MATHEMATICS BOOK

ACCOUNTING PROFESSION

for Rwandan Schools

Senior 4 Student Book

Experimental Version

FOREWORD

Dear Student,

Rwanda Basic Education Board (REB) is honoured to present Senior 4 Mathematics book for the students of Accounting Profession Option which serves as a guide to competence-based teaching and learning to ensure consistency and coherence in the learning of the Mathematics. The Rwandan educational philosophy is to ensure that you achieve full potential at every level of education which will prepare you to be well integrated in society and exploit employment opportunities.

The government of Rwanda emphasizes the importance of aligning teaching and learning materials with the syllabus to facilitate your learning process. Many factors influence what you learn, how well you learn and the competences you acquire. Those factors include the relevance of the specific content, the quality of teachers' pedagogical approaches, the assessment strategies and the instructional materials available. In this book, we paid special attention to the activities that facilitate the learning process in which you can develop your ideas and make new discoveries during concrete activities carried out individually or in groups.

In competence-based curriculum, learning is considered as a process of active building and developing knowledge and meanings by the learner where concepts are mainly introduced by an activity, situation or scenario that helps the learner to construct knowledge, develop skills and acquire positive attitudes and values.

For efficiency use of this textbook, your role is to:

- Work on given activities which lead to the development of skills;
- Share relevant information with other learners through presentations, discussions, group work and other active learning techniques such as role play, case studies, investigation and research in the library, on internet or outside;
- Participate and take responsibility for your own learning;
- Draw conclusions based on the findings from the learning activities.

To facilitate you in doing activities, the content of this book is self-explanatory so that you can easily use it yourself, acquire and assess your competences. The book is made of units as presented in the syllabus. Each unit has the following structure: the unit title and key unit competence are given and they are followed by the introductory activity before the development of mathematical concepts that are connected to real world problems more especially to production, finance and economics.

The development of each concept has the following points:

- Learning activity which is a well set and simple activity to be done by students in order to generate the concept to be learnt;
- Main elements of the content to be emphasized;
- Worked examples; and
- Application activities to be done by the user to consolidate competences or to assess the achievement of objectives.

Even though the book has some worked examples, you will succeed on the application activities depending on your ways of reading, questioning, thinking and handling calculations problems not by searching for similar-looking worked out examples.

Furthermore, to succeed in Mathematics, you are asked to keep trying; sometimes you will find concepts that need to be worked at before you completely understand. The only way to really grasp such a concept is to think about it and work related problems found in other reference books.

I wish to sincerely express my appreciation to the people who contributed towards the development of this book, particularly, REB staff, development partners, Universities Lecturers and secondary school teachers for their technical support. A word of gratitude goes to Secondary Schools Head Teachers, Administration of different Universities (Public and Private Universities) and development partners who availed their staff for various activities.

Any comment or contribution for the improvement of this textbook for the next edition is welcome.

Dr. MBARUSHIMANA Nelson

Director General, REB.

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I wish to express my appreciation to the people who played a major role in the development of this Mathematics book for Senior 4 students in Accounting Profession Option. It would not have been successful without active participation of different education stakeholders.

I owe gratitude to different universities and schools in Rwanda that allowed their staff to work with REB in the in-house textbooks production initiative.

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MURUNGI Joan

Head of CTLR Department

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UNIT 1

BASIC CONCEPTS OF ALGEBRA

Key unit competence: Apply algebraic principles and concepts in solving production, financial and economical related problems

1.0. Introductory activity

Introductory activity

A survey was carried out on Fifty learners at school, and they were asked if they are taking Taxation, Financial accounting or Financial management subject in their learning. The following information was corrected and displayed in the table,

21 are taking a taxation subject,	26 are taking a financial accounting,
19 are taking a Financial Management subject,	9 are taking taxation and financial accounting subjects,
7 are taking a taxation and financial management,	10 are taking financial accounting and financial management subjects,
3 are taking all the three subjects,	7 are taking none

a. Present this information using Venn diagrams
b. How many learners are only taking Taxation subject?
c. Do you think that elements of a set can be written using any other form/method than only a Venn diagram? Provide two different examples.

1.1. Set theory and its applications

1.1.1. Set of numbers

Activity 1.1.1.

1. How many sets of numbers do you know? List them down.
2. Using a mathematical dictionary or the internet, define the sets of numbers you listed in (1)
3. Give an example of element for each set of numbers you listed.
4. Establish the relationship between the set of numbers that you listed above.

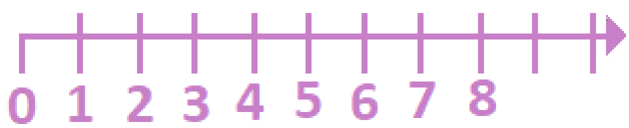
A well-defined collection of objects or elements is called a set. Each member of a set is called an element. All elements of a set follow a certain rule and share a common property amongst them. The capital letter in case of alphabet is used to represent a set while elements are represented by small letters. The elements written in the set can be in any order but cannot be repeated. Some commonly used sets are:

$\mathbb{N}: Set\ of\ all\ natural\ numbers$, $\mathbb{Z}: set\ of\ all$ integers, $\mathbb{Q}: set\ of\ all\ rational\ numbers$, and $\mathbb{R}: set\ of\ all\ real\ numbers$.

Cardinality of a set: a measure of the number of elements of the set. For example, the set $A=\{2,6,8\}$ contains 3 elements, and therefore A has a cardinality of 3. The cardinality of a set A is usually denoted by n(A), card(A), or # A.

1. Set of natural numbers

Usually, when counting, we start by one, followed by two, then three and so on. The numbers we use in counting including zero, are called Natural numbers. The set of natural numbers is denoted by $\mathbb{N}=\{0,1,2,3,4,\ldots\ldots\ldots\ldots\ldots\ldots\}$. On a number line, natural numbers are represented as follows:



Subsets of natural numbers

There are several subsets of natural numbers:

a) Even numbers

Even numbers are numbers which are divisible by 2 or numbers which are multiples of 2.

Example: Even numbers from 0 to 20 are 0, 2, 4, 6, 8, 10,12, 14, 16, 18 and 20.

The set of even numbers is $E=\{0,2,4,6,8,\ldots\}$ and it is a subset of natural numbers.

b) Odd numbers

Odd numbers are numbers which leave a remainder of 1 when divided by 2.

Example: Odd numbers between 0 and 20 are 1,3,5,7,9,11,13,15,17 and 19. The set of odd numbers is $O=\{1,3,5,7,\ldots\}$ and it is a subset of natural numbers.

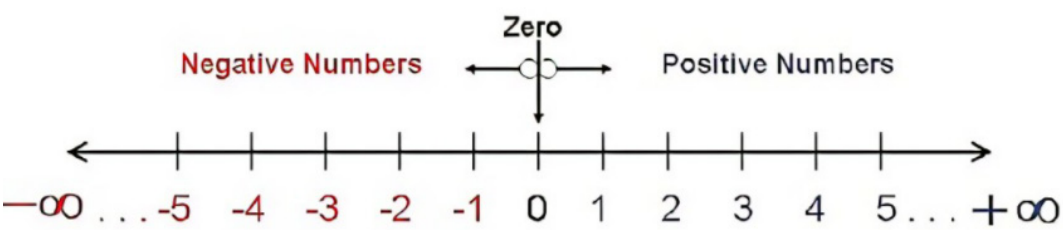
c) Prime numbers

Prime number is a number that has only two divisors 1 and itself.

Example: Prime numbers between 0 and 20 are 2, 3, 5, 7, 11, 13, 17 and 19. The set of prime numbers is $P=\{$üüüüüüüü $\ldots\}$ and it is a subset of natural numbers.

2. Set of integers

Integers are whole numbers which have either negative or positive together with zero and is represented by $\mathbb{Z}$. Integers can be negative {-1, -2, -3, -4, -5, ...}, positive {1, 2, 3, 4, 5, ...}, or zero where zero is neither positive nor negative. The set of integers is represented by using Carly brackets as follows: $\mathbb{Z}:\{\ldots,-5,-4,-3,-2,-1,0,1,2,3,4,5,\ldots\}$. On a number line the set of integers shows both positive and negative integers together with zero



From the above number line, taking 4 steps forward can be considered as $+4$ (positive 4). This means it is 4 steps ahead of the starting point. While, taking 4 steps backward is considered as -4 (negative 4). This means it is 4 steps behind the starting point.

Subsets of integers

Integers have several subsets such as set of natural numbers, set of even numbers, set of odd numbers, set of prime numbers, set of negative numbers and so on.

Some of the **special subsets** of integers include:

- The set of **positive integers denoted** as $\mathbb{Z}^+ = \{1, 2, 3, \ldots\}$
- The set of **negative integers** denoted as $\mathbb{Z}^- = \{\cdots, -4, -3, -2, -1\}$

3. Rational numbers

From any two integers a and b, we deduce fractions expressed in the form of $\frac{a}{b}$, where b is a non-zero integer. **A rational number is a number that can be expressed as a fraction wher**e both the numerator and denominator in the fraction are integers; the denominator in a rational number cannot be zero. As fractions are rational numbers, thus set of fractions is known as a set of rational numbers denoted by $\mathbb{Q}$.

Example: $\left\{..., \frac{51}{3}, \frac{2}{5}, 1.\dot{3}\dot{3}, ...\right\}$

Note: Rational numbers can be represented as decimals. Any decimal number whose terms are terminating or non-terminating, but repeating is a rational number.

Subsets of rational numbers

From the concept of subset and definition of a set of rational numbers we can establish some subsets of rational numbers; among them we have:

- Integers and its subsets: $\mathbb{Z}, \mathbb{Z}^-, \mathbb{Z}^+, ...$
- Natural numbers and its subsets: $\mathbb{N}, \mathbb{N}^+$, square numbers, prime numbers, odd numbers, even numbers,…

4. Irrational numbers

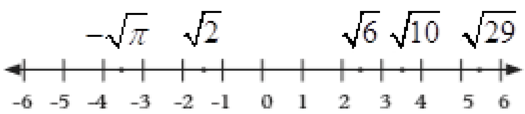
Some decimals are expressed as a rational $\frac{p}{q}$. For example, $0.13 = \frac{13}{100}, 0.25 = \frac{1}{4}$.

The recurring decimals such as $0.3333\ldots = \frac{1}{3}$ fall under rational numbers. There are some decimals which do not recur. Their values keep changing and they go on without an end. For example, $\sqrt{3} = 1.7320508\ldots,\ \sqrt[3]{4} = 1.587401\ldots$

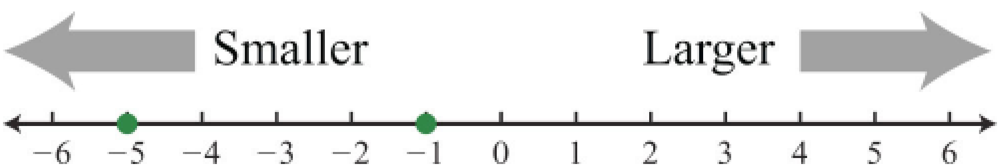
Similarly, there are some numbers which do not have exact values neither can be expressed as fractions, for example π etc. These numbers are called **irrational numbers**. They are numbers which cannot be expressed other than by means of roots or cannot be presented as ratio of two integers. An irrational number can be written as a decimal, but not as a fraction. An irrational number has endless non-repeating digits to the right of the decimal point. Although irrational numbers are not often used in daily life, they do exist on the number line. In fact, between 0 and 1 on the number line, there are an infinite number of irrational numbers.

5. Real numbers

The set of rational numbers and the set of irrational numbers combined form the set of real numbers. The set of real numbers is denoted by $\mathbb{R}$. Real numbers are represented on a number line as infinite points or they are set of decimal numbers found on a number line. This is illustrated on the number line below

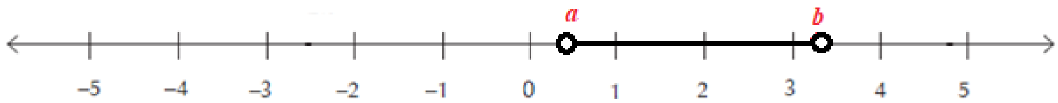


The real numbers are ordered. We say a is ***less than*** b and write $a < b$ if $b - a$ is positive. **Geometrically** this means that a lies to the left of b on the number line. Equivalently, we say b is ***greater than*** a and write $b > a$. The symbol $a \leq b$ $(\text{or}\, b \geq a)$ means that either $a < b$ or $a = b$ and is read a ***less than or equal to*** b. In fact, when comparing real numbers on a number line, the larger number will always lie to the right of the smaller one. It is clear that 5 is greater than 2, but it may not be so clear to see that −1 is greater than −5 until we graph each number on a number line.

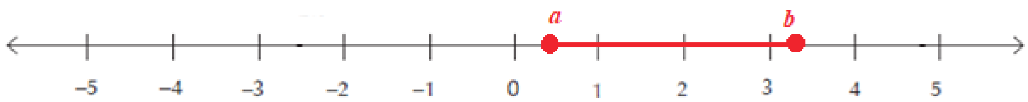


Subsets of real numbers and intervals

The main subsets of real numbers are sets of irrational numbers, rational numbers and its subsets. Certain subsets of real numbers are called **intervals**, they occur frequently in calculus and correspond geometrically to line segments. For example, if $a < b$, the **open interval** from a to b consists of all numbers between a and b. It is denoted as follow: (a,b) or $]a,b[$ and described as: $(a,b) = \{x : a < x < b\}$. For this case, the end points of the interval namely a and b are excluded. This is indicated by the **round brackets** $(\ )$ and by the **open dots** as illustrated in the following figure:



The **closed interval** from a to b is the set described as $[a,b] = \{x : a \le x \le b\}$. Here the endpoints of the interval are included. This is indicated by **square brackets** $[\ ]$ and by the **solid dots** as illustrated in the following figure:



Representation of set of real numbers

Sets can be represented in three forms:

1. Roster Form

Roster form /notation is a method of listing the elements of a set in a row with comma within curly brackets. For example, set of even numbers less than 8 is given by $A = \{2,4,6\}$

2. Statement form

In statement form, a well-defined description of a member of a set is written and enclosed in the curly brackets. For example, the set of even number less than 15 is written as $\{Even\ number\ less\ than\ 15\}$

3. Set builder form

Set builder form / notation is a method of describing the set while describing properties and not listing its elements. It is written in the form of $\{y | (properties\ of\ y)\}$ or $\{y : (properties\ of\ y)\}$. For example, $A = \{X : X = 2n, 1 \le n \le 4\}$.

Application activity 1.1.1.

1. Use roster and set builder forms to write the integers lying between -7 and -3
2. Given a set of numbers with the following elements

   $\left\{-2.479, \frac{2}{3}, \frac{57}{9}, \frac{1}{6}, \pi, \frac{3}{4}, 1.3, \sqrt{2}, 1.\dot{4}2, 1000, \dot{7}, -\sqrt{9}\right\}$, select the appropriate numbers for the following sets of numbers:

   a. $\mathbb{N}$ b. $\mathbb{Z}$ c. $\mathbb{Q}$ d. $\mathbb{R}$

   e. With diagram, show the relationship between these sets of numbers $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$

1.1.2. Operations and properties on numbers

Activity 1.1.2.

1. From the given any three numbers $a, b\ and\ c$, in each set $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$ investigate the following operations and use example to justify your answer:

   i. Addition

   ii. Subtraction

   iii. Multiplication

2. From the given any three numbers $a, b\ and\ c$, in each set of $\mathbb{N}\ and\ \mathbb{R}$ investigate the following properties and use example to justify your answer.

   i. Closure property for addition and subtraction

   ii. Distributive property for addition

   iii. Commutative property

1. Operations and properties on natural numbers

Addition

(i) Closure property: The sum of any two natural numbers is always a natural number. This is called 'Closure property of addition' on natural numbers. Thus, $\mathbb{N}$ is closed under addition. If a and b are any two natural numbers, then $(a+b)$ is also a natural number.

Example: 2 + 4 = 6 is a natural number.

(ii) Commutative property: If a and b are any two natural numbers, then, $a+b=b+a$. Hence, addition of two natural numbers is commutative.

Example: 2+ 4 = 6 and 4 + 2 = 6. Hence, 2 + 4 = 4 + 2, addition of two natural numbers is commutative

(iii) Associative property:

If a, b and c are any three natural numbers, then $a + (b + c) = (a + b) + c$. Addition of natural numbers is associative.

Example: 2 + (4 + 1) = 2 + (5) = 7 and (2 + 4) + 1 = (6) + 1 = 7.

Hence, 2 + (4 + 1) = (2 + 4) + 1

(iv) Identity element: If a is an identity element, then a+b=b+a=a. Zero is an identity element under addition. For example, 2+0=0+2=2.

Subtraction

(i) Closure property: The difference between any two natural numbers needs not to be a natural number. Hence $\mathbb{N}$ is not closed under subtraction.

Example: 2 - 5 = -3 is not a natural number.

(ii) Commutative property: If a and b are any two natural numbers, then $(a - b)\neq(b - a)$. Hence, Subtraction of two natural numbers is not commutative.

Example: 5 - 2 = 3 and 2 - 5 = -3. Hence, 5 - 2 ≠ 2 – 5. Therefore, Commutative property is not true for subtraction.

(iii) Associative property: If $a, b \text{ and } c$ are any three natural numbers, then $a - (b - \mathrm{c}) \neq (a - b) - \mathrm{c}$. Subtraction of natural numbers is not associative.

Example: 2 - (4 - 1) = 2 - 3 = -1 and (2 - 4) - 1 = -2 - 1 = -3

Hence, 2 - (4 - 1) ≠ (2 - 4) – 1. Therefore, Associative property is not true for subtraction.

Multiplication

(i) Closure property: If $a \text{ and } b$ are any two natural numbers, then $a \times b = ab$ is also a natural number. The product of two natural numbers is always a natural number. Hence $\mathbb{N}$ is closed under multiplication.

Example: 5 x 2 = 10 is a natural number.

(ii) Commutative property: If a and b are any two natural numbers, then $a \times b = b \times a$, Multiplication of natural numbers is commutative.

Example: 5 x 9=45 and 9 x 5 = 45. Hence, 5 x 9 = 9 x 5. Therefore, Commutative property is true for multiplication.

(iii) Associative property: If $a, b \text{ and } c$ are any three natural numbers, then $a \times (b \times c) = (a \times b) \times c$. Multiplication of natural numbers is associative.

Example: 2 x (4 x 5) = 2 x 20 = 40 and (2 x 4) x 5 = 8 x 5 = 40.
Hence, 2 x (4 x 5) = (2 x 4) x 5. Therefore, associative property is true for multiplication.

(iv) Multiplicative identity: a is any natural number, then $a \times 1 = 1 \times a = a$. The product of any natural number and 1 is the whole number itself. 'One' is the multiplicative identity for natural numbers.

Example: 5 x 1 = 1 x 5 = 5

Division

(i) Closure property: When we divide of a natural number by another natural number, the result does not need to be a natural number. Hence, $\mathbb{N}$ is not closed under division.

Example: When we divide the natural number 3 by another natural number 2, we get 1.5 which is not a natural number.

(ii) Commutative property: If a and b are two natural numbers, then $a \div b \neq b \div a$. Hence, division of natural numbers is not commutative.

Example: 2 ÷ 1 = 2 and 1 ÷ 2 = 0.5 . Hence, 2 ÷ 1 ≠ 1 ÷ 2.

Therefore, Commutative property is not true for division.

(iii) Associative property: If $a, b \text{ and } c$ are any three natural numbers, then $a \div (b \div c) \neq (a \div b) \div c$. Division of natural numbers is not associative.

Example: 3 ÷ (4 ÷ 2) = 3 ÷ 2 = 1.5 and (3 ÷ 4) ÷ 2 = 0.75 ÷ 2 = 0.375

Hence, 3 ÷ (4 ÷ 2) ≠ (3 ÷ 4) ÷ 2. Therefore, Associative property is not true for division.

Distributive Property

(i) Distributive property of multiplication over addition:

If $a, b \text{ and } c$ are any three natural numbers, then $a \times (b + c) = ab + ac$. Multiplication of natural numbers is distributive over addition.

Example: 2 x (3 + 4) = 2 x 3 + 2 x 4 = 6 + 8 = 14.

2 x (3 + 4) = 2 x (7) = 14. Hence, 2 x (3 + 4) = 2 x 3 + 2 x 4. Therefore, Multiplication is distributive over addition.

(ii) Distributive property of multiplication over subtraction:

If $a, b \text{ and } c$ are any three natural numbers, then $a \times (b - c) = ab - ac$. Multiplication of natural numbers is distributive over subtraction.

Example: 2 x (4 - 1) = 2 x 4 – 2 x 1 = 8 - 2 = 6.

2 x (4 - 1) = 2 x (3) = 6. Hence, 2 x (4 - 1) = 2 x 4 – 2 x 1. Therefore, multiplication is distributive over subtraction.

2. Operations and properties on integers

The properties of operations on integers help us to quickly simplify and answer a series of mathematical questions that are involving integers. All properties for $\mathbb{Z}$ including closure, commutative, associative, and distributive property for addition, subtraction, multiplication, and division are performed in the table below:

Operation	Calculations and explanations	Properties
Addition	For any two integers $a \text{ and } b$, $a+b$ is an integer. For example, (–5) + 8 = 3, the result is an integer.	Closure property under addition: $a+b \in \mathbb{Z}$
	For any two integers $a \text{ and } b$, the order of terms doesn't matter, the result is the same under addition. For example, 3+2=2+3=5	Commutative property under addition: $a+b = b+a$
	For any three integers, $a, b \text{ and } c$, the way of grouping numbers doesn't matter. The results from addition is the same. Parenthesis can be done, irrespective of the order of terms. Example, (2+3)+4=9=2+(3+4)	Addition is associative for integers: $a+(b+c)=(a+b)+c$

	when any integer is added to zero it gives the same number. For example, 2+0=2=0+2	Identity: $a + 0 = a = 0 + a$ Zero is called additive identity
	For any integer a, if there exist an integer b such that a+b=b+a=0, then b is an inverse of a. Example: 2+(-2)=(-2)+2=0	Inverse: a+b=b+a=0 b is called additive inverse of a.
Subtraction	For any two integers $a\ and\ b$, $a-b$ is an integer. For example, 8-3 = 5, the result is an integer.	Closure property under addition: $a-b \in \mathbb{Z}$
	For any two integers $a\ and\ b$, $a-b \neq b-a$ **Example,** 4 – (–6) ≠ (–6) – 4	Commutative property: $(a-b \neq b-a)$, subtraction is not commutative for integers and whole numbers
	For any three integers $a, b\ and\ c$, the difference of them is not equal. for example, $(2-3)-5 \neq 2-(3-5)$	Associative property: $(a-b)-c \neq a-(b-c)$ subtraction of integers is not associative.
	For any integer minus zero is different from zero minus that integer. For example, $3-0 \neq 0-3$	Identity property: $a-0 \neq 0-a$, 0 is not identity for subtraction
Multiplication	For any two integers $a\ and\ b$, $a \times b$ is an integer. For example, 8x3 = 24, the result is an integer.	Closure property: $a \times b \in \mathbb{Z}$
	For any two integers $a\ and\ b$, the order of terms doesn't matter, the result is the same under multiplication. For example, 3x2=2x3=6	Commutative property: $a \times b = b \times a$

	For any three integers, $a, b\ and\ c$, the way of grouping numbers doesn't matter. The results from multiplication is the same. Parenthesis can be done, irrespective of the order of terms. Example, (2x3) x 4=24=2x(3x4)	Associative property: $a\times(b\times c)=(a\times b)\times c$
	Any integer multiplied by 1 gives the integer itself as the product. But If any integer is multiplied by 0, the product is zero. For example, 5x1=1x5=5 but 5x0=0x5=0	Identity property: $a\times 1 = a = 1\times a$, hence, 1 is called the multiplicative identity for multiplication. Zero is an absorbing element for multiplication
	For any three integers $a, b\ and\ c$ are added or subtracted first and multiplied with each number within the bracket and then added or subtracted. For example, $-5\ (2+1) = -15 = (-5\times 2)+(-5\times 1)$	Distributive $a\times(b+c)=a\times b+a\times c$ and $a\times(b-c)=a\times b-a\times c$
Division	The quotient of any two integers $a\ and\ b$ may or may not be an integer. For example, (–3) ÷ (–6) = ½, is not an integer.	Closure property: $a\div b\notin\mathbb{Z}$ Division of integers doesn't follow the closure property
	For any two integers, $a\ and\ b$, the quotient $a\div b$ is differ from $b\div a$ For example, $2\div 6\neq 6\div 2$	Commutative property: $a\div b\neq b\div a$, hence, division of integers doesn't follow commutative property

	For any three integers a, b and c, the quotient between them is different Example, $(2 \div 3) \div 4 \neq 2 \div (3 \div 4)$	Associative property: $(a \div b) \div c \neq a \div (b \div c)$ hence, division of integers does not follow associative
	For any integer a, the quotient $a \div 1$ and $1 \div a$ are different	Identity: $a \div 1 = a \neq 1 \div a$, hence 1 is not identity for division

3. Operations and properties on rational numbers

The following table shows how addition, subtraction, multiplication, and division of rational numbers are performed.

Operation	**Calculations and explanations**	**Properties**
Addition of Rational Numbers	For any two rational numbers *a* and *b*, *a* + *b* is also a rational number.	Closure property under addition
	For any two rational numbers *a* and *b*, *a* + *b* = *b*+ *a*. Two rational numbers can be added in any order and the result remain the same	Commutative property under addition
	For any three rational numbers *a, b* and *c*. *(a + b)* + c = *a + (b + c)*. Rational numbers can be added regardless of how they are grouped, and the result remain the same.	Addition is associative for rational numbers.
	For any rational number, a a+0=0+a=a	Identity property – additive identity is 0.

	For any rational numbers, a a+(-a) =(-a)+a=0	Additive inverse: if a is rational, then -a (for all nonzero a) is an inverse of a.
Subtraction of Rational Numbers	For any two rational numbers *a* and *b*, $a - b$ is also a rational number.	Closure property under subtraction of rational numbers
	For any two rational numbers *a* and *b*, $a - b \neq b - a$.	Subtraction is not commutative for rational numbers.
	For any three rational numbers *a*, *b* and *c*. $(a - b) - c \neq a - (b - c)$.	Subtraction is not associative for rational numbers
Multiplication of Rational Numbers	For any two rational numbers *a* and *b*, $a \times b$ is also a rational number.	Closure Property under multiplication
	For any two rational numbers *a* and *b*, $a \times b = b \times a$ Two rational numbers can be multiplied in any order and the result remain the same	Multiplication is commutative for rational numbers.
	For any three rational numbers *a*, *b* and *c*. $(a \times b) \times c = a \times (b \times c)$. Rational numbers can be multiplied regardless of how they are grouped and the result remain the same.	Multiplication is associative for rational numbers.
	For any three numbers *a*, *b* and *c*. $a\times(b+c) = (a\times b)+(a\times c)$	Distributive property states that for any three numbers *a*, *b* and *c* we have $a\times(b+c) = (a\times b)+(a\times c)$
	For any rational number a, $a \times 1 = 1 \times a = a$	Multiplicative identity is 1

	For any rational number a, $a \times \frac{1}{a} = \frac{1}{a} \times a = 1$	Multiplicative inverse: if a rational number, then 1/a (for all nonzero a) is an inverse of a.
Division of Rational Numbers	For any two rational numbers *a* and *b with b different from zero, a* ÷ *b* is also a rational number. But we know that any rational number *a, a* ÷ 0 is not defined.	Closure Property under division for non-zero rational numbers.
	For any two rational numbers *a* and *b,* *a* ÷ *b* ≠ *b* ÷ *a*. The expressions on both the sides are not equal	Division is not commutative for rational numbers.
	For any three rational numbers *a*, *b* and *c,* *a* ÷(*b*÷ *c*) ≠ (*a* ÷ *b*) ÷*c*. The expressions on both the sides are not equal	Division is not associative for rational numbers

4. Operations and properties on real numbers

The following table shows how addition, subtraction, multiplication, and division of real numbers are performed

Operation	Calculations	Properties
Addition of real Numbers	For any two real numbers *a* and *b, a* + *b* is also a real number.	Closure property under addition
	For any two real numbers *a* an *b,* *a* + *b* = *b*+ *a*. Two real numbers can be added in any order and the result remain the same	Commutative property under addition

	For any three real numbers *a, b* and *c*. $(a+b) + c = a + (b+c)$. Real numbers can be added regardless of how they are grouped, and the result remain the same.	Addition is associative for real numbers.
	For any real number, a a+0=0+a=a	Identity property – additive identity is 0.
	For any real numbers, a a+(-a) =(-a) +a=0	Additive inverse: if a is real number, then -a (for all nonzero a) is an inverse of a.
Subtraction of Real Numbers	For any two real numbers *a* and *b*, $a - b$ is also a real number.	Closure property under subtraction of real numbers
	For any two real numbers *a* and *b*, $a - b \neq b - a$.	Subtraction is not commutative for real numbers.
	For any three real numbers *a*, *b* and *c*. $(a - b) - c \neq a - (b - c)$.	Subtraction is not associative for real numbers
Multiplication of real Numbers	For any two real numbers *a* and*b*, $a \times b$is also a real number.	Closure Property under multiplication
	For any two real numbers *a* and *b*, $a \times b = b \times a$ Two real numbers can be multiplied in any order and the result remain the same	Multiplication is commutative for real numbers.
	For any three real numbers *a*, *b* and *c*. $(a \times b) \times c = a \times (b \times c)$ Real numbers can be multiplied regardless of how they are grouped, and the result remain the same.	Multiplication is associative for real numbers.

	For any three numbers *a, b* and *c*. $a\times(b+c) = (a\times b)+(a\times c)$	Distributive property states that for any three numbers *a, b* and *c* we have $a\times(b+c)=(a\times b)+(a\times c)$
	Any real number multiplied by 1 gives the real number itself as the product. a ×1=1 × a=a	1 is an identity element under multiplication.
	For any real number a, $a\times\frac{1}{a}=\frac{1}{a}\times a=1$	Multiplicative inverse; if a real number, then $\frac{1}{a}$ (for all nonzero a) is an inverse of a.
Division of Real Numbers	For any two real numbers *a* and *b with b different from zero,* $a\div b\neq b\div a$ is also a real number. But we know that any real number *a, a* ÷ 0 is not defined. For any two real numbers *a* and *b,* $a\div b\neq b\div a$. The expressions on both sides are not equal	Closure Property under division for real numbers different from zero. Division is not commutative for real numbers.
	For any three real numbers *a* , *b* and *c,* $a\div(b\div c)\neq(a\div b)\div c$. The expressions on both the sides are not equal	Division is not associative for real numbers

Application activity 1.1.2.

