} \left[e^{j \arctan\left(\frac{-1}{\omega CR}\right)} \right] I_m e^{j\omega t}$$

Taking $\theta = \arctan\left(\dfrac{-1}{\omega CR}\right)$ and $|Z| = \sqrt{R^2 + \dfrac{1}{(\omega C)^2}}$, we find

$$E = Z.I_m.e^{j(\omega t - \theta)}$$

When we apply De Moivre theorem, we find

$$E = Z.I_m.e^{j(\omega t - \theta)} = Z.\mathrm{I}_m \left[\cos(\omega t - \theta) + j \sin(\omega t - \theta) \right]$$

This shows that $|Z| = \sqrt{R^2 + \frac{1}{(\omega C)^2}}$ is the modulus of E and $\theta = \arctan\left(\frac{-1}{\omega CR}\right)$ is the phase between the electromotive force E and the current.

Example 4

An alternating voltage of 240V, 50 Hz is connected across an impedance of $Z = 60 - j100\ \ \Omega$ Determine

a) the resistance

b) the capacitance

c) the magnitude of the impedance and its phase angle

d) the current flowing

Solution

a) Impedance $Z = 60 - j100\ \ \Omega$.

Hence resistance is 60Ω

b) Capacitive reactance, $X_C = 100\Omega$; as $X_C = \frac{1}{2\pi fC}$ then capacitance,

$$C = \frac{1}{2\pi fX_C} = \frac{1}{2\pi(50)(100)} = \frac{10^6}{2\pi(50)(100)}\mu F = 31.83\mu F$$

c) Magnitude of impedance,

$$|Z| = |60 - j100\ \ \Omega| = \sqrt{(60)^2 + (-100)^2} = 116.6\Omega$$

Phase angle, $\arg(Z) = \tan^{-1}\left(\frac{-100}{60}\right) = -59.04^0$

d) Current flowing, $I = \frac{V}{Z} = \frac{240cis0^0}{116.6cis(-59.04^0)} = 2.058cis(59.04^0)A$

The voltage in AC circuit is expressed by $U(t) = U_0 e^{i\omega t}$ where ω is the angular frequency which is related to the frequency f by $\omega = 2\pi f$ and t the time the voltage appears somewhere in the circuit. Write this voltage as a complex in the polar form number and deduce its modulus and argument.

Application activity 1.7.3

1) Determine the resistance *R* and series inductance *L* (or capacitance *C*) for each of the following impedances assuming the frequency to be 50 Hz.

(a) $(3+j8)\Omega$ (b) $(2-j3)\Omega$ (c) $j14\Omega$ (d) $8cis\left(-60^0\right)\Omega$

2) Two impedances, $Z_1=(3+j6)\Omega$ and $Z_2=(4-j3)\Omega$ are connected in series to a supply voltage of $120V$. Determine the magnitude of the current and its phase angle relative to the voltage.

3) If the two impedances in Problem 2 are connected in parallel determine the current flowing and its phase relative to the 120V supply voltage.

Hint: For the *n–branch parallel circuit*, Impedance *Z* is given by: $\frac{1}{Z}=\sum_{k=1}^{n}\frac{1}{Z_k}$

4) A $2.0H$ inductor of resistance 80Ω is connected in series with a 420Ω resistor and a $240V$, $50Hz$ supply. Find

   a) The current in the circuit

   b) The phase angle between the applied p.d. and the current

5) For a transmission line, the characteristic impedance Z_0 and the propagation coefficient γ are given by:

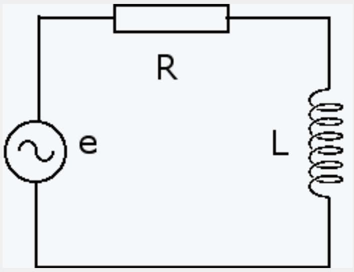
$$Z_0=\sqrt{\frac{R+j\omega L}{G+j\omega C}}$$

and

$$\gamma=\sqrt{(R+j\omega L)(G+j\omega C)}$$

Determine, in polar form, Z_0 and γ, given that $R=25\Omega,\ L=5\times10^{-3}H,$ $G=80\times10^{-6}S,\ C=0.04\times10^{-6}$ and ω=2000ϖ rad/s,

6) Consider the RL series in an alternating current circuits in which the e.m.f. that is supplied to the circuit is distributed between the resistor and the capacitor grouped in series.



If the current $I = I_m.e^{j\omega t}$,

a) Discuss and find the expression of the electromotive force E in the form a complex number.

b) From the answer found in a), deduce the modulus $|E|$ and the expression of the phase θ between the current and the electromotive force E.

d) Given that a100V, 1000/ϖ Hz supply is connected in series with a 30Ω resistor and a 20mH inductor. Take the emf as the reference phase and find:

(1) the complex impedance of the circuit

(2) the complex current, The real (i.e. physical) current which is the imaginary part of the complex current and an equivalent dc current, and

(3) the complex , real (i.e. physical or the imaginary part of the complex) and equivalent dc potential differences across each element.

1.8 End unit assessment

1) Given two complex numbers $z_1 = 6+3i$ and $z_2 = 10+8i$, evaluate the following

a) $z_1 + z_2$ b) $\frac{z_1}{z_2}$ c) $z_1 \cdot z_2$ d) $\left(z_1 - \overline{z_2}\right)\left(z_1 + \overline{z_2}\right)$

2) If $Z = R + j\omega L + \frac{1}{j\omega C}$, express Z in $(a+jb)$ form when

$R = 10, L = 5, C = 0.04$ and $\omega = 4$

3) Given the complex number $z = 3+3i$,

a) Convert z in polar form and in exponential form

b) Evaluate $(3+3i)^5$ and write the answer in algebraic form.

c) Transform the two square roots of $z=3+3i$ into algebraic form

4) Show that multiplication by i rotates a complex number through $\frac{\pi}{2}$ in the anticlockwise direction and division by i rotates it through $\frac{\pi}{2}$ in the clockwise direction.

5) Using Euler's formula, linearize the following:

a) $\sin^2 x\cos x$ b) $\sin x\cos^2 x$ c) $\sin^2 x\cos^2 x$ d) $\sin^3 x$

6) A man travels 12 kms North-East, 20 kms 30° West of North, and then 18 kms 60° South of West. Determine analytically and graphically how far and in what direction he is from his starting point.

UNIT 2

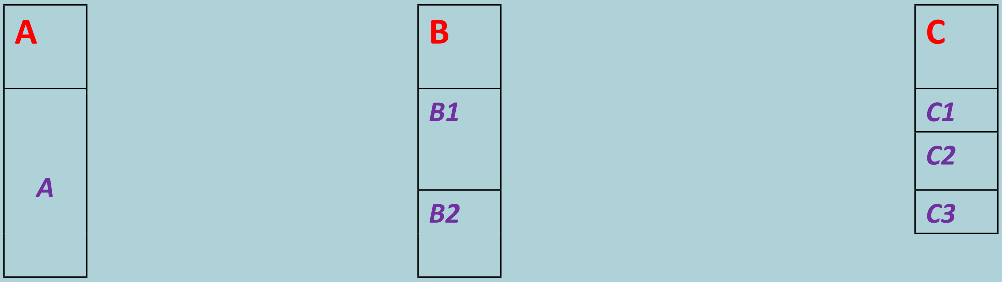
ARRANGEMENT, PERMUTATION AND COMBINATION

Key Unit competence:

Apply formulae of combinatory analysis to count possible outcomes of a random experiment.

2.0 Introductory activity

There are 2 roads joining A and B and 3 roads joining B and C. Write down different roads from A to C via B. How many are they?



2.1 Simple counting techniques

2.1.1 Venn diagram, tree diagrams, contingency table and basic product principle of counting

Activity 2.1.1

In a class of 30 students, 10 got A on the first test, 9 got A on a second test, and 15 did not get an A on either test. Find: the number of students who got:

a) an A on both tests;

b) an A on the first test but not the second;

c) an A on the second test but not the first.

CONTENT SUMMARY

Many problems in real life can be solved by simply counting the number of different ways in which a certain event can occur.

For principle of counting, it is helpful to know and apply the concept of the following terms

Experiment is defined as any human activity.

Trial is small experiment contained in a large experiment.

Outcome is a result of an experiment.

Sample space is the set of all possible outcomes of a given even.

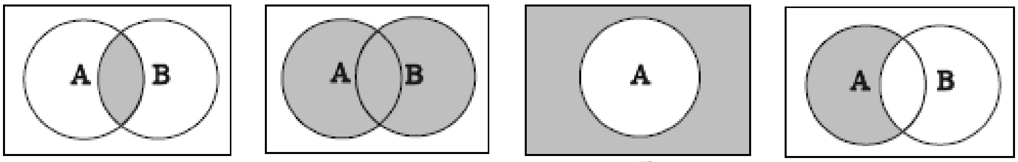
Among different techniques of counting, let us have a recall on Venn diagram, tree diagrams and Contingency table.

1. Use of Venn diagram

As studied in senior two, Venn diagram, Tree diagrams and Contingency table can be used to determine all the possible outcomes of some events.

A **Venn diagram** refers to representing mathematical or logical sets pictorially as circles or closed curves within an enclosing rectangle (the universal set), common elements of the sets represented by intersections of the circles.

In many cases, events can be described in terms of other events through the use of the standard constructions of set theory. We will briefly review definitions of these constructions. The reader is referred to the following figure for Venn diagrams which illustrate these constructions.



$A \cap B$ $A \cup B$ $\bar{A}$ $A-B$

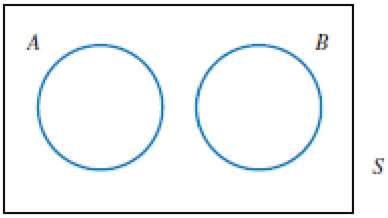
Let A and B be two sets. Then the union of A and B is the set
$A \cup B = \{x / x \in A \, or \, x \in B\}$.

The intersection of a and b is the set $A \cap B = \{x / x \in A \, and \, x \in B\}$

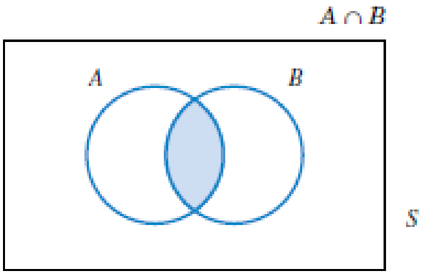
The difference of A and B is the set $A - B = \{x / x \in A \, and \, x \notin B\}$.

The set A is a subset of B, written $A \subset B$, if every element of A is also an element of B.

When $A \cap B = \varnothing$ element, the two events are said to be **mutually exclusive**. This means that they cannot occur at the same time, they do not have outcomes in common.

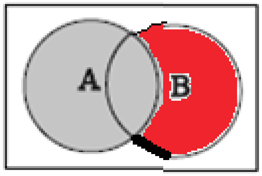


When $A \cap B \neq \varnothing$ element, the two events are not mutually exclusive. This means that they have some outcomes in common.



The complement of an event is A

It is the set of outcomes in the sample space Ω that are not included in the outcomes of event A. The complement of A is denoted A'



Finally, the complement of A is the set $\bar{A} = \{x / x \in \Omega \; and \; x \notin A\}$

Example

Determine which events are mutually exclusive and which are not when a single die is rolled.

a) Getting an odd number and getting an even number.

b) Getting a 3 and getting an odd number.

c) Getting an odd number and getting a number less than4

d) Getting a number greater than 4 and getting a number less than 4.

Solution:

a) Events are mutually exclusive.

b) Events are not mutually exclusive.

c) Events are not mutually exclusive.

d) Events are mutually exclusive

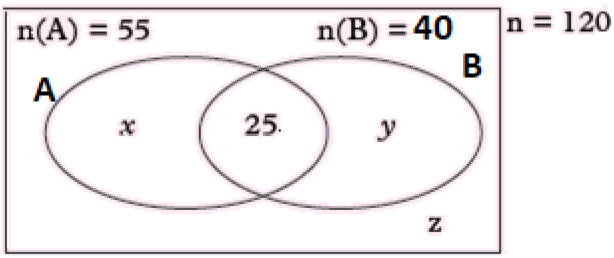
Example

A survey involving 120 people about their preferred breakfast showed that 55 eat eggs for breakfast, 40 drink juice for breakfast, 25 eat eggs and drink juice for breakfast.

Represent the information on a Venn diagram.

Solution:

Let A= set of people who eat eggs, B =set of people who take juice and z represent the number of people who did not take any.



Here, we can now solve for the number of people who didn't take eggs only.

$x = 55 - 25 = 30$

So 30 people took Eggs only

Also, y = 40 – 25 = 15

So, 15 people took Juice only.

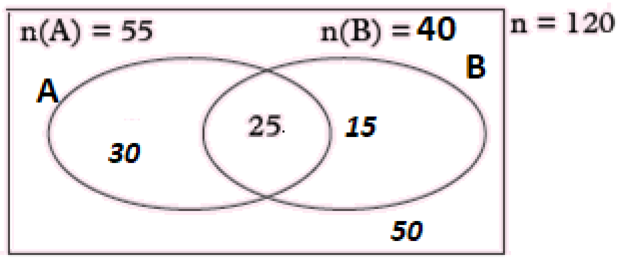
Hence 30 + 25 + 15 + z = 120

Z = 120 – (30+15+25)

Z = 120 – 70

Z = 50

The number of people who did not take anything for breakfast is 50.



2. Use of tree diagrams

A **tree diagram** is simply a way of representing a sequence of events. Tree diagrams are particularly useful in counting since they record all possible outcomes in a clear and uncomplicated manner.

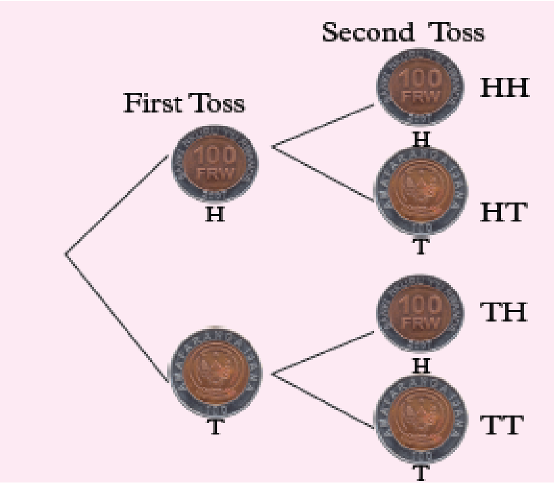
It has branches and sub-branches which help us to see the sequence of events and all the possible outcomes at each stage.

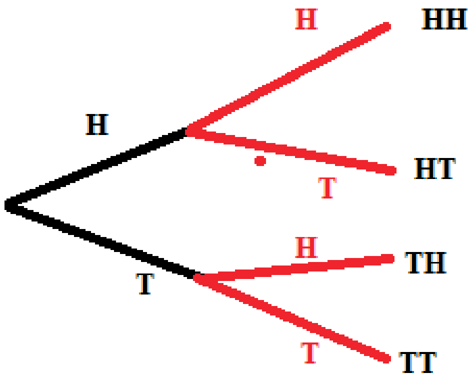
Example

Using a tree diagram, determine all the possible outcomes that can be obtained when a coin is tossed twice.

Solution:

In the first toss, we get either a head (H) or a tail (T). On getting a H in the first toss, we can get a H or T in the second toss. Likewise, after getting a T in the first toss, we can get a H or T in the second toss.





$S = \{HH, HT, TH, TT\}$

3. Use of a table

A **table** is simply a way of representing a sequence of events. It is a rectangular array in which the fist column has elements of the first set while the first low has elements of the second set to be associated with the first.

Example

Find the total number of all possible outcomes while rolling two dice.

Solution:

As each die come land in 6 different ways, and two dice are rolled, the sample space can be presented as a rectangular array.

Die 2 Die 1	1	2	3	4	5	6
1	(1,1)	(1,2)	(1,3)	(1,4)	(1,5)	(1,6)
2	(2,1)	(2,2)	(2,3)	(2,4)	(2,5)	(2,6)
3	(3,1)	(3,2)	(3,3)	(3,4)	(3,5)	(3,6)
4	(4,1)	(4,2)	(4,3)	(4,4)	(4,5)	(4,6)
5	(5,1)	(5,2)	(5,3)	(5,4)	(5,5)	(5,6)
6	(6,1)	(6,2)	(6,3)	(6,4)	(6,5)	(6,6)

Thus the total number of all possible outcomes while rolling two dice is 36.

Application activity 2.1.1

1) In a survey of 100 student-teachers, it was found that 77 student-teachers were studying Mathematics; 47 student-teachers were studying Physics; 44 student-teachers were studying Chemistry; 43 student-teachers were studying both Mathematics and Physics; 37 student-teachers were studying both Mathematics and Chemistry; 12 student-teachers were studying both Physics and Chemistry ; 12 student-teachers were studying all three subjects.

   a) Find the number of student-teachers from among the 100 who were not studying any one of the three sciences.

   b) Find the number of student-teachers from among the 100 who were studying both Physics and Mathematics but not Chemistry.

2) Use a tree diagram to find the gender for 3 children in a family.

3) A coin is tossed and a die is rolled. Find the number of outcomes for the sequence of events.

2.1.2 Basic product principle of counting

Activity 2.1.2

Rwanda requires its residents to register their motor vehicles and display vehicle registration plate. The current series of vehicle registration plate in Rwanda are on a white plate with black lettering whose format is RLLoooL, where "R" denotes Rwanda, "L" denotes a letter from English alphabet and "o" denotes a digit. Following this pattern, how many registrations have they been done from the 1st vehicle registration plate series to the delivering of the plate RAB999Z?

 **Figure 1: First vehicle registration plate of the series RAA** Note that the plate containing only 0 as digit (for example RAA000A) is not registered.	 **Figure 2: Last vehicle registration plate of the series RAB**

Content summary

One of the techniques of counting without necessarily listing the total number of all possible outcomes is to use the product rule namely **Basic product principle of counting:**

"If a sequence of n events in which the first one has n_1 possibilities, the second with n_2 possibilities the third with n_3 possibilities, and so forth until n_k, the total number of possibilities of the sequence will be given by *the product* $n_1 \cdot n_2 \cdot n_3 \cdots n_k$".

Example

A car license plate is to contain three letters of the alphabet, the first of which must be R, S, T or U, followed by three decimal digits. How many different license plates are possible?

Solution

The first letter can be chosen in 4 different ways, the second and third letters in 26 different ways each, and each of the three digits can be chosen in ten ways.

By using basic product principle of counting, we get that there are $4\times26\times26\times10\times10\times10=2\,704\,000$ plates possible.

Example

a) How many numbers of four different digits can be formed?

b) How many of these are even?

Solution

a) There are nine ways to choose the first digit since 0 cannot be the first digit, and nine, eight and seven ways to choose the next three digits since no digit may be repeated.

Therefore there are $9\times9\times8\times7=4536$ numbers possible

b) The last digit must be a $0,2,4,6$ or 8. There are five ways of choosing it. Then the first digit can be chosen in eight different ways since it cannot be a zero or the number chosen for the last digit. The other two digits can be chosen in eight and seven ways respectively.

Therefore the number of even numbers is $8\times8\times7\times5=2240$

Application activity 2.1.2

1. There are 20 teams in the local football competition. In how many ways can the first four places in the premiership table be filled?

2. There are four bus lines between A and B, and three bus lines between B and C. Find the number m of ways that a man can travel by bus:

a) from A to C by way of B;

b) round-trip from A to C by way of B;

c) round-trip from A to C by way of B but without using a bus line more than once.

2.1.3 Basic sum principle of counting (Mutually exclusive situations)

Activity 2.1.5

1) Suppose that you go to a restaurant and you are allowed a soup or juice. Will you pick one, the other or both?

2) How many different four digits numbers, end in a 3 or a 4, can be formed from the figures 3,4,5,6 if each figure is used only once in each number.

Content summary

Two experiments 1 and 2 are mutually exclusive, if when experiment 1 occurs, experiment 2 cannot occur. Likewise, if experiment 2 occurs, experiment 1 cannot occur.

Basic sum principle of counting

In such cases, the number of all outcomes for which either experiment 1 or experiment 2 occur is obtained by adding the number of all outcomes for experiment 1 to the number of outcomes of experiment 2.

This suggests the following result:

"*If Experiment 1 has m possible outcomes, experiment 2 has n possible outcomes, then an experiment which might be experiment 1 or experiment 2, called experiment 1 or 2, has $(m+n)$ possible outcomes.*"

Example

In tossing an object which might be a coin (with two sides H and T) or a die (with six sides 1 through 6), how many possible outcomes will appear?

Solution

The experiment may be tossing a coin (experiment 1) or tossing a die (experiment 2), or just experiment 1 or 2.

So the number of outcomes is $2+6=8$ according to the above basic sum principle of counting.

Notice:

The number of outcomes in which a certain experiment 1 occurs will be clearly mutually exclusive with those permutations in which that experiment does not occur. Thus,

$$\begin{aligned}&\textit{Number of permutations in which experiment 1 does not occur}\\&=\textit{total number of permutation - number of permutations in which experiment 1 occurs}\end{aligned}$$

Generalized sum principle of counting

If experiment 1 ***through*** k ***have respectively*** n_1 ***through*** n_k ***outcomes, then the experiment*** 1 *or* 2 *or* $\dots$ *or* k ***will have*** $n_1+n_2+\cdots+n_k$ ***outcomes.***

Example

How many even numbers containing one or more digits can be formed from the digits 2, 3, 4, 5, 6 if no digit may be repeated?

Solution

Since the required numbers are even, last digit must be 2 or 4 or 6. Note that there are 5 digits.

So we can form a number containing one digit, two digits, three digits, four digits or five digits as follow

From the digits 2, 3, 4, 5, 6,

- the even numbers containing one digit are 2 or 4 or 6. That is 3 numbers;
- for forming even number containing two digits, there are 3 ways of choosing the last digit and 4 ways to choose the first. That is $3\times4=12$ numbers;
- for forming even number containing three digits, there are 3 ways to choose the last digit, 4 ways to choose the first and 3 ways to choose the second. That is $3\times4\times3=36$;

- for forming even number containing four digits, there are 3 ways of choosing the last digit, 4 ways to choose the first, 3 ways to choose the second and 2 ways to choose the fourth. That is $3 \times 4 \times 3 \times 2 = 72$;
- for forming even number containing five digits, there are 3 ways to choose the last, 4 ways to choose the first, 3 ways to choose the second, 2 ways to choose the fourth and 1 way to choose the fifth. That is $3 \times 4 \times 3 \times 2 \times 1 = 72$;

Since these events are mutually exclusive, we can apply basic sum principle of counting;

From which the total number of even numbers that can be formed from the given digits is $3 + 12 + 36 + 72 + 72 = 195$.

Application activity 2.1.3

a) How many numbers containing 4 digits can be formed using only the digits 0, 1, 2, 5, 8 and 9?

b) How many of these numbers are even?

c) How many of these numbers are multiples of 5?

2.2 Permutations

2.2.1 Permutations without repetitions

Activity 2.2.1

Consider three letters R, E and B written in a row, one after another.

Form all possible different words from three letters R, E and B (not necessarily sensible).

In fact, each arrangement is a possible permutation of the letters R, B and E; for example REB, RBE, …

How many arrangements, are they possible for three letters R, E and B?

Content summary

From different arrangement of three letters ***R, E and B***, the first letter to be written down can be chosen in 3 ways. The second letter can then be chosen in 2 ways because there are 2 remaining letters to be written down and the third letter can be chosen in 1 way because it is only one letter remain to be written down. Thus, the three operations can be performed in $3\times2\times1=6$ ways.

This arrangement of letters is the same as sitting different people on the same bench. A **permutation** is an arrangement of n objects in a specific order.

Example

Give all different ways three students: Aloys, Emmanuel and Mary can be sit on the same bench. Two ways were given in the table, complete others.

Aloys	Emmanuel	Mary
Aloys	Mary	Emmanuel
..	..	..

This suggests the following fact:

The number of different permutations of *n* different objects (unlike objects) in a row is

$$n\times(n-1)\times(n-2)\times(n-3)\times...\times2\times1$$

A useful short hand of writing this operation is $n!$ (read n factorial). Then,
$$n!=n\times(n-1)\times(n-2)\times(n-3)\times...\times2\times1$$

Thus, $1!=1$, $2!=2\times1=2$, $3!=3\times2\times1=6$, $4!=4\times3\times2\times1=24$,

$5!=5\times4\times3\times2\times1=120$ and so on.

Note that $0!=1$.

Example

Five children have to be seated on a bench. In how many ways they can be seated? How many arrangements are they, if the youngest child has to sit at the left end of the bench?

Solution

Since there are five children, the first child can be chosen in 5 ways, the next child in 4 ways, the next in 3 ways, the next in 2 ways and the last in 1 way. Then, the number of ways is $5!=5\times4\times3\times2\times1=120$.

Now, if the youngest child has to sit at the left end of the bench, this place can be filled in only 1 way. The next child can then be chosen in 4 ways, the next in 3 ways and so on. Thus, the number of total arrangement is $1\times4!=1\times4\times3\times2\times1=24$.

Example

Three different mathematics books and five other books are to be arranged on a bookshelf. Find:

a) The number of possible arrangements of the books.

b) The number of possible arrangements if the three mathematics books must be kept together?

Solution:

We have 8 books altogether.

a) Since we have 8 books altogether, the first book can be chosen in 8 ways, the next in 7 ways, the next in 6 ways and so on. Thus, the total arrangement is $8!=40320$

b) Since the 3 mathematics books have to be together, consider these bound together as one book. There are now 6 books to be arranged and these can be performed in $6!=720$.

Note that we have taken the three mathematics book as one book; these three books can be arranged in $3!=6$ ways. Thus, the total number of arrangements is $720\times6=4320$.

Activity 2.2.1

a) Using the Cartesian plane, plot the point $A(1,2)$ and $B(-2,4)$; deduce the coordinate of the vector $\overrightarrow{OA}+\overrightarrow{OB}$.

b) Basing on the answer found in a), deduce the affix of the complex number z_1+z_2 if $z_1=1+2i$ and $z_2=-2+4i$.

c) Check your answer using algebraic method/technique.

d) Express your answer in words.

1. Simplify

a. $\frac{5!}{2!}$ b. $\frac{10!}{6!7!}$

2. Four different English books, five different Biology books and ten other books are to be arranged on a bookshelf. Find

   a. The number of possible arrangements of the books.

   b. The number of possible arrangements if the three Biology books must be kept together?

2.2.2 Permutations with repetitions

Activity 2.2.2

1) Make a list of all arrangements formed by 4 numbers: 1,2,3,4. How many arrangements are they possible?

2) Consider the arrangements of four letters in the word "MOON".
   - Write down all possible different arrangements.
   - How many arrangements are they possible of four letters in the word "MOON"?

Content summary

Consider the arrangements of six letters in the word "**AVATAR**" (a title used for the movie).

We see that there are three **A**'s (or 3 alike letters).

- Let the three **A**'s in the word be distinguished as A_1A_1, A_2 A_2 and A_3 A_3 respectively. Then all the six letters are different, so the number of permutations of them (called labeled permutations) is $n! = 6!$.
- However, consider each of the real permutations without distinguishing the three **A**'s, for example **W=RATAVA**.
- The following are all of the 6 (=3!) labeled permutations among the 6! ones, which come from permuting the three labeled **A**'s in **W=RATAVA**:

$$\mathrm{RA_1TA_2VA_3, RA_1TA_3VA_2, RA_2TA_1VA_3, RA_2TA_3VA_1, RA_3TA_1VA_2, RA_3TA_2VA_1}$$

- All these six labeled permutations should be considered as an identical real permutation, which is **W=RATAVA**.
- Since each real permutation has six of such labeled permutations

coming from the three **A**'s, we conclude that the desired number of real permutations is just $\frac{6!}{3!} = \frac{6 \times 5 \times 4 \times 3!}{3!} = 6 \times 5 \times 4 = 120$

This suggests the following fact:

The number of different permutations of n ***indistinguishable*** *objects with* n_1 *alike,* n_2 *alike, …, is* $\frac{n!}{n_1!n_2!\ldots}$.

Note that **alike** means that the objects in a group are indistinguishable from one another.

Example

How many distinguishable six digit numbers can be formed from the digits 5, 4, 8, 5, 5, 4?

Solution

There are 6 letters in total with three 5's and two 4's. Then the required numbers are $\frac{6!}{3!2!} = \frac{720}{12} = 60$

Example

How many arrangements can be made from the letters of the word **TERRITORY**?

Solution

There are 9 letters in total with three **R**'s and two **T**'s.

Thus, we have $\frac{9!}{3! \times 2!} = \frac{9 \times 8 \times 7 \times 6 \times 5 \times 4}{2 \times 1} = \frac{60,480}{2} = 30,240\, arrangements$.

Example

In how many different ways can 4 identical red balls, 3 identical green balls and a yellow ball be arranged in a row?

Solution

There are 8 balls in total with 4 red, 3 green and one yellow.

Thus, we have $\frac{8!}{4! \times 3!} = \frac{8 \times 7 \times 6 \times 5}{3 \times 2 \times 1} = 280\, ways$

Application activity 2.2.2

1. How many different arrangements can be made from the letters of the word

   a) ENGLISH b) MATHEMATICS c) SOCIOLOGICAL

2. How many arrangements can be made from the letters of English alphabet?

3. In how many ways can 4 red, 3 yellow and 2 green discs be arranged in a row, if discs of the same colour are indistinguishable?

2.2.3 Circular arrangements

Activity 2.2.3

Take 5 different note books

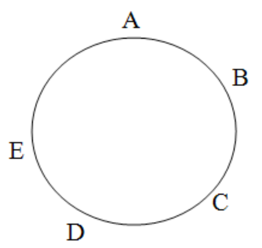
- Put them on a circular table
- Fix one note book, for example A;
- Try to interchange other 4 note books as possible
- How many different way obtained?

Remember that there is one note book that will not change its place.

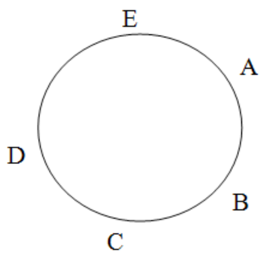
Content summary

We have seen that if we wish to arrange *n* different things in a row, we have $n!$ possible arrangements. Suppose that we wish to arrange *n* things around a circular table. The number of possible arrangements will no longer be $n!$ because there is now no distinction between certain arrangements that were distinct when written in a row.

For example ABCDE is different arrangement from EABCD, but



is not a different arrangement from



With circular arrangement of this type, it is the relative positions of the items being arranged which is important. One item can therefore be fixed and the remaining items arranged around it.

The number of arrangements of n unlike things in a circle will therefore be $(n-1)!$*. In those cases where clockwise and anticlockwise arrangements are not considered to be different, this reduces to* $\frac{1}{2}(n-1)!$*.*

Example

Four men Peter, Rogers, Smith and Thomas are to be seated at a circular table. In how many ways can this be done?

Solution

Suppose Peter is seated at some particular place. The seats on his left can be filled in 3 ways, the next seat can then be filled in 2 ways and the remaining seat in 1 way.

Thus, total number of arrangements is $3!=6$.

Example

In how many ways can five people Betty, James, Elisabeth, Rachel and John, be arranged around a circular table in each of following cases:

a) Betty must sit next Rachel?

b) Betty must not sit next Rachel?

Solution

There are five people.

a) Since Betty and Rachel must sit next to each other, consider these two bound together as one person. There are now, 4 people to seat. Fix on of these, and then the remaining 3 people can be seated in $3\times2\times1=6$ ways relative to the one who was fixed.

In each of these arrangements Betty and Rachel are seated together in a particular way. Betty and Rachel could now change the seats giving another 6 ways of arranging the five people. Total number of arrangements is $6\times2=12$.

b) If Betty must not sit next Rachel, then this situation is a mutually exclusive with the situation in a).

Total number of arrangements of 5 people at a circular table is $(5-1)! = 4! = 24$.

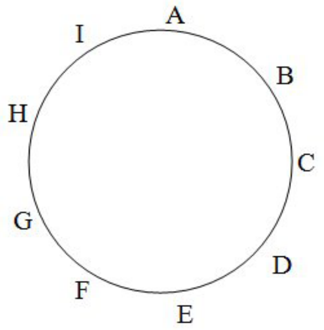
Thus, if Betty must not sit next Rachel, the number of arrangements is $24 - 12 = 12$

Example

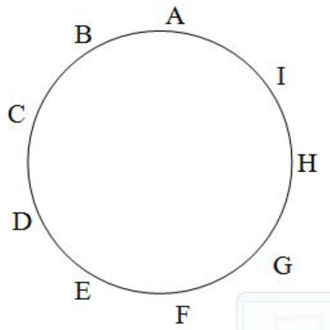
Nine beads, all of different colors are to be arranged on a circular wire. Two arrangements are not considered to be different if they appear the same when the ring is turned over. How many different possible arrangements are there?

Solution

When the ring is turned over, the arrangements



would appear as



When viewed from one side, these arrangements are only different in that one is a clockwise arrangement and the other is anticlockwise. If one bead is fixed, there are $(9-1)! = 8!$ ways of arranging the remaining beads relative to the fixed one.

But, half of these arrangements will appear the same as the other half when the ring is turned over, because for every clockwise arrangement there is a similar anticlockwise arrangement. Hence, number of arrangements is $\frac{1}{2}8! = 20160$.

Application activity 2.2.3

1. Eric, Loyce, John, Jane and Thomas are to be seated at a circular table. In how many ways can this be done?
2. Eleven different books are placed on a circular table. In how many ways can this be done?
3. Find the number of ways that 7 people can arrange themselves:

a) In a row of chairs;

b) Around a circular table.

2.3 Arrangements

2.3.1 Permutations of r unlike objects selected from n distinct objects

Activity 2.3

Make a selection of any three letters from the word "*PRODUCT*" and fill them in 3 empty spaces

Use a box like this for empty spaces		

Write down all different possible permutations of 3 letters selected from the letters of the word "*PRODUCT*". How many are they?

Content summary

Consider the number of ways of placing 3 of the letters A, B, C, D, E, F, G in 3 empty spaces.

The first space can be filled in 7 ways, the second in 6 ways and the third in 5 ways. Therefore there are $7\times6\times5$ ways of arranging 3 letters taken from 7 letters. This is the number of permutations of 3 objects taken from 7 and it is written ${}^7P_3$.

So ${}^7P_3 = 7\times6\times5 = 210$.

Note that the order in which the letters are arranged is important: ABC is a different permutation from ACB.

Now, $7\times6\times5$ could be written $\dfrac{7\times6\times5\times4\times3\times2\times1}{4\times3\times2\times1}$

i.e ${}^7P_3 = \dfrac{7\times6\times5\times4\times3\times2\times1}{4\times3\times2\times1} = \dfrac{7!}{4!} = \dfrac{7!}{(7-3)!}$

This suggests the following fact:

The number of different permutations (ways) of r unlike objects selected from n different objects is $^{n}P_{r}=\dfrac{n!}{(n-r)!}$ *or we can use the denotation* $P_{r}^{n}=\dfrac{n!}{(n-r)!}$

or $P(n,r)=\dfrac{n!}{(n-r)!}$

Note that if $r=n$, we have $^{n}P_{n}=\dfrac{n!}{(n-n)!}=\dfrac{n!}{0!}=n!$ which is the ways of arranging *n* unlike objects.

Example

How many permutations are there of 3 letters chosen from eight letters of the word **RELATION?**

Solution

We see that all those eight letters are distinguishable (unlike). So the required arrangements are given by

$$^{8}P_{3}=\frac{8!}{(8-3)!}=\frac{8!}{5!}=336$$

Example

How many permutations are there of 2 letters chosen from letters A, B, C, D, E?

Solution

There are 5 letters which are distinguishable (unlike). So the required arrangements are given by $^{5}P_{2}=\dfrac{5!}{(5-2)!}=\dfrac{5!}{3!}=20$.

Example

How many different ways can a chairperson and an assistant chairperson be selected for a research project if there are seven scientists available?

Solution:

There are $_{7}P_{2}$ different ways for which a chairperson and an assistant chairperson can be selected for a research project from seven available scientists or $_{7}P_{2}=\dfrac{7!}{(7-2)!}=42$ different ways.

Application activity 2.3.1

1. Simplify the following factorials

a) $\frac{(n-1)!}{n!}$ b) $\frac{(n+2)!}{n!}$ c) $\frac{(n+2)!-n!}{(n-1)!}$

2. Determine the value of n if ${}^{n}P_{2}=72$

3. How many permutations are there of 4 letters chosen from letters of the word **ANGELIUS**?

4. How many permutations are there of 10 letters chosen from English alphabet.

5. Find the number of arrangements of four different letters chosen from the word **PROBLEM** which a) begin with vowel; b) end with consonant.

2.3.2 Permutations of r objects selected from the mixture of n alike and unlike objects

Activity 2.3.2

Make a selection of any three letters from the word "***BOOM***" and fill them in 3 empty spaces

Use a box like this for empty spaces		

Write down all different possible permutations of 3 letters selected from the letters of the word "**BOOM**". How many are they? Comment on your findings.

Content summary

In determining the number of all possible permutations of r objects selected from mixture of n alike and unlike objects, instead of listing and then counting them, you can determine all possible mutually exclusive events that may occur from the given experiment and then apply basic sum principle of counting.

Example

How many different arrangements are there of 3 letters chosen from the word COMBINATION?

Solution

There are 11 letters including two O's, two I's and two N's. To find the total number of different arrangements we consider the possible arrangements as four mutually exclusive situations.

- Arrangements in which all 3 letters are different: there are ${}^{8}P_{3} = 336$
- Arrangements containing two O's and one other letter: the other letter can be one of seven letters (C, M, B, I, N, A or T) and can appear in any of the three positions (before the two O'S, between the two O's, or after the two O's). i.e $3 \times 7 = 21$ arrangements containing two O's and one other letter.
- Arrangements containing two I's and one other letter: by the same reasoning in b) there will be $3 \times 7 = 21$ arrangements containing two I's and one other letter.
- Arrangements containing two N's and one other letter: by the same reasoning in b) there will be $3 \times 7 = 21$ arrangements containing two N's and one other letter.

Thus the total number of arrangements of 3 letters chosen from the word COMBINATION will be $336 + 21 + 21 + 21 = 399$.

Application activity 2.3.2

1. How many permutations are there of 2 letters chosen from letters of the word **RWANDAN**?
2. How many different permutations can be formed from 4 letters chosen from letters of the word **EMMANUEL**?

2.4 Combinations

2.4.1 Combination of distinguishable objects

Activity 2.4.1

Let us assume that you have a part-time job in the weekday evenings where you have to be at work just two evenings out of the five. Let us also assume that your employer is very flexible and allows you to choose which evenings you work provided you ring him up on Sunday and tell him. One possible selection could be $(Monday,\ Tuesday)$; another selection could be $(Thursday,\ Friday)$

a) List all possible arrangements of two working evenings among the five days. How many are they?

b) If your employer asked you to work 3 evenings out of the 5, how many arrangements of three days are there?

Content summary

From permutation of r unlike objects selected from n different objects, we have seen that the order in which those objects are placed is important. But when considering the number of combinations of r unlike objects selected from n different objects, the order in which they are placed is not important.

For example, from activity 2.9, if, on the Sunday, you made your first choice as Tuesday and your second choice as Thursday, this would be the same arrangement as making your first choice as Thursday and your second choice as Tuesday.

So every arrangement or selection is duplicate.

Generally, if you have r identical items to be located in n different places where $n \geq r$,

The first item can be located in any one of n places;

The second item can be located in any one of the remaining $n-1$ places;

The third item can be located in any one of the remaining $n-2$ places;

$\vdots$

The r^{th} item can be located in any one of the remaining $n-(r-1)$ places.

This means that there are $\frac{n!}{(n-r)!}$ arrangements. However, any one arrangement can be rearranged within itself $r!$ times. Thus the total number of selections is given as $\frac{n!}{(n-r)!r!}$.

This suggests the following fact:

The number of different selections (combinations) of r items that could be formed from a set of n distinct objects with the order of selections being ignored is

$$^{n}C_{r}=\frac{n!}{(n-r)!r!} \quad \textit{where } 0\leq r\leq n$$

.

We can write $^{n}C_{r}=\frac{^{n}P_{r}}{r!}$

$^{n}C_{r}$ is sometimes denoted by C_{n}^{r} or $_{n}C_{r}$ or $\binom{n}{r}$ or $C(n,r)$.

Example

How many different selections are there of 9 identical umbrellas on a rack of 16 coat hooks?

Solution

There are different arrangements of 9 identical umbrellas on a rack of 16 coat hooks, where $^{16}C_{9}=\frac{16!}{(16-9)!9!}=\frac{16!}{7!9!}=22880$

Example

From a group of 5 men and 7 women, how many different committees consisting of 2 men and 3 women can be formed?

Solution

From this experiment, we have two trials: select 2 men from 5 and select 3 women from 7.

Number of all possible outcomes from first trial is $^{5}C_{2}=\frac{5!}{2!3!}=\frac{5\times4\times3!}{2\times3!}=10$;

Number of all possible outcomes from second trial is $^{7}C_{3}=\frac{7!}{3!4!}=\frac{7\times6\times5\times4!}{6\times4!}=35$

From the basic product principle of counting, we find that the number of all possible outcomes from first trial and the second one is ${}^{5}C_{2} \times {}^{7}C_{3} = 10 \times 35 = 350$

That is, the desired number of possible outcomes of the experiment of forming a committee is 350.

Example

A committee of three men and one woman is obtained from five men and three women. In how many ways can the members be chosen?

Solution

Three men can be selected from five men, i.e ${}^{5}C_{3} = \dfrac{5!}{(5-3)!3!} = \dfrac{5!}{2!3!}$ ways

One woman can be selected from three women, i.e ${}^{3}C_{1} = \dfrac{3!}{(3-1)!1!} = \dfrac{3!}{2!1!}$ ways

By the basic product principle of counting, there are

${}^{5}C_{3} \times {}^{3}C_{1} = \dfrac{5!}{2!3!} \times \dfrac{3!}{2!1!} = \dfrac{5!}{2!2!} = 30$ ways of selecting the committee.

Example

A woman has 11 close friends. Find the number of ways she can invite 5 of them to dinner where:

a) There are no restrictions;

b) Two of the friends are married to each other and will not attend separately;

c) Two of the friends are not speaking with each other and will not attend together.

Solution

a) That woman has a selection of 5 invitees from her 11 close friends. As there is no restriction, she has ${}^{11}C_{5} = \dfrac{11!}{(11-5)!5!} = 462$ different ways of inviting them to dinner.

b) Since two of the friends are married to each other and will not attend separately, thus there is a restriction of inviting; either this couple is invited or not invited which are mutually exclusive events.

If this couple is invited, the woman has ${}^{9}C_{3} = \dfrac{9!}{(9-3)!3!} = 84$ selections of choosing the three remaining invitees; while the couple is not invited, she has to select 5 invitees from her 9 close friends, that is $^9C_5 = \frac{9!}{(9-5)!5!} = 126$ selections.

Using basic sum principle, we get that the number of ways she can invite 5 of them to dinner if two of her friends are married to each other and will not attend separately is $84+126=210$.

c) If two of her friends will not attend together, the woman invites either only one from these or no one.

If one of the two has been invited, the woman has to select the 4 remaining invitees from 9 to avoid that the second one attends and then she has $2\times {}^9C_4 = 252$ different selections.

If she has not invited none of them, that woman has to select 5 invitees from the 9 remaining that is she has $^9C_5 = 126$ different selections.

From basic sum principle, we get that the number of ways she can invite 5 of her close friends to dinner if two of them will not attend together is $2\times {}^9C_4 + {}^9C_5 = 378$.

Notice:

Some properties of combinations

a) $^nC_n = {}^nC_0 = 1$ and $^nC_1 = n$

b) $^nC_{n-r} = {}^nC_r$

c) $^nC_r + {}^nC_{r+1} = {}^{n+1}C_{r+1}$ or $^nC_r + {}^nC_{r-1} = {}^{n+1}C_r$

Application activity 2.4.1

1. Prove the following theorem known as Pascal's identity: $^{n+1}C_r = {}^nC_r + {}^nC_{r-1}$
2. A committee of four men and two women is obtained from 10 men and 12 women. In how many ways can the members be chosen?
3. A group containing 4 Mathematics books and 5 Physics books is formed from 9 Mathematics books and 10 Physics books. How many groups can be formed?
4. A box contains 8 blue socks and 6 red socks. Find the number of ways two socks can be drawn from the box if:

a) They can be any colour.

b) They must be the same colour.

5. A student-teacher is to answer 10 out of 13 questions. Find the number of his/her choices where he/she must answer:

   a) the first two questions;

   b) the firsst or second question but not both;

   c) exactly 3 out of the first 5 questions;

   d) at least 3 of the first 5 questions

2.4.2 Combination of r objects taken from the mixture of n alike and unlike objects

Activity 2.4.2

Write down all different possible combinations of 3 letters selected from the letters of the word "**BANANA**". How many are they?

Use a box like this for empty spaces		

How many of these selections contain no vowels?

Content summary

In determining the number of all possible combinations of r objects selected from mixture of n alike and unlike objects you consider all possible mutually exclusive events corresponding to the given experiment and then apply basic sum principle of counting.

Example

Find the number of different selections of 2 letters that can be made from the letters of the word **STATISTICS**

Solution

There are 10 letters including three S's, three T's, two I's and two other distinct letters. To find the total number of different selections we consider the all possible mutually exclusive situations and then apply basic sum principle for counting.

- Number of selections in which all 2 letters are different is ${}^{5}C_{2}=\frac{5!}{3!2!}=10$ (from 5 letters S,T, A, I or C);
- Number of selections containing two S's is 1
- Number of selections containing two T's is 1
- Number of selections containing two I's is 1

Thus the total number of selections of 2 letters chosen from the word **STATISTICS** is $10+1+1+1=13.$

Application activity 2.4.2

1. a) Find the number of different selections of 3 letters that can be made from the letters of the word **SUCCESSFUL**?

   b) How many of these selections contain no vowel?

   c) How many of these selections contain at least one vowel?

2. a) Find the number of different selections of 3 letters that can be made from the letters of the word **STATISTICS**?

   b) How many of these selections contain at least one T?

2.4.3 Binomial expansion and Pascal's triangles

Activity 2.4.3

Expand the expressions

$(a+b)^2$

Since $)^{3}=(a+b)^{2}(a+b)\ and\ (a+b)^{4}=(a+b)^{3}($

Expand $(a+b)^3$ and $(a+b)^4$

Once more find the expansion of $(a+b)^5$.

Complete the following table

Power	Coefficient of powers of a and b					Binomial expression
0						$(a+b)^0$
1						$(a+b)^1$
2						$(a+b)^2$
3						$(a+b)^3$
4						$(a+b)^4$

Try and generalize the form of coefficients for each term of a binomial expansion using combinations learnt in the previous lessons.

Content summary

Pascal's Triangle is a triangular array of the binomial coefficients. The rows of Pascal's Triangle are conventionally enumerated starting with row $n=0$ at the top. The entries in each row are numbered from the left beginning with $r=0$ and are usually staggered relative to the numbers in the adjacent rows.

The elements of Pascal's Triangle are the number of combinations of *r* objects chosen from *n* unlike objects. That is ${}^{n}C_{r}$. This triangle is constructed by the **Pascal's identity:**

$${}^{n+1}C_r = {}^{n}C_r + {}^{n}C_{r-1} \quad \text{or} \quad {}^{n}C_r = {}^{n-1}C_{r-1} + {}^{n-1}C_r \quad \text{or} \quad {}^{n+1}C_{r+1} = {}^{n}C_r + {}^{n}C_{r+1}$$

$$\begin{array}{ccc} & \longrightarrow & \\ \boxed{x} & + & \boxed{y} \\ & & \| \quad \downarrow \\ & & \boxed{z} \end{array}$$

Here, $z = {}^{n+1}C_r$,

$y = {}^{n}C_r$ and

$x = {}^{n}C_{r-1}$

r / n	0	1	2	3	4	5	...
0	${}^0C_0 = 1$						
1	${}^1C_0 = 1$	${}^1C_1 = 1$					
2	${}^2C_0 = 1$	${}^2C_1 = 2$	${}^2C_2 = 1$				
3	${}^3C_0 = 1$	${}^3C_1 = 3$	${}^3C_2 = 3$	${}^3C_3 = 1$			
4	${}^4C_0 = 1$	${}^4C_1 = 4$	${}^4C_2 = 6$	${}^4C_3 = 4$	${}^4C_4 = 1$		
5	${}^5C_0 = 1$	${}^5C_1 = 5$	${}^5C_2 = 10$	${}^5C_3 = 10$	${}^5C_4 = 5$	${}^5C_5 = 1$	
$\vdots$							

A simple construction of this triangle proceeds in the following manner:

- On row 0, write only the number 1.
- Then, to construct the elements of following rows, add the number above and to the left with the number above to the right to find the new value.
- If either the number to the right or left is not present, substitute a zero in its place.

For example, the first number in the first row is $0+1=1$, whereas the numbers 1 and 3 in the third row are added to produce the number 4 in the fourth row.

An element of Pascal's Triangle, ${}^nC_r$, is the coefficients of any term in the expansion of $(a+b)^n$ where *r* is the exponent of either *a* or *b*

Consider the product $(a+b)^n = (a+b)(a+b)(a+b)...(a+b)$. If this product is multiplied out, each term of the answer will be of the form $c_1c_2c_3...c_k...c_n$ where, for all k, c_k is either *a* or *b*.

Thus, if $c_k = a$ for all *k* we obtain the term a^k. If $c_k = b$ for one of the terms and $c_k = a$ for the rest, we obtain terms such as $b\times a\times a\times...\times a\times a$, $a\times b\times a\times...\times a\times a$, ..., $a\times a\times a\times...\times b\times a$, $b\times a\times a\times...\times a\times b$, and their sum is $na^{n-1}b$.

If $c_k = b$ for *r* of the terms and $c_k = a$ for the rest we obtain a number of terms of the form $a^{n-r}b^r$.

The number of such terms is the number of ways in which r of the form

$c_1c_2c_3...c_n$ can be selected as equal to b. This number is $\binom{n}{r}$, denoted also as $\binom{n}{r} = {}^nC_r = \frac{n!}{(n-r)!r!}$.

Thus, ${}^nC_r = \frac{n!}{(n-r)!r!}$ is the coefficient of $a^{n-r}b^r$ in the expansion of $(a+b)^n$.

This suggests the following theorem known as **Binomial theorem**:

For every integer $n \geq 1$, $(a+b)^n = \sum_{r=0}^{n} {}^nC_r a^{n-r}b^r$

The following properties of the expansion of $(a+b)^n$ should be observed:

a) There are $n+1$ terms.

b) The sum of the exponents of a and b in each term is *n*.

c) The exponents of a decrease term by term from *n* and 0; the exponent of b increase term by term from 0 to *n*.

d) The coefficient of any term is ${}^nC_r$ where *r* is the exponent of either a or b

e) The coefficients of terms equidistant from the end are equal.

Example

By using binomial theorem, find out

a) $(a+b)^2$ b) $(a-b)^3$

Solution

a) $(a+b)^2 = \sum_{r=0}^{2} {}^2C_r a^{2-r}b^r = {}^2C_0 a^2 b^0 + {}^2C_1 a^{2-1}b^1 + {}^2C_2 a^{2-2}b^2 = a^2 + 2ab + b^2$

b)

$(a-b)^3 = (a+(-b))^2 = \sum_{r=0}^{3} {}^3C_r a^{3-r}(-b)^r$

$= {}^3C_0 a^3(-b)^0 + {}^3C_1 a^{3-1}(-b)^1 + {}^3C_2 a^{3-2}(-b)^2 - {}^3C_3 a^{3-3}(-b)^3 = a^3 - 3a^2b + 3ab^2 - b^3$

Example

Find the coefficient of x^3 in the expansion of $(2x-1)^5$.

Solution

The term in x^r is ${}^5C_r(2x)^{5-r}(-1)^r = {}^5C_r 2^{5-r}x^{5-r}(-1)^r$ and so the term in x^3 has $r=2$.

The coefficient of this term is ${}^5C_3 2^3(-1)^2 = 80$.

Example

Find the coefficient of x^3 in the expansion of $\left(x^2-\frac{1}{x}\right)^6$

Solution

The term in x^r is will be given by ${}^6C_r\left(x^2\right)^{6-r}\left(-\frac{1}{x}\right)^r$ which can be written as ${}^6C_r x^{12-2r}\frac{(-1)^r}{x^r} = {}^6C_r x^{12-3r}(-1)^r$ and so the term in x^3 has $r=3$.

The coefficient of this term is ${}^6C_3(-1)^3 = -20$.

Application activity 2.4.3

1) Given the following n^{th} row of Pascal's triangle, find:

a) the ninth row; b) the tenth row

1	7	21	35	35	21	7	1

Find the coefficient of

a) x^2 in the expansion of $(4x+1)^6$

b) x^3 in the expansion of $\left(x+\frac{1}{x}\right)^4$

c) x^6 in the expansion of $(9x-3)^{10}$

2) Expand a) $(x+4)^7$ b) $(2x-3)^3$

2.5 End unit assessment

1. If there are three different roads joining town A to town B and four different roads joining town B to town C, in how many different ways can I travel from A to C via B and return if

   a) there are no restrictions,

   b) I am not able to return on any road I used on the outwards journey?

2. A survey of 80 car owners shows that 24 own a foreign-made car and 60 own a made in Rwanda car. Find the number of them who own: (a) both a foreign made car and a made in Rwanda car; (b) only a foreign made car; (c) only a made in Rwanda car

3. Suppose a code consists of five characters, two letters followed by three digits. Find the number of:

   a) codes; b) codes with distinct letter;

   c) codes with the same letters.

4. A debating team consists of 3 boys and 3 girls. Find the number of ways they can sit in a row where:

   a) there are no restrictions;

   b) the boys and girls are each to sit together;

   c) just the girls are to sit together.

5. A class has 8 male students and 6 female students. Find the number n of ways that the class can elect:

   a) 1 class representative;

   b) 2 class representatives, 1 male and 1 female;

   c) 1 president and 1 vice president.

6. A restaurant has 6 different desserts. Find the number of ways a customer can choose:

   a) 1 dessert; b) 2 of the desserts; c) 3 of the desserts.

7. Find the term independent of x in the expansion of $\left(\frac{3}{x^2}-2x\right)^6$

8. Find the term involving x^8 in the expansion of $(4+x)^{12}$.

UNIT 3

PROBABILITY

Key Unit competence:

Determine probability of occurrence of an event from random experiment and Apply Bayes' theorem.

3.0 Introductory activity

A woman applying the family planning program considers the assumption that one boy or one girl can be born at each delivery. If she wishes to have 3 children including two girls and one boy, the family knows that this is a case among other cases which can happen for the 3 children they can get.

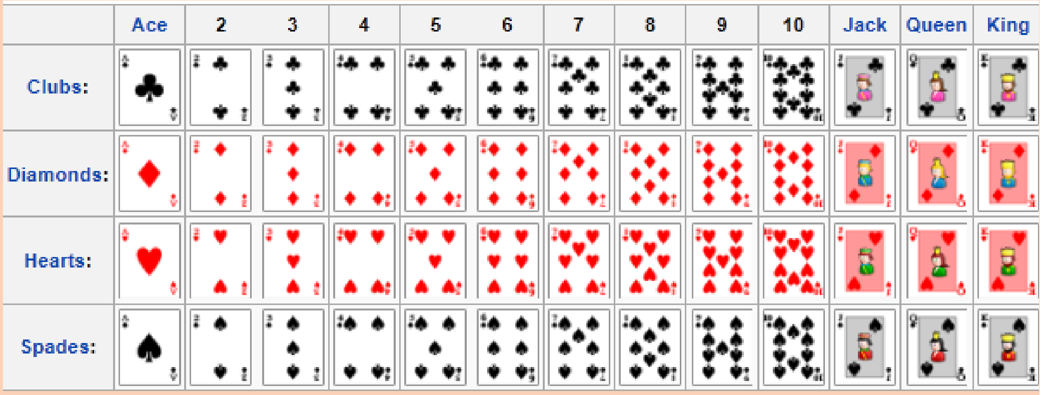
With your colleagues, discuss all those cases and deduce the chance that the woman has for having a girl at the first and the second delivery with a boy at the last delivery.

3.1 Concepts of probability

3.1.1 Definitions

Activity 3.1.1

Consider the deck of 52 playing cards



1. Suppose that you are choosing one card

   a) How many possibilities do you have for the cards to be chosen?

   b) How many possibilities do you have for the kings to be chosen?

   c) How many possibilities do you have for the aces of hearts to be chosen?

2. If "selecting a queen is an example of event, give other examples of events.

Content summary

Probability is the chance that something will happen.

The concept of probability can be illustrated in the context of a game of 52 playing cards. In a park of deck of 52 playing cards, cards are divided into four suits of 13 cards each. If a player selects a card at random (by simple random sampling), then each card has the same chance or same probability of being selected.

When a coin is tossed, it may show Head (H-face with logos) or Tail (T-face with another symbol).

We cannot say beforehand whether it will show head up or tail up. That depends on chance. The same, a card drawn from a well shuffled pack of 52 cards can be red or black. That depends on chance. Such phenomena are called probabilistic. The theory of probability is concerned with this type of phenomena.

Probability is a concept which numerically measures the degree of uncertainty and therefore of certainty of occurrence of events.

In most sampling situations we are generally not concerned with sampling a specific individual but instead we concern ourselves with the probability of sampling certain types of individuals.

Random experiments and Events

A **random experiment** is an experiment whose outcome cannot be predicted or determined in advance.

Example of experiments:

- Tossing a coin,
- Throwing a dice
- Selecting a card from a pack of playing cards, etc.

In all these cases there are a number of possible results (outcomes) which can occur but there is an uncertainty as to which one of them will actually occur.

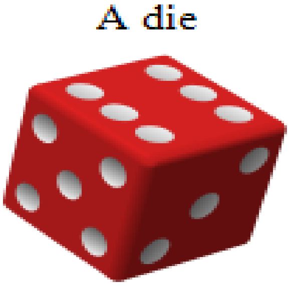
Each performance in a random experiment is called a **trial**. The result of a trial in a random experiment is called an outcome, an elementary event, or a sample point. The totality of all possible outcomes (or sample points) of a random experiment constitutes the sample space which is denoted by Ω. Sample space may be discrete or continuous.

Discrete sample space:

- Firstly, the number of possible outcomes is **finite**.
- Secondly, the number of possible outcomes is **countably infinite**, which means that there is an infinite number of possible outcomes, but the outcomes can be put in a one-to-one correspondence with the positive integers.

Example

If a die is rolled and the number that show up is noted, then $\Omega = \{1, 2, 3, ..., 6\}$.



If a die is rolled until a "6" is obtained, and the number of rolls made before getting first "6" is counted, then we have that $\Omega = \{0, 1, 2, 3, ...\}$.

Continuous sample space: If the sample space contains one or more intervals, the sample space is then **uncountable infinite.**

Example

A die is rolled until a "6" is obtained and the time needed to get this first "6" is recorded. In this case, we have that $\Omega = \{t \in \mathbb{R} : t > 0\} = (0, \infty)$.

An **event** is a subset of the sample space. The null set ϕ is thus an event known as the **impossible event** . The sample space Ω corresponds to the **sure event**.

In particular, every elementary outcome is an event, and so is the sample space itself.

Remarks

- An elementary outcome is sometimes called a **simple event**, whereas a **compound** event is made up of at least two elementary outcomes.
- To be precise we should distinguish between the elementary outcome *w*, which is an element of Ω and the elementary event $\{w\} \subset \Omega$.
- The events are denoted by *A*, *B*, *C* and so on.

Example

Consider the experiment that consists in rolling a die and recording the number that shows up. Let A be the event "the even number is shown" and B be the event "the odd number less than 5 is shown". Define the events A and B.

Solution

We have the sample space $\Omega = \{1,2,3,4,5,6\}$.

$A = \{2,4,6\}$ and $B = \{1,3\}$

Definitions

- Two or more events which have an equal probability (same chance) of occurrence are said to be **equally likely**, i.e. if on taking into account all the conditions, there should be no reason to except any one of the events in preference over the others. Equally likely events may be simple or compound events.
- Two events *A* and *B* are said to be **incompatible** (or **mutually exclusive**) if their intersection is empty. We then write that $A \cap B = \varnothing$.
- Two events, *A* and *B* are said to be **exhaustive** if they satisfy the condition $A \cup B = \Omega$.
- An event is said to be impossible if it cannot occur.

Example

Consider the experiment that consists in rolling a die and recording the number that shows up.

We have that $\Omega = \{1,2,3,4,5,6\}$.

We define the events

$A = \{1,2,4\}$, $B = \{2,4,6\}$, $C = \{3,5\}$, $D = \{1,2,3,4\}$ and $E = \{3,4,5,6\}$

We have

$A \cup B = \{1,2,4,6\}$,

$A \cap B = \{2,4\}$,

$A \cap C = \varnothing$ and

$D \cup E = \Omega$.

Therefore, *A* and *C* are incompatible events.

D and *E* are exhaustive events.

Moreover, we may write that $A' = \{3,5,6\}$, where the symbol A' or $\overline{A}$ denotes the **complement** of the event *A*.

This suggests the following definition:

If *E* is an event, then E' is the event which occurs when *E* does not occur. Events *E* and E' are said to be **complementary events.**

Example of event and sample spaces

- Tossing a coin: there are two possible outcomes, you gain Head up or Tail up. Then, $\Omega = \{H, T\}$ - throwing a dice and noting the number of its uppermost face. There are 6 possible outcomes: one number from 1 to 6 can be up. Then, $\Omega = \{1,2,3,4,5,6\}$.
- Two coins are thrown simultaneously. $\Omega = \{HH, HT, TH, TT\}$
- Tree coins are thrown simultaneously.