1. For each of the following operation, identify the properties being expressed

   a) $3(2x-5)=6x-15$

   b) $(0.08+0.12)+\frac{1}{2}=0.8+\left(0.12+\frac{1}{2}\right)$

   c) $(3\times5)\times2=3\times(5\times2)$ but $(3\div5)\div2\neq3\div(5\div2)$

   d) $\pi-2\neq2-\pi$ but $\pi+2=2+\pi$

2. Discuss whether Closure Property under division for real numbers is satisfied.

3. From the given any three numbers a, b *and* c, in each of the following sets $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$ investigate the following and use example to justify your answers.

   i. Closure property for addition and subtraction

   ii. Identity property

   iii. Commutativity property

   iv. Distributive property for addition

1.1.3. Percentages and ratios

Activity 1.1.3.

Refer to the meaning of decimals and percentage learnt in previous years.

1) Calculate 60:100 and write it in the form of

   a) a fraction, b) a percentage and c) a decimal number.

2) Express $\frac{1}{3}$ *and* $\frac{22}{7}$ in the form of decimal numbers . Is it possible to express these numbers in the form of percentage? Is the pecentage obtained an exact number?

3) 24 students in a class took Auditing test. If 18 students passed the test, what percentage of those who did not pass?

4) Does bank managers, local leaders and shop keepers need the concepts of percentage and ratios in their daily duties ? Explain your answer.

a. Ratios

A ratio is a comparison of two similar quantities. When given two similar quantities or two different numbers/ integers a *and* b, the ratio of a *to* b is represented as $a:b = a/b$, where $b \neq 0$. The two numbers in a ratio can only be compared when they have the same unit.

Example: The working capital of ''X'' Ltd has deteriorated in recent years and now stands as under current assets of 980 000Frw to the current liabilities of 700 000Frw. Compute the current ratio.

Solution: The current ratio is given by $\frac{\text{current assets}}{\text{current liabilities}} = \frac{980\ 000 Frw}{700\ 000 Frw} = 1.4$, Hence the current ratio is 1.4

b. Percentages

Percentage is defined as the proportion, rate or ratio expressed with a denominator of 100.

For example: $\frac{3}{100}, \frac{25}{100}$ **, etc...** Fraction can be expressed in the form of percentage as $\frac{1}{4} \times 100 = 25\%$.

Decimal format is often required for a value that is usually specified as a percentage in everyday usage. For example, interest rates are usually specified as percentages. A percentage format is really just another way of specifying a decimal fraction, for example $62\% = \frac{62}{100} = 0.62$, percentages can easily be converted into decimal fractions by dividing a number by 100. Because some fractions cannot be expressed exactly in decimals, one may need to 'round off' an answer for convenience. In many of the economic problems (of various books) there is not much point in taking answers beyond *two decimal places (2dp)*. Where this is done then we denote, (to 2 dp) is normally putted after the answer. *For example,* $\frac{1}{7}$ as a percentage is 14.29% (to 2 dp). Percentages are widely used in business for computing the profit or loss percentage of a business.

Percentage change (Percent increase or decrease)

Percentage change is the measure of how much a quantity has changed over time. For example, assume that you are keeping track of a security's advertised price. If the price has risen, then you will have a percentage increase while if the price dropped, then you will have a percentage decrease. Percentage change is used to compare the values of different currencies, as well as to follow the prices of individual stocks and huge market indexes.

- To calculate percentage increase, we first work out on difference between the two numbers/ quantities, then we divide the increase by original number and finally we multiply the answer by 100.
- To calculate percentage decrease, we work out difference first, then we divide the obtained decrease by original number and multiply answer by 100.

$$Percent\ increase = \frac{New\,value - original\ value}{original\ value},$$

$$Percent\ decrease = \frac{Original\ value - new\,value}{original\ value}$$

Example 1: In shop, a product was reduced from 33 000Frw to 29 000Frw. What percent reduction is this?

Solution: The percent decrease is given by

$$\frac{33\,000Frw - 29\,000Frw}{33\,000Frw} = \frac{4000}{33\,000} = \frac{4}{33}$$

So, percent decrease is $\frac{4}{33} \times 100 = 12\%$

Example 2: The annual salary of Emmanuel was increased from 180 000Frw to 220 000Frw. Calculate the percentage increase.

Solution: Percentage increase in salary is

$$\frac{220\,000Frw - 180\,000Frw}{180\,000Frw} \times 100 = \frac{40\,000}{180\,000} \times 100 = 22.22\%$$

c. Converting ratio to percentage

The Ratio to percentage formula given by $Percentage = Ratio \times 100$ is applied to convert ratio to percentage. This can be applied when solving problems involving percentages.

In solving word problems involving percentage, 3 steps can help:

1. Make sure you understand the question
2. Sort out the information to make a basic percent problem
3. Apply the operations to find out what asked

Example 1: Mary received her monthly salary. The ratio of her expenditure to savings for schools is 7:3. Calculate the percentage of her spending, and the percentage of her savings.

Solution: The part of savings and expenditure are 3 and 7 respectively. Salary takes 3+7=10 parts. The spends salary takes $\frac{7}{10}$ while saved salary takes $\frac{3}{10}$.

Percentage of expenditure $= \frac{7}{10} \times 100\% = 70\%$

Percentage of savings $= \frac{3}{10} \times 100\% = 30\%$

Example 2: A town council imposes different taxes on different fixed assets as follows: Commercial property 25% per year, Residential property 15% per year, Industrial property 20% per year. An investor owns a residential building on a plot all valued at 80 000 000 Frw an industrial plot worth 75 000 000 Frw and a commercial premise worth 12 500 000 Frw. How much tax does the investor pay annually?

Solution:

Commercial: $\frac{25}{100} \times 12\,500\,000 Frw = 3\,125\,000 Frw$

Resident: $\frac{15}{100} \times 80\,000\,000 Frw = 12\,000\,000 Frw$

Industrial: $\frac{20}{100} \times 75\,000\,000 Frw = 15\,000\,000 Frw$

Total tax the investor pays annually:

$3\,125\,000 Frw + 12\,000\,000 Frw + 15\,000\,000 Frw = 30\,125\,000 Frw$

Application activity 1.1.3.

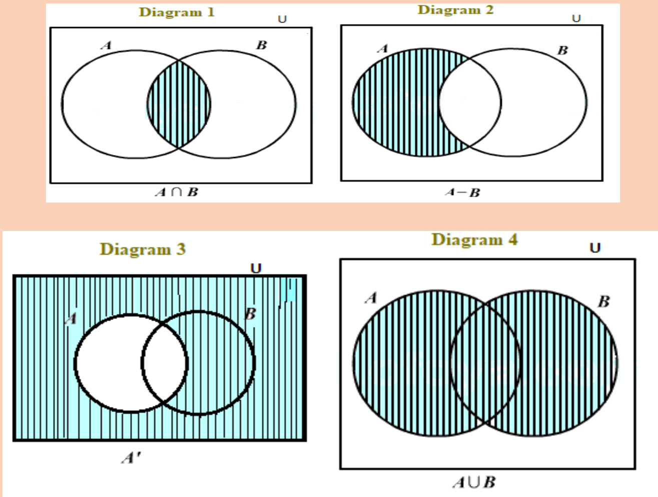
1. In the middle of the first term, the school organize the test and the teacher of Mathematics prepared 20 questions for both section A and B. Peter gets 80% correct. How many questions did Peter missed?
2. Student earned a grade of 80% on Mathematics test that has 20 questions. How many did the student answer correctly? And what percentage of that not answered correctly?
3. A product increased production from 1 500Frw last month to 1 650Frw this month. Calculate the percent increase.

1.1.4. Venn diagrams and operations on sets of numbers

Activity 1.1.4.

1. Define Venn diagram
2. Place each of the following numbers: in the correct regions on the following Venn diagrams

3. Consider a class of students as a universal set. Set A is the set of all students who were present in the Auditing class, while Set B is the set of all students who were present in the Taxation class. It is obviously that there were students who were present in both classes as well as those who were present in one class but not in the other class. The shaded part in the Venn diagrams below shows different scenarios on the presence of the students in class.



Observe the diagrams and identify which one to represent the following:

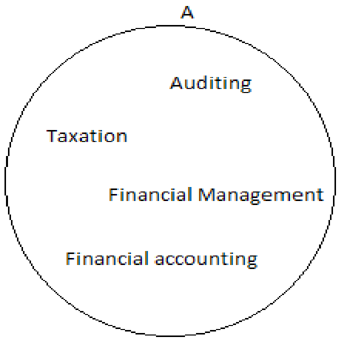
- All students who were absent in the Auditing class
- All students who were present in at least one of the two classes.
- All the students who were present for both Auditing as well as Taxation classes.
- All the students who have attended only the Auditing class and not the Taxation class

A. Venn Diagram

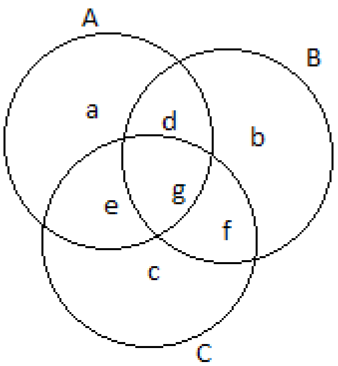
A Venn Diagram is an illustration that shows logical relationships between two or more sets (grouping items) using closed geometrical figures. Commonly, Venn diagrams show how given items are similar and different. The capital letter outside the circle denotes the name of the set while the small letters inside the circle denote the elements of the set. When drawing Venn diagrams, some important facts like "**intersection**", "**union**" and "**complement**" should be well considered and represented.

Example 1: The given set $A = \{Auditing, Taxation, Financial\ Management, Financial\ \ Accounting\}$ is representing subjects found in Accounting option. Represent on Venn diagram

Solution:



Example 2: The following diagram is a Venn diagram representing set A, B and C



The above figure is a representation of a Venn diagram where each of the circles *A*, *B* and *C* represents a set of elements.

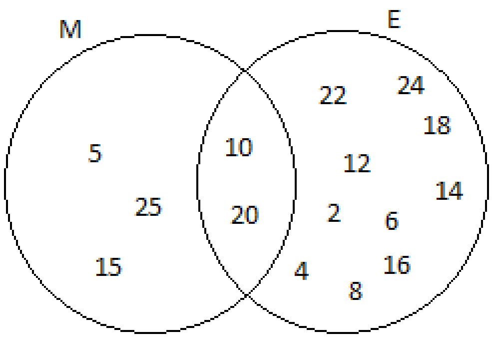
- Set *A* has the elements *a*, *d*, *e* and *g*
- Set *B* has the elements *b*, *d*, *g* and *f.*
- Set *C* has the elements *e*, *g*, *f* and *c*.
- Both *A* and *B* have the elements *d* and *g*.
- Both *B* and *C* have the elements *g* and *f*.
- Both *C* and *A* have the elements *e* and *g*.
- *A*, *B* and *C* all have the element *g*.

Example 3:

Present a Venn diagram showing the correlation between set contains even number from 1 to 25 and the set contains numbers of multiples 5 from 1 to 25.

Solutions

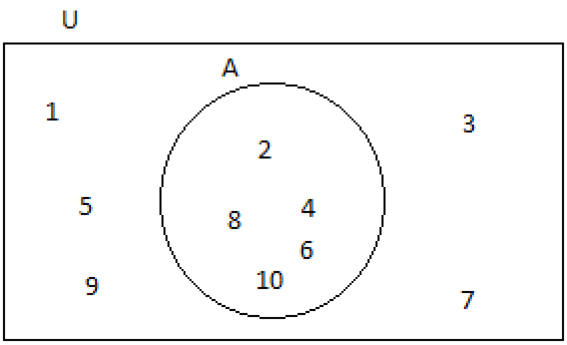
The following Venn diagram shows the correlation between set contains even numbers from 1 to 25 and the other set contains numbers in the multiples of 5 from 1 to 25. Set M represent multiple of 5 from 1 to 25 while set E represent even number from 1 to 25



In the above Venn diagram, the elements 10 and 20 are both even and multiple of 5 from 1 to 25.

Universal set

The universal set is the set of all elements or members of all related sets in each scenario. It consists of all the elements of its subsets, including its own elements. It is usually denoted by the symbol U. A universal set can be either a finite or infinite set. The universal set "U" is represented by a rectangle and its subsets "A, B, C, …"represented by circles. For example,



The above diagram shows the universal set U with the elements

$\{1,2,3,4,5,6,7,8,9,10\}$ and the subset of this universal set is the set

$A=\{2,4,6,8,10\}$

Example 1: A school is the universal set of all learners in the school. The set of all senior four learners can be considered as a subset of this universal set.

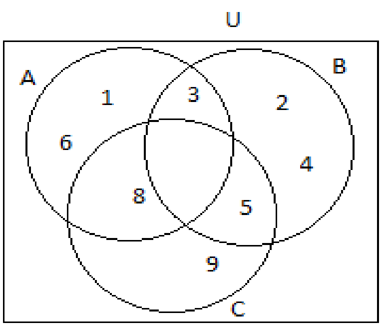
Example 2: Given the three sets named A, B, and C. The elements of all sets are defined as:

$A = \{1,3,6,8\}$, $B = \{2,3,4,5\}$, $C = \{5,8,9\}$

a. Represent the information on Venn diagram
b. Find out the universal set
c. Comment on your findings by showing the nature of each element in the universal set.

Solutions:

a. The given information is presented on Venn diagram as follows:



b. By definition, a universal set includes all elements of the given sets. Therefore, a Universal set for sets A, B and C is U= {1,2,3,4,5,6,8,9}
c. From this example, the elements of sets A, B and C are altogether available in Universal set "U". Also, there is no repeated element in the universal set as all the elements are unique.

B. Operations on sets of numbers

From the above activity, two or more sets can be represented using one Venn diagram and from the representations, different sets can be determined. These consist of ways or operations whereby sets are combined to obtain other sets of interest. The operations on sets are:

- Intersection of sets
- Union of sets
- Simple difference of sets
- Symmetric difference of sets
- Complement of sets

a. The intersection of sets

The common elements which appear in two or more sets form the intersection of sets. The symbol used to denote the intersection of sets is $\cap$.

The intersection of sets A and B is denoted by $A \cap B$ and consists of those elements which belong to A and B that is $A \cap B = \{x | x \in A \text{ and } x \in B\}$

Example:

Given that set A $= \{\text{the first 5 letters of the alphabet}\}$ and set B $= \{\text{all the vowels}\}$;

a) List the elements of each set.

b) Find $A \cap B$

c) Draw a Venn diagram to represent set $A \cap B$

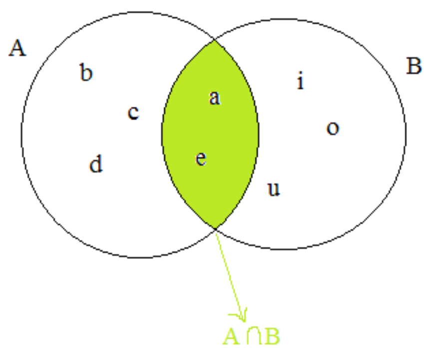
Solution:

A $= \{a, b, c, d, e\}$

B $= \{a, e, i, o, u\}$

A $\cap B = \{a, b, c, d, e\} \cap \{a, e, i, o, u\} = \{a, e\}$

The Venn diagram for intersection of sets A and b is as shown below.



b. The union of sets

Elements of two or more sets can be put together to form a set. The set formed is known as **the union of sets**. The symbol for the union of sets is $\cup$.

The union of two set A and B, is denoted by $A \cup B$ and consists of all the elements which are members of either A or B or both A and B that is $A \cup B = \{x | x \in A \text{ or } x \in B\}$

Example

Given the following sets $A = \{a, b, c, d, e, f\}$ and $B = \{a, b, c, h, i\}$

a. Find the number of elements of the following sets: $A,\ B,\ A\cup B$:

$(i)\ n(A)$

$(ii)\ n(B)$

$(iii)\ n(A\cup B)$

b. Draw Venn diagrams to represent $A\cup B$

Solution

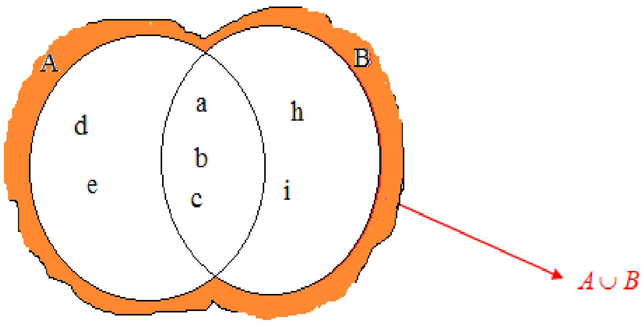
a.

$(i)\ n(A) = 6$

$(ii)\ n(B) = 5$

$$(iii)\ A\cup B = \{a,\ b,\ c,\ d,\ e,\ f\}\cup\{a,\ b,\ c,\ h,\ i\}$$
$$= \{a,\ b,\ c,\ d,\ e,\ f,\ h,\ i\}$$
$$\Rightarrow n\ (A\cup B) = 8$$

b. The Venn diagram for union of sets A and b is as shown below



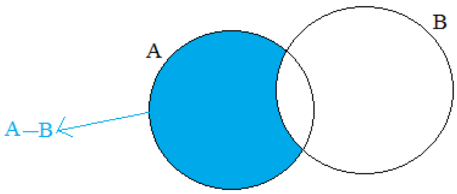
c. Difference and Symmetric difference of sets

i) Difference of sets

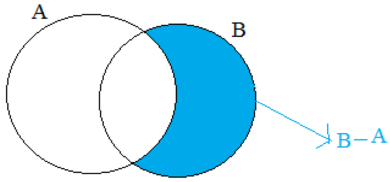
Difference between sets A and B written as $A-B$ or $A\setminus B$ is the set of the elements of set A which are not in set B. It means that $A-B=\{x|x\in A, x\notin B\}$

Likewise, $B-A\ or\ B\setminus A$ is difference between sets B and A. This is the set of elements that are in set B and not in set A. It means that $B-A=\{x|x\in B, x\notin A\}$

A−B and B−A can be shown using a Venn diagram as follows:



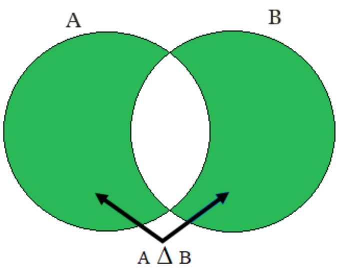
and



ii) Symmetric difference of sets

The union of sets $A-B \text{ and } B-A$ is known as **the symmetric difference between sets A and B**. It is written in symbols as $A \Delta B$ to mean $A \Delta B = (A-B) \cup (B-A)$

$A \Delta B$ can be shown using a Venn diagram as follows



Example:

Given that $A = \{3,\ 4,\ 5,\ 6,\ 7,\ 8\}$ and $B = \{2,\ 4,\ 8,\ 12\}$, find:

$(a)\ A - B$

$(b)\ B - A$

$(c)\ A \Delta B$

Solution

$(a)\ A - B = \{3,\ 5,\ 6,\ 7\}$

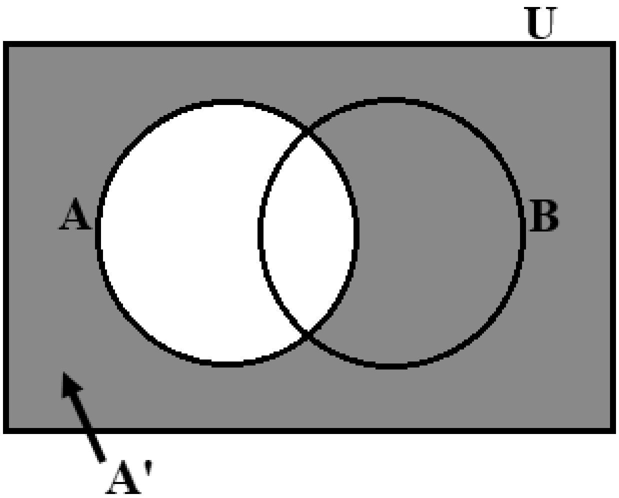
$(b)\ B - A = \{2,\ 12\}$

$(c)\ A \Delta B = (A-B) \cup (B-A = \{2,3,5,6,7,12\}$

d. The complement of a set

Complement of a set is the set of all elements in the universal set that are not members of a given set. The symbol for the universal set is U.

The complement of A is denoted by A' and consist of all those elements in the universal set which do not belong to A that is $A' = \{x | x \in U, x \notin A\}$



Note that $A - B = A \cap B'$ where B' is the complement of B

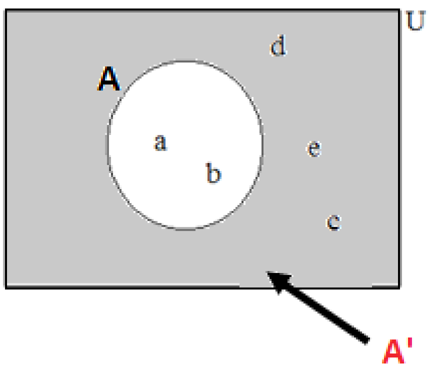
Example

Given that $U = \{a, b, c, d, e\}$ and $A = \{a, b\}$, find A' where $A' = U \setminus A$ **is the complement of A.**

Solution

$U = \{a, b, c, d, e\}, A = \{a, b\} \Rightarrow A' = U \setminus A = \{c, d, e\}$.

This can be represented on a Venn diagram as shown by the diagram.



Application activity 1.1.4.

1. Consider these two sets $A=\{2,4,6,8,10\}$ *and* $B=\{2,3,5,7\}$. Represent them in a Venn diagram

2. Let U be the universal set containing all the natural numbers between 0 and 11. Hence, U = {1, 2, 3, 4, 5, 6, 7, 8, 9, 10}. Let P be the set containing all prime numbers between 0 and 11. Thus, P= {2,3,5, 7}. Let E be the set containing all the even numbers between 0 and 11. Hence, E= {2, 4, 6, 8, 10}.

   a. Represent the above situations using a Venn diagram
   b. Comment on the Venn diagram in (a)

3. Consider the sets $A=\{1,\ 2,\ 3,\ 5,\ 6,\ 8\}$, $B=\{2,\ 4,\ 6,\ 8\}$ and $C=\{1,\ 3,\ 5,\ 7\}$. Find and draw the Venn diagrams for:

   $(a)\ B\cap C$ $(b)\ A\cap C$
   $(c)\ A\cap B$ $(d)\ B\cup C$
   $(e)\ A\cup B.$

4. Given $U=\{letters\ of\ the\ word\ elephatiasis\}$,

   Set $A=\{all\ vowels\}$,

   Set $B=\{first\ five\ letters\ of\ the\ English\ alphabet\}$.Find:

   $a)\ A$
   $b)\ B$
   $c)\ A-B$
   $d)\ B-A$
   $e)\ \mathrm{A}\Delta\mathrm{B}$

5. If $U=\{a,\ e,\ i,\ o,\ u,\ c,\ d\}$, $X=\{a,\ b,\ e\}$ *and* $Y=\{c,\ d,\ e\}$, find:

   $(a)\ (X\cap Y)'$
   $(b)\ (X\cup Y)'$

1.1.5. Applications of set theory in finance and economics related problems

Activity 1.1.5.

1. A survey was carried out in a shop to find out the number of customers who bought bread or milk or both or neither. Out of a total of 79 customers for the day, 52 bought milk, 32 bought bread and 15 bought neither.

   a) Without using a Venn diagram, find the number of customers who:

      i. bought bread and milk

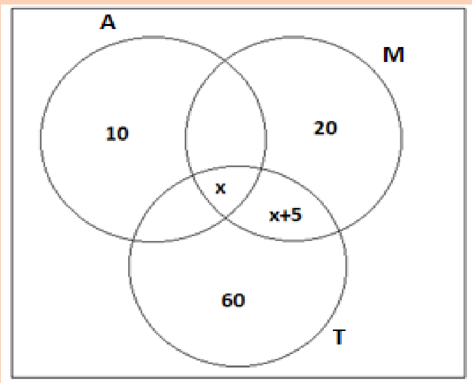
      ii. bought bread only

      iii. bought milk only

   b) With the aid of a Venn diagram, work out (i), (ii) and (iii) in the question (a) above.

   c) Which of the methods in (a) and (b) above is easier to work with? Give reasons for your answer.

2. The Venn diagram below shows the number of senior four students in a school who like Mathematics (M), Auditing (A) and Taxation (T) at the end of 3rd term. Some like more than one subjects and in total 55 students like Mathematics.



   a) How many students like the three subjects?

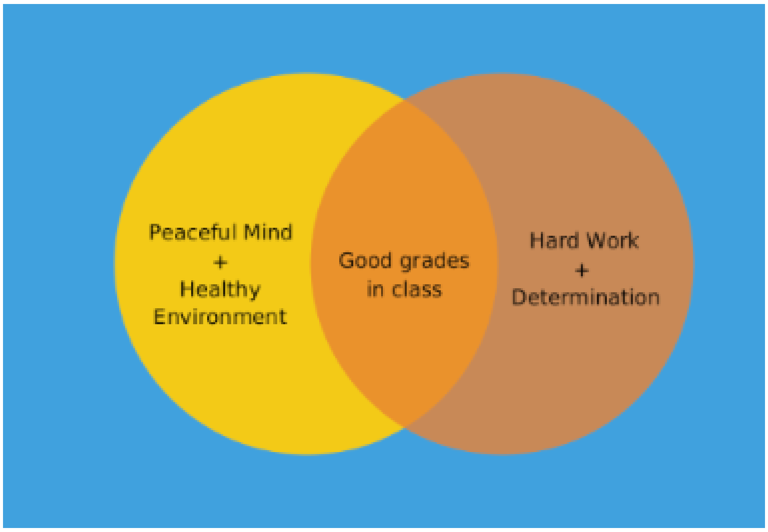
   b) Find the total number of senior four students in the school.

   c) How many students like Auditing and Taxation only?

1. **Analysis, interpretation, and presentation of a problem using Venn diagram**

Venn diagrams are great for comparing things in a visual manner and to quickly identify overlaps. They are diagrams containing circles that show the logical relations between a collection of sets or groups.

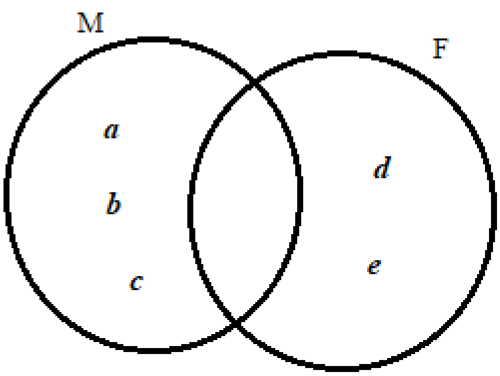
Understanding a student's interaction in class



Venn diagrams are used in many areas of life such as finance, production, and economics where people need to categorize or group items, as well as compare and contrast different items. Venn diagrams are primarily a thinking tool, and they can also be used for assessment.

Example: In a workshop about small business, there are 5 people a,b,c,d,e attended the workshop. Out of these people, $a,b\ and\ c$ are Males while $d\ and\ e$ are Females. Also, $a\ and\ e$ study Accounting while $b,c\ and\ d$ study Financial management.

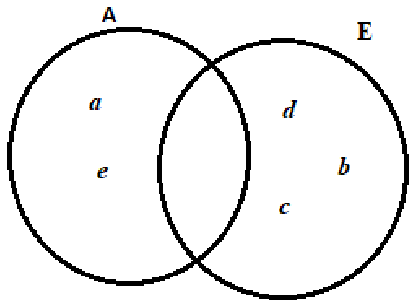
- The set of males is M = {a, b, c} and the set of females is F = {d,e}



If we consider sets M and F, there is no common element between them. Hence, M ∩ F = ϕ

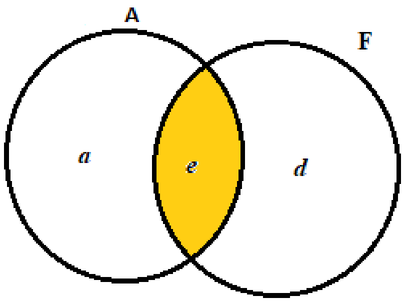
Such sets which have no elements in common are called disjoint sets.

- The set of people in Accounting is A = {a, e} and the set of people in Financial management (E) is E = {b, c, d}



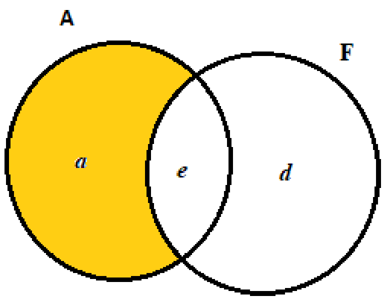
- If we wish to find out all female students who have taken Accounting, we need to find out what is common in set F and set A. This is called an intersection of set F and set A, and is denoted by F ∩ A.

  Here, F ∩ A = {e}



Thus, an intersection of two sets is formed by the elements which are common to both sets.

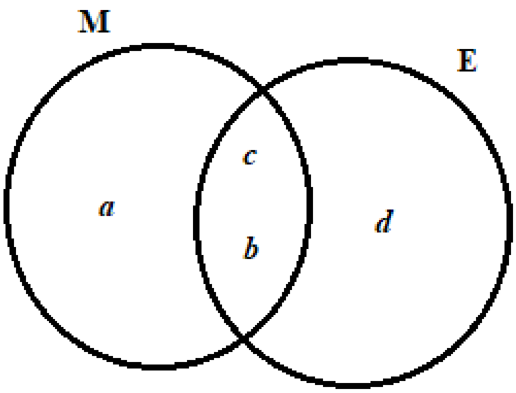
- If we wish to find out those females who have not taken Accounting. Here, we have to check the set F and remove all elements of set A presented in this set. This is called the difference between two sets. A - F = {a}



Thus, difference of set A and set F is defined as the set of all elements present in A but not in F.

A - F = {x | x (A and x ∉ F}

- If we wish to represent a set containing" either males or Financial management students (E) or both". This would mean taking all the elements from set M and set E together into one set. This is called the union of set M and set E and is denoted by $M \cup E$. Thus, $M \cup E = \{a, b, c, d\}$



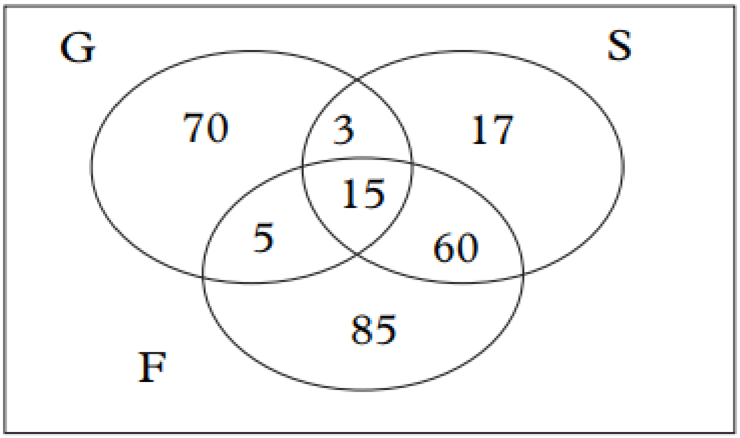
Note:

Though b and c exist in both sets E and M , they are written only once while writing the union. This is because no element is ever written twice in a set.