  $\Omega = \{HHH, HHT, HTH, THH, HTT, THT, TTH, TTT\}$
- Two dice are thrown simultaneously. The sample space consists of 36 points: .....
- $\Omega = \{(1,1),(1,2),....\}$. Please complete other points!

Note: The determination of sample space for some events such as the one for dice thrown simultaneously requires the use of complex reasoning but it can be facilitated by different counting techniques.

Application activity 3.1.1

1. A box contains 5 red, 3 blue and 2 green pens. If a pen is chosen at random from the box, then which of the following is an impossible event?

   a. Choosing a red pen

   b. Choosing a blue pen

   c. Choosing a yellow pen

   d. None of the above

2. Which of the following are mutually exclusive events when a day of the week is chosen at random?

a. Choosing a Monday or choosing a Wednesday

b. Choosing a Saturday or choosing a Sunday

c. Choosing a weekday or choosing a weekend day

d. All of the above

3. A die is tossed, indicate if the following events are exhaustive or not.

a. X = Get prime number; Y = Get multiple of 2 ; Z = Get 1.

b. X = Get prime numbers; Y = Get composite numbers; Z = Get 1.

c. X = an odd number; Y = an even number

4. Two dice are thrown simultaneously and the sum of points is noted, determine the sample space.

3.1.2 PROPERTIES AND FORMULAS

Activity 3.1.2

Consider the letters of the word “PROBABILITY”.

a. How many letters are in this word?

b. How many vowels are in this word? What is the ratio of numbers of vowels to the total number of letters?

c. How many consonants are in this word? What is the ratio of numbers of consonants to the total number of letters?

d. Let A be the set of all vowels and B the set of all consonants. Find

i. $A \cap B$

ii. $A \cup B$

iii. A'

iv. B'

Content summary

The probability of an event $A \subset \Omega$, is the real number obtained by applying to *A* the function *P* defined by

$$P(A) = \frac{number\ of\ favorable\ outcomes}{number\ of\ possible\ outcomes} = \frac{\#A}{\#\Omega}$$

Theorem 1

Suppose that an experiment has only a finite number of equally likely outcomes. If *E* is an event, then $0 \le P(A) \le 1$.

Note that if $A = \Omega$, then $P(A) = 1$ and $P(\Omega) = 1$ (the event is certain to occur), and

if $A = \varnothing$ then $P(A) = 0$ (the event cannot occur).

Example

A letter is chosen from the letters of the word "**MATHEMATICS**". What is the probability that the letter chosen is an "A"?

Solution

Since two of the eleven letters are *A*'s, we have two favorable outcomes.

There are eleven letters, so we have 11 possible outcomes.

Thus, the probability of choosing a letter *A* is $\frac{2}{11}$.

Theorem 2

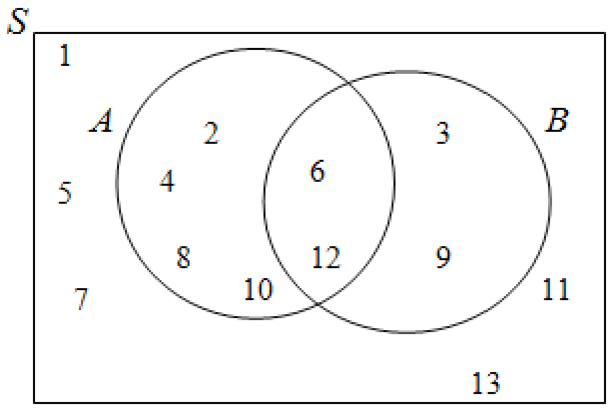
$P(E) = 1 - P(E')$ where *E* and E' are complementary events.

Note that if $A = A_1 \cup A_2 \cup A_3 \cup ... \cup A_n$, where $A_1, A_2, A_3, ..., A_n$ are **incompatible events,** then we may write that $P(A) = \sum_{i=1}^{n} P(A_i)$ for $n = 2, 3, ...$

Example

An integer is chosen at random from the set $S = \{x : x \in \mathbb{Z}^+, x < 14\}$. Let A be the event of choosing a multiple of 2 and let B be the event of choosing a multiple of 3. Find $P(A \cup B)$, $P(A \cap B)$ and $P(A - B)$.

Solution



From the diagram, $\#S = 13$

$A \cup B = \{2,3,4,6,8,9,10,12\} \Rightarrow \#(A \cup B) = 8, \text{ thus } P(A \cup B) = \frac{8}{13}$

$A \cap B = \{6,12\} \Rightarrow \#(A \cap B) = 2, \text{ thus } P(A \cap B) = \frac{2}{13}$

$A - B = \{2,4,8,10\} \Rightarrow \#(A - B) = 4, \text{ thus } P(A - B) = \frac{4}{13}$

Application activity 3.1.2

1. A letter is chosen from the letters of the word "**MATHEMATICS**". What is the probability that the letter chosen is

   a. M?

   b. T?

2. An integer is chosen at random from the set $S = \{\text{all positive integers less than } 20\}$. Let A be the event of choosing a multiple of 3 and let B be the event of choosing an odd number. Find

   a. $P(A \cup B)$

   b. $P(A \cap B)$

   c. $P(A - B)$

3.1.3 Additional law of probability

Activity 3.1.3

Consider a machine which manufactures car components. Suppose each component falls into one of four categories: **top quality**, **standard, substandard, reject**

After many samples have been taken and tested, it is found that under certain specific conditions the probability that a component falls into a category is as shown in the following table.

The probability of a car component falling into one of four categories.

Category	*Probability*
Top quality	0.18
Standard	0.65
Substandard	0.12
Reject	0.05

The four categories cover all possibilities and so the probabilities must sum to 1. If 100 samples are taken, then on average 18 will be top quality, 65 of standard quality, 12 substandard and 5 will be rejected.

Using the data in table determine the probability that a component selected at random is **either** standard or top quality.

Content summary

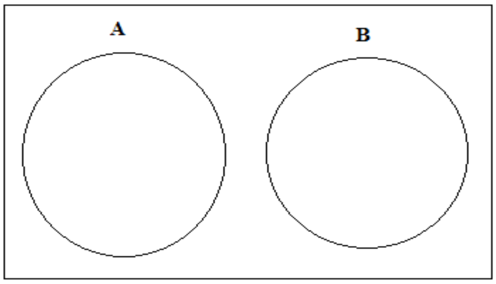
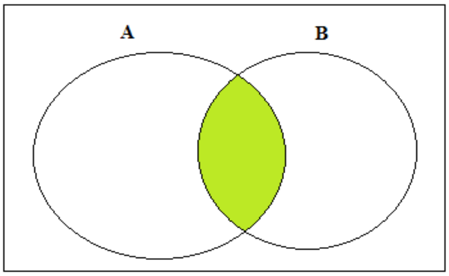
For any event A and B from a sample space E,

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

This is known as the **addition law** of probability from which we deduce that if A and B are mutually exclusive events, then $P(A \cup B) = P(A) + P(B)$.

If E_i and E_j are mutually exclusive we denote this by $E_i \cap E_j = \phi$ that is the compound event $E_i \cap E_j$ is an **impossible event** and so will never occur.

On Venn diagram A and B shown as mutually exclusive (**disjoint sets) events** and shown as no mutually exclusive.

Mutually exclusive (disjoint) events A and B	Non-mutually exclusive events A and B
	

Suppose that $E_1, E_2, E_3, ..., E_n$ are n events and that in a single trial only one of these events can occur. The occurrence of any event, E_i , excludes the occurrence of all other events. Such events are mutually exclusive.

Generally,

For **mutually exclusive events**, the addition law of probability applies:

$$P(E_1 \text{ } or \text{ } E_2 or \text{ } E_3 \cdots or \text{ } E_n) = P(E_1 \cup E_2 \cup E_3 \cdots \cup E_n) = P(\cup_i^n E_i)$$

$$= P(E_1) + P(E_1) + P(E_3) + \cdots + P(E_n) = \sum_{i=1}^{n} P(E_i)$$

For **inclusive events** the addition law of probability applies:

$$P(E_1 \text{ } or \text{ } E_2 or \text{ } E_3 \cdots or \text{ } E_n) = P(E_1 \cup E_2 \cup E_3 \cdots \cup E_n) = P(\cup_i^n E_i)$$

$$= \sum_{i=1}^{n} P(E_i) - \sum_{j>i=1}^{n} P(E_i \cap E_j) + \sum_{k>j>i=1}^{n} P(E_i \cap E_j \cap E_k) + \cdots + (-1)^n P(E_1 \cap E_2 \cap \cdots \cap E_n)$$

The sum of the probability of outcomes

If two events *A* and *B* are such that $A \cup B = \Omega$ then $P(A \cup B) = 1$ and then these two events are said to be **exhaustive**.

In the sample space, the sum of the probability of outcomes is 1.

Generally, given a finite sample space, say $\Omega = \{a_1, a_2, a_3, ..., a_n\}$, we can find a finite probability by assigning to each point $a_i \in \Omega$ a real number P_i, called the probability of a_i, satisfying the following:

a) $P_i \geq 0$ for all integers i, $1 \leq i \leq n$;

b) $\sum_{i=1}^{n} P_i = 1$.

Example

Suppose a student is selected at random from 100 students where 30 are taking Mathematics, 20 are taking Chemistry, and 10 are taking Mathematics and chemistry. Find the probability that the student is taking mathematics or chemistry.

Solution

Let M be the event "students taking mathematics" and C the event "students taking chemistry". Since the space is equiprobable,

$$P(M) = \frac{30}{100} = \frac{3}{10}, P(C) = \frac{20}{100} = \frac{1}{5} \text{ and } P(M \cap C) = \frac{10}{100} = \frac{1}{10}.$$

Thus, by the addition principle,

$$P(M \, or \, C) = P(M \cup C) = P(M) + P(C) - P(M \cap C) = \frac{3}{10} + \frac{1}{5} - \frac{1}{10} = \frac{4}{10} = \frac{2}{5}.$$

Example

In a competition in which there are no dead heats, the probability that John wins is 0.3,the probability that Mike wins is 0.2 and the probability that Putin wins is 0.4. Find the probability that :

(a) John or Mike wins

(b) John or Putin or Mike wins,

(c) Someone else wins.

Solution:

Since only one person wins, the events are mutually exclusive.

(a) $P(John \ or \ Mike \, win) = 0.3 + 0.2 = 0.5$

(b) $P(\, John \ or \ Putin \ or \, Mike \ win) = 0.3 + 0.2 + 0.4 = 0.9$

c) $P(Someone \ else \ wins) = 1 - 0.9 = 0.1$

Example

Machines A and B make components. Machine A makes 60% of the Components. The probability that a component is acceptable is 0.93 when made by machine A and 0.95 when made by machine B. A component is picked at random. Calculate the probability that it is:

a) Made by machine A and is acceptable.

b) Made by machine B and is acceptable.

c) Acceptable.

Solution:

(a) We know that 60% of the components are made by machine A and 93% of these are acceptable. Converting these percentages to decimal numbers we have.

$$P(\textit{component is made by machine A and is acceptable}) = \frac{60}{100} \times \frac{93}{100} = 0.60 \times 0.93 = 0.558$$

(b) We know that 40% of the components are made by machine B and 95% of these are acceptable.

$$P(\textit{component is made by machine } \mathrm{B} \textit{ and is acceptable}) = \frac{40}{100} \times \frac{95}{100} = 0.40 \times 0.95 = 0.38$$

(c) Note that the events described in (a) and (b) are mutually exclusive and so the addition law can be applied.

P(component is acceptable) = *P*(component is made by machine A and is acceptable)

+*P*(component is made by machine B and is acceptable) = 0.558 + 0.38 = 0.938

application activity 3.1.3

1. A fair die is rolled, what is the probability of getting an even number or prime number?

2. Events A and B are such that they are both mutually exclusive and exhaustive. Find the relation between these two events.

3. In a class of a certain school, there are 12 girls and 20 boys. If a teacher want to choose one student to answer the asked question

a) What is the probability that the chosen student is a girl?

b) What is the probability that the chosen student is a boy?

c) If teacher doesn't care on the gender, what is the probability of choosing any student?

3.2. Independent events

Activity 3.2

A box contains 3 red pens, 4 green pens and 5 blue pens. One pen is taken from the box and then replaced. Another pen is taken from the box. Let *A* be the event "the first pen is red" and *B* be the event the second pen is blue."

Is the occurrence of event *B* affected by the occurrence of event *A*? Explain.

Content summary

If probability of event *B* is not affected by the occurrence of event *A*, events *A* and *B* are said to be **independent** and $P(A \cap B) = P(A) \times P(B)$

This rule is the simplest form of the **multiplication law** of probability.

Example

A die is thrown twice. Find the probability of obtaining a 4 on the first throw and an odd number on the second throw.

Solution

Let A be the event: "a 4 is obtained on the first throw", then $P(A)=\frac{1}{6}$. That is $A=\{4\}$

B be the event: "an odd number is obtained on the second throw". That is $B=\{1,3,5\}$

Since the result on the second throw is not affected by the result on the first throw, A and B are independent events.

There are 3 odd numbers, then

$$P(B)=\frac{3}{6}=\frac{1}{2}$$

Therefore,

$$\begin{aligned} P(A\cap B) &= P(A)P(B) \\ &= \frac{1}{6}\times\frac{1}{2} \\ &= \frac{1}{12} \end{aligned}$$

Example

A factory runs two machines. The first machine operates for 80% of the time while the second machine operates for 60% of the time and at least one machine operates for 92% of the time. Do these two machines operate independently?

Solution

Let the first machine be M_1 and the second machine be M_2, then

$P(M_1)=80\%=0.8$, $P(M_2)=60\%=0.6$ and $P(M_1\cup M_2)=92\%=0.92$

Now,

$$P(M_1\cup M_2)=P(M_1)+P(M_2)-P(M_1\cap M_2)$$

$$P(M_1 \cap M_2) = P(M_1) + P(M_2) - P(M_1 \cup M_2)$$
$$= 0.8 + 0.6 - 0.92$$
$$= 0.48$$
$$= 0.8 \times 0.6$$
$$= P(M_1) \times P(M_2)$$

Thus, the two machines operate independently.

Example

A coin is weighted so that heads is three times as likely to appear as tails. Find $P(H)$ and $P(T)$.

Solution

Let $P(T) = p_1$, then $P(H) = 3p_1$.

But $P(H) + P(T) = 1$

Therefore $p_1 + 3p_1 = 1 \Leftrightarrow 4p_1 = 1 \Rightarrow p_1 = \frac{1}{4}$

Thus, $P(H) = \frac{3}{4}$ and $P(T) = \frac{1}{4}$.

Application activity 3.2

1. A dresser drawer contains one pair of socks with each of the following colors: blue, brown, red, white and black. Each pair is folded together in a matching set. You reach into the sock drawer and choose a pair of socks without looking. You replace this pair and then choose another pair of socks. What is the probability that you will choose the red pair of socks both times?

2. A coin is tossed and a single 6-sided die is rolled. Find the probability of landing on the head side of the coin and rolling a 3 on the die.

3. A husband and a wife appear in an interview for two vacancies in the same post. The probability of husband's selection is $\frac{1}{7}$ and that of wife's selection is $\frac{1}{5}$. What is the probability that
   a) both of them will be selected
   b) only one of them will be selected,
   c) None of them will be selected?

3.3 Dependent events and conditional probability

Activity 3.3

Suppose that you have a deck of cards; then draw a card from that deck, not replacing it, and then draw a second card.

a) What is the sample space for each event?

b) Suppose you select successively two cards, what is the probability of selecting two red cards? c) Explain if there is any relationship (Independence or dependence) between those two events considering the sample space. Does the selection of the first card affect the selection of the second card?

Content summary

When the outcome or occurrence of the first event affects the outcome or occurrence of the second event in such a way that the probability is changed, the events are said to be ***dependent***.

Suppose A is an event in a sample space S with $P(A) > 0$. The probability that an event B occurs once A has occurred, written as $P(B \mid A)$ is called the **conditional probability of B given A** and is defined as follows:

$$P(B / A) = \frac{P(A \cap B)}{P(A)}$$

From this result, we have general statement of the **multiplication law**:

$$P(A \cap B) = P(A) \times P(B \mid A)$$

This shows us that the probability that two events will both occur is the product of the probability that one will occur and the conditional probability that the other will occur given that the first has occurred. We can also write $P(A \cap B) = P(B) \times P(A \mid B)$.

Notice:

If A and B are **independent**, then the probability of B is not affected by the occurrence of A and so $P(B \mid A) = P(B)$ giving $P(A \cap B) = P(A) \times P(B)$ as discussed in 3.2.

Example

Suppose a card is drawn from a deck and not replaced, and then the second card is drawn. What is the probability of selecting an ace on the first card and a king on the second card?

Solution:

The probability of selecting an ace on the first draw is $\frac{4}{52}$. But since that card is not replaced, the probability of selecting a king on the second card is $\frac{4}{51}$, since there are 51 cards remaining.

The outcomes of the first draw has affected the outcome of the second. By multiplication rule , the probability of both events occurring is :

$$\frac{4}{52}\times\frac{4}{51}=\frac{16}{2652}=\frac{4}{663}=0.006.$$

Example

A die is tossed. Find the probability that the number obtained is a 4 given that the number is greater than 2.

Solution

Let *A* be the event: "the number is a 4", then $A=\{4\}$

B be the event: "the number is greater than 2", then $B=\{3,4,5,6\}$ and

$$P(B)=\frac{4}{6}=\frac{2}{3}$$

But $A\cap B=\{4\}$ and $P(A\cap B)=\frac{1}{6}$

Therefore,

$$P(A\mid B)=\frac{P(A\cap B)}{P(B)}$$

$$P(A\mid B)=\frac{\frac{1}{6}}{\frac{2}{3}}$$

$$P(A|B)=\frac{1}{6}\times\frac{3}{2}$$
$$=\frac{1}{4}$$

Example

At a middle school, 18% of all students play football and basketball, and 32% of all students play football. What is the probability that a student who plays football also plays basketball?

Solution

Let *A* be a set of students who play football and *B* a set of students who play basketball then the set of students who play both games is $A\cap B$. We have $P(A)=32\%=0.32$, $P(A\cap B)=18\%=0.18$. We need the probability of *B* known that *A* has occurred.

Therefore,

$$P(B|A)=\frac{P(A\cap B)}{P(A)}$$
$$=\frac{0.18}{0.32}$$
$$=0.5625$$
$$=56\%$$

Notice: Contingency table

Contingency table (or **Two-Way table**) provides a different way of calculating probabilities. The table helps in determining conditional probabilities quite easily. The table displays sample values in relation to two different variables that may be dependent or contingent on one another.

Below, the contingency table shows the favorite leisure activities for 50 adults, 20 men and 30 women. Because entries in the table are frequency counts, the table is a **frequency table.**

	Dance	**Sports**	**TV**	**Total**
Men	2	10	8	20
Women	16	6	8	30
Total	18	16	16	50

Entries in the total row and total column are called **marginal frequencies** or the **marginal distribution**. Entries in the body of the table are called **joint frequencies**.

Example

Suppose a study of speeding violations and drivers who use car phones produced the following fictional data:

	Speeding violation in the last year	No speeding violation in the last year	Total
Car phone user	25	280	305
Not a car phone user	45	405	450
Total	70	685	755

Calculate the following probabilities using the table:

a) P(person is a car phone user)

b) P(person had no violation in the last year)

c) P(person had no violation in the last year AND was a car phone user)

d) P(person is a car phone user OR person had no violation in the last year)

e) P(person is a car phone user GIVEN person had a violation in the last year)

f) P(person had no violation last year GIVEN person was not a car phone user)

Solution

a) $\text{P(person is a car phone user)} = \dfrac{\text{number of car phone users}}{\text{total number in study}} = \dfrac{305}{755}$

b) $\text{P(person had no violation in the last year)} = \dfrac{\text{number that had no violation}}{\text{total number in study}} = \dfrac{685}{755}$

c) $\text{P(person had no violation in the last year AND was a car phone user)} = \dfrac{280}{755}$

d) P(person is a car phone user OR person had no violation in the last year)

$$= \left(\frac{305}{755} + \frac{685}{755}\right) - \frac{280}{755} = \frac{710}{755}$$

e) The sample space is reduced to the number of persons who had a violation. Then

$$P(\text{person is a car phone user GIVEN person had a violation in the last year}) = \frac{25}{70}.$$

f) The sample space is reduced to the number of persons who were not car phone users. Then

$$P(\text{person had no violation last year GIVEN person was not a car phone user}) = \frac{405}{450}$$

Application activity 3.3

1) The world-wide Insurance Company found that 53% of the residents of a city had home owner's Insurance with its company of the clients, 27% also had automobile Insurance with the company. If a resident is selected at random, find the probability that the resident has both home owner's and automobile Insurance with the world wide Insurance Company.

2) Calculate the probability of a 6 being rolled by a die if it is already known that the result is even.

3) A jar contains black and white marbles. Two marbles are chosen without replacement. The probability of selecting a black marble and then a white marble is 0.34, and the probability of selecting a black marble on the first draw is 0.47. What is the probability of selecting a white marble on the second draw, given that the first marble drawn was black?

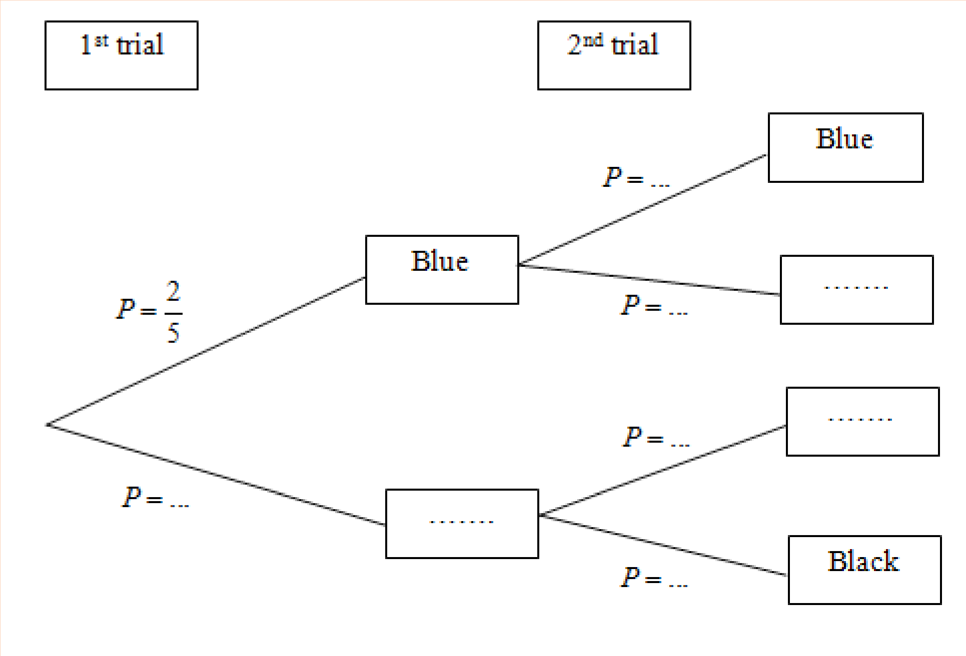
4) A bag contains five discs, three of which are red. A box contains six discs, four of which are red. A card is selected at random from a normal pack of 52 cards, if the card is a club a disc is removed from the bag and if the card is not a club a disc is removed from the box. Find the probability that, if the removed disc is red it came from the bag.

3.4 Successive trials and Tree diagram

Activity 3.4

A box contains 4 blue pens and 6 black pens. One pen is drawn at random, its color is noted and the pen is replaced in the box. A pen is again drawn from the box and its color is noted.

1) For the 1st trial, what is the probability of choosing a blue pen and probability of choosing a black pen?
2) For the 2nd trial, what is the probability of choosing a blue pen and probability of choosing a black pen? Remember that after the 1st trial the pen is replaced in the box.
3) In the following figure complete the missing colours and probabilities



Content summary

A **tree diagram** can be used to show the probabilities of certain **outcomes** occurring when two or more **trials** take place in succession.

The **outcome** is written at the end of the branch and the fraction on the branch gives the probability of the outcome occurring.

For each **trial** the number of branches is equal to the number of possible outcomes of that trial. In the diagram there are two possible outcomes, A and B, of each trial.

Example

A bag contains 8 balls of which 3 are red and 5 are green. One ball is drawn at random, its color is noted and the ball replaced in the bag. A ball is again drawn from the bag and its color is noted. Find the probability the ball drawn will be

a) red followed by green,

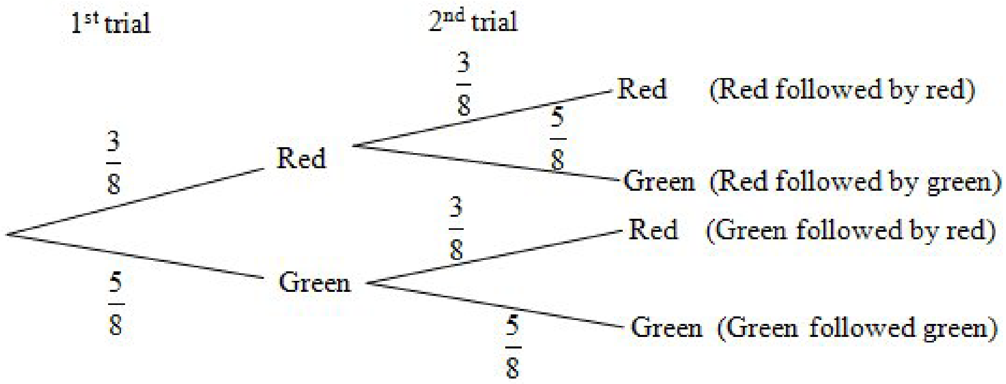
b) red and green in any order,

c) of the same color.

Solution

Since there are 3 red balls and 5 green balls, for the 1st trial, the probability of choosing a red ball is $\frac{3}{8}$ and probability of choosing a green ball is $\frac{5}{8}$ and since after the 1st trial the ball is replaced in the bag, for the second trial the probabilities are the same as in the first trial.

Draw a tree diagram showing the probabilities of each outcome of the two trials.



a) $P(\text{Red followed by green}) = \frac{3}{8} \times \frac{5}{8} = \frac{15}{64}$

b) $P(\text{Red and green in any order}) = \frac{3}{8} \times \frac{5}{8} + \frac{5}{8} \times \frac{3}{8} = \frac{15}{32}$

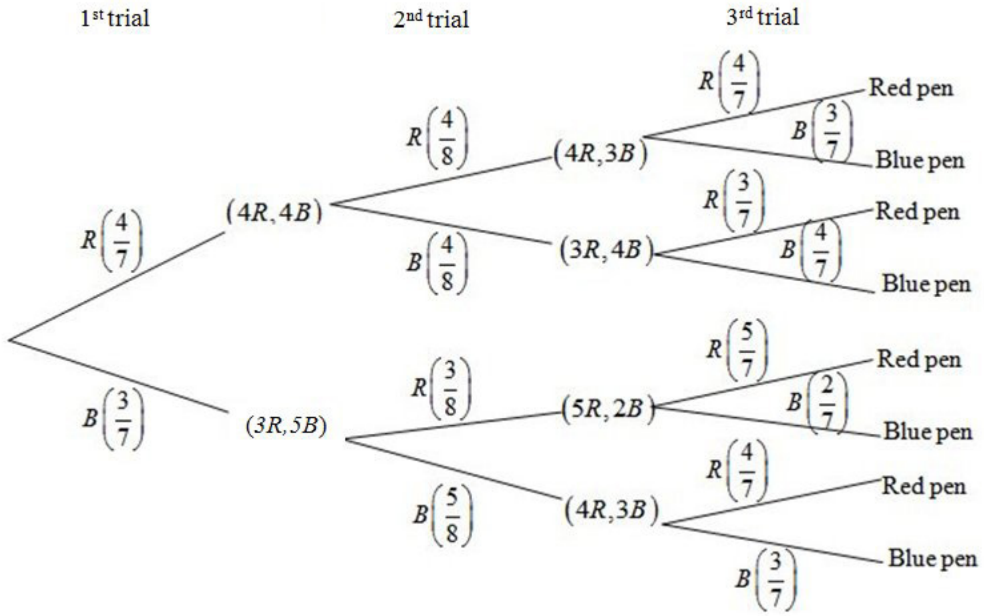
c) $P(\text{both of the same colors}) = \frac{3}{8} \times \frac{3}{8} + \frac{5}{8} \times \frac{5}{8} = \frac{17}{32}$

Example

A bag (1) contains 4 red pens and 3 blue pens. Another bag (2) contains 3 red pens and 4 blue pens. A pen is taken from the first bag (1) and placed into the second bag (2). The second bag (2) is shaken and a pen is taken from it and placed in the first bag (1). If now a pen is taken from the first bag, use the tree diagram to find the probability that it is a red pen.

Solution

Tree diagram is given below:



From tree diagram, the probability to have a red pen is

$$P(R) = \frac{4}{7}\times\frac{4}{8}\times\frac{4}{7}+\frac{4}{7}\times\frac{4}{8}\times\frac{3}{7}+\frac{3}{7}\times\frac{3}{8}\times\frac{5}{7}+\frac{3}{7}\times\frac{5}{8}\times\frac{4}{7}$$

$$= \frac{64}{392}+\frac{48}{392}+\frac{45}{392}+\frac{60}{392}$$

$$= \frac{31}{56}$$

Application activity 3.4

1) Calculate the probability of three coins landing on: Three heads.

2) A class consists of six girls and 10 boys. If a committee of three is chosen at random, find the probability of:

   a) Three boys being chosen.

   b) Exactly two boys and a girl being chosen.

   c) Exactly two girls and a boy being chosen.

   d) Three girls being chosen.

3) A bag contains 7 discs, 2 of which are red and 5 are green. Two discs are removed at random and their colours noted. The first disk is not replaced before the second is selected. Find the probability that the discs will be

   a) both red

   b) of different colours

   c) the same colours.

4) Three discs are chosen at random, and without replacement, from a bag containing 3 red, 8 blue and 7 white discs. Find the probability that the discs chosen will be

   a) all red b) all blue c) one of each colour.

3.5. Bayes theorem and its applications

Activity 3.5

Suppose that entire output of a factory is produced on three machines. Let B_1 denote the event that a randomly chosen item was made by machine 1, B_2 denote the event that a randomly chosen item was made by machine 2 and B_3 denote the event that a randomly chosen item was made by machine 3. Let A denote the event that a randomly chosen item is defective.

1) Use conditional probability formula and give the relation should be used to find the probability that the chosen item is defective, $P(A)$, given that it is made by machine 1 or machine 2 or machine 3.

2) If we need the probability that the chosen item is produced by machine 1 given that is found to be defective, i.e $P(B_1 | A)$, give the formula for this conditional probability. Recall that $P(B_i \cap A)$ can be written as $P(A|B_i)P(B_i)$. Do the same if the item is produced by machine 2 and by machine 3. Give the general formula if the item is produced by machine i (i from 1 to 3)

Content summary

Let $B_1, B_2, B_3, ..., B_n$ be incompatible and exhaustive events and let A be an arbitrary event.

We have:

$$P(B_i | A) = \frac{P(B_i \cap A)}{P(A)} = \frac{P(A|B_i)P(B_i)}{\sum_{i=1}^{n} P(A|B_i)P(B_i)}$$

This formula is called **Bayes' formula**.

Remark

We also have (**Bayes' rule)**

$$P(B|A) = \frac{P(A|B)P(B)}{P(A)}$$

Example

Suppose that machines M_1, M_2, and M_3 produce, respectively, 500, 1000, and 1500 parts per day, of which 5%, 6%, and 7% are defective. A part produced by one of these machines is taken at random, at the end of a given workday, and it is found to be defective. What is the probability that it was produced by machine M_3?

Solution

Let A_i be the event "the part taken at random was produced by machine M_i," for $i = 1,2,3$; and let D be "the part taken at random is defective."

Using Bayes' formula, we seek

$$P(A_3 \mid D) = \frac{P(D \mid A_3)P(A_3)}{\sum_{i=1}^{3} P(D \mid A_i)P(A_i)}$$

$$= \frac{(0.07)\left(\frac{1500}{3000}\right)}{(0.05)\left(\frac{1}{6}\right) + (0.06)\left(\frac{1}{3}\right) + (0.07)\left(\frac{1}{2}\right)}$$

$$= \frac{105}{190}$$

$$= \frac{21}{38}$$

Example

Two machines A and B produce 60% and 40% respectively of total output of a factory. Of the parts produced by machine A, 3% are defective and of the parts produced by machine B, 5% are defective. A part is selected at random from a day's production and found to be defective. What is the probability that it came from machine A?

Solution

Let E be the event that the part came from machine A

C be the event that the part came from machine B and

D be the event that the part is defective.

We require $P(E \mid D)$.

Now, $P(E)\times P(D|E)=0.6\times 0.03=0.018$ and

$$P(D)=P(E\cap D)+P(C\cap D)$$

$$=0.018+0.4\times 0.05$$
$$=0.038$$

Therefore, the required probability is $\frac{0.018}{0.038}=\frac{9}{19}$.

Application activity 3.5

1. 20% of a company's employees are engineers and 20% are economists. 75% of the engineers and 50% of the economists hold a managerial position, while only 20% of non-engineers and non-economists have a similar position. What is the probability that an employee selected at random will be both an engineer and a manager?

2. The probability of having an accident in a factory that triggers an alarm is 0.1. The probability of its sounding after the event of an incident is 0.97 and the probability of it sounding after no incident has occurred is 0.02. In an event where the alarm has been triggered, what is the probability that there has been no accident?

3.6 End unit assessment

1. Which of the following experiments does not have equally likely outcomes?

   a) Choose a number at random between 1 and 7,

   b) Toss a coin,

   c) Choose a letter at random from the word SCHOOL,

   d) None of the above.

2. Tickets numbered 1 to 20 are mixed up and then a ticket is drawn at random. What is the probability that the ticket drawn has a number which is a multiple of 3 or 5?

3.cIn a class, there are 15 boys and 10 girls. Three students are selected at random. What is the probability that 1 girl and 2 boys are selected?

4. One card is drawn at random from a pack of 52 cards. What is the probability that the card drawn is a face card (Jack, Queen and King only)?

5. The letters of the word FACETIOUS are arranged in a row. Find the probability that

   a) the first 2 letters are consonants,

   b) all the vowels are together.

6. At TTC Rubengera, the probability that a student takes Technology and French is 0.087. The probability that a student takes Technology is 0.68. What is the probability that a student takes French given that the student is taking Technology?

7. A car dealership is giving away a trip to Akagera National Park to one of their 120 best customers. In this group, 65 are women, 80 are married and 45 married women. If the winner is married, what is the probability that it is a woman?

8. For married couples living in a certain suburb the probability that the husband will vote on a bond referendum is 0.21, the probability that his wife will vote in the referendum is 0.28, and the probability that both the husband and wife will vote is 0.15. What is the probability that

   a. at least one member of a married couple will vote ?

   b. a wife will vote, given that her husband will vote ?

   c. a husband will vote, given that his wife does not vote ?

9. The probability that a patient recovers from a delicate heart operation is 0.8. What is the probability that

   a. exactly 2 of the next 3 patients who have this operation will survive ?

   b. all of the next 3 patients who have this operation survive ?

10. In a certain college, 5% of the men and 1% of the women are taller than 180 *cm*. Also, 60% of the students are women. If a student is selected at random and found to be taller than 180 *cm*, what is the probability that this student is a woman?

UNIT 4

LOGARITHM AND EXPONENTIAL FUNCTIONS

Key Unit competence:

Extend the use of concepts and definitions of functions to determine the domain and sketch the graphs of logarithmic and exponential functions.

4.0 Introductory activity

An economist created a business which helped him to make money in an interesting way so that the money he/she earns each day doubles what he/she earned the previous day. If he/she had 200USD on the first day and by taking t as the number of days, discuss the money he/she can have at the t^{th} day through answering the following questions:

a) Draw the table showing the money this economist will have on each day starting from the first to the 10^{th} day.

b) Plot these data in rectangular coordinates

c) Based on the results in a), establish the formula for the economist to find out the money he/she can earn on the n^{th} day. Therefore, if t is the time in days, express the money $F(t)$ for the economist.

d) Now the economist wants to possess the money F under the same conditions, discuss how he/she can know the number of days necessary to get such money from the beginning of the business.

From the discussion, the function $F(t)$ found in c) and the function $Y(F)$ found in d) are respectively exponential function and logarithmic functions that are needed to be developed to be used without problems. In this unit, we are going to study the behaviour and properties of such essential functions and their application in real life situation.

4.1 LOGARITHM FUNCTIONS

4.1.1 Definition of logarithmic functions

Activity 4.1.1

In a class of 30 students, 10 got A on the first test, 9 got A on a second test, and 15 did not get an A on either test. Find: the number of students who got:

a) an A on both tests;

b) an A on the first test but not the second;

c) an A on the second test but not the first.

CONTENT SUMMARY

Let the function $y = \log_a x$, it is proven that if $x > 0$ and a is a constant $(a > 0, a \neq 1)$, then $\log_a x$ is a real number called the **"logarithm to the base a of x"**

For a positive constant a $(a \neq 1)$, we call logarithmic function, the function $f : \mathbb{R} \to \mathbb{R} : x \mapsto \log_a x$.

In the expression $y = \log_a x$, y is referred to as **the logarithm**, is **the base**, and is the **argument.**

If the base is 10, it is not necessary to write the base, and we say **decimal logarithm or common logarithm or Brigg's logarithm**. So, the notation will become $y = \log x$.

If the base is (where =2.718281828), we say" **Neperean logarithm or natural logarithm".**

The **natural logarithm** is usually written using the shorthand notation $y = \ln x$ instead of $y = \log_e x$.

The function $y = \log_a x$ is defined on positive real numbers, $]0, +\infty[$ and its range is all real numbers.

Particularly,

- If $x = 1$, then $\log_a x = \log_a 1$. That is, $\log_a x = 0$
- If $x > 1$, then $\log_a x > \log_a 1$ or $\log_a x > 0$
- If $0 < x < 1$, then $\log_a x < \log_a 1$ or $\log_a x < 0$

It means that: $\forall x \in]1,+\infty[$, $\log_a x > 0$ and $\forall x \in]0,1[$, $\log_a x < 0$ as shown in the graph

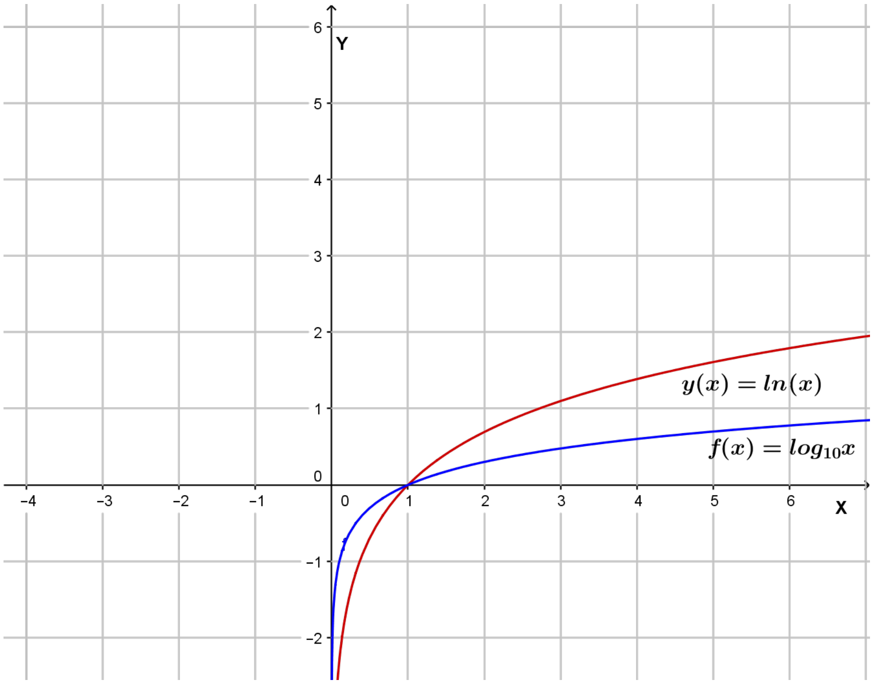


Figure 4.1: Graphs of Common and Natural logarithmic functions $f(x) = \log_{10}(x)$ *and* $y(x) = \ln(x)$.

Generally,

The domain of common and the logarithm functions is the set of positive real numbers $\left(dom\, f = \{x \in \mathbb{R} : x > 0\} =]0,+\infty[= \mathbb{R}_0^+\right)$ and **the range** is the line of all real numbers $\left(Range\ f = \mathbb{R} =]-\infty,+\infty[\right)$.

The logarithmic function is neither **even** nor **odd**. If $u:\mathbb{R} \to \mathbb{R} : x \mapsto u(x)$ is any other function we can compose u and the logarithmic function as $y = \log_a(u(x))$ defined for x such that $u(x) > 0$.

Example 1

Find the domain and range for the function

a) $f(x) = \log(x-4)$

b) $g(x) = \ln(x+6)$

Solution

a) Because $\log x$ defined only for positive values of x . So in this problem $y=\log(x-4)$, is defined if and only if $x-4>0 \Leftrightarrow x>4$ and gives that $x\in]4,+\infty[$.

The range of y is still all real number $\mathbb{R}$ $or\ y\in]-\infty,+\infty[$.

$$Dom\ f = \{x\in\mathbb{R} : x-4>0\} = \{x\in\mathbb{R} : x>4\} =]4,+\infty[.\ Range\ f = \mathbb{R}..$$

b) The function $y=\ln(x+6)$, is defined if and only if $x+6>0 \Leftrightarrow x>-6$ and gives that $x\in]-6,+\infty[$ which is the domain. The range is $\mathbb{R}$

$$Dom\ g = \{x\in\mathbb{R} : x+6>0\} = \{x\in\mathbb{R} : x>-6\} =]-6,+\infty[.\ Range\ g = \mathbb{R}.$$

Example 2

Find the domain of definition of $f(x)=\ln(2x-3)$

Solution

Condition: $2x-3>0$

$$2x-3>0 \Leftrightarrow x>\frac{3}{2}$$

Thus, $Domf = \left]\frac{3}{2},+\infty\right[$ and $Range\ f = \mathbb{R}$.

Example 3

Find the domain of definition of $f(x)=\ln(x+3)(x+2)$

Solution

Condition:$(x+3)(x+2)>0$

$(x+3)(x+2)>0$ if $x\in]-\infty,-3[\cup]-2,+\infty[$ (sign table can be used)

Thus, $Domf =]-\infty,-3[\cup]-2,+\infty[$ and $Range\ f = \mathbb{R}$

Example 4

Find the domain of $f(x)=\log_3(1-x)+\log_2 x$

Solution

Conditions: $1-x>0$ and $x>0$

$1-x>0 \Rightarrow x<1$

Domain is the intersection of $x<1$ and $x>0$

Thus, $Domf =]0,1[$ and $Range\ f = \mathbb{R}$

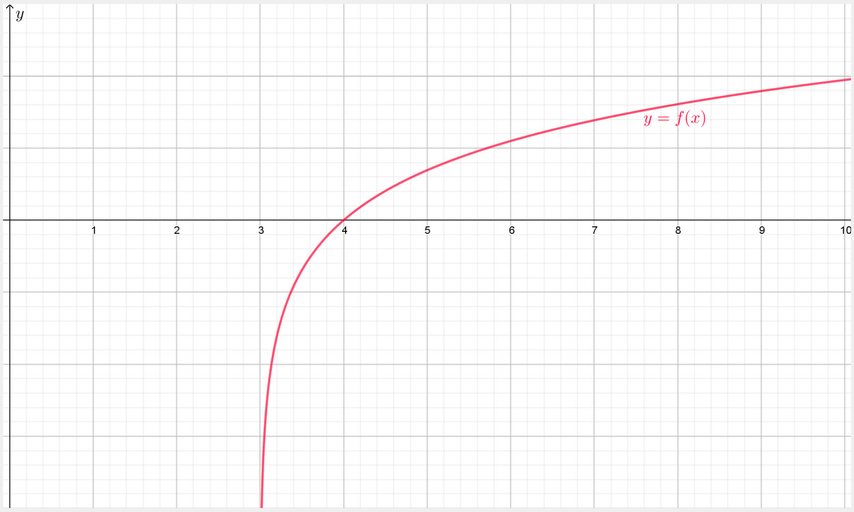
Application activity 4.1.1

1. State the domain and range of the following functions:

   a) $y=\log_3(x-2)+4$

   b) $y=\log_5(8-2x)$

2. Observe the following graph of a given logarithmic function , then state its domain and range . Justify your answers.



3. Find the domain of definition for each of the following functions

a) $f(x)=\log_2\sqrt{x}$ b) $f(x)=\log_3(x^2-1)$

c) $f(x)=\log_{\frac{1}{2}}\dfrac{x+1}{x-4}$ d) $f(x)=\log_4\dfrac{x}{x^2+7x+10}$

4. Find the values of x for which the expressions below are defined

$a)\log_2(2x-3)$ $c)2\log(5x-1)+\log 2x$ $e)\log_{x^2-4}2+\log(x+6)$

$b)\log_{\frac{1}{2}}\left(\dfrac{1-x}{x}\right)$ $d)\log_3\left(\dfrac{1}{x^2}\right)-\log x$ $f)4\log_{\frac{1}{4}}\left(\dfrac{1}{x-1}\right)-\log\left(\dfrac{x-1}{x}\right)$

4.1.2 Properties and operations on logarithmic function with any base

Activity 4.1.2

a) Use calculator to complete the following table

x	-2	0	1	2	4	$\frac{2}{4}$	2×4	$\frac{1}{4}$	2^4
$\ln x$									

b) Using the answer from (a), Determine the value of

i) $\ln 2+\ln 4$ ii) $\ln 2-\ln 4$ iii) $\ln 1-\ln 4$ iv) $4\ln 2$

c) Observe the answer from (a) and (b); then compare and show the equal values.

d) What can you conclude about:

$a)\ \ln(xy)$ $c)\ \ln\left(\frac{1}{y}\right)$

$b)\ \ln\left(\frac{x}{y}\right)$ $d)\ \ln(x)^n$

CONTENT SUMMARY

From activity 4.1.2, we observe that $\forall x, y \in \left]0,+\infty\right[$ then:

CASE I: Properties and operations on logarithmic function with base e

a) $\ln 1=0$

b) $\ln(xy)=\ln x+\ln y$

c) $\ln\left(\frac{x}{y}\right)=\ln x-\ln y$

d) $\ln\left(\frac{1}{y}\right)=-\ln y$

e) $\ln(x)^n=n\ln x$

f) $\ln\sqrt[n]{x}=\frac{1}{n}\ln x$

g) $\ln\sqrt[n]{x^m}=\frac{m}{n}\ln x$

CASE II: Properties and operations on logarithmic function with base a

$\forall x, y \in \left]0,+\infty\right[, a \in \left]0,+\infty\right[\setminus \{1\}:$

a) $\log_a xy = \log_a x + \log_a y$

b) $\log_a \frac{1}{y} = -\log_a y$

c) $\log_a \frac{x}{y} = \log_a x - \log_a y$

d) $\log_a x^r = r\log_a x$

e) $\log 10^n = n\log 10 = n$

f) $\log_a \sqrt[m]{x^n} = \log_a x^{\frac{n}{m}} = \frac{n}{m}\log_a x$

g) $\log_{a^m} x = \frac{1}{m}\log_a x$

h) $\log_{a^m} x^n = \frac{n}{m}\log_a x$

Notice

By Changing logarithm from one base to another

$$\log_b x = \frac{\log_a x}{\log_a b}$$

Example 1

Change logarithm from one base to another

Given that $y = \log_b x$, express y in function of $\log_a$

Solution

$$y = \log_b x = \frac{\ln x}{\ln b}$$

$$= \frac{\ln x}{\ln b} \cdot \frac{\ln a}{\ln a}$$

$$= \frac{\ln x}{\ln a} \cdot \frac{\ln a}{\ln b}$$

$$=\frac{\ln x}{\ln a}\cdot\frac{1}{\frac{\ln b}{\ln a}}$$

$$=(\log_a x)\frac{1}{\log_a b}$$

$$=\frac{\log_a x}{\log_a b}$$

Thus, $\log_b x=\dfrac{\log_a x}{\log_a b}$ (This relation is used to change logarithm from one base to another)

Example 2

Change $\log_4(x+1)$ to base 2

Solution

$$\log_4(x+1)=\frac{\log_2(x+1)}{\log_2 4}$$

$$=\frac{\log_2(x+1)}{\log_2 2^2}$$

$$=\frac{\log_2(x+1)}{2\log_2 2}$$

$$=\frac{1}{2}\log_2(x+1)$$

Example 3

Given that $\ln 2=0.693; \ln 3=1.098$ and $\ln 5=1.609$. Calculate:

$a)\ln 6$ $b)\ln 10$ $c)\ln 0.045$

Solution

$a)\ln 6=\ln 2\times 3=\ln 2+\ln 3=0.693+1.098=1.791$

$b)\ln 10=\ln 2\times 5=\ln 2+\ln 5=0.693+1.609=2.302$

$c)\ln 0.045=\ln 3^2\times 5\times 10^{-3}=2\ln 3+\ln 5-3\ln 10$

$$=2\ln 3+\ln 5-3(\ln 2+\ln 5)=2(1.098)+1.609-3(2.302)$$

$$=3.805-6.906=-3.101$$

Application activity 4.1.2

1. Given that $\ln 2=0.693; \ln 3=1.098$ and $\ln 5=1.609$. Calculate:

$a)\ln 9 \quad b)\ln 0.3 \quad c)\ln 9 \quad d)\ln 20 \quad e)\ln 3^5 \quad f)\ln 15 \quad g)\ln 0.15 \quad h)\ln 75$

2. Calculate

$a)\log_2\left(5+\sqrt{21}\right)-2\log_2 4+\log_2\left(5-\sqrt{21}\right)$

$b)\log_3\left(17-\sqrt{19}\right)+\log_3\left(17+\sqrt{19}\right)-\log_3 27+\log_3 0.1$

3. Show that

$i)\ln\dfrac{a^3c}{b}=3\ln a-\ln b+\ln c \quad ii)\ln\sqrt[5]{x^2y^{-5}z^{10}}=\dfrac{2}{5}\ln x-\ln y+2\ln z$

4. Use the properties of logarithms to write the expression as a sum, difference, or multiple of logarithms.

5.

$\imath)\ln\dfrac{2}{3} \quad b)\ln\dfrac{xy}{z} \quad c)\ln\sqrt{x^2+1} \quad d)\ln\dfrac{3x(x+1)}{(2x+1)^2} \quad e)\ln\dfrac{2x}{\sqrt{x^2-1}} \quad f)\ln\left(x\sqrt[3]{x^2+1}\right)$

6. Write the expression as the logarithm of a single quantity.

$a)\ln(x-2)-\ln(x+2)$

$b)\ln(2x+1)+\ln(2x-1)$

$c)3\left[\ln x+\ln(x+3)-\ln(x+4)\right]$

$d)\dfrac{1}{3}\left[2\ln(x+3)+\ln x-\ln\left(x^2-1\right)\right]$

7. Express each of the following as single logarithm

$a)\log_3 x+\log_3 y-\log_3 y^3$

$b)\log(x+1)-\log\left(x^2-1\right)$

$c)5\log_2 x+3\log_2 2x+3\log_2 x^2$

$d)\log\left(a^3+b^3\right)-\log(a+b)$

$e)\dfrac{\log 5-4\log 3+3\log 9+\log 2}{\log 4-\log 2}$

4.1.3. Logarithmic equations

Activity 4.1.3

Let three functions be defined by

$f(x)=\ln x;\quad h(x)=\ln(x+2);\quad g(x)=\ln(x^2-5x+6)$

For which value(s) of x, each function is defined.

2. Use the properties for logarithm to determine the value of x in the following expressions:

a) $\log x=2$ b) $\ln x=\ln 10$

c) $\ln x=3$ d) $\log(100x)=2+\log 4$

Content summary

Logarithmic equation in $\mathbb{R}$ is the equation containing the unknown within the logarithmic expression.

To solve logarithmic equations the steps below are proceeded:

- Set existance conditions for solution(s) of equation.
- Express logarithms in the same base
- Use logarithmic properties to obtain:

$\log_a u(x)=\log_a v(x)\Leftrightarrow u(x)=v(x)$; where $u(x)$ and $v(x)$ are the functions in x.

- Make sure that the value(s) of unknown verifies (y) the conditions above.

Noted

- $y=\ln x=\log_e x \Leftrightarrow e^y=x$ (as inverse)
- The equation $\ln x=1$ has a unique solution; the rational number $2.71828182845904523536...$ and this number is represented by letter e.

Thus, $e=2.71828182845904523536...$

Hence, if $\ln x=1$ then $x=e$.

The properties logarithms can be used to solve logarithmic equations, as shown in the next examples.

Example 1

Solve each equation

$a)\ \ln x - \ln 5 = 0 \qquad b)\ 3 + 2\ln x = 7 \qquad c)\ \ \ln 2x + \ln(x+2) = \ln 6$

Solution

$a)\quad \ln x - \ln 5 = 0$

$Then;\ \ln x = \ln 5 \Leftrightarrow x = 5$

- $S = \{5\}$

$b)\quad 3 + 4\ln x = 7$

$condition\ of\ validity : x > 0$

Solving:

$4\ln x = 7 - 3 \Leftrightarrow \ln x = 1 \quad \Leftrightarrow \ln x = \ln e \Leftrightarrow x = e$

$S = \{e\}$

$c)\quad \ln 2x + \ln(x+2) = \ln 6$

$Conditions\ of\ validity : 2x > 0\ and\ x + 2 > 0$

$$\Leftrightarrow x > 0\ and\ x > -2$$

$$Domain\ of\ validity : x \in \left]-2, +\infty\right[\cap \left]0, +\infty\right[\Leftrightarrow x \in \left]0, +\infty\right[$$

Solving:

$$\ln 2x + \ln(x+2) = \ln 6 \Leftrightarrow \ln\lfloor 2x(x+2)\rfloor = \ln 6$$

$$\Leftrightarrow 2x(x+2) = 6$$

$$\Leftrightarrow 2x^2 + 4x - 6 = 0$$

$$\Leftrightarrow x^2 + 2x - 3 = 0$$

$$\Leftrightarrow (x-1)(x+3) = 0$$

$$\Leftrightarrow x = 1\ or\ x = -3$$

As $x \in \left]0, +\infty\right[;\ then\ S = \{1\}$

Example 2

Solve each equation

$a)\log_3(x+1)=\log_3 2$ $b)\log_{x-2}3=1$ $c)\log_2(x+14)+\log_2(x+2)=6$

Solution

$a)\log_3(x+1)=\log_3 2$

Condition

$x+1>0 \Leftrightarrow x>-1$

Then, $\log_3(x+1)=\log_3 2 \quad \Leftrightarrow x+1=2 \quad \Leftrightarrow x=1 \qquad \therefore S=\{1\}$

$b)\log_{x-2}3=1$

Condition

$x-2>0 \Leftrightarrow x>2$ and $x-2\neq 1 \Leftrightarrow x\neq 3$ this means $x\in]2,3[\cup]3,+\infty[$

Then, $\log_{x-2}3=1 \Leftrightarrow \log_{x-2}3=log_{x-2}(x-2)$

$$\Leftrightarrow 3=x-2$$
$$\Leftrightarrow x=5$$
$$\therefore S=\{5\}$$

$c)\log_2(x+14)+\log_2(x+2)=6$

Condition

$x+14>0$ and $x+2>0 \;\Leftrightarrow\; x>-14$ and $x>-2$

$$\Leftrightarrow x\in]-2,+\infty[$$

$$\log_2(x+14)+\log_2(x+2)=6 \Leftrightarrow \log_2(x+14)(x+2)=6\log_2 2$$

$$\Leftrightarrow \log_2(x+14)(x+2)=\log_2 2^6$$

$$\Leftrightarrow (x+14)(x+2)=64$$

$$\Leftrightarrow x^2+16x+28-64=0 \Leftrightarrow x^2+16x-36=0$$

$$\Leftrightarrow (x+18)(x-2)=0 \ \Leftrightarrow x=2 \text{ or } \ x=-18$$

$$\therefore S=\{2\}$$

Application activity 4.1.3

1. Solve each equation

$a)\ln x=0$ $\quad d)\ln\left(x^2-1\right)=\ln(4x-1)-2\ln 2$

$b)\ln x+\ln 4=0$ $\quad e)\ln 2x=\ln 2.4$

$c)2\ln x=\ln 36$ $\quad f)2\ln 4x=7$

2. Solve in the following equations in the set of real numbers:

$a)\ \ln^2 x=3-2\ln x$

$b)\begin{cases}2\ln x+3\ln y=-2\\3\ln x+5\ln y=-4\end{cases}$ $\quad c)\begin{cases}\ln(xy)=7\\ \ln\dfrac{x}{y}=1\end{cases}$

3. Solve each equation

$a)\log(x+2)=2$ $\quad c)\log\left(35-x^3\right)=3\log(5-x)$

$b)\log x+\log\left(x^2+2x-1\right)-\log 2=0$ $\quad d)\log(1-x)=-1$

$e)\log(3x-2)+\log(3x-1)=\log(4x-3)^2$

$f)2\log(2x-1)-\log\left(2x+3x^2\right)=\log(3x-7)-\log x$

$g)\log x=\log\left(5-6\sqrt{3}\right)-18\log\left(\sqrt{3}-\sqrt{2}\right)+2\log\left(4-3\sqrt{3}\right)-17\log\left(\sqrt{3+\sqrt{2}}\right)$

$h)\log_a(x+1)+\log_a(x+2)=\log_a 20$

4. Solve in $\mathbb{R}^2$, the systems below

$a)\begin{cases}x+y=9\\ \log x+\log y=\log 14\end{cases}$ $\quad c)\begin{cases}x-y=-8\\ \log_2 x-\log_2 y=\log_2\dfrac{3}{7}\end{cases}$

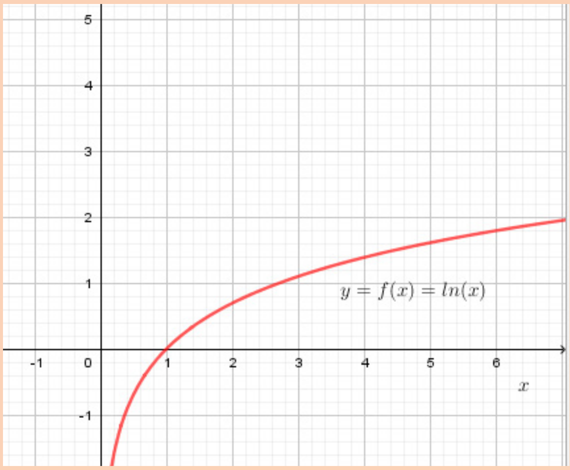
$b)\begin{cases}x^2+y^2=221\\ \log_5 x+\log_5 y=\log_5 110\end{cases}$ $\quad d)\begin{cases}x^3+y^3=35\\ \log_9 x-\log_9 y=\log_9 6\end{cases}$

4.1.4 Limit of logarithmic functions

4.1.4.1 Limit of logarithmic functions with base e

Activity 4.1.4.1

The graph below represents natural logarithmic function $f\left(x\right)=\ln x$



Considering the form of this graph and the logarithm of the following numbers in the table below:

x	0.5	0.001	0.001	0.0001	2	100	1001	10000
$\ln x$								

1) Discuss the values of $\ln x$ when x takes values closer to 0 from the right and deduce $\lim_{x\to 0^+}\ln x$.

2) Discuss the values of $\ln x$ when x take greater values and conclude about the
$\lim_{x\to +\infty}\ln x$.

3) Explain why it is senseless to discuss $\lim_{x\to 0^-}\ln x$.

CONTENT SUMMARY

The limit $\lim_{x\to 0^+}\ln x=-\infty$ shows that the line OY with equation $x=0$ is the vertical asymptote. This means that as the independent variable x takes values approaching 0 from the right, the graph of the function approaches the line of equation $x=0$ without intercepting. In other words, the dependent variable y takes "bigger and bigger" negative values.

Then, $\lim\limits_{x\to+\infty} \ln x = +\infty$, which implies that there is no horizontal asymptote.

The $\lim\limits_{x\to0^-} \ln x$ does not exist because values closer to 0 from the left are not included in the domain of the given function.

Example

Determine each of the following limit

a) $\lim\limits_{x\to e} \ln x$ b) $\lim\limits_{x\to2} (1-\ln x)$

Solution

a) $\lim\limits_{x\to e} \ln x = 1$

b) $\lim\limits_{x\to2} (1-\ln x) = 1-\ln 2$

Application activity 4.1.4.1

I. Evaluate the following limits

1) $\lim\limits_{x\to0^+} \frac{1+2\ln x}{x}$ 2) $\lim\limits_{x\to+\infty} \frac{1+2\ln x}{x}$

3) $\lim\limits_{x\to-\infty} \ln(x^2-4x+1)$ 4) $\lim\limits_{x\to+\infty} \ln(x^2-4x+1)$

II. Evaluate the following limits

) $\lim\limits_{x\to+\infty} \ln(7x^3-x^2+1)$ 2) $\lim\limits_{x\to1^+} \left(\ln\frac{1}{x-1}\right)$ 3) $\lim\limits_{a\to4^+} \ln\frac{a}{\sqrt{a-4}}$

III. Observe the graph of the function $p(x)=\frac{\ln x}{x}$ and deduce $\lim\limits_{x\to+\infty} \frac{\ln x}{x}$, $\lim\limits_{x\to0^+} \frac{\ln x}{x}$, $\lim\limits_{x\to1} \frac{\ln x}{x}$. Calculate $\lim\limits_{x\to\frac{1}{5}} \left(\frac{\ln x}{x}\right)$

4.1.4.2 Limit of logarithmic functions with any base

Activity 4.1.4.2

Let us consider the logarithmic function $f:\mathbb{R}^+ \to \mathbb{R}, y = f(x) = \log_2(x)$

Complete the following table:

$x = x_0$	$y = \log_2 x$	$\lim\limits_{x\to x_0} \log_2 x$
$\frac{1}{4}$		
$\frac{1}{2}$		
1		
2		
4		

CONTENT SUMMARY

In general the limit of any logarithmic function can be determined in the same way as the limit of the natural function. If you feel more comfortable with the natural logarithmic function, use the change base formula:

$$f(x)=\log_a u(x)=\frac{\ln(u(x))}{\ln a} \text{ provided } a>0, a\neq 1$$

Example 1

Determine each of the following limit

$$\lim_{x\to+\infty}\log_3\left(\frac{x-4}{2x+6}\right)$$

Solution

$$\lim_{x\to+\infty}\log_3\left(\frac{x-4}{2x+6}\right)=\log_3\frac{1}{2}=-\log_3 2$$

$$\text{Since } \lim_{x\to\infty}\frac{x-4}{2x+6}=\frac{1}{2}$$

Alternatively, using natural logarithmic function, we have

$$\lim_{x\to+\infty}\log_3\left(\frac{x-4}{2x+6}\right)=\lim_{x\to+\infty}\frac{\ln\left(\frac{x-4}{2x+6}\right)}{\ln 3}=\frac{1}{\ln 3}\lim_{x\to+\infty}\ln\left(\frac{x-4}{2x+6}\right)=\frac{1}{\ln 3}\times\ln\frac{1}{2}=-\frac{\ln 2}{\ln 3}=-\log_3 2=\log_3\frac{1}{2}$$

Example 2

Evaluate $\lim_{x\to 0}\frac{1}{x}\log_a(1+x)$

Solution

$$\lim_{x\to 0}\frac{1}{x}\log_a(1+x)=\lim_{x\to 0}\log_a(1+x)^{\frac{1}{x}}$$

$$=\log_a\left[\lim_{x\to 0}(1+x)^{\frac{1}{x}}\right]$$

We saw that $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x=e$

Let $y=\frac{1}{x}\Rightarrow x=\frac{1}{y}$. If $x\to\infty$, $y\to 0$

$$\lim_{x\to 0}(1+x)^{\frac{1}{x}}=\lim_{x\to+\infty}\left(1+\frac{1}{y}\right)^y=e$$

Then, $\log_a\left[\lim_{x\to 0}(1+x)^{\frac{1}{x}}\right]=\log_a e$

Therefore, $\lim_{x\to 0}\frac{1}{x}\log_a(1+x)=\log_a e$

Application activity 4.1.4.2

1. Evaluate the following limits

$a)\ \lim_{x\to 2^-}\log_5\left(x^2-5x+6\right)$ $\qquad b)\ \lim_{x\to +\infty}\frac{2+4\log x}{x}$

2. Evaluate the following limits

a) $\lim_{x\to 0^+}\log_2\frac{1}{x}$ $\qquad$ b) $\lim_{x\to -2^-}\log_2\frac{x+1}{x+2}$

c) $\lim_{x\to -1^+}\log_2\frac{x+1}{x+2}$ $\qquad$ d) $\lim_{x\to -2^+}\log_3\frac{1}{x^2-4}$

4.1.4.3 Asymptotes of the graph of logarithmic functions (including base e and any base)

Activity 4.1.4.3

Let $f(x)=\log_a x=\frac{\ln x}{\ln a}$

1) Evaluate limits at the boundaries of the domain of $f(x)$ for $a>1$. Hence deduce the asymptotes, if any.

2) Evaluate limits at the boundaries of the domain of $f(x)$ for $0<a<1$. Hence deduce the asymptotes, if any.

CONTENT SUMMARY

The graph of the logarithmic function $f:\mathbb{R}^+\to\mathbb{R}, f(x)=\log_a(x),\ a>1$ has the following characteristics:

- The domain is $]0,+\infty[$ and $f(x)$ is continuous on this interval.
- The range is $\mathbb{R}$
- The graph intersects the $x-axis$ at $(1,0)$
- As $x \to 0^+, y \to -\infty$, so the line of equation $x=0$ (the y- axis) is an asymptote to the curve.
- As x increases, the graph rises more steeply for $x \in [0,1]$ and is flatter for $x \in [1,+\infty[$
- The logarithmic function is increasing and takes its values (range) from negative infinity to positive infinity.

Example

Let us consider the logarithmic function $y = \log_2(x-3)$

a) What is the equation of the asymptote line?

b) State the domain and range

c) Find the $x-$ intercept.

d) Determine another point through which the graph passes

e) Sketch the graph

Solution

a) The basic graph of $y = \log_2 x$ has been translated 3 units to the right, so the line $L \equiv x = 3$

is the **vertical asymptote**.

b) The function $y = \log_2(x-3)$ is defined for $x-3>0$

So, the domain is $]3,+\infty[$ .The range is $\mathbb{R}$

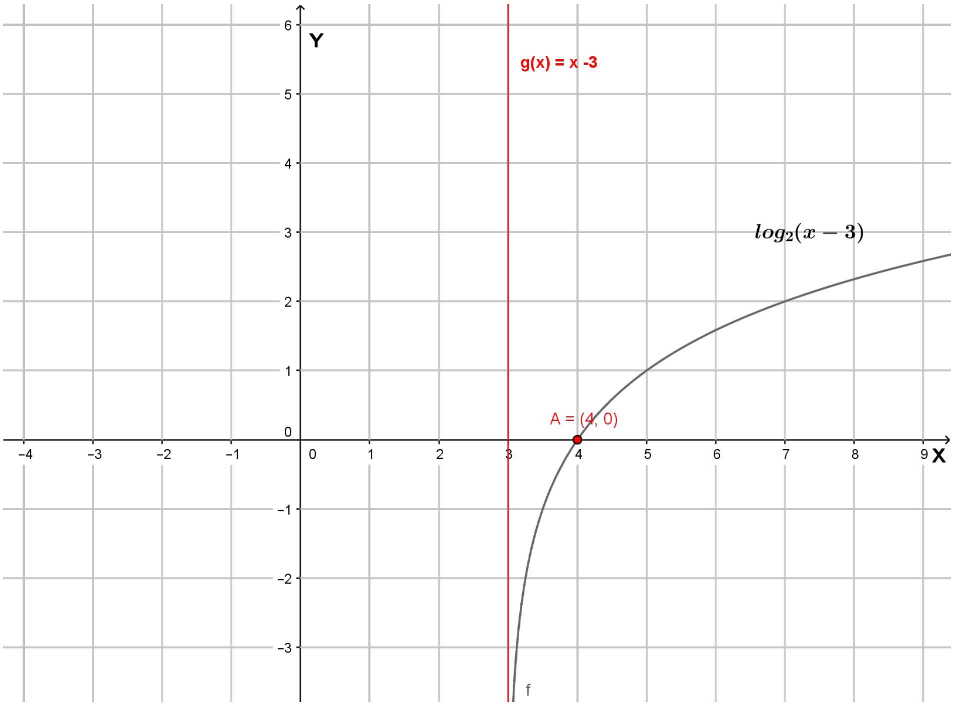
c) The intercept is (4, 0) since $\log_2(x-3) = 0 \Leftrightarrow x = 4$

d) Another point through which the graph passes can be found by allocating an arbitrary value to x in the domain then compute y.

For example, when $x = 4, y = \log_2(4-2) = \log_2 2 = 1$ which gives the point $(4,1)$.

Note that the graph does not intercept *y*-axis because the value 0 for *x* does not belong to the domain of the function.

The graph of $y = f(x) = \log_2(x-3)$



Application activity 4.1.4.3

Evaluate the following limits and state the asymptotes if any.

1) $\lim\limits_{x \to 0^+} \log_2 \frac{1}{x}$

2) $\lim\limits_{x \to -2^-} \log_2 \frac{x+1}{x+2}$

3) $\lim\limits_{x \to -1^+} \log_2 \frac{x+1}{x+2}$

4) $\lim\limits_{x \to -2^+} \log_3 \frac{1}{x^2-4}$

4.1.5 Derivative of logarithmic function

4.1.5.1 Derivative of logarithmic functions of base e and of any base *a*

Activity 4.1.5.11

1. Let $f(x) = \ln x$

   a) Find $f(x+h)$ and $f(2+h)$

   b) Complete the following table

h	$\frac{\ln(2+h)-\ln 2}{h}$
-0.1	
-0.001	
-0.00001	
0.1	
0.001	
0.00001	

2. Let $f(x) = \log_2 x$ and knowing that $\log_2 x = \frac{\ln x}{\ln 2}$. Find the derivative of $f(x)$

a) If $u = x^2$ is another differentiable function in x, use rule for differentiating composite functions to find the derivative of $g(x) = \log_2 u$.

Hint: $(f \circ g)' = f'(g)g'$

From the results found in the above table approximate the value of

$$f'(2) = \lim_{h \to 0} \frac{\ln(2+h) - \ln 2}{h}$$

And deduce the expression of $f'(\mathrm{x})$.

Based on your existing knowledge, provide any interpretation of the number $f'(2)$.

CONTENT SUMMARY

The definition of derivative shows that if $y = \ln x$,

$$y' = \lim_{h\to 0}\frac{\ln(x+h)-\ln x}{h} = \lim_{h\to 0}\ln\left(\frac{x+h}{x}\right)^{\frac{1}{h}}$$

$$= \lim_{h\to 0}\ln\left(1+\frac{h}{x}\right)^{\frac{1}{h}} = \ln \lim_{h\to 0}\left(1+\frac{h}{x}\right)^{\frac{1}{h}} = \ln e^{\frac{1}{x}} = \frac{1}{x}$$

If $f(x) = \log_a x \Leftrightarrow f(x) = \frac{\ln x}{\ln a}$ then $f'(x) = \frac{1}{x\ln a}$

Thus, $(\log_a x)' = \frac{1}{x\ln a}$

Also, if u is another differentiable function of x, then

$$(\log_a u)' = \frac{u'}{u\ln a}$$

Then, the natural logarithmic function $y = \ln x$ is differentiable on $]0,+\infty[$ and $\frac{d}{dx}(\ln x) = \frac{1}{x}$, $(x > 0)$.

Noted: Certain functions containing more complicated products and quotients, differentiation is often made easier if the logarithm of the function is taken before differentiating. This technique, called **'logarithmic differentiation'** is achieved with knowledge of

(i) The laws of logarithms,

(ii) The derivative of logarithmic functions, and

(iii) The differentiation of implicit functions.

Example 1

Differentiate each of the following functions with respect to

$$y = \log_2(5x^3)$$

Solution

$$\frac{d}{dx}\log_2(5x^3) = \frac{d}{dx}\left(\frac{\ln 5x^3}{\ln 2}\right) = \frac{1}{\ln 2}\frac{d}{dx}(\ln 5 + 3\ln x)$$

$$=\frac{1}{\ln 2}\left(\frac{d}{dx}\ln 5+\frac{d}{dx}(3\ln x)\right)=\frac{1}{\ln 2}\left[0+3\frac{d}{dx}\ln x\right]$$

$$=\frac{1}{\ln 2}\times 3\times\frac{1}{x}=\frac{3}{x\ln 2}$$

Example 2

Find the slope of the line tangent to the graph of $y=\log_2(3x+1)$ at $x=1$

Solution

To find the slope, we must evaluate $\frac{dy}{dx}$ at $x=1$

$$\frac{d}{dx}\log_2(3x+1)=\frac{d}{dx}\left(\frac{\ln(3x+1)}{\ln 2}\right)=\frac{1}{\ln 2}\frac{d}{dx}(\ln 3x+1)=\frac{3}{(3x+1)\ln 2}$$

By evaluating the derivative at $x=1,$ we see that the tangent line to the curve at the point $(1,\log_2 4)=(1,2)$ has the slope

$$\left.\frac{dy}{dx}\right|_{x=1}=\frac{3}{4\ln 2}=\frac{3}{\ln 16}.$$

Example 3

Find derivative of $\log_2\left(4x^2-3x\right)$

Solution

$$\left[\log_2\left(4x^2-3x\right)\right]'=\frac{\left(4x^2-3x\right)'}{\left(4x^2-3x\right)\ln 2}$$

$$=\frac{8x-3}{\left(4x^2-3x\right)\ln 2}$$

Example 4

Find derivative of $\log_a\left(\ln|\sin x|\right)$

Solution

Example5

Given that $|u|'=\frac{u}{|u|}\cdot u'$, where u is the function;

$$\left[\log_a\left(\ln|\sin x|\right)\right]' = \frac{\left(\ln|\sin x|\right)'}{\left(\ln|\sin x|\right)\ln a} = \frac{\cot x}{\left(\ln|\sin x|\right)\ln a}.$$

Example 5

Given that $y = \frac{(x+1)(x-2)^3}{(x-3)}$.

1) Take $\ln$ on both sides and applying laws of logarithms
2) Using derivative of logarithmic function, find the derivative of expression found in (1) and deduce the value of $\frac{dy}{dx}$.

Solution

$$y = \frac{(x+1)(x-2)^3}{x-3}$$

1) Taking ln on both sides yields $\ln y = \ln\frac{(x+1)(x-2)^3}{x-3}$

or $\ln y = \ln(x+1) + \ln(x-2)^3 - \ln(x-3)$

$$\Leftrightarrow \ln y = \ln(x+1) + 3\ln(x-2) - \ln(x-3)$$

2) Differentiating both sides gives

$$\frac{d(\ln y)}{dx} = \frac{d\left[\ln(x+1)\right]}{dx} + \frac{d\left[3\ln(x-2)\right]}{dx} - \frac{d\left[\ln(x-3)\right]}{dx}$$

$$\Leftrightarrow \frac{1}{y}\frac{dy}{dx} = \frac{1}{x+1} + \frac{3}{x-2} - \frac{1}{x-3}$$

The derivative of expression found in (1) is $\frac{1}{y}\frac{dy}{dx} = \frac{1}{x+1} + \frac{3}{x-2} - \frac{1}{x-3}$

Hence $\frac{dy}{dx} = y\left(\frac{1}{x+1} + \frac{3}{x-2} - \frac{1}{x-3}\right)$

Or $\frac{dy}{dx} = \frac{(x+1)(x-2)^3}{x-3}\left(\frac{1}{x+1} + \frac{3}{x-2} - \frac{1}{x-3}\right)$

Example 6

Differentiate each of the following functions with respect to x

a) $f(x) = \ln(x^3 + 3x - 4)$ b) $f(x) = x^2 \ln x$

c) $f(x)=\sin x \ln x$

Solution

a) $\dfrac{d}{dx}\ln\left(x^3+3x-4\right)=\dfrac{1}{x^3+3x-4}\left(x^3+3x-4\right)'=\dfrac{3x^2+3}{x^3+3x-4}$

b) $\dfrac{d}{dx}\left(x^2\ln x\right)=\ln x\dfrac{d}{dx}x^2+x^2\dfrac{d}{dx}\ln x=2x\ln x+x^2\left(\dfrac{1}{x}\right)=2x\ln x+x$

c) $\dfrac{d}{dx}\left(\sin x\ln x\right)=\ln x\dfrac{d}{dx}\sin x+\sin x\dfrac{d}{dx}\ln x=(\cos x)\ln x+\dfrac{1}{x}\sin x=(\cos x)\ln x+\dfrac{\sin x}{x}$

Example 7

Find the derivative of $f(x)=\ln\sqrt{x^2+1}$

Solution

$$f'(x)=\left(\ln\sqrt{x^2+1}\right)'$$

$$=\frac{\left(\sqrt{x^2+1}\right)'}{\sqrt{x^2+1}}$$

$$=\frac{\dfrac{2x}{2\sqrt{x^2+1}}}{\sqrt{x^2+1}}$$

$$=\frac{x}{\left(\sqrt{x^2+1}\right)^2}$$

$$=\frac{x}{x^2+1}$$

Then $f'(x)=\dfrac{x}{x^2+1}$

Example 8

Differentiate the function $g(x)=\ln\left(x+\cos x\right)$

Solution

$$g'(x)=[\ln(x+\cos x)]'$$

$$=\frac{(x+\cos x)'}{x+\cos x}$$

$$=\frac{1-\sin x}{x+\cos x}$$

Then, $g'(x)=\dfrac{1-\sin x}{x+\cos x}$

Application activity 4.1.5.1

I. Differentiate $y=\ln\sqrt{\dfrac{1+x}{1-x}}$ with respect to x.

II. Find derivative of the following functions

1) $f(x)=(\ln x)^2$ 2) $g(x)=\ln(\tan x)$ 3) $h(x)=\ln\sqrt{x^2+1}$

4) $k(x)=\ln\dfrac{1-x}{1+x}$ 5) $f(x)=\dfrac{\ln(\sin x)}{x}$ 6) $g(x)=\ln x+\ln(\cos x)$

III. Differentiate each of the following functions

1) $f(x)=\log(x^2+2x+1)$ 2) $g(x)=\log_2\dfrac{x+1}{x-5}$

IV. Use logarithmic differentiation to differentiate each of the following functions

1) $y=\dfrac{(x-2)(x+1)}{(x-1)(x+3)}$ 2) $y=\dfrac{(2x-1)\sqrt{x+2}}{(x-3)\sqrt{(x+1)^3}}$

3) $y=3\theta\sin\theta\cos\theta$ 4) $y=\dfrac{x^3\ln 2x}{e^x\sin x}$ 5) $y=\dfrac{2x^4\tan x}{e^{2x}\tan x}$

4.1.5.2 Variation and graphical representation of logarithmic functions (for base e and for any base)

Activity 4.1.5.2

Given two functions $f(x)=\ln x$ and $g(x)=\log_{10} x$,

1. Compare $f(2)$ and $f(10)$, $g(2)$ and $g(10)$ and deduce whether those functions are increasing or decreasing on $[2,10]$.
2. Use the tables of signs for $f'(x)$ and $g'(x)$ to establish the intervals and the variation of those functions.
3. Which function f or g is increasing or decreasing faster than another on [2,10]

CONTENT SUMMARY

The logarithmic function $f(x)=\log_a x$, $a>0, a\neq 1$ varies in the following way:

a) If $x>0$, then ,

$f'(x)=\dfrac{1}{x\ln a}$. The sign of $f'(x)$ depends therefore on the value of the base a.

If $a>1$, $\ln a>0$ then $f'(x)$ is always positive.

Thus $f(x)=\log_a x$ is strictly increasing on $\mathbb{R}_0^+$

Variation table for $y=f(x)=\log_a x$ **for** $a>1$

x	0		1		a		$+\infty$
y'		$+$	$\frac{1}{\ln a}$	$+$	$\frac{1}{a\ln a}$	$+$	
y	$-\infty$	$\nearrow$	0	$\nearrow$	1	$\nearrow$	$+\infty$

If $0<a<1$, $\ln a<0$. Therefore $f'(x)$ is always negative.

Thus $f(x)=\log_a x$ is strictly decreasing on $\mathbb{R}_0^+$. This implies the absence of extrema (maxima or minima) values.

Table of variation for $y=f(x)=\log_a x$ for $0<a<1$

x	0		a		1		$+\infty$
y'		-	$\frac{1}{a\ln a}$	-	$\frac{1}{\ln a}$	-	
y	$+\infty$	$\searrow$	1	$\searrow$	0	$\searrow$	$-\infty$

Example 1

For the following function, find relative asymptotes (if any), study the variation and sketch the curve: $f(x)=\log_2 \sqrt{x+1}$

Solution

Asymptotes:

Domain:

$$x+1>0 \Rightarrow x>-1$$

$$Domf =]-1,+\infty[$$

$$\lim_{x\to -1} f(x) = \lim_{x\to -1} \log_2 \sqrt{x+1}$$
$$= -\infty$$

$x=-1$ is a vertical asymptote.

$$\lim_{x\to +\infty} f(x) = \lim_{x\to +\infty} \log_2 \sqrt{x+1}$$
$$= +\infty$$

No horizontal asymptote

$$\lim_{x\to+\infty}\frac{f(x)}{x}=\lim_{x\to+\infty}\frac{\log_2\sqrt{x+1}}{x}$$

$$=\frac{\infty}{\infty}\quad IF$$

$$\lim_{x\to+\infty}\frac{\log_2\sqrt{x+1}}{x}=\lim_{x\to+\infty}\frac{1}{2(x+1)\ln 2}\qquad[\text{Hôpital rule}]$$

$$=0$$

No oblique asymptote.

Variation:

$$f'(x)=\left(\log_2\sqrt{x+1}\right)'$$

$$=\frac{\left(\sqrt{x+1}\right)'}{\sqrt{x+1}\ln 2}$$

$$=\frac{\frac{1}{2\sqrt{x+1}}}{\sqrt{x+1}\ln 2}$$

$$=\frac{1}{2\sqrt{x+1}\sqrt{x+1}\ln 2}$$

$$=\frac{1}{2(x+1)\ln 2}$$

Since $\forall\in Domf$, $x+1>0$ and $\ln 2>0$ then $\forall x\in Domf$, $f'(x)>0$

Hence, $\forall x\in Domf$, $f(x)$ increases.

Curve:

Intersection with axes:

Intersection with $x-axis$:

$$f(x)=0\Leftrightarrow\log_2\sqrt{x+1}=0$$

$$\log_2 \sqrt{x+1} = \log_2 1$$

$$\sqrt{x+1} = 1 \Rightarrow x = 0$$

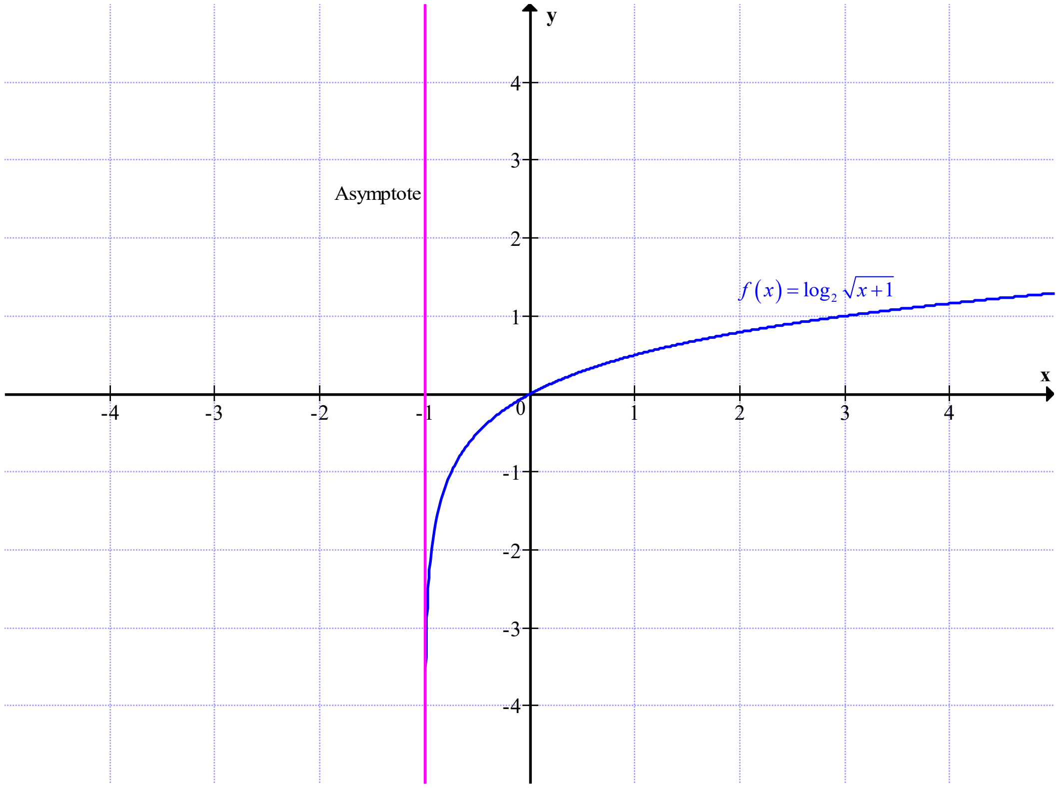
Thus, $f(x) \cap ox = \{(0,0)\}$

Intersection with $y-axis$:

$$f(0) = \log_2 \sqrt{0+1} = 0$$

Thus, $f(x) \cap oy = \{(0,0)\}$

Additional points



Example 2

Discuss variations of the logarithmic function $f(x) = x - \ln x$

Solution

$f(x) = x - \ln x$ is defined for all $x > 0$

$$f'(x)=\frac{d}{dx}(x-\ln x)=1-\frac{1}{x}$$

$$f'(x)=0 \Leftrightarrow 1-\frac{1}{x}=0$$
$$\Leftrightarrow \frac{1}{x}=1$$
$$\Leftrightarrow x=1$$

If $x=1,\ y=f(1)=1-\ln 1=1$, thus(1,1) is a point of the graph.

Then; $f''(x)=0 \Leftrightarrow 0=\frac{1}{x^2}$; $x\neq 0$ $f''(x)=\frac{d}{dx}\left(1-\frac{1}{x}\right)=\frac{1}{x^2}$.

It means that $f''(x)$ is positive.

Variation table of $y=f(x)=x-\ln x$.

x	0		1		$+\infty$
y'		-	0	+	
$f''(x)$	+	+	+	+	
y	$+\infty$	$\searrow$	1	$\nearrow$	$+\infty$

From the table, one can observe that the function is **decreasing** for values when x lies in]0,1] and **increasing** for x greater than 1. The point (1,1) is **minimum** or equivalently the function takes the minimum value equal for $x=1$. The minimum value that is equal to 1 is absolute.

There is **no inflection point,** and the graph **curves up/turns up.**

Example 3

For the following function, find relative asymptotes (if any), study the variation, concavity and sketch the curve: $f(x)=\frac{1+2\ln x}{x}$

Solution

Asymptotes:

First we need domain of definition:

Condition: $x > 0 \Rightarrow Domf = \mathbb{R}_0^+$

$\lim_{x \to 0^+} f(x) = \frac{-\infty}{0} = -\infty \Rightarrow x = 0$ is a vertical asymptote

$$\lim_{x \to +\infty} f(x) = \lim_{x \to +\infty} \frac{1 + 2\ln x}{x} = \frac{\infty}{\infty} \quad IF$$

$$= \lim_{x \to +\infty} \frac{\frac{2}{x}}{1} \qquad [\text{Hôpital rule}]$$

$$= 0$$

$\Rightarrow y = 0$ is horizontal asymptote.

Since there is horizontal asymptote for $x \to +\infty$, there is no oblique asymptote for $x \to +\infty$

Variation:

First derivative

$$f'(x) = \frac{\frac{2}{x}x - (1 + 2\ln x)}{x^2} \text{ with } x > 0$$

$$= \frac{1 - 2\ln x}{x^2}$$

Roots of first derivative

$$f'(x) = 0 \Leftrightarrow \frac{1 - 2\ln x}{x^2} = 0$$

$$\Rightarrow 1 - 2\ln x = 0$$

$$\Leftrightarrow 1 - \ln x^2 = 0$$

$$\Leftrightarrow \ln x^2 = 1$$

$$\Leftrightarrow x^2 = e$$

$$\Rightarrow x = \pm\sqrt{e}$$

As $x>0$, $x=-\sqrt{e}$ is to be rejected.

The root of $f'(x)$ is $x=\sqrt{e}$

x	0	$\sqrt{e}$	$+\infty$
$f'(x)$	$+$		$-$
$f(x)$	$-\infty \nearrow$	0	$\searrow +\infty$

For $x\in\left]0,\sqrt{e}\right[$, $f(x)$ increases while for $x\in\left]\sqrt{e},+\infty\right[$, $f(x)$ decreases

Concavity

$$f''(x)=\left[\frac{1-2\ln x}{x^2}\right]'$$

$$=\frac{2x(1-2\ln x)-x^2\left(-\frac{2}{x}\right)}{x^4}$$

$$=\frac{2-4\ln x+2}{x^3}$$

$$=\frac{4-4\ln x}{x^3}$$

Roots of second derivative

$$f''(x)=0\Leftrightarrow=\frac{4-4\ln x}{x^3}=0$$

$$\Rightarrow 4-4\ln x=0$$

$$\Leftrightarrow 4-\ln x^4=0$$

$$\Leftrightarrow \ln x^4=4$$

$$\Leftrightarrow x^4=e^4$$

$$\Rightarrow x=\pm e$$

As $x>0$, $x=-e$ is to be rejected.

The root of $f''(x)$ is $x=e$

Curve:

Intersection with axes of coordinates:

Intersection with $x-axis$

$$f(x)=0 \Leftrightarrow 1+2\ln x=0$$

$$\ln x=-\frac{1}{2} \Rightarrow e^{\ln x}=e^{-\frac{1}{2}} \Rightarrow x=-\frac{1}{2}.$$

This, $f(x)\cap \mathrm{ox}=\left\{\left(e^{-\frac{1}{2}},0\right)\right\}$

Intersection with $y-axis$

$$f(0)=\frac{1+2\ln 0}{0}\ \ impossible$$

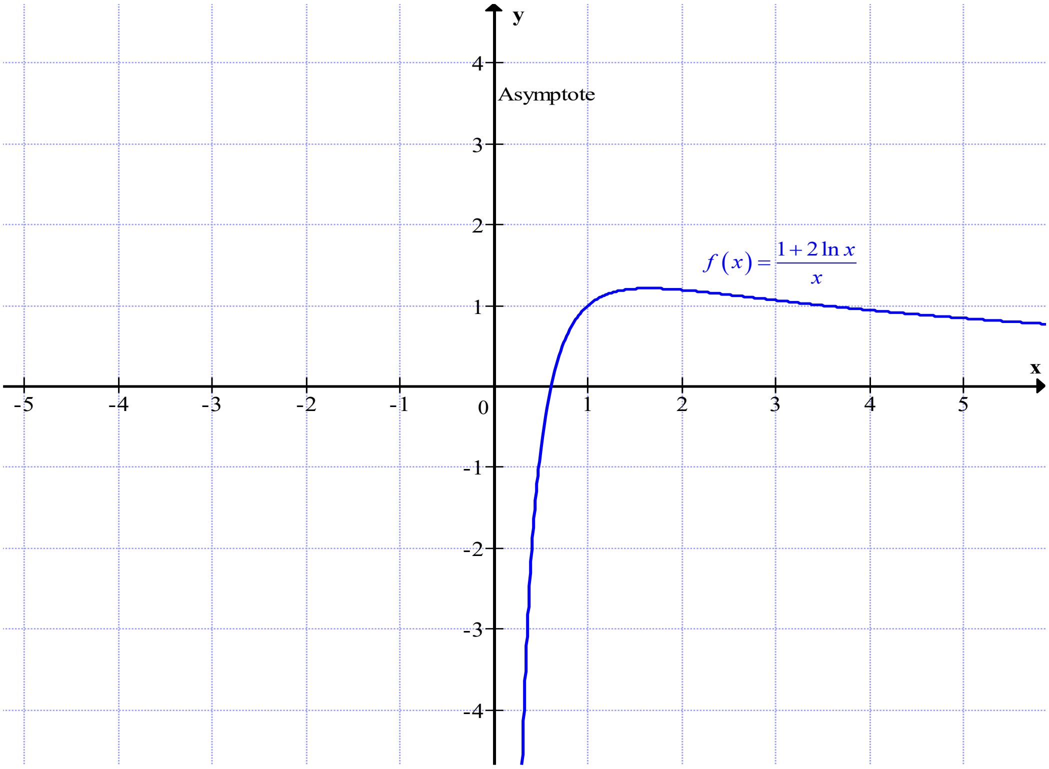
Thus, no intersection with $y-axis$

Additional points:

x	0.1	0.4	0.7	1.0	1.3	1.6	1.9	2.2	2.5	2.8
$f(x)$	-36.05	-2.08	0.40	1.0	1.17	1.21	1.20	1.17	1.13	1.09

x	3.1	3.4	3.7	4.0	4.3	4.6	4.9	5.2	5.5	5.8
$f(x)$	1.05	1.01	0.97	0.94	0.91	0.88	0.85	0.82	0.80	0.77

Sketch graph:



Application of differentiation: limits involving indeterminate forms

Example 4

Evaluate a) $\lim\limits_{x\to+\infty}\frac{\ln x}{x}=0$ b) $\lim\limits_{x\to0^+}\frac{\ln(1+x)}{x}$

Solution

$\lim\limits_{x\to+\infty}\frac{\ln x}{x}$ takes indeterminate form $\frac{\infty}{\infty}$;

$$\lim_{x\to+\infty}\frac{\ln x}{x}=\lim_{x\to+\infty}\frac{\frac{1}{x}}{1}=\lim_{x\to+\infty}\frac{1}{x}=0$$

$b)\ \lim\limits_{x\to0^+}\frac{\ln(1+x)}{x}$ (indeterminate form $\frac{0}{0}$)

Apply Hospital rule: $\lim\limits_{x\to0^+}\frac{\ln(1+x)}{x}=\lim\limits_{x\to0^+}\frac{1}{1+x}=1$

Application activity 4.1.5.2

1. Discuss variations of the function $f(x)=\dfrac{\ln(x-2)}{x-2}$

2.

a) Suppose a satellite has been shot upward with an initial velocity of $20\,m/s$. Then the equation $h(t)=100\ln(t+1)$ gives the height of the satellite in m after t $seconds$. The derivative of the function for the height of the satellite gives the rate of change of the height or the velocity of the satellite. Find the velocity function.

b) Find the velocity function after 2 $seconds$

c) Is the velocity increasing or decreasing?

3. For each of the following functions, find relative asymptotes (if any), *study the variation, concavity and sketch the curve*

a) $f(x)=\ln(x^2)$ b) $g(x)=\ln(x+1)$ c) $f(x)=\log_2(x+1)$

d) $g(x)=\log_3(2x-4)$ e) $h(x)=\log_{\frac{1}{2}}x^2$ f) $k(x)=\log_{\frac{1}{2}}\sqrt{x}$

c) $h(x)=\dfrac{\ln x}{x}$ d) $k(x)=\ln(x^2-3x+1)$

4.2 EXPONENTIAL FUNCTIONS

4.2.1 Definition, Domain and range of exponential functions

Activity 4.2.1

1. Let $f(x) = \ln x$ and $g(x) = e^x$ denotes the inverse function of $f(x)$.

   i. Complete the following table:

x	0	1				
$g(x) = f^{-1}(x)$					3	4

ii. Discuss and find out the set of all values of

2. Consider the function and complete the following table

x	-10	-1	0	1	10
$h(x) = 3^x$					

a) Discuss whether $\forall x \in \mathbb{R}, h(x) \in \mathbb{R}$ and deduce the domain of $h(x)$

b) Discuss whether $h(x)$ can be negative or not and deduce the range of $h(x)$.

CONTENT SUMMARY

Remember that for $a > 0, a \neq 1$ the logarithmic function is defined as

$\log : \mathbb{R}_0^+ \to \mathbb{R}$ *or* $x \to y = \log_a x$.

The inverse of logarithmic function is called **exponential function** and defined as $\exp_a : \mathbb{R} \to \mathbb{R}_0^+ : x \mapsto y = \exp_a x$ or for simplicity we write $\exp_a x = a^x$. Therefore $a^x = y$ if and only if $\log_a y = x$.

In the expression $a^x = y$, a is the **base**, x the **exponent** and y the **exponential of *x* in base *a***.

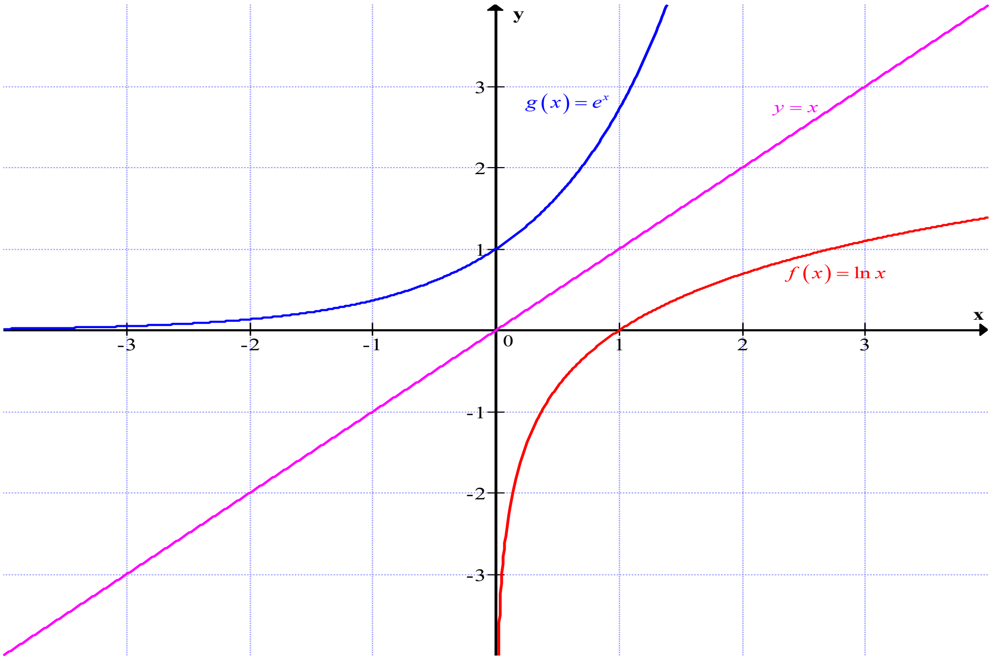
Similarly to logarithmic function, if the base "*a*" is the number "e", we have exponential function $y = e^x$ as the inverse of natural logarithm $y = \ln x$.

Generally, if $u(x)$ is a defined function of x, the function $f(x)=a^{u(x)}$ has the range $]0,+\infty[$ and its domain is the domain of $u(x)$.

The domain of definition of $y=e^x$ is $]-\infty,+\infty[$ and its range is $]0,+\infty[$.

Since e^x is the inverse of $\ln x$, the curve of $g(x)=e^x$ is the image of the curve of $f(x)=\ln x$ with respect to the first bisector, $y=x$. Then the coordinates of the points for $f(x)=\ln x$ are reversed to obtain the coordinates of the points for $g(x)=e^x$

The curve of $g(x)=e^x$ is as follows



Obviously, **the domain** of the exponential function $y=f(x)=a^x$ is $]-\infty,+\infty[$ and **its range** is $]0,+\infty[$.

Example 1

Determine the domain and the range of the function

$f(x)=3^{\sqrt{2x}}$

Solution

Condition for the existence of $\sqrt{2x}$ in $\mathbb{R}$: $x\geq 0$.

Thus, $Domf=[0,+\infty[$ and the range is $[1,+\infty[$

Example 2

Find the domain and the range of $f\left(x\right)=2^{\ln x}$

Solution

Condition: $x>0$

Thus, $Domf=\left]0,+\infty\right[$ and the range is is $\left]0,+\infty\right[$.

Example 3

Find the domain and the range of $f\left(x\right)=3^{\frac{x+1}{x-2}}$

Solution

Condition: $x-2\neq 0\Rightarrow x\neq 2$

Thus, $Domf=\mathbb{R}\setminus\left\{2\right\}$ and the range is $]0,3[\cup]3,+\infty[$.

Example 4

Find the domain and the range of $f\left(x\right)=4^{\sqrt{x^2-4}}$

Solution

Condition: $x^2-4\geq 0\Rightarrow x\in\left]-\infty,-2\right]\cup\left[2,+\infty\right[$

Thus, $Domf=\left]-\infty,-2\right]\cup\left[2,+\infty\right[$ and the range is $[1,+\infty[$

Example5

Determine the domain and the range of each of the following functions:

1. $g\left(x\right)=e^{\frac{x+2}{x-3}}$

2. $h\left(x\right)=e^{\sqrt{x^2-4}}$

Solution

1.Condition for the existence of $\dfrac{x+2}{x-3}$ in $\mathbb{R}$: $x\neq 3$.

Therefore $Dom\ g=\mathbb{R}\setminus\left\{3\right\}=]-\infty,3[\cup]3,+\infty[$

and range is $]0,e[\cup]e,+\infty[$

2. Condition $x^2-4\geq 0\Rightarrow x\in\left]-\infty,-2\right]\cup\left[2,+\infty\right[$.

Thus, $Dom\ h=\left]-\infty,-2\right]\cup\left[2,+\infty\right[$ and range is $[1,+\infty[$.

Example 6

Find the domain and the range of $f(x)=e^{\sqrt{x}}$

Solution

Condition: $x\geq 0$

Thus, $Domf=[0,+\infty[$ and range is $[1,+\infty[$.

Example 7

Find the domain and the range of $g(x)=e^{\frac{x+1}{x-2}}$

Solution

Condition: $x-2\neq 0\Rightarrow x\neq 2$

Thus, $Domg=\mathbb{R}\setminus\{2\}$ and range is $]0,e[\cup]e,+\infty[$.

Example 8

Find the domain of $h(x)=e^{\sqrt{x^2-1}}$

Solution

Condition: $x^2-1\geq 0\Rightarrow x\in]-\infty,-1]\cup[1,+\infty[$

Thus, $Domh=]-\infty,-1]\cup[1,+\infty[$ and range is $[1,+\infty[$.

Application activity 4.2.1

I. Discuss and determine the domain and range of the following functions

ı) $f(x)=5e^{2x}$ b) $f(x)=e^{2x+3}$ c) $f(x)=e^{\sqrt{2x-1}}$ d) $f(x)=e^{\frac{x-3}{x^2-}}$

e) $f(x)=e^{\frac{2x}{x^2-7x+10}}$ f) $g(x)=e^{\frac{4x+7}{x^2-x+10}}$

g) $h(x)=\dfrac{3x+1}{1-e^{\sqrt{3x}}}$ h) $k(x)=\dfrac{e^x+1}{\log\left(e^{\sqrt{x-4}}\right)}$

II. Discuss and determine the domain and range of the following functions

a) $h(x)=2^{\ln x}$ b) $f(x)=3^{\frac{x+1}{x-2}}$

III. Find the domain of the following functions

1) $f(x)=3^{\frac{4}{x^2+7x+10}}$ 2) $g(x)=2^{\frac{3x+1}{x^2-x+10}}$

3) $h(x)=4^{\sqrt{\frac{x+1}{x-3}}}$ 4) $k(x)=3^{\log(x^2+5x+6)}$

4.2.2 Properties and operations on exponential functions (for base e and for any base)

Activity 4.2.2

1. Evaluate each expression for $-3 \le x < 3$

$a) e^{3x}$ $b) e^{x+3}$ $c) e^{-x}$

2. Use calculator to obtain approximations of

$a) 2^{-0.6}$ $b) \pi^{0.75}$ $c) (1.56)^{\sqrt{2}}$

CONTENT SUMMARY

When working with exponential functions, the properties of exponents, shown below, are useful.

$\forall x \in]0,+\infty[, y \in]-\infty,+\infty[: y = \ln x \Leftrightarrow x = e^y$.

- $\forall x \in \mathbb{R}, \ln e^{x} = x$ and $\forall y \in]0,+\infty[, e^{\ln y} = y$
- $e^0 = 1$
- $e^1 = e$

Properties of Exponents for base e

a) $e^x e^y = e^{x+y}$

b) $\left(e^a\right)^n = e^{na}$

c) $\frac{1}{e^a} = e^{-a}$

d) $\frac{e^a}{e^b} = e^{a-b}$

e) $e^{\ln x} = x$

f) $e^{-\ln x} = \dfrac{1}{x}$

g) $\ln e^{-x} = -x$

Properties of Exponents for any base

Let a and b be positive numbers. Then

$1.a^0 = 1$

$2.a^x a^y = a^{x+y}$

$3.\left(a^x\right)^y = a^{xy}$

$4.a^{-x} = \dfrac{1}{a^x}$

$8.\sqrt[n]{a^x} = a^{\frac{x}{n}}$

$9.\left(\sqrt[n]{a^x}\right)^y = a^{\frac{xy}{n}}$

$10.\left(\sqrt[n]{a^m}\right)^{-x} = \dfrac{1}{a^{\frac{mx}{n}}}$

Example 1

Simplify each expression using the properties of exponents.

$a.(0.5)^0$ $\quad b.2^x 2^4$ $\quad c.\left(3^4\right)^m$

Solution

$a.(0.5)^0 = 1$ $\quad c.\left(3^4\right)^m = 3^{4m}$

Example 2

Evaluate

$a)e^{-\ln x} \quad b)\left(e^2\right)^{\ln x} \quad c)e^{1-\ln x}$

Solution

$a) \quad e^{-\ln x} = e^{\ln x^{-1}} = e^{\ln\frac{1}{x}} = \dfrac{1}{x}$ $\qquad e^{\ln x} = x$

$b)\left(e^2\right)^{\ln x} = e^{2\ln x} = e^{\ln x^2} = x^2$

$c)\, e^{1-\ln x} = e^1 e^{-\ln x} = e^1 e^{\ln x^{-1}} = e^1 e^{\ln\frac{1}{x}} = e^1 \times \dfrac{1}{x} = \dfrac{e}{x}$

Application activity 4.2.2

1. Use the properties of exponents to simplify the expression.

$a)\left(e^{\frac{3}{2}}\right)\left(\frac{1}{e}\right)^{\frac{3}{2}}$ $\qquad$ $b)\left[\left(e^{-1}\right)\left(e^{\frac{2}{3}}\right)\right]^{3}$

2. If $x=-5, y=3, z=\frac{1}{2}$, simplify each expression

$a) e^{z}$ $\qquad$ $c)\left(e^{x}e^{y}\right)^{z}$

$b) e^{\frac{3}{5}}\times e^{x}$ $\qquad$ $d)\left(e^{2}e^{z}\right)^{3}$

3. Use the properties of exponents to simplify the expression.

$a)\left(32^{\frac{3}{2}}\right)\left(\frac{1}{2}\right)^{\frac{3}{2}}$ $\qquad$ $b)\left[\left(8^{-1}\right)\left(8^{\frac{2}{3}}\right)\right]^{3}$

4. Evaluate each expression for $-3\le x<3$

$a)(0.005)^{x}$ $\qquad$ $b) 2^{x}3^{x}$ $\qquad$ $c)\left(\frac{1}{5}\right)^{x}$

5. If $x=-5, y=3, z=\frac{1}{2}$, simplify each expression

4.2.3 Exponential equations

Activity 4.2.3

1 For which value(s), each function $f(x)$ below can be defined. Explain.

$a) f(x)=e^{(x+2)}$

$b) f(x)=e^{x^{2}-5x+6}$

2. Detect the value of x , if $2^{1-x}=6$

CONTENT SUMMARY

Equations that involves powers as terms of their expressions are refered to as **exponentiol equations.** Such equations can some times be solved by approprietely applying the properties of exponents as shown in the next examples.

Some exponential equations are solved by introducing logarithms within expression.

Example 1

Solve each equation

$a) e^x = 5$ $\qquad$ $b) 10 + e^{0.1t} = 14$

Solution

$a) e^x = 5$

- $Domain\ of\ validity: x \in \mathbb{R}$

$\ln e^x = \ln 5$

$x \ln e = \ln 5; \quad where \quad \ln e = 1$

$x = \ln 5$

- $S = \{\ln 5\}$

$b) 10 + e^{0.1t} = 14$

- $Domain\ of\ validity: \text{t} \in \mathbb{R}$

$e^{0.1t} = 4$

$\ln e^{0.1t} = \ln 4$

$0.1t = \ln 4$

$t = 10 \ln 4$

- $S = \{10 \ln 4\}$

Example 2

Solve $e^{-x^2} = \left(e^x\right)^2 . \frac{1}{e^3}$

Solution

$e^{-x^2} = \left(e^x\right)^2 . \frac{1}{e^3}$

- $Domain\ of\ validity: x \in \mathbb{R}$

$e^{-x^2} = e^{2x}.e^{-3} = e^{2x-3}$

$e^{-x^2} = e^{2x-3}$

$-x^2 = 2x - 3$

$x^2 + 2x - 3 = 0$

The solution of this equation gives $x = -3 \text{ or } x = 1$

- The solution set is $S = \{-3, 1\}$

Example 3

Solve each equation

$a)2^{x-4}=8$ $\quad b)5+3^{t-4}=7$ $\quad c)2^{2y}+3\left(2^{y}\right)=4$ $\quad d)3\left(2^{4x}\right)-7\left(2^{2x}\right)+4=0$

Solution

$$\begin{aligned}&a)2^{x-4}=8\\&\Rightarrow 2^{x-4}=2^{3}\\&\Rightarrow x-4=3\\&\Rightarrow x=7\\&\therefore S=\{7\}\end{aligned}$$

$$\begin{aligned}&b)5+3^{t-4}=7\\&\Rightarrow 3^{t-4}=2\\&\Rightarrow \ln 3^{t-4}=\ln 2\\&\Rightarrow (t-4)\ln 3=\ln 2\\&\Rightarrow t=4+\frac{\ln 2}{\ln 3}\end{aligned}$$

$$c)2^{2y}+3\left(2^{y}\right)=4 \Leftrightarrow \left(2^{y}\right)^{2}+3\left(2^{y}\right)=4$$

Let $2^{y}=x$, then $x^{2}+3x=4$

$$\Rightarrow x^{2}+3x-4=0$$

$$\Rightarrow (x+4)(x-1)=0$$

$\Rightarrow x=-4$ or $x=1$

Replacing the value of x in the equation $2^{y}=x$

For $x=-4$: $2^{y}=-4$ doesn't exist

For $x=1$: $\begin{aligned}&2^{y}=1\\&\Rightarrow y=0\end{aligned}$ $\qquad \therefore S=\{0\}$

$$d)3\left(2^{4x}\right)-7\left(2^{2x}\right)+4=0 \Leftrightarrow 3\left(2^{2x}\right)^{2}-7\left(2^{2x}\right)+4=0$$

Let $2^{2x}=k$, then $3k^{2}-7k+4=0$

$$3k^{2}-7k+4=0$$

$\Delta=1 \qquad \Rightarrow k=\frac{4}{3}$ or $k=1$

Replacing the value of k in the equation $2^{2x}=k$

For $k=\frac{4}{3}$:

$$2^{2x}=\frac{4}{3}$$

$$2x\ln 2=\ln\frac{4}{3}$$

$$x=\frac{\ln 4-\ln 3}{2\ln 2}$$

For $k=1$: $2^{2x}=1 \Rightarrow 2x\ln 2=\ln 1 \Rightarrow x=0$ $\quad \therefore S=\left\{0,\frac{\ln 4-\ln 3}{2\ln 2}\right\}$.

Application activity 4.2.3

1) Solve each equation for x or t.

$a)\, e^{x}=6 \quad b)\, 5+e^{0.2t}=10 \quad c)\, e^{2x}=3e^{x} \quad d)\, e^{2x}=e^{x}+12 \quad e)\, e^{t}=12-32e^{-t}$

2) Solve $a)\, 2e^{-x+1}-5=9 \quad b)\, \frac{50}{1+12e^{-0.02x}}=10.5 \quad c)\, e^{\ln x^{2}}-9=0 \quad d)\, e^{x}-12=\frac{-5}{e^{-x}}$

3) Solve in the following equations in the set of real numbers:

$$\frac{e^{x}+e^{-x}}{2}=1,\left(\text{Hint: multiply by } \mathrm{e}^{x}\right)$$

4) Find the value of marked letter in each equation.

$a)\, 9^{t}+3^{t}=12 \quad c)\, \frac{2^{x}}{4}-\frac{3^{x}}{9}=0 \quad e)\, 4^{x}-10\cdot 2^{x}+16=0$

$b)\, 2^{x}+2^{x-1}=\frac{3}{2} \quad d)\, \left(\frac{5}{2}\right)^{x}=0.16 \quad f)\, 5^{m}\sqrt[m]{8^{m-1}}=500$

5) Solve in $\mathbb{R}$

$a)\, (4)^{1-2x}-\left(\frac{1}{16}\right)^{\frac{x}{2}}=0 \quad c)\, \begin{cases}5^{3x}=25^{2y-2}\\ 9^{y}=3^{x+1}\end{cases} \quad e)\, \begin{cases}5^{x}=3y\\ 3^{x}=5y\end{cases}$

$b)\, \begin{cases}16\cdot 2^{x}=4^{x+y}\\ 5\cdot 25^{2x+y}=5^{x+1}\end{cases} \quad d)\, \begin{cases}3^{x+1}=243\\ 2^{y}=64\end{cases} \quad f)\, \begin{cases}5^{3x-2y}=1\\ 11^{6x-7y}=14641\end{cases}$

6) Solve

$a) 2^{4x}-6\cdot 2^{3x}+6\cdot 2^{x}-1=0$

$b) 4^{x+1}+31\cdot 2^{x-1}=2$

$c) 2^{x}+\dfrac{1}{2^{x}-7}=9$

$d) 81^{x}+81^{1-x}=30$

4.2.4 Limit of exponential functions

4.2.4.1 Limit of exponential functions with base e

Activity 4.2.4.1

Let $y=e^{x}$

1) Complete the following table

x	e^x
-1	
-2	
-5	
-15	
-30	

x	e^x
1	
2	
5	
15	
30	

2. From the table in 1), deduce $\lim\limits_{x\to-\infty} e^{x}$ and $\lim\limits_{x\to+\infty} e^{x}$.

CONTENT SUMMARY

In general

$$\lim_{x\to-\infty} e^{x}=0 \text{ and } \lim_{x\to+\infty} e^{x}=+\infty$$

Example 1

Evaluate $\lim\limits_{x\to\infty} e^{1-4x-5x^{2}}$

Solution

$$\lim_{x\to\infty} e^{1-4x-5x^{2}}$$

We know that $\lim\limits_{x\to\infty} 1-4x-5x^{2}=-\infty$

Therefore, as the exponent goes to minus infinity in the limit and so the exponential must go to zero in the limit using the ideas from the previous formula.

Hence, $\lim\limits_{x\to\infty} e^{1-4x-5x^2} = e^{-\infty} = 0$

Example 2

Consider $f(x) = \dfrac{e^{-3x} - 2e^{8x}}{9e^{8x} - 7e^{-3x}}$, evaluate each of the following:

$\lim\limits_{x\to-\infty} f(x)$ and $\lim\limits_{x\to+\infty} f(x)$.

Solution:

$$a) \lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \frac{e^{-3x} - 2e^{8x}}{9e^{8x} - 7e^{-3x}} = \lim_{x\to-\infty} \frac{e^{-3x}\left(1-2e^{11x}\right)}{e^{-3x}\left(9e^{11x}-7\right)} = \lim_{x\to-\infty} \frac{1-2e^{11x}}{9e^{11x}-7} = -\frac{1}{7}$$

$$b) \lim_{x\to+\infty} f(x) = \lim_{x\to+\infty} \frac{e^{-3x} - 2e^{8x}}{9e^{8x} - 7e^{-3x}} = \lim_{x\to+\infty} \frac{e^{8x}\left(e^{-11x}-2\right)}{e^{8x}\left(9-7e^{-11x}\right)} = -\frac{2}{9}$$

Application activity 4.2.4.1

For each given function, evaluate limit at $+\infty$ and $-\infty$

1. $f(x) = e^{8+2x-x^3}$
2. $f(x) = e^{\frac{6x^2+x}{5+3x}}$
3. $f(x) = 2e^{6x} - e^{-7x} - 10e^{4.}$
4. $f(x) = 3e^{-x} - 8e^{-5x} - e^{10:}$
5. $f(x) = \dfrac{e^{-3x} - 2e^{8x}}{9e^{8x} - 7e^{-3x}}$

4.2.4.2 Limit of exponential functions for any base $(a > 0 \text{ and } a \neq 1)$

Activity 4.2.4.2

1) Discuss $\lim\limits_{x\to-\infty} \left(\dfrac{1}{2}\right)^x$ *and* $\lim\limits_{x\to+\infty} \left(\dfrac{1}{2}\right)^x$.

2) Generalize above results to $\lim\limits_{x\to-\infty} a^x$ *and* $\lim\limits_{x\to+\infty} a^x$

CONTENT SUMMARY

In calculating limit of exponential functions for any base, you have to take care on the given base;

If $a > 1$, $\lim_{x \to -\infty} a^x = 0$ and $\lim_{x \to +\infty} a^x = +\infty$

If $0 < a < 1$, $\lim_{x \to -\infty} a^x = +\infty$ and $\lim_{x \to +\infty} a^x = 0$

Example 1

Evaluate a) $\lim_{x \to 1^+} \left(\frac{3}{5}\right)^{\frac{1}{x-1}}$ b) $\lim_{x \to +\infty} 3^{\frac{1}{x}}$ c) $\lim_{x \to 1} 3^{\frac{1}{x-1}}$

Solution

a) The exponent goes to infinity in the limit and so the exponential will also need to go to zero in the limit since the base is less than 1.

Hence, $\lim_{x \to 1^+} \left(\frac{3}{5}\right)^{\frac{1}{x-1}} = 0$

b) $\lim_{x \to -\infty} 3^{\frac{1}{x}} = 3^{\lim_{x \to -\infty} \frac{1}{x}} = 3^0 = 1$

c) $\lim_{x \to 1} 3^{\frac{1}{x-1}} = 3^{\lim_{x \to 1} \frac{1}{x-1}} = 3^{\frac{1}{0}}$

Study one side limit:

x	$3^{\frac{1}{x-1}}$
0	0.33
0.2	0.25
0.4	0.16
0.6	0.06
0.8	0.004
0.9	0.00001

x	$3^{\frac{1}{x-1}}$
2	3
1.8	3.948
1.6	6.24
1.4	15.59
1.2	243
1.1	59049

$\lim_{x \to 1^-} 3^{\frac{1}{x-1}} = 0$ and $\lim_{x \to 1^+} 3^{\frac{1}{x-1}} = +\infty$

Hence, $\lim_{x \to 1} 3^{\frac{1}{x-1}}$ does not exist.

Alternatively:

Since $\lim_{x\to 1^+}\frac{1}{x-1}=+\infty$ and $\lim_{x\to 1^-}\frac{1}{x-1}=-\infty$, apply results on

$\lim_{x\to\pm\infty} a^x$ for $a>1$ to have:

$\lim_{x\to 1^-} 3^{\frac{1}{x-1}}=0$ and $\lim_{x\to 1^+} 3^{\frac{1}{x-1}}=+\infty$

Hence, $\lim_{x\to 1} 3^{\frac{1}{x-1}}$ does not exist.

Example 2

Evaluate $\lim_{x\to -\infty} 2^{\frac{1}{x}}$

Solution

$\lim_{x\to -\infty} 2^{\frac{1}{x}}=2^0=1$

Indeterminate form $0^0, 1^\infty, \text{and } \infty^0$

These indeterminate forms are found in functions of the form $y=\left[f(x)\right]^{g(x)}$

To remove these indeterminate forms we change the function in the form

$y=\left[f(x)\right]^{g(x)}=e^{g(x)\ln f(x)}$

Also $\lim_{x\to k} e^{f(x)\ln g(x)}=e^{\lim_{x\to k} f(x)\ln g(x)}$

Example 3

Show that $\lim_{x\to +\infty}\left(1+\frac{1}{x}\right)^x=e$

Solution

$\lim_{x\to +\infty}\left(1+\frac{1}{x}\right)^x=1^\infty \quad IF$

$$\lim_{x\to+\infty}\left(1+\frac{1}{x}\right)^x = \lim_{x\to\infty} e^{x\ln\left(1+\frac{1}{x}\right)}$$

$$= e^{\lim\limits_{x\to\infty} x\ln\left(1+\frac{1}{x}\right)}$$

But,

$$\lim_{x\to\infty} x\ln\left(1+\frac{1}{x}\right) = \infty\cdot 0 \ \ IF$$

$$\lim_{x\to\infty} x\ln\left(1+\frac{1}{x}\right) = \lim_{x\to\infty}\frac{\ln\left(1+\frac{1}{x}\right)}{\frac{1}{x}}$$

$$= \frac{0}{0} \quad IF$$

$$\lim_{x\to\infty}\frac{\ln\left(1+\frac{1}{x}\right)}{\frac{1}{x}} = \lim_{x\to\infty}\frac{-\frac{1}{x^2}}{\left(1+\frac{1}{x}\right)\left(-\frac{1}{x^2}\right)} \quad \text{[By Hôpital rule]}$$

$$= \lim_{x\to\infty}\frac{1}{1+\frac{1}{x}}$$

$$= 1$$

Thus, $\lim\limits_{x\to+\infty}\left(1+\frac{1}{x}\right)^x = e^1 = e$

Example 4

Evaluate $\lim\limits_{x\to 0} x^x$

Solution

$$\lim_{x\to 0} x^x = 0^0 \ \ IF$$

$$\lim_{x\to 0} x^x = \lim_{x\to 0} e^{x\ln x}$$
$$= e^{\lim\limits_{x\to 0} x\ln x}$$
$$= e^0 \qquad \left[\text{since } \lim_{x\to 0} x\ln x = 0\right]$$
$$= 1$$

Alternative method

When finding limits of the function of the form $y = \left[f(x)\right]^{g(x)}$, the following relation may be used: $\lim\limits_{x\to k}\left[f(x)\right]^{g(x)} = \lim\limits_{x\to k} e^{\left[f(x)-1\right]g(x)}$

Example 5

Evaluate $\lim\limits_{x\to 0} x^x$

Solution

$$\lim_{x\to 0} x^x = 0^0 \ IF$$

$$\lim_{x\to 0} x^x = \lim_{x\to 0} e^{(x-1)x}$$
$$= e^{\lim\limits_{x\to 0}(x-1)x}$$
$$= e^0$$
$$= 1$$

Example 6

Evaluate $\lim\limits_{x\to +\infty}\left(\dfrac{x}{x+1}\right)^{x+2}$

Solution

$$\lim_{x\to +\infty}\left(\frac{x}{x+1}\right)^{x+2} = 1^\infty$$

$$\lim_{x\to +\infty}\left(\frac{x}{x+1}\right)^{x+2} = \lim_{x\to +\infty} e^{\left(\frac{x}{x+1}-1\right)(x+2)}$$

$$= \lim_{x\to +\infty} e^{\left(\frac{x-x-1}{x+1}\right)(x+2)}$$

$= \lim_{x\to+\infty} e^{\frac{-x-2}{x+1}}$

$= e^{\lim_{x\to+\infty}\frac{-x-2}{x+1}}$

$= e^{-1} \quad \left[\text{since } \lim_{x\to+\infty}\frac{-x-2}{x+1} = -1\right]$

$= \frac{1}{e}$

Application activity 4.2.4.2

Evaluate

1) $\lim_{x\to\infty}\left(\frac{x^2-1}{x^2+1}\right)^{\frac{x-1}{x+1}}$

2) $\lim_{x\to\infty}\left(\frac{x}{x-1}\right)^{4x}$

3) $\lim_{x\to0}\left(1+x^2\right)^{\frac{1}{e^x-1-x}}$

4) $\lim_{x\to0}(1+x)^{\frac{1}{x}}$

5) $\lim_{x\to\infty}\left(1+\frac{k}{x}\right)^{x}$

6) $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{kx}$

4.2.4.3 Asymptotes of the graph of exponential functions

Activity 4.2.4.3

Given the function $f(x) = e^{(x-2)}$ and $g(x) = 2^{(x-2)}$,

a. Find the domain and range of f and g.

b. Determine $\lim_{x\to-\infty} f(x)$ and $\lim_{x\to-\infty} g(x)$. Hence, deduce the equation of horizontal asymptote for the graphs.

c. Evaluate the value of $f(x)$ *and* $g(x)$ *for* $x = 0$ and deduce y-intercept.

d. i) Determine $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty}\frac{f(x)}{x}$

ii) Determine $\lim_{x\to\infty} g(x)$ and $\lim_{x\to-\infty}\frac{g(x)}{x}$

e. i) Evaluate $\lim_{x\to 0^+} f(x)$ and $\lim_{x\to 0^-} f(x)$. Discuss the continuity of this function at $x=0$.