A Venn diagram plays a very important role in analyzing the set problem and helps in solving the problem very easily. To well perform the task of solving problems using Venn diagrams, we first express the data in terms of set notations and then fill the data in the Venn diagram for easy solution. Some important facts like "intersection", "union" and "complement" should be well considered and represented when drawing Venn diagrams.

2. Modelling and solving problems that involve Set operations using Venn diagram

Example 1: Consider the Venn diagram showing the numbers of people doing business about selling foods like fishes (F), Green tomatoes (G) and Sardines (S) in the district. Apply Venn diagrams to find out total number of people who doing business of selling food in the district



The total number of people doing business of selling food in the district is given by the union of the three sets as shown by the following formula.

$$n(G\cup S\cup F)=n(G)+n(S)+n(F)-\{n(G\cap S)+n(G\cap F)+(S\cap F)+n(G\cap S\cap F)\}$$

From the Venn diagram above, it is clear that:

$$n(G\cup S\cup F)=93+95+165-(18+20+20+75)+15=353-113+15=255$$

So, the total number of people who doing business of selling food in the district is 255

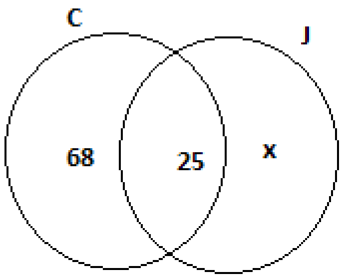
Example 2:

At a shop, 93 people chose to buy coffee and 47 people chose to buy juice, 25 chose to buy both coffee and juice. If each person chose at least one of these beverages, how many people visited the shop. Present the answer by using a Venn diagram

Solution:

Number of people who chose coffee is 93, number of people who chose juice is 47 and number of people who chose both coffee and juice is 25

Let x represent the number of people who chose to buy juice only

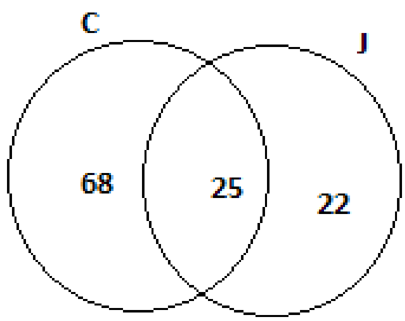


Now, we need to subtract 25 from both classifications.

Number of people who chose to buy only coffee is 93 - 25 = 68

Number of people who chose to buy only juice is 47 - 25 = 22

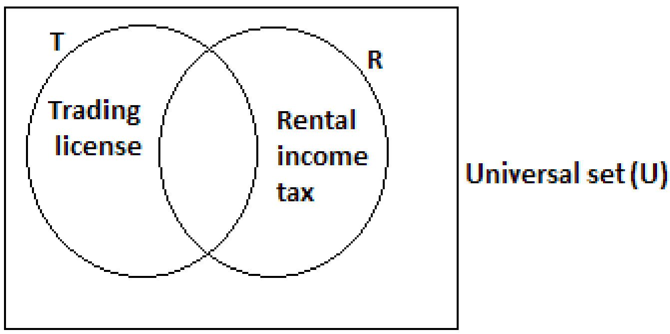
Representation of answer on a Venn diagram



People visited the shop are 68+25+22=115 People

Example 3:

150 small business tax payers were interviewed about taxes payment. 85 were registered for a trading license, 70 were registered for rental income tax while 50 were registered for both trading license and rental income.



Model this problem using variables and Venn diagram to find out the following:

- How many taxpayers signed up only for trading license?
- How many taxpayers signed up only for rental income taxes?
- How many taxpayers signed up for trading license or rental income taxes?
- How many taxpayers signed up for neither trading license nor rental income taxes?

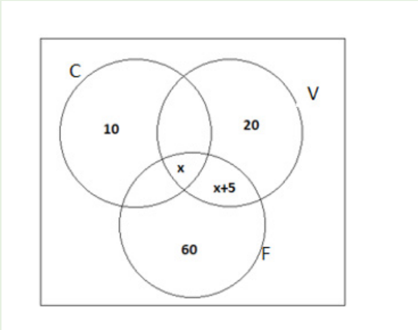
Solution

- Let x be the number of taxpayers who signed up for both trading license and rental income taxes.
- The number of taxpayers who signed up only for trading license $85 - x$. Knowing that $x = 50$, taxpayers who signed up only for trading license is 35.
- The number of taxpayers who signed up only for rental income tax is $70 - x$. Knowing that $x = 50$, taxpayers who signed up only for rental income taxes is 20.
- The number of taxpayers who signed up for trading license or rental income tax is given by the total number of all taxpayers in both sets. This is 35 +50 + 20 = 105
- The number of taxpayers who signed up for neither trading license or rental income tax is given by the total number of all tax payers who were interviewed minus the total number of all tax payers in both sets. This is given by 150 - 105 = 45

Application activity 1.1.5.

1. In a competition about taxpayers, an institution in charge awarded different categories of taxpayers, 36 medals in rental income tax payers, 12 medal in taxes on property and 18 in taxes on goods and services. If these medals went to a total of 45 persons and only 4 persons got medals in all the three categories, how many received medals in exactly two of these categories? Present on Venn diagram

2. The Venn diagram below shows the number of customers in shop who buys different goods include vegetables (V), Cell phones (C) and fruits (F). Some bought more than one good. In total 55 customers buy vegetables.



a) How many customers who buy the three goods?

b) Find the total number of customers.

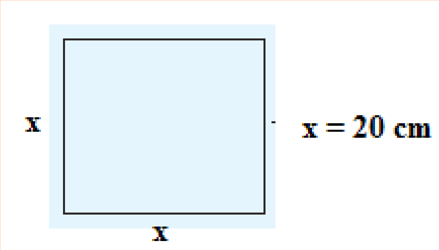
c) How many customers who bought cell phone and fruits only?

1.2. Indices/powers/exponents, Surds and absolute value, decimal and Naperean logarithms, and applications

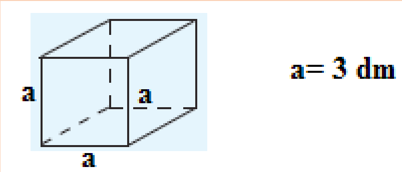
1.2.1. Indices/powers/exponents

Activity 1.2.1.

1. How can you find the area of the following paper in the form of a square?



2. Given a cube of the following form:



Determine the volume of this cube.

1. Meaning of power/exponent/indices

We call n^{th} power of a real number *a* $(a \neq 0)$ that we note a^n, the product of *n* factors of *a.* that is

$$a^n = \underbrace{a.a.a...a}_{n\ factors} \begin{cases} n \text{ is an exponent} \\ a \text{ is the base} \end{cases}$$

Examples:

i. $2^4 = \underbrace{2.2.2.2}_{4\ factors}$ ii. $3^3 = \underbrace{3.3.3}_{3\ factors} = 27$

Note:

- $a^1 = a$
- $a^0 = 1, a \neq 0$
- If $a = 0, a^0$ is not defined

2. Properties of powers/indices/exponents

Let $a,b \in \mathbb{R}$ but $a,b \neq 0$ and $m,n \in \mathbb{R}$

a) $a^m \cdot a^n = a^{m+n}$

In fact, $a^m \cdot a^n = \underbrace{a \cdot a \cdot a \cdots a}_{m\ factors} \times \underbrace{a \cdot a \cdot a \cdots a}_{n\ factors} = \underbrace{a \cdot a \cdot a \cdots a}_{m+n\ factors} = a^{m+n}$

b) $\left(a^m\right)^n = a^{mn}$

In fact, $\left(a^m\right)^n = \underbrace{\underbrace{a \cdot a \cdot a \cdots a}_{m\ factors} \times \underbrace{a \cdot a \cdot a \cdots a}_{m\ factors} \cdots \underbrace{a \cdot a \cdot a \cdots a}_{m\ factors}}_{n\ factors} = a^{mn}$

c) $\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$

In fact,

$$\left(\frac{a}{b}\right)^m = \underbrace{\frac{a}{b} \cdot \frac{a}{b} \cdot \frac{a}{b} \cdots \frac{a}{b}}_{m\ factors} = \frac{\overbrace{a \cdot a \cdot a \cdots a}^{m\ factors}}{\underbrace{b \cdot b \cdot b \cdots b}_{m\ factors}} = \frac{a^m}{b^m}$$

d) $\frac{1}{b^m} = b^{-m}$

In fact,

$$\frac{1}{b^m} = \frac{1}{b^m}^m = \left(\frac{1}{b}\right)^m = \left(b^{-1}\right)^m = b^{-m}$$

e) $\frac{a^m}{a^n} = a^{m-n}$

In fact,

$$\frac{a^m}{a^n} = a^m \frac{1}{a^n} = a^m a^{-n} = a^{m-n}$$

f) $(ab)^m = a^m b^m$

In fact,

$$(ab)^m = \underbrace{ab \cdot ab \cdots ab}_{m\ factors} = \underbrace{a \cdot a \cdots a}_{m\ factors} \times \underbrace{b \cdot b \cdots b}_{m\ factors} = a^m b^m$$

Note: There is no general way to simplify the sum of powers, even when the powers have the same base. For instance, $2^5 + 2^3 = 32 + 8 = 40$, and 40 is not an integer power of 2. But some products or ratios of powers can be simplified using repeated multiplication models of a^n.

Example:

a) $2^4 \cdot 2^3 \cdot 4 = 2^4 \cdot 2^3 \cdot 2^2 = 2^9 = 512$

b) $a^4 \cdot b^3 \cdot a^5 \cdot b^8 = a^4 \cdot a^5 \cdot b^3 \cdot b^8 = a^9 \cdot b^{11}$

$a^9 \cdot b^{11}$ cannot be simplified further because the bases are different.

$$\frac{y^9}{y^2} = y^{9-2} = y^7$$

Application activity 1.2.1.

3) Simplify

a) x^3x^2 b) $(xy^3)^2 + 4x^2y^6$ c) $\frac{6xy^2}{3xy}$ d) $\frac{ab}{a^3} - \frac{a^3b^2}{a^5b}$ e) $\frac{yx}{4xy}$

2) Referring to your real-life experience, where powers are used?

1.2.2. Surds and absolute value

1.2.2.1. Surds

Activity 1.2.2.1.

1) Evaluate the following powers without using a calculator

a) $\sqrt{81}$ b) $(216)^{\frac{1}{3}}$ c) $(-27)^{\frac{1}{3}}$ d) $\sqrt[4]{16}$

2) Simplify the following expressions

a. $\frac{\sqrt{3}-\sqrt{2}}{\sqrt{3}+\sqrt{2}}$

b. $\frac{4-\sqrt{6}}{\sqrt{2}}$

1. Definition

The n^{th} root of a real number is $\frac{1}{n}$ power of that real number. It is noted by $\sqrt[n]{b}$,

$b \in \mathbb{R}, n \in \mathbb{N} \setminus \{1\}$. $\forall a, b \in \mathbb{R}, \sqrt[n]{b} = a \Leftrightarrow b^{\frac{1}{n}} = a \Leftrightarrow b = a^n$.

$$\forall a, b \in \mathbb{R}, \sqrt[n]{b} = a \Leftrightarrow b^{\frac{1}{n}} = a \Leftrightarrow b = a^n \begin{cases} n \text{ is called index} \\ b \text{ is called base or radicand} \\ \sqrt[n]{\ } \text{ is called the radical sign} \end{cases}$$

If $n = 2$, we say square root $\sqrt[2]{b}$ and is written as $\sqrt{b}$. Here *b* must be a positive real number or zero.

If $n = 3$, we say cube root and is written as $\sqrt[3]{b}$. Here *b* can be any real number.

If $n = 4$, we say 4th root and is written as $\sqrt[4]{b}$. Here *b* must be a positive real number or zero.

Generally, for any n ($n \neq 0$ and $n \neq 1$), we say n^{th} root and is written as $\sqrt[n]{b}$. Here if *n* is even, *b* must be a positive real number or zero and if n is odd *b* can be any real number.

Examples

a) $\sqrt[3]{27} = (27)^{\frac{1}{3}} = \left(3^3\right)^{\frac{1}{3}} = 3^{3 \times \frac{1}{3}} = 3$

b) $\sqrt[4]{16} = (16)^{\frac{1}{4}} = \left(2^4\right)^{\frac{1}{4}} = 2$

Many numbers are not exact powers. Their roots are, therefore, irrational. Expressions containing roots of such numbers are called **surds**. **Surds** are the square roots of numbers that cannot be further simplified into whole or rational number. The examples of surds are $\sqrt{2}, \sqrt{3}, \sqrt{5}, 1 + \sqrt{2}$,etc., as these values cannot be further simplified. There are different types of surds:

- **Simple Surds:** Surds that has only one term. Example: √2, √5, …
- **Pure Surds:** Surds which are completely irrational. Example: √3
- **Similar Surds:** Surds having the same common surds factor.
- **Mixed Surds:** Surds that are not completely irrational and can be expressed as a product of a rational number and an irrational number.
- **Compound Surds:** An expression which is the addition or subtraction of two or more surds.
- **Binomial Surds:** Surds that are made of two other surds.

A surd is said to be in its simplest form when the number under the radical $\sqrt{\ }$ is a prime number.

2. Surds properties

a. $\sqrt{a \times b} = \sqrt{a} \times \sqrt{b}$

Example: $\sqrt{18} = \sqrt{9 \times 2} = \sqrt{9} \times \sqrt{2} = 3\sqrt{2}$

b. $\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$

Example: $\sqrt{\frac{12}{121}} = \frac{\sqrt{12}}{\sqrt{121}} = \frac{\sqrt{4 \times 3}}{\sqrt{11 \times 11}} = \frac{2\sqrt{3}}{11}$

3. Operations on surds

Addition and subtraction of surds

To be able to add or subtract surds, they must contain roots of the same number. In general,

$a\sqrt{x} \pm b\sqrt{x} = (a+b)\sqrt{x}$.

Example: $2\sqrt{5} + 10\sqrt{5} = (2+10)\sqrt{5} = 12\sqrt{5}$ and $3\sqrt{2} - 5\sqrt{2} = (3-5)\sqrt{2} = -2\sqrt{2}$

But in $7\sqrt{2} + 2\sqrt{5}$; there is no common factor just like there is no common factor 7x+2y. Therefore, $7\sqrt{2} + 2\sqrt{5}$ cannot be simplified further.

Multiplication of surds

When two monomial surds must be multiplied together.

1. First simplify each surd where possible, and then
2. Multiply whole numbers together and surds together.

Example: $\sqrt{32} \times \sqrt{75} = \sqrt{16 \times 2} \times \sqrt{25 \times 3} = 4\sqrt{2} \times 5\sqrt{3} = 20\sqrt{2} \times \sqrt{3} = 20\sqrt{6}$

Division of surds and rationalizing the denominator

If a fraction has a surd in the denominator, it is usually better to **rationalize the denominator. Rationalizing the denominator** means making the denominator a rational number, so that we divide by a rational number rather than divide by a surd. When rationalizing, we multiply both the numerator and the denominator of the fraction by a surd which makes the denominator rational.

It is easier to divide by a rational number than a surd.

Consider the fraction $\frac{a}{\sqrt{b}}$. To rationalize the denominator, we multiply both the numerator and denominator by $\sqrt{b}$ i.e. $\frac{a}{\sqrt{b}} = \frac{a \times \sqrt{b}}{\sqrt{b} \times \sqrt{b}} = \frac{a\sqrt{b}}{b}$.

Generally, to rationalize a denominator, we multiply both the numerator and the denominator by the conjugate of the denominator. Some examples of conjugate are:

The conjugate of $a \pm \sqrt{b}$ is $a \mp \sqrt{b}$

The conjugate of $\sqrt{a}$ is $\sqrt{a}$

The conjugate of $a\sqrt{b}$ is $\sqrt{b}$. Remember that $(a+b)(a-b) = a^2 - b^2$

Example:

a) $\frac{1}{1+\sqrt{2}} = \frac{1-\sqrt{2}}{(1+\sqrt{2})(1-\sqrt{2})} = \frac{1-\sqrt{2}}{1-2} = \frac{1-\sqrt{2}}{-1} = \sqrt{2}+1$

b) $\frac{\sqrt{2}}{\sqrt{5}-\sqrt{3}} \aleph \frac{\sqrt{2}(\sqrt{5}+\sqrt{3})}{(\sqrt{5}-\sqrt{3})(\sqrt{5}+\sqrt{3})} \quad \frac{\sqrt{10}+\sqrt{6}}{5-3} \quad \frac{\sqrt{10}+\sqrt{6}}{2}$

$\frac{\sqrt{3}+\sqrt{7}}{4\sqrt{2}} = \frac{(\sqrt{3}+\sqrt{7})\sqrt{2}}{(4\sqrt{2})\sqrt{2}} = \frac{\sqrt{6}+\sqrt{14}}{8}$

Application activity 1.2.2.1.

Rationalize the denominator

a) $\frac{2\sqrt{2}}{4+3\sqrt{3}}$ b) $\frac{a-\sqrt{b}}{\sqrt{d}}$ c) $\frac{3\sqrt{3}+2\sqrt{2}}{1+2\sqrt{2}}$

1.2.2.2. Absolute value

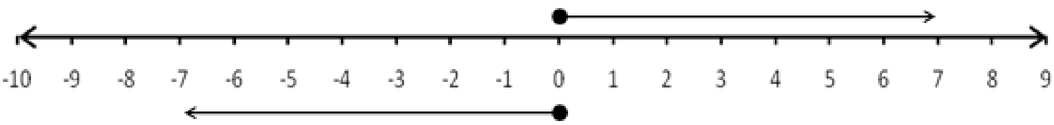
Activity 1.2.2.2.

1) Draw a number line and state the number of units is between
   a) 0 and -8 b) 0 and 8
   c) 0 and $\frac{1}{2}$ d) 4 and 17
2) Do you think that a distance can be expressed by a negative number? Explain

Absolute value of a number is the distance of that number from the origin (zero point) on a number line. The symbol $|\ |$ is used to denote the absolute value.

Example:

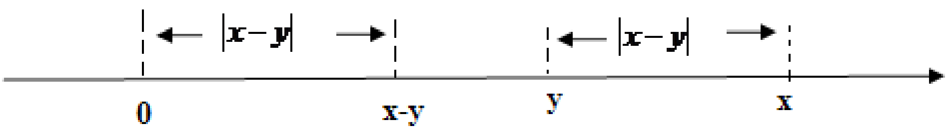
7 is at 7 units from zero, thus the absolute value of 7 is 7 or $|7| = 7$. Also -7 is at 7 units from zero, thus the absolute value of -7 is 7 or $|-7| = 7$. So $|-7| = |7| = 7$ since -7 and 7 are on equal distance from zero on number line.



Note:

- The absolute value of zero is zero
- The absolute value of a non-zero real number is a positive real number.
- Given that $|x| = k$ where k is a positive real number or zero, then $x = -k$ or $x = k$.
- $|x| = \begin{cases} x, & if\ x \geq 0 \\ -x, & if\ x < 0 \end{cases}$
- **Geometrically**, $|x|$ represents the nonnegative distance from x to 0 on the real line. More generally, $|x-y|$ represents the nonnegative distance between the point x and y on the real line, since this distance is the same as that from the point $x - y$ to 0.(See the following figure)

$$|x-y| = \begin{cases} x-y, if\ x \geq y \\ -(x-y) = y-x, if\ x \leq y. \end{cases}$$



Example:

Find x in the following:

a) $|x| = 5$ b) $|x| + 5 = 1$ c) $|x-4| = 10$

Solution

a) $|x| = 5,\ x = -5 \quad or \quad x = 5$

b) $|x| + 5 = 1$

$\Leftrightarrow |x| = 1 - 5 \Rightarrow |x| = -4$

There is no value of *x* since the absolute value of *x* must be a positive real number.

c) $|x - 4| = 10$

$x - 4 = -10 \;\; or \;\; x - 4 = 10$

$x = -10 + 4 \;\; or \;\; x = 10 + 4$

$x = -6 \;\; or \;\; x = 14$

Example:

Simplify

a) $-|40 - 12|$ b) $|4(-3) - (2)(5)|$ c) $|-4(-2)|$

Solution

a) $-|40 - 12| = -|28| = -28$

b) $|4(-3) - (2)(5)| = |-12 - 10| = |-22| = 22$

c) $|-4(-2)| = |8| = 8$

Properties of the Absolute Value

1. Opposite numbers have equal absolute value. $|a| = |-a| = a$

Example: $|5| = |-5| = 5$

2. The absolute value of a product is equal to the product of the absolute values of the factors.

$|ab| = |a||b|$

Example:

$|4(-6)| = |4||-6|$

$|4(-6)| = |-24| = 24$

$|4||-6| = 4 \times 6 = 24$

3. The absolute value of a sum is less than or equal to the sum of the absolute values of the ends.

$|a+b| \le |a| + |b|$

Example:

$|-3+2| \le |-3| + |2|$

$|-1| \le 3 + 2$

$1 \le 5$

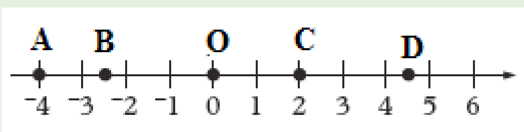
Application activity 1.2.2.2.

1. Find the value(s) of x: a) $|x| = 6$ b) $|x-3| - 4 = 2$
2. Simplify

1) $|-4||-5|$ 2) $|-7| + |4|$ 3) $-|4 \times 6|$ 4) $-|-6+8|$

3. Let a and b be the coordinates of points A and B, respectively, on a coordinate line. The distance between A and B, denoted by d (A, B) is defined by $d(A,B) = |b-a|$.

   Refer to the figure below, determine the distance d(C,B) and d(O,A).



1.2.3. Decimal and Naperian logarithms

Activity 1.2.3.

1) What is the real number at which 10 must be raised to obtain:
   a) 1 b) 10 c) 100 d) 1000 e)10000
   f) 100 000
2) Explain to your classmate how you can find the number x if $x^3 = 64$.

Decimal logarithms

1. Definition

The **decimal logarithm** of a positive real number *x* is defined to be a real number *y* for which 10 must be raised to obtain *x*. we write $\forall x>0, y=\log x$. $\log x$ is the same as $\log_{10} x$ and is defined for all positive real numbers only. 10 is the base of this logarithm. In general notation we do not write this base for decimal logarithm.

In the notation $y=\log x$, *x* is said to be the antilogarithm of *y*. Since logarithms are defined using exponentials, any "log *x*" has an equivalent "exponent" form, and vice-versa.

Example:

a. $\log_{10} 0.001=-3$ since $10^{-3}=0.001$ $\log 100=2$ since $10^2=100$

Therefore, $y=\log x$ means $10^y=x$

Note: $\log 2x+1 \neq \log(2x+1)$ but $\log 2x+1=(\log 2x)+1$

2. Properties of decimal logarithms

$\forall a,b \in]0,\infty[$, the following properties displayed

a) $\log ab=\log a+\log b$

b) $\log \frac{a}{b}=\log a-\log b$

c) $\log \frac{1}{b}=\log 1-\log b=-\log b$

Note: $10^0=1$ and $10^1=10$, therefore $\log 1=0$ and $\log 10=1$. Furthermore, for any base b we have $b^0=1$ and $b^1=b$. Hence $\log_b 1=0$ and $\log_b b=1$. The logarithm of negative number and zero are not defined.

a) $\log a^n=n\log a$

b) $\log \sqrt{a}=\log a^{\frac{1}{2}}=\frac{1}{2}\log a$

c) $\log \sqrt[n]{a}=\log a^{\frac{1}{n}}=\frac{1}{n}\log a$

Examples:

1) Calculate in function of $\log a, \log b$ *and* $\log c$

a) $\log a^2b^2$

b) $\log \frac{ab}{c}$

c) $\log \frac{ab}{\sqrt{c}}$

Solution

a) $\log a^2b^2 = \log(ab)^2$

$= 2\log ab$

$= 2(\log a + \log b)$

b) $\log \frac{ab}{c} = \log ab - \log c$

$= \log a + \log b - \log c$

c) $\log \frac{ab}{\sqrt{c}} = \log ab - \log \sqrt{c}$

$= \log a + \log b - \log(c)^{\frac{1}{2}}$

$= \log a + \log b - \frac{1}{2}\log c$

2) Given that $\log 2 = 0.30$, $\log 3 = 0.48$ and $\log 5 = 0.7$. Without using a calculator, find

a) $\log 6$

b) $\log 0.9$

Solution

a) $\log 6 = \log(2\times 3)$

$= \log 2 + \log 3$

$= 0.30 + 0.48$

$= 0.78$

b) $\log 0.9 = \log \frac{9}{10}$

$= \log 9 - \log 10$

$= \log 3^2 - \log(2\times 5)$

$= 2\log 3 - \log 2 - \log 5$

$= 2(0.48) - 0.30 - 0.7$

$= -0.04$

Co-logarithm

Co-logarithm, sometimes shortened to **colog** of a number is the logarithm of the reciprocal of that number, equal to the negative of the logarithm of the number itself, $co\log x = \log\left(\frac{1}{x}\right) = -\log x$

Example:

$\text{colog } 200 = -\log 200 = -2.3010$

Change of base formula

If u $(u > 0)$ and if a and b are positive real numbers different from 1, $\log_b u = \dfrac{\log_a u}{\log_a b}$. This means that if you have a logarithm in any other base, you can convert it in the decimal logarithm in the following way where $a = 10$:

$\log_b u = \dfrac{\log_{10} u}{\log_{10} b} = \dfrac{\log u}{\log b}$. This is for example: $\log_2 5 = \dfrac{\log_{10} 5}{\log_{10} 2} \approx 2.322$

Note: Let b be a positive number. The logarithm of any positive number x to be the base b, written as $\log_b x$ represents the exponent to which b must be raised to obtain x. That is, $y = \log_b x$ and $b^y = x$

Example:

a. $\log_2 8 = 3$ since $2^3 = 8$

b. $\log_2 64 = 6$ since $2^6 = 64$

B. Natural logarithm (Naperian Logarithm)

1. Definition

Natural logarithm is another special logarithm which has the base of a number $e \approx 2.71828$. This logarithm is written as $\log_e x = \ln x$. *When* $e^y = x$ *then base* e *logarithm of* x *is* $\ln(x) = \log_e(x) = y$.

2. Properties of Natural logarithms

a. Product rule: $ln(x.y) = ln(x) + ln(y)$.

Example: $ln(3\times7) = ln(3) + ln(7)$

b. Quotient rule: $ln(\frac{x}{y}) = ln(x) - ln(y)$.

Example: $ln(3/7) = ln(3) - ln(7)$

c. Power rule: $lnx^y = ylnx$.

Example: $ln3^8 = 8ln3$

d. $\ln x$ is defined only for positive numbers greater than zero.

e. $ln1 = 0$

f. $\ln e = 1$

Note: The rules for using natural logarithms are the same as for logarithms to any other base

Application activity 1.2.3.

1. Without using calculator, compare the numbers *a* and *b*.
   a) $a = 3\log 2$ and $b = \log 7$
   b) $a = \log 2 + \log 40$ and $b = 4\log 2 + \log 5$
   c) $a = 2\log 2$ and $b = \log 16 - \log 3$
2. Given that $\log 2 = 0.30$, $\log 3 = 0.48$ and $\log 5 = 0.69$. Calculate
   a. $\log 150$ b. $\log \frac{9}{2}$ c. $\log 0.2 + \log 10$
3. Find co-logarithm of
   a) 100 b) 42 c)15
4. Calculate $\ln \frac{1}{\sqrt{e}}$
5. Write as one logarithm $\frac{1}{3}\ln(x-1) - \frac{1}{2}(x+1)$

1.2.4. Application of exponents and logarithms

Activity 1.2.4.

1. Make a research in the library or on internet the use of logarithms in Economics and Finance
2. How long will 70 000Frw take to accumulate to 100 000Frw if it is invested at 11%?
3. Mr. Mateso was surprised when he was told that his money 200Frw in the Bank was raised to a certain power and became 8 000 000Frw after 3 years. Explain how he can discover that power which raised his money.

Both exponents and logarithms are useful in solving financial and economical related problems such as determination of continuous growth rates and compound interest of invested money. This may have done to find out the best time for certain economic activities to take place.

Determination of continuous growth rates

Growth rates refer to the percentage change of a specific variable within specific time period. For example, investors in business, growth rates represent the compounded annual rates of growth of company' s revenue or earning interest. When growth rates continue compounding infinitely at similar rate, it is continuous growth rates.

Example

Over the last 15 years a country's population has risen continuously at the same annual growth rate from 8.2 million to 11.9 million. Calculate this rate of growth?

Solution:

Using the formula for finding a continuous growth rate and entering the known values gives the rate of growth as

$$r = \frac{1}{t}\ln\left(\frac{y}{A}\right) = \frac{1}{15}\ln\left(\frac{11.9}{8.2}\right)$$

$$r = \frac{1}{15}\ln(1.45122) = \frac{1}{15}(0.3724) = 0.02443 = 2.48\%$$

Calculation of compound interest

Compound interest is defined as the interest on the loan or deposit that accumulates on both initial and accumulated interest from previous periods. In making a sum of invested money grow at a faster rate you will earn returns on the invested money as well as on returns at the end of every compounding period, compound interest is needed.

Considering the case in which P is invested in a bank for *n* interest periods per year at the rate r expressed as a decimal. The accumulated amount A after the time t (number of years P is invested) is given by:

$A = P\left(1 + \frac{r}{n}\right)^{nt}$ where *n* is the number of interest periods per year, r is the interest rate expressed as decimal, A is the amount after t years.

Note: When the interest rate is compounded per year, $A = P(1+r)^n$ where r is expressed as a decimal for example $r = 9\% = 0.09$. When the interest rate is compounded monthly, $A = P\left(1 + \frac{r}{12}\right)^{12t}$ where t is the number of years

Example:

For how many Years must 1 000 Frw be invested at 10% in order to accumulate 1 600Frw?

Solution

P= 1 000 Frw, $A = 1\ 600$ Frw, $r = 10\% = 0.1$

As the interest compounded monthly, $A = P\left(1+r\right)^n$ substituting the given values into $\frac{A}{P} = (1+r)^n$ we get, $\frac{1\,600}{1\,000} = (1+0.1)^n$, $1.6 = (1.1)^n$

To find the n^{th} power of a number, logarithm must be multiplied. Our equation becomes $\log 1.6 = \log\left(1.1\right)^n$

$\log(1.6) = n\log(1.1)$ and $n = \frac{\log(1.6)}{\log(1.1)} = 4.93$.

If investments must be made for whole years, then the answer is 5 years. This answer can be checked using the final sum formula $\mathrm{A} = \mathrm{P}(1+r)^n = 1\,000(1.1)^5 = 1\,610.51 \approx 1\,600$. If the 1 000Frw is invested for a full 5 years then it accumulates to just over 1 600 Frw, which checks out with the answer above. A general formula to solve for *n* can be derived as follows from the final sum formula:

$A = P(1+r)^n$, $\frac{A}{P} = (1+r)^n$ and $n = \frac{\log(A/P)}{\log(1+r)}$

An alternative approach is to use the iterative method and plot different values on a spreadsheet. To find the value of *n* for which $1.6 = (1.1)^n$. This entails setting up a formula to calculate the function $y = (1.1)^n$ and then computing it for different values of *n* until the answer 1.6 is reached.

Example 2:

In a bank, a customer saved 10 000Frw on his/her Saving account. The bank promised him/her an interest on savings. How long will it take this amount of 10 000Frw to double it on his/her account earning 2% if compounded quarterly?

Solution

For this problem, we'll use the compound Interest formula, $F = P\left(1+i\right)^n$ where F is the final value, P the initial value of investment.

Since we want to know how long it will take, let t represent the time in years. The number of compounding periods is four times the time or $n = 4t$. The original amount is $P = 10\,000 Frw$ and the future value is the double to become $F = 20\,000 Frw$.

The interest rate per period is $i = \frac{0.02}{4} = 0.005$

When these values are substituted into the compound interest formula, we get the exponent.

$$20\,000Frw = 10\,000Frw(1+0.005)^{4t}$$

To solve for *t*, isolate the exponential factor by dividing both sides by $10\,000Frw$ to give $2=(1.005)^{4t}$

By introducing a decimal logarithm on both sides, then $\log 2 = 4t\log(1.005)$.

Therefore, $4t = \dfrac{\log 2}{\log 1.005} = 138.975$ which gives $t = \dfrac{138.975}{4} \approx 34.7\,years$.

It is interesting to note that the starting amount is irrelevant when doubling. If we started with $P\,Rwandan\ francs$ and wanted to accumulate 2P at the same interest's rate and compounding periods, we would need to solve $2P = P(1.005)^{4t}$. This means it takes about 37.4 years to double any amount of money at an interest rate of 2% compounded quarterly.

Application activity 1.2.4.

1. An initial investment of £50 000 increases to £56 711.25 after 2 years. What interest rate has been applied?
2. How long does it take to double $1 000 at an annual interest rate of 6.35% compounded monthly?

1.3. End unit assessment

End of unit assessment

1. Simplify the following: $\sqrt[n]{a^{2n+1}}$
2. simplify and turn into radical the following $\sqrt[4]{(4)^{-3}.(2)^{7}} + (6)^{-0.2}.(36)^{2}$
3. Use the properties of logarithms to rewrite each expression as a single logarithm:

   a. $2\log_b x + \frac{1}{2}\log_b(x+4)$

   b. $4\log_b(x+2) - 3\log_b(x-5)$

   c. $\log_b\left(\dfrac{x\sqrt{y}}{z^5}\right)$

4. In a group of 100 tax payers, 72 tax payers can pay their taxes through cell telephone and 43 can pay through Bank. It is obviously clear that a certain number of tax payers pay using both cell phone and bank. How many can pay their taxes through cell phone only? How many can pay their taxes through Bank only and how many can pay their taxes using both cell telephone and Bank? Present findings
5. How long will it take $30 000 to accumulate to $110 000 in a trust that earns a 10% annual return compounded semi-annually?
6. How long will it take our money to triple in a bank account with an annual interest rate of 8.45% compounded annually?

UNIT 2 NUMERICAL FUNCTIONS, EQUATIONS, AND INEQUALITIES

Key unit competence: Solve production, financial and economical related problems using numerical functions, equations, and inequalities

2.0. Introductory activity

Introductory activity

1. If x is the number of pens for a learner, the teacher decides to give him/her two more pens. What is the number of pens will have a learner with one pen?

   a) Complete the following table of value to indicate the number $y = f(x) = x + 2$ of pens for a learn who had x pens for $x \geq 0$.

x	-2	-1	0	1	2	3	4
$y = f(x) = x + 2$			2				
(x,y)			(0,2)				

   b) Use the coordinates of points obtained in the table and plot them on the Cartesian plan.

   c) Join all points obtained. What is the form of the graph obtained?

   d) Suppose that instead of writing $f(x) = x + 2$ you write the equation $y = x + 2$. Is this equation a linear equation or a quadratic equation? What is the type of the inequality "$x + 2 \geq 0$"?

2. Suppose that average weekly household expenditure on food C depends on average net household weekly income Y according to the relationship $C = 12 + 0.3Y$.

   a) Can you find a value of Y for which C is not a real number?

   b) Complete a table of value from Y= 0 to Y= 10 and use it to draw the graph of $C = 12 + 0.3Y$

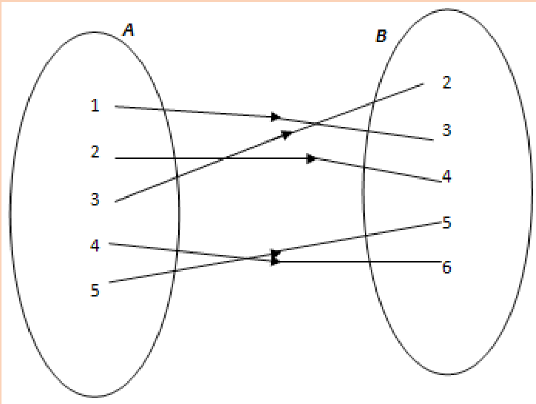
   c) If Y=90, what is the value of C?

2.1. Numerical functions

2.1.1. Generalities on Numerical functions

Activity 2.1.1.

In the following arrow diagram, for each element of set A, states which element of B is mapped to it.



a) What is the set of elements of A which have images in B?

b) Determine the set of elements in B which have antecedent in A.

c) Is there any element of A which has more than one image?

A function is a rule that assigns to each element in a set *A* one and only one element in set *B*. We can even define a function as any relationship which takes one element of one set and assigns to it one and only one element of second set.

If *x* is an element in the domain of a function *f*, then the element that *f* associates with *x* is denoted by the symbol $f(x)$ (**read *f* of *x***) and is called the **image of *x* under *f*** or the **value of *f* at *x***.

The set of all possible values of $f(x)$ as *x* varies over the domain is called the **range** of *f* and it is denoted $R(f)$. The set of all values of A which have images in B is called Domain of f and denoted $Domf$.

We shall write $f(x)$ to represent the image of *x* under the function *f*. The letters commonly used for this purpose are *f*, *g* and *h*.

Example

Given that $f(x)=x^2$, *find the values of* $f(0), f(2), f(3), f(4)$ *and* $f(5)$

Solution

$f(0)=0^2=0$ $f(2)=2^2=4$

$f(3)=3^2=9$ $f(4)=4^2=16$

$f(5)=5^2=25$

Note:

$f(x)=x^2$ can also be written as $f:x\to x^2$ which is read as "*f* is a function which maps *x* onto x^2"

Application activity 2.1.1.

If $f(x)=2x+4$, find

a. $f(2)$
b. $f(-2)$
c. $f(d)$

2.1.2.Types of functions

Activity 2.1.2.

1. Make your own research to differentiate rational from irrational numbers.
2. Refer to the following functions and try to classify them into constant, linear, quadratic, rational or irrational functions. Justify your working steps.

   a. $f(x)=2$
   b. $f(x)=\sqrt{x^2+x-2}$
   c. $f(x)=2x+1$
   d. $h(x)=\dfrac{x^3+2x+1}{x-4}$
   e. $f(x)=x^2$

a) A constant function

A function that assigns the same value to every member of its domain is called **a constant function**. This is $f(x)=c$ where c is a given real number.

Example: The function given by $f(x)=4$ is constant.

b) Polynomial Function

A function that is expressible as the sum of finitely many monomials in *x* is called **polynomial in *x***.

Example: *f(x)* = x^3+4x+7 is a polynomial function. Also *f(x)* = $(x-2)^3$ is a polynomial function in x because it is expressible as a sum of monomials.

In general, *f* is a polynomial function in *x* if it is expressible in the form $f(x)=a_0+a_1x+a_2x^2+......+a_nx^n$ where *n* is a nonnegative integer and $a_0, a_1, ..., a_n$ are real constants.

A polynomial function is called:

- **Monomial** if it has the form cx^n, where *c* is constant and *n* a nonnegative integer.

  Example: $f(x)=2x^2$

- **linear** if it has the form $a_0+a_1x,\ a_1\neq 0$, with degree 1

  Example: $f(x)=2x+1$

- **Quadratic** if it has the form $a_0+a_1x+a_2x^2,\ a_2\neq 0$, with degree 2

  Example: $f(x)=x^2$

- **Cubic** if it has the form $a_0+a_1x+a_2x^2+a_3x^3,\ a_3\neq 0$, with degree 3
- **nth degree polynomial if it has the form**

  $a_0+a_1x^1+a_2x^2+a_3x^n+...+a_nx^n; a_n\neq 0$, **with degree n**

c) Rational function

A function that is expressible as ratio of two polynomials is called **rational function**. It has the form *f(x)*= $\dfrac{a_0+a_1x+a_2x^2+......+a_nx^n}{b_0+b_1x+b_2x^2+......+b_mx^m}$ where *n and m* are nonnegative integers and $a_0, a_1, ..., a_n; b_0, b_1, b_2, ..., b_n$ are real constants.

Example:

$f(x)=\dfrac{x^2+4}{x-1},\ g(x)=\dfrac{1}{3x-5}$ are rational functions

d) Irrational function

A function that is expressed as root extractions is called irrational function. It has the form $\sqrt[n]{f(x)}$, where $f(x)$ is a polynomial and *n* is a positive integer greater or equal to 2.

Example:

$f(x)=\dfrac{\sqrt{x^2+4}}{\sqrt[3]{x-1}}$, $g(x)=\sqrt{\dfrac{x}{x-5}}$ are irrational functions

Application activity 2.1.2.

Classify the following functions into polynomial, rational and irrational functions

1. $f(x)=x^3+2x^2-2$
2. $f(x)=\sqrt{x-1}$
3. $h(x)=\dfrac{x^3+2x^2-2}{x-5}$

2.1.3. Domain of definition for numerical functions

Activity 2.1.3.

For which value(s) the following functions are not defined

1. $f(x)=x^3+2x+1$
2. $f(x)=\dfrac{1}{x}$
3. $g(x)=\dfrac{x+2}{x-1}$
4. $f(x)=\sqrt{2x+1}$

Case 1: The given function is a polynomial: linear, quadratic, cubic, etc.

Given that $f(x)$ is polynomial, then the domain of definition is the set of real numbers. That is $Domf=\mathbb{R}$

Example:

The domain of the function $f(x)=3x^5+2x^4+4x+6$ is $\mathbb{R}$ since it is a polynomial.

Case 2: The given function is a rational function

Given that $f(x)=\dfrac{g(x)}{h(x)}$ where $g(x)$ and $h(x)$ are polynomials, then the domain of definition is the set of real numbers excluding all values where the denominator is zero. That is $Domf=\{x\in\mathbb{R}: h(x)\neq 0\}$

Example:

Given $f(x)=\frac{x+1}{3x+6}$, find the domain of definition.

Solution

Condition: $3x+6\neq 0$

$3x+6=0\Rightarrow x=-2$

Then, $Domf=\mathbb{R}\setminus\{-2\}$ or $Domf=]-\infty-2[\cup]-2,+\infty[$

Case 3: The given function is an irrational function

Given that $f(x)=\sqrt[n]{g(x)}$ where $g(x)$ is a polynomial, there are two cases

a) If *n* is odd number, then the domain is the set of real numbers. That is $Domf=\mathbb{R}$

b) If *n* is even number, then the domain is the set of all values of *x* such $g(x)$ is positive or zero. That is $Domf=\{x\in\mathbb{R}: g(x)\geq 0\}$

Example:

Given $f(x)=\sqrt{x^2-1}$, find domain of definition

Solution

Condition: $x^2-1\geq 0$.

We need to construct sign table to see where x^2-1 is positive

$x^2-1=0\Rightarrow x=\pm 1$

x	$-\infty$		-1		1		$+\infty$
x^2-1		+	0	-	0	+	

Thus, $Domf=]-\infty,-1]\cup[1,+\infty[$

Application activity 2.1.3.

Classify the following functions into polynomial, rational and irrational functions

1. $f(x)=\sqrt{4x-8}$ 2. $f(x)=x^2+5x-6$ 3. $f(x)=\frac{x-2}{x^2-25}$

2.1.4. Graph of linear and quadratic functions

Activity 2.1.4.

Refer to the tables below in the sub-question 1 and work out the questions that follow:

1. Copy and complete the tables below.

x	-3	-2	-1	0	1	2	3
$y = 2x - 1$							

x	-3	-2	-1	0	1	2	3
$y = x^2 - 1$							

2. Use the coordinates from each table to plot the graphs on separate Cartesian planes.
3. What is your conclusion about the shapes of the graphs?

Linear function

Any function of the form $f(x) = mx + b$, where m is not equal to 0 is called a linear function. The ***domain*** of this function is the set of all real numbers. The ***range*** of the linear function is the set of all real numbers. The graph of f is a line with slope m and y intercept b.

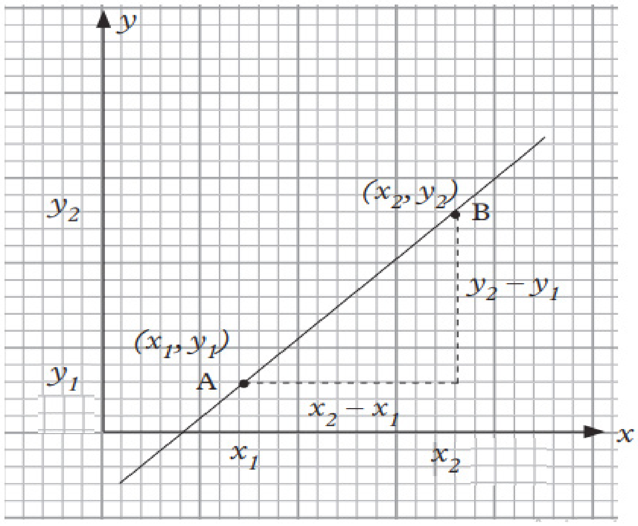
Note: A function $f(x) = b$, where b is a real number is called a constant function. Its graph is a horizontal line at $y = b$.

Examples of linear functions are $y = x + 1, \quad y = 2x - 3, \quad y = -3x + 4, \ldots$

Graphs of linear functions.

The ordered pair (x, y) represents coordinates of any point on the Cartesian plane.

Consider a line passing through the points $A(x_1, y_1)$ and $B(x_2, y_2)$.



From A to B, the change in the x-coordinate (horizontal change) is $x_2 - x_1$ and the change in the y-coordinate (vertical change) is $y_2 - y_1$.

Gradient or slope is equal to $\dfrac{change\ in\ y-coordinates}{change\ in\ x-coordinates} = \dfrac{y_2 - y_1}{x_2 - x_1}$

In the Cartesian plane, the gradient of a line is the measure of its slope or inclination to the x-axis. It is defined as the ratio of the change in y-coordinate (vertical) to the change in the x-coordinate (horizontal).

When drawing a graph of a linear function, it is sufficient to plot only two points and these points may be chosen as the x and y intercepts of the graph. In practice, it is wise to plot three points. If the three points lie on the same line, the working is probably correct, if not you have a chance to check whether there could be an error in your calculation.

If we assign *x* any value, we can easily calculate the corresponding value of *y*.

Determine the *x*-intercept, set *f(x) = 0* and solve for x and then determine the y intercept, set *x = 0* to find *f (0)*.

Consider the function *y = 2x + 3.*

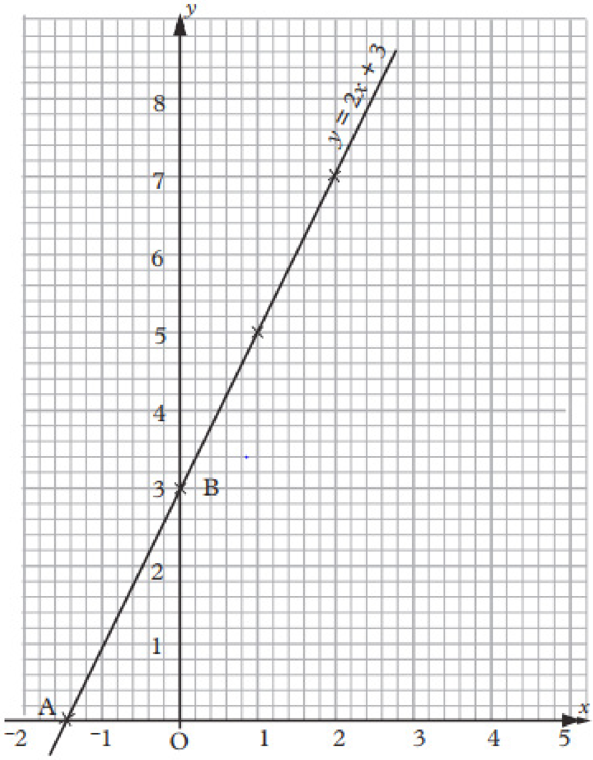
- When x = 0, y = 2 × 0 + 3 = 3
- When x = 1, y = 2 × 1 + 3 = 5
- When x = 2, y = 2 × 2 + 3 = 7 and so on.

For convenience and ease while reading, the calculations are usually tabulated as shown below in the table **of values for *y = 2x + 3***:

x	0	1	2	3	4
$2x$	0	2	4	6	8
$+3$	3	3	3	3	3
$y = 2x + 3$	3	5	7	9	11

From the table, the coordinates *(x, y)* are (0, 3), (1, 5), (2, 7), (3, 9), (4, 11)....

When drawing the graph, the dependent variable is marked on the vertical axis generally known as the y – axis. The independent variable is marked on the horizontal axis also known as the x – axis.



Quadratic function

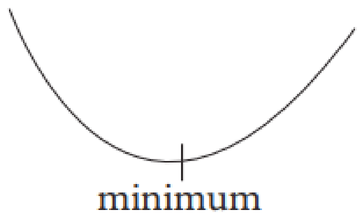
A polynomial in which the highest power of the variable is 2 is called a quadratic function. The expression $y = ax^2 + bx + c$, where a, b and c are constants and $a \neq 0$, is called a quadratic function of x or a function of the second degree (highest power of x is two).

Table of values are used to determine the coordinates that are used to draw the graph of a quadratic function. To get the table of values, we need to have the domain (values of an independent variable) and then the domain is replaced in each quadratic function to find range (values of dependent variables). The values obtained are useful for plotting the graph of a quadratic function. All quadratic function graphs are parabolic in nature.

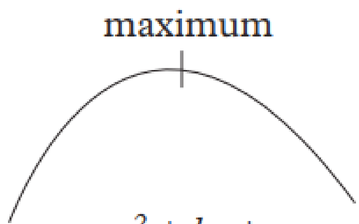
Any quadratic function has a graph which is symmetrical about a line which is parallel to the y-axis i.e., a line $x = h$ where h is constant value. This line is called **axis of symmetry**.

For any quadratic function $f(x) = ax^2 + bx + c$ whose axis of symmetry is the line $x = -\frac{b}{2a}$, we can get the y-coordinate of the vertex by substituting the x-coordinate of axis of symmetry. The vertex becomes $\left(-\frac{b}{2a}, f(-\frac{b}{2a})\right)$. The vertex of a quadratic function is the point where the function crosses its axis of symmetry.

If the coefficient of the x^2 term is positive, the vertex will be the lowest point on the graph, the point at the bottom of the U-shape. If the coefficient of the term x^2 is negative, the vertex will be the highest point on the graph, the point at the top of the ∩-shape. The shapes are as below.



$y = ax^2 + bx + c$
where a is positive



$y = ax^2 + bx + c$
where a is negative

The intercepts with axes are the points where a quadratic function cuts the axes. There are two intercepts i.e. x-intercept and y-intercept. x-intercept for any quadratic function is calculated by letting $y = 0$ and y- intercept is calculated by letting $x = 0$

Graph of a quadratic function

The graph of a quadratic function can be sketched without table of values if the following are known.

- The vertex
- The x-intercepts
- The y-intercept

Example: Find the vertex and axis of symmetry of the parabolic curve $y = 2x^2 - 8x + 6$

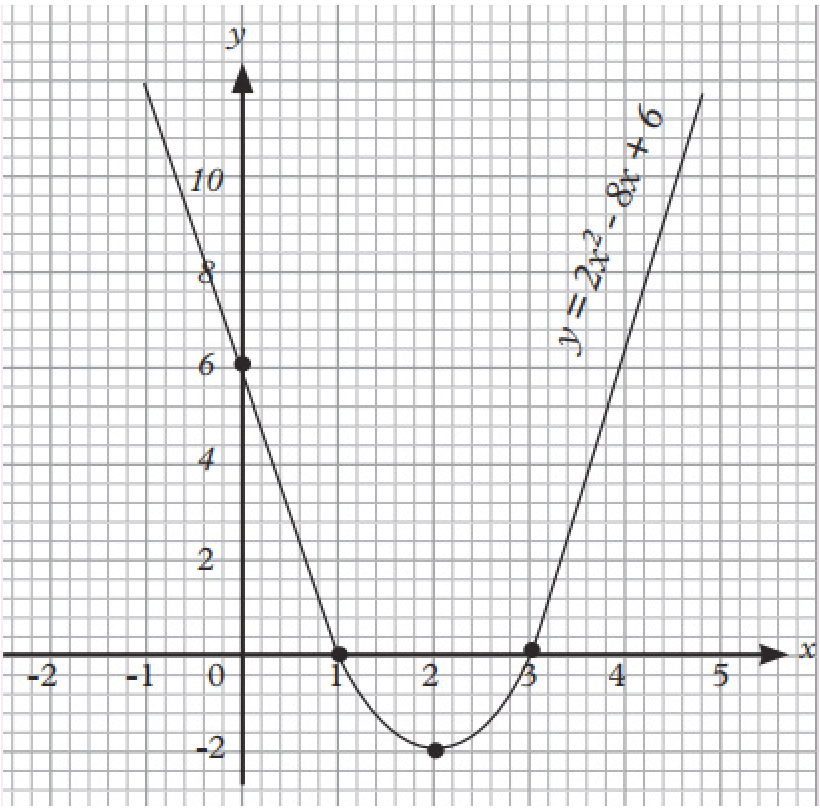
Solution

- The coefficients are a = 2, b = −8 and c = 6
- The x-coordinate of the vertex is $h = -\frac{b}{2a} = \frac{-(-8)}{2(2)} = \frac{8}{4} = 2$

- The y-coordinate of the vertex is obtained by substituting the x-coordinate of the vertex to the quadratic function. We get $y = 2(2)^2 - 8(2) + 6 = -2$
- The vertex is (2, −2) and the axis of symmetry is $x = 2$.
- When x = 0, $y = 2(0)^2 - 8(0) + 6 = 6$.
- The y-intercept is (0, 6)

When y = 0, $0 = 2x^2 - 8x + 6$, we therefore solve the quadratic equation for the values of x and we find the x-intercepts are (1,0) or (3,0)

The graph is as below.



Application activity 2.1.4.

1. Using the table of values sketch the graph of the following functions
   a. y = − 3x + 2
   b. $y = x^2 - 3x + 2$

2. Without tables of values, state the vertices, intercepts with axes, axes of symmetry, and sketch the graphs.

   a) $y = 2x^2 + 5x - 1$

   b) $y = 3x^2 + 8x - 6$

3. The cost C, in Rwandan francs, to produce x units of a product in a certain company is given by $C(x) = 80x + 150$ for $x \geq 0$.

   a. Find and interpret C (10).

   b. How many products can be produced for 15 000 Frw?

   c. Explain the significance of the restriction on the domain, $x \geq 0$.

   d. Find and interpret C (0).

   e. Find and interpret the slope of the graph of $y = C(x)$

2.1.5. Parity of numerical function (odd or even)

Activity 2.1.5.

For each of the following functions, find $f(-x)$ and $-f(x)$. Compare $f(-x)$ and $-f(x)$ using comparison signs ($=$ or $\neq$)

1) $f(x) = x^2 + 3$ 2) $f(x) = \sqrt[3]{x^3 + x}$ 3) $f(x) = \dfrac{x^2 - 3}{x^2 + 1}$

Even function

A function $f(x)$ is said to be **even** if the following conditions are satisfied

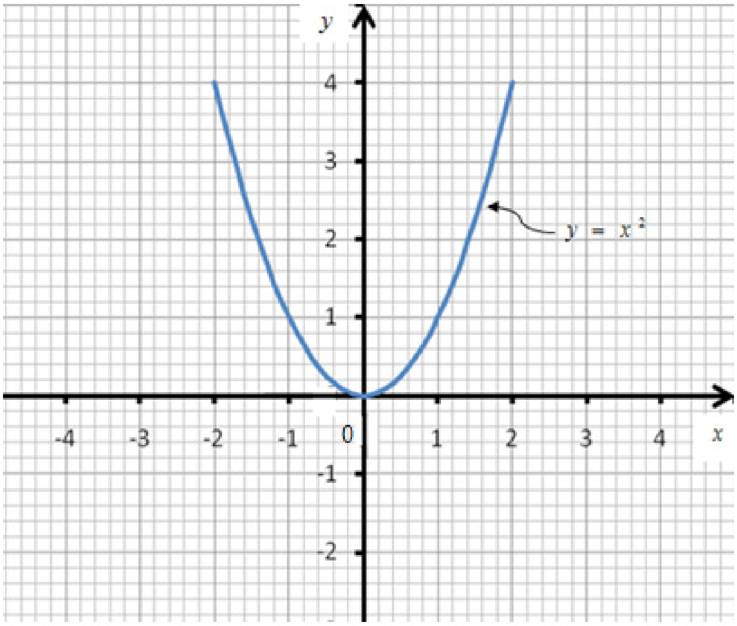
- $\forall x \in Domf, -x \in Domf$
- $f(-x) = f(x)$

The graph of such function is **symmetric about the vertical axis.** i.e $x = 0$

Example:

The function $f(x) = x^2$ is an even function since $\forall x \in Domf = \mathbb{R}, -x \in Domf = \mathbb{R}$ and $f(-x) = (-x)^2 = x^2 = f(x)$

Here is the graph of $f(x) = x^2$



Odd function

A function $f(x)$ is said to be **odd** if the following conditions are satisfied

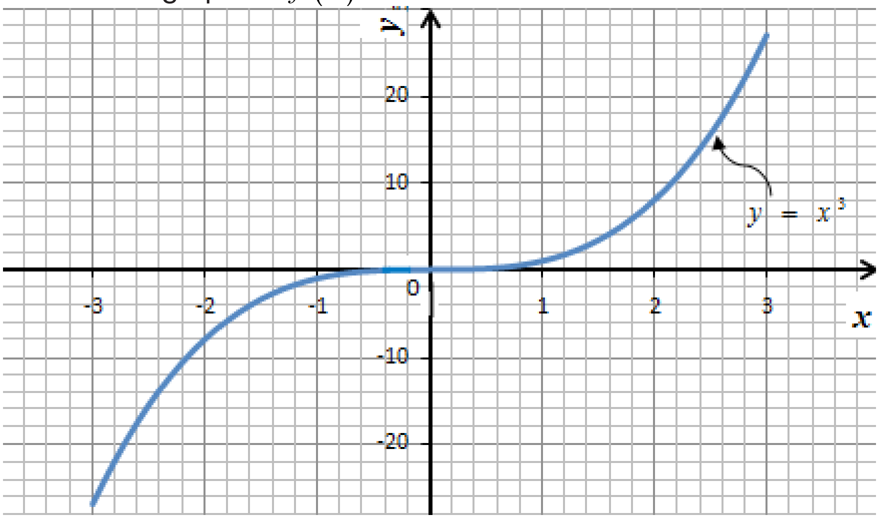
- $\forall x \in Domf, -x \in Domf$
- $f(-x) = -f(x)$

The graph of such function looks the same when rotated through half a revolution about 0. This is called **rotational symmetry**.

Example:

1. $f(x) = x^3$ is an odd function since $\forall x \in Domf = \mathbb{R}, -x \in Domf = \mathbb{R}$ and $f(-x) = (-x)^3 = -x^3 = -f(x)$

Here is the graph of $f(x)=x^3$



Application activity 2.1.5.

Study the parity of the following functions

1. $f(x)=2x^2+2x-3$
2. $h(x)=\dfrac{x^2+4}{x^2-4}$

2.2. Equations and inequalities

2.2.1. Linear equations and system of equations

Activity 2.2.1.

1) The following mathematical statements are always true for only one value of x. For each statement, find out the real value of x
   a) $x+1=5$ b) $2x-4=0$
   c) $2x+1=-5$ d) $x-4=10$
2) Given the two linear equations in 2 unknowns $x-y=1$ and $x+y=1$
   a. For each linear equation, choose any two values of x and use them to find the values of y. This gives you two points in the form of (x,y)
   b. Plot the obtained points in *XY* plane and join them to obtain two different lines.
   c. What is the point of intersection for the two lines?

1. Linear equations

An equation is a statement in which the values of two mathematical expressions are equal. Consider the statement $ax+b=0$, where $a,b\in\mathbb{R}$ and $a\neq 0$. This statement is true when $x=-\frac{b}{a}$ (the solution or the root of the equation $ax+b=0$). Thus, to find a solution to the given equation is to find the value that satisfy that equation.

In general, linear equation in one unknown is of the form $ax+b=0$, with a and b constant and $a\neq 0$

Examples: Solve in set of real numbers

a) $x+6=14$

b) $4x+5=20+x$

c) $x=14-x$

Solutions

$x+6=14$ $\Leftrightarrow x=14-6$ $\Rightarrow x=8$ $S=\{8\}$	$4x+5=20+x$ $\Leftrightarrow 4x-x=20-5$ $\Leftrightarrow 3x=15$ $\Leftrightarrow x=\frac{15}{3}$ $\Rightarrow x=5$ $S=\{5\}$	$x=14-x$ $\Leftrightarrow 2x=14$ $\Rightarrow x=7$ $S=\{7\}$

2. System of linear equations

Solving a system of linear equations by equating two same variables

To find the value of unknown from simultaneous equation by equating the same variable in terms of another, we do the following steps:

i. Find out the value of one variable in first equation

ii. Find out the value of another variable in second equation

iii. Equating the obtained two same variables

iv. Solve the equation to find out the unknown variables

v. Substitute the obtained value of one unknown in one equation to get the second value.