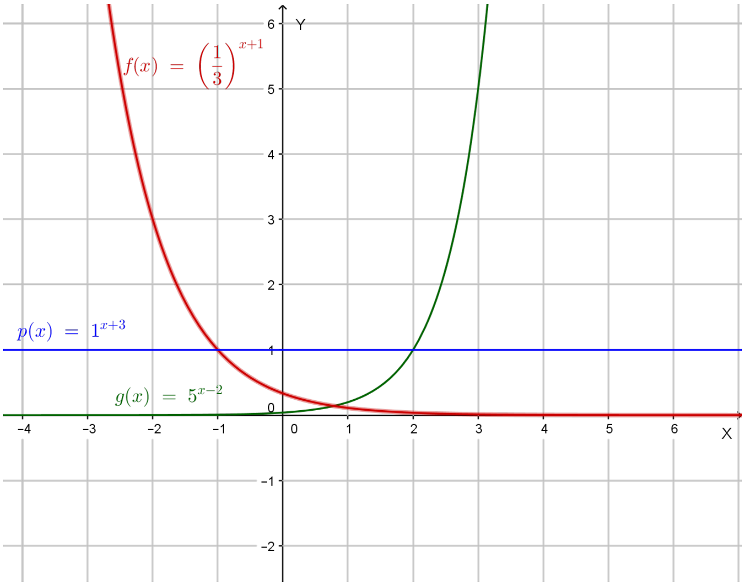
ii) Evaluate $\lim_{x\to 0^+} g(x)$ and $\lim_{x\to 0^-} g(x)$. Discuss the continuity of this function at $x=0$.

f. Sketch the graph of $f(x)$ *and* $g(x)$.

CONTENT SUMMARY

For $a>0, a\neq 1$, the exponential function $f(x) = a^x$ is continuous on $\mathbb{R}$ and takes always nonnegative values. Its graphs admits the line of equation $y = 0$ as horizontal symptote and intercepts y-axis at (0,1).. The function f is increasing from 0 to $+\infty$ if a is greater than 1 and decreasing from $+\infty$ to 0 if a is smaller than 1. The function is the constant 1 if $a=1$ and its graph is the horizontal line of equation $y=1$.

Graphs of $g(x)=5^{x-2}$ **,** $f(x)=\left(\frac{1}{3}\right)^{x+1}$ **and** $p(x)=1^{x+3}$



Example 1

Let $f(x)=3^{x+1}-1$.

Find the domain, range and equation of the horizontal asymptote of the graph of

f. Precise intercepts (if any) of the graph with axes. .

Solution:

The domain of f is the set of all real numbers since the expression $x+1$ is defined for all real values.

To find the range of f, we start with the fact that 3(x+ 1) > 0 as exponential function.

Then, subtract 1 to both sides to get $3^{x+1}-1>-1$.

Therefore, for any value of x, $f(x)>-1$. in other words, the range of f is $]-1,\infty[$.

As x decreases without bound, $f(x)=3^{x+1}-1$ approaches -1, in other words

$\lim\limits_{x\to-\infty} f\left(x\right)=-1$. Thus, the graph of f has horizontal asymptote the line of

equation $y=-1$. To find the x intercept we need to solve the equation $f(x)=0$

This is $3^{(x+1)}-1=0$.

Solving yields to $x=-1$. The x- intercept is the point (-1,0).

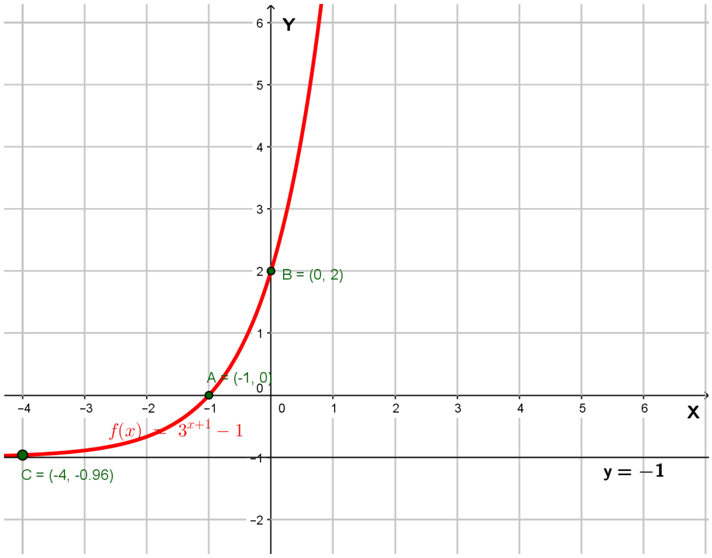
The y- intercept is given by $(0,f(0))=\left(0,3^{(0+1)}-1\right)=(0,2)$.

Extra points:

$$\left(-2,f(-2)\right)=\left(-2,3^{(-2+1)}-1\right)=\left(-2,-\frac{4}{3}\right)$$

$$\left(-4,f(-4)\right)=\left(-4,3^{(-4+1)}-1\right)=\left(-4,-\frac{26}{27}\right)$$

We can now use all the above information to plot $f(x)=3^{(x+1)}-1$:



Example 2

For the following function, find relative asymptotes (if any), study the variation and sketch the curve of $f(x) = \frac{1}{2}x^2e^{x+1}$.

Solution

Asymptotes:

First we need domain of definition: $Domf = \mathbb{R}$

$\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty} \frac{1}{2}x^2e^{x+1} = (+\infty)(+\infty) = +\infty$, no horizontal asymptote at $x \to +\infty$

$\lim_{x\to+\infty} \frac{f(x)}{x} = \lim_{x\to+\infty} \frac{1}{2}xe^{x+1} = +\infty$, no oblique asymptote at $x \to +\infty$

$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \frac{1}{2}x^2e^{x+1} = +\infty \times 0$

Remove this indeterminate case:

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \frac{1}{2}x^2e^{x+1}$$

$$= \frac{1}{2}\lim_{x\to-\infty} \frac{x^2}{e^{-x-1}}$$

$$= \frac{1}{2}\frac{-\infty}{+\infty}\ I.C.$$

$$\lim_{x\to-\infty} f(x) = \frac{1}{2}\lim_{x\to-\infty}\frac{x^2}{e^{-x-1}}$$

By Hôpital's rule

$$\lim_{x\to-\infty} f(x) = \frac{1}{2}\lim_{x\to-\infty}\frac{2x}{-e^{-x-1}}$$

$$= \lim_{x\to-\infty}\frac{x}{-e^{-x-1}}$$

$$= \frac{-\infty}{+\infty}\ I.C.$$

Apply again Hôpital's rule

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty}\frac{1}{e^{-x-1}} = 0$$

There is horizontal asymptote $y=0$ for $x\to-\infty$.Hence no oblique asymptote

Also there is no vertical asymptote according to the boundaries of the domain.

Application activity 4.2.4.3

Given the function $f(x)=e^x+1$ and $f(x)=2^x+1$

a) Determine domain and range for each function.

b) Write the equation of horizontal asymptote of the graph for each function..

c) Find the x and y intercepts of each graph if there are any.

d) In the same $xy-plane$, sketch the graphs.

4.2.5 Derivative of exponential functions

4.2.5.1 Derivative of exponential functions of base e and of any base

Activity 4.2.5.1

Given functions $f(x)=e^x$ and $g(x)=2^x$

1. Determine the inverse of $f(x)=e^x$ and $g(x)=2^x$

2. i) Use the derivative of logarithmic function $p(x)=\ln x$, then apply the rule of differentiating inverse functions to find the derivative of $f(x)=e^x$

   ii) Use the derivative of logarithmic function $k(x)=\log_2 x$, then apply the rule of differentiating inverse functions to find the derivative of $g(x)=2^x$

CONTENT SUMMARY

- The derivative of $g(x)=a^x$ is $g'(x)=a^x \ln a$.
- Therefore, if u is a function of x, the derivative of $g(x)=a^{u(x)}$ is $g'(x)=u'(x)a^{u(x)}\ln a$
- The derivative of $f(x)=e^x$ is noted by $\frac{d(e^x)}{dx}=e^x$ or $f'(x)=e^x$.
- If u is a function of x, the derivative of $y=e^{u(x)}$ is

$$y'=d\left(\frac{e^{u(x)}}{dx}\right)=e^{u(x)}\frac{du(x)}{dx}=u'e^{u(x)}.\ \text{Thus, } y'=\left(e^{u(x)}\right)'=u'(x)e^{u(x)}$$

Example 1

1. Given the function $f(x)=3^x$

   i. Complete the table below

x	$f(x)$	$f'(x)$
2		
4		

5		

ii. Deduce the slope of the tangent line at each value of x from the table above

iii. Graph the function $f(x)=3^x$ indicating the slope of the tangent line at $x=4$

Solution:

i. The derivative of $f(x)=3^x$ is $f'(x)=3^x \ln 3$

x	$f(x)$	$f'(x)$
2	9	1.81
4	81	88.29
5	243	264.87

The equation of tangent line at (x_0, y_0) is $T \equiv y-y_0 = f'(x_0)(x-x_0)$.

ii. For $x=2$, y-value is $3^2=9$ and $f'(2)=3^2\ln 3=1.81$, then the slope of the tangent line at $x=2$ is also 1.81.

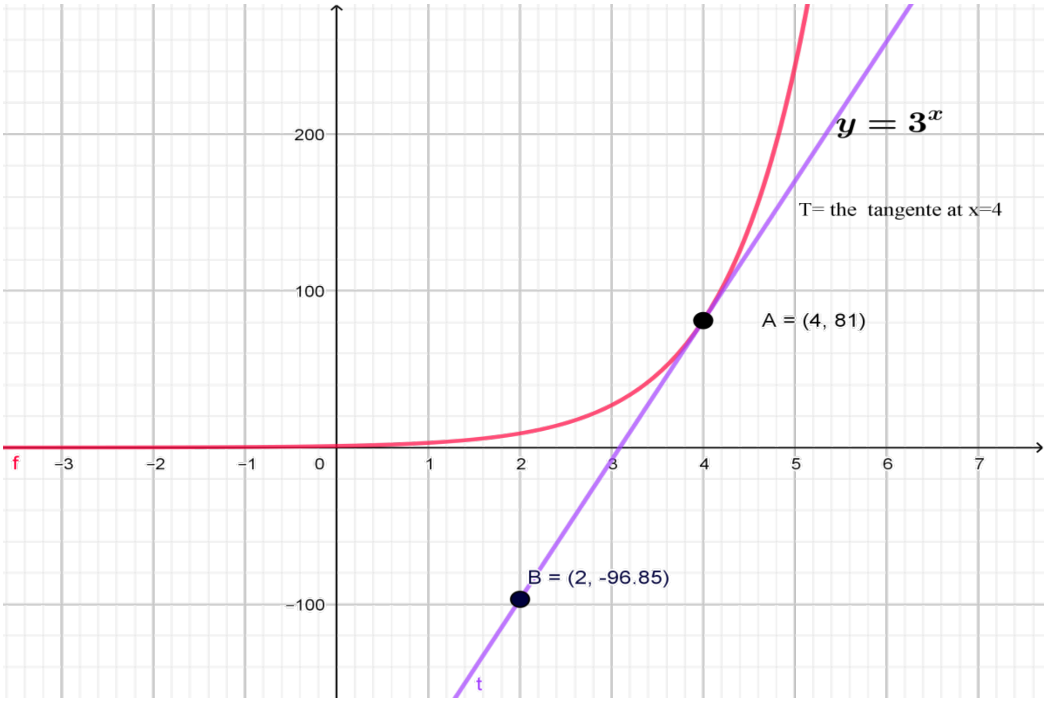
For $x=4$, y-value is $3^4=81$ and the slope of the tangent line at $x=4$ is also $f'(4)=3^4\ln 3=88.29$.

For $x=5$, y-value is $3^5=243$ and the slope of the tangent line at $x=5$ is $f'(5)=264.87$.

Graph the function $f(x)=3^x$ indicating the slope of the tangent line at $x=4$.

From above calculations, the tangent line at the point $(4, 88.29)$ has the equation $y=88.98x-274.92$.

Graph of $f(x) = 3^x$ and its tangent at $x = 4$



Observe that the slope of the tangent line at a given point of the graph is the same as the derivative of the function at the *x*-coordinate of the same point.

Example 2

Given that $f(x) = e^x$ and $f'(x) = e^x$

i. Complete the table below

x	$f(x)$	$f'(x)$
2		
4		
5		

ii. Deduce the slope of the tangent line at each value of x from the table above

iii. Graph the function $y = e^x$ indicating the slope of the tangent line at $x = 2$

Solution:

i. Complete the table below

x	$f(x)$	$f'(x)$
2	7.38	7.38
4	54.59	54.59
5	148.41	148.41

The equation of tangent line $T \equiv Y - y_o = f'(x_0)(x - x_0)$. Remember that $f'(x_0)$ is the slope of the line tangent to the graph of the function $f(x)$ at (x_0, y_0).

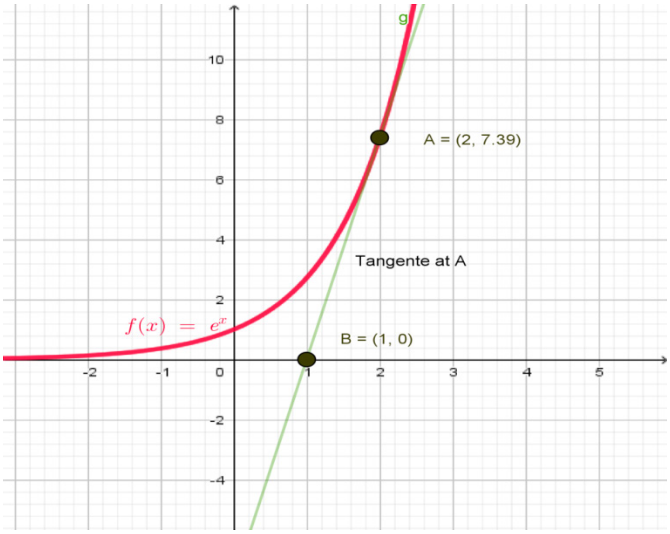
So for $x = 2$, *y*-value is $e^2 \approx 7.38$

Since the derivative of e^x is e^x then the slope of the tangent line at $x = 2$ is also $e^2 \approx 7.38$

For $x = 4$, $y - value = 54.59$, the slope of the tangent line at $x = 4$ is also 54.59

For $x = 5$, $y - value = 148,41$, the slope of the tangent line at $x = 5$ is also 148.41

Graph of $f(x) = e^x$ indicating the slope of the tangent line at $x = 2$



Example 3

Find the derivative of $f(x) = e^{x^2}$

Solution

$f(x)=e^{x^2}$

From the formula of derivative

$(e^u)'=u'e^u \Rightarrow (e^{x^2})'=(x^2)'.e^{x^2} \quad 2xe^{x^2}$

Example 4

Find the derivative of the function $f(x)=\frac{1}{2}x^2e^{x+1}$

Solution

$$f'(x)=\frac{1}{2}(2x)e^{x+1}+\frac{1}{2}x^2(1)e^{x+1}$$
$$=\frac{1}{2}xe^{x+1}(x+2)$$

Example 5

Find the second derivative of the function $f(x)=e^{\frac{1}{x-1}}$

Solution

$$f'(x)=-\frac{1}{(x-1)^2}e^{\frac{1}{x-1}}$$

$$f''(x)=\frac{2(x-1)}{(x-1)^4}e^{\frac{1}{x-1}}-\frac{1}{(x-1)^2}\left(-\frac{1}{(x-1)^2}e^{\frac{1}{x-1}}\right)$$

$$=\frac{2x-2}{(x-1)^4}e^{\frac{1}{x-1}}-\frac{e^{\frac{1}{x-1}}}{(x-1)^4}$$

$$=\frac{(2x-1)e^{\frac{1}{x-1}}}{(x-1)^4}$$

Application of derivatives to remove indeterminate form $0^0, 1^\infty$, and ∞^0

These indeterminate forms are found in functions of the form $y=\left[f(x)\right]^{g(x)}$

To remove these indeterminate forms we change the function in the form

$$y=\left[f\left(x\right)\right]^{g(x)}=e^{g(x)\ln f(x)}$$

Also $\lim\limits_{x\to k} e^{f(x)\ln g(x)}=e^{\lim\limits_{x\to k} f(x)\ln g(x)}$

Example 6

Show that a) $\lim\limits_{x\to 0^+} x^x=1$ b) $\lim\limits_{x\to +\infty}\left(1+\frac{1}{x}\right)^x=e$

Solution

a)

$\lim\limits_{x\to 0^+} x^x \quad \left(0^0 \quad IF\right)$

$\lim\limits_{x\to 0^+} x^x=\lim\limits_{x\to 0^+} e^{x\ln x}=e^{\lim\limits_{x\to 0^+} x\ln x}$

$\lim\limits_{x\to 0^+} x\ln x \ (0\times\infty \ IF)$

$\lim\limits_{x\to 0^+} x\ln x=\lim\limits_{x\to 0^+}\frac{\ln x}{\frac{1}{x}}(\frac{\infty}{\infty} IF)$

$\lim\limits_{x\to 0^+}\frac{\ln x}{\frac{1}{x}}=\lim\limits_{x\to 0^+}\frac{\frac{1}{x}}{-\frac{1}{x^2}}=0$ (Hospital rule).

Finally, $\lim\limits_{x\to 0^+} x^x=1$

b)

$\lim\limits_{x\to +\infty}\left(1+\frac{1}{x}\right)^x \quad \left(1^\infty \quad IF\right)$

$\lim\limits_{x\to +\infty}\left(1+\frac{1}{x}\right)^x=\lim\limits_{x\to +\infty} e^{x\ln\left(1+\frac{1}{x}\right)}=e^{\lim\limits_{x\to +\infty} x\ln\left(1+\frac{1}{x}\right)}$

But,

$\lim\limits_{x\to +\infty} x\ln\left(1+\frac{1}{x}\right)=\lim\limits_{x\to +\infty}\frac{\ln\left(1+\frac{1}{x}\right)}{\frac{1}{x}} \quad \left(\frac{0}{0} IF\right)$

$\lim\limits_{x\to +\infty}\frac{\ln\left(1+\frac{1}{x}\right)}{\frac{1}{x}}=\lim\limits_{x\to +\infty}\frac{-\frac{1}{x^2}}{\left(1+\frac{1}{x}\right)\left(-\frac{1}{x^2}\right)}$

(Hospital rule)

$=\lim\limits_{x\to +\infty}\frac{1}{1+\frac{1}{x}}=1$

Thus, $\lim\limits_{x\to +\infty}\left(1+\frac{1}{x}\right)^x=e^1=e$.

1. Find the derivative of the function $f(x)=xe^{x^2+1}$

2. Find the derivative of

a) $f\left(x\right)=e^{2x-1}$ b) $g\left(x\right)=e^{2x}-e^{-2x}$ c) $h\left(x\right)=e^{\tan x}$ d) $k\left(x\right)=\frac{e^x}{|x-1|}$

3. Given the function $f(x)=4^x$

i. Find $f'(x)$ the derivative function of $f(x)$

ii. Find $f(5)$

iii. Deduce the slope of the tangent line at $x=5$

iv. Plot the function $f(x)$ at $x=4$ by indicating the slope of the tangent line

4. Find the derivative of each of the following functions

(a) $f(x)=10^{3x}$ (b) $f(x)=\frac{3^{4x+2}}{x}$

4.2.5.2 Variation and Graphical representation of exponential functions

Activity 4.2.5.2

I. Given two functions $f(x)=e^x$

1. Compare $f(1)$ and $f(10)$, deduce whether the function $f(x)$ is increasing or decreasing on the interval $[1,10]$.

2. Use derivatives $f'(x)$ to discuss variations of each of the function.

3. Plot the graphs of $f(x)$

4. Express in your own words the variations of exponential function

II. Given two functions $f(x)=2^x$ and $g(x)=0.5^x$,

a) Compare $f(1)$ and $f(10)$, deduce whether the function $f(x)$ is increasing or decreasing on the interval $[1,10]$.

b) Compare $g(1)$ and $g(10)$ and deduce whether the function $g(x)$ is increasing or decreasing on the interval $[1,10]$.

c) Use derivatives $f'(x)$ and $g'(x)$ to discuss variations of each of the functions.

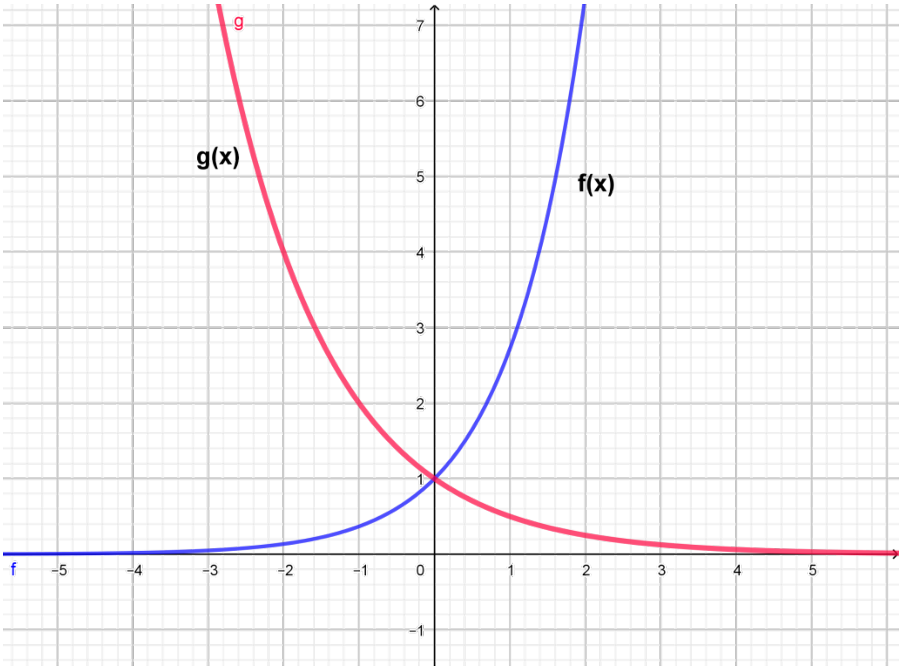
d) Plot the graphs of $f(x)$ and $g(x)$.

e) Express in your own words the variations of exponential function

CONTENT SUMMARY

The function $g(x)=a^x, a>1$ defined on $\mathbb{R}$ is always increasing. When $0<a<1$, the function $g(x)=a^x$ is always decreasing. This means the exponential functions $g(x)=a^x$ does not have extremum (maximum or minimum); this means that the function increases or decreases "indefinitely" as shown in the example of graph.

Graph of $g(x)=(0.5)^x$ **and** $f(x)=2^x$



Example

Given the function $f(x)=xe^x$

i. Find the derivative of $f(x)=xe^x$

ii. Solve $f'(x)=0$

iii. Discuss extrema of the function.

iv. Establish the sign diagram of $f'(x)$ and variations of $f(x)$

v. Plot the graph of the function.

Solution

i. The domain of the function is $\mathbb{R}$.

ii. The derivative of $f(x)=xe^x$ is defined by

$f'(x) = e^x + xe^x = (1+x)e^x$

$f'(x) = 0$ if $x = -1$

iii. $f''(x) = \left((1+x)e^x\right)' = (1+x)'e^x + (1+x)\left(e^x\right)' = e^x + (1+x)e^x = (2+x)e^x$

$f''(x) = 0 \quad if \quad (2+x)e^x = 0 \Leftrightarrow 2+x = 0$

$\Leftrightarrow x = -2$

iv. Sign diagram for $f(x)$

There is need to find limit of the function at the boundaries of the domain: $\lim\limits_{x \to -\infty} xe^x = 0$ and $\lim\limits_{x \to +\infty} xe^x = +\infty$. The limit at $-\infty$ tells us that line of equation $y = 0$ is an horizontal asymptote when x is taking “indefinitely” negative values.

x	$-\infty$		-2		-1		0		$+\infty$
$f'(x)$		$-$			0	$+$		$+$	
$f''(x)$		$-$	0		$+$				
$f(x)$	0	$\searrow$			$-\frac{1}{e}$	$\nearrow$	0	$\nearrow$	$+\infty$

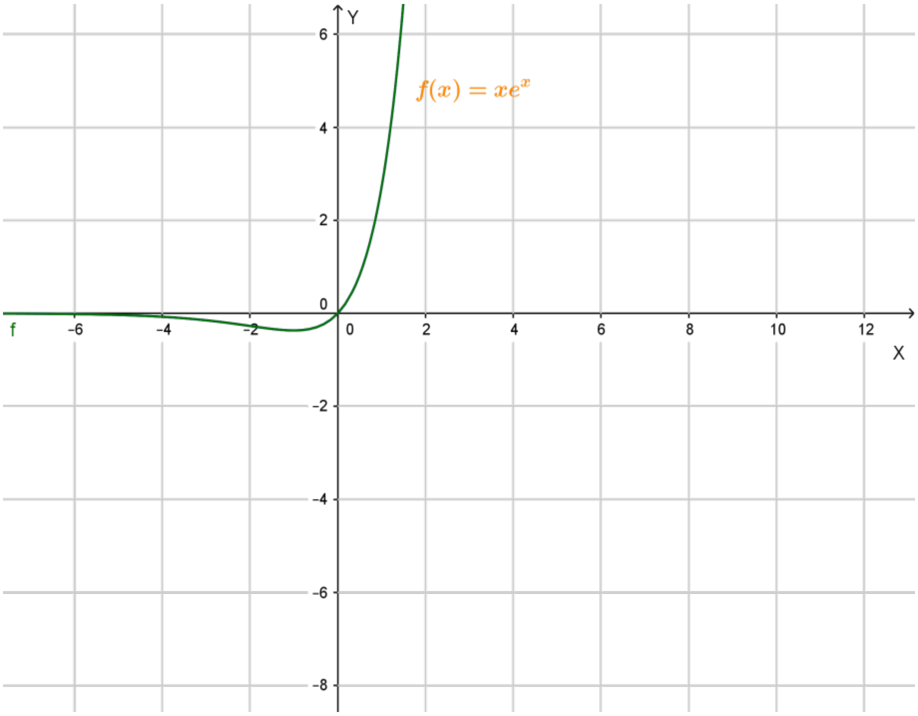
Therefore, $f(x)$ is **decreasing** from 0 to $\frac{-1}{e}$ on the interval $]-\infty,-1]$ and **increasing** from $-\frac{1}{e}$ to $+\infty$ on $[-1,+\infty[$. The function has **minimum**

(absolute) equal to $-\frac{1}{e}$ when $x = -1$.

The **inflection point** is $\left(-2, -\frac{2}{e^2}\right)$

The concavity is turning down on the interval $]-\infty,-2]$ and is turning up on $[-2,+\infty[$.

v. Graph of $f(x) = xe^x$



Example

For the following function, find relative asymptotes (if any), study the variation and sketch the curve of $f(x) = \frac{1}{2}x^2e^{x+1}$

Solution

Asymptotes:

First we need domain of definition: $Domf = \mathbb{R}$

$\lim\limits_{x\to+\infty} f(x) = \lim\limits_{x\to+\infty} \frac{1}{2}x^2e^{x+1} = (+\infty)(+\infty) = +\infty$, no horizontal asymptote at $x \to +\infty$

$\lim\limits_{x\to+\infty} \frac{f(x)}{x} = \lim\limits_{x\to+\infty} \frac{1}{2}xe^{x+1} = +\infty$, no oblique asymptote at $x \to +\infty$

$\lim\limits_{x\to-\infty} f(x) = \lim\limits_{x\to-\infty} \frac{1}{2}x^2e^{x+1} = +\infty \times 0$

Remove this indeterminate case:

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \frac{1}{2}x^2e^{x+1}$$

$$= \frac{1}{2}\lim_{x\to-\infty} \frac{x^2}{e^{-x-1}}$$

$$= \frac{1}{2}\frac{-\infty}{+\infty}\ I.C.$$

$$\lim_{x\to-\infty} f(x) = \frac{1}{2}\lim_{x\to-\infty} \frac{x^2}{e^{-x-1}}$$

By Hôpital's rule

$$\lim_{x\to-\infty} f(x) = \frac{1}{2}\lim_{x\to-\infty} \frac{2x}{-e^{-x-1}}$$

$$= \lim_{x\to-\infty} \frac{x}{-e^{-x-1}}$$

$$= \frac{-\infty}{+\infty}\ I.C.$$

Apply again Hôpital's rule

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \frac{1}{e^{-x-1}} = 0$$

There is horizontal asymptote $y=0$ for $x\to-\infty$.Hence no oblique asymptote

Also there is no vertical asymptote according to the boundaries of the domain.

Variation:

$$f'(x) = \frac{1}{2}2xe^{x+1} + \frac{1}{2}x^2e^{x+1}$$
$$= \frac{1}{2}e^{x+1}\left(x^2+2x\right)$$

$f'(x) > 0 \Leftrightarrow x^2+2x > 0$ or $x \in]-\infty,-2[\cup]0,+\infty[$

$f'(x) < 0 \Leftrightarrow x^2+2x < 0$ or $x \in]-2,0[$

Thus, if $x \in]-\infty,-2[\cup]0,+\infty[$, $f(x)$ increases and if $x \in]-2,0[$, $f(x)$ decreases.

Curve:

Intersection with axes of coordinate:

Intersection with $x-axis$

$f(x)=0 \Leftrightarrow \frac{1}{2}x^2e^{x+1}=0 \Rightarrow x=0$

Thus, intersection with $x-axis$ is $\{(0,0)\}$.

Intersection with $y-axis$

$f(0)=\frac{1}{2}0^2e^{0+1}=0$

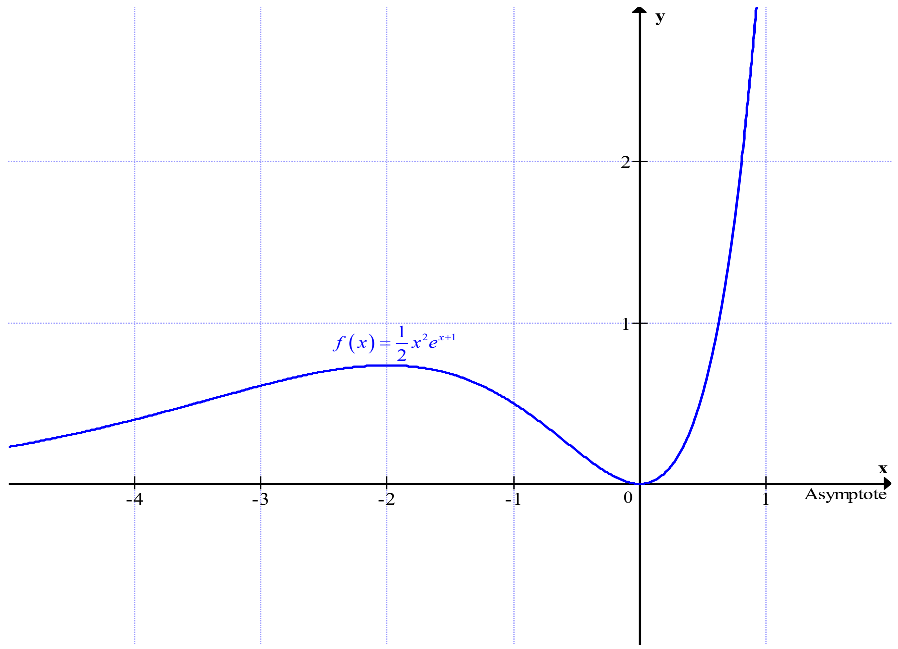
Thus, intersection with $y-axis$ is $\{(0,0)\}$.

Additional points:

x	-5.00	-4.70	-4.40	-4.10	-3.80	-3.50	-3.20	-2.90	-2.60	-2.30	-2.00
$f(x)$	0.23	0.27	0.32	0.38	0.44	0.50	0.57	0.63	0.68	0.72	0.74

x	-1.70	-1.40	-1.10	-0.80	-0.50	-0.20	0.10	0.40	0.70	1.00
$f(x)$	0.72	0.66	0.55	0.39	0.21	0.04	0.02	0.32	1.34	3.69

Curve



Application activity 4.2.5.1

1. Given the function $f(x)=xe^{x^2}$

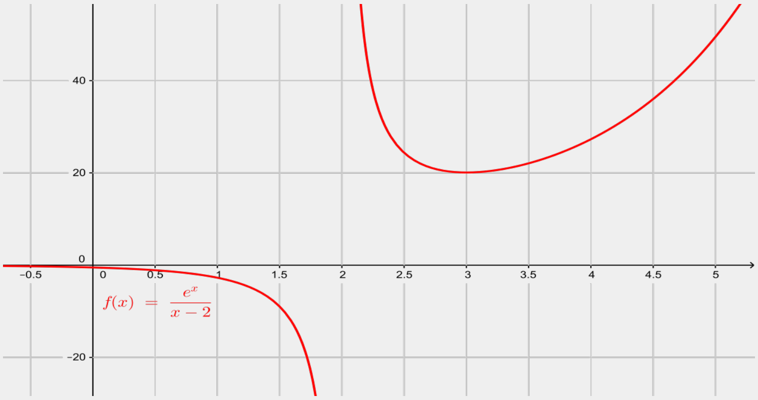
   a. Find the derivative of $f(x)$

   b. Discuss the table of variation of $f(x)$, deduce whether $f(x)$ is increasing or decreasing and write down the interval where the function increasing or decreasing.

   c. Indicate the extremum points and asymptotes to the graph

   d. Using calculator, complete a table of values and plot the graph of $f(x)$ and compare this graph with the one you find by the use of a mathematical software if possible.

2. Observe the following graph representing a certain function $f(x)=\dfrac{e^x}{x-2}$ and answer the proposed questions.



   a. Precise the domain and the range of the function

   b. Discuss continuity of the function and existence of asymptotes to the graph

   c. Discuss variations of the function

3. For each of the following functions, find relative asymptotes (if any), study the variation of the function and sketch the curve

a) $f(x)=\dfrac{1}{2}e^{x+1}$ b) $g(x)=\dfrac{e^x}{x}$ c) $h(x)=e^{2x-3}$ d) $k(x)=\dfrac{e^x}{x+2}$

Evaluate

a) $\frac{d}{dx}\left(2^{\tan\frac{1}{x}}\right)$ at $x=3$ b) $\frac{d}{dx}\left(\sqrt[x]{x-2}\right)$ at $x=3$ c) $\frac{d}{dx}\left(\sqrt[x]{x-1}\right)$ at $x=2$

4.3 Modelling and solving problems involving logarithms or exponents for any base

4.3.1 Modelling and solving Interest rate problems

Activity 4.3.1

An amount of 2 000 dollars is invested at a bank that pays an interest rate of 10% compounded once annually. Find the total amount at the end of t years by proceeding as follows:

Complete the table below:

At the end of	The total amount
The first year	2000+0.1(2000) =2000(1+0.1)
The second year	2000(1+0.1)+0.1[2000(1+0.1)] $=2000(1+0.1)^2$
The third year	$2000(1+0.1)^{\cdots} + \ldots = 2000(1+01)^3$
The fourth year	…
The fifth year	…
...	…
The t^{th} year	…

CONTENT SUMMARY

If a principal P (the money you put in) is invested at an interest rate *r* for a period of *t* years, then the amount A (how much you make) of the investment can be calculated by the following generalized formula of the interest rate problems:

a) $A = P(1+r)$ Simple interest for one year

b) $A = P\left(1+\frac{r}{n}\right)^{nt}$ Interest compounded n times per year

c) $A = Pe^{rt}$ Interest compounded continuously.

Example

An amount of 500 000 FRW is invested at a bank that pays an interest rate of 12% compounded annually.

a) How much will the owner have at the end of 10 years, in each of the following cases?

The interest rate is compounded:

i. once a year.

ii. twice a year

b) What type of interest rate among the two would the client prefer? Explain why.

Solution

a). i. For once a year, at the end of 10 years the owner will have

$$A = P(1+r)^t = 500000(1+0.12)^{10}$$
$$= 500000(1.12)^{10} = 1552924.10 Frw$$

ii. For twice a year, at the end of 10 years the owner will have

$$A = P(1+\frac{r}{2})^{2t} = 500000(1+\frac{0.12}{2})^{2(10)}$$
$$= 500000(1.06)^{20} = 1603567 Frw$$

b) Since $1603567 > 1552924.10$, the client will prefer compounding many times per year as it results in more money.

Application activity 4.3.1

Your aunt would like to invests 300 000 Frw at a bank. The Bank I pays an interest rate of 10% compounded once annually. The Bank II pays an interest rate of 9.8% compounded continuously. Your aunt will withdraw the money plus interest after 10 years.

At which bank do you advice your aunt to invest her money so as to get much money at the end of 10 years?

4.3.2 Modelling and solving Mortgage problems

Activity 2.3

Make a selection of any three letters from the word "*PRODUCT*" and fill them in 3 empty spaces

Use a box like this for empty spaces		

Write down all different possible permutations of 3 letters selected from the letters of the word "*PRODUCT*". How many are they?

1. Go to conduct a research in the library, on internet or conduct a conversation with a bank officer to write down the meaning of the following when you get a loan from the bank:

i. the periodic payment P

ii. the annual interest rate r

iii. the mortgage amount M

iv. the number t of years to cover the mortgage

v. the number n of payments per year.

vi. Among all these elements/components, what is the most useful for the client to be informed about by the bank once he/she is given the mortgage loan ?

2. Your elder brother is newly employed at a company and earns 500 000 FRW per month. He would like to know if he can afford monthly payments on a mortgage of 20 000 000 FRW with an interest rate of 6% that runs for 20 years. Given that the quantities above are governed by the relation

$$P=\frac{\frac{rM}{n}}{1-\left(1+\frac{r}{n}\right)^{-nt}}$$ show your brother that he can afford the monthly payments

by determining the following :

i. the monthly payment, that will be retained at the bank

ii. the balance that your brother can withdraw each month from the bank

iii. how much interest your brother will pay to the bank by the end of 20 years.

CONTENT SUMMARY

When a person gets a loan (mortgage) from the bank, the mortgage amount ***M***, the number of payments or the number ***t*** of years to cover the mortgage, the amount of the payment ***P***, how often the payment is made or the number ***n*** of payments per year, and the interest rate ***r,*** it is proved that all the 5 components are related by the following formula: $P=\dfrac{\frac{rM}{n}}{1-\left(1+\frac{r}{n}\right)^{-nt}}$

If we let $i=\dfrac{r}{n}$, the amount of payment $P=M\left[\dfrac{i}{1-(1+i)^{-nt}}\right]$.

Example

A business woman wants to apply for a mortgage of 75 000 US dollars with an interest of 8% that runs for 20 years. How much interest she will pay over the 20 years?

Solution:

Substituting for *M=75 000, r =0.08, t=20,n=12* in the equation

$P=\dfrac{\frac{rM}{n}}{1-\left(1+\frac{r}{n}\right)^{-nt}}$, we have

$$P=\frac{\frac{(0.08)(75000)}{12}}{1-\left(1+\frac{0.08}{12}\right)^{-(12)(20)}}$$

$$=627.33$$

Each month she will be paying 627.33 US dollars.

The total amount she will pay is $627.33\times 12\times 20\text{US dollars}=150559.2$ US dollars

The interest will be $(150559.2-75000)\text{US dollars}=75559.2$ US dollars

Example

You deposit P dollars in an account whose annual interest rate is r, compounded continuously. How long will it take for your balance to double?

Solution

The balance in the account after t years is $A = Pe^{rt}$

So, the balance will have double, when $Pe^{rt} = 2P$

Then ,

$$
\begin{aligned}
Pe^{rt} &= 2P \\
e^{rt} &= 2 \\
\ln e^{rt} &= \ln 2 \\
rt &= \ln 2 \\
t &= \frac{1}{r}\ln 2
\end{aligned}
$$

From this result, you can see that the time it takes for the balance to double is inversely proportional to the interest rate .

Notice that the doubling time decreases as the rate increases.

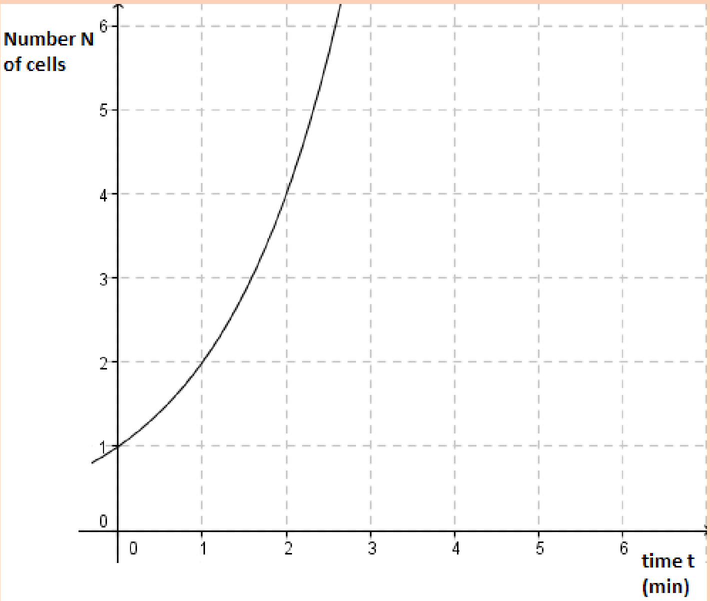
Application activity 4.3.2

A bank can offer a mortgage at 10 % interest rate to be paid back with monthly payments for 20 years. After analysis, a potential borrower finds that she can afford monthly payment of 200000FRW. How much of mortgage can she ask for?

4.3.3 Modelling and solving Population growth problems

Activity 4.3.3

Analyze the graph below showing the number of cells recorded by a student in a biology laboratory of his/her school during an experiment as function of time t .



a.. Complete the table below:

Time t(minutes)	0	1	2	3	4			
Number of cells	…	…	…					

b. Given that $N(t) = N_0 e^{kt}$, where N (t) is the quantity at time t , No is the initial quantity and k is a positive constant, what is the value of N_0 ? Predict the number of cells after 5 minutes

c. What happens to the number of cells as the time becomes larger and larger? Is the number of cells growing or not? Explain your answer.

CONTENT SUMMARY

If P_0 is the population at the beginning of a certain period and $r\%$ is the constant rate of growth per period, the population for *n* periods will be $P_n = P_0\left(1+r\right)^n$. This is similar to the final value (*F*) of an initial investment (*A*) deposited for *t* discrete time periods at an interest rate of $i\%$ *which is* calculated using the formula

$F = A(1+i)^t$

To derive a formula that will give the final sum accumulated after a period of continuous growth, we first assume that growth occurs at several discrete time intervals throughout a year. We also assume that A is the initial sum, r is the nominal annual rate of growth, n is the number of times per year that increments are accumulated and y is the final value. This means that after t years of growth the final sum will be:

$$y = A\left(1+\frac{r}{n}\right)^t$$

Growth becomes continuous as the number of times per year that increments in growth are accumulated increases towards infinity.

When $n \to \infty$, we get $\lim\limits_{x\to\infty} y = \lim\limits_{x\to\infty} A\left(1+\frac{r}{n}\right)^t = Ae^{rt}$.

This is similar to $N(t) = N_0e^{kt}$ where A and N_0, r and k take respectively the same meanings.

Therefore, the final value $A(t)$ of any variable growing continuously at a known annual rate r from a given original value A_0 is given by the following formula.

$$A(t) = A_0e^{rt}$$

Example

The number of bacteria in a culture increases according to an equation of the type $N(t) = N_0e^{kt}$ Given that the number of bacteria triples in 2 hours,

a. find an equation free of N_0 and solve the equation for k

b. How long would it take for the number of bacteria to be 5 times the initial number?

Solution:

$$a) N(2) = 3N_0 \Leftrightarrow a) 3N_0 = N_0e^{k(2)} \Leftrightarrow e^{2k} = 3 \Rightarrow 2k = \ln 3 \Leftrightarrow k = \frac{\ln 3}{2} = 0.5493$$

$$b) 5N_0 = N_0e^{0.5493t} \Leftrightarrow e^{0.5493t} = 5 \Rightarrow t = \frac{\ln 5}{0.5493} \approx 2.93.$$

It will take 2.93 hours for the number of bacteria to be 5 times the initial number.

Example

Population in a developing country is growing continuously at an annual rate of 3%. If the population is now 4.5 million, what will it be in 15 years' time?

Solution

The final value of the population (in millions) is found by using the formula $y = Ae^{rt}$ and substituting the given numbers: initial value A = 4.5; rate of growth r = 3% = 0.03; number of time periods t = 15, giving $y = 4.5e^{0.03(15)} = 7.0574048\text{ million}$

Application activity 4.3.3

1. The population of a city increases according to the law of uninhibited growth. If the population doubles in 5 years and the current population is one million, what will be the size of the population in ten years from now?
2. A country economy is forecast to grow continuously at an annual rate of 2.5%. If its Gross National product (GNP) is currently56 billion of USD, what will the forecast for GNP be after 1.75 years (at the end of the third quarter the year after Next) ?
3. One town of a given country had a population of 11,000 in 2000 and 13,000 in 2017. Assuming an exponential growth model, determine the constant rate of growth per year.

4.3.4 Modelling and solving uninhibited decay and radioactive decay problems

Activity 4.3.2

The annual catch of fish from a specific dam is declining continually at a constant rate. Five years ago the total catch was 1000 kilograms, if the rate of decline is 20%, graph the process and deduce the total catch of this year considering that the number of fish reduces with $N(t) = N_0e^{kt}$, *(Where* N_0 *is the total catch at initial time period and* $k = -20\%$ *the constant rate of decline).*

CONTENT SUMMARY

A phenomena that can be modelled by an equation of the type $N(t) = N_0e^{kt}$, where N (t) is the quantity at time t , N° is the initial quantity and k is a negative constant, is said to follow the law of uninhibited decay. Radioactive materials follow the law of uninhibited decay.

Example

1. Suppose that you start an experiment in the biology laboratory of your school with 5 000 000 cells. The cells die according to the equation of the type $N(t) = N_0 e^{kt}$

   After one minute you observe that there are 2 750 000 cells.

   a) How many cells will be remaining after two minutes?

   b) How long will it take for the number of cells to be less than 1 000?

Solution:

a) $N(t) = N_0 e^{kt}$

$$N(0) = N_0 e^{k(0)} = N_0 = 5,000,000$$

$$N(t) = 5,000,000 e^{kt}$$

$$N(1) = 5,000,000 e^{k} = 2,750,000$$

$$\Leftrightarrow e^{k} = \frac{2,750,000}{5,000,000} = 0.55 \Rightarrow k = \ln 0.55 = -0.597837$$

$$N(t) = 5,000,000 e^{-0.597837t}$$

After 2 minutes: $N(2) = 5,000,000 e^{-0.597837(2)} = 1,512,500$

There will be 1 512 500 cells remaining after two minutes

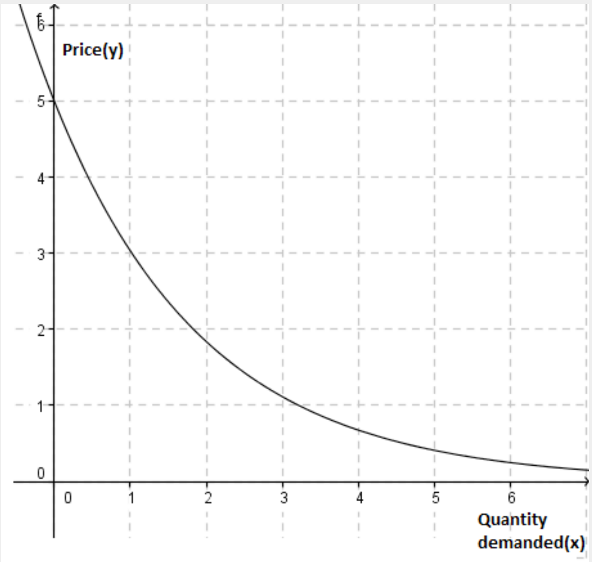
b) $1,000 = 5,000,000 e^{-0.597837t} \Leftrightarrow e^{-0.597837t} = \dfrac{1,000}{5,000,000} = \dfrac{1}{5,000}$

$$\Rightarrow -0.597837t = -\ln 5000 \Leftrightarrow t = \frac{\ln 5000}{0.597837} \approx 14.25$$

The time is 14.25 minutes.

Application activity 4.3.4

1. Analyze the graph below showing the price for a particular commodity:



   a. What is the fixed price ?

   b. What happens to the price as the time becomes larger and larger?

   c. Model the problem by an equation of the type $N(t) = N_0 e^{kt}$ Write down the value of N° and the value of k. Precise the sign of k.

2. The normal healing of a wound is modelled by the equation $W(t) = 50e^{-0.2t}$, where $w(t)$ is the surface area ,in cm2,of the wound t days following the injury when there is no infection to retard the healing.

   a. What is the initial surface area of the wound?

   b. Use the model to predict how large should the area of the wound be after 4 days if the healing is taking place

3. An object is heated to 80°C and then allowed to cool in a room whose temperature is20°C.

Given that $u(t) = t + (u_0 - T)e^{kt}$,$k < 0$,u is the temperature of the heated object at a given time t and T is the constant temperature of the surrounding medium, if the temperature of the object is 60°C after 4 minutes, when will the temperature be 25°C?

4.3.5 Modelling and solving Earthquake problems

Activity 4.3.5

Do the research in the library or explore internet to find out how Charles Richter tried to compare the magnitude of two earthquakes by the use of logarithmic function.

CONTENT SUMMARY

An earthquake is characterized by its epicenter and its magnitude.

Seismographic readings are made at a distance of 100 kilometers from the epicenter of an earthquake. If there is no earthquake, the seismographic reading is $x_0 = 0.001$ millimeter.

For an earthquake, the Richter's scale converts the seismographic reading x millimeters into magnitude through the formula $M(x) = \log \frac{x}{x_0}$, where $M(x)$ is the magnitude of the earthquake, x is the intensity of the earthquake (measured by the amplitude of a seismograph reading taken 100 km from the epicenter of the earthquake), and x_0 the intensity of a standard earthquake. The ratio of the seismographic readings is used to compare two earthquakes

Example

Two earthquakes took place at A and at B. Their magnitudes, on Richter's scale, were 8.9 and 8.3, respectively. Compare the two earthquakes by finding the ratio of their seismographic readings.

Solution:

Let x and y be the seismographic readings of the earthquakes at A and at B, respectively.

Then, $\log \frac{x}{0.001} = 8.9$ and $\log \frac{y}{0.001} = 8.3$

This is equivalent to: $\frac{x}{0.001} = 10^{8.9}$ and $\frac{y}{0.001} = 10^{8.3}$

Dividing side by side, $\frac{\frac{x}{0.001}}{\frac{y}{0.001}} = \frac{10^{8.9}}{10^{8.3}} \Leftrightarrow \frac{x}{y} = 10^{8.9-8.3} = 10^{0.6} = 3.981$

This means that the earthquake at A is about 4 times heavy than the one happened at B.

Application activity 4.3.5

The earthquake that took place in Ecuador in April 2016 was of magnitude 7.8 on Richter's scale. How intense was that earthquake compared to the one that took place in:

a. the Mexico city in 1985, which was of magnitude 8.1 on Richter's

b. San Francisco in 1906, which was of magnitude 6.9 on Richter's scale.

4.3.6 Modelling and solving Carbon dating problems

Activity 4.3.6

Carbon-14, a radioactive isotope of the element that, unlike other more stable forms of carbon, decays away at a steady rate. Organisms capture a certain amount of carbon-14 from the atmosphere when they are alive. By measuring the ratio of the radio isotope to non-radioactive carbon, the amount of carbon-14 decay can be worked out, thereby giving an age for the specimen in question. Scientists found that the time necessary for the amount of carbon-14 to reduce to its half is about 5700 years, and the radioactive material decays according to

the equation $N(t)=N_0e^{\left(\frac{kt}{t_{1/2}}\right)}$,where $N(t)$ is the

level of Carbon-14 in the remains, and t is the time (in years) from the

moment of the death of the human, $t=t_{1/2}$ is the half-life of carbon-14 (5,700 ±30 years), and the constant $k=-0.693$

Discuss how to find the time t elapsed from the death of the human to the moment of the discovery of the remains.

CONTENT SUMMARY

The half-life of a substance is the amount of time it takes for half of that substance to decay. It is only a property of substances that decay at a rate proportional to their mass. Through research, scientists have agreed that the half-life of C^{14} is approximately 5700 years.

A formula to calculate how old a sample is by carbon-14 dating is:

$$t=\frac{\ln\left(\frac{N(t)}{N_0}\right)}{-0,693}t_{1/2}$$ where $t=t_{1/2}$ is the half-life of carbon-14 that is (5,700 ±30 years) and $\frac{N(t)}{N_0}$ the per cent of carbon-14 in the sample compared to the amount in living tissue.

Example

A scientist determines that a sample of petrified wood has a carbon-14 decay rate of 8.00 counts per minute per gram. What is the age of the piece of wood in years? The decay rate of carbon-14 in fresh wood today is 13.6 counts per minute per gram, and the half- life of carbon-14 is 5730 years.

Solution

$$t=\frac{\ln\left(\frac{8}{13.6}\right)}{-0.693}.5730=4387.4 \text{ years.}$$

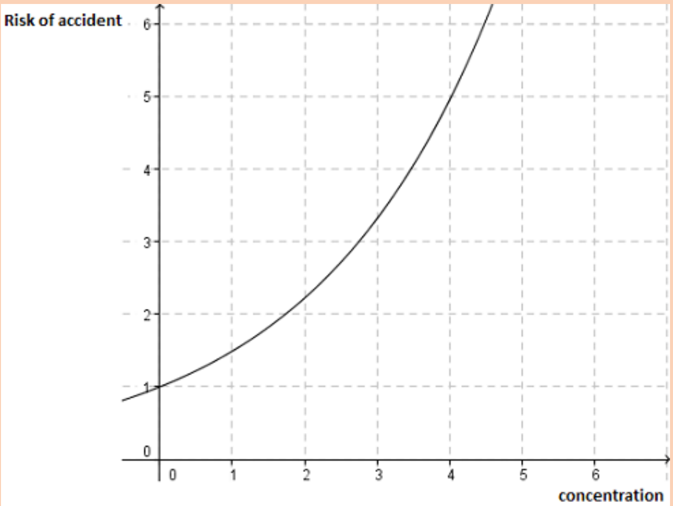
Application activity 4.3.6

1A scrap of paper taken from the Dead Sea animal was found to have a C^{14}/C^{12} ratio of 0.79 times that found in plants living today. Estimate the age of the animal given that the half-life of carbon-14 is 5700 years.

4.3.7 Problems about alcohol and risk of car accident

Activity 4.3.7

Do the research in the library or explore internet to find out how Charles Richter tried to compare the magnitude of two earthquakes by the use of logarithmic function.



i. What is the risk when there is no alcohol in the blood? Why is that risk not 0?

ii. Comment on the variation of the risk with respect to the concentration of alcohol in the driver's blood.

c. Write down approximately the type of the equation that can be used to model the risk.

CONTENT SUMMARY

Science shows that the concentration of alcohol in a person's blood is measurable. Recent medical research suggests that the risk R (given as a present) of having an accident while driving can be modelled by an equation of the type $R(x) = R_0 e^{kx}$ where x is the variable concentration of alcohol in the blood and k is a constant.

Example

The risk R of having an accident while driving is modelled by the equation $R(x) = 2e^{kx}$, where x is the concentration of alcohol in the driver's blood.

Suppose that a concentration of alcohol in the blood of 0.06 results in 4% risk

(R=4) of an accident.

a. Find the value of the constant k in the equation $R(x) = 2e^{kx}$

b. Using this value of k, what is the risk if the concentration of alcohol is 0.08?

c. Using the same value of k, what concentration of alcohol corresponds to a risk of 100%?

d. If the law stipulates that anyone with a risk of having an accident of 10% or more should not drive, at what concentration of alcohol should the driver be arrested and charged?

Solution:

a. From the equation $R(x) = R_0 e^{kx}$, substituting,

$$4 = 2e^{k(0.06)} \Leftrightarrow e^{0.06k} = 2 \Rightarrow 0.06k = \ln 2 \Leftrightarrow k = \frac{\ln 2}{0.06} = 11.552453$$

b. The equation becomes $R(x) = 2e^{11.552453x}$. Now, $R(0.08) = 2e^{(11.552453)(0.08)} = 5.0$

Therefore, the risk is 5%

c. $100 = 2e^{11.552453x} \Leftrightarrow e^{11.552453x} = 50 \Rightarrow x = \dfrac{\ln 50}{11.552453} = 0.33$.

Application activity 4.3.7

Suppose that the risk R of having accident while driving a car is modelled by the equation $R(x) = 4e^{kx}$. Suppose the concentration of alcohol of 0.05 results in 8% of risk of accident, what is the risk if the concentration is 0.18 and what concentration yields to 100% of risk of accident

4.4 End unit assessment

1. Determine the domain and range of the following functions

$a)\ f(x)=\log_2(3x-2)$ $c)\ f(x)=2e^{3x+1}$

$b)\ f(x)=\ln(x^2-1)$ $d)\ f(t)=4^{\sqrt{3t+1}}$

2. Evaluate the limit and give the equation of the asymptotes if any of $\lim_{x\to 0^+}\frac{\ln x}{x}$

3. Differentiate with respect to x the following functions

a) $f(x)=\log_2\sqrt{\frac{x^2-4}{x+2}}$ b) $h(x)=\frac{1}{3}\left(4^{2x+5}\right)$

4. The tangent line touches the function $y=e^{2x+1}$ at the point A whose abscissa equal to $\frac{-1}{2}$, i) Determine the coordinates of the intersection point of the graph and its tangent line ;

ii) Determine the equation of the tangent line and the equation of the normal line at A;

iii) Sketch the graph of the function and the tangent in the same *xy*-coordinates system using a table of values and/or mathematical software if possible.

5. A deposit of $1000 is made in an account that earns interest at an annual rate of 5%. How long will it take for the balance to double if the interest is compounded

(a) annually, (b) monthly, (c) daily, and (d) continuously?

UNIT 5

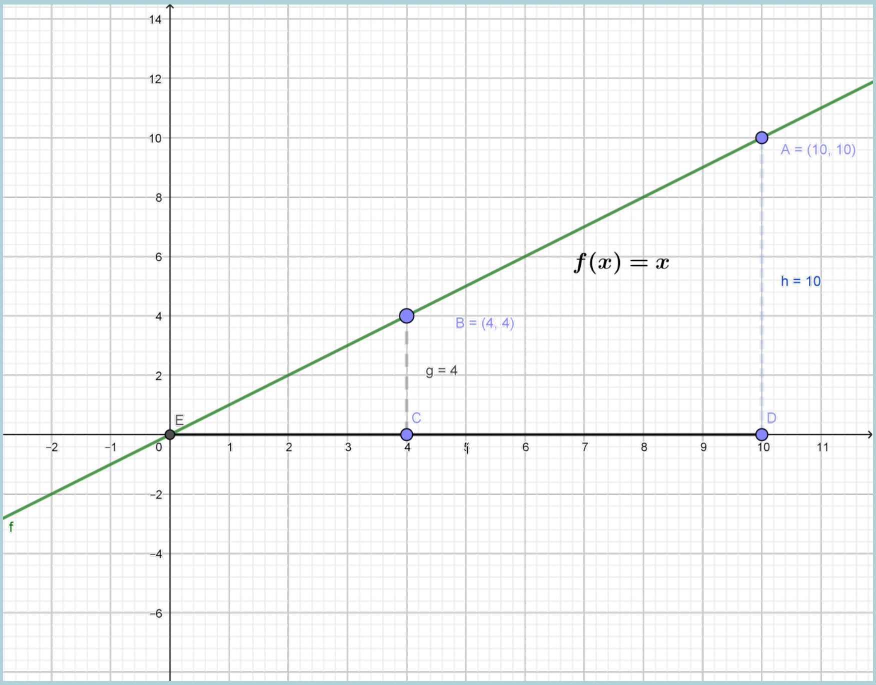
INTEGRATION

Key Unit competence:

Determine correctly integration as the inverse of differentiation or limit of a sum and apply it to find area of plane surfaces, volumes of solid of revolution and lengths of curved lines.