Example

1) Algebraically, solve the simultaneous linear equation by equating the same variables.

$$\begin{cases} 4x + 5y = 2 \\ x + 2y = -1 \end{cases}$$

Solution:

$$\begin{cases} 4x + 5y = 2 \\ x + 2y = -1 \end{cases}$$

From equation (1) $4x+5y=2 \Rightarrow x=\frac{2-5y}{4}$, from equation (2) $x + 2y = -1 \Longrightarrow x = -1 - 2y$

Equalize the values of x from equation (1) and (2)

$$\frac{2-5y}{4}=-1-2y$$

$$2-5y=-4-8y$$

$$-5y + 8y = -4 - 2$$

$$y = -2$$

$x=\frac{2-5y}{4}$, $x=\frac{2-5(-2)}{4}=\frac{12}{4}=3$ then, $S=\{(3,-2)\}$

Solving a system of linear equations by Elimination method

To eliminate one of the variables from either of equations to obtain an equation in just one unknown, make one pair of coefficients of the same variable in both equations negatives of one another by multiplying both sides of an equation by the same number. Upon adding the equations, that unknown will be eliminated.

Example

Solve the system of equations using elimination method.

$$\begin{cases} x+y=1 \\ 2x+3y=2 \end{cases}$$

Solution

$$\begin{cases} x+y=1 \\ 2x+3y=2 \end{cases}\left|\begin{matrix} -2 \\ 1 \end{matrix}\right. \Leftrightarrow \begin{cases} -2x-2y=-2 \\ 2x+3y=2 \end{cases}$$

$$y=0$$

$x+y=1 \Leftrightarrow x=1-y=1$ $S=\{(1,0)\}$

Solving a system of linear equations by Graphical Method

One way to solve a system of linear equations is by graphing. The intersection of the graphs represents the point at which the equations have the same x-value and the same y-value. Thus, this ordered pair represents the solution common to both equations. This ordered pair is called the solution to the system of equations.

The following steps can be applied in solving system of linear equation graphically:

1. Find at least two points for each equation.
2. Plot the obtained points in XY plane and join these points to obtain the lines. Two points for each equation give one line.
3. The point of intersection for two lines is the solution for the given system

Example

Solve the following system by graphical method

$$\begin{cases} x+y=4 \\ x-y=2 \end{cases}$$

Solution

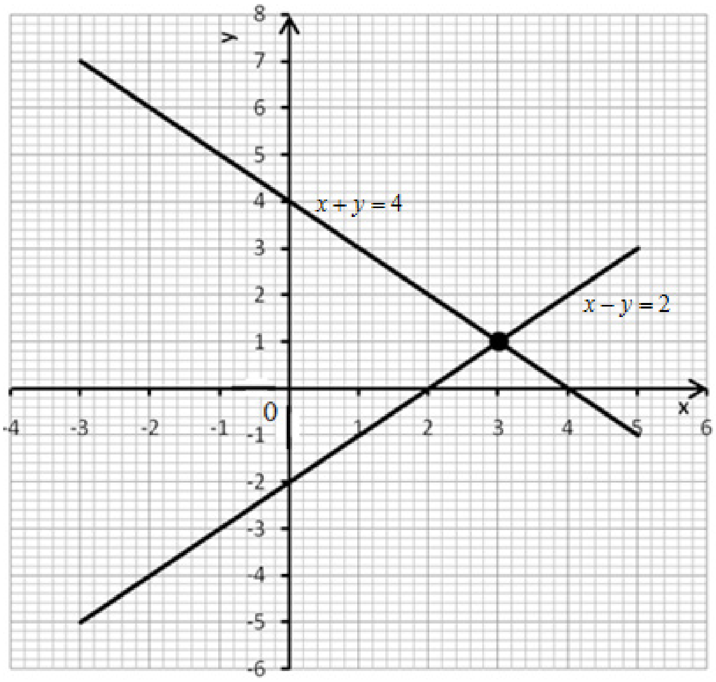
For $x+y=4$

x	-3	5
y	7	-1

For $x-y=2$

x	-3	5
y	-5	3

Graph



The two lines intersect at point $(3,1)$. Therefore, the solution is $S=\{(3,1)\}$.

Application activity 2.2.1.

1. Solve the following linear equations in the set of real numbers

a) $x+5=9$ b) $6x+5=5$

c) $x-2=3$ d) $25=2x-5$

e) $-5=x-1$ e) $3x-4=2x+1$

g) $x+5=9x+1$ h) $-6x-5=9$

i) $x+100=99$ j) $6x-51=9$

2. Solve the following system of linear equations

a. $\begin{cases} 4y+4x=8 \\ -x+y=2 \end{cases}$ b. $\begin{cases} 3x+y=1 \\ x-y=-1 \end{cases}$ c. $\begin{cases} 2x+y=3 \\ 6x+3y=9 \end{cases}$

2.2.2.Quadratic equations

Activity 2.2.1.

1. The mathematical statement $x^2+2x-24=0$ is always true for only some values of *x*. Find out the real values of *x*.

2. Suppose that you own a plot of land and want to build a fence. Let the area of the plot be 900 m². You want your fence to have a length equal to twice its width. Let x be the width, then the length is 2x. Knowing that the Length times width is equal to the area, how will you optimize the area of land you have? Do you think the statement $2x^2=900$ can help you to find out the length and width of your fence? Explain your answer.

Equations of the type $ax^2+bx+c=0$ $(a\neq 0)$ are called quadratic equations.

There are three main ways of solving such equations:

a) By factorizing or finding square roots

b) By using the formula $x=\dfrac{-b\pm\sqrt{b^2-4ac}}{2a}$

c) By completing the square

1. Solving Quadratic equations by factorizing or finding square roots

The method of solving quadratic equations by factorization should only be used if is readily factorized by inspection.

To factorize a quadratic equation, one can use the sum and product of its roots.

Let x and y be two real numbers such that $x+y=s$ and $xy=p$.

Here $y=s-x$ and $x(s-x)=p$. Or $sx-x^2=p$ or $x^2-sx+p=0$. This equation is said to be quadratic equation and *s*, *p* are the sum and product of the two roots respectively.

Quadratic equation or equation of second degree has the form $ax^2+bx+c=0$, where the sum of two roots is $s=-\frac{b}{a}$ and their product is $p=\frac{c}{a}$.

Example 1: *Solve in* $\mathbb{R}$: $x^2+2x-24=0$	**Example 2:** *Solve in* $\mathbb{R}$: $5x^2+7x-6=0$
Solution $x^2+2x-24=0 \Leftrightarrow (x+6)(x-4)=0$ So, either $x+6=0$ or $x-4=0$ giving $x=-6$ or $x=4$. $S=\{-6,4\}$	**Solution** $5x^2+7x-6=0$ $\Leftrightarrow 5x^2-3x+10x-6=0$ $\Leftrightarrow x(5x-3)+2(5x-3)=0$ $\Leftrightarrow (5x-3)(x+2)=0$ So, either $5x-3=0$ or $x+2=0$ giving $x=\frac{3}{5}$ or $x=-2$. $S=\left\{-2,\frac{3}{5}\right\}$

Example 3: *Solve in* $\mathbb{R}$ *:* $x^2-7x+5=-5$

Solution

$x^2-7x+5=-5$

$\Leftrightarrow x^2-7x+10=0$

In this equation the sum of two roots is -7 and the product is 10. To find those roots we can think about two numbers such that their sum is -7 and their product is 10. Those numbers are -2 and -5. Thus $S=\{2,5\}$

2. Solving Quadratic equations by the formula

To solve this equation, first we find the discriminant (delta): $\Delta = b^2 - 4ac$

In fact,

$$ax^2 + bx + c = 0$$

$$\Leftrightarrow ax^2 + bx = -c$$

$$\Leftrightarrow a\left(x^2 + \frac{b}{a}x\right) = -c \text{ as } a \neq 0$$

$$\Leftrightarrow x^2 + \frac{b}{a}x = -\frac{c}{a} \text{ as } a \neq 0$$ (making the coefficient of x^2 one)

$$\Leftrightarrow \left(x + \frac{b}{2a}\right)^2 = -\frac{c}{a} + \frac{b^2}{4a^2} \leftarrow \frac{b^2}{4a^2} \text{ is the square of half the coefficient of } x, \left(\frac{b}{a}\right), \text{ in } x^2 + \frac{b}{a}x$$

$$\Leftrightarrow \left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2} \quad \Rightarrow x + \frac{b}{2a} = \pm\sqrt{\frac{b^2 - 4ac}{4a^2}}$$

$$\Leftrightarrow x + \frac{b}{2a} = \pm\sqrt{\frac{b^2 - 4ac}{4a^2}} \quad \Leftrightarrow x + \frac{b}{2a} = \pm\frac{\sqrt{b^2 - 4ac}}{2|a|}$$

$$\Leftrightarrow x + \frac{b}{2a} = \pm\frac{\sqrt{b^2 - 4ac}}{2a}, \text{if } a > 0$$

$$\text{or } x + \frac{b}{2a} = \mp\frac{\sqrt{b^2 - 4ac}}{2a}, \text{if } a < 0$$

Simply,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Let $\Delta = b^2 - 4ac$, There are three cases:

- If $\Delta > 0$, there are two distinct real roots:

  $$x_1 = \frac{-b + \sqrt{\Delta}}{2a} \text{ and } x_2 = \frac{-b - \sqrt{\Delta}}{2a}$$

- If $\Delta = 0$, there is one repeated real root (one double root):

  $$x_1 = x_2 = \frac{-b}{2a}$$

- If $\Delta < 0$, there is no real root.

Example 1: *Solve in* $\mathbb{R}$ *:* $x^2+2x+1=0$ ***Solution*** $x^2+2x+1=0$ $\Delta=2^2-4(1)(1)=0$ $x_1=x_2=\frac{-2}{2}=-1$; $S=\{-1,-1\}$	**Example 2:** *Solve in* $\mathbb{R}$ *:* $x^2-7x+5=-5$ ***Solution*** $x^2-7x+5=-5$ $\Leftrightarrow x^2-7x+10=0$; $\Delta=(-7)^2-4(1)(10)=9$ $x_1=\frac{-(-7)+\sqrt{9}}{2}=5,\ x_1=\frac{-(-7)-\sqrt{9}}{2}=2$ $S=\{2,5\}$
Example 3: *Solve in* $\mathbb{R}$ *:* $2x^2+3x+4=0$ ***Solution*** $2x^2+3x+4=0$ $\Delta=3^2-4(2)(4)=-23$ Since $\Delta<0$, there is no real root. Then, $S=\varnothing$	**Example 4:** *For what value of k will the equation* $x^2+2x+k=0$ *have one double roots? Find that root.* ***Solution*** For one double root $\Delta=0$. $\Delta=4-4k$ $4-4k=0\Rightarrow k=1$ Thus, the value of k is 1. That root is $x=-\frac{2}{2}=-1$. $S=\{-1\}$

3. Solving Quadratic equations by completing the square

Before solving quadratic equations by completing the square, let's look at some examples of expanding a binomial by squaring it.

- $(x+3)^2=x^2+6x+9$.
- $(x-5)^2=x^2-10x+25$

Notice that the constant term (k^2) of the trinomial is the square of half of the coefficient of trinomial's x-term. Thus, to make the expression x^2+kx a perfect square, you must add $\left(\frac{1}{2}k\right)^2$ to the expression.

When completing the square to solve quadratic equation, remember that you must preserve the equality. When you add a constant to one side of the equation, be sure to add the same constant to the other side of equation.

Example 1: Solve $x^2-4x+1=0$ by completing the square	**Example 2:** Solve $4x^2+2x-5=0$ by completing the square
Solution $x^2-4x+1=0$ Rewrite original equation $x^2-4x=1$ Subtract 1 from both sides. $x^2-4x+(-2)^2=1+(-2)^2$ Add $(-2)^2=4$ to both sides. $(x-2)^2=3$ Binomial squared. $x-2=\pm\sqrt{3}$ Take square roots. $x=2\pm\sqrt{3}$ Solve for x. ***The equation has two solutions:*** $x=2+\sqrt{3}$ ***and*** $x=2-\sqrt{3}$ $S=\left\{2+\sqrt{3},2-\sqrt{3}\right\}$	***Solution*** $4x^2+2x-5=0$ Rewrite original equation. $4x^2+2x=5$ Add 5 to both sides. $x^2+\frac{1}{2}x=\frac{5}{4}$ Divide both sides by 4. $x^2+\frac{1}{2}x+\left(\frac{1}{4}\right)^2=\frac{5}{4}+\frac{1}{16}$ Add $\left(\frac{1}{4}\right)^2=\frac{1}{16}$ to both sides. $\left(x+\frac{1}{4}\right)^2=\frac{21}{16}$ Binomial squared. $x+\frac{1}{4}=\pm\frac{\sqrt{21}}{2}$ Take square roots. $x=-\frac{1}{4}\pm\frac{\sqrt{21}}{2}$ ***then*** $x=-\frac{1}{4}+\frac{\sqrt{21}}{2}$ ***and*** $x=-\frac{1}{4}-\frac{\sqrt{21}}{2}$ $S=\left\{-\frac{1}{4}+\frac{\sqrt{21}}{2},-\frac{1}{4}-\frac{\sqrt{21}}{2}\right\}$

Application activity 2.2.2.

A. Solve in set of real numbers the following equations by factorization

1. $x^2+6x+8=0$
2. $x^2-2x=3$

B. Solve in set of real numbers the following equations using the quadratic formula

1. $x^2-12x+11=0$
2. $x^2+2x=35$

C. Solve in set of real numbers the following equations by completing the square

1. $x^2 + 5x - 24 = 0$
2. $x^2 - 13x + 36 = 0$

D. A manufacturer develops a formula to determine the demand for its product depending on the price in Rwandan franc. The formula is $D = 2000 + 100P - 6P^2$, where P is the price per unit, and D is the number of units in demand. At what price will the demand drop to 1000 units?

2.2.4. Linear inequalities

Activity 2.2.3.

Find at least 5 value(s) of x such that the following statements are true

1) $x < 5$ 2) $x > 0$ 3) $-4 < x < 12$

The statement $x + 3 = 10$ is true only when $x = 7$. If x is replaced by another number (for example 5), the statement is false. To be true we may say that 5+3 is less than 10 or in symbol $5 + 3 < 10$. In this case we no longer have equality but **inequality**. Suppose that we have the inequality $x + 3 < 10$, in this case we have an inequality with one unknown. Here the real value of x to satisfy this inequality is not unique. For example, 1 is a solution but 3 is also a solution. In general, all real numbers less than 7 are solutions. In this case we will have many solutions combined in an interval.

Now, the solution set of $x + 3 < 10$ is an open interval containing all real numbers less than 7 whereby 7 is excluded. Solving the inequality $x + 3 < 10$, one can easily find that the values of x are less than 7. Mathematically is written as follow: $S =]-\infty, 7[$

In general, inequalities in one unknown are of the forms:

$ax + b > 0,\quad ax + b < 0,\quad ax + b \geq 0,\quad ax + b \leq 0$, with a and b constant and $a \neq 0$

Recall that

- When the same real number is added or subtracted from each side of inequality the direction of inequality is not **changed**.
- The direction of the inequality is not **changed** if both sides are multiplied or divided by the same **positive real number** and is **reversed** if both sides are multiplied or divided by the **same negative real** number.

Examples: *Algebraically solve the following inequalities in the set of real numbers*

Example 1:	Example 2:	Example 3:
$-2x+5\le 0$ ***Solution*** $-2x+5\le 0$ $\Leftrightarrow -2x\le -5$ $\Leftrightarrow x\ge \frac{5}{2}$ $S=\left[\frac{5}{2},+\infty\right[$	$x-4>0$ ***Solution*** $x-4>0$ $\Leftrightarrow x>4$ $S=]4,+\infty[$	$x>2x-4$ ***Solution*** $x>2x-4$ $\Leftrightarrow -x>-4$ $\Leftrightarrow x<4$ $S=]-\infty,4[$

Example 4:

$2x+5\le 2x+4$

Solution

$2x+5\le 2x+4$ $0x\le -1$

Since any real number times zero is zero and zero is not less or equal to -1 then the solution set is the empty set. $S=\varnothing$

Example 6:

$2(x+5)>2x-8$

Solution

$2(x+5)>2x-8$

$\Leftrightarrow 2x+10>2x-8$

$\Leftrightarrow 0x>-18$

Since any real number times zero is zero and zero is greater than -18, then the solution set is the set of real numbers. $S=\mathbb{R}=]-\infty,+\infty[$

Application activity 2.2.3.

Solve the following inequalities

1) $x+6<15$ 2) $2x-4<16$ 3) $5x\le 25$ 4) $3x-5>21$

2.2.4. Inequalities products / quotients

Activity 2.2.4.

Find at least 5 value(s) of x such that the following statements are true

1. $(x+1)(x-1)<0$
2. $\frac{x+2}{x-1}\le 0$

- To solve the inequality of the form $(ax+b)(cx+d)<0$ or $(ax+b)(cx+d)\le 0$ one can find out all real numbers that make the left-hand side negative.
- If we find the product of the left-hand side, the result will be a quadratic expression of the form ax^2+bx+c. Then to solve the inequality of the form $ax^2+bx+c>0$, we need to put the expression ax^2+bx+c in factor form and use the method to solve inequality product described below.
- If the expression to be transformed in factor form has no factor form, we find its sign by replacing the unknown by any chosen real number. We may find that the expression is always positive or always negative. If the expression to be transformed in factor form has a repeated root, it is zero at that root and positive elsewhere.
- To solve the inequality of the form $\frac{ax+b}{cx+d}<0$ or $\frac{ax+b}{cx+d}\le 0$, one can find out all real numbers that make the left-hand side negative.
- To solve the inequality of the form $(ax+b)(cx+d)>0$ or $(ax+b)(cx+d)\ge 0$, one can find out all real numbers that make the left-hand side positive.
- To solve the inequality of the form $\frac{ax+b}{cx+d}>0$ or $\frac{ax+b}{cx+d}\ge 0$ one can find out all real numbers that make the left-hand side positive.

The following steps are essential:

a) First, we solve for $\frac{ax+b}{cx+d}=0$ or $(ax+b)(cx+d)=0$

b) We construct the table of signs, by finding the sign of each factor and then the sign of the product or quotient.

For the quotient the value that makes the denominator to be zero is always excluded in the solution. For that value we use the symbol ‖ in the row of quotient sign.

c) Write the interval considering the given inequality sign.

Examples: *Solve inequalities in the set of real numbers*

1. $(3x+7)(x-2)<0$

   Start by solving $(3x+7)(x-2)=0$

   $3x+7=0 \Leftrightarrow x=-\frac{7}{3}$ or $x-2=0 \Leftrightarrow x=2$

The next is to find the sign table.

x	$-\infty$		$-\frac{7}{3}$		2		$+\infty$
$3x+7$		-	0	+	+	+	
$x-2$		-		-	0	+	
$(3x+7)(x-2)$		+	0	-	0	+	

Since the inequality is $(3x+7)(x-2)<0$ we will take the interval where the product is negative. Thus, $S=\left]-\frac{7}{3},2\right[$

2. $(x-3)(-2x+4)(x+1)\geq 0$

   $x-3=0 \Rightarrow x=3, \quad -2x+4=0 \Rightarrow x=2, \ x+1=0 \Rightarrow x=-1$

x	$-\infty$		-1		2		3		$+\infty$
$x-3$		-	-	-	-	-	0	+	
$-2x+4$		+	+	+	0	-	-	-	
$x+1$		-	0	+	0	+		+	
$(x-3)(2x-4)(x+1)$		+	0	-	0	+	0	-	

$S=]-\infty,-1]\cup[2,3]$

3. $\frac{x+4}{2x-1} \geq 0$

$x+4=0 \Rightarrow x=-4$

$2x-1=0 \Rightarrow x=\frac{1}{2}$

x	$-\infty$		-4		$\frac{1}{2}$		$+\infty$
$x+4$		-	0	+		+	
$2x-1$		-		-	0	+	
$\frac{x+4}{2x-1}$		+	0	-	‖	+	

$S=]-\infty,-4] \cup \left]\frac{1}{2},+\infty\right[$

4. $\frac{3x+6}{-x-1} > 0$

$3x+6=0 \Rightarrow x=-2$

$-x-1=0 \Rightarrow x=-1$

x	$-\infty$		-2		-1		$+\infty$
$3x+6$		-	0	+		+	
$-x-1$		+		+	0	-	
$\frac{3x+6}{-x-1}$		+	0	+	‖	-	

$S=]-2,-1[$

Examples on quadratic inequalities of the form:

$ax^2+bx+c>0$; $ax^2+bx+c<0; ax^2+bx+c \leq 0; ax^2+bx+c \geq 0$

Solve the following quadratic inequalities

$x^2-2x+1 \leq 0$

Solution

$x^2-2x+1=0$

$x_1=x_2=1$ and $x^2-2x+1=(x-1)(x-1)$

The expression x^2-2x+1 is zero for $x=1$ otherwise it is positive since $x^2-2x+1=(x-1)(x-1)=(x-1)^2$

The solution is only $x=1$ since we are given $x^2-2x+1\le 0$.

$x^2-5x+6\ge 0$

Solution

$x^2-5x+6=0$, either $x=2$ or $x=3$ and then $x^2-5x+6=(x-2)(x-3)$

x	$-\infty$		2		3		$+\infty$
$x-2$		-	0	+		+	
$x-3$		+		-	0	+	
x^2-5x+6		+	0	-	0	+	

$S=]-\infty,2]\cup[3,+\infty[$

$3x^2+x-14<0$

Solution

$3x^2+x-14=0$, $x_1=-\frac{7}{3}$ or $x_2=2$,

$3x^2+x-14=3\left(x+\frac{7}{3}\right)(x-2)=(3x+7)(x-2)$

x	$-\infty$		$-\frac{7}{3}$		2		$+\infty$
$3x+7$		-	0	+		+	
$x-2$		-		-	0	+	
$3x^2+x-14$		+	0	-	0	+	

Thus, $S=\left]-\frac{7}{3},2\right[$

$x^2-4x+4\ge 0$

Solution

$x^2-4x+4=0$.

$x_1=x_2=2$ and $x^2-4x+4=(x-2)(x-2)$ or $x^2-4x+4=(x-2)^2$

The expression x^2-4x+4 is zero for $x=2$ and positive elsewhere.

Then, $S=\mathbb{R}$

$2x^2+2x\le 4x-10$ ***Solution*** $2x^2+2x\le 4x-10$ $\Leftrightarrow 2x^2+2x-4x+10\le 0$ $\Leftrightarrow 2x^2-2x+10\le 0$ $\Leftrightarrow x^2-x+5\le 0$ $x^2-x+5=0$ $\Delta=1-20=-19$ The expression x^2-x+5 cannot be factorized. Let $x=0$, $0^2-0+5=5>0$. Then the expression x^2-x+5 is always positive and the solution set is an empty set.
$-2x^2+2x-10<0$ ***Solution*** $-2x^2+2x-10<0$ $\Leftrightarrow x^2-x+5>0$ $x^2-x+5=0$ $\Delta=1-20=-19$ The expression x^2-x+5 cannot be factorized. Let $x=0$, $0^2-0+5=5>0$. Then the expression x^2-x+5 is always positive and the solution set is the set of real numbers.

Application activity 2.2.4.

Solve the following inequalities

1. $(x-3)(x+3)>0$
2. $\dfrac{2x-6}{x+2}\ge 0$
3. $x^2-10x-20>0$
4. $6x^2-5x+1<0$
5. $x^2+2x+12>0$

2.3. Application of linear and quadratic functions in production, finance, and economics

Activity 2.3.

1. Considering that C is the dependent variable, measured in the vertical axis, and Y is the independent variable, measured on the horizontal axis, Draw the graph of the function

   $C = 200 + 0.6Y$ Where C is consumer spending and Y is income. Note that the income cannot be negative.

   Determine the point (Y, C) at which the line cuts the vertical axis.

   What graph does tell about consumer spending and income?

2. Earl's Biking Company manufactures and sells bikes. Each bike costs $40 to make, and the company's fixed costs are $5000. In addition, Earl knows that the price of each bike comes from the price function $P(x) = 300 - 2x$ Find:

   i. The company's cost function, C(x).

   ii. The company's revenue function, R(x).

   iii. The company's profit function, P(x)

A function is a relationship between two or more variables such that a unique value of one variable is determined by the values taken by the other variables in the function.

Example: Suppose that average weekly household expenditure on food (C) depends on average net household weekly income (Y) according to the relationship $C = 12 + 0.3Y$. For any given value of Y, one can evaluate what C will be. For example, if Y = 90 then C = 39

2.3.1. Cost function

The **cost function**, C(q), gives the total cost of producing a quantity q of some good.

Cost is the total cost of producing output. The cost function consists of two different types of cost:

a) Variable costs, which depend on how many units are produced.

b) Fixed costs, which are incurred even if nothing is produced,

Variable cost varies with output (the number of units produced). The total variable cost can be expressed as the product of variable cost per unit and number of units produced. If more items are produced cost is more.

Fixed costs normally do not vary with output. In general, these costs must be incurred whether the items are produced or not.

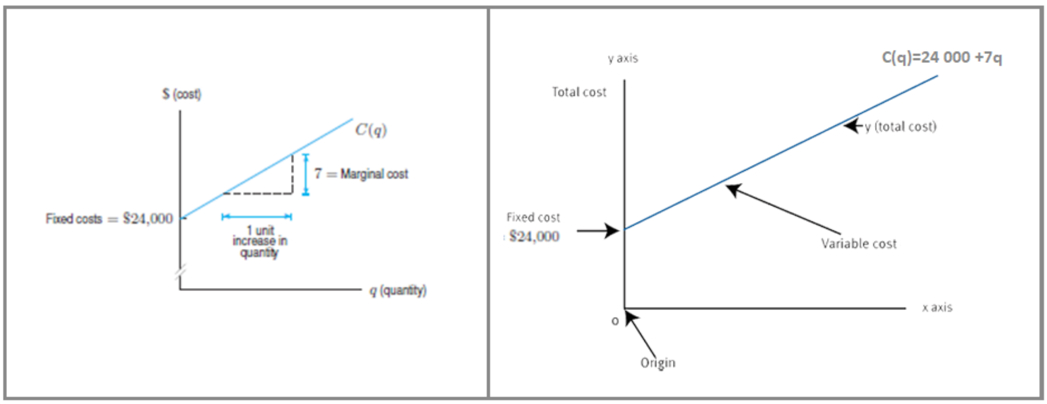
Cost Function C(q) = F +Vq, it is called a linear cost function where,

- C = Total cost
- F = Fixed cost
- V = Variable cost Per unit
- q = No of units produced and sold

Example 1:

Let's consider a company that makes radios. The factory and machinery needed to begin production with fixed costs which are incurred/earned even if no radios are made. The costs of labor and raw materials are variable costs since these quantities depend on how many radios are made. The fixed costs for this company are $24 000 and the variable costs are $7 per radio. Then, Total costs for the company = Fixed costs + Variable costs = 24 000 + 7$\times$(Number of radios), so, if q is the number of radios produced, C(q) = 24 000 + 7q.

This is the equation of a line with slope 7 and vertical intercept 24 000.



If C(q) is a linear cost function,

- Fixed costs are represented by the vertical intercept.
- Marginal cost is represented by the slope.

Example 2:

If a firm faces the total cost function $TC = 6 + x^2$, where *x* is output, what is TC when *x* is (a) 14? (b) 1? (c) 0?

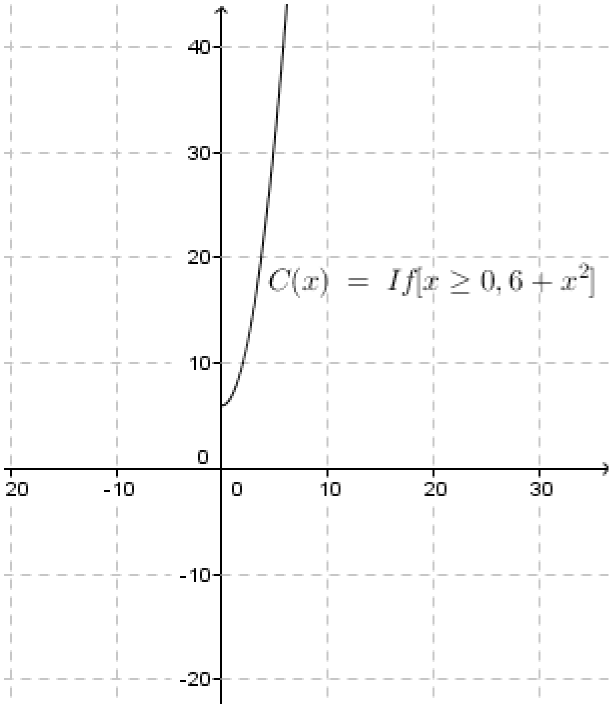
(d) What restrictions on the domain of this function would it be reasonable to make?

(a) When $x = 14$, the value of TC is 202

(b) When $x = 1$, the value of TC is 7

(c) When $x = 0$, the value of TC is 6,

(d) The restrictions on the domain to make the cost function $TC = 6 + x^2$ being reasonable is only considering the values of $x \geq 0$.



3.2.2. Revenue function

Revenue is the total payment received from selling a good or performing a service. The **Revenue function**, R(q), gives the total revenue received by a firm from selling a quantity, q, of some good. If the good sells for a price of p per unit, and the quantity sold is q, then

Revenue = Price x Quantity, so **R = pq.**

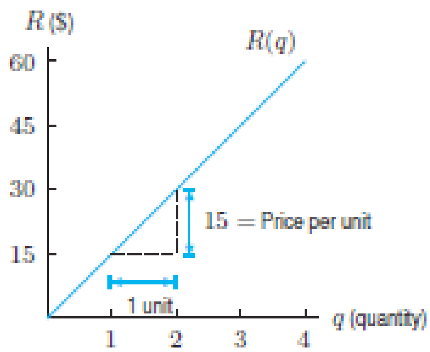
If the price does not depend on the quantity sold, so p is a constant, the graph of revenue as a function of q is a line through the origin, with slope equal to the price p.

Example:

1. If radios sell for \$15 each, sketch the manufacturer's revenue function. Show the price of a radio on the graph.

Solution:

Since R(q) = pq = 15q, the revenue graph is a line through the origin with a slope of 15. See the figure. The price is the slope of the line.



2. Graph the cost function C(q) = 24 000 + 7q and the revenue function R(q) = 15q on the same axes. For what values of q does the company make money?

Solution:

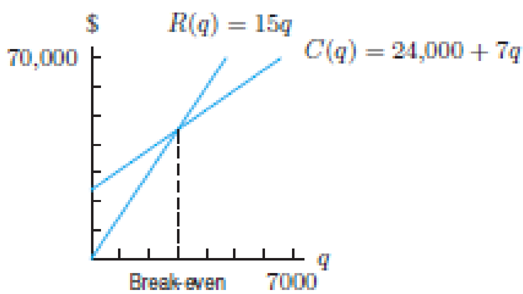
The company makes money whenever revenues are greater than costs, so we find the values of q for which the graph of R(q) lies above the graph of C(q). We find the point at which the graphs of R(q) and C(q) cross: **Revenue = Cost**

15q = 24 000 + 7q

8q = 24 000

q = 3 000

The company makes a profit if it produces and sells more than 3 000 radios. The company loses money if it produces and sells fewer than 3 000 radios.



2.3.3. Profit function

The Profit Function $P(x)$ is the difference between the revenue function R(x) and the total cost function $C(x)$ Thus $P(x) = R(x) - C(x)$

We have: **Profit = Revenue – Cost.**

The *break-even point* for a company is the point where the profit is zero and revenue equals cost.

Example:

Let's consider a company that makes radios. The factory and machinery needed to begin production with fixed costs which are incurred/earned even if no radios are made. The costs of labor and raw materials are variable costs since these quantities depend on how many radios are made. The fixed costs for this company are $24 000 and the variable costs are $7 per radio while each radio is sold for $15.

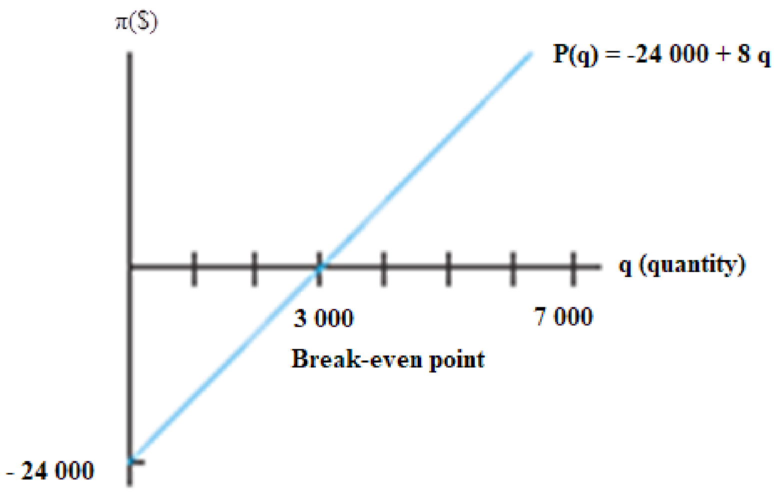
Find a formula for the profit function of the radio manufacturer. Graph it, marking the break-even point

Solution:

Since R(q) = 15q and C(q) = 24 000 + 7q, we have

$$P(q) = R(q) - C(q)$$
$$= 15q - (24\,000 + 7q) = -2\,400 + 8q$$

Notice that the negative of the fixed costs is the vertical intercept and the break-even point is the horizontal intercept. See the figure;



2.3.4. Demand function

The demand function is in the form $P = a - bQ$, where a and b are parameters, P is the price and Q is the quantity demanded.

Example:

Consider the function P = 60−0.2Q where P is price and Q is quantity demanded. Assume that P and Q cannot take negative values, determine the slop of this function and sketch its graph.

Solution:

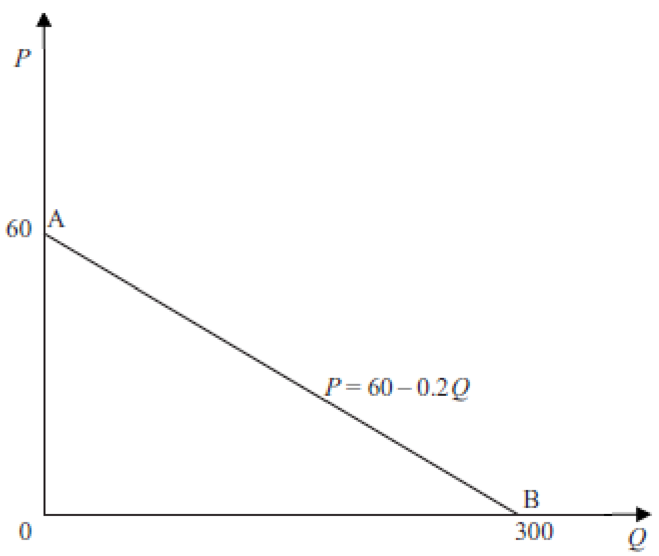
When Q = 0 then P = 60

When P = 0 then 0 = 60 − 0.2Q

0.2Q = 60

$$Q = \frac{60}{0.2} = 300$$

Using these points: $(0, 60)$ and $(300, 0)$, we can find the graph as follows:



The slope of a function which slopes down from left to right is found by applying the formula $slope = (-1)\dfrac{height}{base}$ to the relevant right-angled triangle. Thus, using the triangle 0BA, the slope of our function is $(-1)\dfrac{60}{300} = -0.2$

This is the same as the coefficient of Q in the function $P = 60 - 0.2Q$.

Remember that in Finance and economics the usual convention is to measure P on the vertical axis of a graph. If you are given a function in the format $Q = f(P)$ then you would need to derive the inverse function to read off the slope.

Example:

What is the slope of the demand function $Q = 830 - 2.5P$ when P is measured on the vertical axis of a graph?

Solution:

If $Q= 830-2.5P$; then $2.5P = 830-Q$

$P = 332-0.4Q$

Therefore, the slope is the coefficient of Q, which is −0.4.

Point elasticity of demand

Elasticity can be calculated at a specific point on a linear demand schedule. This is called '*point elasticity of demand*' and is defined as

$$e = (-1)\left(\frac{P}{Q}\right)\left(\frac{1}{slope}\right)$$

where P and Q are the price and quantity at the point in question. The slope refers to the slope of the demand schedule at this point although, of course, for a linear demand schedule the slope will be the same at all points.

Example:

Calculate the point elasticity of demand for the demand schedule $P = 60 - 0.2Q$ where price is

(i) zero, (ii) \$20, (iii) \$40, (iv) \$60.

Solution

This is the demand schedule referred to earlier and illustrated above as demand function. Its slope must be −0.2 at all points as it is a linear function and this is the coefficient of Q.

To find the values of Q corresponding to the given prices we need to derive the inverse function.

Given that $P = 60 - 0.2Q$ then $0.2Q = 60 - P$

$Q = 300 - 5P$

i) When P is zero, at point B, then $Q = 300 - 5(0) = 300$. The point elasticity will therefore be

$$e = (-1)\frac{P}{Q}\left(\frac{1}{slope}\right) = -1\frac{0}{30}\left(\frac{1}{-0.2}\right) = 0$$

ii) When $P = 20$, then $Q = 300 - 5(20) = 200$.

$$e = (-1)\frac{P}{Q}\left(\frac{1}{slope}\right) = -1\frac{20}{200}\left(\frac{1}{-0.2}\right) = 0.5$$

iii) When $P = 40$, then $Q = 300 - 5(40) = 100$

$$e = (-1)\frac{P}{Q}\left(\frac{1}{slope}\right) = -1\frac{40}{100}\left(\frac{1}{-0.2}\right) = 2$$

iv) When $P = 60$, then $Q = 300 - 5(60) = 0$.

If Q = 0, then $\frac{P}{Q} \to \infty$.

Therefore, $e = (-1)\frac{P}{Q}\left(\frac{1}{slope}\right) = -1\frac{60}{0}\left(\frac{1}{-0.2}\right) \to \infty$

2.3.5. Supply function

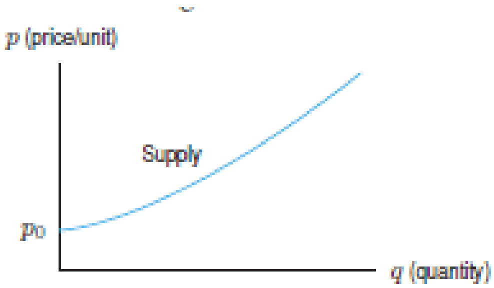
A. Price as function of quantity supplied

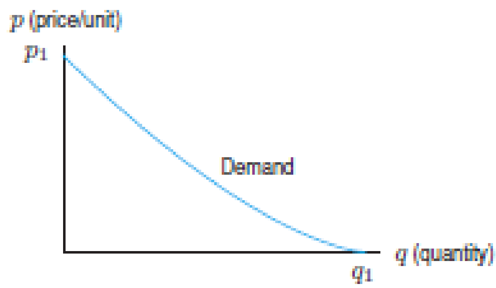
The quantity, q, of an item that is manufactured and sold depends on its price, p. As the price increases, manufacturers are usually willing to supply more of the product, whereas the quantity demanded by consumers falls.

The supply curve, for a given item, relates the quantity, q, of the item that manufacturers are willing to make per unit time to the price, p, for which the item can be sold.

The demand curve relates the quantity, q, of an item demanded by consumers per unit time to the price, p, of the item.

Economists often think of the quantities supplied and demanded Q as functions of price P. However, for historical reasons, the economists put price (the independent variable) on the vertical axis and quantity (the dependent variable) on the horizontal axis. (The reason for this state of affairs is that economists originally took price to be the dependent variable and put it on the vertical axis





B. Consumption as function of income

It is assumed that consumption C depends on income Y and that this relationship takes the form of the linear function $C = a + bY$.

Example:

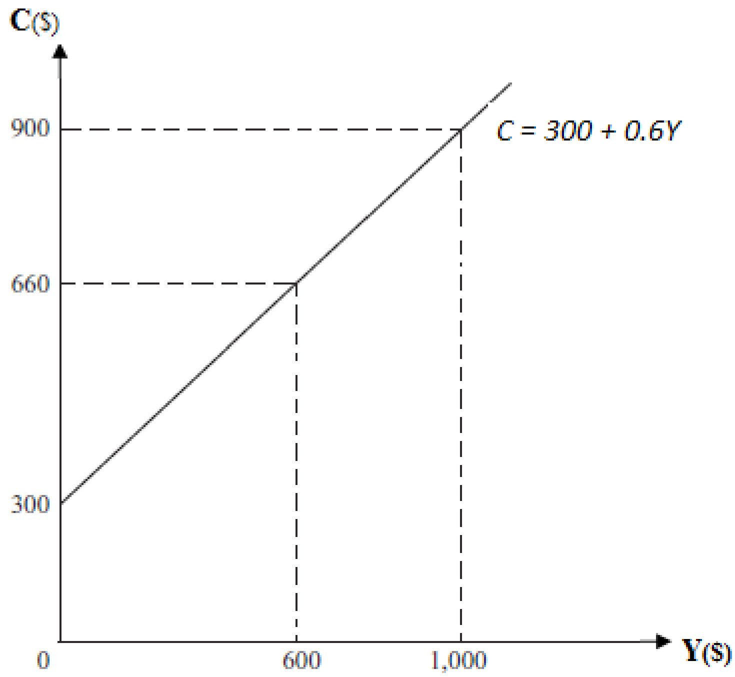
When *the income* is \$600, the consumption observed is \$660. When *the income* is \$1 000, the consumption observed is \$900. Determine the "consumption function of income".

Solution:

As the consumption C depends on income Y and the linear function $C = a + bY$ gives their relationship, we expect *b* to be positive, i.e. consumption increases with income, and the function will slope upwards. As this is a linear function, then equal changes in Y will cause the same changes in C.

Therefore,

- A decrease in Y of \$400 from \$,000 to \$600 causes C to fall by \$240 from \$900 to \$660.
- If Y is decreased by a further \$600 (i.e. to zero), then the corresponding fall in C will be 1.5 times the fall caused by an income decrease of \$400, since \$600 = 1.5 × \$400.
- The fall in C is 1.5 × \$240 = \$360. This means that the value of C when Y is zero is \$660 – \$360 = \$300. Thus *a* = 300.
- A rise in Y of \$400 causes C to rise by \$240.
- A rise in Y of \$1 will cause C to rise by $\$\frac{240}{400} = \$0.6$. Thus, *b* = 0.6.
- The consumption function of income is specified as $C = 300 + 0.6Y$.



- The graph shows that when the income increases, the consumption increases also.

C. Graphical representation of demand and supply functions

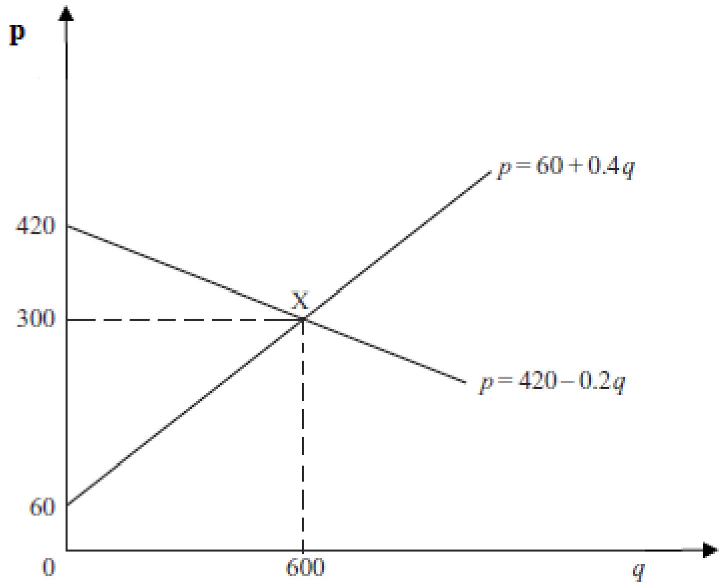
When only two or single variables and functions are involved, graphical solutions can be related to linear functions such as supply and demand analysis.

Example:

Assume that in a competitive market the demand schedule is given by $p=420-0.2q$ and the supply schedule is given by $p = 60 + 0.4q$, *solve* graphically for p and q and determine the point at which the market is in equilibrium. If this market is in equilibrium, the equilibrium price and quantity will be where the demand and supply schedules intersect. This requires to graphically solve the demand and supply functions at the same time. Its solution will correspond to a point which is on both the demand schedule and the supply schedule. Therefore, the equilibrium values of p and q will be such that both functions (1) and (2) hold.

Solution:

The two functional relationships $p = 420 - 0.2q$ and $p = 60 + 0.4q$ are plotted in the figure below and both hold at the intersection point $X(600, 300)$.



Application activity 2.3.

1. Assume that consumption C depends on income Y according to the function

   $C = a + bY$, where a and b are parameters. If C is \$60 when Y is \$40 and C is \$90 when Y is \$80, If a= 30, sketch the graph of C(Y) and interpret it.

2. Suppose that q = f(p) is the demand curve for a product, where p is the selling price in dollars and q is the quantity sold at that price.

   a) What does the statement f(12) = 60 tell you about demand for this product?

   b) Do you expect this function to be increasing or decreasing? Why?

3. A demand curve is given by 75p + 50q = 300, where p is the price of the product, in dollars, and q is the quantity demanded at that price. Find p^* and q* intercepts and interpret them in terms of consumer demand.

4. If a retail store has fixed cost of 15 000 Frw and variable cost per unit is 17 500 Frw and sells its product at 50 000 Frw per unit.

   i. Find the cost function C(x)

   ii. What would the revenue function be?

   iii. What would the profit function be?

2.4. End unit assessment

End unit assessment

1. The total cost C for units produced by a company is given by $C(q) = 50\,000 + 7q$ where q is the number of units produced.
   a) What does the number 50 000 represent?
   b) What does the number 8 represent?
   c) Plot the graph of C and indicate the cost when $q = 5$.
   d) Determine the real domain and the range of $C(q)$.
   e) Is $C(q)$ an odd function?
2. Solve the following equations and inequalities in set of real numbers
   a. $x + 5 = 2x - 8$
   b. $2x - 8 \geq 0$
   c. $(x+3)(x-2) < 0$
   d. $(3x+7)(x-2) < 0$
   e. $x^2 - 10x + 1 = 0$
   f. $6x^2 - 5x + 1 \leq 0$
   g. $x^4 - 3x^3 + 4x^2 - 3x + 1 = 0$
3. Suppose your cell phone plan is 3000 Frw per month plus 20 Frw per minute. Your bill is 7 000 Frw. Use the equation $3\,000 + 20x = 7\,000$, to find out how many minutes are on your bill.
4. A company produces and sells a product and fixed costs of the company are $ 6 000 and variable cost is S 25 per unit and sells the product at $ 50 per unit.
   a) Find the total cost function.
   b) Find the total revenue function.
   c) Find the profit function and determine the profit when 1 000 units are sold.
   d) How many units must be produced and sold to yield a profit of $10 000?

UNIT 3 EXPONENTIAL AND LOGARITHMIC FUNCTIONS AND EQUATIONS

Key unit competence: Solve production, financial and economical related problems using logarithmic and exponential functions and equations.

3.0. Introductory activity

Introductory activity

An economist created a business which helped him to make money in an interesting way so that the money he/she earns each day doubles what he/she earned the previous day. If he/she had 200USD on the first day and by taking t as the number of days, discuss the money he/she can have at the t^{th} day through answering the following questions:

a) Draw the table showing the money this economist will have on each day starting from the first to the 10^{th} day.

b) Plot these data in rectangular coordinates

c) Based on the results in (a), establish the formula for the economist to find out the money he/she can earn on the n^{th} day. If t is the time in days, express the money $F(t)$ for the economist.

d) Now the economist wants to possess the money F under the same conditions, discuss how he/she can know the number of days necessary to get such money from the beginning of the business.

3.1. Exponential and logarithmic functions and equations

Many quantities increase or decrease with time in proportion to the amount of the quantity present. Exponential and logarithmic functions can be seen in mathematical concepts in finance, especially in annuities. This relationship is illustrated by the exponential function and its natural logarithmic inverse.

3.1.1. Exponential functions

Activity 3.1.1.

Consider the function $h(x)=3^x$

a) Complete the following table

x	-10	-1	0	1	10
$h(x)=3^x$					

b) Discuss whether $\forall x\in\mathbb{R}, h(x)\in\mathbb{R}$ and deduce the domain of $h(x)$

c) Discuss whether $h(x)$ can be negative or not and deduce the range of $h(x)$.

1. Definition

With $a\neq 0$ and $a\neq 1$, an exponential function is written as function in the form $f(x)=a^x$, where a is the base and x is an exponent. An exponential function is also defined as:

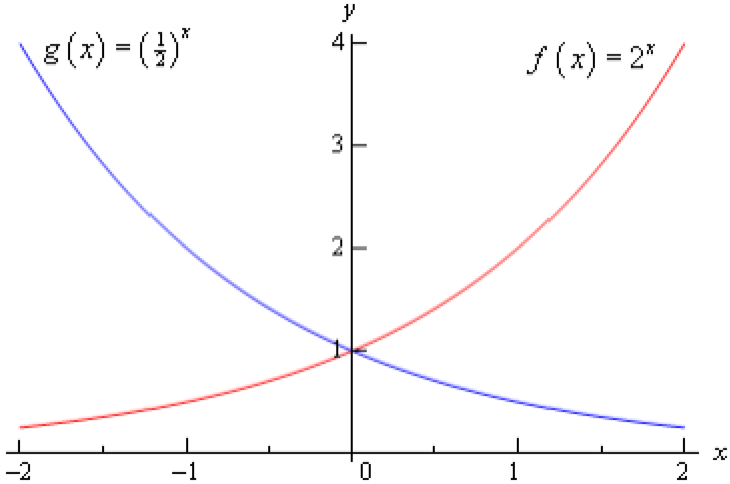
$\exp_a:\mathbb{R}\to\mathbb{R}_0^+ : x\mapsto y=\exp_a x$. For simplicity we write $\exp_a x=a^x$. If the base a of exponential function $f(x)=a^x$ is equal to e where $e=2.71\ldots$, we have natural exponential function denoted by $f(x)=e^x$ or $f(x)=exp(x)$. The exponential function $y=e^x$ has graph which is like the graph of $f(x)=a^x$ where $a>1$.

Example 1

On the same graph, sketch the graph of $f(x)=2^x$ and $g(x)=\left(\frac{1}{2}\right)^x$

x	-2	-1	0	1	2
$f(x)$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4
$g(x)$	4	2	1	$\frac{1}{2}$	$\frac{1}{4}$

Solution:



From this, graph $f(x)$ is increasing while $g(x)$ decrease and they meet at point $(0,1)$. It is very important to note that exponential function $f(x) = a^x$ is:

- Ever zero
- Always positive
- Only taking value 1 when x=0

2. Domain and range of exponential functions

With the definition $y = f(x) = a^x$ and the restrictions that $a > 0$ and that $a \neq 1$, the domain of an exponential function is the set of all real numbers or $]-\infty, +\infty[$. The range is the set of all positive real numbers or $]0, +\infty[$ because there is no exponent that can turn $y = f(x) = a^x$ into a zero nor into a negative result.

Note: Generally, if $u(x)$ is a defined function of x, the range and domain of the function $f(x) = a^{u(x)}$ and its domain will depend on $u(x)$.

Example 1

Determine the domain and the range of the function $f(x) = 3^{\sqrt{2x}}$

Solution

Condition for the existence of $\sqrt{2x}$ in $\mathbb{R}$: $x \geq 0$. Thus, $Domf = [0, +\infty[$ and the range is $[1, +\infty[$.

Example 2

Find the domain and the range of $f(x) = 3^{\left(\frac{x+1}{x-2}\right)}$

Solution

Condition: $x-2\neq 0 \Rightarrow x\neq 2$. *Thus, Domf* $=\mathbb{R}\setminus\{2\}$ *and the range is* $]0,3[\cup]3,+\infty[$

Example 3

Find the domain and the range of $f(x)=4^{\sqrt{x^2-4}}$

Solution

Condition: $x^2-4\geq 0 \Rightarrow x\in]-\infty,-2]\cup[2,+\infty[$. Thus, $Domf=]-\infty,-2]\cup[2,+\infty[$ and the range is $[1,+\infty[$

Example 4

Determine the domain and the range of each of the following functions:

1. $g(x)=e^{\frac{x+2}{x-3}}$
2. $h(x)=e^{\sqrt{x^2-4}}$

Solution

1. Condition for the existence of $\frac{x+2}{x-3}$ in $\mathbb{R}$: $x\neq 3$.

   Therefore $Dom\ g=\mathbb{R}\setminus\{3\}=]-\infty,3[\cup]3,+\infty[$ and range is $]0,e[\cup]e,+\infty[$

2. Condition $x^2-4\geq 0 \Rightarrow x\in]-\infty,-2]\cup[2,+\infty[$.

   Thus, $Dom\ h=]-\infty,-2]\cup[2,+\infty[$ and range is $[1,+\infty[$.

Example 5

Find the domain and the range of $f(x)=e^{\sqrt{x}}$

Solution

Condition: $x\geq 0$. Thus, $Domf=[0,+\infty[$ and range is $[1,+\infty[$.

Example 6

Find the domain and the range of $g(x)=e^{\frac{x+1}{x-2}}$

Solution

Condition: $x-2\neq 0 \Rightarrow x\neq 2$. Thus, $Domg=\mathbb{R}\setminus\{2\}$ and range is $]0,e[\cup]e,+\infty[$.

Example 8

Find the domain of $h(x)=e^{\sqrt{x^2-1}}$

Solution

Condition: $x^2-1\geq 0 \Rightarrow x\in \left]-\infty,-1\right]\cup\left[1,+\infty\right[$. Thus, $Domh=\left]-\infty,-1\right]\cup\left[1,+\infty\right[$ and range is $\left[1,+\infty\right[$.

Application activity 3.1.1.

Discuss and determine the domain and range of the following functions

1) $f(x)=5e^{2x}$ 2) $f(x)=3^{\frac{x+1}{x-2}}$ 3) $f(x)=3^{(x+4)}$

3.1.2. Logarithmic functions

Activity 3.1.2.

For the following values of x: 50, 100, $\frac{1}{2}$, 0.7, 0.8, -30, -20, -5, 0.9, 10, 20, and 40

a) Draw and complete the table of values for $\log_{10}(x)$. What do you notice about the value of $\log_{10}(x)$ for $x<0$?

b) Discuss the values of $\log_{10}(x)$ for $0<x<1$, $x=1$ and $x>1$

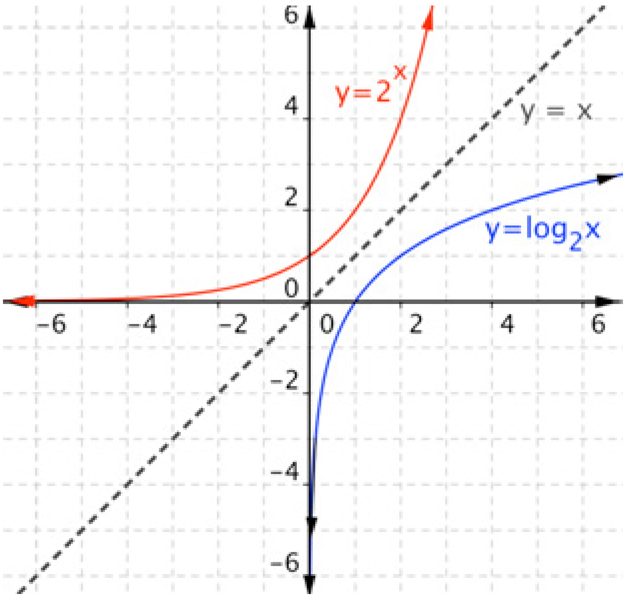
c) Using the findings in a) plot the graph of $\log_{10}(x)$ for $x>0$

d) Explain in your own words what are the values of x for which $\log_{10}(x)$ is defined (the domain) and what are output values (the range).

Given the function $y=\log_a x$, if $x>0$ and a is a constant $(a>0, a\neq 1)$, then $\log_a x$ is a real number called the "**logarithm of** x **with base** a".

1. Definition of logarithmic function

For a positive constant a $(a\neq 1)$, we call logarithmic function, the function $f:\mathbb{R}\rightarrow\mathbb{R}:x\mapsto \log_a x$. In the expression $y=\log_a x$, y is referred to as the logarithm, aa is the base, and xx is the argument. $\log_a x$ means the power to which a must be raised to produce x. If the base is 10, it is not necessary to write the base, and we say **decimal logarithm** or **common logarithm** or **Brigg's logarithm**. So, the notation will become $y=\log x$. $\log x$ means the power to which 10 must be raised to produce x. If the base is e, then $\log_e x$ is usually written $\ln x$ which is called **natural logarithm**. $\ln x$ means the power to which e must be raised to produce x. For every $a > 0\, with\, a \neq 1$, the graph of logarithmic function $\log_b x$ is the inverse of exponential function $\exp_b x$, it follows that the graph of $\log_b x$ and that of $\exp_b x$ are symmetric about the line $x = y$. The following graph shows $y = \log_2 x$ as logarithmic function which is the inverse of exponential function $y = 2^x$, here the base is 2.



2. Domain of definition and range of logarithmic function

The domain of the logarithmic function is the set of positive real numbers, and the range is the line of all real numbers. This means that

$$dom\, f = \{x \in \mathbb{R} : x > 0\} =]0, +\infty[= \mathbb{R}_0^+ \text{ and range } f = \mathbb{R} =]-\infty, +\infty[.$$

The logarithmic function is neither even nor odd. If $u : \mathbb{R} \to \mathbb{R} : x \mapsto u(x)$ is any other function we can compose u and the logarithmic function as $y = \log_a(u(x))$ defined for x such that $u(x) > 0$.

Example

Find the domain and range for the function $f(x) = \log(x-4)$

Solution

To find the domain and the range of the function $y = \log(x-4)$, recalling that:

- **Domain:** Includes all values of x for which the function is defined.
- **Range:** Includes all values y for which there is some x such that $y = \log(x-4)$.

Because $\log x$ defined only for positive values of x . So in this problem $y = \log(x-4)$, is defined if and only if $x-4>0 \Leftrightarrow x>4$ and gives that $x \in]4,+\infty[$.
The range of y is still all real number $\mathbb{R}$.

$Dom\, f = \{x \in \mathbb{R} : x-4>0\} = \{x \in \mathbb{R} : x>4\} =]4,+\infty[.\ Range\, f = \mathbb{R}.$

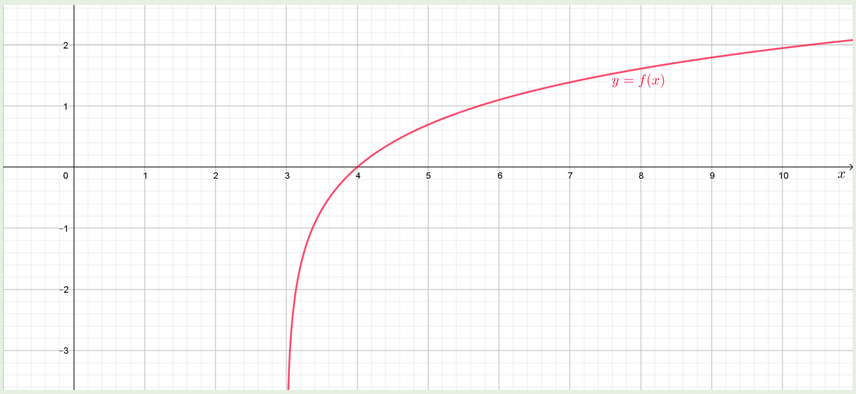
Application activity 3.1.2.

1. State the domain and range of the following functions:

   a. $y = \log_3(x-2)+4$

   b. $y = \log_5(8-2x)$

2. Observe the following graph of a given logarithmic function, then state its domain and range. Justify your answers.



3.1.3. Natural logarithmic functions

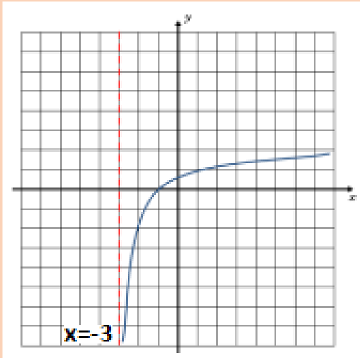
Activity 3.1.3.