5.0 Introductory activity

Two groups of students were asked to calculate the area of a quadrilateral field BCDA shown in the following figure:



The first group calculated the difference of the area for two triangles EDA and ECB

$$A_1 = area(\Delta EDA) - area(\Delta ECB)$$

The second group with high critical thinking skills used a function $F(x)$ that was differentiated to find $f(x)=x$ (which means $F'(x)=f(x)$) and the x-coordinate d of D and the x- coordinate c of C in the following way: $A_2=F(d)-F(c)$.

1. Determine the area A_1 found by the first group.

2. Discuss and determine the function $F(x)$ used by the second group. What is the name of $F(x)$ if you relate it with $f(x)$?

3. Determine A_2 the area found by the second group using $F(x)$

4. Compare A_1 and A_2. Discuss if it is possible to find the area bounded by a function $f(x)$, the x-axis and lines with equation $x=x_1$ and $x=x_2$?

In this unit we are going to study the anti-derivatives of a given function $f(x)$ generally called integrals and their application in other sciences and real life situations such as the calculation of area of plane regions, etc.

5.1 Differentials

5.1.1 Increment and differential of a function

Activity 5.1.1

The total consumption of a company is modeled by the function $y=f(x)=4+0.5x+0.1\sqrt{x}$, where x is the total disposable income (one unit representing 10^6 Rwandan Francs). If $x=24$ with a maximum error of 0.2;

What is the consumption of the company at $x=2$ and at $x=10$?

If Δx is the increment of x from 2 to 10, what is the corresponding increment Δy of the consumption of the company? Represent graphically this situation

Discuss the increment of f if x changes from x_0 to x_1 where $x_1>x_0$.

Given that the variation of f when x changes from x_0 *to* $x_0+\Delta x$ is $\Delta y=f'(x_0)\Delta x$ determine the limit of Δy *as* Δx becomes very small.

Represent graphically the increment on x and the increment on f showing x_0 and x_1 and compare Δy and its limit when $\Delta x\to 0$.

Content summary

Let be given a function $y=f(x)$ continuous on a certain real interval. When the variable x changes from x to $x+h$ within the interval, $f(x)$ changes from $f(x)$ to $f(x+h)$.

The variation in x **called increment of** x is $\Delta x=h$ while the corresponding variation in y becomes $\Delta y=f(x+\Delta x)-f(x)$.

In this case, the **increment of the function** $y=f(x)$ is $\Delta y=f(x+\Delta x)-f(x)$.

When Δx becomes very small, the change in y can be approximated by the differential of y, that is, $\Delta y\approx dy$ *and* $\Delta x=dx$.

The rate of change $\frac{\Delta y}{\Delta x}=\frac{f(x+h)-f(x)}{h}$ means that

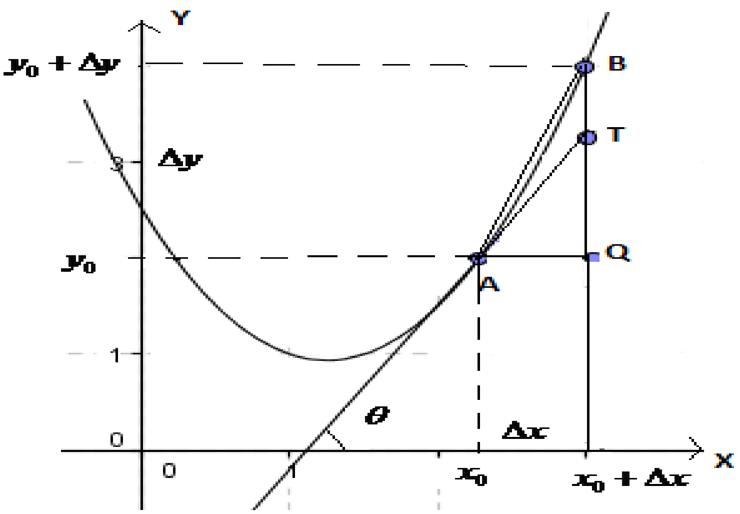
$\Delta y=\frac{\Delta y}{\Delta x}\Delta x\approx f'(x)\Delta x$ and $\Delta y\approx f'(x)\Delta x$

Therefore, $dy=f'(x)dx$.

The differential of a function $f(x)$ is the approximated increment of that function when the variation in x becomes very small. It is given by $dy=f'(x)dx$ .f' (x) {\displaystyle f'(x)}

Geometrically, the ratio $\frac{\Delta y}{\Delta x}=\frac{f(x_0+\Delta x)-f(x_0)}{\Delta x}$ represents the slope of the line AB passing through $A(x_0, f(x_0))$ *and* $B(x_0+\Delta x, f(x_0+\Delta x))$ as illustrated in Figure bellow.

Figure 5.1: Increment of a function



When the change in x becomes smaller and smaller, that is Δx approaches 0, the line L becomes the tangent line (T) to the graph at the point $(x_0, f(x_0))$ where $f(x_0) = y_0$.

In addition, the graph shows that dy is the change in y along the tangent line, while Δy is the resulting change in y along the curve of the function.

This means that the ratio $\dfrac{\Delta y}{\Delta x}$ approaches the slope of this tangent which is

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = f'(x_0)$$

The figure 5.1 shows that the derivative of function $y = f(x)$ at x_0 is the slope of the geometric tangent line to the graph of the function at the point A; it is

such that $y' = \tan\theta = \dfrac{QT}{AQ} = \dfrac{QT}{dx}$

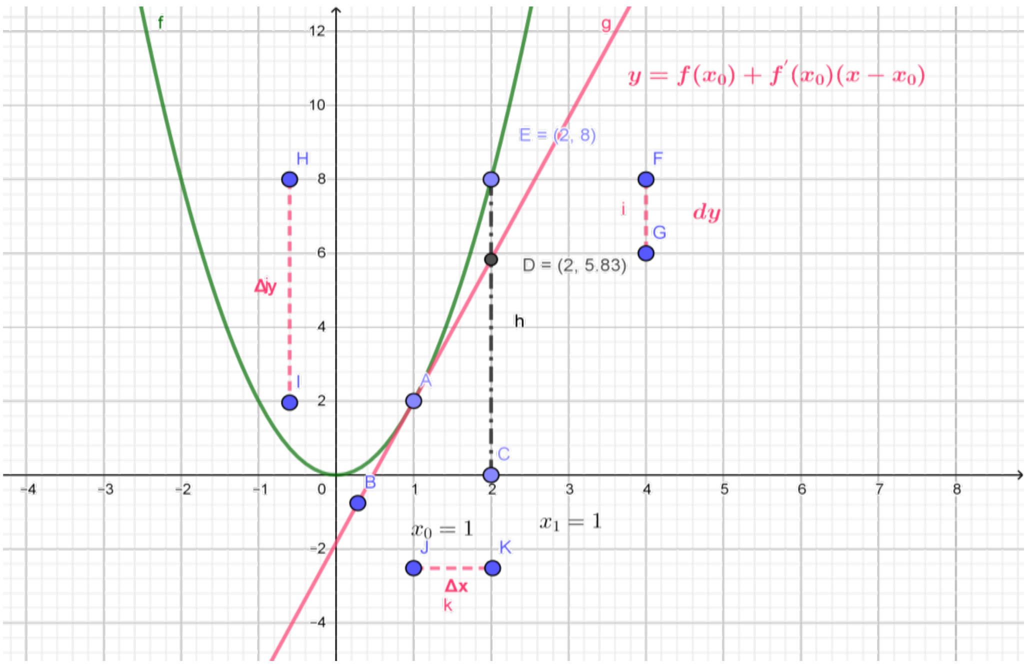
Example

Consider the function $y = f(x) = 2x^2$.

Illustrate the increment of y when increases from 1 to 2 .

Solution

Figure 5.2: illustration of increment of the function $y = f(x) = 2x^2$



This graph shows that from the point $x = x_0 = 1\ to\ x = x_1 = 2$ where the increment is $\Delta x = 2-1=1$, the function $y = f(x) = 2x^2$ varies from $y_0 = f(1) = 2$ to $y_0 = f(2) = 8$. That means the increment of the function $\Delta y = 8-2 = 6$ which is different to the differential dy measured from the tangent to the graph as it illustrated on the above figure 5.2

Application activity 5.1.1

Determine the differential of each of the following function:

$a)\quad f(x) = x^2e^x \qquad b)\quad f(x) = \dfrac{\ln x}{x}$

5.1.2 Operation on increments

Activity 5.1.2

Consider two functions $f(x) = 2x+5$ and $g(x) = x^2 - 5$.

a) Determine $h(x) = f(x) + g(x)$

b) When x varies from $x = x_0 = 1\ to\ x = x_1 = 1.5$, Calculate the increment of $f(x),\ g(x) \text{and}\ h(x)$

c) Deduce how to find the increment of $h(x)$ in function of the increment of $f(x) \text{ and } g(x)$

Content summary

Increment of the sum of functions

Given that the increment of the function $y = f(x)$ is $\Delta y = f(x+\Delta x) - f(x)$, this can be extended to a composite function, a sum of function or a product of function.

Indeed, let us consider two functions $f(x) = x$ and $g(x) = 3x \Rightarrow (f+g)(x) = 4x$

$\Delta f = (x+\Delta x) - x = \Delta x$, $\Delta g = 3(x+\Delta x) - 3x = 3\Delta x$

$\Delta(f+g) = 4(x+\Delta x) - 4x = 4\Delta x$

On the other hand, $\Delta f + \Delta g = 4\Delta x$

Therefore $\Delta(f+g) = \Delta f + \Delta g$.

Similarly, $\Delta(f-g) = \Delta f - \Delta g$.

In addition, $\forall k \in \mathbb{Z}$ and $f(x)$ a real function, $\Delta(kf) = k\Delta f$

Increment of the product of functions

Let us consider $y = f(x) = x^2$

$\Delta y = (x+\Delta x)^2 - x^2 = x^2 + 2x\Delta x + (\Delta x)^2 - x^2 \Rightarrow \Delta y = 2x\Delta x + \Delta x$ (1)

In the other hand, $y = f(x) = x^2$ can be looked as $y = x \times x$ as a product of two identique functions.

In this case we have:

$\Delta y = (\Delta x)^2$ (2)

The result in the equality (1) is different to the result on the equality (2) where

$\Delta y = 2x\Delta x + (\Delta x)^2 \neq (\Delta x)^2$

Therefore, $\Delta(f.g) \neq (\Delta f).(\Delta g)$

Similarly, $\Delta\left(\frac{f}{g}\right) \neq \frac{(\Delta f)}{(\Delta g)}$.

Example

Given two functions $f(x) = x+1$ *and* $g(x) = 2x$. When x varies from 2 *to* 2.7. Find the increment of:

a) $(f+g)(x)$

b) $(f-g)(x)$

c) $(f \times g)(x)$

Solution:

a) We have $(f+g)(x) = 3x+1$

then, $\Delta(f+g) = 3(2+0.7)+1-(3(2)+1) = 2.1$

b) $(f-g)(x) = -x+1$

$\Delta(f-g)=-0.7$

c) $(f.g)(x)=2x^2+2x$

$\Delta(f.g)=2(2.7)^2+2(2.7)-(8+4)=0.69$

Application activity 5.1.2

Find the increment of each of the following function when the variable changes from 5 to 5.08:

a) $f(x)=x^2e^x$ b) $f(x)=\dfrac{\ln x}{x}$

5.1.3 Properties of differentials and their applications

Activity 5.1.3

Consider two functions $f(x)=2x+5$ and $g(x)=x^2-5$.

a) Determine $h(x)=f(x)+g(x)$

b) Calculate the differential of $f(x)$, $g(x)$ and $h(x)$

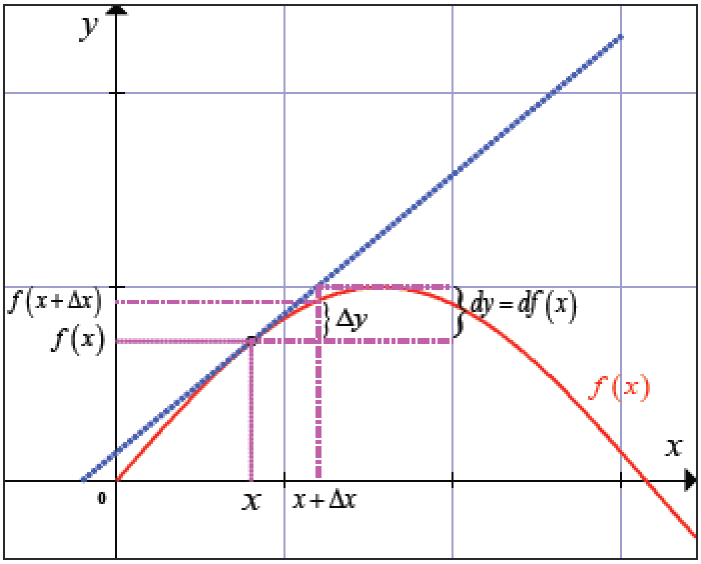
c) Deduce how to find the differential of $h(x)$ in function of the increment of $f(x)$ and $g(x)$.

Content summary

Properties of differentials

It has been shown that $\Delta y \approx f'(x)\Delta x$

If we denote the change in *x* by δx instead of Δx, then the change Δy in *y*, is approximated by the differential δx, that is $\Delta y \approx \delta y \approx f'(x)\,\mathrm{d}x$.



Given the functions f and g, it is easy to show that when f and g are differentiable functions on an interval I of $\mathbb{R}$, and $k \in \mathbb{R}$, the following properties hold:

a) $d(f+g) = df + dg$

b) $d(f.g) = gdf + fdg$

c) $d(kf) = kdf$

d) $d\left(\frac{f}{g}\right) = \frac{gdf - fdg}{g^2}$

e) $d(g \circ f) = g'(u)df$ where $g(u) = g\left[f(x)\right]$

f) $d\left(\sqrt{f}\right) = \frac{df}{2\sqrt{f}}$

Application of differentials on approximation

It was highlighted above that if the variable in *x changes* by δx instead of Δx, then the change for the value of the function $y = f(x)$ is Δy which is approximately $\Delta y \approx f'(x)\mathrm{d}x$.

Example

The demand function of an item is modeled by the equation $y = \frac{2}{\sqrt[4]{x}}$, where x the number of units is demanded and y is the price in thousands of Frw.

Given that $x = 16$, with a maximum error of 2, use differentials to approximate the maximum error in y and interpret your result.

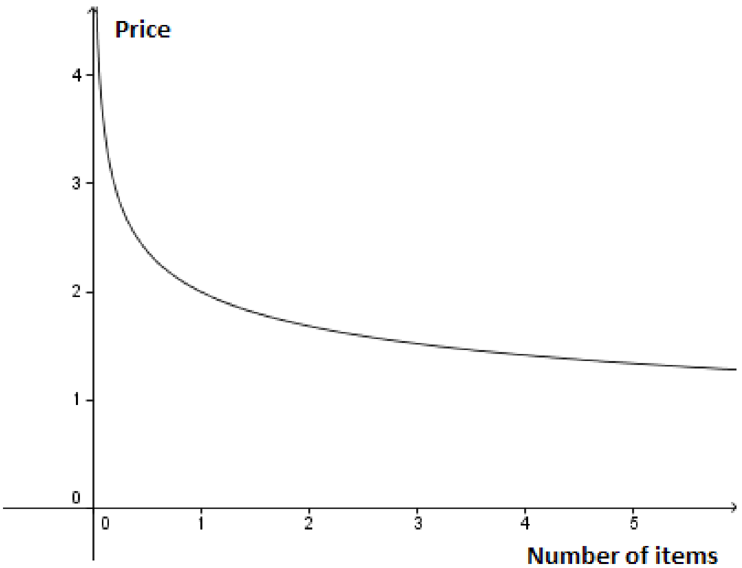
Solution

$dy = \frac{-dx}{2\sqrt[4]{x^5}}$. For $x = 16$ and $dx = 2$, we have $dy = \frac{-2}{2\sqrt[4]{16^5}} = -\frac{1}{32} = -0.03125$

For 16 units demanded ($x = 16$), with an error of 2, the corresponding price is $y = \frac{2}{\sqrt[4]{16}} = 1$ thousand (1 000Frw), with an approximate error of 31 Frw or

$(0.03125 \times 1000 = 31.25 Frw)$.

Figure 5.3: The price in thousands of Frw as function of the number of units demanded



X	0.5	1	2	3	4	....
Y	2.4	2	1.7	1.5	1.4	....

One can observe that as the number of demanded units increases, the price decreases.

Example

By approximately what percentage does the area of a circle increase if the radius increases by: 2% ?

Solution

The area A of a circle is given in terms of the radius r by $A = \pi r^2$

$$\Delta A = dA = \frac{dA}{dr} = 2\pi r dr$$

We divide this approximation by $A = \pi r^2$ to get an approximation that links the relative changes in A and r:

$$\frac{\Delta A}{A} \approx \frac{dA}{A} = \frac{2\pi r dr}{\pi r^2} = 2\frac{dr}{r}$$

If r increases by 2%, then $dr = \frac{2r}{100}$, so $\frac{\Delta A}{A} \approx 2\times\frac{2}{100} = \frac{4}{100}$

Thus, A increases by approximately 4%.

Application of differentials on the calculation of error

It was proved that the increment Δy is given by $\Delta y \approx f'(x)\,\mathrm{d}x$. When small error dx is made on the variable x, the error made on the value of the function $f(x)$ is Δy.

Example

While measuring the diameter of a circular garden, Felix made an error of 0.02m. Given that he obtained 15m of diameter, estimate the error made on the area of this garden.

Solution:

Let x be the diameter of the garden,

The error made on the diameter is dx.

The area of the garden is $A = \frac{\pi x^2}{4}$. The error made on the area of this garden is $dA = \frac{\pi x}{2}dx$.

Then, for $x = 15m$ and $dx = 0.02$, we have $dA = \frac{\pi x}{2}dx = \frac{\pi .15.(0.02)}{2}m^2 = 0.47m^2$

Example

The deflection at the centre of a road of length l and diameter d supported at its ends and loaded at the centre with a weight w varies as wl^3d^{-4}. What is the percentage increase in the deflection corresponding to the percentage increase in w, l and d of $3, 2$ and 1 respectively?

Solution

Let the deflection of the road at the centre be D

$$D = \frac{kwl^3}{d^4}$$

$$\ln D = \ln \frac{kwl^3}{d^4}$$

$$\Rightarrow \ln D = \ln k + \ln w + 3\ln l - 4\ln d$$

$$\Rightarrow \frac{\Delta D}{D} = \frac{\Delta w}{w} + 3\frac{\Delta l}{l} - 4\frac{\Delta d}{d}$$

$$\Rightarrow 100\frac{\Delta D}{D} = 100\frac{\Delta w}{w} + 3\times 100\frac{\Delta l}{l} - 4\times 100\frac{\Delta d}{d}$$

$$\Rightarrow 3 + 3\times 2 - 4\times 1 = 5\%$$

Example

While determining the volume of a cube of 7dm of edge, Anathalia committed an error of $0.03dm^3$. What is the corresponding error made on the edge?

Solution:

Let x be the edge of the cube and V its volume.

We have $x = 7dm$ and $dV = 0.03dm^3$. However $V = x^3$ and $dV = 3x^2dx$.

Therefore, $0.03 = 3x^2dx \Leftrightarrow 0.03 = 3.49dx$

$$\Leftrightarrow dx = \frac{0.03}{3.49}$$

$$dx = 0.00002$$

The error made on the edge for the cube is $0.00002dm$.

Application activity 5.1.3

1) Apply properties of differentials to calculate the differential of each of the following function:

a) $f(x)=2x^2e^x$ b) $f(x)=2x^3-x+5$

2. A company designed a tank in the shape of a cube. It claims that the side measures 4 meters, with an error of 0.02. Approximate, in liters, the capacity of the container and use differentials to approximate the error on the measurement of the volume.

3) Find the percentage error in the area of a rectangle when an error of $+1$ per cent is made in measuring its length and breadth.

4) The period T of a simple pendulum is $T=2\pi\sqrt{\frac{l}{g}}$. Find the maximum error in T due to possible errors up to 1% in l and 2.5% in g

5.2 Definition and properties of Indefinite integrals

5.2.1 Definition of indefinite integral

Activity 5.2.1

Suppose that three caterpillars are moving on a straight line with constant velocity $v=2m\,\text{min}^{-1}$ (in meters per min).

1) Write down the position of each caterpillar at time t if their respective initial positions are:

i) 1 meter ii) 2 meters iii) 4 meters.

2) If $e(t)$ is the position in function of time, draw the graph of $e(t)$ for the third caterpillar and verify whether or not $e'(t)=v(t)$ where $v(t)$ is the velocity.

3) In the same way:

i) Find a function $F(x)$ whose derivative is $f(x)=2x$, that is, $F'(x)=f(x)=2x$

ii) Discuss the number of possibilities for $F(x)$ which are there and the relationship among them.

iii) How do functions $F(x)$ differ?

Content summary

Anti-derivatives

Let $y = f(x)$ be a continuous function of variable x. An anti-derivative of $f(x)$ is any function $F(x)$ such that $F'(x) = f(x)$. A function has infinitely many anti-derivatives, all of them differing by an additive constant. It means that if $F(x)$ is an anti-derivative of $f(x)$, $F(x)+c$ (where c is an arbitrary constant) is also an anti-derivative of function $f(x)$.

Example

Given the function $f(x) = x\ln x - x$,

a) Find the derivative of $f(x)$

b) From the answer in (a), deduce the anti-derivative of $g(x) = \ln x$ whose graph passes through point $(e,1)$. Plot the graph of the function g and its anti-derivative on the same rectangular coordinate.

Solution:

a) $f'(x) = (x\ln x - x)' = (x\ln x)' - (x)' = x'\ln x + x(\ln x)' - x' = \ln x$.

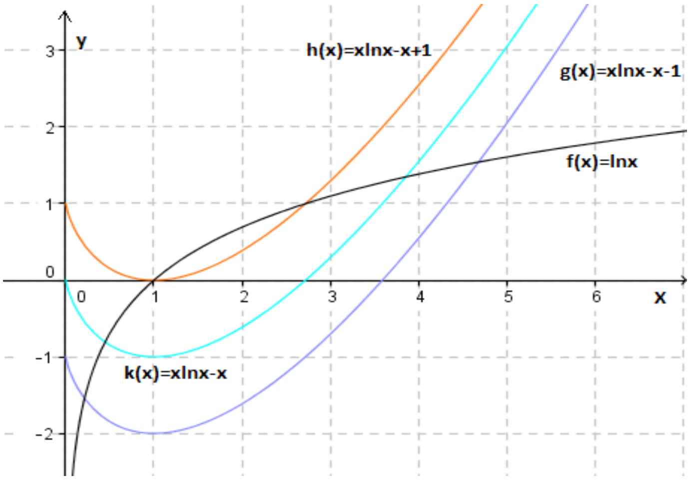
b) The anti-derivatives of $g(x) = \ln x$ are of the type $F(x) = x\ln x - x + c$

$F(e) = e\ln e - e + c = 1 \Leftrightarrow c = 1$

Therefore, the required anti-derivative is $F(x) = x\ln x - x + 1$

The figure below shows function $g(x) = \ln x$ and three of its anti-derivatives.

Figure 5.2: Graph of the function $f(x) = \ln x$ and 3 of its anti-derivatives



Indefinite integrals

Let $y = f(x)$ be a continuous function of variable x. The **indefinite integral** of $f(x)$ is the set of all its anti-derivatives.

If $F(x)$ is any anti-derivative of function $f(x)$, then the indefinite integral of $f(x)$ is denoted and defined as follows:

$\int f(x)dx = F(x) + c$ Where c an arbitrary constant is called the **constant of integration**.

Thus, $\int f(x)dx = F(x) + c$ if and only if $\left[F(x) + c\right]' = F'(x) = f(x)$.

The process of finding the indefinite integral of a function is called **integration.**

The symbol $\int$ is the sign of integration while $f(x)$ is **the integrand.** Note that the integrand is a differential, dx shows that one is integrating with respect to variable x.

Example

Evaluate the following indefinite integrals:

a) $\int 5dx$ b) $\int e^t dt$ c) $\int \frac{1}{x} dx \text{ where } x > 0$

Solution:

a) $\int 5dx = 5x + c$

b) $\int e^t dt = e^t + c$

c) $\int \frac{1}{x} dx = \ln x + c$ for $x > 0$.

Application activity 5.2.1

Evaluate the following integrals

a) $\int \cos x dx$ b) $\int 3x dx$ c) $\int x^2 dx$

5.2.2 Properties of indefinite integrals

Activity 5.2.2

Let $f(x)=5$ and $g(x)=\frac{1}{x}$,

a) Determine $\int f(x)dx$ and $\int g(x)dx$

b) Evaluate $\int (f+g)(x)dx$

c) Compare $\int (f+g)(x)dx$ and $\int f(x)dx+\int g(x)dx$

Content summary

Let $y=f(x)$ and $y=g(x)$ be continuous functions and k a constant. Integration obeys the following properties:

$\int kf(x)dx=k\int f(x)dx$: The integral of the product of a constant by a function is equal to the product of the constant by the integral of the function.

$\int [f(x)\pm g(x)]dx=\int f(x)dx\pm\int g(x)dx$: The indefinite integral of the algebraic sum or difference of two functions is equal to the algebraic sum or difference of the indefinite integrals of those functions.

The derivative of the indefinite integral is equal to the function to be integrated.

$$\frac{d}{dx}\int f(x)dx=f(x)$$

Example

Evaluate:

a) $\int (3x^2+4x-5)dx$ b) $\int 8e^{-2x}dx$ c) $\int (3^{2x}-\frac{1}{x})dx,\ where\ x>0$

Solution:

a) $\int (3x^2+4x-5)dx=x^3+2x^2-5x+c$

b) $\int 8e^{-2x}dx=-4e^{-2x}+c$

c) $\int (3^{2x}-\frac{1}{x})dx=\frac{1}{2\ln 3}e^{2x}-\ln x+c$

Note that: integration is a process which is the inverse of differentiation. In differentiation, we are given a function and we are required to find its derivative or differential coeffitent. In integration, we are to find a function whose differential coeffient is given.

The process of finding a function is colled integration and it reverses the operation of differentiation.

If the function $F(x)$ is an antiderivative of the function f(x) , then the expression $F(x)+C$ is called the indefinte integration of $f(x)$ and is usually denoted by $\int f(x)dx$.

Thus, $\int f(x)dx = \left[F(x)+c\right]' = F'(x) = f(x)$

The function $f(x)$ is called **integrand**.

The symbol $\int$ is usually referred to as **the sign of integration** or **integral sign**.

The symbol dx indicates that integration is to be performed **with respect** to the variable x.

$\int \ldots dx$ means the integral of…with respect to x. For the indefinite integral, we get a complete integral by adding an unkown constant.

Application activity 5.2.2

1. Evaluate:

a) $\int (x^3 + 3\sqrt{x} - 7)dx$ b) $\int (4x - 12x^2 + 8x^6 - 9)dx$ c) $\int (\frac{1}{x^2} + e^{-x} - \frac{2}{x})dx$

2. A student calculated $\int \frac{x^3-2}{x^3}dx$ as follows: $\frac{\int (x^3-2)dx}{\int x^3 dx} = \frac{\frac{1}{4}x^4 - 2x}{\frac{1}{4}x^4} + c$, which is not correct.

Show the mistake and suggest the correct working step and solution.

3. Function $y = f(x)$ is such that $\frac{dy}{dx} = \frac{x^3-5}{x^2}$. Find the expression of $y = f(x)$ if $f(1) = \frac{1}{2}$

4.In Economics, if $f(x)$ is the total cost of producing x units of a certain item, then the marginal cost is the derivative, with respect to x, of the total cost.

Given that the marginal cost is $M(x) = 1 + 50x - 4x^2$, graph $f(x)$ and $M(x)$ on the same diagram.

5.3 Techniques of integration

5.3.1 Basic integration formulas (or immediate integration)

Activity 5.3.1

Given $\left(Arc\tan x\right)'=\frac{1}{1+x^2}$ and $=\left(\frac{a^x}{\ln a}\right)'=a^x$

Discuss how to find $\int\frac{1}{1+x^2}dx$ and $\int a^x dx$

What is the formula that can be used to find these integrals?

Content summary

Given any anti-derivative $F(x)$ of a function $f(x)$, every possible anti-derivative of $f(x)$ can be written in the form of $F(\mathrm{x})+c$,where c is any constant

This means that when you remember the formulae used to differentiate some functions, it is easy to determine integrals. Roughly speaking, the integration is backward of the differentiation.

This table shows some basic integration formulas

$f(x)$	$\phi(x)=\int f(x)dx+c$
$x^k\ (k\neq -1)$	$\frac{x^{k+1}}{k+1}+c$
$\frac{1}{x}$	$\ln x+c$
$\sin x$	$-\cos x+c$
$\cos x$	$\sin x+c$
$\tan x$	$-\ln\lvert\cos x\rvert=\ln\lvert\sec x\rvert+c$
$\sec x$	$\ln\lvert\sec x+\tan x\rvert+c$
$\mathrm{co}\sec x$	$\ln\lvert\mathrm{co}\sec x-co\tan x\rvert=\ln\left\lvert\tan\frac{x}{2}\right\rvert+c$
$\cot x$	$\ln\lvert\sin x\rvert+c$

$\sin^2 x$	$\frac{1}{2}\left(x-\frac{1}{2}\sin 2x\right)+c$
$\cos^2 x$	$\frac{1}{2}\left(x+\frac{1}{2}\cos 2x\right)+c$
$e^{ax}\ (a\neq 0)$	$\frac{1}{a}e^{ax}+c$
$a^x\ (a>0, a\neq 1)$	$\frac{a^x}{\ln x}+c$
$\ln x$	$x\ln x - x + c$
$\frac{1}{\sqrt{a^2-x^2}}, a>0$	$\arcsin\frac{x}{a}+c$
$\frac{1}{a^2+x^2}, a>0$	$\frac{1}{a}\arctan\frac{x}{a}+c$
$\frac{1}{x^2-a^2}, a>0$	$\frac{1}{2a}\ln\left\|\frac{x-a}{x+a}\right\|+c$
$\frac{u'}{u^2}$	$-\frac{1}{u}+C$
$\frac{1}{x^2+k^2}$	$\frac{1}{k}\arctan\frac{x}{k}+C$
$\frac{1}{x^2-k^2}$	$\frac{1}{2k}\ln\left\|\frac{x-k}{x+k}\right\|+C$

Application activity 5.3.1

Compute the following indefinite integrals:

1. $\int e^{3x+1}dx$
2. $\int 3^x dx$
3. $\int (10+\sin x)dx$
4. $\int (8-x^5)dx$

5.3.2 Integration by substitution or change of variable

Activity 5.3.2

Consider the integral $\int e^{5x+2}dx$. By letting $u=5x+2$ and differentiating u with respect to x, deduce dx in function of u and discuss how to determine $\int e^{5x+2}dx$ using expression of u.

Content summary

Integration by substitution is based on rule for differentiating composite functions. The formula for integration by substitution is

$\int f(x)dx=\int f(x(t))x'(t)dt$, referring to the table above of basics formulas of integrations.

Example

Find $\int e^{2x}dx$

Solution

Let $t=2x$, so $dt=2dx \Rightarrow dx=\frac{1}{2}dt$

We have $\int e^{2x}dx=\frac{1}{2}\int e^{t}dt=\frac{1}{2}e^{t}+c=\frac{1}{2}e^{2x}+c$

Example

Find $\int (2x+1)^4\, dx$

Solution

Let $u=2x+1$, so $du=2dx \Rightarrow dx=\frac{1}{2}du$

We have $\int (2x+1)^4\, dx=\frac{1}{2}\int u^4 du \Rightarrow \frac{1}{2}\times\frac{1}{5}u^5+c \Rightarrow \frac{1}{10}u^5+c \Rightarrow \frac{1}{10}(2x+1)^5+c$.

Application activity 5.3.2

1) Evaluate

a) $\int (x^2+1)2xdx$;

b) $\int x^2 e^{x^3} dx$;

c) $\int (2x+1)e^{x^2+x+2}dx$;

d) $\int e^{3\cos 2x}\sin 2xdx$

2) A particle moves in a straight line such that its velocity at time t seconds is given by $v=\frac{100t}{(t^2+1)^3}ms^{-1}$. Find the distance travelled by the particle in the first two seconds of motion.

5.3.3 Integration by parts

Look at the function $f(x)=(x-1)e^x$.

1) Differentiate $f(x)$ using the product rule.

2) From 1) determine the value of $\int xe^x dx$.

3) Is it true that $\int uvdx = \int udx \int vdx$?

Content summary

From Activity 5.3.3 we see that the integral of a product of two functions does not equal the product of the integrals of the two functions. To develop a rule, we start with the product rule for differentiation:

$$\frac{d}{dx}(uv)=v\frac{du}{dx}+u\frac{dv}{dx}$$

Integrating both sides with respect to x yields

$$uv=\int v\frac{du}{dx}dx+\int u\frac{dv}{dx}dx$$

$$\Rightarrow uv=\int vdu+\int udv$$

$$\Rightarrow \int udv = uv-\int vdu \text{ or } \int vdu = uv-\int udv$$

This is the formula for **integration by parts.**

To apply the integration by parts to a given integral, we must first factor its integrand into two parts.

An effective strategy is to choose for dv the most complicated factor that can readily be integrated. Then we differentiate the other part u to find du.

The following table can be used:

u	v' or dv
Logarithmic function	Polynomial function
Polynomial function	Exponential function
Polynomial function	Trigonometric function
Exponential function	Trigonometric function
Inverse trigonometric function	Polynomial function

Example

Find $\int \ln x dx$

Solution

Let

$$u = \ln x \Rightarrow du = \frac{dx}{x}$$

$$dv = dx \Rightarrow v = x$$

Then, $\int \ln x dx = x \ln|x| - \int x \frac{dx}{x} = x \ln|x| - \int dx = x \ln|x| - x + c$

Example

Find $\int xe^x dx \int xe^x dx$

Solution

Let

$$u = x \Rightarrow du = dx$$

$$dv = e^x dx \Rightarrow v = e^x$$

Then,

$$\int xe^x dx = xe^x - \int e^x dx$$

$$\Rightarrow xe^x - e^x + c$$

Example

Find $\int x \sin 2x dx$

Solution

Let

$$u = x \Rightarrow du = dx$$

$$dv = \sin 2x dx \Rightarrow v = -\frac{1}{2}\cos 2x$$

Then,

$$\int x \sin 2x dx = -\frac{1}{2}x\cos 2x - \int -\frac{1}{2}\cos 2x dx$$

$$\Rightarrow -\frac{1}{2}x\cos 2x + \frac{1}{2}\int \cos 2x dx$$

$$\Rightarrow -\frac{1}{2}x\cos 2x + \frac{1}{4}\sin 2x + c$$

Example

Find $I = \int e^x \sin 2x dx$

Solution

Let

$$u = \sin 2x \Rightarrow du = 2\cos 2x dx$$

$$dv = e^x dx \Rightarrow v = e^x$$

$$I = \int e^x \sin 2x dx = e^x \sin 2x - 2\int e^x \cos 2x dx$$

$$let\ I_1 = \int e^x \cos 2x dx$$

$$u = \cos 2x \Rightarrow du = -2\sin 2x dx$$

$$dv = e^x dx \Rightarrow v = e^x$$

$$I_1 = \int e^x \cos 2x dx = e^x \cos 2x + 2\int e^x \sin 2x\, dx$$

$$I = e^x \sin 2x - 2\left(e^x \cos 2x + 2\int e^x \sin 2x dx\right)$$

$$I = e^x\left(\sin 2x - 2\cos 2x\right) - 4I$$

$$5I = e^x\left(\sin 2x - 2\cos 2x\right)$$

$$I = \frac{1}{5}\left(\sin 2x - 2\cos 2x\right)$$

Example

Find

$$I = \int \arcsin x dx$$

Solution

$$u = \arcsin x \Rightarrow du = \frac{dx}{\sqrt{1-x^2}}$$

$$dv = dx \Rightarrow v = x$$

$$I = x \arcsin x - \int \frac{xdx}{\sqrt{1-x^2}}$$

$$I_1 = \int \frac{xdx}{\sqrt{1-x^2}}$$

$$let\ k = 1 - x^2 \Rightarrow dk = -2xdx \Rightarrow xdx = -\frac{dk}{2}$$

$$I_1 = -\frac{1}{2}\int k^{-\frac{1}{2}} dk = -\frac{1}{2}\left(\frac{k^{1-\frac{1}{2}}}{1-\frac{1}{2}}\right) + c$$

$$I_1 = -\frac{1}{2}\left(\frac{2}{1}\right)\left(k^{\frac{1}{2}}\right) + c$$

$$I_1 = -1k^{\frac{1}{2}} + c \Rightarrow I_1 = -\sqrt{1-x^2} + c$$

$$I = x \arcsin x + \sqrt{1-x^2} + c$$

Application activity 5.3.3

Use integration by parts method to find:

1) $\int x\cos 2x dx$ 2) $\int xe^{3x}dx$ 3) $\int x\sin 4x dx$ 4) $\int xe^{-2x}dx$

5.4 Integration of rational functions

5.4.1 Integration of rational function where numerator is expressed in terms of derivative of denominator

Activity 5.4.1

From derivative of reciprocal functions and logarithmic derivative, find:

1) $\int \frac{xdx}{\left(1-x^2\right)^2}$ 2) $\int \frac{(2x-1)dx}{3x^2-3x+1}$

Content summary

When the numerator is expressed in terms of derivative of denominator, the integration by substitution is adequate. Reference can also be made on the table of derivatives and integration.

Example

Find: $\int\left(2-3x^2+\frac{1}{x}-\frac{4}{1+x^2}\right)dx$

Solution:

$$2\int dx-3\int x^2dx+\int\frac{dx}{x}-4\int\frac{dx}{1+x^2}$$

$$I=2x-x^3+\ln|x|-4\arctan x+c$$

Example

Find $\int\frac{x+1}{x^2+2x+3}dx$

Solution

Let $k=x^2+2x+3\Rightarrow dk=2(x+1)dx$

$(x+1)\,dx = \frac{dk}{2}$

$I = \frac{1}{2}\int \frac{dk}{k} = \frac{1}{2}\ln|k| + c$

$I = \frac{1}{2}\ln\left|x^2+2x+3\right| + c$

$I = \ln\sqrt{x^2 2x+3} + c$

Example

Find: $\int \frac{dx}{x^2+2x+1}$

Solution

$x^2+2x+1 = (x+1)^2$

$I = \int \frac{dx}{(x+1)^2}$

$let\ k = x+1 \Rightarrow dk = dx$

$I = \int \frac{dk}{k^2} = \int k^{-2}dk = -k + c$

$I = -x-1+c$

Application activity 5.4.1

Evaluate:

1) $\int \frac{(x+1)\,dx}{\left(x^2+2x+3\right)^2}$

2) $\int \frac{xdx}{\left(1-x^2\right)^5}$

3) $\int \frac{x^2dx}{\left(2x^3+3\right)^2}$

4) $\int \frac{x+1}{\left(x^2+2x+5\right)^3}dx$

5) $\int \frac{x^2}{\left(2x^3+5\right)^2}dx$

5.4.2 Integration of rational functions for which the degree of the numerator is greater or equal to the degree of the denominator

Activity 5.4.2

Recall that if quotient of the division $\frac{f(x)}{g(x)}$ is $q(x)$ and remainder is $r(x)$, then : $\frac{f(x)}{g(x)}=q(x)+\frac{r(x)}{g(x)}$. Use long division to write the equivalent expression for:

1) $\frac{2x+4}{5x-3}$ 2) $\frac{x^2-3x+2}{x^2+2}$ 3) $\frac{x^2+1}{x-1}$ 4) $\frac{x^3+2x-4}{x^2+2}$

Hence deduce their antiderivatives.

Content summary

When we have rational functions for which the degree of the numerator is greater or equal to the degree of the denominator, we proceed by long division.

Example

Find: $\int \frac{x^2 dx}{x+1}$

Solution:

Using long division , we get:

$x^2 : x+1 = x-1+\frac{1}{x+1}$, then $I=\int\left(x-1+\frac{1}{x+1}\right)dx$

$I=\frac{x^2}{2}-x+\ln|x+1|+c$

Example

Find: $\int\left(\frac{x+1}{x-1}\right)dx$

Solution:

$$\int\frac{(x+1)}{(x-1)}dx \Rightarrow x+1:x-1=1+\frac{2}{x-1}$$

$$\int dx+2\int\frac{dx}{x-1}$$

$$x+2\ln|x-1|+c$$

Application activity 5.4.2

Evaluate the following integrals:

1) $\int\left(\frac{x^3-2}{x^2+1}\right)dx$ 2) $\int\left(\frac{x^2-2}{x^2+x-2}\right)dx$

3) $\int\left(\frac{x^2+1}{6x-9x^2}\right)dx$ 4) $\int\left(\frac{x^3+1}{x^2+7x+12}\right)dx$

5.4.3 Integration of rational function where the degree of the numerator is less than degree of the denominator

In this case, we reduce the fraction in simple fractions. The first step is to factorize the denominator.

Four cases arise:

5.4.3.1 The denominator is factorized into linear factors

Activity 5.4.3.2

Factorize completely the denominator and then decompose the given fraction into partial fractions:

1) $\frac{x-2}{x^2+2x}$ 2) $\frac{x}{x^2+3x+2}$ 3) $\frac{2}{x^2-4}$ 4) $\frac{2x-3}{x^2-x-2}$

Hence or otherwise find their anti-derivatives

Content summary

To each factor: $ax+b$ occurring once in the denominator of a proper rational fraction, there corresponds a single partial fraction of the form $\frac{A}{ax+b}$. Where A is a constant to be found. But to each factor $ax+b$ occurring n terms in the denominator of a proper rational fraction, there corresponds a sum of n partial form : $\frac{A_1}{ax+b}+\frac{A_2}{(ax+b)^2}+\frac{A_3}{(ax+b)^3}+\ldots+\frac{A_n}{(ax+b)^n}$. Where A_n are constants to be determined.

Example

Find: $\int \frac{x+3}{x^2-5x+4}dx$

Solution:

We need to factorize: $x^2-5x+4=(x-4)(x-1)$

$$\frac{x+3}{x^2-5x+4}=\frac{x+3}{(x-4)(x-1)}=\frac{A}{x-4}+\frac{B}{x-1}$$

$$\Rightarrow x+3=Ax-A+Bx-4B$$

$$\Rightarrow \begin{cases} x(A+B)=x(1) \\ -A-4B=3 \end{cases} \Rightarrow \begin{cases} A+B=1 \\ -A-4B=3 \end{cases} \Rightarrow \begin{cases} A=\frac{7}{3} \\ B=-\frac{4}{3} \end{cases}$$

$$\int \frac{Adx}{x-4}+\int \frac{Bdx}{x-1} \Rightarrow \frac{7}{3}\int \frac{dx}{x-4}-\frac{4}{3}\int \frac{dx}{x-1}$$

$$\Rightarrow \frac{7}{3}\ln|x-4|-\frac{4}{3}\ln|x-1|+c$$

$$\Rightarrow I=\ln \sqrt[3]{\frac{(x-4)^7}{(x-1)^4}}+c$$

Example

Find: $\int \frac{dx}{x^2-4}$

Solution:

$$x^2-4=(x-2)(x+2)$$

$$\frac{1}{x^2-4}=\frac{A}{x-2}+\frac{B}{x+2}=\frac{A(x+2)+B(x-2)}{(x-2)(x+2)}$$

$$\Rightarrow Ax+2A+Bx-2B=1$$

$$\Rightarrow \begin{cases} A+B=0 \\ 2A-2B=1 \end{cases} \Rightarrow \begin{cases} A=\frac{1}{4} \\ B=-\frac{1}{4} \end{cases}$$

$$\Rightarrow \frac{1}{4}\int\frac{dx}{x-2}-\frac{1}{4}\int\frac{dx}{x+2}$$

$$\Rightarrow \frac{1}{4}\ln|x-2|-\frac{1}{4}\ln|x+2|+c$$

$$\Rightarrow \ln\sqrt[4]{\frac{x-2}{x+2}}+c$$

Example

Find: $\int\frac{(2x+2)\,dx}{x^2+2x+1}$

Solution

$$x^2+2x+1=(x+1)(x+1)=(x+1)^2$$

$$\Rightarrow \frac{2x+2}{(x+1)^2}=\frac{A}{x+1}+\frac{B}{(x+1)^2}=\frac{Ax+A+B}{(x+1)^2}$$

$$\Rightarrow \begin{cases} A=2 \\ A+B=2 \end{cases} \Rightarrow \begin{cases} A=2 \\ B=0 \end{cases}$$

$$\Rightarrow 2\int\frac{dx}{x+1}+0$$

$$\Rightarrow 2\ln|x+1|+c$$

Application activity 5.4.3.1

Evaluate the following integrals

1) $\int \frac{2dx}{x^2-1}$ 2) $\int \frac{xdx}{x^2+3x+2}$

3) $\int \frac{(x-2)dx}{2x-x^2}$ 4) $\int \frac{xdx}{x^2+4x+4}$

5.4.3.2 The denominator is a quadratic factor

Activity 5.4.3.2

Given that, for $a \neq 0, b, c \in \mathbb{R}$, $x^2+bx+c = a\left[\left(x+\frac{b}{2a}\right)^2 - \frac{b^2-4ac}{4a^2}\right]$

Use this relation to transform the denominator of each of the following integrals and then integrate taking care that, where necessary, you can let

$u = x+\frac{b}{2a}$ and get $\int \frac{u'}{u^2}du = -\frac{1}{u}+C$.

1) $\int \frac{dx}{x^2+3x+2}$ 2) $\int \frac{dx}{x^2-4x+4}$ 3) $\int \frac{dx}{x^2-6x+18}$

Given that, for $a \neq 0, b, c \in \mathbb{R}$, $x^2+bx+c = a\left[\left(x+\frac{b}{2a}\right)^2 - \frac{b^2-4ac}{4a^2}\right]$

Use this relation to transform the denominator of each of the following integrals and then integrate taking care that, where necessary, you can let

$u = x+\frac{b}{2a}$ and get $\int \frac{u'}{u^2}du = -\frac{1}{u}+C$.

1) $\int \frac{dx}{x^2+3x+2}$ 2) $\int \frac{dx}{x^2-4x+4}$ 3) $\int \frac{dx}{x^2-6x+18}$

Content summary

For the integral of the form: $\int \frac{dx}{ax^2+bx+c}$

If $b^2-4ac=0$, then $\int \frac{dx}{ax^2+bx+c}=\frac{1}{a}\int \frac{dx}{\left(x+\frac{b}{2a}\right)^2}, u=x+\frac{b}{2a}$

If $b^2-4ac>0, then \int \frac{dx}{ax^2+bx+c}=\frac{1}{a}\int \frac{dx}{\left(x+\frac{b}{2a}\right)^2+\frac{b^2-4ac}{4a^2}}$

, we let $u=x+\frac{b}{2a}, k^2=\frac{b^2-4ac}{4a^2}$, after we use standard integral:

$$\int \frac{dx}{x^2k^2}=\frac{1}{2k}\ln\left|\frac{x-k}{x+k}\right|+C$$

If $b^2-4ac<0$, then $\int \frac{dx}{ax^2+bx+c}=\frac{1}{a}\int \frac{dx}{\left(ax^2+\frac{b}{2a}\right)^2-\frac{b^2-4ac}{4a^2}}$

we let $u=x+\frac{b}{2a}; and -k^2=\frac{b^2-4ac}{4a^2}$ and use the standard integral

$$\int \frac{dx}{x^2+k^2}=\frac{1}{k}\arctan\frac{x}{k}+C$$

Notice

If there are other factors, to each irreducible quadratic factor: ax^2+bx+c occurring once in the denominator of a proper fraction there correspond a single partial fraction of the form: $\frac{Ax+B}{ax^2+bx+c}$, where A and B are constants to be found

To each irreducible quadratic factor ax^2+bx+c occurring n times in the denominator of a proper fraction, there corresponds a sum of n partial fractions of the form :

$$\frac{A_1x+B_1}{ax^2+bx+c}+\frac{A_2x+B_2}{\left(ax^2+bx+c\right)^2}+\ldots+\frac{A_nx+B_n}{\left(ax^2+bx+c\right)^n}$$

Where A_n and B_n are constants to be found.

Example

Find: $\int \frac{dx}{x^2 - x + 1}$ Solution :

$$x^2 - x + 1 = \left(x - \frac{1}{2}\right)^2 + 1 - \frac{1}{4} = \left(x - \frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2$$

$$\int \frac{dx}{x^2 - x + 1} = \int \frac{dx}{\left(x - \frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}$$

$$\Rightarrow \frac{2\sqrt{3}}{3} \arctan\left(\frac{\sqrt{3}(2x-1)}{3}\right) + C$$

Example

Find: $\int \frac{x^3 dx}{x^4 + 2x^2 + 1}$

Solution:

$$x^4 + 2x^2 + 1 = \left(x^2 + 1\right)^2$$

$$\frac{x^3}{x^4 + 2x^2 + 1} = \frac{Ax + B}{x^2 + 1} + \frac{Cx + D}{\left(x^2 + 1\right)^2}$$

$$\Rightarrow (Ax + B)\left(x^2 + 1\right) + (Cx + D) = x^3$$

$$\Rightarrow Ax^3 + Bx^2 + x(A + C) + B + D = x^3$$

$$\begin{cases} A = 1 \\ B = 0 \\ A + C = 0 \\ B + B = 0 \end{cases} \Rightarrow \begin{cases} A = 1 \\ B = 0 \\ C = -1 \\ D = 0 \end{cases}$$

$$\int \frac{x^3 dx}{x^4 + 2x^2 + 1} = \int \frac{\frac{1}{2}(2x)}{x^2 + 1} dx - \int \frac{\frac{1}{2}(2x)}{\left(x^2 + 1\right)^2} dx$$

$$I_1 = \int \frac{\frac{1}{2}(2x)}{x^2+1}dx \Rightarrow k = x^2+1 \Rightarrow dk = 2xdx \Rightarrow I_1 = \frac{1}{2}\ln|k|$$

$$I_2 = \int \frac{\frac{1}{2}(2x)}{\left(x^2+1\right)^2}dx \Rightarrow let\ k = x^2+1 \Rightarrow 2xdx = dk$$

$$I_2 = \frac{1}{2}\int \frac{dk}{k^2} \Rightarrow -\frac{1}{2k}$$

$$I = I_1 + I_2 \Rightarrow \frac{1}{2}\ln\left|x^2+1\right| + \frac{1}{2}\left(\frac{1}{x^2+1}\right) + C$$

Or $\ln\sqrt{x^2+1} + \frac{1}{2}\left(\frac{1}{x^2+1}\right) + C$

Example

Find: $\int \frac{x^2+2}{x^3-1}dx$

Solution:

$$x^3-1 = (x-1)\left(x^2+x+1\right)$$

$$\frac{x^2+2}{x^3-1} = \frac{A}{x-1} + \frac{Bx+C}{x^2+x+1}$$

$$\Rightarrow Ax^2 + Ax + A + Bx^2 + Cx - Bx - C = x^2 + 2$$

$$\Rightarrow \begin{cases} A+B=1 \\ A+C-B=0 \\ A-C=2 \end{cases} \Rightarrow \begin{cases} A=1-B \\ 1-B-B+C=0 \\ 1-B-C=2 \end{cases} \Rightarrow \begin{cases} C-2B=-1 \\ -C-B=1 \end{cases} \Rightarrow \begin{cases} B=0 \\ C=-1 \\ A=1 \end{cases}$$

By replacing A, B, and C we get:

$$\int \frac{x^2+2}{x^3-1}dx = \int \frac{dx}{x-1} - \int \frac{dx}{x^2+x+1}$$

$$I_1 = \int \frac{dx}{x-1} = \ln|x-1|$$

$$I_2 = -\int \frac{dx}{x^2+x+1} = -\int \frac{dx}{\left(x+\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} \Rightarrow -\frac{2\sqrt{3}}{3}\arctan\left(\frac{\sqrt{3}(2x+1)}{3}\right) + C$$

Finally,

$$I = I_1 + I_2 = \ln|x-1| - \frac{2\sqrt{3}}{3}\arctan\left(\frac{\sqrt{3}(2x+1)}{3}\right) + c$$

Example

Find: $\int \frac{dx}{6x^2-5x+1}$

Solution:

$$6x^2-5x+1 = \frac{1}{6}\left(x^2-\frac{5x}{6}+\frac{1}{6}\right) \Rightarrow \frac{1}{6}\left(\left(x-\frac{5}{12}\right)^2-\left(\frac{1}{12}\right)^2\right)$$

$$\frac{1}{6}\int\frac{dx}{x^2-\frac{5}{6}x+\frac{1}{6}} \Rightarrow \frac{1}{6}\int\frac{dx}{\left(x-\frac{5}{12}\right)^2-\left(\frac{1}{12}\right)^2}$$

$$\Rightarrow \frac{1}{6}\times\frac{1}{\frac{2\times 1}{12}}\ln\left|\frac{x-\frac{5}{12}-\frac{1}{12}}{x-\frac{5}{12}+\frac{1}{12}}\right|+C$$

$$\Rightarrow \ln\left|\frac{2x-1}{3x-1}\right|+C$$

Application activity 5.4.3.2

Evaluate the following:

1) $\int \frac{dx}{x^2+x+2}$ 2) $\int \frac{xdx}{9x^2+6x+2}$ 3) $\int \frac{6x^2-x+5}{(x-2)(2x^2+1)}dx$

4) $\int \frac{x-4}{(2x+)(x^2+2)}dx$

5.5 Integration of trigonometric functions

5.5.1 Integral of the form: $\int \sin mx \cos nx dx$ *or* $\int \cos mx \cos nx dx$ *or* $\int \sin mx \sin nx dx$

Activity 5.5.1

For each of the following, transform the product into sum and hence find the integral of $\int f(x)dx$.

1) $f(x)=\sin 2x\cos x$ 2) $f(x)=\sin x\sin 5x$ 3 $f(x)=\cos 2x\cos 3x$)

Content summary

To evaluate the integral of the form $\int \sin mx\cos nxdx$ **or** $\int \cos mx\cos nxdx$ **or** $\int \sin mx\sin nxdx$, we use the corresponding identities:

$$\sin A\cos B=\frac{1}{2}\left[\sin(A-B)+\sin(A+B)\right]$$

$$\cos A\sin B=\frac{1}{2}\left[\sin(A+B)-\sin(A-B)\right]$$

$$\sin A\sin B=\frac{1}{2}\left[\cos(A-B)-\cos(A+B)\right]$$

$$\cos A\cos B=\frac{1}{2}\left[\cos(A-B)+\cos(A+B)\right]$$

Example

Find: $\int \cos 3x\sin 5xdx$

Solution :

$$\cos 3x\sin 5x=\frac{1}{2}\left[\sin(3x+5x)-\sin(3x-5x)\right]=\frac{1}{2}\sin 8x+\frac{1}{2}\sin 2x$$

$$\frac{1}{2}\int \sin 8xdx+\frac{1}{2}\int \sin 2xdx$$

$$-\frac{1}{16}\cos 8x-\frac{1}{4}\cos 2x+C$$

Example

Find: $\int \sin x\sin 2x\sin 3xdx$

Solution:

$$\sin x \sin 2x = \frac{1}{2}\left(\cos x - \cos 3x\right)$$

$$\frac{1}{2}\left(\cos x - \cos 3x\right)\sin 3x = \frac{1}{2}\left(\cos x \sin 3x - \cos 3x \sin 3x\right)$$

$$\Rightarrow \frac{1}{4}\left(\sin 4x + \sin 2x\right) - \frac{1}{4}\left(\sin 6x - \sin 0\right)$$

$$\Rightarrow \frac{1}{4}\sin 4x + \frac{1}{4}\sin 2x - \frac{1}{4}\sin 6x$$

Then,

$$\int \sin x \sin 2x \sin 3x dx = \frac{1}{4}\int \sin 4x dx + \frac{1}{4}\int \sin 2x dx - \frac{1}{4}\int \sin 6x dx = -\frac{1}{16}\cos 4x - \frac{1}{8}\cos 2x + \frac{1}{24}\cos 6x + c$$

Example

Find: $\int \cos^2 x \cos x dx$

Solution:

$$\cos^2 x = \frac{1}{2} + \frac{1}{2}\cos 2x$$

$$\cos^2 x \cos x = \frac{1}{2}\cos x + \frac{1}{2}\cos x \cos 2x$$

$$\frac{1}{2}\cos x \cos 2x = \frac{1}{4}\cos 3x + \frac{1}{4}\cos x$$

$$\cos^2 x \cos x = \frac{1}{2}\cos x + \frac{1}{4}\cos x + \frac{1}{4}\cos 3x$$

$$\Rightarrow \frac{3}{4}\cos x + \frac{1}{4}\cos 3x$$

$$\frac{3}{4}\int \cos x dx + \frac{1}{4}\int \cos 3x dx = \frac{3}{4}\sin x + \frac{1}{12}\sin 3x + C$$

Application activity 5.5.1

Find:

1) $\int \sin 3x \cos 2x dx$ 2) $\int \sin 2x \cos 3x dx$

3) $\int \sin 3x \sin 3x dx$ 4) $\int \sin x \cos x dx$ 5) $\int \cos 3x \cos 3x dx$

5.5.2 Integral of the form: $\int \sin^m x \cos^n x dx, \left(m, n \in \mathbb{Z}^+\right)$

Activity 5.5.2

By letting: $u = \cos x$, and using the identities $\sin^2 x = \frac{1}{2} - \frac{1}{2}\cos 2x$ and $\cos^2 x = \frac{1}{2} + \frac{1}{2}\cos 2x$ where necessary,

evaluate: a) $\int \sin x \cos^2 x dx$; b) $\int \sin^2 x \cos^2 x dx$

Content summary

To evaluate integrals of the form $\int \sin^m x \cos^n x dx, \left(m, n \in \mathbb{Z}^+\right)$, we have to consider the value of m and n.

1) If *m* or *n* is odd, save one cosine factor (or one sine factor) and use the relation:

$$\sin^2 x = 1 - \cos^2 x$$
$$\cos^2 x = 1 - \sin^2 x$$

Let $u = \sin x \Rightarrow du = \cos x dx$

$or\ u = \cos x \Rightarrow du = -\sin x dx$

2) If m and n are even, we use the identities: $\sin^2 x = \frac{1}{2} - \frac{1}{2}\cos 2x$ and

$$\cos^2 x = \frac{1}{2} + \frac{1}{2}\cos 2x$$

It is sometimes helpful to use the identity: $\sin x \cos x = \frac{1}{2}\sin 2x$.

Example

Find: $\int \sin^3 x \cos^2 x dx$

Solution:

$$\int \sin^3 x \cos^2 x dx = \int \sin^2 x \sin x \cos^2 x dx$$
$$\int \left(1-\cos^2 x\right)\cos^2 x \sin x dx$$

Let: $u = \cos x \Rightarrow du = -\sin x dx$

$$-\int \left(1-u^2\right)u^2 du \Rightarrow -\int u^2 du + \int u^4 du$$

$$-\frac{1}{3}u^3 + \frac{1}{5}u^5$$
$$-\frac{1}{3}\cos^3 x + \frac{1}{5}\cos^5 x + C$$

Example

Find: $\int \cos^5 x dx$

Solution:

$$\Rightarrow \int \cos x dx - \int \sin^2 \cos x dx + \int \sin^4 x \cos x dx$$

$$\int \cos^5 x dx = \int \cos^4 x \cos x dx$$
$$\Rightarrow \int \left(1-\sin^2 x\right)^2 \cos x dx$$
$$\Rightarrow \int \left(1-2\sin^2 x + \sin^4 x\right)\cos x dx$$

$$I_1 = \int \cos x dx = \sin x$$
$$I_2 = \int \sin^2 x \cos x dx$$
$$u = \sin x \Rightarrow du = \cos x dx$$

$$I_2 = \int u^2 du \Rightarrow \frac{1}{3}u^3 \Rightarrow \frac{1}{3}\sin^3 x$$

$$I_3 = \int sin^4 x \cos x dx \Rightarrow \int u^4 du$$
$$I_3 = \frac{1}{5}u^5 \Rightarrow \frac{1}{5}\sin^5 x$$

$I = I_1 + I_2 + I_3$

$I = \sin x - \frac{2}{3}\sin^3 x + \frac{1}{5}\sin^5 x + C$

Example

Find: $\int \sin^4 x \cos^2 x dx$

Solution

Let us linearize the expression:

$$\sin^4 x \cos^2 x = \left(\frac{u - \overline{u}}{2i}\right)^4 \left(\frac{u + \overline{u}}{2}\right)^2$$

$$\frac{1}{16}\frac{1}{4}\left(u^2 + \overline{u^2} + 2\right)\left(u^4 - 4u^2 + 6 - 4\overline{u^2} + \overline{u^4}\right)$$

$$\frac{1}{64}\left(u^6 - 4u^4 + 6u^2 - 4 + \overline{u^2} + u^2 - 4 + 6\overline{u^2} - 4\overline{u^4} + \overline{u^6} + 2u^4 - 8u^2 + 12 - 8\overline{u^2} + 2\overline{u^4}\right)$$

$$\frac{1}{32}\left(\frac{u^6 + \overline{u^6}}{2}\right) - \frac{1}{16}\left(\frac{u^4 + \overline{u^4}}{2}\right) - \frac{1}{32}\left(\frac{u^2 + \overline{u^2}}{2}\right) + \frac{1}{16}$$

$$\Rightarrow \frac{1}{32}\cos 6x - \frac{1}{16}\cos 4x - \frac{1}{32}\cos 2x + \frac{1}{16}$$

Then, $\int \sin^4 x \cos^2 x dx = \frac{1}{32}\int \cos 6x dx - \frac{1}{16}\int \cos 4x dx - \frac{1}{32}\int \cos 2x dx + \frac{1}{16}\int dx$

$$\Rightarrow \frac{1}{192}\sin 6x - \frac{1}{64}\sin 4x - \frac{1}{64}\sin 2x + \frac{1}{16}x + C$$

Application activity 5.5.2

Find:

1) $\int \cos^3 x \sin x dx$ 2) $\int \sin^4 2x \cos 2x dx$ 3) $\int \sin^3 x dx$

4) $\int \cos^3 4x dx$ 5) $\int \sin^3 x \cos^3 x dx$

5.6 Definite integrals

5.6.1 Riemann sum approximation and definition of define integrals

Activity 5.6.1

A learner in Senior Six is preparing to sit for an end year exam of Mathematics. He/she draws on the same axes the linear function defined by: $f(x)=2x,\ y=0,$ and two vertical lines, $x=0,\ and\ x=4$.

Draw the shape obtained and prove that it is in the form of a triangle.

By using the formula for the area of a triangle, calculate the area enclosed by the functions $y=2x, y=0, and\ x=4$

Let consider the function $F(x)$ as an anti-derivative of $f(x)=2x$. Find $F(x)$ and carry out $F(4)-F(0)$

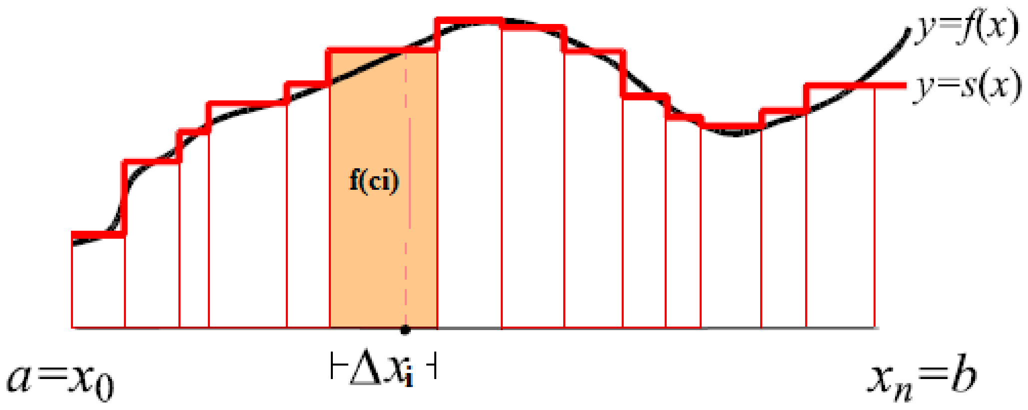
Compare the findings of b) to the area obtained in c).

Content summary

Riemann sum approximation

Let f be a continuous function defined on the closed interval $[a,b]$. To begin, partition the interval into

n subintervals, each of width $\Delta x=\dfrac{b-a}{n}$ as shown.



$a=x_0<x_1<x_2<...<x_{n-1}<x_n=b$

In each sub-interval $[x_{i-1}, x_i]$, choose an arbitrary point c_i and form the sum

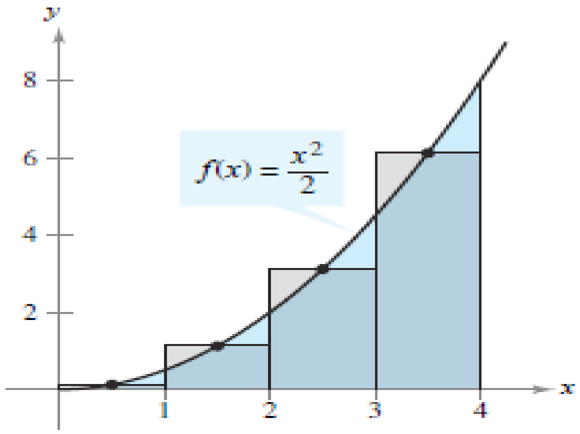
$$S = f(c_1)\Delta x_1 + f(c_2)\Delta x_2 + \ldots + f(c_{n-1})\Delta x_{n-1} + f(c_n)\Delta x_n.$$

This type of summation is called a Riemann sum, and is often written using summation notation as shown below.

$$S = \sum_{i=1}^{n} f(c_i)\Delta x_i, \quad x_{i-1} \le c_i \le x_i$$

Example

Use the four rectangles to approximate the area of the region lying between the graph of $f(x) = \dfrac{x^2}{2}$ and the x axis , between $x = 0$ and $x = 4$.



Solution:

You can find the heights of the rectangles by evaluating the function f at each of the midpoints of the subintervals

$[0,1]$,$[1,2]$, $[2,3]$ and $[3,4]$.

Because the width of each rectangle is 1, the sum of the areas of the four rectangles is

$S = 1f\left(\frac{1}{2}\right) + 1f\left(\frac{3}{2}\right) + 1f\left(\frac{5}{2}\right) + 1f\left(\frac{7}{2}\right)$ where the width is and $f\left(\frac{c_i}{2}\right)$ is the corresponding height.

Thus, $S = \left(\frac{1}{8}\right) + \left(\frac{9}{8}\right) + \left(\frac{25}{8}\right) + \left(\frac{49}{8}\right) = \left(\frac{84}{8}\right) = 10.5$

So, you can approximate the area of the region to be 10.5 square units.

Definition of definite integral

Let f be a continuous function defined on a close interval $[a,b]$.

The Riemann sum of the function f on this interval is

$$S=\sum_{i=1}^{n} f(c_i)\Delta x_i, \quad x_{i-1}\le c_i \le x_i$$
.

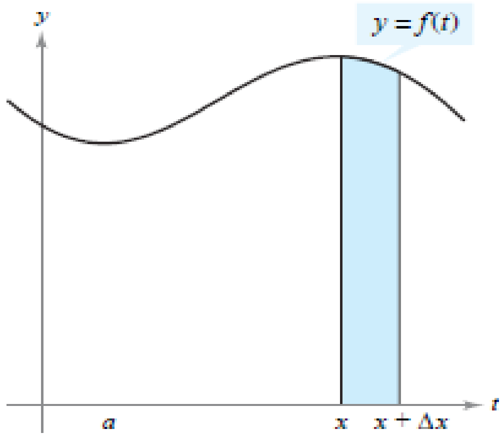
If Δx becomes very small, x_i corresponds to x, Δx_i becomes dx and the area is expressed by the Riemann sum. $S=\lim_{\Delta x\to 0}\sum_{i=1}^{n} f(c_i)\Delta x_i=\int_a^b f(x)dx$ where the limit is over Δx -finite partition of $[a,b]$, when the limit exists.

Thus, The definite integral of the function $f(x)$ over $[a,b]$ is defined as

$$S=\lim_{\Delta x\to 0}\sum_{i=1}^{n} f(c_i)\Delta x_i=\int_a^b f(x)dx.$$

The numbers a, b are called the limits of integration and the function $f(x)$ is called the integrand of the definite integral.

On the other hand, Let $F(x)$ be the area of the region under the graph of $f(x)$ from a to x as indicated in the figure bellow:



The area under the shaded region in Figure is $F(x+\Delta x)-F(x)$.

If Δx is small, this area is approximated by the area of a rectangle of height $f(x)$ and width Δx. So we have: $F(x+\Delta x)-F(x)\approx f(x)\Delta x$

Dividing by Δx, we have $f(x) \approx \dfrac{F(x+\Delta x)-F(x)}{\Delta x}$

Taking the limit as Δx approaches 0, you get $f(x) = \lim\limits_{\Delta x \to 0} \dfrac{F(x+\Delta x)-F(x)}{\Delta x} = F'(x)$

It means that $f(x)$ is the anti-derivative of $F(x)$.

Therefore, for any anti-derivative $F(x)$ of $f(x)$ on $[a,b]$ the difference $F(b)-F(a)$ has a unique value. This value is defined as a definite integral of $f(x)$ for $a \le x \le b$. We write, $\int_a^b f(x)\,dx = F(b)-F(a)$.

Fundamental theorem of integral calculus:

Let $F(x)$ and $f(x)$ be functions defined on an interval $[a,b]$. If $f(x)$ is continuous

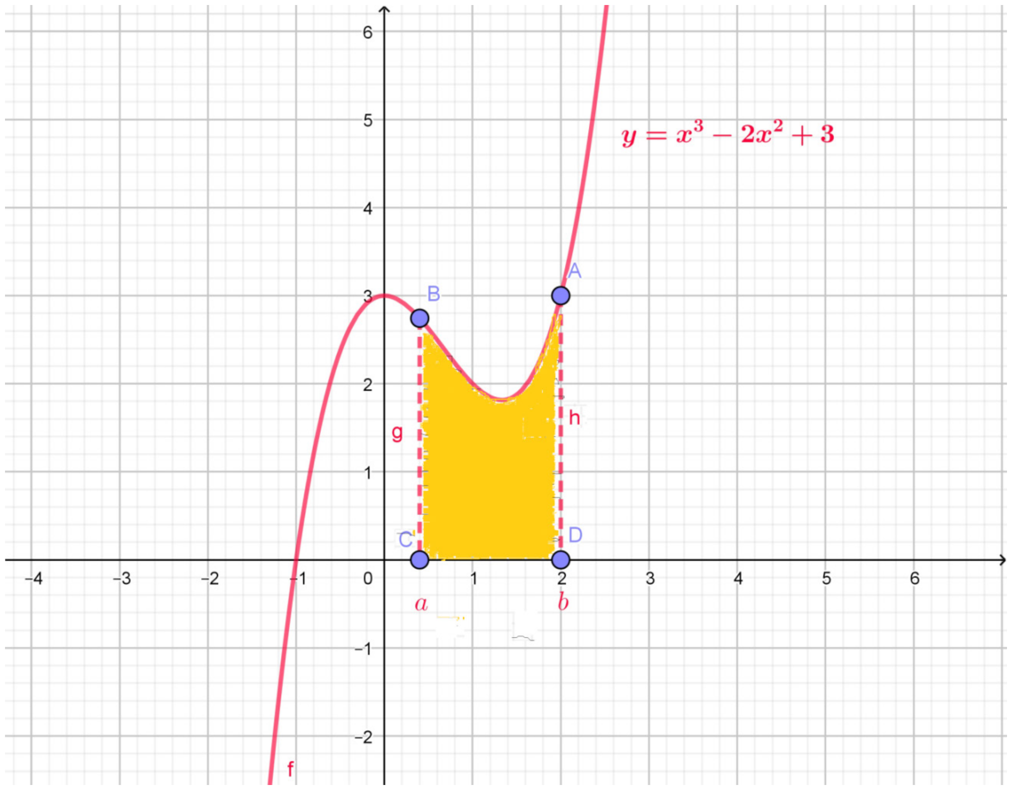
and $F'(x) = f(x)$, *then* $\int_a^b f(x)\,dx = \left[F(x)\right]_a^b = F(b)-F(a)$.

$\int_a^b f(x)\,dx$ is read as the **"integral from a to b of $f(x)$"**, a is called **lower limit**

and b is called **upper limit**. The interval $[a,b]$ is called **the range of integration**.

Geometrically, the definite integral $\int_a^b f(x)\,dx$ is the area of the region enclosed by the curve $y = f(x)$, the vertical lines $x = a$, $x = b$ and the $x-axis$ as illustrated in the following figure.

Figure3.8: Definite integral of a function $f(x)$ on a given interval $[a,b]$.



The area of coloured region is given by $\int_a^b (x^3 - 2x^2 + 3)\,dx$.

If measurement units are provided for axes, then the area of the region is the product of this definite integral and the area of square unit.

Example

Determine:

1) $\int_1^2 (x^3+3)\,dx$ 2) $\int_0^2 (2x^2+3)\,dx$ 3) $\int_{-1}^1 (3x^2-2)\,dx$

Solution:

1) $\int_1^2 (x^3+3)\,dx = \left[\frac{x^4}{4}+3x\right]_1^2 = \left(\frac{2^4}{4}+6\right)-\left(\frac{1}{4}+3\right) = 4+3-\frac{1}{4} = \frac{27}{4}$

2) $\int_0^2 (2x^2+3)\,dx = \left[\frac{2x^3}{3}+3x\right]_0^2 = \left(\frac{2}{3}(2)^3+6\right)-0 = \frac{34}{3}$

3) $\int_{-1}^{1}(3x^2-2)dx=\left[x^3-2x\right]_{-1}^{1}=(1-2)-(-1+2)=-2$

Application activity 5.6.1

Find a function $F(x)$ satisfying $F'(x)=5x^2+1$ *and* $F(0)=2$, then plot its graph by joining its main points.

5.6.2 Properties of definite integrals

Activity 5.6.2

Given that $f(x)=(x^3+3)$ and $g(x)=-2x^2$,

1) Evaluate a) $\int_1^2 f(x)dx$ b) $\int_1^2 g(x)dx$ and c) $\int_1^2 (f+g)(x)dx$

2) Compare $\int_1^2 (f+g)(x)dx$ and $\int_1^2 f(x)dx+\int_1^2 g(x)dx$ and conclude on how to find $\int_a^b (f+g)(x)dx$.

Content summary

If $f(x)$ *and* $g(x)$ are continuous functions on a closed interval $[a,b]$ then:

1. $\int_a^b odx=0$

2. $\int_a^b f(x)dx=-\int_b^a f(x)dx$ (Permutation of bounds)

3. $\int_a^b[\alpha f(x)\pm\beta g(x)]dx=\alpha\int_a^b f(x)dx\pm\beta\int_a^b g(x)dx, \alpha \text{ and } \beta\in\mathbb{R}\ (linearity)$

4. $\int_a^a f(x)dx=0(bounds\, are\, equal)$

5. $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)dx$ *with* $a < c < b$ (Charles relation)

6. $\forall x \in [a,b], f(x) \le g(x) \Rightarrow \int_a^b f(x)\,dx \le \int_a^b g(x)\,dx$ it follows that

$$f(x) \ge 0 \Rightarrow \int_a^b f(x)dx \ge 0$$

and

$$\left|\int_a^b f(x)\,dx\right| \le \int_a^b |f(x)|\,dx \quad \text{(positivity)}$$

$$\int_{-a}^{a} f(x)dx = \begin{cases} 2\int_0^a f(x)\,dx, \text{ if } f(x) \text{ is even function} \\ 0, \text{ if } f(x) \text{ is odd function} \end{cases}$$

Example

Calculate the definite integral: $\int_1^2 x^3 dx$

Solution

Fist we calculate $\int_1^2 x^3 dx = \left[\frac{1}{4}x^4\right]_1^2$

Then $\int_1^2 x^3 dx = \frac{1}{4}\left[x^4\right]_1^2 = \frac{1}{4}\left(2^4 - 1^4\right) = \frac{15}{4}$. Therefore, $\int_1^2 x^3 dx = \frac{15}{4}$

Example

Calculate $\int_0^1 x^2 dx$

Solution

As $\int_0^1 x^2 dx = \left[\frac{1}{3}x^3\right]_0^1$, we have $\int_0^1 x^2 dx = \frac{1}{3}\left[x^3\right]_0^1 = \frac{1}{3}\left(1^3 - 0^3\right) = \frac{1}{3}$.

Application activity 5.6.2

Calculate

1) $\int_{0}^{3} x dx$ 2) $\int_{1}^{2} (x^2 - x)dx$ 3) $\int_{1}^{2} (3x^2 - 6x)dx$ 4) $\int_{-1}^{2} (x^3 + 3x^2 - 4)dx$

5.6.3 Techniques of Integration of definite integrals

Activity 5.6.3

1) Consider the continuous function $f(x) = e^{x^2}$ on a closed interval $[a,b]$

i. Let $t = x^2$, determine the value of t when $x = 0$ and when $x = 2$

ii. Determine the value of dx in function of dt

iii. Evaluate the integral $\int_{a}^{b} 2xe^{x^2} dx$ using expression of t, considering the results found in i) and ii).

iv. Explain what happens to the boundaries of the integral when you apply the substitution method

2) Evaluate $\int_{1}^{e} x^2 \ln x dx$

Content summary

Many times, some functions can not be integrated directly.

In that case we have to adopt other techniques in finding the integrals. The fundamental theorem in calculus tells us that computing definite integral of *f(x)* requires determining its antiderivtive, therefore the techniques used in determining indefinite integrals are also used in computing definite integrals.

a) Integration by substitution

The method in which we change the variable to some other variable is called "**Integration by substitution".**

When definite integral is to be found by substitution then change the lower and upper limits of integration. If substitution is $\varphi(x)$ and lower limit of integration is a and upper limit is b then new lower and upper limits will be $\varphi(a)$ and $\varphi(b)$ respectively.

Example

Evaluate

a) $\int_0^2 x\sqrt{5-x^2}\,dx$ b) $\int_0^3 6xe^{x^2+1}\,dx$

Solution

a) $\int_0^2 x\sqrt{5-x^2}\,dx$

Let $5-x^2=t$, then $-2xdx=dt$, $or\ xdx=-\frac{1}{2}dt$

$when\ x=0,\ t=5,\ when\ x=2,\ t=5-4=1$

$$\int_0^2 x\sqrt{5-x^2}\,dx=\int_5^1 -\sqrt{t}\frac{dt}{2}=\frac{1}{2}\int_1^5 \sqrt{t}\,dt$$

$$=\frac{1}{2}\int_1^5 t^{\frac{1}{2}}dt=\frac{1}{2}\left[\frac{t^{\frac{1}{2}+1}}{\frac{1}{2}+1}\right]_1^5=\frac{1}{2}\left[\frac{t^{\frac{3}{2}}}{\frac{3}{2}}\right]_1^5=\frac{1}{2}\times\frac{2}{3}\left[5^{\frac{3}{2}}-1^{\frac{3}{2}}\right]=\frac{1}{3}\left(\sqrt{125}-1\right)$$

b) $\int_0^3 6xe^{x^2+1}\,dx$

Let $x^2+1=t$, then $2xdx=dt\ or\ xdx=\frac{1}{2}dt$

$when\ x=0\ ,\ t=1\ and\ when\ x=3,\ t=10$

$$\int_0^3 6xe^{x^2+1}dx=\int_1^{10} 6e^t\frac{dt}{2}=3\int_1^{10} e^t\,dt=3\left[e^t\right]_1^{10}=3\left(e^{10}-e^1\right)=3\left(e^{10}-e\right)$$

b) Integration by parts

To compute the definite integral of the form $\int_a^b f(x)g(x)dx$ using integration by parts, simply set $u=f\left(x\right)\ and\ dv=g(x)dx$. Then $du=f'(x)dx$ and $v=G(x)$, antiderivative of $g(x)$ so that the integration by parts becomes:

$$\int_a^b u\,dv = [uv]_a^b - \int_a^b v\,du$$

Example

Evaluate the following definite integrals:

1. $\int_0^3 xe^x dx$

2. $J = \int_0^{\frac{\pi}{6}} \left(4+5x^2\right)\cos 3x\,dx$

Solution

1. $I = \int_0^3 xe^x dx$

Let $\begin{cases} u = x, & du = dx \\ dv = e^x dx, & v = \int e^x dx = e^x + c \end{cases}$

Applying the integration parts formula $I = uv\Big|_a^b - \int_a^b v\,du$ yields to

$$I = xe^x\Big|_0^3 - \int_0^3 e^x dx$$

$$= xe^x\Big|_0^3 - e^x\Big|_0^3$$

$$= \left[3e^3 - 0e^0\right] - \left[e^3 - e^0\right]$$

$$= 3e^3 - e^3 + 1 = 2e^3 + 1$$

2. $J = \int_0^{\frac{\pi}{6}} \left(4+5x^2\right)\cos 3x\,dx$

Let $\begin{cases} u = 4+5x^2, & du = 10x\,dx \\ dv = \cos 3x\,dx, & v = \int \cos 3x\,dx \Rightarrow v = \frac{1}{3}\sin 3x + c \end{cases}$

Applying the integration by parts formula $I = uv\Big|_a^b - \int_a^b v\,du$ to get

$$J = \left[\left(4+5x^2\right)\times\frac{1}{3}\sin 3x\right]_0^{\frac{\pi}{6}} - \int_0^{\frac{\pi}{6}}\left(\frac{1}{3}\sin 3x\right)(10x)\,dx$$

$$= \left[\frac{1}{3}\left(4+5\frac{\pi^2}{36}\right)\sin\frac{3\pi}{6} - 0\right] - \frac{10}{3}\int_0^{\frac{\pi}{6}} x\sin 3x\,dx$$

$$= \frac{1}{3}\left[4+\frac{5\pi^2}{36} - 10\int_0^{\frac{\pi}{6}} x\sin 3x\,dx\right]$$

$$= \frac{1}{3}\left[\left(4+\frac{5\pi^2}{36}\right) - 10\left[\frac{-x\cos 3x}{3}\Bigg|_0^{\frac{\pi}{6}} + \frac{1}{3}\int_0^{\frac{\pi}{6}} 1\times\cos 3x\,dx\right]\right]$$

$$= \frac{1}{3}\left[\left(4+\frac{5\pi^2}{36}\right) - 10\left[\frac{-x\cos 3x}{3} + \frac{1}{9}\sin 3x\right]_0^{\frac{\pi}{6}}\right]$$

$$= \frac{1}{3}\left[\left(4+\frac{5\pi^2}{36}\right) + 0 - \frac{10}{9}\right]$$

$$= \frac{1}{27}\left(\frac{5\pi^2}{4} + 26\right)$$

Application activity 5.6.3

Evaluate the following definite integrals by using the indicated technique

1) $\int_0^{\pi} \cos x e^{\sin x}\,dx$ (Use integration by substitution)

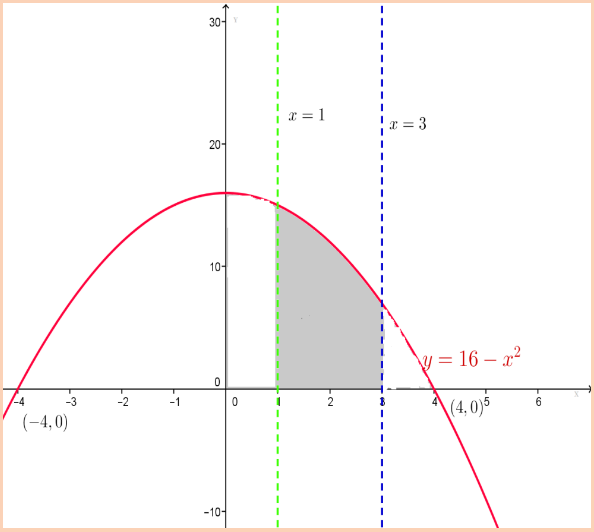
2) $\int_0^{1} \ln(1+x)\,dx$ (Use integration by parts)

5.7 Applications of definite integrals

5.7.1 Calculation of area of a plane surface

Activity 5.7.1

The plane region bounded by the curve $y = 16 - x^2$, $the\ x - axis\ and\ x = 1, x = 3$ is shown in the following diagram.



Write down the definite integral which represents the measure of this surface area.

Hence, calculate the area

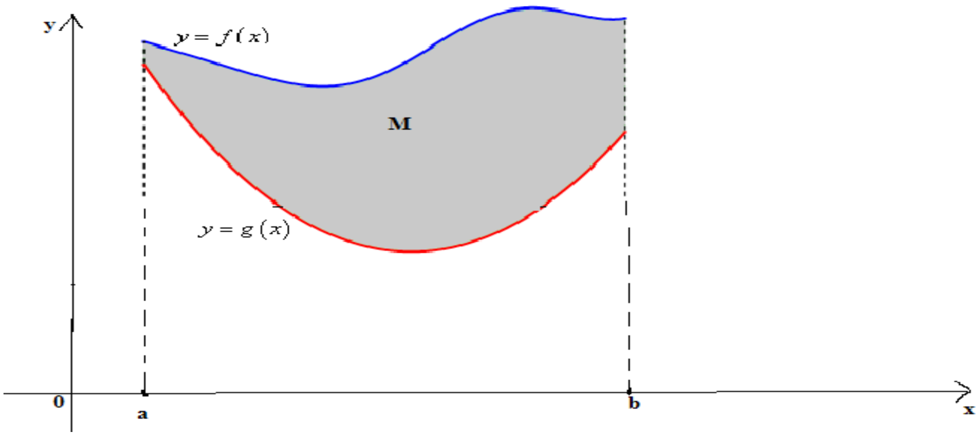
Content summary

Area of a region between two curves

We can apply the definite integrals to evaluate the area bounded by the graph of function and lines : $x = a, x = b\ and\ y = 0$ on the interval where the function is defined.

Suppose that a plane region M is bounded by the graphs of two **continuous functions**

$y = f(x)\ and\ y = g(x)$ and the vertical straight lines $x = a\ and\ x = b$ as shown in figure below



Assume that $a<b$ and that $f(x)\geq g(x)\ on\,[a,b]$, so the graph of f lies above the graph of g If $g(x)\geq 0\ on\,[a,b]$, then the area $A\ of\ M$ is the area above the $x-axis$ under the graph of f minus the area above the $x-axis$ under the graph of g : $A=\int_a^b f(x)dx-\int_a^b g(x)dx=\int_a^b\left[f(x)-g(x)\right]dx$

From the above figure the area $A=\int_a^b\left[f(x)-g(x)\right]dx$ is calculated as follows

$$A=\int_a^b\left[\begin{pmatrix}upper\\ function\end{pmatrix}-\begin{pmatrix}lower\\ function\end{pmatrix}\right]dx\ with\ f(x)\geq g(x)\ for\ a\leq x\leq b$$

$$f(y)\geq g(y)$$

Even if $f\ and\ g$ can take negative values on $[a,b]$, this interpretation and resulting area formula

$A=\int_a^b\left[f(x)-g(x)\right]dx$ remain valid, provided that $f(x)\geq g(x)\ on\,[a,b]$.

Hence the total area lying between the graphs $y=f(x)\ and\ y=g(x)$ and between the vertical lines $x=a\ and\ x=b$ is given by using the absolute value form:

$$A=\left|\int_a^b f(x)-g(x)dx\right|$$

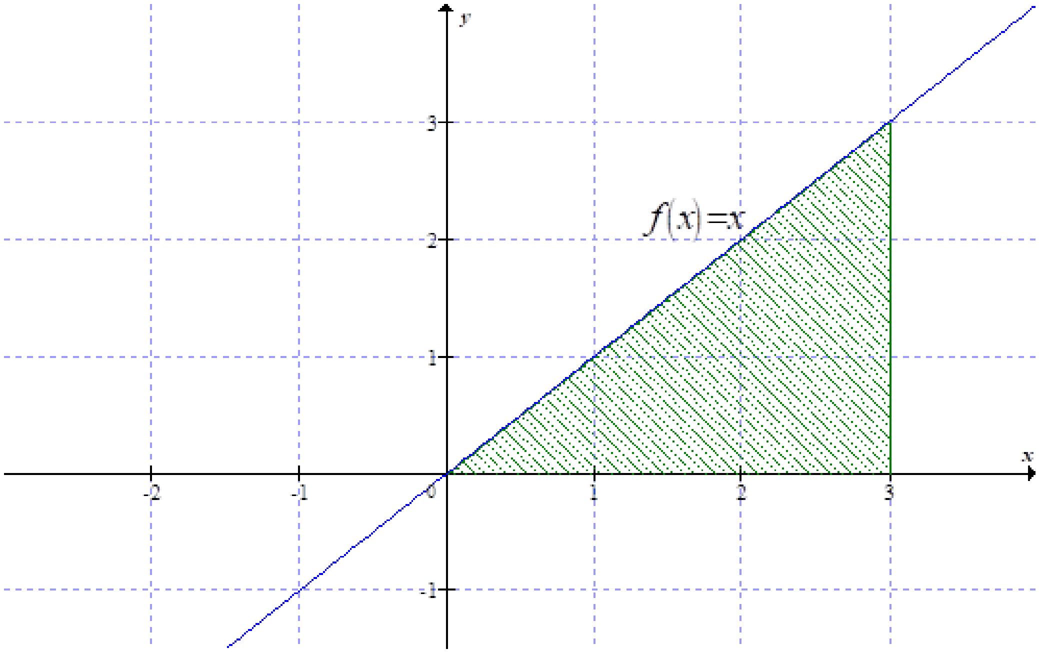
Remember to plot graphs of the functions to locate the upper and lower function before starting calculations.

Example

Calculate the area between the curve representing $f(x) = x$ and $x - axis$.

Solution

$x - axis$ is represented by the function $g(x) = 0$ while $f(x) = x$ is the first



The area is given by

$$A = \int_0^3 (x-0)\,dx = \left[\frac{1}{2}x^2\right]_0^3 = \frac{9}{2}U.A$$

Alternative method

From the figure, the shaded area is a triangle with base 3 units and height 3 units. So, the area is

$$a = \frac{1}{2} base \times height = \frac{1}{2} \times 3 \times 3 = \frac{9}{2}U.A$$

Example

Find the area of plane region M lying between the curve

$f(x) = 8 - 3x^2$ *and* $g(x) = x^2 - 4x$

Solution

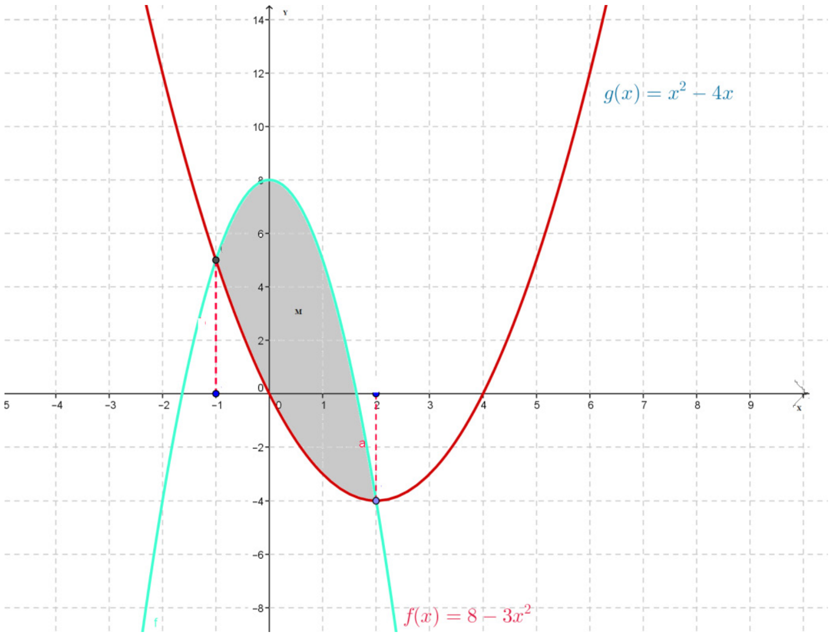
First, we have to solve the equation $f(x) = g(x)$ to find the intersections of the curves.

We get now : $8 - 3x^2 = x^2 - 4x$.

Simple calculations lead to $x^2 - x - 2 = 0$ (Quadratic equation)

Solve the equation to get $x = 2 \; or \; x = -1$.

So the two curves intersect at two points of respective abscissa $x = 2 \; and \; x = -1$. The graphs of the two functions are parabola.



$f(x) \geq g(x)$The bounded region M between $f(x) = 8 - 3x^2 \; and \; g(x) = x^2 - 4x$ is shaded.

Since $f(x) \geq g(x) \; for \; -1 \leq x \leq 2$, the area $A \; of \; M$ is given by:

$$A = \int_{-1}^{2} \left[f(x) - g(x) \right] dx = \int_{-1}^{2} \left[\left(8 - 3x^2\right) - \left(x^2 - 4x\right) \right] dx$$

$$\Rightarrow \int_{-1}^{2} \left(8 - 4x^2 + 4x\right) dx = \left[8x - \frac{4}{3}x^3 + \frac{4}{2}x^2 \right]_{-1}^{2}$$

$$\Rightarrow \left[8(2) - \frac{4}{3}(2)^3 + 2(2)^2 \right] - \left[8(-1) - \frac{4}{3}(-1)^3 + 2(-1)^2 \right] = \frac{54}{3} = 18$$

The final answer is expressed in term of surface area as: $A = 18\ square\ units$

For example, if the unit on axis stands for $2\ cms$. The area of the square unit is then $4\ cm^2$. In this case the area of the region is $18 \times 4\ cm^2 = 72cm^2$.

Example

Determine the area of the region K bounded by $y = 2x^2 + 10\ and\ y = 4x + 16$

Solution

To find the intersection points, we have to equate the two equations:

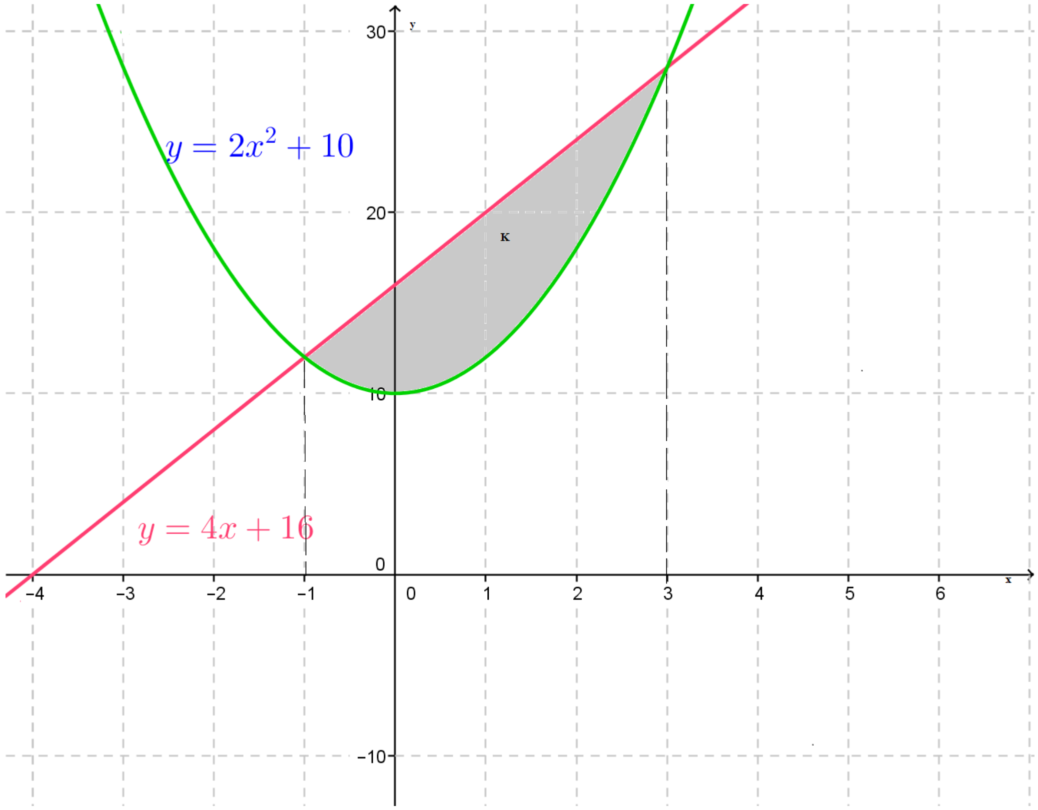
$$2x^2 + 10 = 4x + 16$$
$$\Rightarrow 2x^2 - 4x - 6 = 0$$
$$\Rightarrow 2(x+3)(x-3) = 0$$

So, the two curves will intersect when

$x = -1\ and\ x = 3$, The graph of g is a parabola while the graph of f is a line.

Graph of $x = -1\ g(x) = 2x^2 + 10\ and\ f(x) = 4x + 16\ x = 3$

$$A = \int_a^b \begin{pmatrix} \text{upper} \\ \text{function} \end{pmatrix} - \begin{pmatrix} \text{lower} \\ \text{function} \end{pmatrix} dx$$
$$= \int_0^2 x + 1 - xe^{-x^2}\, dx$$
$$= \left(\frac{1}{2}x^2 + x + \frac{1}{2}e^{-x^2} \right) \Bigg|_0^2$$
$$= \frac{7}{2} + \frac{e^{-4}}{2} = 3.5092$$



$$A = \int_a^b \left[\begin{pmatrix} upper \\ function \end{pmatrix} - \begin{pmatrix} lower \\ function \end{pmatrix}\right] dx$$

$$A = \int_{-1}^{3} \left[(4x+16) - (2x^2+10)\right] dx$$

$$A = \int_{-1}^{3} \left(-2x^2 + 4x + 6\right) dx$$

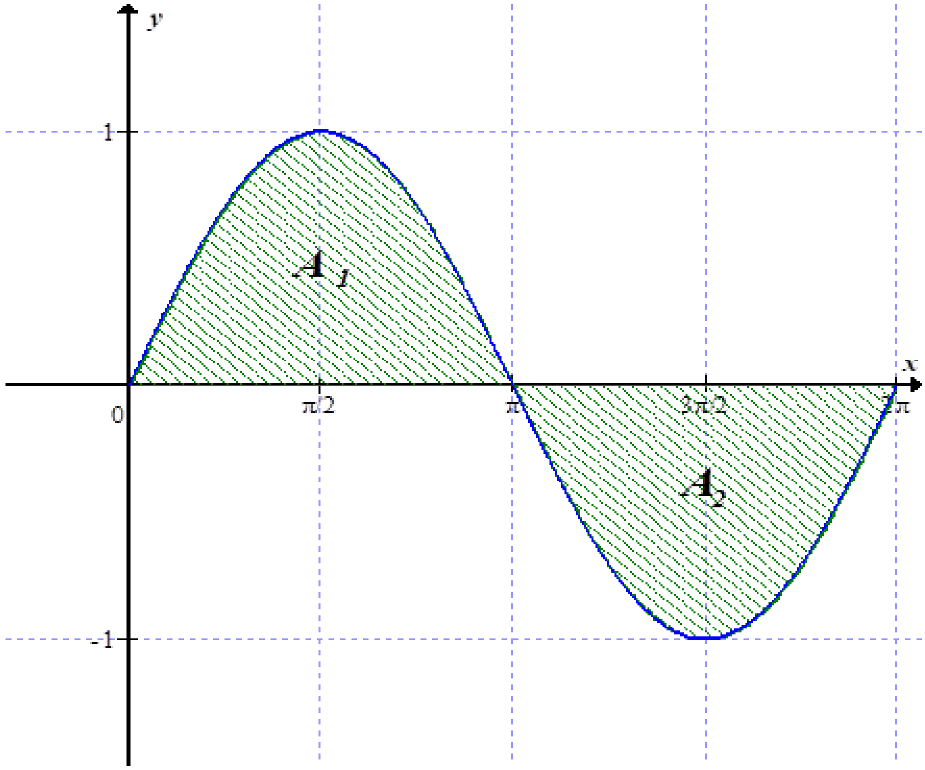
$$A = \left[-\frac{2}{3}x^3 + \frac{4}{2}x^2 + 6x\right]_{-1}^{3}$$

$$A = \frac{64}{3} U.A$$

Where: **U.A** stands for unit of Area

Example

Find the area of a sinusoidal in $[0, 2\pi]$



For A_1, we have 2 functions $g(x)=\sin x$ *and* $f(x)=0$,

Then

$$A_1=\int_0^\pi(\sin x-0)\,dx=\int_0^\pi\sin x\,dx=-[\cos x]_0^\pi=-(\cos\pi-\cos 0)=2$$

For A_2, we have $g(x)=0$ *and* $f(x)=\sin x$,

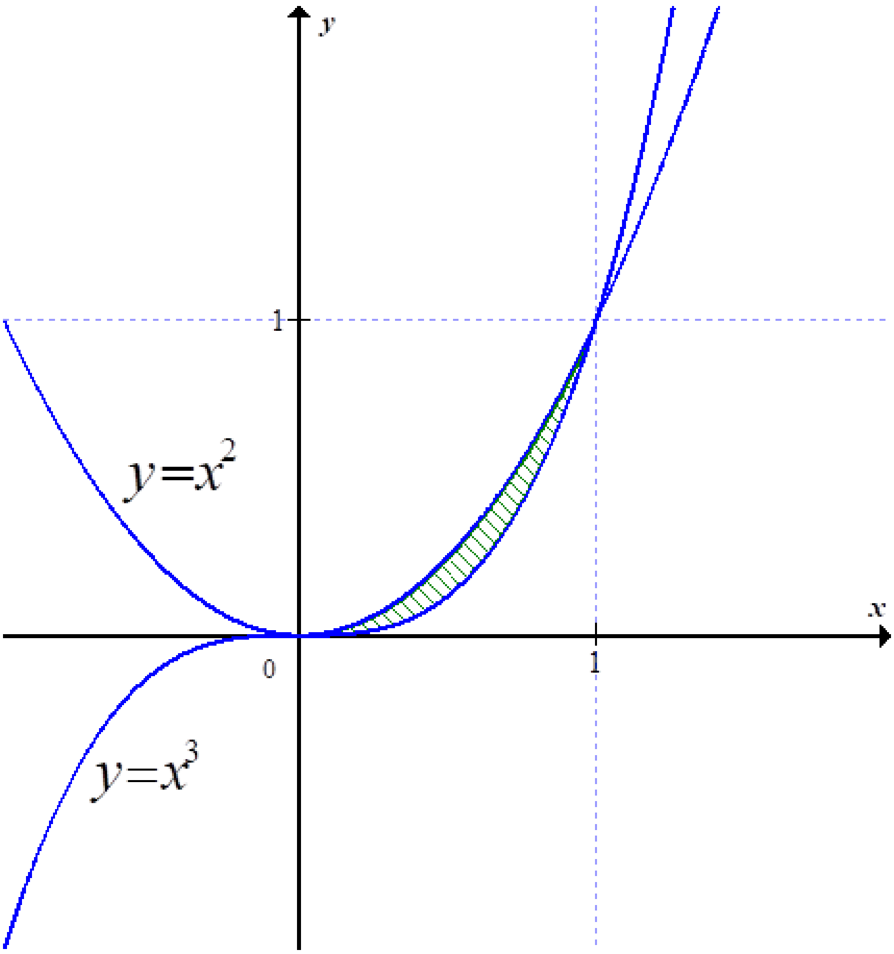
Then $A_2=\int_\pi^{2\pi}(0-\sin x)\,dx=-\int_\pi^{2\pi}\sin x\,dx=-[-\cos x]_\pi^{2\pi}=\cos 2\pi-\cos\pi=2$

The total area is $A=4U.A$

Example

Find the area enclosed by the curves $y=x^3$ *and* $y=x^2$

Solution



We need to know the intersection points of two curves.

To do this we solve for $x^2 = x^3 \Rightarrow x^2 - x^3 = 0 \Rightarrow x^2(1-x) = 0 \Rightarrow \begin{cases} x^2 = 0 \\ 1-x=0 \end{cases} \Rightarrow \begin{cases} x=0 \\ x=1 \end{cases}$

The area is given by : $A = \int_0^1 (x^2 - x^3)\,dx = \left[\frac{1}{3}x^3 - \frac{1}{4}x^4\right]_0^1 = \frac{1}{3} - \frac{1}{4} = \frac{1}{12} U.A$

Application activity 5.7.1

1) Calculate the area enclosed by the curve and the straight line in each of the following:

a) $y=(2x-1)(2x+1)$ and $x-axis$

b) $y=x(x-1)(x-2)$ and $x-axis$

c) $y=x^2-3x-4$ and $y=x+1$

d) $y^2=x,\ y=\frac{1}{2}x$

e) $y^2=4x,\ y=2x-4$

2) Find the area enclosed between the curve $y=2a^2x^2-x^4, a>0$, and the line joining its local maxima.

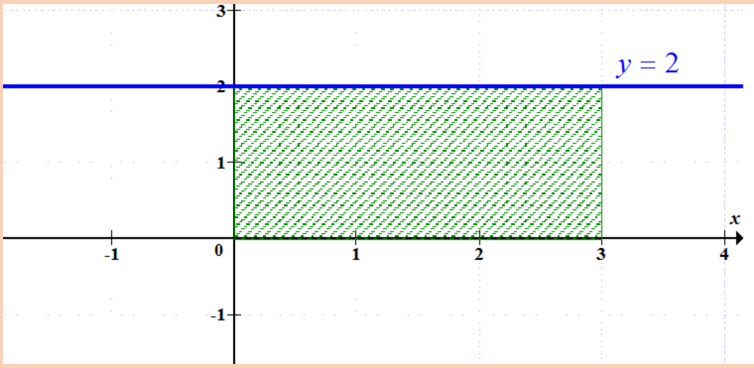
3) Find the area enclosed between the curves $y=-x^3+6x^2+2x-3$ and $y=(x-3)^2$

4) Find the area bounded by $y=\frac{1}{x^2}, y=-27x$ and $y=-\frac{1}{8}x$.

5.7.2 Calculation of volume of a solid of revolution

Activity 5.7.2

Consider the graph of the function $y = f(x) = 2 \ for \ 0 \le x \le 3$



a) If $f(x)$ is rotated about $x - axis$, illustrate the volume of revolution obtained.

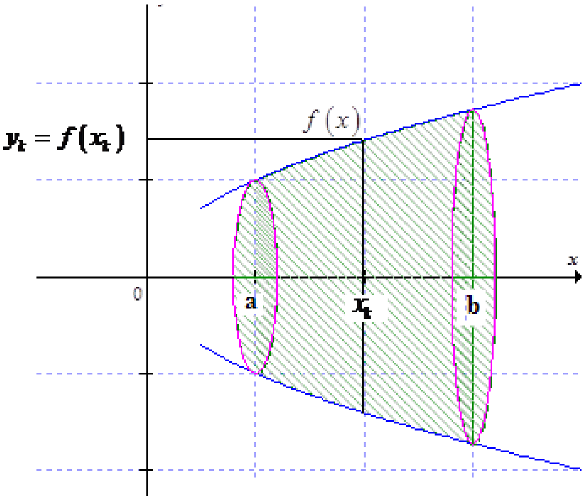
b) What is the radius and the height of the cylinder obtained?

c) Determine the volume of this cylinder: algebraically, by the use of integration and compare the results.

d) Try to generalize how to calculate the volume of the solid of revolution obtained when the graph of the function $y = f(x)$ is rotated about OX for $a \le x \le b$

Content summary

Consider the curve for $y = f(x) \ for \ a \le x \le b$.



If function $f(x)$ is rotated about the $x-axis$, we obtain volume of revolution as shown on figure above.

Suppose that the element ΔV which is the volume limited by the planes $x = x_k$ *and* $x = x_{k+1}$. It is approximately equal to the volume of cylinder of height $\Delta x_k = x_{k+1} - x_k$ and radius $y_k = f(x_k)$. Its volume is $\Delta V_k = \pi y_k^2 \Delta x_k$.

The total volume is the sum of all elements $\Delta V_k = \pi y_k^2 \Delta x_k$ along the interval $[a,b]$.

This is $V = \sum_{k=1}^{\infty} \pi y_k^2 \Delta x_k,\ x_k \in [a,b]$. Given that $y_k = f(x_k)$, the volume is

$$V = \sum_{k=1}^{\infty} \pi \left[f(x_k) \right]^2 \Delta x_k,\ x_k \in [a,b].$$

As the step size is made smaller and smaller, $\Delta x_k \to dx$ and $V = \int_a^b \pi \left[f(x) \right]^2 dx$. Therefore,

the volume of the solid of revolution bounded by the curve $f(x)$ about the $x-axis$ calculated from $x = a$ *to* $x = b$, is given : $V = \pi \int_a^b f^2(x)\,dx$.

This method is called **disc method**.

When using this method, it is necessary to integrate along the axis of revolution. If the region is revolved about a horizontal line, integrate by x, and if the region is revolved about a vertical line, integrate by y.

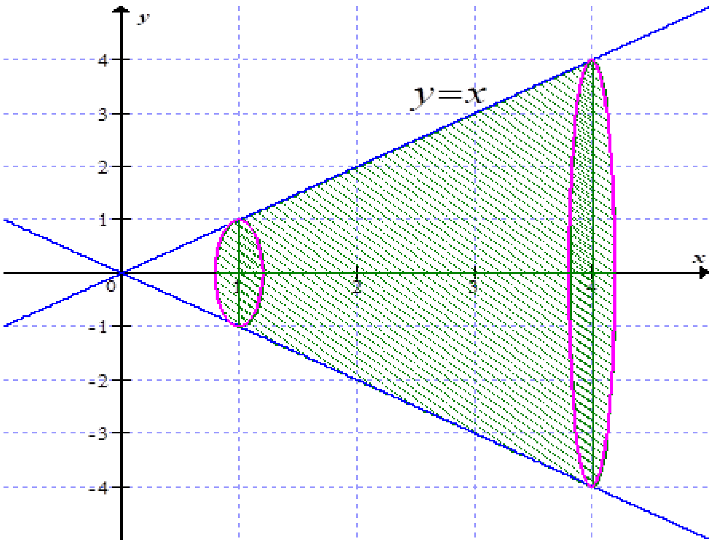
In this last case, if we have $y = f(x)$, we must express it in function of y, i.e we find $x = g(\mathrm{y})$ and we make change on boundaries to get the volume

$$V = \pi \int_{y_1}^{y_2} g^2(y)\,dy$$

Example

Use integration to find the volume of the solid generated when the line $y = x$ for $1 \le x \le 4$ is revolved around the $x-axis$.

Solution

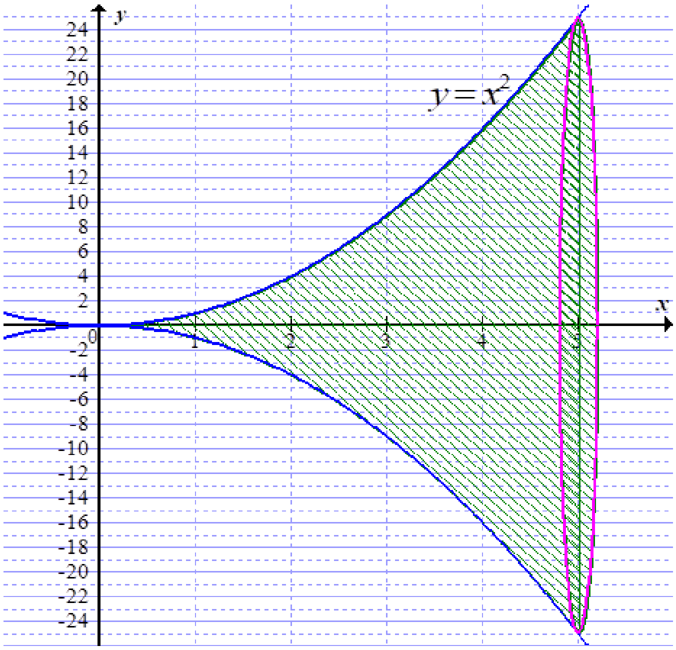


$$V = \pi\int_a^b y^2 dx \Rightarrow \pi\int_1^4 x^2 dx \Rightarrow \pi\left[\frac{1}{3}x^2\right]_1^4 \Rightarrow 21\pi U.V$$

Example

Find the volume of the solid formed when the graph of the function $y = x^2$ *for* $0 \le x \le 5$ is revolved about the $x - axis$.

Solution



Volume is:

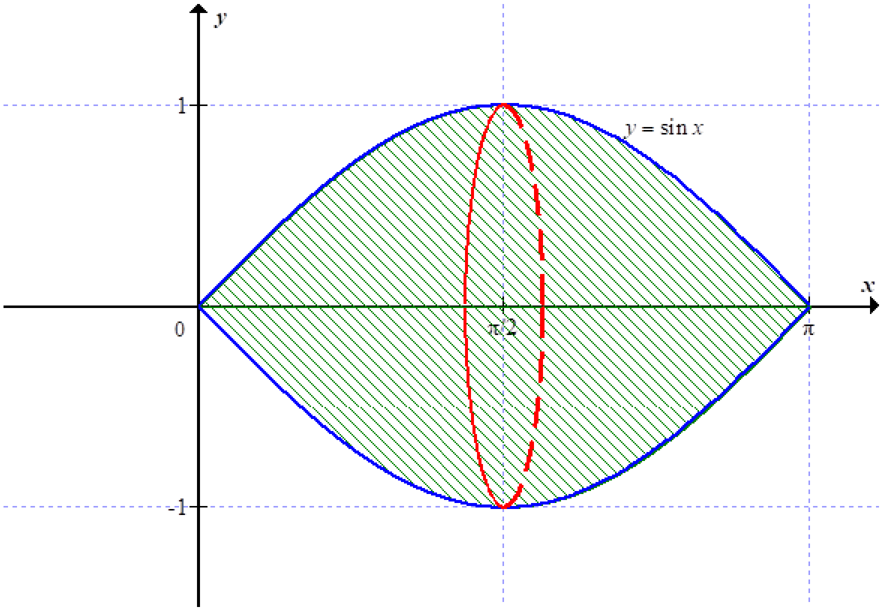
$$V=\pi\int_a^b f^2dx=\pi\int_0^5\left(x^2\right)^2 dx \Rightarrow \pi\int_0^5 x^4dx=\pi\left[\frac{1}{5}x^5\right]_0^5=625\pi U.V$$

Where U.V=unit of volume

Example

One arch of $y=\sin x$ is rotated about the $x-axis$. What is the volume of revolution?

Solution



$$V=\pi\int_a^b y^2dx \Rightarrow \pi\int_0^\pi \sin^2 xdx \Rightarrow \pi\int_0^\pi\left(\frac{1}{2}-\frac{1}{2}\cos 2x\right)dx$$

$$\Rightarrow \pi\left[\frac{1}{2}x-\frac{1}{4}\sin 2x\right]_0^\pi \Rightarrow \frac{1}{2}\pi^2 U.V$$

Application activity 5.7.2

a) Find the volume of the solid that results when the region enclosed by the given curves is revolved about the $x-axis$

1. $y=2x, \quad for\ 0\le x\le 5$

2. $y=x^2, \quad x=0, \quad x=2, \quad y=0$

3. $y = 1 + x^3, \quad x = 1, \quad x = 2, \quad y = 0$

4. $y = 9 - x^2, y = 0$

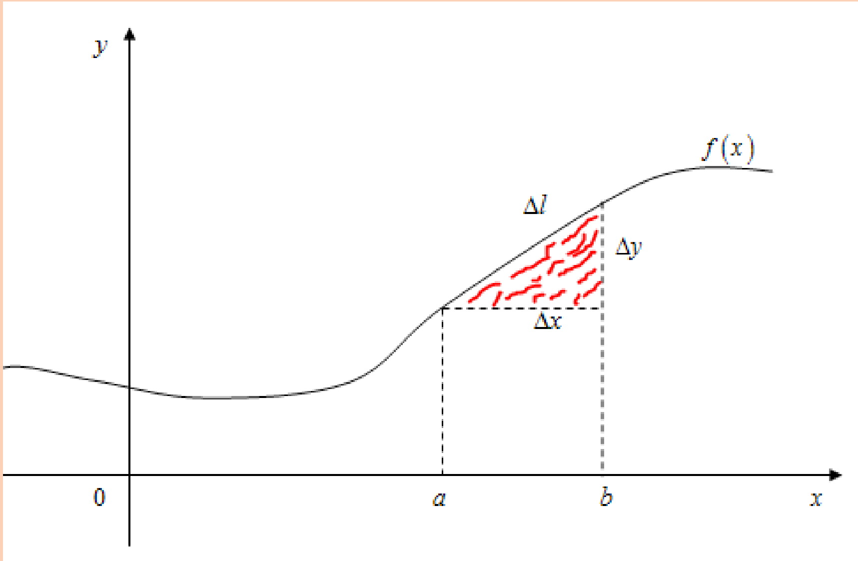
b) Find the volume of the solid that results when the region enclosed by the given curves is revolved about the $y-axis$

1) $y = x^3, x = 0, y = 1$ 2) $x = \sqrt{y+1}, x = 0, y = 3$

5.7.3 Calculation of the arc length of curved surface

Activity 5.7.3

Consider a curve given by the function $y = f(x)$, for $a < x < b$



In the triangle shown (shaded region),

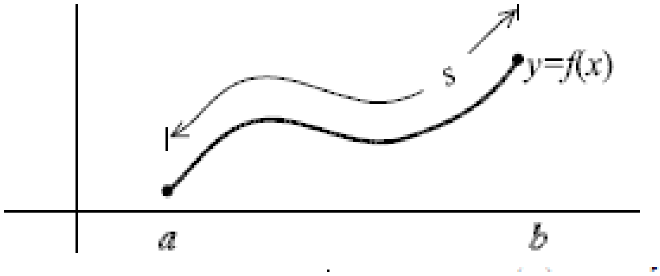
1) By the Pythagorean Theorem find Δl

2) As the step size is made smaller and smaller $\Delta x \to dx$, $\Delta y \to dy$ and $\Delta l \to dl$. From result obtained in 1), write expression equivalent to dl using $f'(x)$ where $f(x) = (x-1)^{\frac{3}{2}}$.

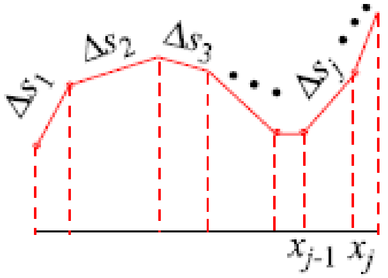
3) Take integral from a to b both sides of the relation obtained in (2) for $2 \le x \le 5$, to find L

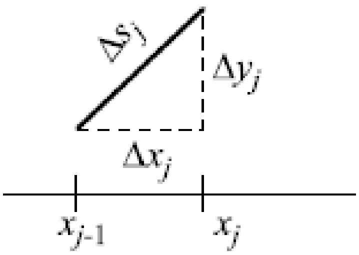
Content summary

If $f(x)$ is piecewise differentiable over $[a,b]$, then the arc length of the graph of $f(x)$ is the distance S from one end of the graph to the other end.



Let $\Delta s_1, \Delta s_2 \ldots \Delta s_n$ denote the lengths of the individual sections of the graph of $f(x)$.





Since a run of length Δx_j results in a rise of distance $\Delta y_j = f(x_j) - f(x_{j-1})$, the Pythagorean theorem implies that $\Delta S_j = \sqrt{\Delta x_j^2 + \Delta y_j^2}$.

If we approximate $y = S(x)$, then $S_{app} = \sum_{j=1}^{n} S_j = \sum_{j=1}^{n} \sqrt{\Delta x_j^2 + \Delta y_j^2}$.

In other hand, the mean value theorem says that we can choose the tags t_j such that $\Delta y_j = f'(t_j)\Delta x_j$.

As a result, $S_{app} = \sum_{j=1}^{n} \sqrt{\Delta x_j^2 + \left[f'(t_j)\Delta x_j\right]^2} = \sum_{j=1}^{n} \sqrt{1 + \left[f'(t_j)\right]^2}\,\Delta x_j$.

The result is a Riemann sum. Thus in the limit,

$S = \lim_{h \to 0} S_{app} = \lim_{h \to 0} \sum_{j=1}^{n} \sqrt{1 + \left[f'(t_j)\right]^2}\,\Delta x_j$ where the limit is over h-fine partition of

$[a,b]$.

Therefore, if $f(x)$ is continuous on $[a,b]$ and differentiable on $]a,b[$, then the length of the curve $y=f(x)$ over the interval $[a,b]$ is

$$S=\int_a^b \sqrt{1+\left[f'(x)\right]^2}\,dx$$

Example

Find the length of a line whose slope is -2 given that line extends from $x=1 \, to \, x=5$

Solution

We need the equation of the line in order to find $f'(x)$ but for the purpose of this example, we are given that the slope is -2 and we know that the slope is given by the derivative of the function, then $f'(x)-2$.

Hence

$$L=\int_1^5 \sqrt{1+\left(f'(x)\right)^2}\,dx \Rightarrow \int_1^5 \sqrt{1+(-2)^2}\,dx \Rightarrow \int_1^5 \sqrt{5}dx \Rightarrow \sqrt{5}\,[x]_1^5 = 4\sqrt{5}U.L$$, where

U.L: unit of length

Example

Find the length of the circle of radius R and center $(0,0)$.