1. For each of the following function, when are they defined? Determine their range

   a. $g(x) = \ln(6+x)$

   b. $f(x) = \ln(x-5)^2$

2. Observe the following graph then show the region where the function is defined



Definition of Natural logarithmic function

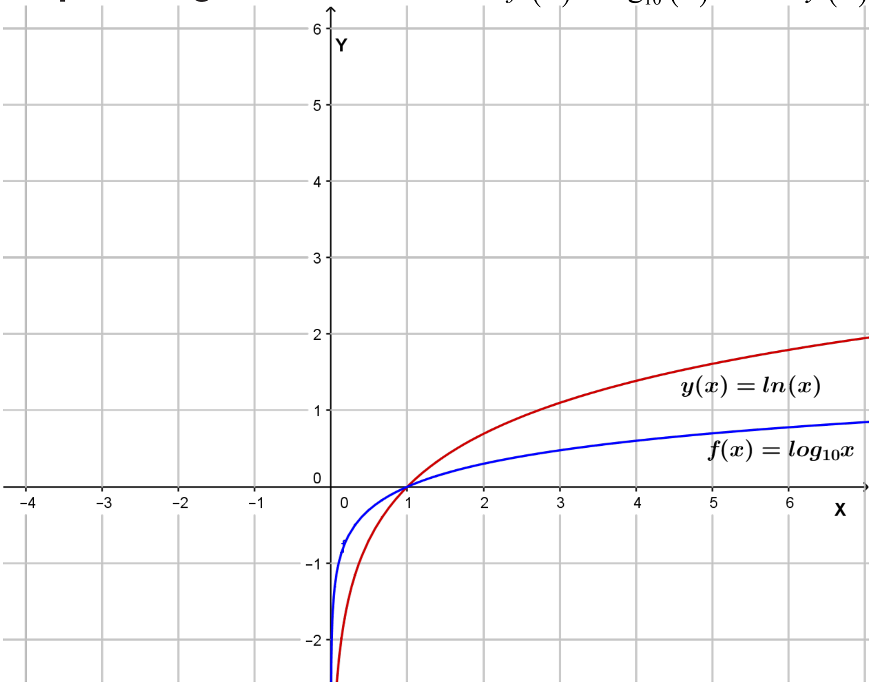
When the base "a" of the logarithmic function $y = \log_a x$ is replaced with natural number e (where $e = 2.718281828$), we have the natural logarithmic function . The natural logarithmic function is usually written using the shorthand notation $y = \ln x$ instead of $y = \log_e x$ as we might expect. The natural logarithmic function $y = \ln x$ is defined on positive real numbers, $]0, +\infty[$ and its range is all real numbers.

Particularly,

- If $x = 1$, then $\ln x = \ln 1$. That is, $\ln x = 0$
- If $x > 1$, then $\ln x > \ln 1$ or $\ln x > 0$
- If $0 < x < 1$, then $\ln x < \ln 1$ or $\ln x < 0$
- For any $x > 0$, then $e^{\ln x} = x$
- $\ln e = 1$

It means that: $\forall x \in]1, +\infty[$, $\ln x > 0$ and $\forall x \in]0, 1[$, $\ln x < 0$. The graph of natural logarithmic function $y = \ln x$ is similar to the graph of $y = \log_a x$ where $a > 1$ as shown in the graph below.

Graphs of logarithmic functions $f(x)=\log_{10}(x)$ **and** $y(x)=\ln(x)$



Domain of definition and range of natural logarithmic function

The logarithmic functions in the form of $f(x)=\ln x$ have domain consisting of positive real numbers $]0,+\infty[$ since exist if and only if $x>0$ and a range consisting the set of all real numbers $]-\infty,+\infty[$. This means that $dom\, f=\{x\in\mathbb{R}: x>0\}=]0,+\infty[=\mathbb{R}_0^+$ and range f $=\mathbb{R}=]-\infty,+\infty[$.

Example 1: Find the domain and range for the function

a. $g(x)=\ln(x+6)$
b. $f(x)=\ln(3-x)$
c. $h(x)=\ln(x-3)^2$

Solutions

a. The function $y=\ln(x+6)$, is defined if and only if $x+6>0 \Leftrightarrow x>-6$ and gives that $x\in]-6,+\infty[$ which is the domain. The range is $\mathbb{R}$

$Dom\ g=\{x\in\mathbb{R}: x+6>0\}=\{x\in\mathbb{R}: x>-6\}=]-6,+\infty[.\ Range\ g=\mathbb{R}.$

b. The domain of f consists of all x for which $3-x>0$. $\Rightarrow x<3$. Thus, the domain of f is given by $\{x|x<3\}$ or $]-\infty,3[$ and Range is given by $]-\infty,+\infty[$ or $\mathbb{R}$

c. The domain of h consists of all x for which $(x-3)^2 > 0$. Thus, the domain of h is given by $\{x | x \neq 3\}$ or $]-\infty, 3[\cup]3, +\infty[$ and Range is given by $]-\infty, +\infty[$ or $\mathbb{R}$

Application activity 3.1.3.

Find the domain and range of the following function

a. $f(x) = \ln(4-x)$

b. $h(x) = \ln x^2$

c. $f(x) = \ln(-4+x)^2$

3.2. Exponential and logarithmic equations

3.2.1.Exponential equations

Activity 3.2.1.

1. For which value(s), each function $f(x)$ below can be defined. Explain.

   a. $f(x) = e^{(x+2)}$

   b. $f(x) = e^{x^2-5x+6}$

2. Detect the value of x, if $2^{1-x} = 6$

Equations that involves powers (exponents) as terms of their expressions are refered to as exponential equations. Such equations can some times be solved by appropriately applying the properties of exponents or introducing logarithms within expression.

1. Solve exponential equations using one- to- one property

To solve exponential function using properties follow the three steps:

a. Rewrite both sides of the equation as an exponential expression with the same base.

b. Since the bases are equal, then the exponents must be equal. Set the exponents equal to each other

c. Solve the equations and check answers

Example 1: Solve the following exponential equations

a. $3^{x+2} = 27^x$

b. $4^{x-1} = 16$

Solutions

a. Domain of validity $x \in \mathbb{R}$,

$$3^{x+2} = 27^x \Rightarrow 3^{x+2} = \left(3^3\right)^x$$

$$3^{x+2} = 3^{3x}$$

$$x + 2 = 3x \Rightarrow 2 = 2x, \text{ then, } x = 1$$

Therefore, solution $S = \{1\}$

b. Domain of validity $x \in \mathbb{R}$,

$$4^{x-1} = 16^x \Rightarrow 4^{x-1} = (4)^2$$

$$4^{x-1} = 4^2$$

$$x - 1 = 2 \Rightarrow x = 3$$

Therefore, solution $S = \{3\}$

Example 2: Solve the following exponential equations

$a)\ e^{-x^2} = \left(e^x\right)^2 . \frac{1}{e^3}$ b) $2^{2y} + 3\left(2^y\right) = 4$

Solution

$a)\ e^{-x^2} = \left(e^x\right)^2 . \frac{1}{e^3}$

- *Domain of validity* $: x \in \mathbb{R}$

$e^{-x^2} = e^{2x} . e^{-3} = e^{2x-3}$

$e^{-x^2} = e^{2x-3} \Rightarrow -x^2 = 2x - 3$

then, $x^2 + 2x - 3 = 0$

The solution of this equation $x^2 + 2x - 3 = 0$ gives $x = -3$ or $x = 1$

The solution set is $S = \{-3, 1\}$

b) $2^{2y}+3(2^y)=4 \Leftrightarrow (2^y)^2+3(2^y)=4$

Let $2^y=x$, then $x^2+3x=4$

$\Rightarrow x^2+3x-4=0$

$\Rightarrow (x+4)(x-1)=0$

$\Rightarrow x=-4$ or $x=1$

Replacing the value of x in the equation $2^y=x$

For $x=-4$: $2^y=-4$ doesn't exist

For $x=1$: $2^y=1$ $\Rightarrow y=0$ $\therefore S=\{0\}$

2. **Solve an exponential equations by taking logarithms for both sides**

To solve an exponential equation using logarithms the four steps followed:

a. Isolate the exponential expression
b. Take logarithms for both sides
c. Rewrite exponential side as a linear expression
d. Solve the obtained equation and check the answer

Example 1: solve an exponential equation $2e^{x+1}-4=12$

Solution: let first isolate the exponential expression, $2e^{x+1}-4=12$

$2e^{x+1}=16 \Rightarrow e^{x+1}=8$ apply logarithm for both sides to get $\ln e^{x+1}=\ln 8$

$(x+1)\ln e=\ln 8 \Rightarrow x+1=\ln 8$

$x=\ln 8-1$ then, $x\approx 1.079$ therefore solution $S=\{1.079\}$

Example 2

Solve each of the following equations

$a) e^x=5$ $\qquad b) 10+e^{0.1t}=14$

Solution

$a) e^x=5$

- $Domain\ of\ validity: x\in\mathbb{R}$

$\ln e^x=\ln 5$

$x\ln e=\ln 5;\quad where\quad \ln e=1$

$x=\ln 5$

- $S=\{\ln 5\}$

$b) 10+e^{0.1t}=14$

- *Domain of validity* : $t \in \mathbb{R}$

$e^{0.1t}=4$

$\ln e^{0.1t}=\ln 4$

$0.1t=\ln 4$

$t=10\ln 4$

- $S=\{10\ln 4\}$

Example 3

Solve each equation

a) $5+3^{t-4}=7$ b) $3\left(2^{4x}\right)-7\left(2^{2x}\right)+4=0$

Solution

a) $5+3^{t-4}=7$

$3^{t-4}=2 \Rightarrow \ln 3^{t-4}=\ln 2$

$\Rightarrow (t-4)\ln 3=\ln 2 \Rightarrow t=4+\dfrac{\ln 2}{\ln 3}$

Therefore, $t=4+\dfrac{\ln 2}{\ln 3}$

b) $3\left(2^{4x}\right)-7\left(2^{2x}\right)+4=0 \Leftrightarrow 3\left(2^{2x}\right)^2-7\left(2^{2x}\right)+4=0$

Let $2^{2x}=k$, then $3k^2-7k+4=0$

$3k^2-7k+4=0$

$\Delta=1 \Rightarrow k=\dfrac{4}{3}$ or $k=1$

Replacing the value of k in the equation $2^{2x}=k$

For $k=\dfrac{4}{3}$:

$2^{2x}=\dfrac{4}{3}$

$2x\ln 2=\ln\dfrac{4}{3}$

$x=\dfrac{\ln 4-\ln 3}{2\ln 2}$

For $k=1$: $2^{2x}=1$

$\Rightarrow 2x\ln 2=\ln 1$

$\Rightarrow x=0$

$\therefore S=\left\{0,\dfrac{\ln 4-\ln 3}{2\ln 2}\right\}.$

Application activity 3.2.1.

1) Solve each equation for x or t.

$a)5+e^{0.2t}=10$ $\quad b)e^{2x}=3e^{x}$ $\quad c)e^{2x}=e^{x}+12$ $\quad d)e^{t}=12-32e^{-t}$

2) Solve $a)2e^{-x+1}-5=9$ $\quad b)\frac{50}{1+12e^{-0.02x}}=10.5$ $\quad c)e^{\ln x^{2}}-9=0$

$d)e^{x}-12=\frac{-5}{e^{-x}}$

3) Solve in the following equations in the set of real numbers:

$\frac{e^{x}+e^{-x}}{2}=1,\left(\text{Hint: multiply by } e^{x}\right)$

4) Find the value of marked letter in each equation.

$a)9^{t}+3^{t}=12$

$b)2^{x}+2^{x-1}=\frac{3}{2}$

$c)\frac{2^{x}}{4}-\frac{3^{x}}{9}=0$

$d)\left(\frac{5}{2}\right)^{x}=0.16$

$e)4^{x}-10\cdot 2^{x}+16=0$

$f)5^{m}\sqrt[m]{8^{m-1}}=500$

5) Solve

$a)2^{4x}-6\cdot 2^{3x}+6\cdot 2^{x}-1=0$

$b)4^{x+1}+31\cdot 2^{x-1}=2$

$c)2^{x}+\frac{1}{2^{x}-7}=9$

$d)81^{x}+81^{1-x}=30$

3.2.2. Logarithmic equations

Activity 3.2.2.

1) Use the properties for logarithm to determine the value of x in the following expressions:

   a) $\log x=2$

   b) $\log(100x)=2+\log 4$

   c) $\log_{2} x=-3$

2) For which values of x satisfies the following expression

$\log_{2} x=5-\log_{2}(x+4)$

1. Logarithmic equations involving one unknown

Logarithmic equation in $\mathbb{R}$ is the equation containing the unknown within the logarithmic expression. To solve logarithmic equations the following steps are followed:

- Set existance conditions for solution(s) of equation.
- Express logarithms in the same base
- Use logarithmic properties to obtain:

  $\log_a u(x) = \log_a v(x) \Leftrightarrow u(x) = v(x)$; where $u(x)$ and $v(x)$ are the functions in x.
- Make sure that the value(s) of unknown verifies the conditions above.

The properties of logarithms can be used to solve logarithmic equations.

Example 1: Solve the following equation $\log_x 49 = 2$

Solution: $\textit{condition of validity} : x > 0, x \neq 1$

From the equation $\log_x 49 = 2$, we change logarithmic equation to exponential equation to get $x^2 = 49$.

$x = \pm 7$ $\textit{for } x > 7$ therefore, solution set is $\{7\}$

Example 2: Solve each equation

$a) \log_3(x+1) = \log_3 2$ $\quad b) \log_{x-2} 3 = 1$ $\quad c) \log_2(x+14) + \log_2(x+2) = 6$

Solution

$a) \log_3(x+1) = \log_3 2$

Condition: $x+1>0 \Leftrightarrow x>-1$,

Then, $\log_3(x+1) = \log_3 2$ (simplify) to obtain

$\Leftrightarrow x+1=2$ $\quad \textit{then}, x=1$ $\quad \therefore S=\{1\}$

$b) \log_{x-2} 3 = 1$

Condition: $x-2>0 \Leftrightarrow x>2$ and $x-2 \neq 1 \Leftrightarrow x \neq 3$ this means $x \in]2,3[\cup]3,+\infty[$

Then, $\log_{x-2} 3 = 1 \Leftrightarrow \log_{x-2} 3 = 1og_{x-2}(x-2)$

$$\Leftrightarrow 3 = x-2$$
$$\Leftrightarrow x = 5$$
$$\therefore S = \{5\}$$

$c) \log_2(x+14)+\log_2(x+2)=6$

Condition: $x+14>0$ and $x+2>0 \Leftrightarrow x>-14$ and $x>-2$

$$\Leftrightarrow x\in]-2,+\infty[$$

$$\log_2(x+14)+\log_2(x+2)=6 \Leftrightarrow \log_2(x+14)(x+2)=6\log_2 2$$

$$\Leftrightarrow \log_2(x+14)(x+2)=\log_2 2^6$$

$$\Leftrightarrow (x+14)(x+2)=64$$

$$\Leftrightarrow x^2+16x+28-64=0 \Leftrightarrow x^2+16x-36=0$$

$$\Leftrightarrow (x+18)(x-2)=0 \Leftrightarrow x=2 \text{ or } x=-18$$

$$\therefore S=\{2\}$$

2. Systems of equations involving logarithms

In solving systems of equations using logarithms, like one-variable logarithmic equations require the same set of techniques like logarithmic identities and exponents, which help to rephrase the logarithms in ways that make it easier to solve for the variables. Algebraic procedures like substitution and elimination can used in the creation of a one-variable equation that is simple to solve.

Example: solve the following system of equations

$$\begin{cases}\log x+\log y=1\\ \log(10x)-\log y=2\end{cases}$$

Solution

Let us apply elimination method by eliminating the variable y,

$$\begin{cases}\log x+\log y=1\\ +(\log(10x)-\log y=2)\end{cases} \Rightarrow \frac{\begin{cases}\log x+\log y=1\\ \log(10x)-\log y=2\end{cases}}{\log x+\log(10x)=3}$$

Use logarithmic property of addition to solve for x the new equation $\log x+\log 10x=3$,

$\log(10x.x)=3 \Rightarrow \log 10x^2=3$

$\log 10x^2=3\log 10 \Rightarrow \log 10x^2=\log 10^3$

$10x^2=10^3 \Rightarrow x=\pm 10$, since for -10 is undefined, we reject it and keep $x=10$.

Substituting the value of $x=10$ in any of the equations in the system, we get $\log(10)+\log y=1 \Rightarrow \log y=0$. This gives the value of $y=1$.

The solution is $S=\{(10,1)\}$

Example 2: solve the following system of equations

$$\begin{cases}\log x+\log y=1\\ y=2x+1\end{cases}$$

Solution

Let use substitution method, take equation (1), $\log x+\log y=1$ and apply property to get $\log(x.y)=1 \Rightarrow xy=10 \Rightarrow x=\frac{10}{y}$. Replace the value of x in equation (2) to get $y=2\left(\frac{10}{y}\right)+1 \Rightarrow y^2-y-2=0$. By solving this quadratic equation $y=5, y=-4$. Since at -4 is not defined then, we take $y=5$. By replacing $y=5$ in equation (2) we get $x=2$. The solution for the given system of equation is given by $S=\{2,5\}$.

Application activity 3.2.2.

1. For each of the following function solve for x.

   a) $\log(x+2)=2$

   b) $\log x+\log(x^2+2x-1)-\log 2=0$

   c) $\log(35-x^3)=3\log(5-x)$

   d) $\log(1-x)=-1$

   e) $\log(3x-2)+\log(3x-1)=\log(4x-3)^2$

2. In $\mathbb{R}^2$, solve the following equations

   a. $\begin{cases}x+y=9\\ \log x+\log y=\log 14\end{cases}$

   b. $\begin{cases}x^2+y^2=221\\ \log_5 x+\log_5 y=\log_5 110\end{cases}$

3.2.3. Natural logarithmic equations

Activity 3.2.3.

1. Let three functions be defined by

   $a. f(x) = \ln x$ $\quad b. h(x) = \ln(x+2)$ $\quad c. \ln(x^2 - 5x + 6)$

   For which value(s) of x, each function is defined.

2. Use the properties for logarithm to determine the value of x in the following expressions:

   $a. \ln x = 10$ $\quad b. \ln x = 3$

The Natural logarithmic equation in $\mathbb{R}$ is the equation containing the unknown within the natural logarithmic expression. To solve Natural logarithmic equations the following steps are followed:

- Set existance conditions for solution(s) of equation.
- Use logarithmic properties to obtain:
  $\ln u(x) = \ln v(x) \Leftrightarrow u(x) = v(x)$; where $u(x)$ and $v(x)$ are the functions in x.
- Make sure that the value(s) of unknown verifies the conditions above.

Note:

- $y = \ln x = \log_e x \Leftrightarrow e^y = x$ (as inverse)

  The equation $\ln x = 1$ has a unique solution; the rational number $2.71828182845904523536...$ and this number is represented by letter e. Thus, if $\ln x = 1$ then $x = e$

The properties of natural logarithms can be used to solve natural logarithmic equations.

Example 1:

Solve each equation

$a)\ \ln x - \ln 5 = 0$ $\qquad b)\ 3 + 2\ln x = 7$ $\qquad c)\ \ln 2x + \ln(x+2) = \ln 6$

Solution

$a)\quad \ln x - \ln 5 = 0$

$\textit{condition of validity}: x > 0$

$\textit{Then};\ \ln x = \ln 5 \Leftrightarrow x = 5$

Therefore, the solution $S=\{5\}$

$b)\quad 3+4\ln x=7$

$condition\ of\ validity: x>0$

By Solving:

$4\ln x=7-3 \Leftrightarrow \ln x=1 \quad \Leftrightarrow \ln x=\ln e \Leftrightarrow x=e$

Hence, $S=\{e\}$

$c)\quad \ln 2x+\ln(x+2)=\ln 6$

$Conditions\ of\ validity: 2x>0\ and\ x+2>0$

$$\Leftrightarrow x>0\ and\ x>-2$$

$$Domain\ of\ validity: x\in\,]-2,+\infty[\, \cap\,]0,+\infty[\, \Leftrightarrow x\in\,]0,+\infty[$$

Solving:

$$\ln 2x+\ln(x+2)=\ln 6 \Leftrightarrow \ln[2x(x+2)]=\ln 6$$
$$\Leftrightarrow 2x(x+2)=6$$
$$\Leftrightarrow 2x^2+4x-6=0$$

$\Leftrightarrow x^2+2x-3=0$

$\Leftrightarrow (x-1)(x+3)=0$

$\Leftrightarrow x=1\ or\ x=-3$

As $x\in\,]0,+\infty[\,; then\ S=\{1\}$

Systems of equations involving natural logarithms

In solving systems of equations using natural logarithms, like one-variable natural logarithmic equations require the same set of techniques like logarithmic identities and exponents, which help to rephrase the natural logarithms in ways that make it easier to solve for the variables. Algebraic procedures like substitution and elimination can used in the creation of a one-variable equation that is simple to solve.

Example: solve the following system of equations

$$\begin{cases}\ln(xy)=7\\ \ln\left(\dfrac{x}{y}\right)=1\end{cases}$$

Solution

$$\begin{cases}\ln(xy)=7\\ \ln\left(\dfrac{x}{y}\right)=1\end{cases}\Rightarrow\begin{cases}\ln x+\ln y=7\\ \ln x-\ln y=1\end{cases},$$

solve the system of equations started eliminating variable y

$$\Rightarrow\begin{cases}\ln x+\ln y=7\\ +(\ln x-\ln y=1)\end{cases}\quad\Rightarrow\frac{\begin{cases}\ln x+\ln y=7\\ +(\ln-\ln y=1)\end{cases}}{2\ln x=8}$$

$2\ln x=8\Rightarrow\ln x=4$

$\ln x=4\ln e\quad\Rightarrow x=e^4$ replacing the value of x in equation (1), $\ln e^4+\ln y=7$ then by solving this equation the solution for y is $y=e^3$. Solution of the system is $S=\{e^4,e^3\}$

Application activity 3.2.3.

1. Solve each of the following equations

   a. $\ln x=0$

   b. $\ln(x^2-1)=\ln(4x-1)-2\ln 2$

   c. $2\ln 4x=7$

   d. $\ln x+\ln 4=0$

   e. $\ln 2x=\ln 2.4$

2. Solve the following system

$$\begin{cases}2\ln x+3\ln y=-2\\ 3\ln x+5\ln y=-4\end{cases}$$

3.3. Applications of exponential and logarithmic functions

Activity 3.3.

Considering that you go in bank to open a savings account that will generate an interest on the deposited money, the bank promises you that the interest will be calculated on the previous accumulated interest. If you need to check your final value after 5 years, explain how you can proceed? Does exponential or logarithmic functions/equations helpful to this situation? Justify your point of view.

In this section, we are interested on four of the most common applications of exponential and logarithmic functions such as: compound Interest, population growth, Depreciation/discounting, and amortization

1. Compound interest

A compound interest plan pays interest on interest already earned. The value of an investment depends not only on the interest rate, but how frequently the interest is compounded. If, for example, a $100 investment is made with 5% interest compounded annually, after one year, the investment will be worth $105. The next year, the interest added to the value of the investment will be 5% of the $105. Compound interest causes the amount of interest earned to increase with every compounding period.

Let $A(t)$ be the accumulated amount at time t. The formula $A(t)=P\left(1+\frac{r}{n}\right)^{nt}$, where P is the principal, r is the interest rate, n is the number of times the interest is compounded each year, and t is the number of years since the investment was made.

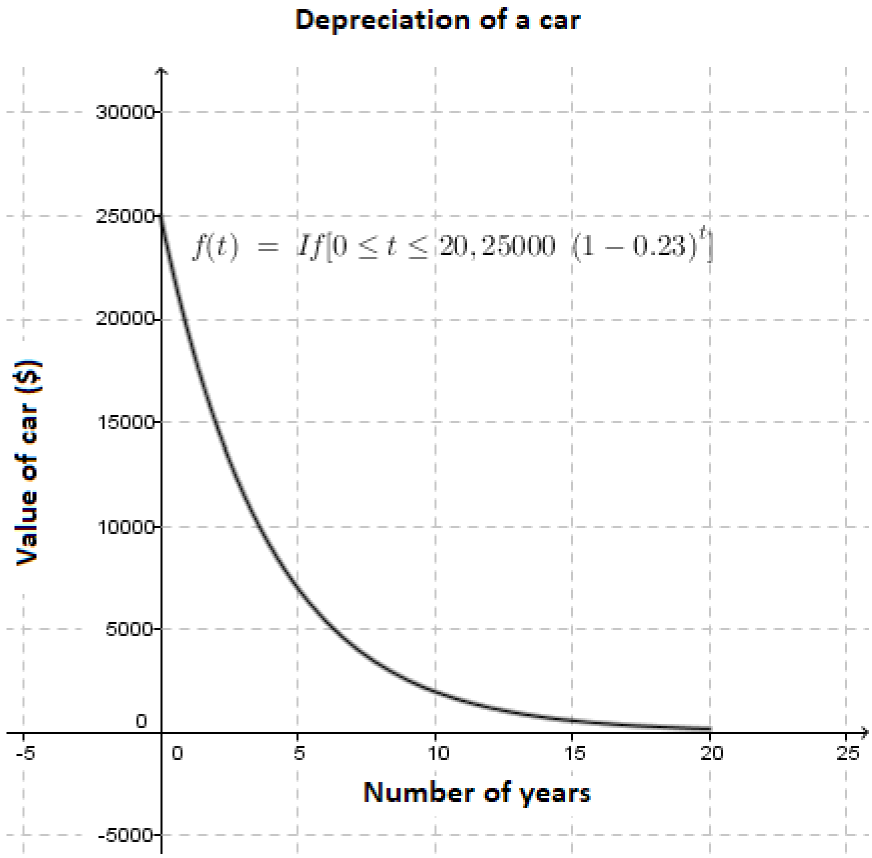
When the interest on an investment is compounded continuously, a natural exponential function is used. Let the function $A(t)$ model the value of an investment made with continuous compounding. $A(t) = Pe^{rt}$, where P is the principal, r is the interest rate, and t is the number of years since the investment was made. Continuously compounded interest allows for the fastest growth of the value of an investment.

2. Depreciation

Businesses and individuals typically own assets. Over time, these assets sometimes lose value due to age or use. This loss of value is called depreciation. One type of depreciation is called declining balance depreciation. It is like reverse compound interest in that the value of the asset is at its highest at the time of purchase and then continuously decreases over time. The depreciation formula, $D=P\left(1-\frac{r}{100}\right)^{n}$, where D is the final value of asset, P is the initial value of the asset, r is the rate of depreciation per period in % and n is the number of depreciation periods. Notice that they are like those for calculation of compound interest.

For example, suppose we purchased a car valued at $25 000 and we know that cars depreciate at 22.5% each year, what would the value of the car be after a period. The depreciation of this car can be shown by the calculations for 5 years. If we graph the function determined in this example, we will see that it is typical exponential decay shape if taken over 20 years.

The graph showing depreciation of a car is shown below



3. Population Growth

When a population has a constant relative growth rate, its size can be calculated using a natural exponential function. The population *P* after *t* units of time $P(t) = P(0)e^{kt}$, where *k* is the constant relative growth rate, and *P*(0) is the initial population, measure at time zero. The units of time used in problems like these usually are proportional to the life span of the organisms of the population. For populations of bacteria, hours or days are common, and for people, years are common. Populations can also be decrease. In this case, the value of *k* is negative and everything else remains the same

4. Amortization

Amortization refers to how loan payments are applied to specific types of loans. Typically, the monthly payment stays the same and is divided between interest costs (what your lender gets paid for the loan), reducing your loan balance (paying off the loan amount), and other expenses like property taxes. Your Last

Loan Payment will pay off the remaining balance of your debt. For example, you paid off a 30-year mortgage after exactly 30 years (or 360 monthly payments). Amortization tables help us to understand how a loan works and can help us to predict our outstanding balances or interest costs at any point in the future. Before you take out a loan, you want to know whether the monthly instalments fit into your budget. Therefore, calculating the payment amount per period is of paramount importance. The following formula is used for calculating amortization, $A=P\frac{i(1+i)^n}{(1+i)^n-1}$, where A is Periodic payment amount, P is the principal, i is periodic interest rate and n is total number of payments. For example, you want to calculate the monthly payment on a five-year car loan of $20 000, which has an interest rate of 7.5 %. Assuming that the initial price was $21 000 and a down payment of $1 000 has already been made. So, the above formula can help us to estimate the amortization period and the monthly payment amount.

Application activity 3.3.

Make research in library or on internet to find out two more applications of exponential and logarithmic functions in real life

3.4. End unit assessment

End unit assessment 3

1. Solve each equation

   a. $10+e^{0.1t}=14$ b. $3+2\ln x=7$ c. $(6.5)^x= 44$

   d. $\log_3(x-1)^2=0$

2. Solve the following system of equations

   a. $\begin{cases} x-y=8 \\ \log_2 x-\log_2 y=1 \end{cases}$ b. $\begin{cases} \ln x-\ln y=1 \\ \ln 2x+\ln y=3 \end{cases}$

3. Determine the domain and range of the following functions

   a. $f(x)=\log_2(3x-2)$ b. $f(x)=\ln(x^2-1)$

4. Solve the following equations

   a. $\log(2x^2+3)=\log(x^2+5x-3)$ b. $\ln x+\ln(x-3)=\ln 10$

5. Considering that you apply for a (P) loan in a bank that will compounded continuously and pay 11% of interest every year. You used the received loan to buy a house in countryside, after n years the cost of that house starts to decrease at 12% every year. Its value continues to reduce over 15 years.

   a. How can you calculate the final value that you will pay? Justify your answer?

   b. How do we call this kind of reduction? How can we calculate the final value after reduction over 15 years?

   c. How can you estimate the amortization period and monthly payment amount?

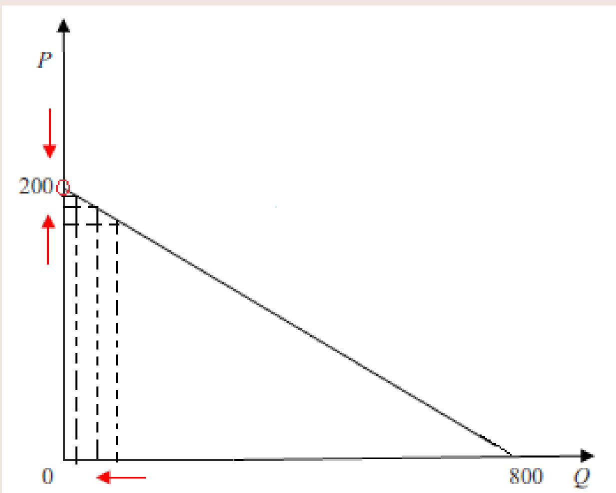
UNIT 4 LIMIT OF POLYNOMIAL, LOGARITHMIC, EXPONENTIAL FUNCTIONS AND APPLICATIONS

Key unit competence: Apply limits in solving production, financial and economical related problems.functions and equations

4.0. Introductory activity

Introductory activity

Let us take a function denoted as $P(Q) = 200 - 0.24Q$, where P, and Q stand for price and quantity respectively. The function of quantity $f(Q) = P$ is given by the following sketch/graph.



Graph of f (Q, P)

1) Complete the table and add the approximate value of price (P) when the quantity (Q) approaches to 0.

Q	1	0.5	0.1	0.01	0.001	0.0001	…	0
P								

The price, P, gets "closer and closer "to ….., When the quantity, Q, approaches to 0.

2) Complete the table and approximate the value of price, P, when the quantity, Q, approaches 20.

Q	19.5	19.9	19.999	20	20.1	20,2	20,5	21
P								

When Q approaches to 20, the price gets "closer and closer" to ……, This can be written as $\lim_{Q \to 20} P = ...$

4.1. Limits of numerical functions

4.1.1. Definition of Limit and Neighbourhood of a Real Number

Activity 4.1.1.