Solution

The circle of radius R and center $(0,0)$ has equation:

$$C \equiv x^2+y^2=R^2 \Rightarrow y^2=R^2-x^2$$

$$\Rightarrow y=\pm\sqrt{R^2-x^2}$$

$$\Rightarrow y'=\pm\frac{x}{\sqrt{R^2-x^2}} \Rightarrow (y')^2=\frac{x^2}{R^2-x^2}$$

$L=\int_a^b \sqrt{1+[y']^2}\,dx$; hence $a=-R \, and \, b=R \Rightarrow L=2\int_{-R}^{R}\left(1+\frac{x^2}{\sqrt{R^2-x^2}}\right)dx$

We multiplied by 2 because we have two parts one above $x-axis$ $x-axis$ and another below $x-axis$.

$$L=2\int_{-R}^{R}\sqrt{1+\frac{x^2}{R^2-x^2}}dx \Rightarrow 2\int_{-R}^{R}\sqrt{\frac{R^2-x^2+x^2}{R^2-x^2}}dx \Rightarrow 2R\int_{-R}^{R}\frac{dx}{\sqrt{R^2-x^2}}$$

$$2R\left[\arcsin\frac{x}{R}\right]_{-R}^{R} \Rightarrow 2R\left(\arcsin\frac{R}{R}-\arcsin\frac{-R}{R}\right)=2R\left(\frac{\pi}{2}+\frac{\pi}{2}\right)=2\pi R$$

Notice

For a curve expressed in the form: $x=g(y)$ where g' is continuous on $[c,d]$,

the arc length from $y=c\ to\ y=d$ is given by $L=\int_{c}^{d}\sqrt{1+[g'(y)]^2}\,dy$

Example

Find the arc length of the curve $y=x^{\frac{3}{2}}\ from\ (1,1)\ to\ \left(2,2\sqrt{2}\right)$

Solution

$y=x^{\frac{3}{2}} \Rightarrow x=y^{\frac{2}{3}}$. Hence $g(y)=y^{\frac{2}{3}}\ and\ g'(y)=\frac{2}{3}y^{-\frac{1}{3}}$

The arc length is: $L=\int_{1}^{2\sqrt{2}}\sqrt{1+\left[\frac{2}{3}y^{-\frac{1}{3}}\right]^2}\,dy$

$$\int_{1}^{2\sqrt{2}}\sqrt{1+\frac{4}{9y^{\frac{2}{3}}}}dy \Rightarrow \int_{1}^{2\sqrt{2}}\sqrt{\frac{9y^{\frac{2}{3}}+4}{9y^{\frac{2}{3}}}}dy \quad \Rightarrow \int_{1}^{2\sqrt{2}}\frac{1}{3y^{\frac{1}{3}}}\sqrt{9y^{\frac{2}{3}}+4}dy \Rightarrow L=\frac{1}{3}\int_{1}^{2\sqrt{2}}y^{-\frac{1}{3}}\sqrt{9y^{\frac{2}{3}}+4}dy$$

Let $k=9y^{\frac{2}{3}}+4 \Rightarrow dk=9\times\frac{2}{3}y^{\frac{2}{3}-1}dy \Rightarrow y^{-\frac{1}{3}}dy=\frac{1}{6}dk$

If $y=1 \to k=13$

$y=2\sqrt{2} \to k=22$

$$L=\frac{1}{18}\int_{13}^{22}k^{\frac{1}{2}}dk \Rightarrow \frac{1}{27}\left[k^{\frac{3}{2}}\right]_{13}^{22} \Rightarrow \frac{1}{27}\left[(22)^{\frac{3}{2}}-(13)^{\frac{3}{2}}\right] \Rightarrow \frac{1}{27}\left[22\sqrt{22}-13\sqrt{13}\right]U.L$$

Application activity 5.7.3

1) Find the arc length of the curve $y = 3x^{\frac{3}{2}} - 1$ from $x = 0$ to $x = 1$.

2) Find the arc length of the curve $y = x^{\frac{2}{3}}$ $from\ x = 1\ to\ x = 8$

3) Find the arc length of the curve $y = x^{\frac{2}{3}}$ $from\ x = -1\ to\ x = 8$

5.7.4 Application of integrals in real life or other sciences

Activity 5.7.4

1) Given that the marginal cost (MC) is the rate of change of the total cost (TC) function or $MC = \frac{dTC}{dq}$, at a certain factory, the marginal cost is $3(q-4)^2$ dollars per unit of q when the level of production is q units. By how much will the total manufacturing cost increase if the level of production is raised from 6 units to 10 units.

2) The force when the spring is compressed by x units is given by $F(x) = 16x\ Newton$. If the displacement of the Force F on the distance dx produces a work $dW = F.dx$, Determine the work done on a spring when it is compressed from its natural length of 1m to the length of 0.75.

Content summary

Determination of the work done in Physics

When a force acts on an object to move that object, it is said to have done work on the object. The amount of the work done by a constant force is measured by the product of the magnitude of the force and distance moved in the direction of the force. This assumes that the force is in the direction of the motion.

$$work = (force) \times (displacement)$$

Suppose that a force in the direction of the $x-axis$ moves an object from $x = a\ to\ x = b$ on that axis and that force varies continuously with the position

x of the object, that is $F = F(x)$ is a continuous function. The element of work done by the force in moving the object through a very short distance from $x \text{ to } x + dx \text{ is } dW = F(x)dx$, so the total work done by the force is

$$W = \int_{x=a}^{x=b} dW = \int_a^b F(x)dx$$

The unit of work is the joule J, the force is in Newton (N) and the displacement is in metres (m).

Example

A variable force $F(Newton)$ modelled by the equation $F = 4x - 3$ is applied over a certain displacement (x). What is the work $(W \text{ in Joules})$ done in moving the object for a displacement of $(2m \text{ to } 4m)$.

Solution

$$W = \int_2^4 (4x-3)dx = \left[\frac{4}{2}x^2 - 3x\right]_2^4 = 18J$$

The work done is 18 joules.

Determination of cost function in Economics

In economics, the marginal function is obtained by differentiating the total function. Now, when marginal function is given and initial values are given, the total function can be obtained using integration.

If (C) denotes the total cost and $M(x) = \frac{dC}{dx}$ is the marginal cost, we can write $C = C(x) = \int M(x)dx + K$, where the constant of integration K represents the fixed cost.

Example

The marginal cost function of manufacturing $x \text{ units}$ of a product is

$5 - 16x + 3x^2 \ RwF$. Find the total cost of producing $5 \text{ up to } 20 \text{ items}$.

Solution

$$C = \int_5^{20} (5 - 16x + 3x^2)dx = \left[5x - \frac{16}{2}x^2 + \frac{3}{3}x^3\right]_5^{20} = 4950RwF$$

The required cost is 4950 RwF

Given that the marginal cost (MC) of an industry is $7.5q^2-26q+50$, determine the total cost function

5.8 End unit assessment

QUESTION ONE

Calculate the following integrals

A) $\int (9x^7+\frac{1}{x-1}+\frac{2}{\cos^2 x}-\frac{1}{2}e^x)dx=$

B) $\int \frac{x}{\sqrt{x+3}}dx=$

c) $\int \frac{1}{4}\sin 3\,\text{xdx}=$

QUESTION TWO

Discuss and solve the following problems

a) The marginal cost function of producing x units of soft drink is given by the function

$$MC=\frac{x}{\sqrt{x^2+1600}}$$

Given that the fixed cost is $500FRW$,

Determine

The total cost function

An average cost function

b). Consider the function f defined by $f(x)=4-\sqrt{x}$

a) Plot the graph of $y=f(x)$ showing the intercepts with the coordinate axes.

b) On the diagram, shade the area which is bounded by the curve and the coordinate axes.

c) Express the shaded area in terms of a definite integral

d) Calculate the area of the shaded part.

QUESTION THREE

Discuss how this unit inspired you in relation to learning other subjects or to your future. If no inspiration at all, explain why.

UNIT 6

DIFFERENTIAL EQUATION OF FIRST ORDER

Key Unit competence:

Use differential equations to solve related problems that arise in daily life.

6.0 Introductory activity

1. A quantity $y(t)$ is said to have an exponential growth model if it increases at a rate that is proportional to the amount of the quantity present, and it is said to have an exponential decay model if it decreases at a rate that is proportional to the amount of the quantity present.

Thus, for an exponential growth model, the quantity $y(t)$ satisfies an equation of the form $\frac{dy}{dt} = ky$ (k is a non-negative constant called annual growth rate).

Given that $\frac{dy}{dt} = ky$ can be written as $\frac{dy}{y} = kdt$, solve this equation and apply the

answer $y(t)$ obtained in the following problem:

The size of the resident Rwandan population in 2018 is estimated to 12,089,721 with a growth rate of about 2.37% comparatively to year 2017 (www.statistics.gov.rw/publication/demographic-dividend).

Assuming an exponential growth model and constant growth rate,

1. Estimate the national population at the beginning of the year 2020, 2030, 2040 and 2050.
2. Discuss your observations on the behavior of the national population along these 4 years.
3. What are pieces of advice would you provide to policy makers?
4. Draw a graph representing your observations mentioned in 2.

6.1 Definition and classification

Activity 6.1

For each of the following equations, form a differential equation by using derivatives in eliminating arbitrary constants k and b.

Discuss and write down the highest order of the derivative that occurs in the obtained equation:

1) $y = 4kx$

2) $y = kx + bx^2$

3) $y = k\cos 2x - b\sin 2x$

Content summary

A differential equation is any equation which contains derivatives of the unknown function; it shows the relationship between an **independent variable**, a **dependent variable** (unknown) and one or more differential coefficients of *a dependent variable* with respect to *independent variable.*

An **ordinary differential equation** (ODE) for a **dependent variable** y (unknown) in terms of an **independent variable** x is any equation which involves first or higher order derivatives of y with respect to x, and possibly x and y.

The general ordinary differential equation of the n^{th} order is

$$F\left(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, \ldots\ldots, \frac{d^ny}{dx^n}\right) = 0$$

or $F\left(x, y, y', y', \ldots\ldots, y^{(n)}\right) = 0$.

The general differential equation of the 1st order is

$F\left(x, y, \frac{dy}{dx}\right) = 0$ or $\frac{dy}{dx} = f(x, y)$

Order of a differential equation: Differential equations are classified according to the highest derivative which occurs in them.

The **order** of a differential equation is the highest derivative present in the differential equation.

The **degree** of a differential equation is the algebraic degree of its highest ordered derivative after simplification.

Example

1) $2(y')^2 + 2x + y''' = 0 \; or \; 2\left(\frac{dy}{dx}\right)^2 + 2x + \frac{d^3y}{dx^3}$; the order is 3 and the degree is 1.

2) $y'' + 2y' + x^2 = 0$; the order is 2 and the degree is 1

3) $y'' + 2x + y^2 = 0$; the order is 2 and the degree is 1.

4) $L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{q}{C} = E\sin\omega t \; or \; Lq'' + Rq' + \frac{q}{c} = E\sin\omega t$;
order is 2, the degree is 1.

5) $\frac{dy}{dx} = x^2$ is of order 1 and degree 1

6) $y\frac{d^2y}{dx^2} + \cos x = 0$, Order 2 and degree 1

7) $\left(\frac{dy}{dx}\right)^2 + y = x$, Order 1 and degree 2

Note: Differential equations is an important branch of Mathematics which has wide applications in: Physics, Chemistry, Astronomy and Economics. Many practical problems like decay of a certain substance, change in temperature of a body, increase in population can be solved with help of differential equations.

An equation involving the independent variable x, dependent variable y and the differential coefficients: $\frac{dy}{dx}, \frac{d^2y}{dx^2}, ..., \frac{d^ny}{dx^n}$ is called a **differential equation.**

Examples of differential equations:

$i)\; \frac{dy}{dx} = x$

$ii)\; \frac{dy}{dx} = \cot x$

$iii)\; \frac{d^4y}{dx^4} - 4\frac{dy}{dx} + 4y = 5\cos 3x$

$iv)\ \frac{d^2y}{dx^2}+16y=0$

$v)\ \sin xdx+\cos ydy=0$

Order and degree of a differential equation:

1. **Order:** the order of a differential equation is the order of the highest derivative occurring in the differential equation.

Examples:

a) $\frac{d^2y}{dx^2}-5\frac{dy}{dx}+6y=0$ this is a differential equation of second order, because the highest derivative occurring in it is 2.

b) $\frac{d^3y}{dx^3}+5\frac{dy}{dx}-y=0$ this is a differential equation of third order, because the highest derivative occurring in it is 3.

c) $\left(\frac{d^2y}{dx^2}\right)^3+\left(\frac{dy}{dx}\right)^4=e^x$ this is a differential equation of second order, because the highest derivative occurring in it is 2.

2. **Degree:** the degree of the differential equation is the degree of the highest derivative when differential coefficients are rational and free from fraction.

Examples:

i) $x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+y=0$ this is a differential equation of second order, and degree one.

ii) $y=\frac{dy}{dx}+\frac{c}{\frac{dy}{dx}}$ in this differential equation derivatives are not free from fractions.

Removing the fractions it takes the form: $y\frac{dy}{dx}=x\left(\frac{dy}{dx}\right)^2+c$ this is a differential equation of first order, and degree two.

Application activity 6.1

Work out the following:

Discuss and state the order and the degree of each of the following differential equations. Explain your answer.

a) $\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^4-4x+y=1$

b) $\left(\frac{dy}{dx}\right)^3-2x=\cos y-2\sin x$

c) $(y'')^3+(y')-2y=x$

d) $y\frac{d^2y}{dx^2}=-\cos x$

e) $x^2\left(\frac{d^2y}{dx^2}\right)^4+y\left(\frac{dy}{dx}\right)+y^4=0$

6.2 First order differential equations

6.2.1 Differential equations with separable variables

Activity 6.2.1

1. Consider $4y'-2x=0$,

a) Express y' in a differential form and then integrate both sides to deduce the value of the dependent variable y.

b) What can you say if you were given $4\frac{dy}{dx}-2x=0$?

c) Check whether y is solution of the given equation.

2. Apply the technique so-called separation of variables used in (1) to solve the following:

a) $\sin xdx-\sin ydy=0$ b) $x\frac{dy}{dx}=1$

3. Discuss how to solve $f(y)\frac{dy}{dx}=g(x)$

Content summary

A **separable differential equation** is an equation of the form $\frac{dy}{dx}=f(x)h(y)$.

These are called separable variables because the expression for $\frac{dy}{dx}$ or y' can be separated into a product of separate functions of x and y alone. This means that they can be rewritten so that all terms involving y are on one side of the equation and all terms involving x are on the other.

That is: $\dfrac{dy}{h(y)} = f(x)\,dx$

Hence, solving the equation requires simply integrating both sides with respect to their respective variables;

$$\int \frac{dy}{h(y)} = \int f(x)dx + c$$

Of course the left-hand side is now an integral with respect to *y*, the right-hand side with respect to *x*. Note that we only need one arbitrary constant.

In particular if $h(y) = m$ (a constant), the differential equation of the form $\dfrac{dy}{dx} = mf(x)$ is solved by direct integration. That is:

$$dy = mf(x)\,dx \Leftrightarrow \quad y = m\int f(x)\,dx + c.$$

Similarly, equation of the form $\dfrac{dy}{dx} = mf(y)$ is solved by direct integration:

$$\frac{dy}{f(y)} = mdx \quad \Leftrightarrow \int \frac{dy}{f(y)} = mx + c$$

A **solution** to a differential equation on an interval $\alpha < x < \beta$ is any function which satisfies the differential equation in question on that interval. It is important to note that solutions are often accompanied by intervals and these intervals can impart some important information about the solution.

Example

Show that $y = \dfrac{x}{2}\left(\dfrac{x^2}{6} + 1\right)$ is a solution of $\dfrac{dy}{dx} = \dfrac{x^2+2}{4}$ on $]-\infty, +\infty[$.

Solution

Given that $y = \dfrac{x}{2}\left(\dfrac{x^2}{6} + 1\right) = \dfrac{x^3 + 6x}{12}$, we have $\dfrac{dy}{dx} = \dfrac{3x^2+6}{12} = \dfrac{x^2+2}{4}$.

Therefore, $y = \dfrac{x}{2}\left(\dfrac{x^2}{6} + 1\right)$ is solution of $\dfrac{dy}{dx} = \dfrac{x^2+2}{4}$ on $]-\infty, +\infty[$.

It is easily checked that for any constant c, $y=\frac{x}{2}\left(\frac{x^2}{6}+1\right)+c$ is also a solution to the equation called general solution of the given equation.

A **general solution** to a given differential equation is the most general form that the solution can take and doesn't take any initial conditions into account. In this way, there are an infinite number of solutions to a differential equation depending on the value of the constant; it is better (especially in applied problems) to precise conditions which lead to a particular solution.

Initial Conditions are conditions or set of conditions imposed to the general solution that will allow us to determine one **particular solution** also called **actual solution** that we are looking for.

In other words, initial conditions are values of the solution and/or its derivative(s) at specific points which help to determine values of arbitrary constants that appear in the general solution. Since the number of arbitrary constants in general solution to a given differential equation is equal to the order of the ODE, it follows that it requires *n* conditions to determine values for all *n* arbitrary constants in the general solution of an n^{th}-order differential equation (one condition for each constant). For a first order equation, the single arbitrary constant can be determined by specifying the value of the unknown function *y(x)* at an arbitrary *x*-value x_0, say $y(x_0)=y_0$.

Geometrically, initial condition of a first order differential equation ($\frac{dy}{dx}=f(x,y)$) enables us to identify a specific function $(y=y(x))$ whose curve passes through the point (x_0,y_0) and the slope is $f(x_0,y_0)$.

A differential equation along which an appropriate number of initial conditions are given is called **Initial Value Problem** (or **IVP**). Therefore, the **actual solution or particular solution** to a differential equation is the specific solution that not only satisfies the differential equation, but also satisfies the given initial condition(s).

Example

Solve (a) $\frac{dy}{dx}=xy$ (b) $\frac{dy}{dx}=\frac{x^2+1}{4}$

Solution

(a) $\frac{dy}{dx}=xy$ we separate to give $\frac{dy}{y}=xdx$ so, $\int\frac{1}{y}dy=\int xdx$ and integrating both sides with respect to their respective variables gives $\ln y = \frac{x^2}{2} + c$ or $y = e^{\frac{x^2}{2}+c}$.

Then, $y = e^{\frac{x^2}{2}+c} \Rightarrow y = e^c \times e^{\frac{x^2}{2}}$

(b) $\frac{dy}{dx} = \frac{x^2+1}{4} \Leftrightarrow 4dy = \left(x^2+1\right)dx$

$\int 4dy = \int (x^2+1)dx \Rightarrow 4y = \frac{x^3}{3} + x + c \Rightarrow y = \frac{x^3}{12} + \frac{x}{4} + k$ (where we set $k = \frac{c}{4}$).

Example

Solve the IVP: $\frac{dy}{dx} = \frac{y}{x-3}$ with $y(0) = -3$

Solution

IVP: $\frac{dy}{dx} = \frac{y}{x-3}$; $y(0) = -3$

$\int \frac{1}{y} dy = \int \frac{1}{x-3} dx$

Simple integrations yield to $\ln y = \ln(x-3) + c$.

For simplicity and aesthetic purpose, let $c = \ln k$.

Thus $\ln y = \ln(x-3) + \ln k$ or equivalently $y = k(x-3)$.

(general solution).

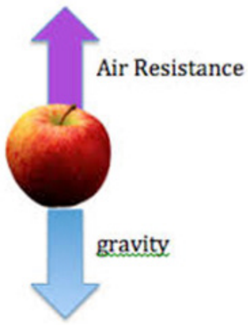
Let's apply initial condition to the general solution. If $y(0) = -3$,

it follows that $-3 = k(0-3)$ giving $k = 1$. Therefore, $y = x-3$ is the required particular solution that represents equation of the unique line passing through (0,-3) and whose slope is $\frac{y_0}{x_0 - 3} = \frac{-3}{0-3} = 1$.

Example

Consider a falling apple with mass m illustrated on the figure 6.1, assume that only gravity and air resistance with the coefficient $\alpha = 0.38$ are acting upon it when falling.

Figure 6.1: forces acting upon the falling object



a) Derive a differential equation expressing Newton's Second Law of motion for the apple and solve it to determine its velocity at any time t.

b) If the apple weights 0.2 kg and the coefficient for air resistance $\alpha = 0.38$ and supposing that at the initial time $(t=0)$ the velocity was null $(v=0)$, determine the velocity of the apple at any time t.

c) Plot the function velocity and discuss its limit as the time increases towards infinity.

Solution

Recall that Newton's Second Law of motion can be written as $m\frac{dv}{dt} = F(t,v)$ where $F(t,v)$ is the sum of forces that act on the apple and may be a function of the time t and its velocity v.

For this situation, we have two forces acting on the apple:

Force of gravity $F_G = mg$ (where g is gravitational acceleration) acting in the downward positive direction, and

Air resistance $F_A = -\alpha v$ (where α is a coefficient and v the velocity) acting in the upward direction and hence in negative direction .

Putting all of these together into Newton's Second Law, we find the following $m\frac{dv}{dt} = mg - \alpha v$ and the acceleration $\frac{dv}{dt} = g - \frac{\alpha}{m}v$. This is a differential equation with separable variables: the time t and the velocity v.

Separating variables, we have $\frac{dv}{g - \frac{\alpha}{m}v} = dt$

b) So, let's assume that the apple has a mass of 0.2 kg and that $\alpha = 0.38$. Plugging these values into (1) gives the following differential equation

$$\frac{dv}{9.8-\frac{0.38}{0.2}v}=dt \text{ or equivalently } \frac{dv}{9.8-1.9v}=dt$$

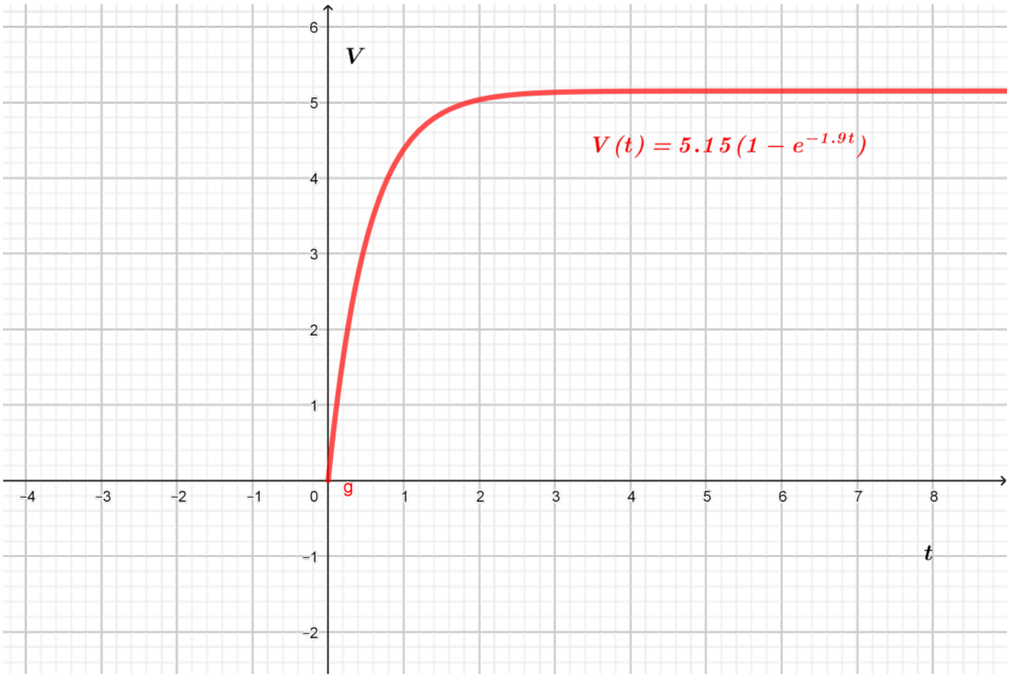
Integrating both sides, the velocity of the falling object at the time t becomes:

$v=5.15+ke^{-1.9t}$ where k is an arbitrary constant.

c) Supposing that at the initial time $(t=0)$ the velocity was null $(v=0)$, the constant becomes $k=-5.15$.

Therefore, the velocity v of the apple of 0.2kg released with the initial velocity $v_0=0$ in the air of resistance $\alpha=0.38$, is given by: $v=5.15\left(1-e^{-1.9t}\right)$ Unit of velocity.

Figure: Velocity of a falling object



The figure shows that the velocity increases with time towards $V=5.15\ m/s$ as time increases indefinitely.

Example

Solve $\frac{dy}{dx}=\frac{x^2+1}{4}$

Solution

$$\frac{dy}{dx}=\frac{x^2+1}{4} \Leftrightarrow 4dy=\left(x^2+1\right)dx$$

$$\int 4dy=\int\left(x^2+1\right)dx$$

$$\Rightarrow 4y=\frac{x^3}{3}+x+c$$

$$y=\frac{x^2}{12}+\frac{x}{4}+c$$

Notice:

In general a DE can have more than one such solution. Sometimes a single function can be found which incorporates all possible solutions of the DE

– this is then referred to as the **general solution**. Such solutions contain one or more arbitrary constants, usually depending on the order of the DE. Any other solution than the general solution is called a **particular solution**.

The arbitrary constant(s) occurring in the general solution can be determined by supplementing the DE by specified conditions in which *y*, or a sufficient number of its derivatives, is given for some particular value(s) of *x*. Such conditions are called **initial** or **boundary conditions**.

Example

Determine the general solution of $x\frac{dy}{dx}=2-4x^3$

Solution

Rearranging $x\frac{dy}{dx}=2-4x^3$ gives

$$\frac{dy}{dx}=\frac{2-4x^3}{x}=\frac{2}{x}-4x^2$$

Integrating both sides gives:

$$y=\int\left(\frac{2}{x}-4x^2\right)dx=2\ln x-\frac{4}{3}x^3+c$$

Thus the required solution is $y = 2\ln x - \frac{4}{3}x^3 + c$

Example

Find the particular solution of the differential equation $5\frac{dy}{dx} + 2x = 3$ given boundary conditions $y = \frac{7}{5}$ when $x = 2$

Solution

$$5\frac{dy}{dx} + 2x = 3 \Leftrightarrow 5\frac{dy}{dx} = 3 - 2x$$

$$\Leftrightarrow \frac{dy}{dx} = \frac{3-2x}{5}$$

Direct integration gives $y = \int \frac{3-2x}{5}dx = \frac{3}{5}x - \frac{1}{5}x^2 + c$

$y = \frac{7}{5}$ when $x = 2 \Rightarrow \frac{7}{5} = \frac{6-4}{5} + c \Rightarrow c = 1$

Hence particular solution is $y = \frac{3}{5}x - \frac{1}{5}x^2 + 1$

Example

Determine the general solution of

$$\frac{dy}{dx} = x\cos x$$

Solution

Direct integration gives $y = \int x\cos x dx + c$;

Integration by part gives

$y = x\sin x - \int \sin x dx + c = x\sin x + \cos x + c$ which is the required solution.

Example

Determine the particular solution of $\left(y^2 - 1\right)\frac{dy}{dt} = 3y$ given that $y = 1$ when $t = 2\frac{1}{6}$.

Solution

Rearranging gives $\frac{y^2-1}{3y}\frac{dy}{dt}=dt \Leftrightarrow \left(\frac{y}{3}-\frac{1}{3y}\right)\frac{dy}{dt}=dt$

Direct integration yields

$\frac{y^2}{6}-\frac{\ln y}{3}=t+c$ or $t=\frac{y^2}{6}-\frac{\ln y}{3}-c$ that is general equation.

Given that $y=1$ when $t=2\frac{1}{6}$, thus

$$2\frac{1}{6}=\frac{(1)^2}{6}-\frac{\ln 1}{3}-c \Leftrightarrow 2\frac{1}{6}=\frac{1}{6}-c \Rightarrow c=\frac{1-13}{6}=-2$$

Hence particular solution is $t=\frac{y^2}{6}-\frac{\ln y}{3}+2$.

Example

The variation of resistance R ohms, of an Aluminum conductor with temperature ϑ°C is given by $\frac{dR}{d\theta}=\alpha R$, where α is the temperature coefficient of resistance of Aluminum.

If $R=R_o$ when $\theta=0^oC$, solve the equation for R.

(b) If $\alpha=38\times10^{-4}/{}^oC$, determine the resistance of an Aluminum conductor at 50^oC, correct to 3 significant figures, when its resistance at 0^oC is 24.0Ω

Solution

$$\frac{dR}{d\theta}=\alpha R \Leftrightarrow d\theta=\frac{dR}{\alpha R}$$

$$\int d\theta=\int\frac{dR}{\alpha R}\Rightarrow \theta=\frac{1}{\alpha}\ln R+c$$

The general solution is $\theta=\frac{1}{\alpha}\ln R+c$.

$R=R_o$ when $\theta=0^oC$, thus $0=\frac{1}{\alpha}\ln R_o+c \Rightarrow c=-\frac{1}{\alpha}\ln R_o$

Hence the particular solution is $\theta=\frac{1}{\alpha}\ln R-\frac{1}{\alpha}\ln R_o \Leftrightarrow \theta=\frac{1}{\alpha}\ln\frac{R}{R_o}$

$\Rightarrow \alpha\theta=\ln\frac{R}{R_o}$ or $e^{\alpha\theta}=\frac{R}{R_o}$.

Hence $R = R_o e^{\alpha\theta}$

(b) Substituting $\alpha = 38\times10^{-4}$, $\theta = 50$ and 24.0Ω into $R = R_o e^{\alpha\theta}$ gives the resistance at 50^oC

$$R = 24.0e^{38\times10^{-4}\times50} = 29.0\Omega$$

Application activity 6.2.1

1. Determine the general solution for: a) $\frac{dy}{dx} = x\cos x$ b) $x\frac{dy}{dx} = 2-4x^3$.

2. Solve the following initial value problem: $(x+1)\frac{dy}{dx} = x(y^2+1)$, $y(0)=0$

3. (a) The graph of a differentiable function y = y(x) passes through the point (0, 1) and at every point P (x, y) on the graph the tangent line is perpendicular to the line through P and the origin. Find an initial-value problem whose solution is y(x).

(b) Explain why the differential equation in part (a) is separable.

Solve the initial-value problem using either separation of variables and describe the curve.

4. Determine the particular solution of $(y^2-1)\frac{dy}{dt} = 3y$ given that $y=1$ when $t = 2\frac{1}{6}$.

5. (a) The variation of resistance R in ohms of an aluminum conductor with temperature $\theta^\circ C$ is given by $\frac{dR}{d\theta} = \alpha R$, where α is the temperature coefficient of resistance of aluminum.

If $R = R_o$ when $\theta = 0^oC$, solve the equation for R.

(b) If $\alpha = 38\times10^{-4} / {}^oC$, determine the resistance of an aluminum conductor at 50^oC, correct to 3 significant figures, when its resistance at 0^oC is 24.0Ω.

6.2.2 The solution of equation of the form $\frac{dy}{dx} = f(x)h(y)$

Activity 6.2.2

Express each of the following equations in the form $f(y)dy = g(x)dx$ and integrate both sides

1) $4xy\frac{dy}{dx} = y^2 - 1$

2) $\frac{y^2+1}{x^2+1} = \frac{y}{x}\frac{dy}{dx}$

Content summary

A differential equation of the form $\frac{dy}{dx} = f(x)h(y)$ where $f(x)$ is a function of x only and $f(y)$ is a function of y only, may be rearranged as

$\frac{dy}{f(y)} = f(x)dx$, and then the solution is obtained by direct integration,

i.e. $\int \frac{dy}{f(y)} = \int f(x)dx$

Example

Solve the equation $(x+1)\frac{dy}{dx} = x(y^2+1)$.

Solution

Separating the variables gives:

$(x+1)dy = x(y^2+1)dx$ or $\frac{1}{y^2+1}dy = \frac{x}{x+1}dx$

Integrating both sides gives

$$\int \frac{1}{y^2+1}dy = \int \frac{x}{x+1}dx$$

Or $\int \frac{1}{1-y^2}dy = \int \left(1 - \frac{1}{x+1}\right)dx$

Or $\int \frac{1}{1-y^2}dy = \int dx + \int \frac{1}{x+1}dx$

Or $\tan^{-1} y = x - \ln(x+1) + c$

Example

Determine the particular solution of $xy=\left(1+x^2\right)\frac{dy}{dx}$ given that $y=1$ when $x=0$.

Solution

Separating the variables gives:

$$\frac{dy}{y}=\frac{x}{1+x^2}dx$$

Integrating both sides gives

$$\ln y=\frac{1}{2}\ln\left(1+x^2\right)+c$$

Given that $y=1$ when $x=0$, thus $\ln 1=\frac{1}{2}\ln 1+c\Rightarrow c=0$

The particular solution is $\ln y=\frac{1}{2}\ln\left(1+x^2\right)$ or $y=\sqrt{1+x^2}$

Application activity 6.2.2

1) Find the general solution of the following differential equations

a) $\frac{dy}{dx}=2y\cos x$

b) $(x+1)\frac{dy}{dx}=x\left(y^2+1\right)$ satisfying y(1) = 0

c) $(2y-1)\frac{dy}{dx}=3x^2+1$, given that $x=1$ and $y=2$

d) $\frac{d\theta}{dt}=2e^{3t-2\theta}$ with $\theta=0$ for $t=0$

e) $\left(xy^2+x\right)dx+\left(yx^2+y\right)dy=0$

2) Determine the value of p, given that $x^3\frac{dy}{dx}=p-x$ and that $y=0$ when $x=2$ and when $x=6$.

6.3 Simple homogeneous differential equations

Letting $z=\frac{y}{x}$, solve $\frac{dy}{dx}=\frac{xy}{x^2-y^2}$

Content summary

A function $f(x,y)$ is called homogeneous of degree n if $f(tx,ty)=t^n f(x,y)$ for all suitably restricted x,y and t this means that if x and y are replaced with tx,ty and t^n, factors out of the resulting function.

Example

Show that $\sqrt{x^2+y^2}$ is homogeneous of degree one.

Solution:

$\sqrt{(tx)^2+(ty)^2} \Rightarrow \sqrt{t^2\left(x^2+y^2\right)} \Rightarrow t^1\sqrt{x^2+y^2}$; where n=1

Example

Show that $\sin\frac{x}{y}$ is homogeneous of degree zero.

Solution:

$\mathrm{f}(tx,ty)=\sin\frac{tx}{ty}=\sin t^0\left(\frac{x}{y}\right)=\sin\frac{x}{y}=t^0\sin\frac{x}{y}=t^0 f(x,y)$; where n=0

Note: Differential equation $M(x,y)dx+N(x,y)dy=0$ is said to be homogeneous if M and N are homogeneous functions of the same degree.

This equation can then be written as, $\frac{dy}{dx}=f(x,y)$ where $f(x,y)=-\frac{M(x,\mathrm{y})}{N(x,\mathrm{y})}$

is clearly homogeneous of degree zero and we solve this equation by letting $z=\frac{y}{x}$, which reduces the equation to variable separable.

Example

Solve $(x+y)dx-(x-y)dy=0$

Solution:

$$\frac{dy}{dx}=\frac{x+y}{x-y}$$

$$let\ z=\frac{y}{x}\Rightarrow y=xz$$

$$\frac{dy}{dx}=z\frac{dx}{dx}+x\frac{dz}{dx}\Rightarrow\frac{dy}{dx}=z+x\frac{dz}{dx}$$

$$\frac{dy}{dx}=\frac{\frac{x}{x}+\frac{y}{x}}{\frac{x}{x}-\frac{y}{x}}\Rightarrow\frac{dy}{dx}=\frac{1+z}{1-z}$$

$$\Rightarrow z+x\frac{dz}{dx}=\frac{1+z}{1-z}\Rightarrow x\frac{dz}{dx}=\frac{1+z}{1-z}-\frac{z-z^2}{1-z}\Rightarrow x\frac{dz}{dx}=\frac{1+z^2}{1-z}$$

$$\Rightarrow\frac{dx}{x}=(\frac{1-z}{1+z^2})dz\Rightarrow\int\frac{dx}{x}=\int\frac{dz}{1+z^2}-\int\frac{zdz}{1+z^2}$$

$$\Rightarrow\ln|x|=\arctan z-\frac{1}{2}\ln\left|1+z^2\right|+c \qquad z=\frac{y}{x}$$

$$\Rightarrow\arctan\frac{y}{x}=\ln c\left|\sqrt{x^2+y^2}\right|$$

Application activity 6.3

Letting $z=\frac{y}{x}$, solve

a) $\frac{dy}{dx}=\frac{x^2+xy}{xy+y^2}$

b) $2xy\frac{dy}{dx}=y^2-x^2$

6.4 Linear differential equations of the first order

Activity 6.4

Consider the equation $\frac{dy}{dx}+2xy=x$

(1) Assume that there exists a function $I(x)$ called an **integrating factor** that must help us to solve the equation (1).

1) Compute $I(x)=e^{\int 2xdx}$. For the time being, set the integration constant to 0.

2) Multiply both sides in the differential equation(1) by $I(x)$ and verify that the left side becomes the product rule $(I(x).y(x))'$ and write it .

3) Integrate both sides, make sure you properly deal with the constant of integration

4) Solve for the function $y(x)$

Verify if the value of $y(x)$ obtained in 4) is solution of (1).

Content summary

An ordinary differential equation (ODE) in which the only power to which y or any of its derivatives occurs is zero or one is called a **linear ODE.** Any other ODE is said to be **nonlinear.**

Thus, if p and q are functions in x or constants the general linear equation of first order can take the form $\frac{dy}{dx}+py=q$ (2)

There exists a "magical" function $I(x)$ called **integrating factor** that helps to solve the equation (2).

The solution process for a first order linear differential equation is as follows:

a) Determine an **integrating factor** $I(x)=e^{\int pdx}$ taking the integrating constant $c\ =0$.

b) Multiply both sides in the differential equation (2) by $I(x)$ and verify that the left side becomes the product rule $\left(I(x).y(x)\right)'$ and write it as such.

c) Integrate both sides, make sure you properly deal with the constant of integration

d) Solve for the solution $y(x)=\frac{\int I(x)q(x)+C}{I(x)}$

This process can be simplified by letting $y=uv$ where u and v are functions in x to be determined in the following ways:

$v=e^{-\int pdx}$ by taking the constant $c=0$ and $u=\int q\,e^{\int pdx}\,dx$

Therefore, the solution of the equation $\frac{dy}{dx}+py=q$ becomes $y=uv$ where $u=\int q\,e^{\int pdx}\,dx$ and $v=e^{-\int pdx}$.

Example

State the order and degree of each ODE, and state which are linear or non linear:

i) $\frac{dy}{dx}+y=x$

ii) $x''+3t^2=0$

iii) $R\frac{dq}{dt}+\frac{q}{C}=3$

iv) $\frac{d^3y}{dx^3}+\frac{d^2y}{dx^2}-2\frac{dy}{dt}x\sin 2x=y+4x^4$

Solution

i) $\frac{dy}{dx}+y=x$ is a linear differential equation in y of first order and its degree is 1

ii) $x''+3t^2=0$ is linear differential equation of second order in x and the degree is 1.

iii) $R\frac{dq}{dt}+\frac{q}{C}=3$ is linear differential equation of first order in q and its degree is 1.

iv) $\frac{d^3y}{dx^3}+\frac{d^2y}{dx^2}-2\frac{dy}{dt}x\sin 2x=y+4x^4$ is linear differential equation of third order in y and its degree is 1.

Example

Consider the following equation from the falling apple previously seen and solve using integrating factor:

$$\frac{dv}{dt}+1.9v=9.8, \quad v(0)=0$$

Compare the two processes.

Solution

$$\frac{dv}{dt}+1.9v=9.8, \quad v(0)=0$$

Relating this equation to $\frac{dy}{dx}+py=q$, the functions $p(x)$ and $q(x)$ are

constant such that $p=1.9$ and $q=9.8$.

Therefore, the integrating factor $I(t)=e^{\int 1.9dt}=e^{1.9t}$

And $v(t)=\frac{\int I(t)q(t)dt+C}{I(t)}$.

Given that $\int I(t)q(t)dt+C=\int 9.8e^{1.9t}dt+C=\frac{9.8}{1.9}e^{1.9t}+c$,

$$v(t)=\frac{5.15e^{1.9t}+c}{e^{1.9t}}=5.15+ce^{-1.9t}$$

Applying the initial condition we get $v(t)=5.15\left(1-e^{-1.9t}\right)$ which is the same as the answer previously seen.

The solving process to get the general solution seems to involve more steps than the method for separable variables

Example

Use integrating factor to solve $y'-x^2y=x^2$.

Solution

$$y'-x^2y=x^2$$

The equation becomes $\frac{dy}{dx}-x^2y=x^2$, $p=-x^2$, $\quad q=x^2$

Therefore, $I(t)=e^{\int -x^2dx}=e^{-\frac{x^3}{3}}$

$$y(x)=\frac{\int I(x)q(x)dx+C}{I(x)}=\frac{\int e^{-\frac{x^3}{3}}\left(x^2\right)dx+C}{e^{-\frac{x^3}{3}}}$$

$$\frac{\int e^{-\frac{x^3}{3}}.\left(x^2\right)dx+c}{e^{-\frac{x^3}{3}}}=\frac{-e^{-\frac{x^3}{3}}+c}{e^{-\frac{x^3}{3}}}=-1+ce^{\frac{x^3}{3}}$$

$y(x)=-1+ce^{\frac{x^3}{3}}$ where c is a constant.

Example

State the order and degree of each DE, and state which are nonlinear.

i) $\left(\frac{dy}{dx}\right)^2+y=x$

ii) $x''+3t^2=0$

iii) $R\frac{dq}{dt}+\frac{q}{C}=3$

iv) $\frac{d^3y}{dx^3}+\frac{d^2y}{dx^2}-2\frac{dy}{dx}x\sin 2x = y+4x^4$

Solution

i) is nonlinear differential equation of first order in y and degree is 2

ii) is linear differential equation of second order in x and degree is 1

iii) is linear differential equation of first order in q and degree is 1

iv) is linear differential equation of third order in y and degree 1.

Application activity 6.4

1. Determine the general solution of the following equations

a) $y'+\frac{y}{x}=1$ b) $y'+xy=x$

c) $y'+\frac{y}{x}=x$ d) $y'+2y=e^x$

e) $y'-2x\,y=e^{x^2}$ f) $y'+\frac{3}{x}y=\frac{\sin x}{x^3}$

2. The p.d., V, between the plates of a capacitor C charged by a steady voltage E through a resistor R is given by the equation $CR\frac{dV}{dt}+V=E$.

a) Solve the equation for V given that at $t=0, V=0$.

b) Calculate V, correct to 3 significant figures, when $E=25V$, $C=20\times10^{-6}F$, $R=200\times10^3\Omega$ and $t=3.s$.

6.5 Some applications of differential equations

6.5.1 Differential equations and the population growth

Activity 6.5.1

The number of individuals (population) P present at a given time *t* is a function of

time. Given that the rate of change (in time) of this population $\frac{dP}{dt}$ is proportional to the population P present,

1) Write a differential equation expressing this model assuming that the constant of proportionality K is positive;

2) Solve the obtained equation considering that $K = 5\%$ and the population was P_0 at initial time $t = 0$;

3) If the population of a given town respecting the same variation is now $P_0 = 11,500,000$; what will be its size after 5 years? Plot the related graph and give your interpretation in your own words.

What are pieces of advice would you provide to policy makers of that town?

Content summary

The growth of a population is usually modeled with an equation of the form $\frac{dP}{dt} = kP$, where P represents the number of individuals on a given time *t.* The constant k is a positive when the population grows and negative when the population decreases.

Separating variables and integrating both sides of $\frac{dP}{dt} = kP$, we get:

$$\int \frac{dP}{P} = k\int dt \Rightarrow \ln P = kt + c \Rightarrow P = ce^{kt}$$

If the initial population at time $t = 0$ is P_0 , then $P_0 = ce^{(0.02)(0)} = c$

Therefore, $P_0 = c$ and we have $P = P_0 e^{k}$

A solution to this equation is the exponential function: $P(t) = P_0 e^{kt}$ where P_0 is the initial population at time $t = 0$, k the annual growth rate or the annual decay rate.

Example

Consider the population P of a region where there is no immigration or emigration. The rate at which the population is growing is often proportional to the size of the population. This means larger populations grow faster, as we expect since there are more people to have babies.

If the population has a continuous growth rate of 2% per unit time, what is its population at any time t?

Solution

We know that $\frac{dP}{dt} = kP$,

Separating variables and integrating both sides to get:

$\int \frac{dP}{P} = k\int dt \Rightarrow \ln P = kt + c \Rightarrow P = ce^{kt}$ for $k = 0.02$ and has the general solution $P = ce^{0.02t}$

If the initial population at time $t = 0$ is P_0, then $P_0 = ce^{(0.02)(0)} = c$

So, $P_0 = c$ and we have $P = P_0 e^{0.02t}$

Application activity 6.5.1

1) The population of a given city is double in 20 years. We suppose that the rate of increasing is proportional to the number of population. In how many time the population will be three times, if on $t = 0$ we have the population P_0.

2) In laboratory, it is observed that the population of bacteria is increasing from 1000 to 3000 during 10 hours. If the rate of increasing of bacteria is supposed to be proportional to the present number of bacteria on the time t ,find the number of bacteria after 5 hours.

6.5.2 Differential equations and Crime investigation

Activity 6.5.2

Consider the object of temperature T that is cooling with time t in a given environment. Newton's Law of cooling states that the rate of change (in time) of the temperature is proportional to the difference between the temperature T of the object and the temperature Te of the environment surrounding the object considering that the temperature is reducing.

1) Write the differential equation expressing this model;

2) Solve the obtained equation considering that the constant of proportionality $K=\ln\left(\frac{1}{2}\right)$;

3) Assume that the object that had a normal temperature of $98.6^0 F$ at the initial time was put in a room with a constant temperature 70.0°F; plot the graph of T and give its interpretation in your own words.

Content summary

The time of death of a murdered person can be determined with the help of modeling through differential equation. Some police personnel discover the body of a dead person presumably murdered and the problem is to estimate the time of death.

The Newton's Law of Cooling stating that the rate of change (in time) of the temperature is proportional to the difference between the temperature T of the object and the temperature Te of the environment surrounding the object is very essential in solving such a problem.

Therefore, the time of death of a murdered person can be determined with the help of the differential equation:

$$\frac{dT}{dt}=-K\left(T-T_e\right)$$

The given equation is equivalent to $\frac{dT}{dt}+KT=KT_e$

This is a first order linear differential equation, its solution

$T\left(t\right)=T_e+Be^{Kt}$ where B is a constant.

Example

Police discovers a murder victim in a hotel room at $9:00\,am$ one morning. The temperature of the body is $80.0^0 F$. One hour later, at $10:00\,am$, the body has cooled to $75.0^0 F$. The room is kept at a constant temperature of 70.0◦F. Assume that the victim had a normal temperature of $98.6^0 F$ at the time of death.

a) Formulate the differential equation for the temperature of the body as function of time.

b) Solve the differential equation.

c) Use your solution in b) to estimate the time the murder took place.

Solution:

Let T be the temperature of the body after t hours. By Newton's Law of Cooling we have the differential equation $\frac{dT}{dt} = K(T-70)$ where K is a constant.

$$\frac{dT}{T-70} = Kdt$$

$$\int \frac{dT}{T-70} = \int Kdt \Rightarrow \ln(T-70) = Kt + c$$

$T - 70 > 0$ because the body will never be cooler than the room.

$T - 70 = e^{\ln(T-70)} = e^{kt+c} = e^c e^{kt}$ Let $A = e^c$ (positive constant),

Then, $T = 70 + Ae^{kt}$.

Take $t = 0$ when the body was found at 9:00am. Plug in $t = 0$ and $T = 80.0^0 F$ and solve for

c (It's easier to solve for $A = e^c$ and use this in your formula).

$80 = 70 + Ae^{k\times 0} = 70 + A$.

Then, $A = 10$ and $T = 70 + 10e^{kt}$.

Plug $t = 1$ hour and $T = 75.0^0 F$ in the equation and solve for K. (This'll take some log tricks, Unit 2.)

$$75 = 70 + 10e^{k\times 0}\ so\ \frac{1}{2} = e^k\ \ and\ \ K = \ln\left(\frac{1}{2}\right) = -\ln 2$$

Thus, $T = 70 + 10e^{-t\ln 2}$

Therefore, $98.6 = 70 + 10e^{-t\ln 2}$, so $2.86 = e^{-t\ln 2}$ *and* $t = -\dfrac{\ln 2.86}{\ln 2} \approx -1.516$

Finally, 1.516 hours is about 1 hour and 31 minutes. The murder took place about $7:29\,a.m.$

Application activity 6.5.2

The body is located in a room that is kept at a constant 70 degree F. For some time after the death, the body will radiate heat into the cooler room, causing the body's temperature to decrease assuming that the victim's temperature was normal 98.6F at the time of death.

a) Solve the related differential equation.

b) Use your solution in to estimate the time the murder took place if $K = 28.10^{-4}$ and the officer arrived at 10.40 p.m. and the body temperature was 94.4 degrees.

6.5.3 Differential equations and the quantity of a drug in the body

Activity 6.5.3

To fight against the infection to a human body, appropriate dose of medicine is essential. Because the amount of the drug in the human body decreases with time, the medicine must be given in multiple doses. The rate $\dfrac{dQ}{dt}$ at which the level of the drug in a patient's blood decays is proportional to the quantity Q of the drug left in the body. If initially, that is, at $t = 0$ a patient is given an initial dose Q_0,

1) Establish an equation for modeling the situation

2) Solve the obtained equation and find the quantity of drug $Q(t)$ left in the body at the time t

3) Draw $Q(t)$ and interpret the graph given that the drug provided was 100mg at $t = 0$.

4) Discuss what happens when the patient does not respect the dose of medicine as prescribed by the Doctor.

Content summary

The rate at which a drug leaves a patient's body is proportional to the quantity of the drug left in the body. If we let Q represent the quantity of drug left, then

$$\frac{dQ}{dt} = -kQ$$

The negative sign indicates that the quantity of drug in the body is decreasing.

The solution to this differential equation is $Q = Q_0 e^{-kt}$ and the quantity decreases exponentially.

The constant k depends on the drug and Q_0 is the amount of drug in the body at time zero. Sometimes physicians convey information about the relative decay rate with a *half -life*, which is the time it takes for Q to decrease by a factor of 1/2.

Example

A patient having major surgery is given the antibiotic vancomycin (an antibiotic used to treat a number of bacterial infections) intravenously at a rate of $85\,mg\ per\ hour$. The rate at which the drug is excreted from the body is proportional to the quantity present, with proportionality constant 0.1 if time is in hours. Write a differential equation for the quantity, Q in mg, of vancomycin in the body after t hours.

Solution:

The quantity of vancomycin, Q, is increasing at a constant rate of

$85\,mg\ per\ hour$ and is decreasing at a rate of 0.1 times Q. The administration of $85\,mg\ per\ hour$ makes a positive contribution to the rate of change $\frac{dQ}{dt}$. The excretion at a rate of $0.1Q$ makes a negative contribution to $\frac{dQ}{dt}$. Putting these together, we have:

rate of change of a quantity = rate in − rate out,

So, $\frac{dQ}{dt} = 85 - 0.1Q$.

Application activity 6.5.3

Valproic acid is a drug used to control epilepsy; its half-life ($Q=\frac{1}{2}Q_0$) in the human body is about 15 hours.

(a) Use the half-life as initial condition to find the constant K in the differential equation;

(b) At what time will 10% of the original dose remain?

6.5.4 Differential equations in economics and finance

Activity 6.5.4

Assume that in a perfectly competitive market the speed with which price P adjusts towards its equilibrium value depends on how much excess demand there is. Given that the rate of change of the price $P(t)$ of a product at time t is proportional to the difference of the demand and the supply for the commodity $(Q_d - Q_s)$,

1) Write a differential equation modeling the rate of change of the price if the constant of proportionality $k=0.08$ is in proportion to excess demand.

b) Assuming that $Q_d = 280-4P(t)$ and $Q_s = -35+8P(t)$ solve the equation obtained in (1).

c) Determine and plot $P(t)$ at the time t if the price is currently 19.

d) Compare the price at $t=1$ and the price as t gets larger i.e. $\lim_{t\to\infty} P(t)$.

Content summary

In a perfectly competitive market the speed with which price *P* adjusts towards its equilibrium value depends on how much excess demand there is. This is quite a reasonable proposition. If consumers wish to purchase a lot more produce than suppliers are willing to sell at the current price, then there will be great pressure for price to rise, but if there is only a slight shortfall then price adjustment may be sluggish. If excess demand is negative this means that quantity supplied exceeds quantity demanded, in which case price would tend to fall.

To derive the differential equation that describes this process, assume that the demand and supply functions are Q_d and Q_s

If r represents the rate of adjustment of P in proportion to excess demand, then we can write

$$\frac{dP}{dt}=r(Q_d-Q_s)$$

The solution of this equation leads to the function of the price $P(t)$ that changes over time.

Example

A perfectly competitive market has the demand and supply functions $Q_d=170-8P$ and $Q_s=-10+4P$.

When the market is out of equilibrium the rate of adjustment of price is a function of excess demand such that $r=\frac{1}{2}$. Given that in the initial time period price P_0 is -10 which is not its equilibrium value, express P as function of t and comment on the stability of this market.

Solution:

$$\frac{dP}{dt}=r(Q_d-Q_s)$$

$$\Rightarrow\frac{dP}{dt}=\frac{1}{2}\left(170-8P-(-10+4P)\right)$$

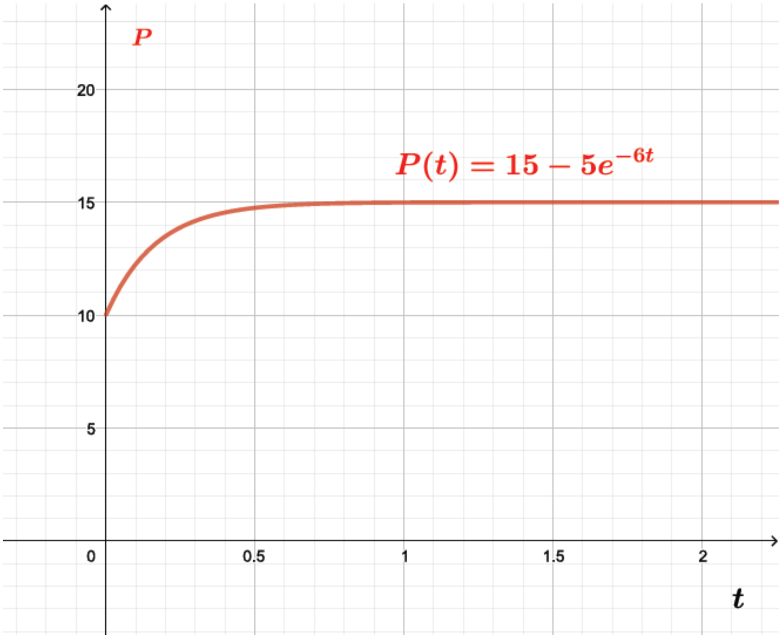
$\frac{dP}{dt}=-6P+90$ which is a linear first-order differential equation .

$$P(t)=Ae^{-6t}+15$$

Considering the initial condition, $P(0)=10$ we find $A=-5$.

Therefore, $P(t)=-5e^{-6t}+15$

Graph of $P(t) = -5e^{-6t} + 15$



The coefficient of t in this exponential function is the negative number -6. This means that the first term of the solution called the complementary function, will get closer to zero as t gets larger and so $P(t)$ will converge on its equilibrium value of 15. This market is therefore stable.

You can check this by using the above solution to calculate $P(t)$ for example taking $t = 3$, you can see that this is extremely close to the equilibrium price of 15 and so we can say that price returns to its equilibrium value within the first few time periods in this particular market.

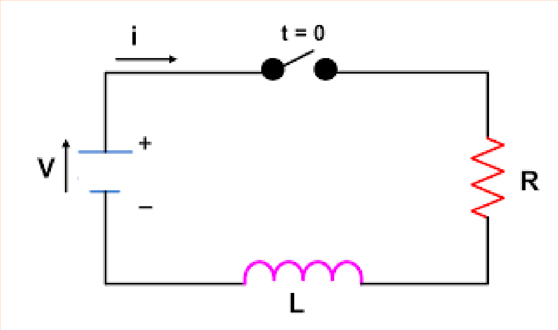
Application activity 6.5.4

If the demand and supply functions in a competitive market are $Q_d = 50 - 0.2P$ and $Q_s = -10 + 0.3P$. Given that the rate $r = 0.04$, derive and solve the relevant difference equation to get a function for $P(t)$ given that price is 100 in time period 0. Comment on the stability of this market.

6.5.5 Differential equations in electricity (Series Circuits)

Activity 6.5.5

Let a series circuit contain only a resistor and an inductor as shown in the Figure of RL series circuit



By Kirchhoff's second law the sum of the voltage drop $L\frac{di}{dt}$ across the inductor L and the voltage drop iR across the resistor R is the same as the impressed voltage $E(t)$ on the circuit.

a) write down the equation modeling the situation where the current $i(t)$ varies with time t,

b) Solve the obtained equation considering that the voltage is constant and equals 110 volts and the current was zero before switching on.

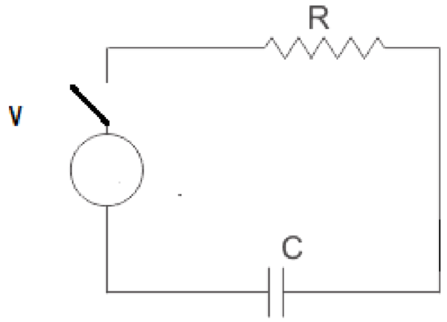
c) What can you say about the value of the current as t gets larger i.e. $\lim_{t\to\infty} i(t)$

Content summary

The voltage in the circuit is modeled by Kirchhoff's second law saying that the voltage in the circuit is the sum of the voltage drop across the components of the circuit. It is known from physics that the voltage drops across the resistor, inductor, and capacitor are respectively RI , $L\frac{dI}{dt}$ and $\frac{Q}{C}$.

Example

The voltage drop across a capacitor with capacitance C is given by $\frac{Q(t)}{C}$, where Q is the charge on the capacitor. Hence, for the series circuit composed of a resistor and a capacitor as shown in the Figure of the RC series circuit



a) Determine the differential equation modeling the voltage of the circuit taking the charge $Q(t)$ as the dependent variable;

b) Solve the equation obtained if 100-volt electromotive force is applied to the circuit

c) If the resistance is 200 ohms and the capacitance is 10^{-4} farads. Find the charge $Q(t)$ on the capacitor if $Q(0)=0$. Deduce the current $i(t)$.

Solution

a) Applying Kirchhoff's second law in the circuit composed of the capacitor and the resistor, we get

$$Ri+\frac{Q}{C}=E(t)$$

Since $i=\frac{dQ}{dt}$, our differential equation can be written as

$$R\frac{dQ}{dt}+\frac{Q}{C}=E(t)$$

b) $$R\frac{dQ}{dt}+\frac{Q}{C}=100$$

$$R\frac{dQ}{dt}+\frac{1}{C}Q=100$$

$$\Rightarrow\frac{dQ}{dt}+\frac{1}{RC}Q=\frac{100}{R}$$

This is a first order linear differential equation of the form $\frac{dQ}{dt}+pQ=q$ where P and Q are constant $p=\frac{1}{RC}$, $q=\frac{100}{R}$

The integrating factor is $I(t)=e^{\int \frac{1}{RC}dt}=e^{\frac{1}{RC}t}$

And $Q(t)=\frac{\int I(t)q(t)dt+K}{I(t)}=\frac{\int e^{\frac{1}{RC}t}\frac{100}{R}dt+K}{e^{\frac{1}{RC}t}}=\frac{RC\frac{100}{R}e^{\frac{1}{RC}t}+K}{e^{\frac{1}{RC}t}}$

$Q(t)=100C+Ke^{-\frac{1}{RC}t}$ where K is a constant of integration.

The charge is $Q(t)=100C+Ke^{-\frac{1}{RC}t}$ where K is a constant.

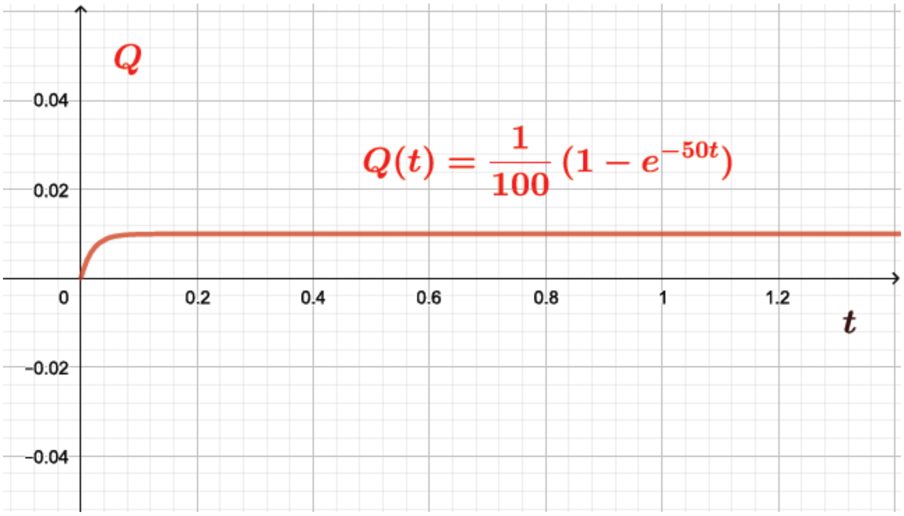
c) Given that $R=200$, $C=10^{-4}$ and $Q(0)=0$ we have

$Q(0)=0\Rightarrow 100C+K=0$

$\Rightarrow K=-100C$

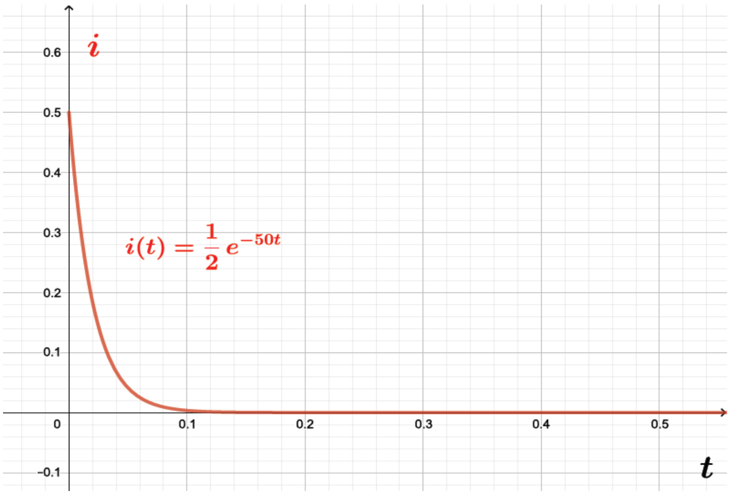
Therefore, $Q(t)=\frac{1}{100}\left(1-e^{-50t}\right)$

Figure: Graph of $Q(t)=\frac{1}{100}\left(1-e^{-50t}\right)$



The charge is increasing towards $Q=\frac{1}{100}$ as the time increases indefinitely

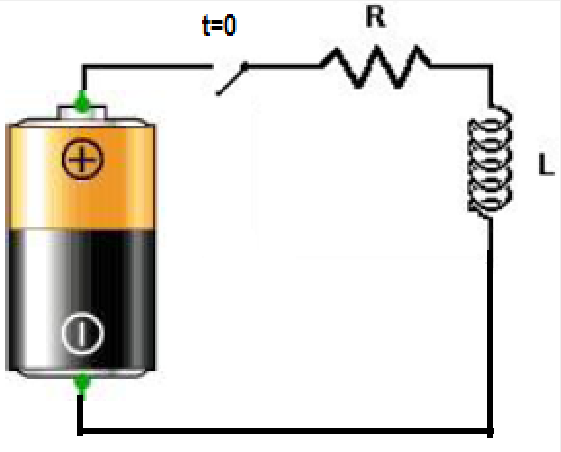
Given that $i=\frac{dQ}{dt}$, The current in the circuit is $i=\frac{dQ}{dt}=\frac{1}{2}e^{-50t}$



The current is decreasing.

Application activity 6.5.5

Let a series circuit contain only a resistor and an inductor as shown on the following figure:



a) Establish a differential equation modeling the current i in the closed circuit

b) If $R = 12\ ohms$, $L = 4H$ are connected to a battery that gives a constant voltage of 60 V and the switch is closed when $t = 0$ (i.e the current starts with $i(0) = 0$);

Find the current after 1 s.

c) Find what happens to the current after a long time.

6.6 End unit assessment

1. Discuss and state the order and the degree of each of the following differential equations. Explain your answer.

a) $\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^4-4x+y=1$

b) $\left(\frac{dy}{dx}\right)^3-2x=\cos y-2\sin x$

c) $(y'')^3+(y')-2y=x$

d) $y\frac{d^2y}{dx^2}=-\cos x$

e) $x^2\left(\frac{d^2y}{dx^2}\right)^4+y\left(\frac{dy}{dx}\right)+y^4=0$

2. Solve the following differential equations

a) $\frac{dy}{dx}=x+1$

b) $\frac{dy}{dx}=y^2$

c) $y'=x^2y$

In a) –c) give the particular solutions satisfying the condition $y(0) = 1$.

3. The rate of cooling of a body is given by $\frac{d\theta}{dt}=k\theta,$ where k is a constant.

If $\theta=60^0C$ when $t=2$ minutes and $\theta=50^0C$ when $t=5$ minutes, determine the time taken for θ to fall to 40^0C, correct to the nearest second.

4. If the demand and supply functions in a competitive market are $Q_d=35-5P$ and $Q_s=-23+6P$ and the rate of adjustment of price when the market is out of equilibrium is $\frac{dp}{dt}=0.2(Q_d-Q_s)$,

a) Solve the relevant differential equation to get a function for P in terms of t given that the price is 100 in time period 0.

b) Plot the graph of P and comment on the stability of this market.

UNIT 7

DIFFERENTIAL EQUATION OF SECOND ORDER

Key Unit competence:

Use differential equations to solve related problems that arise in a variety of practical contexts.

7.0 Introductory activity

Carry out the following activities:

Give two examples of second order differential equations with;

i) degree greater than 1

ii) degree 1

In each case, identify their similarities and differences.

7.1 Second order differential equations

Activity 7.1

Given the differential equation $2\frac{d^2y}{dt^2}+0.1y=0$

a) Verify that $y=A\cos\sqrt{\frac{1}{20}}t+B\sin\sqrt{\frac{1}{20}}t$ is its general solution.

b) Compare the two differential equations: $\frac{d^2y}{dt^2}+0.1y=0$ and

$$\frac{d^2y}{dx^2}+p(x)\frac{dy}{dx}+q(x)y=r(x)$$

Content summary

The general second order linear differential equation is of the form:

$$\frac{d^2y}{dx^2}+p(x)\frac{dy}{dx}+q(x)y=r(x)$$

or more simply, $y''+p(x)y'+q(x)y=r(x)$ where $p(x), q(x) \, and \, r(x)$ are functions of x

alone (or constants).

If $r(x)$ is identically zero, the differential equation is said to be **homogeneous** equation $y''+p(x)y'+q(x)y=0$; otherwise it is said to be **non-homogeneous or inhomogeneous.**

A second order differential equation which cannot be written in the form $\frac{d^2y}{dx^2}+\frac{dy}{dx}+q(x)y=r(x)$ is said to be **non-linear.**

In this part, we limit our study to the particular type of linear equation of second order of the form $a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=0$

where $a, b \, and \, c$ are constants (and $a \neq 0$, otherwise it wouldn't be second order).

This equation is called second order linear homogeneous differential equation with constant coefficients and occurs everywhere in science and engineering, most notably in the modeling of vibrating springs in a resisting medium, and in electrical circuits.

Example

Discuss the characteristics of the following equations and identify linear homogeneous differential equations among them.

a) $\frac{d^2y}{dx^2}-9y=x$ b) $x\frac{d^2y}{dx^2}-4\frac{dy}{dx}+x^2y=0$

c) $\frac{dy}{dx}\frac{d^3y}{dx^3}+2y^2=0$ d) $(y'')^3+(y')^5-2y=x^4$

Solution

We are going to compare each equation with $\frac{d^2y}{dx^2}+p(x)\frac{dy}{dx}+q(x)y=r(x)$ where

$p(x)$ is the function coefficient of $\frac{dy}{dx}$, $q(x)$ the function coefficient of y and $r(x)$ the right hand side function.

a) $\frac{d^2y}{dx^2}-9y=x$, the function coefficient of $\frac{dy}{dx}$ is zero but the right hand side is different from zero; thus the equation is a linear but not homogeneous.

b) The equation $x\frac{d^2y}{dx^2}-4\frac{dy}{dx}+x^2y=0$ can be written as $\frac{d^2y}{dx^2}-\frac{4}{x}\frac{dy}{dx}+xy=0$,

the coefficient of $\frac{dy}{dx}$ is function of x only and the second hand side function is zero; it is a linear homogeneous differential equation.

c) $\frac{dy}{dx}\frac{d^3y}{dx^3}+2y^2=0$,cannot be written in the form of

$\frac{d^2y}{dx^2}+p(x)\frac{dy}{dx}+q(x)y=r(x)$ This

is not a second order linear differential equation.

d) $(y'')^3+(y')^5-2y=x^4$, It is a second order differential equation of third degree.

It is not **a second order** linear differential equation.

Application activity 7.1

Given the following equations, indicate which are linear homogeneous differential equations

a) $\frac{d^2y}{dx^2}-9y=0$ b) $x\frac{d^2y}{dx^2}+\cos x=0$ c) $x\frac{d^2y}{dx^2}+\frac{dy}{dx}+xy=0$

7.2 Homogeneous linear equations with constant coefficients

7.2.1. Linear independence and superposition principle

Activity 7.2.1

Given the following differential equation $\frac{d^2y}{dx^2}+3\frac{dy}{dx}+2y=0$.

1) Verify that e^{-x} and e^{-2x} are two solutions to the given equation.

2) Multiply e^{-x} by any constant and e^{-2x} by another constant and take the function y as the sum of these results.

Substitute the function obtained in (2) in the given homogeneous differential equation. Write down your observations and discuss the results.

Content summary

In order to discuss the solutions to a second-order linear homogeneous differential equation, it is very useful to introduce some terminology.

Two functions are said to be **linearly dependent** if one is a **constant** multiple of the other. Otherwise, they are called **linearly independent.**

Thus, $f(x)=e^{2x}$ and $g(x)=3e^{2x}$ are linearly dependent, but $f(x)=e^{2x}$ and $h(x)=xe^{2x}$ are linearly independent.

There is a useful result (known as superposition or linearity principle) which states that "if

y_1 and y_2 are two solutions of the homogeneous linear differential equations $a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=0$, then any other linear combination $(y=Ay_1+By_2)$ of these two solutions is also a solution of the equation".

Two solutions y_1 and y_2 are called basis of solution of $a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=0$, if y_1 and y_2 are linearly independent. In this case the corresponding general solution is $y=c_1y_1+c_2y_2$ where c_1 and c_2 are arbitrary constant.

A mathematical theorem enables us to check whether or not a set of given functions is linearly independent. Two differentiable functions $y_1(x)$ and $y_2(x)$ are linearly independent if and only if the determinant called the Wronskian of y_1 and y_2 denoted and defined by

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1y_2' - y_1'y_2$$ is not zero.

Example

Verify that $y = \cos 2x$ and $y = \sin 2x$ are solutions of $y'' + 4y = 0$ and that their sum is also a solution. Deduce the general solution of $y'' + 4y = 0$.

Solution

First of all, the two functions are linearly independent since

$$\begin{vmatrix} \cos 2x & \sin 2x \\ -2\sin 2x & 2\cos 2x \end{vmatrix} = 2\cos^2 2x + 2\sin^2 2x = 2 \neq 0$$

From $y = \cos 2x$, we have $y' = -2\sin 2x, y'' = -4\cos 2x$.

Hence, $y'' + 4y = (\cos 2x)'' + 4\cos 2x = -4\cos 2x + 4\cos 2x = 0$.

Thus, $y = \cos 2x$ is a solution of $y'' + 4y = 0$.

Similarly, for $y = \sin 2x$, we have

$$y'' + 4y = (\sin 2x)'' + 4\sin 2x = -4\sin 2x + 4\sin 2x = 0$$

Hence, $y = \sin 2x$ is a solution of $y'' + 4y = 0$.

Their sum is $y = \cos 2x + \sin 2x$, then

$y' = -2\sin 2x + 2\cos 2x$ and $y'' = -4\cos 2x - 4\sin 2x$.

Hence, $y'' + 4y = (\cos 2x + \sin 2x)'' + 4(\cos 2x + \sin 2x)$

$$= -4\cos 2x - 4\sin 2x + 4\cos 2x + 4\sin 2x = 0.$$

Finally, the corresponding general solution is $y = c_1 \cos 2x + c_2 \sin 2x$, where c_1 and c_2 are arbitrary constant.

Application activity 7.2.1

1. Verify that $y_1 = 1+\cos x$ and $y_2 = 1+\sin x$ are solutions of $y''+y=1$ but their sum is not a solution. Explain why.

2. Are the following functions linearly independent?

a) $\cos^2 x$ and $\sin^2 x$ b) e^{-x} and e^{2x} c) e^{ax} and $5e^{ax}$

d) $5\sin x\cos x$ and $4\sin 2x$ e) $e^{ax}\cos 2x$ and $e^{ax}\sin 2x$

f) $\ln x$ and $\ln\sqrt{x}$ g) e^{ax} and xe^{ax} h) $2\sin^2 x$ and $1-\cos^2 x$

7.2.2 Characteristic equation of a second order differential equations

Activity 7.2.2

1) Find the solution of the equation $y'-ky=0, k \textit{ is a constant}$

2) Put the solution obtained in 1) in the equation $y''-3y'-4y=0$ and give the condition so that the solution obtained in 1) is a solution of $y''-3y'-4y=0$. What can you say about the solution of $y''-3y'-4y=0$?

Content summary

From differential equation $a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=0$, if we put a = 0 then we are back to the simple first order linear equation and we know that this has an exponential solution.

This encourages us to try a similar exponential function for the second order equation.

We therefore take a trial solution $y=e^{\lambda x}$ where λ is certain constant parameter to be determined.

Substituting $y'=\lambda e^{\lambda x}$, $y''=\lambda^2 e^{\lambda x}$ into the equation $a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=0$,

we get $\left(a\lambda^2+b\lambda+c\right)e^{\lambda x}=0$,

$\Leftrightarrow a\lambda^2+b\lambda+c=0$, since $e^{\lambda x}\neq 0$

So λ satisfies a quadratic equation with the same coefficients as the DE itself.

This equation in λ is called the **auxiliary** or **characteristic** equation (AE) of

$$a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=0\ .$$

Example

For a given differential equation, write down the characteristic equation, verify by substitution whether the given functions are solutions and deduce the general solution.

a) $y''-9y=0,\ y=e^{3x}\ and\ \ y=e^{-3x}$

b) $y''+2y'+2y=0,\ y=e^{-x}\cos x\ and\ \ y=e^{-x}\sin x$

Solution

a) We get characteristic equation by considering the order of derivative as the power of λ that is $\lambda^2=y'', \lambda=y'$ and $\lambda^0=y$.

Thus, characteristic equation of $y''-9y=0$ is $\lambda^2-9\lambda^0=0$ or $\lambda^2-9=0$.

For $y=e^{3x}$, $y''-9y=\left(e^{3x}\right)''-9\left(e^{3x}\right)=9\left(e^{3x}\right)-9\left(e^{3x}\right)=0$

Therefore, $y=e^{3x}$ is a solution of $y''-9y=0$.

For $y=e^{-3x}$, we have

$$y''-9y=\left(e^{-3x}\right)''-9\left(\ e^{-3x}\ \right)=-3\left(e^{-3x}\right)'-9\left(\ e^{-3x}\ \right)=9\left(e^{-3x}\right)-9\left(\ e^{-3x}\ \right)=0.$$

Thus, $y=e^{-3x}$ is a solution of $y''-9y=0$.

As, $y=e^{3x}$ and $y=e^{-3x}$ are linearly independent from superposition principle,

the general solution is $y=c_1e^{3x}+c_2e^{-3x}$ where c_1 and c_2 are arbitrary constant.

b) Characteristic equation of $y''+2y'+2y=0$ is $\lambda^2+2\lambda+2\lambda^0=0$ or

$$\lambda^2+2\lambda+2=0.$$

For $y=e^{-x}\cos x,$ we have

$$y''+2y'+2y=\left(e^{-x}\cos x\right)''+2\left(e^{-x}\cos x\right)'+2e^{-x}\cos x$$
$$=\left(-e^{-x}\cos x-e^{-x}\sin x\right)'+2\left(-e^{-x}\cos x-e^{-x}\sin x\right)+2e^{-x}\cos x$$
$$=\left(-e^{-x}\cos x-e^{-x}\sin x\right)'-2e^{-x}\sin x$$
$$=e^{-x}\cos x+e^{-x}\sin x+e^{-x}\sin x-e^{-x}\cos x-2e^{-x}\sin x$$
$$=0$$

Therefore, $y=e^{-x}\cos x$ is a solution of $y''+2y'+2y=0$, similarly, if

$y=e^{-x}\sin x,$ we get

$$y''+2y'+2y=\left(e^{-x}\sin x\right)''+2\left(e^{-x}\sin x\right)'+2\left(e^{-x}\sin x\right)$$
$$=\left(-e^{-x}\sin x+e^{-x}\cos x\right)'+2\left(-e^{-x}\sin x+e^{-x}\cos x\right)+2e^{-x}\sin x$$
$$=\left(e^{-x}\cos x-e^{-x}\sin x\right)'+2e^{-x}\cos x$$
$$=-e^{-x}\cos x-e^{-x}\sin x+e^{-x}\sin x-e^{-x}\cos x+2e^{-x}\cos x$$
$$=-2e^{-x}\cos x+2e^{-x}\cos x-e^{-x}\sin x+e^{-x}\sin x$$
$$=0$$

Hence $y=e^{-x}\sin x$ is a solution of $y''+2y'+2y=0$

$y=e^{-x}\cos x$ and $y=e^{-x}\sin x$ are linearly independent

Since $\dfrac{e^{-x}\cos x}{e^{-x}\sin x}=\tan x$, thus the general solution of $y''+2y'+2y=0$ is

$y=c_1e^{-x}\cos x+c_2e^{-x}\sin x$ or $y=e^{-x}(c_1\cos x+c_2\sin x)$.

The basic tools that you need in solving **linear homogeneous differential equations** with constant coefficients, are simply **solution of quadratic** equations (including complex roots) and **superposition principle.** Since the characteristic equation is quadratic, it may have either two distinct real solutions, or repeated real solutions or two complex solutions (Unit 1).

Therefore solutions of ODE: $ay''+by'+cy=0$ depends on solutions of the characteristic equation: $a\lambda^2+b\lambda+c=0$. This is the purpose of the following paragraphs.

Application activity 7.2.2

Verify by substitution whether the given functions are linearly independent; if so, determine the general solution of a corresponding differential equation. Write down and solve the characteristic equation for each ODE.

a) $y'' + 4y = 0,\ y = \cos 2x \ and \ \ y = 2\sin x \cos x$

b) $y'' - 2y' + y = 0,\ y = e^x \ and \ \ y = 3e^x$

c) $4y'' + 4y' + y = 0,\ y = e^{-\frac{x}{2}} \ and \ \ y = xe^{-\frac{x}{2}}$

7.3 Solving an homogeneous Differential equation of second order

7.3.1 Solving DE whose Characteristic equation has two distinct real roots

Activity 7.3.1

Determine the roots λ_1 and λ_2 of characteristic equation of the following differential equation $y'' + 7y' + 6y = 0$.

From λ_1 and λ_2 , review previous example and use superposition principle to find the general solution of $y'' + 7y' + 6y = 0$.

Content summary

From auxiliary equation of differential equation $ay''+by'+cy=0$ having two distinct real roots λ_1 and λ_2, we get $y_1 = e^{\lambda_1 x}$ and $y_2 = e^{\lambda_2 x}$ as the basis of solution of differential equation $ay''+by'+cy=0$.

By superposition principle, the corresponding general solution is $y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$

Example

Solve the following differential equation: $y''+y'-2y=0$

Solution

$y''+y'-2y=0$, the characteristic equation is $r^2+r-2=0$

Resolution of characteristic equation: simple factorization yields to $(r-1)(r+2)=0$. Then, $r_1 = 1$ and $r_2 = -2$ are solutions of the characteristic equation.

Therefore, the general solution is $y = c_1 e^{x} + c_2 e^{-2x}$.

Example

Determine the particular solution of $y'' + 3y' + 2y = 0$, for which $y(0) = y'(0) = 1$ and plot the curve of this solution.

Solution

$y'' + 3y' + 2y = 0$, the characteristic equation is $m^2 + 3m + 2 = 0$. Solving the equation gives $m_1 = -1$ and $m_2 = -2$ as solutions.

The general solution is $y = c_1 e^{-x} + c_2 e^{-2x}$.

Let's apply initial conditions: as $y = c_1 e^{-x} + c_2 e^{-2x}$, $y' = -c_1 e^{-x} - 2c_2 e^{-2x}$.

From $y(0) = y'(0) = 1$, we have $c_1 + c_2 = 1$ and $-c_1 - 2c_2 = 1$.

Solving these simultaneous equations gives $c_1 = 3$ and $c_2 = -2$.

Hence, the particular solution is $y = 3e^{-x} - 2e^{-2x}$.

Graphical representation

x	-1	-0.8	-0.6	-0.4	-0.2	0	0.3	0.6	0.8	1	2	3
y	-6.6	-3.2	-1.2	0.0	0.7	1.0	1.1	1.0	0.9	0.8	0.4	0.1

Graphical presentation of $y = 3e^{-x} - 2e^{-2x}$

Application activity 7.3.1

1. Solve the following differential equations

a) $y'' - 3y' = 0$

b) $y'' - 8y = 0$

c) $y'' + 7y' + 6y = 0$

d) $y'' + y' - 2y = 0$

e) $y'' - 5y' + 6y = 0$

f) $2y'' + 3y' - 2y = 0$

2. Solve the following initial value problem and represent the solution graphically.

a) $y'' + 5y' + 6y = 0, y(0) = 1; y'(0) = 2$

b) $3y'' + 5y' - 2y = 0, \quad y(0) = 2, y'(0) = 3$

7.3.2 Solving DE whose Characteristic equation has a real double root/repeated root

Activity 7.3.2

Suppose that the auxiliary equation $y'' + py' + qy = 0$ has distinct real roots m and n,

a) Show that the function $f(x) = \dfrac{e^{mx} - e^{nx}}{m - n}$ is a solution of the equation $y'' + py' + qy = 0$.

b) Using Hospital rule, show that $\lim\limits_{m \to n} f(x) = xe^{nx}$ and check if $\lim\limits_{m \to n} f(x)$ is a solution of $y'' + py' + qy = 0$.

Check whether $y = \lim\limits_{m \to n} f(x)$ and $y = e^{nx}$ are linearly independent or not and then deduce the general solution of $y'' - 2y' + y = 0$.

Content summary

When the roots of auxiliary equation of the differential equation $ay''+by'+cy=0$ are equal, that is, $\lambda_1=\lambda_2=-\frac{b}{2a}$, we obtain only one solution $y_1=e^{\lambda x}$.

The second linearly independent solution will be $y_2=xe^{\lambda x}$; and then, the basis of $ay''+by'+cy=0$ is made by $y_1=e^{\lambda x}$ and $y_2=xe^{\lambda x}$;

Therefore, the corresponding general solution is $y=c_1e^{\lambda x}+c_2xe^{\lambda x}$.

Example

Solve $y''+4y'+4y=0$ to find its general solution

Solution

The characteristic equation of $y''+4y'+4y=0$ is $m^2+4m+4=0$,

and $m_1=m_2=-2$.

The basis of solution $y''+4y'+4y=0$ is made by $y_1=e^{-2x}$ and $y_2=xe^{-2x}$;

Hence, the general solution is $y=c_1e^{-2x}+c_2xe^{-2x}$.

Example

Solve the following initial value problem and give its graphical interpretation.

$y''-6y'+9y=0$, $y(0)=-2$ and $y'(0)=-12$.

Solution

The characteristic equation of $y''-6y'+9y=0$ is $m^2-6m+9=0$, and $m_1=m_2=3$.

The basis of solution of $y''-6y'+9y=0$ is composed of $y=e^{3x}$ and $y=xe^{3x}$;

The corresponding general solution of $y''-6y'+9y=0$ is $y=c_1e^{3x}+c_2xe^{3x}$.

Considering initial conditions $y(0)=-2$ and $y'(0)=-12$,

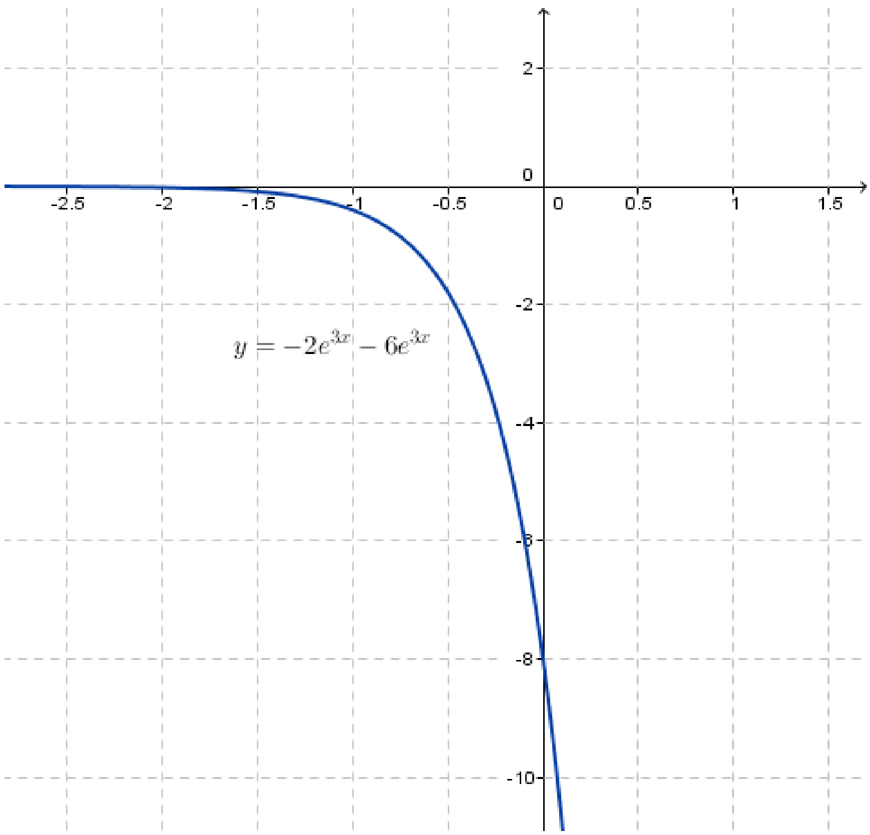
$y(0)=-2$ gives $c_1=-2$; from $y'(0)=-12$ and $y'=3c_1e^{3x}+c_2\left(e^{3x}+3xe^{3x}\right)$ we get $3c_1+c_2=-12 \Leftrightarrow c_2=-12+6=-6$.

We got $c_2=-6,\quad c_1=-2$ and then, particular solution is $y=-2e^{3x}-6xe^{3x}$.

Table of values for $y=-2e^{3x}-6xe^{3x}$.

X	Y
-1.5	-0.09
-1	-0.4
-0.5	-1.79
0	-8
0.5	-35.85
1.33	-432.44
1.5	-720.14

The graph for $y=-5e^{3x}+3xe^{3x}$



From the graph for $y=-2e^{3x}-6xe^{3x}$., it is clear that the solution for the initial valueproblem $y''-6y'+9y=0$, $y(0)=-2$ and $y'(0)=-12$

Application activity 7.3.2

1. Solve the following differential equations

1) $y'' + 8y' + 16 = 0$

2) $4y'' - 12y' + 9y = 0$

3) $4y'' + 4y' + y = 0$

4) $\frac{d^2y}{dx^2} - \frac{1}{3}\frac{dy}{dx} + \frac{1}{36}y = 0$

5) $4\frac{d^2y}{dx^2} - 4\pi\frac{dy}{dx} + \pi^2 y = 0$

2. Solve the following initial value problem and represent the solution graphically.

a) $y'' + 2y' + y = 0, \quad y(0) = 3, \ y'(0) = -6.5$

$10y'' + 5y' + 0.625y = 0, \ y(0) = 2 \ \ and \ \ y'(0) = -4.5$

7.3.3 Solving DE whose characteristic equation has complex roots

Activity 7.3.3

Assume that the roots m_1 and m_2 of the characteristic equation of the differential equation $y'' - 2y' + 5y = 0$ are conjugate complex numbers. In this case m_1 and m_2 can be written in the form $a \pm ib$

1) Write down the two independent solutions y_1 and y_2

2) Use polar form of a complex number to write the two solutions.

Since the obtained solutions are not real valued functions, find the two real valued functions that form a real basis by suitably combining the two solutions obtained in 2) by the use of **Euler formula**. Hence determine the general solution of the differential equation $y'' - y' + 5y = 0$.

Content summary

For auxiliary equation having complex roots, the basis of the given differential equation are $y_1 = e^{(\alpha+i\beta)x}$ and $y_2 = e^{(\alpha-i\beta)x}$ giving general solution $y = e^{\alpha x}\left(c_1 e^{i\beta x} + c_2 e^{-i\beta x}\right)$

From **Euler's formula**, the basis of real solution are $y_1 = e^{\alpha x}\cos\beta x$

and $y_2 = e^{\alpha x}\sin\beta x$, and then, the corresponding general solution is

$y = e^{\alpha x}\left(A\cos\beta x + B\sin\beta x\right)$.

Example

Find the general solution of $y'' - 2y' + 2y = 0$

Solution

Characteristic equation of $y'' - 2y' + 2y = 0$ is $r^2 - 2r + 2 = 0$;

$\Delta = 4 - 8 = -4$ and $\sqrt{\Delta} = 2i$

$r_1 = \dfrac{2+2i}{2} = 1+i$ and $r_2 = \dfrac{2-2i}{2} = 1-i$. Here $\alpha = 1$ and $\beta = 1$, then the general solution is $y = e^x\left(c_1\cos x + c_2\sin x\right)$.

Example

Solve the initial value problem

$y'' + 4y' + 13y = 0$, $y(0) = 0, y'\left(\dfrac{\pi}{2}\right) = 1$ and present solution graphically for $x \geq 0$

Solution

Characteristic equation of $y'' + 4y' + 13y = 0$ is $r^2 + 4r + 13 = 0$; $\Delta = 16 - 52 = -36$

Then, $r_1 = \dfrac{-4+6i}{2} = -2+3i$, $r_2 = \dfrac{-4-6i}{2} = -2-3i$. Here $\alpha = -2$ and $\beta = 3$,

Therefore, the corresponding general solution is $y = e^{-2x}\left(c_1\cos 3x + c_2\sin 3x\right)$.

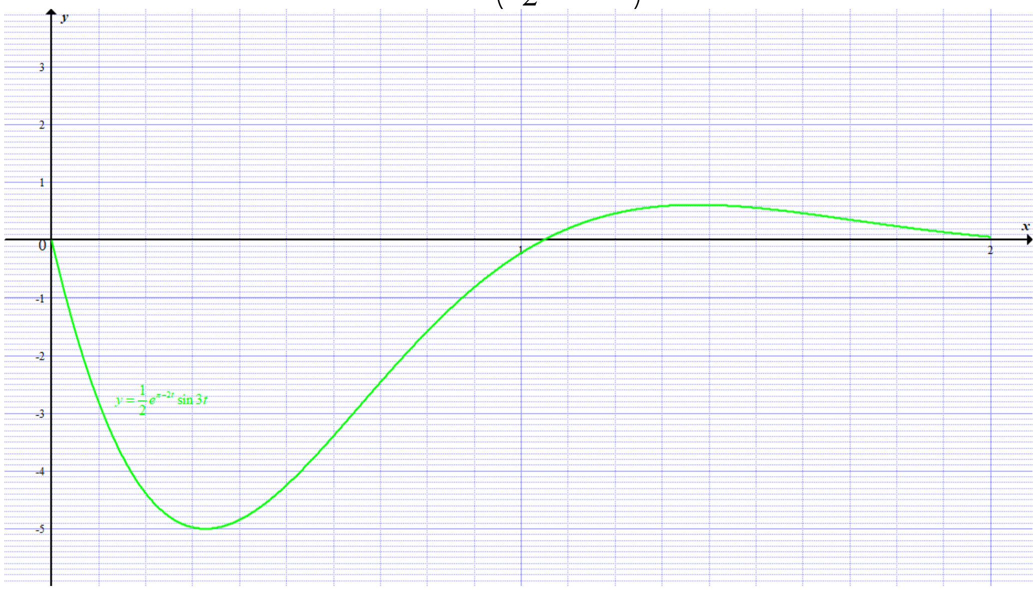
As $y(0) = 0$ it follows $c_1 = 0$.

$y' = -2e^{-2x}\left(c_1\cos 3x + c_2\sin 3x\right) + e^{-2x}\left(-3c_1\sin 3x + 3c_2\cos 3x\right)$

Thus, $y'\left(\frac{\pi}{2}\right)=1 \Leftrightarrow 1=2c_2e^{-\pi}+3c_1e^{-\pi} \Leftrightarrow 1=2c_2e^{-\pi} \Leftrightarrow c_2=\frac{1}{2}e^{\pi}$.

Hence, particular solution is $y=e^{-2x}\left(\frac{e^{\pi}}{2}\sin 3x\right)$ or $y=\frac{1}{2}e^{\pi-2x}\sin 3x$.

Graphical presentation of $y=e^{-2x}\left(\frac{e^{\pi}}{2}\sin 3x\right)$



Physically, the solution represents an oscillation with inconsistent amplitude.

Application activity 7.3.3

1. Determine the solution of the following equations

1) $y''+25y=0$

2) $y''-4y'+5y=0$

3) $y''+4y'+13y=0$

4) $y''+6y'+11y=0$

5) $y''-2y'+10y=0$

6) $10y''+2y'+1.7y=0$

2. Solve the initial value problem and give particular solution graphically.

a) $20y''+4y'+y=0,\ y(0)=2\ \ and\ \ y'(0)=-4.5$

b) $y''+2y'+2y=0,\ y(0)=2\ and\ \ y'(0)=-3$

3. In solving second order linear homogeneous differential equation $a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=0$, you get different forms of solution depending on the number of roots of auxiliary equation. Make a summary by completing the following table $e^{\alpha x}$:

Case	Roots	Basis	General solution
1	Distinct real: λ_1 and λ_2		
2	Real double : λ		
3	Complex conjugate $\alpha \pm i\omega$		

7.4 Non-homogeneous linear equations with constant coefficients

7.4.1 The right hand side is a polynomial function

Activity 7.4.1

Indicate the type of the following differential equations and solve if possible

1) $\frac{d^2y}{dx^2}-4\frac{dy}{dx}=5y$

2) $\frac{d^2y}{dx^2}-4\frac{dy}{dx}-5y=x^2$

CONTENT SUMMARY

The general solution of the second order non-homogeneous linear equation $y''+py'+qy=r(x)$ can be expressed in the form $y=\overline{y}+y^*$

where $\overline{y}$ is any specific function that satisfies the non-homogeneous equation, and $\overline{y} = c_1y_1 + c_1y_1$ is a general solution of the corresponding homogeneous equation $y'' + py' + qy = 0$.

The term $\overline{y} = c_1y_1 + c_1y_1$ is called the **complementary solution** (or the **homogeneous solution**) of the non-homogeneous equation.

The term y^* is called the **particular solution** (or the **non-homogeneous solution**) of the same equation.

Application activity 7.4.1

Calculate:

1. $y'' - 2y' + y = x + 1$;
2. $y'' - 3y' = 3x^2 + 2x + 1$;
3. $y'' = 2x + 4$

7.4.2 The right hand side is a product of the form: $r(x) = Pe^{\alpha x}$

Activity 7.4.2

Consider the equation $y'' - 2y' + y = e^x$

1) Find the general solution, say $\overline{y}$, of $y'' - 2y' + y = 0$

2) Put the right hand side of the given equation in the form $r(x) = Pe^{\alpha x}$. Suppose that the given equation has particular solution $y^* = x^kQ(x)e^{\alpha x}$

Where

α is the coefficient of x in $e^{\alpha x}$ in the right hand side of the given equation, k is the number of roots of the characteristic equation obtained in 1) equals to α and $Q(x)$ is the polynomial with the same degree as the degree of the polynomial found in right hand side of the given equation.

Write down y^*

Put the value of in the given equation to find new expression for .

Content summary

If the right hand side of the equation $y''+py'+qy=r(x)$ is $r(x)=Pe^{\alpha x}$ where P is a polynomial, we take the particular solution to be
$y^*=x^kQ_n(x)e^{\alpha x}, Q_n=a_0x^n+a_1x^{n-1}+...+a_n$

Here: k - is the number of roots of the associated homogeneous equation equals to α.

n - Degree of $Q(x)$, the same as degree of $P(x)$ in right hand side.

α - Coefficient of x in $e^{\alpha x}$ in the right hand side

3 cases arise

- If α is not a root of characteristic equation $k=0$
- If α is a simple root of characteristic equation $k=1$
- If α is a double root of characteristic equation $k=2$

Note that the simple root or double root in the last 2 cases must be real numbers.

Example

Find the general solution of $y''+y=e^x$

Solution

Characteristic equation:

$m^2+0m+1=0\ \ \Delta=0-4=-4$

$$m_1=\frac{0-2i}{2}=-i,\ m_2=\frac{0+2i}{2}=i$$

$\bar{y}=c_1\cos x+c_2\sin x$, which is the general solution of the homogeneous equation.

$\alpha=1$ is not a solution of the characteristic equation so $k=0$

Take $y^*=Ae^x;Q(x)=A\ as\ P(x)=1$
$y^*{}''=Ae^x$

Put this in the given equation:

$Ae^x+Ae^x=e^x$

$$2Ae^x = e^x$$

$$2A = 1$$

$$A = \frac{1}{2} \Rightarrow y^* = \frac{1}{2}e^x$$

The general solution of the given equation is

$$y = \overline{y} + y^*$$

$$y = c_1 \cos x + c_2 \sin x + \frac{1}{2}e^x.$$

Example

Find the general solution of $y'' - 7y' + 6y = e^x(x-2)$

Solution

Characteristic equation: $m^2 - 7m + 6 = 0$

$$\Delta = 49 - 24 = 25 > 0$$

$$\sqrt{\Delta} = \pm 5$$

$$m_1 = \frac{7+5}{2} = 6,\ m_2 = \frac{7-5}{2} = 1$$

We see that $\alpha = 1$ which is one of the roots of characteristic equation, so $k = 1$

Then $y^* = x(Ax+B)e^x$,

$$Q(x)\,Ax + B\ as\ P(x) = x - 2$$

But $\overline{y} = c_1e^x + c_2e^{6x}$

$$y^* = Ax^2e^x + Bxe^x$$

$$y^{*\prime} = 2Axe^x + Ax^2e^x + Be^x + Bxe^x$$

$$y^{*\prime\prime} = 2Ae^x + 2Axe^x + 2Axe^x + Ax^2e^x + Be^x + Be^x + Bxe^x$$

Put this in the given equation:

$$2Ae^x + 2Axe^x + 2Axe^x + Ax^2e^x + Be^x + Be^x + Bxe^x - 14Axe^x - 7Ax^2e^x$$
$$-7Be^x - 7Bxe^x + 6Ax^2 + 6Bxe^x = xe^x - 2e^x$$

$$\begin{cases} x^2e^x(7A-7A)=x^2e^x \\ xe^x(4A-14A+7B-7B)=xe^x(1) \\ e^x(2A-5B)=e^x(-2) \end{cases}$$

$$\begin{cases} -10A=1 \\ 2A-5B=-2 \end{cases}$$

$$\begin{cases} A=-\dfrac{1}{10} \\ B=\dfrac{5}{25} \end{cases}$$

$$y^*=-\frac{1}{10}x^2e^x+\frac{9}{25}xe^x$$

And $y=\overline{y}+y^*=c_1e^x+c_2e^{6x}+xe^x\left(-\dfrac{x}{10}+\dfrac{9}{25}\right)$.

Application activity 7.4.2

Solve :

1) $y''+6y'+9y=5e^{3x}$

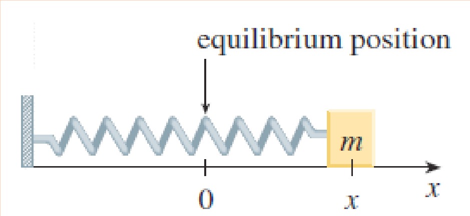
2) $y''-3y'+2y=e^{3x}$

3) $y''+3y'+2y=3e^{2x}$

7.5 Applications of second order linear homogeneous differential equation

Activity 7.5

1. Consider the motion of an object with mass m at the end of a spring as illustrated by the following figure of a non damped spring



If we ignore any external resisting forces (due to air resistance or friction) then, by Newton's Second Law (force equals mass times acceleration), the mass m

make a simple harmonic motion with the equation $m\frac{d^2x}{dt^2}=-kx$ where k is a positive constant called the spring constant.

a) Write the equation of motion in the form of $a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=0$ and deduce the value of a, b and c.

b) Solve $m\frac{d^2x}{dt^2}=-kx$ to determine the general solution.

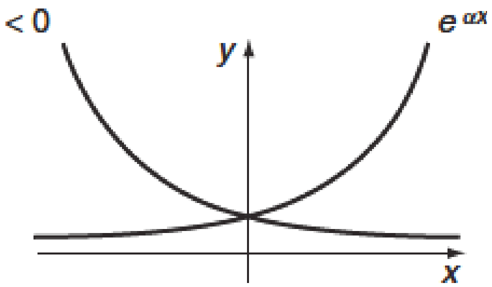
c) If the mass of 2 kg is moving on a spring with the spring constant k equals 128, determine and illustrate the position $x(t)$ of the mass at any time t given that at $t=0$, it is initially stretched at 0.2m and released with initial velocity 0

Content summary

In solving second order linear homogeneous differential equations, we essentially have just three distinct types of functions: $e^{\alpha x}$, $xe^{\alpha x}$ and $e^{\alpha x}\sin\omega x$ Each of these functions has a particular type of behavior for which you can identify the class it belongs to, and the type of physical system it represents:

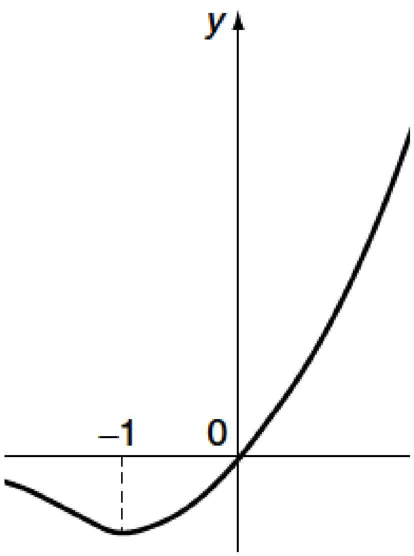
1) $e^{\alpha x}$ gives an increasing ($\alpha > 0$) or decreasing ($\alpha < 0$) exponential function

Graph of $y = e^{\alpha x}$



2) $xe^{\alpha x}$ gives a similar type of function, and it also passes through the origin.

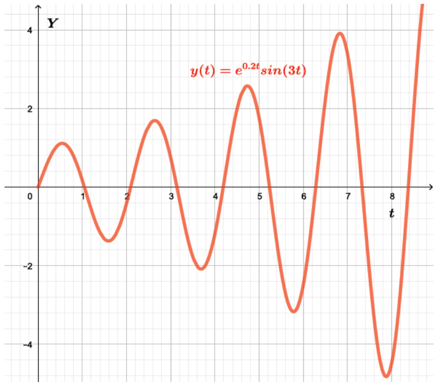
Graph of $y = xe^{\alpha x}$



3) $e^{\alpha x} \sin \omega x$ gives either a simple oscillating wave ($\alpha = 0$), or a sinusoidal wave with amplitude that decreases ($\alpha < 0$) or increases ($\alpha > 0$) as x increases.

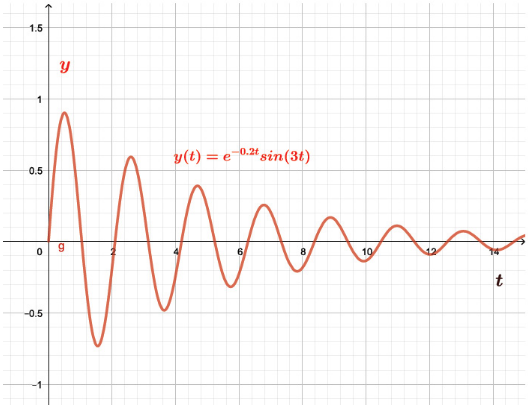
This could represent an oscillating system in a resisting medium where the sign of the exponent of the exponential function determines the amplitude which is increasing or decreasing with the time as illustrated on the following figures.

Graph of $y = e^{0.2t} \sin(3t)$



Amplitude is increasing with time t

Graph of $y = e^{-0.2t}\sin(3t)$



Amplitude is decreasing with time t

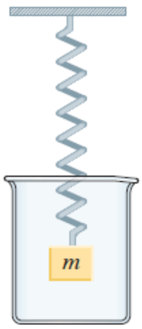
There are many areas of science and engineering where second order linear differential equations provide useful models.

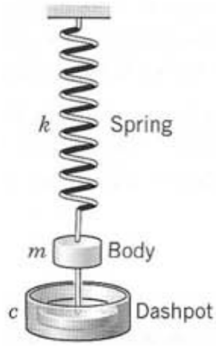
In practice, the amplitude of vibration in **simple harmonic motion** does not remain constant but becomes progressively smaller as the time increases. Such vibration is said to be **damped**.

The differential equations learnt above are used to describe some type of **oscillatory** behavior, with some degree of **damping** as they are illustrated bellow.

Consider the motion of a spring that is subject to a frictional force or a damping force (in the case where a vertical spring moves through a fluid as on the figure below.

spring moves through a fluid **the body to a dashpot**





Physically this can be done by connecting the body to a dashpot (which acts to resist displacement) as illustrated on the figure.

An example is the damping force supplied by a shock absorber in a car or a bicycle.

Apart from the restoring force $-kx$, the damping force proportional to the velocity

of the mass and acts also in the direction opposite to the motion and equals $-c\frac{dx}{dt}$ where c is a positive constant, called the damping constant.

Therefore, Newton's Second Law of motion gives

$$m\frac{d^2x}{dt^2} = -kx - c\frac{dx}{dt} \quad \text{or} \quad m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = 0$$

This is a second-order linear differential equation; Its auxiliary equation is

$m\lambda^2 + c\lambda + k = 0$. Therefore, $\lambda_1 = \frac{-c+\sqrt{c^2-4mk}}{2m}$ and $\lambda_2 = \frac{-c-\sqrt{c^2-4mk}}{2m}$

Depending on the value of $c^2 - 4mk$, 3 cases can occur:

a) $c^2 - 4mk > 0$, in this case, the spring is said to be over damping

In this case λ_1 and λ_2 are distinct real roots and $x(t) = c_1e^{\lambda_1 t} + c_2e^{\lambda_2 t}$

As $t > 0$, both exponents are negative, hence both terms approach zero as $t \to +\infty$, physically speaking, after a sufficiently long time the mass will be at rest at the static equilibrium position $(y = 0)$. This is requires a strong damping force (high-viscosity oil or grease) compared with a weak spring or small mass.

b) $c^2 - 4mk = 0$, in this case, the spring is said to be in critical damping

It corresponds to equal roots $\lambda_1 = \lambda_2 = -\frac{c}{2m}$ and the solution is given by

$$x(t)=(c_1+c_2t)e^{-\frac{c}{2m}t}$$

It is similar to the first case, but it is the border case between non oscillatory motion (over damping) and oscillations; the damping is just sufficient to suppress vibrations.

The solution $x(t)=(c_1+c_2t)e^{-\frac{c}{2m}t}$ can pass through the equilibrium position $(x=0)$ at most once since $e^{-\frac{c}{2m}t}$ is never zero and c_1+c_2t can have at most one root.

c) $c^2-4mk<0$ in this case, the spring is said to be under damping

It is the most interesting case and it occurs if the damping coefficient c is so small that $c^2<4mk$.

The roots of the auxiliary equation are complex.

$$\lambda_1=\frac{-c}{2m}+\omega i;\quad \lambda_2=\frac{-c}{2m}-\omega i$$

$$\omega=\frac{\sqrt{4mk-c^2}}{2m}$$

The corresponding general solution is given by $x=e^{-\frac{c}{2m}t}\left(A\cos(\omega t)+B\sin(\omega t)\right)$

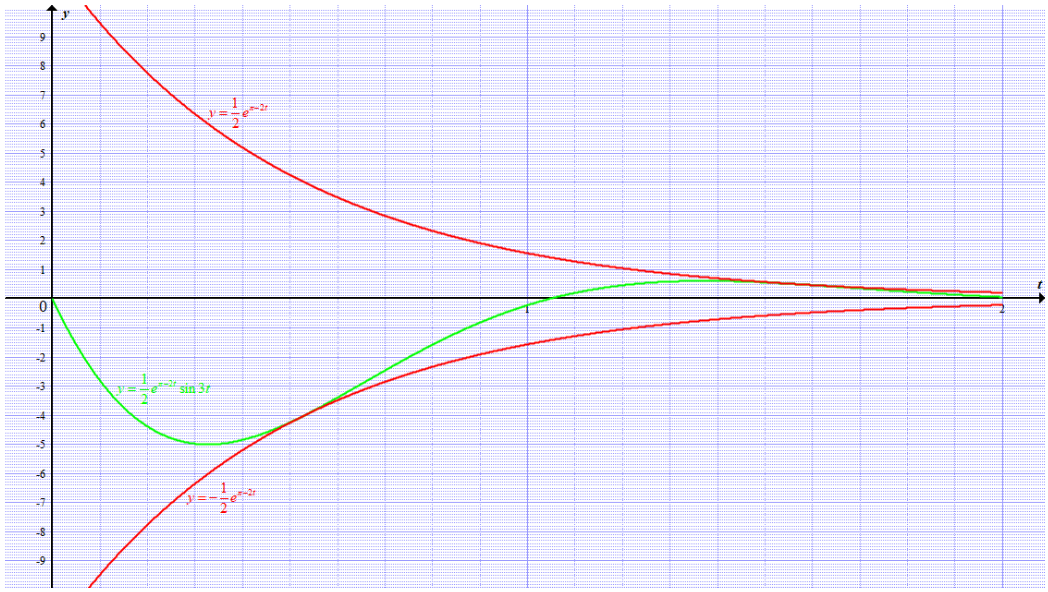
This sum can be combined into a phase-shifted cosine, $y=Ce^{-\frac{c}{2m}t}\cos(\omega t-\delta)$,

with amplitude $C=A^2+B^2$ and phase angle $\delta=\arctan\frac{B}{A}$.

Equation $y=Ce^{-\frac{c}{2m}t}\cos(\omega t-\delta)$, is physically more informative since it exhibits the amplitude and phase of oscillation. These oscillations are damped by factor $y=e^{-\frac{c}{2m}t}$ which means graphically that the oscillations are limited by the curves representing

$y=Ce^{-\frac{c}{2m}t}$ and $y=-Ce^{-\frac{c}{2m}t}$.

Figure: comparison of the graph of $y=\frac{1}{2}e^{-2t}$, $y=\frac{1}{2}e^{-2t}\sin 3x$ **and** $y=-\frac{1}{2}e^{-2t}$



Example

Solve graphically, the initial value problem

$$\frac{d^2y}{dt^2}+0.4\frac{dy}{dt}+9.04y=0, \qquad y(0)=0,\ y'(0)=3$$

In the same diagram, plot the graph of the functions $f(t)=e^{-0.2t}$ and $g(t)=-e^{-0.2t}$.

What is your observation?

Solution

$$\frac{d^2y}{dt^2}+0.4\frac{dy}{dt}+9.04y=0$$

Characteristic equation: $\lambda^2+0.4\lambda+9.04=0$

$\Delta=0.16-36.16=-36$

$\lambda_1=\frac{-0.4+6i}{2}=-0.2+3i$, $\lambda_2=\frac{-0.4-6i}{2}=-0.2-3i$

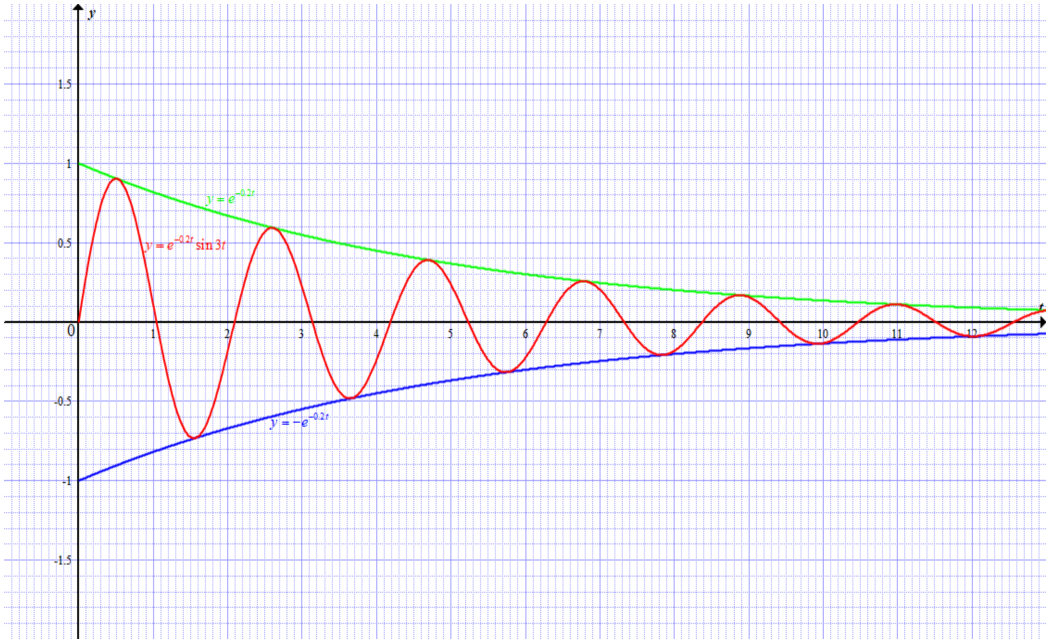
The corresponding general solution is $y(t)=e^{-0.2t}(c_1\cos 3t+c_2\sin 3t)$.

Given the initial conditions $\begin{cases} y(0)=0 \\ y'(0)=3 \end{cases}$

We find $c_1 = 0$ and $c_2 = 1$.

Therefore, $y(t) = e^{-0.2t}\sin 3t$

Graph of $y(t) = e^{-0.2t}\sin 3t$ **,** $f(t) = e^{-0.2t}$ **and** $g(t) = -e^{-0.2t}$

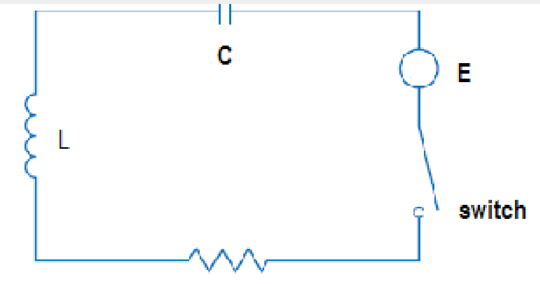


Application activity 7.5

1) A mass of 2k is oscillating on a spring of constant $k = 128$in a fluid with damping constant $c = 40$. Find and plot the graph for the position of the mass at any time t if it starts from the equilibrium position and is given a push to start it with an initial velocity of $0.6m/s$.

2) An electric Circuit contains an electromotive force E (supplied by a battery or generator), a resistor R, an inductor L, and a capacitor C in series. If the charge on the capacitor at time is $Q = Q(t)$, then the current is the rate of change of $Q(t)$ with respect to the time t, which means $I = \frac{dQ}{dt}$.

Electric circuit



It is known from physics that the voltage drops across the resistor, inductor, and capacitor are $RI, L\frac{dI}{dt}$ and $\frac{Q}{C}$.

The Kirchhoff's voltage law says that the sum of these voltage drops is equal to the supplied voltage $E(t)$ and expressed in the following equation:

$$L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{1}{C}Q = E(t)$$

a) Write this equation considering that $R = 40$, $L = 1$, $C = 16.10^{-4}$ and

$E(t) = 100\cos 10t$

b) What type of equation obtained if you consider $E(t)$ for $t = \frac{\pi}{20}$?

c) Determine the general solution for the equation obtained in (b).

7.6 End unit assessment

1. Let $y = e^{5x}$ calculate $y', y'', y''', y^{(iv)}$

Check if i) $y'' - 5y = 0$; ii) $y'' - 4y' - 5y = 0$

2. Solve the following differential equations

a) $y'' + 2y' + y = x + 2$

b) $y'' + 2y' + 5y = 0$

c) $y'' - 4y' + 4y = 0$

d) $y'' + y' - 2y = 0$

e) $y'' + y = e^x$

3. Given the differential equation $\frac{d^2y}{dx^2} + \frac{dy}{dx} - 6y = 0,\ y(0) = 10, y'(0) = 0$

a) What is the order and the degree of the equation?

b) Solve it to establish the general solution

c) If $y(0) = 10, y'(0) = 0$, deduce the solution $y(x)$

d) Plot the graph of the solution $y(x)$ and discuss its variation in your own words.

REFERENCES

1. J. Sadler, D. W. S. Thorning: Understanding Pure Mathematics, Oxford University Press 1987.

2. Arthur Adam Freddy Goossens: Francis Lousberg. Mathématisons 65. DeBoeck,3e edition 1991.

3. Charles D. Hodgman, M.S., Samuel M. Selby, Robert C.Weast. Standard Mathematical Table. Chemical Rubber Publishing Company, Cleveland, Ohio 1959.

4. David Rayner, Higher GCSE Mathematics, Oxford University Press 2000

5. Direction des Progammes de l'Enseignement Secondaire. Géometrie de l'Espace 1er Fascule. Kigali, October1988

7. Direction des Progammes de l'Enseignement Secondaire. Géometrie de l'Espace 2ème Fascule. Kigali, October1988

8. Frank Ebos, Dennis Hamaguchi, Barbana Morrison & John Klassen, Mathematics Principles & Process, Nelson Canada A Division of International Thomson Limited 1990

9. George B. Thomas, Maurice D. Weir & Joel R. Hass, Thomas' Calculus Twelfth Edition, Pearson Education, Inc. 2010

10. Geoff Mannall & Michael Kenwood, Pure Mathematics 2, Heinemann Educational Publishers 1995

11. H.K. DASS...Engineering Mathematics. New Delhi, S. CHAND&COMPANY LTD, thirteenth revised edition 2007.

12. Hubert Carnec, Genevieve Haye, Monique Nouet, ReneSeroux, Jacqueline Venard. Mathématiques TS Enseignement obligatoire. Bordas Paris 1994.

13. James T. McClave, P.George Benson. Statistics for Business and Economics. USA, Dellen Publishing Company, a division of Macmillan, Inc 1988.

14. J CRAWSHAW, J CHAMBERS: A concise course in A-Level statistics with worked examples, Stanley Thornes (Publishers) LTD, 1984.

15. Jean Paul Beltramonde, VincentBrun, ClaudeFelloneau, LydiaMisset, Claude Talamoni. Declic 1re S Mathématiques. Hachette-education, Paris 2005.

16. JF Talber & HH Heing, Additional Mathematics 6th Edition Pure & Applied,

Pearson Education South Asia Pte Ltd 1995

17. J.K. Backhouse, SPTHouldsworth B.E.D. Copper and P.J.F. Horril. Pure Mathematics 2. Longman, third edition 1985, fifteenth impression 1998.

18. Mukasonga Solange. Mathématiques 12, AnalyseNumérique. KIE, Kigali 2006.

19. N. PISKOUNOV, Calcul Différential et Integral tom II 9ème édition. Editions MIR. Moscou, 1980.

20. Paule Faure- Benjamin Bouchon, Mathématiques Terminales F. Editions Nathan, Paris 1992.

21. Peter Smythe: Mathematics HL & SL with HL options, Revised Edition, Mathematics Publishing Pty. Limited, 2005.

22. Robert A. Adms & Christopher Essex, Calculus A complete course Seventh Edition, Pearson Canada Inc., Toronto, Ontario 2010

23. Seymour Lipschutz. Schaum's outline of Theory and Problems of Finite Mathematics. New York, Schaum Publisher, 1966

24. Seymour Lipschutz. Schaum's outline of Theory and Problems of linear algebra. McGraw-Hill 1968.

25. Shampiyona Aimable : Mathématiques 6. Kigali, Juin 2005.

26. Yves Noirot, Jean–Paul Parisot, Nathalie Brouillet. Cours de Physique Mathématiques pour la Physique. Paris, DUNOD, 1997.

27. Swokowski, E.W. (1994). Pre-calculus: Functions and graphs, Seventh edition. PWS Publishing Company, USA.

29. Allan G. B. (2007). Elementary statistics: a step by step approach, seventh edition, *Von Hoffmann Press,* New York.

30. David R. (2000). Higher GCSE Mathematics, revision and Practice. Oxford University Press, UK.

31. Ngezahayo E.(2016). Subsidiary Mathematics for Rwanda secondary Schools, Learners' book 4, Fountain publishers, Kigali.

32. Ngezahayo E.(2016). Advanced Mathematics for Rwanda secondary Schools, Learners' book 4, Fountain publishers, Kigali.

34. Ngezahayo E.(2017). Subsidiary Mathematics for Rwanda secondary Schools,

Learners' book 5, Fountain publishers, Kigali.

35. Ngezahayo E.(2017). Advanced Mathematics for Rwanda secondary Schools, Learners' book 5, Fountain publishers, Kigali.

36. Ngezahayo E.(2018). Advanced Mathematics for Rwanda secondary Schools, Learners' book 6, Fountain publishers, Kigali.

37. REB. (2015). Subsidiary Mathematics Syllabus, MINEDUC, Kigali, Rwanda.

38. REB. (2019). Mathematics Syllabus for TTC-Option of LE, MINEDUC, Kigali Rwanda.

39. Peter S. (2005). Mathematics HL&SL with HL options, Revised edition. Mathematics Publishing PTY. Limited.

40. Elliot M. (1998). Schaum's outline series of Calculus. MCGraw-Hill Companies, Inc. USA.

41. Frank E. et All. (1990). Mathematics. Nelson Canada, Scarborough, Ontario (Canada)

42. Gilbert J.C. et all. (2006). Glencoe Advanced mathematical concepts, MCGraw-Hill Companies, Inc. USA.

43. Robert A. A. (2006). Calculus, a complete course, sixth edition. Pearson Education Canada, Toronto, Ontario (Canada).

44. Sadler A. J & Thorning D.W. (1997). Understanding Pure mathematics, Oxford university press, UK.

45. J. CRAWSHAW and J. CHAMBERS 2001. A concise course in Advanced Level Statistics with worked examples 4th Edition. Nelson Thornes Ltd, UK.

46. Ron Larson and David C (2009). Falvo. Brief Calculus, An applied approach. Houghton Mifflin Company.

47. Michael Sullivan, 2012. Algebra and Trigonometry 9th Edition. Pearson Education, Inc

48. Swokowski & Cole. (1992). Preaclaculus, Functions and Graphs. Seventh edition.

49. Glencoe. (2006). Advanced mathematical concepts, Precalculus with Applications.

50. Seymour Lipschutz, PhD. & Marc Lipson, PhD. (2007). Discrete mathematics. 3rd edition.

51. K.A. Stroud. (2001). Engineering mathematics. 5th Edition. Industrial Press, Inc, New York

52. John bird. (2005). Basic engineering mathematics. 4th Edition. Linacre House, Jordan Hill, Oxford OX2 8DP