When finding the value of a function $f(x)$ when x approaches 2, a student used a calculator and drew a table as follows:

x	$f(x)$	x	$f(x)$
2.5	3.4	1.5	5.0
2.1	3.857142857	1.9	4.157894737
2.01	3.985074627	1.99	4.015075377
2.001	3.998500750	1.999	4.001500750
2.0001	3.999850007	1.9999	4.000150008
2.00001	3.999985000	1.99999	4.000015000

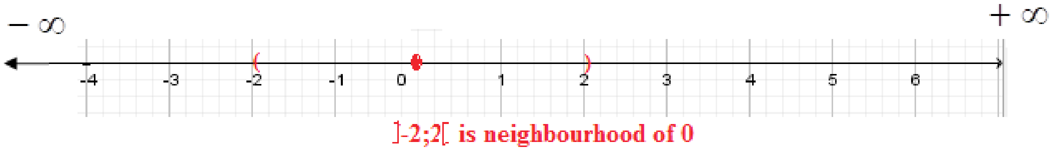
a) Is it possible to put the values of x on a number line? Try to do it and locate the point $x = 2$

b) Write 2 possible open intervals of the number line such that their centre is x =2.

c) Try to approximate the value of $f(x)$ when x approaches 2.

i) Neighbourhood of a Real Number

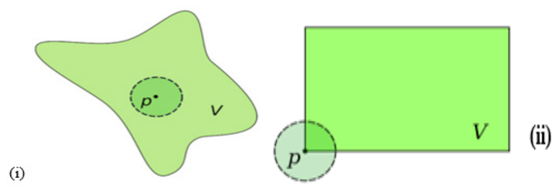
Let W be a number which is represented by a point on the real line. A neighbourhood of W is any open interval centered at W. A neighbourhood can be large or small; for example, the intervals $]-2,2[$, $]-100,100[$ and $]-0.001,0.001[$ are all neighbourhoods of 0, while the intervals $]-2,0[$ and $]-1.01;-0.99[$ are neighbourhoods of -1.



Considering the above number line, we picked values of x that were located on both sides of zero, take $x=-2$ in the left hand side, $x=2$ at the right hand side to make sure that any trends that we might be seeing are in fact correct. Mathematically, a set I, where $I \subset \mathbb{R}$, is called a neighbourhood of point P if there exists an open interval, with center at P and the interval is contained in I. The collection of all neighbourhoods of a point is called the **neighbourhood system** at the point.

Examples:

1. A set *V* in the plane is a neighbourhood of a point *p* if a small disk around *p* is contained in *V* as illustrated in (i) and a rectangle is not a neighbourhood of any of its corner as represented in (ii) below:



The interval $]-1,1[=\{y;-1<y<1\}$ is a neighborhood of $x=0$, so the set $]-1;0[\cup]0;1[=]-1,1[-\{0\}$ is a deleted neighbourhood of 0.

ii) Definition of limit

Consider a function $f(x)$ defined on some reduced neighbourhood of a certain point a. We say that a real number L is a limit of $f(x)$ as x tends to a, and value $f(a)$ does not need to be defined. If for every $\varepsilon>0$ there exists some $\delta>0$ so that $\forall x \in Domf$ satisfying $0<|x-a|<\delta$ we have $|f(x)-L|<\varepsilon$. This is called Cauchy definition for limit, is denoted as $\lim_{x\to a} f(x)=L$

Example:

By using Cauchy definition for limit, show that $\lim_{x \to 3}(3x-2)=7$

Solution:

Let $\varepsilon > 0$ be a positive real number, and $\delta = \frac{\varepsilon}{3}$ then $0 < |x-3| < \delta$,

$$|f(x)-L| = |(3x-2)-7| = |3x-9| = 3|3x-3| < 3\delta = 3\times\frac{\varepsilon}{3} = \varepsilon$$

Thus, by $|f(x)-L| < \varepsilon$ the limit is proved.

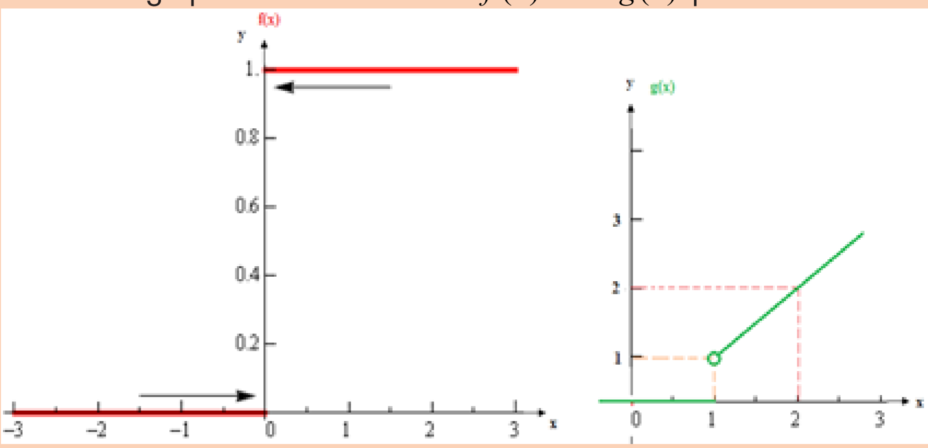
Application activity 4.1.1.

1. A part of Vatican in Europe, give an example of African country that is surrounded by a single country.
2. Give three examples of intervals that are neighbourhoods of -5
3. Is a circle a neighbourhood of each of its points? Why?
4. Draw any plane and show three points on that plane for which the plane is their neighbourhood.

4.1.2. One-sided limits, existence of limit

Activity 4.1.2.

Consider the graphs of two functions $f(x)$ and $g(x)$ plotted below:



Complete:

a) $f(x)=\begin{cases} ....; x<0 \\ .....; x \geq 0 \end{cases}$ and $g(x)=\begin{cases} ....; x \leq 1 \\ ....; x>1 \end{cases}$

b) If we stay to the left side, as x approaches 0, $f(x)$ gets closer to .....;

c) If we stay to the right side, as x approaches 0, $f(x)$ gets closer to ......

d) If we stay to the left side, as x approaches 1, $g(x)$ gets closer to .....;

e) If we stay to the right side, as x approaches 1, $g(x)$ gets closer to .....;.

i) Right-hand sided limit

We say that $\lim_{x\to a^+} f(x) = L$ is t*he right hand side limit, and we can make f(x) as close to L as we want for all x sufficiently close to* a *and* $x > a$ *without actually letting* x *be* a*.*

ii) Left-hand side limit

We say that $\lim_{x\to a^-} f(x) = L$ is t*he left hand side limit* if we can make *f(x)* as close to *L* as we want for all x sufficiently close to a and $x < a$ *without actually letting* x be a.

For the right-hand sided limit, we have $x \to a^+$ (note that the "+") which means that we will only look at *x>a*. Likewise for the left-handed limit we have $x \to a^-$ (note the "-") which means that we will only be looking at *x<a*. So when calculating a limit, it is important to know whether it is a one-sided limit or not. .

iii) Condition of existence for a limit

If the value of $f(x)$ approaches L_1 as *x* approaches x_0 from the right-hand side, we write that $\lim_{x\to x_0^+} f(x) = L_1$ and we read "**the limit of $f(x)$ as *x* approaches x_0 from the right side equals to** L_1. If the value of $f(x)$ approaches L_2 as *x* approaches x_0 from the left-hand side, we write that $\lim_{x\to x_0^-} f(x) = L_2$ and we read it as "**the limit of $f(x)$ as *x* approaches x_0 from the left equals** L_2
If the limit from the left side is the same as the limit from the right side, say $\lim_{x\to x_0^-} f(x) = \lim_{x\to x_0^+} f(x) = L$**,** then we write $\lim_{x\to x_0} f(x) = L$ and we read "**the limit of $f(x)$ as *x* approaches x_0 equals** L. **Note that** $\lim_{x\to x_0} f(x)$ exists if and only if $\lim_{x\to x_0^-} f(x) = \lim_{x\to x_0^+} f(x)$

Always recall that the value of a limit at a point does not actually depend upon the value of the function at that point. The value of the limit of a function at a point only depends on the behaviors of the values of the function around, or in the neighborhood of the given point. Therefore, even if the function is not defined at the point, the limit can still exist.

Example

If, $f(x)=\frac{|x-2|}{x^2+x-6}$, then find the $\lim_{x\to 2^-} f(x)$ and $\lim_{x\to 2^+} f(x)$

Answer:

$f(x)=\frac{|x-2|}{x^2+x-6}$, apply the absolute value properties, and observe that $|x-2|=\begin{cases} x-2, if\ x\geq 2 \\ -(x-2), if\ x<2 \end{cases}$ Therefore, this question is written under (i) and (ii):

i. $\lim_{x\to 2^+} f(x)=\lim_{x\to 2^+}\frac{x-2}{x^2+x-6}$

$=\lim_{x\to 2^+}\frac{x-2}{(x-2)(x+3)}$

$=\lim_{x\to 2^+}\frac{1}{x+3}$

$=\frac{1}{5}$

ii. $\lim_{x\to 2^-} f(x)=\lim_{x\to 2^-}\frac{-(x-2)}{x^2+x-6}$

$=\lim_{x\to 2^-}\frac{-(x-2)}{(x-2)(x+3)}$

$=\lim_{x\to 2}\frac{-1}{x+3}$

$=\frac{-1}{5}$

Since the $\lim_{x\to 2^-} f(x)\neq \lim_{x\to 2^+} f(x)$, then $\lim_{x\to 2} f(x)$ does not exist.

Determine numerically how the function $f(x)=\frac{x^2-9}{x-3}$ behaves near $x=3$.

Note that $f(x)=\frac{x^2-9}{x-3}$ is defined for all real numbers *x* except for $x=3$. For any $x\neq 3$ we can simplify the expression for $f(x)$ by factoring the numerator and cancelling common factors: $f(x)=\frac{(x+3)(x-3)}{x-3}=x+3 \quad for\ x\neq 3$.

Even though $f(3)$ is not defined, as x assumes values closer and closer to 3 $f(x)$ assumes values closer and closer to 6 . Therefore, the limit of function $f(x)=\frac{x^2-9}{x-3}$ as *x* approaches 3,is 6.

We write this as $\lim_{x\to 3} f(x)=6$ or $\lim_{x\to 3}\frac{x^2-9}{x-3}=6$

Formally,

If $f(x)$ is defined for all *x* near *a*, except possibly at *a* itself, and if we can ensure that $f(x)$ is as close as we want to *L* by taking *x* close enough to *a*, but not equal to *a*, we say that the function *f* approaches the limit *L* as *x* approaches *a*, and we write the $\lim_{x\to a} f(x)=L$.

Practically, to find the limit of a function $f(x)$ as *x* approaches *a*, first we evaluate the function at $x=a$, that is we find, if possible $f(a)$ substitute that value a in the function and see what happen. The limit can exist or not.

Examples:

1. $\lim_{x\to 2}(2x+1)=2(2)+1=5$
2. *Since a constant function* $f(x)=k$ *has the same value k everywhere, it follows that at each point* $\lim_{x\to a}k=k$. *For example* $\lim_{x\to 4}5=5$
3. *The limit* $\lim_{x\to a}x=a$. *For example,* $\lim_{x\to -5}x=-5$, $\lim_{x\to 0}x=0$
4. $\lim_{x\to 2}\sqrt{x^2-2x+1}=\sqrt{4-4+1}=1$
5. $\lim_{x\to 3}\frac{\sqrt{2x+1}}{\sqrt[3]{3x-1}}=\frac{\sqrt{7}}{\sqrt[3]{8}}=\frac{\sqrt{7}}{2}$

For any rational function $f(x)=\frac{g(x)}{h(x)}$

If x approaches $a\in\mathbb{R}$, we have three possibilities:

1) $h(x)\neq 0$ for $x=a\Rightarrow \lim_{x\to a}f(x)=f(a)$
2) $g(x)\neq 0$, $h(x)=0$ for $x=a\Rightarrow \lim_{x\to a}f(x)=\infty$;the sign is to be determined
3) $g(x)=0$, $h(x)=0$ for $x=a\Rightarrow \lim_{x\to a}f(x)=\frac{0}{0}$ (Indeterminate form), An indeterminate form, or an indeterminate case, is a case that cannot be calculated by the normal rules of the basic operations (addition ,subtraction, multiplication and division) .An indeterminate case hides the true value of the limit. It is therefore, necessary to remove the indetermination in order to find the true value, if any, hidden by the indetermination.

Examples:

1) $\lim_{x\to 2}\frac{x+4}{2+x}=\frac{2+4}{2+2}=\frac{6}{4}=\frac{3}{2}$
2) $\lim_{x\to 0}\frac{x^2-2x-3}{x+6}=\frac{0-0-3}{0+6}=-\frac{1}{2}$

Application activity 4.1.2.

Find $\lim_{x\to 3}f(x)$ for $f(x)=\begin{cases}x^2-5 & if\ x\le 3\\ \sqrt{x+13} & if\ x>3\end{cases}$

4.1.3. Limits of numerical functions, properties and operations

Activity 4.1.3.

Evaluate

1) $f(2)$ if $f(x)=\frac{x+1}{x+2}$

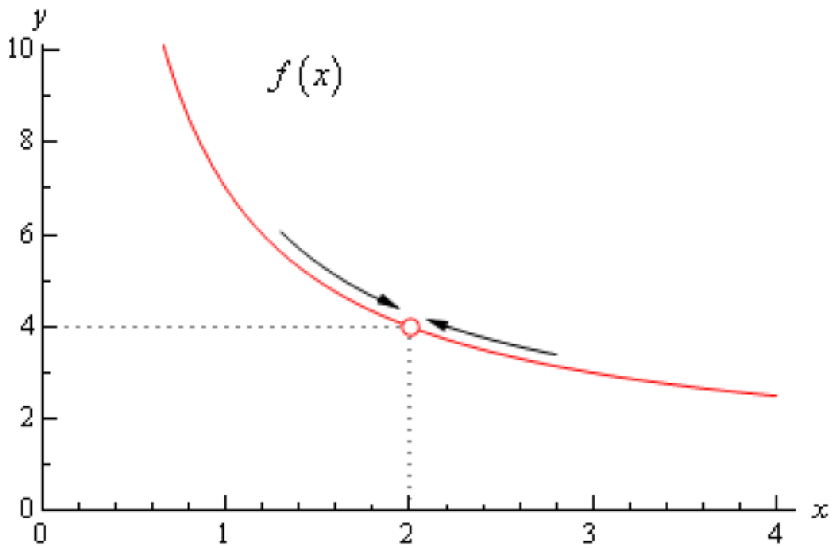
2) $f(3)$ if $f(x)=4x^3-2x^2+3x-1$

2) $f(1)$ if $f(x)=\frac{\sqrt{x+3}}{\sqrt[3]{3x-2}}$

a) Note and example

Limits are used to describe how a function behaves as the independent variable moves towards a certain value.

Let us estimate the values of $f(x)=\frac{x^2+4x-12}{x^2-2x}$ as x approaches 2. As x assumes values closer and closer to 2, from left and from right, the corresponding point moves on the graph as shown on the diagram by the arrows .



The question is what is the $y-coordinate$ of the limiting point on the graph.

In our case we can see that as x moves in towards 2 (from both sides) the function is approaching $y=4$ even though the function itself is not defined at $x=2$.

Therefore, we can say that the limit of the function $f(x)=\frac{x^2+4x-12}{x^2-2x}$ is 4 and we write: $\lim_{x\to 2}\frac{x^2+4x-12}{x^2-2x}=4$

b) Operations and Properties on limits

Consider numerical functions $f(x), g(x)$ and real number k

a) $\lim_{x \to a} k = k$: the limit of a constant equals the constant itself, irrespective to the value of a

b) $\lim_{x \to a} (f \pm g)(x) = \lim_{x \to a} f(x) \pm \lim_{x \to a} g(x)$, for a finite or infinite: the limit of a sum of functions equals the sum of the limits of the terms.

c) $\lim_{x \to a} [kf(x)] = k \lim_{x \to a} f(x)$:the limit of the product of a function by a real number equals the product of the number by the limit of the function.

d) $\lim_{x \to a} (f \cdot g)(x) = \lim_{x \to a} f(x) \cdot \lim_{x \to a} g(x)$:the limit of a product of functions equals the product of the limits of the factors.

e) $\lim_{x \to a} \left(\frac{f}{g}\right)(x) = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}$: the limit of a quotient equals the quotient of the limits of the numerator by the denominator, provided $\lim_{x \to a} g(x) \neq 0$

f) $\lim_{x \to a} [f(x)]^n = [\lim_{x \to a} f(x)]^n$,where n is a positive integer: the limit of the nth power of a function equals the nth power of the limit of the base function, provided it is meaningful .

Examples

1. $\lim_{x \to 3} x^4 = \left[\lim_{x \to 3} x\right]^4 = 3^4 = 81$;

2. Find $\lim_{x \to 5}\left(x^4 - 4x + 3\right)$, this is computed as follows:

$$\begin{aligned}\lim_{x \to 5}\left(x^2 - 4x + 3\right) &= \lim_{x \to 5} x^2 - \lim_{x \to 5} 4x + \lim_{x \to 5} 3 \\ &= \lim_{x \to 5} x^2 - 4\lim_{x \to 5} x + \lim_{x \to 5} 3 \\ &= 5^2 - 4\lim_{x \to 5} x + \lim_{x \to 5} 3 \\ &= 25 - 4(5) + 3 \\ &= 8\end{aligned}$$

3. Find $\lim_{x \to 2} \frac{f(x)}{g(x)}$ *if* $f(x) = 5x^3 + 4$ *and* $g(x) = x - 3$

Solution:

Given $f(x)=5x^3+4$ *and* $g(x)=x-3$

$$\lim_{x\to 2}\frac{f(x)}{g(x)}=\lim_{x\to 2}\frac{5x^3+4}{x-3}$$

$$\lim_{x\to 2}\frac{5x^3+4}{x-3}=\frac{\lim_{x\to 2}(5x^3+4)}{\lim_{x\to 2}(x-3)}$$

$$=\frac{5(2)^3+4}{2-3}$$

$$=-44$$

4. Find $\lim_{x\to 0} f(x)g(x)$ *if* $f(x)=6x^2+2$ *and* $g(x)=x+2$

Solution:

Given $f(x)=6x^2+2$ *and* $g(x)=x+2$,

$$\lim_{x\to 0} f(x)g(x)=\lim_{x\to 0}(6x^2+2)\lim_{x\to 0}(x+2)=2\times 2=4$$

Application activity 4.1.3.

1. Given that $-x^2\le g(x)\le x^2$. Find $\lim_{x\to 0} g(x)$

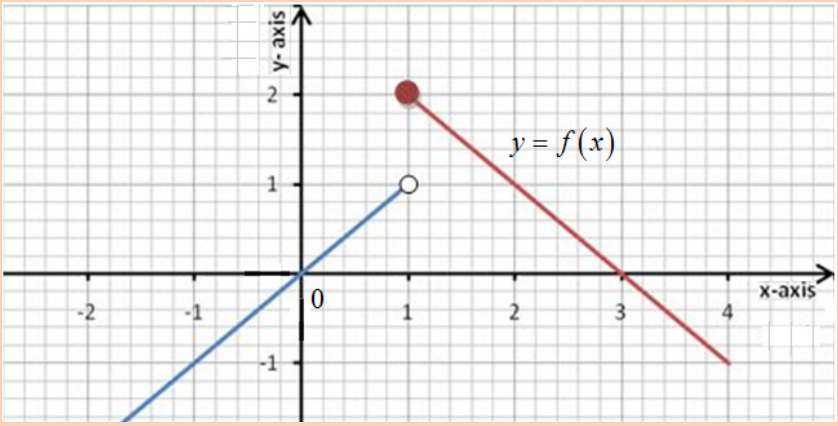
2. If $\lim_{x\to 3} f(x)=6$ and $\lim_{x\to 3} g(x)=-3$. Find
   i. $\lim_{x\to 3}\left[f(x)+g(x)\right]$
   ii. $\lim_{x\to 3}\left[f(x)g(x)\right]^3$

3. Explain why the following calculation is incorrect
   a. $\lim_{x\to 0^+}\left(\frac{1}{x}-\frac{1}{x^2}\right)=\lim_{x\to 0^+}\frac{1}{x}-\lim_{x\to 0^+}\frac{1}{x^2}=\infty-\infty=0$
   b. $\lim_{x\to 0^+}\left(\frac{1}{x}-\frac{1}{x^2}\right)=-\infty$

4.1.4. Graphical interpretation of limit of a function

Activity 4.1.4.

Consider the following curve of function $f(x)$



Using this graph estimate, by inspection:

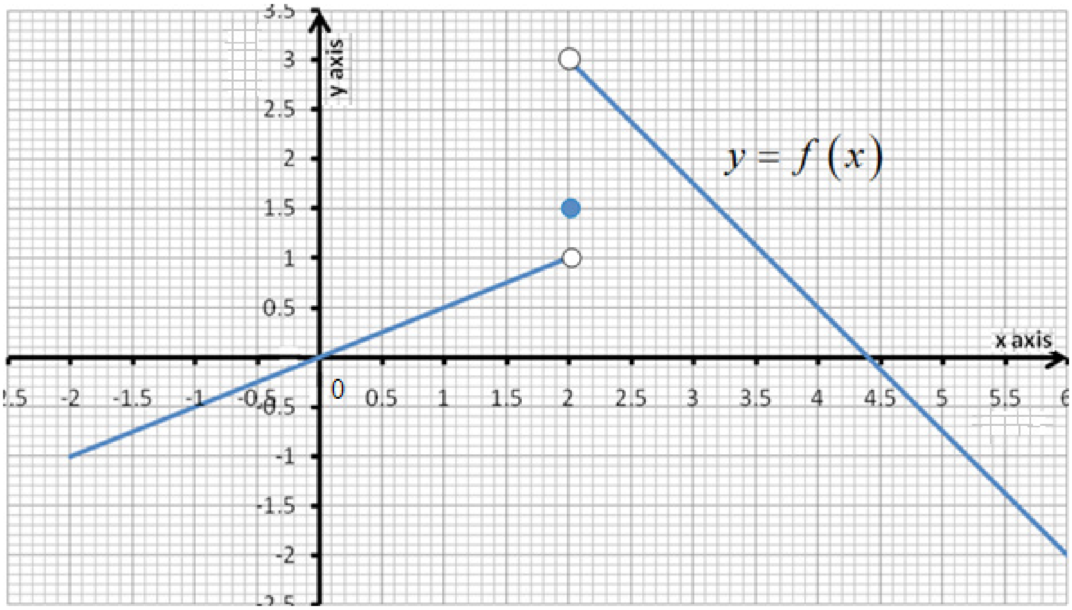
1) $\lim_{x \to 1^-} f(x)$
2) $\lim_{x \to 1^+} f(x)$
3) What can you say about $\lim_{x \to 1} f(x)$?

To determine if a left-hand limit exists, observe the branch of the graph to the left of $x = a$ but near $x = a$. This is where $x < a$. The outputs are close to some real number L, so there is a left-hand limit.

To determine if a right-hand limit exists, observe the branch of the graph to the right of $x = a$ but near $x = a$. This is where $x > a$. The outputs are getting close to some real number L, so there is a right-hand limit. If the two one-sided limits are equal, the function $f(x)$ has a limit.

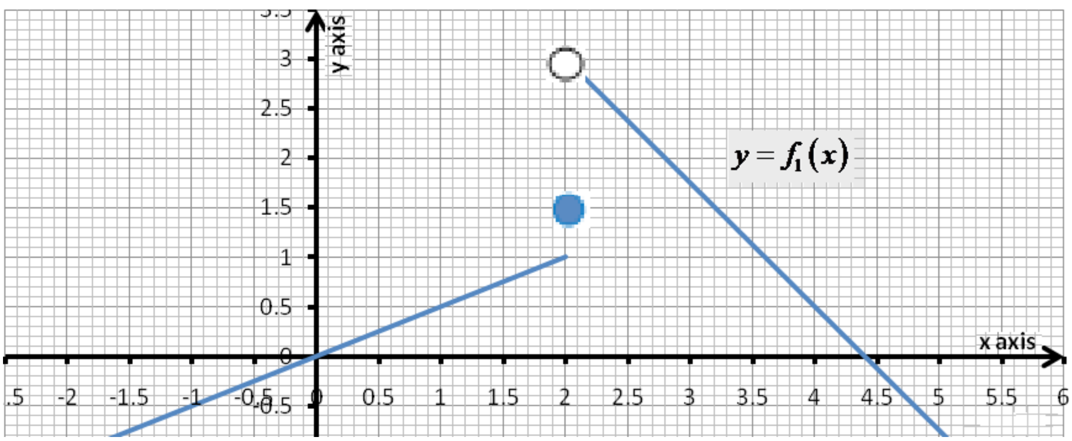
Example:

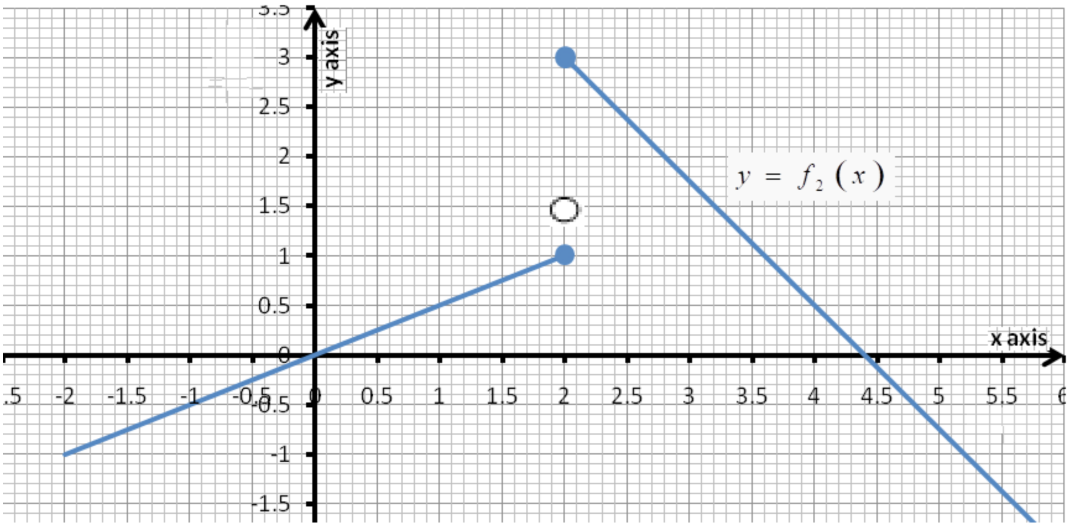
1. Let *f* be the function whose graph is shown below



As x approaches 2 from the left-hand side, $f(x)$ approaches 1, so, the $\lim_{x\to 2^-} f(x) = 1$. As x approaches 2 from the right-hand side, $f(x)$ approaches 3, so $\lim_{x\to 2^+} f(x) = 3$ but $f(2) = 1.5$

Therefore, the value of a function at a point, and the left- and right-hand limits at the point can all be different.

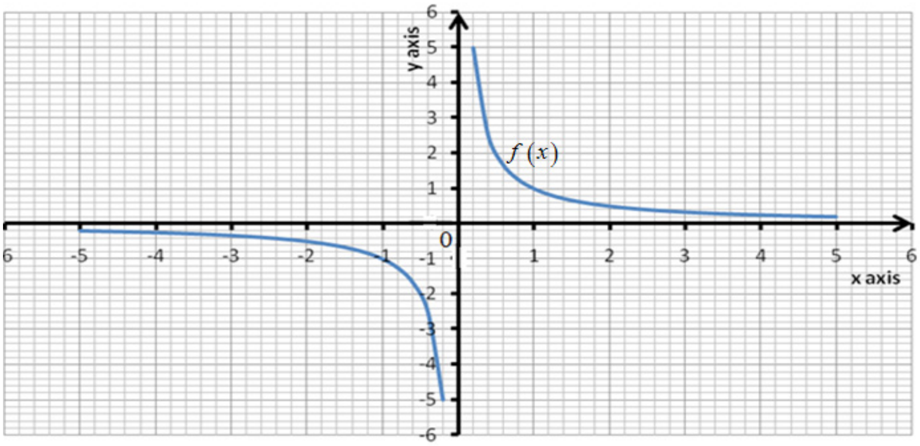




If we compare f_1, f_2 and f, we can find that $f(2)=1.5$ while $f_1(2)=1$ but $\lim\limits_{x\to 2^-} f(x) = \lim\limits_{x\to 2^-} f_1(x) = \lim\limits_{x\to 2^-} f_2(x) = 1$ and

$\lim\limits_{x\to 2^+} f(x) = \lim\limits_{x\to 2^+} f_1(x) = \lim\limits_{x\to 2^+} f_2(x) = 3$.

2. Let f be the function whose graph is shown hereafter:

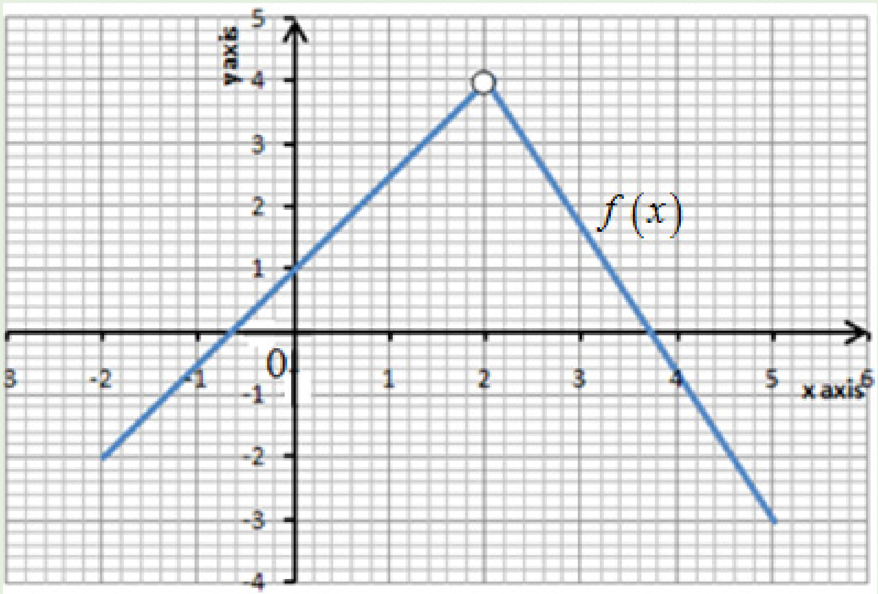


As *x* approaches 0 from the right-hand side, $f(x)$ gets larger and larger without bound. and consequently, approaches no fixed real value.

Thus $\lim\limits_{x\to 0^+} f(x) = +\infty$. As *x* approaches 0 from the left-hand side, $f(x)$ becomes more and more negative without bound and consequently $\lim\limits_{x\to 0^-} f(x) = -\infty$. As *x* gets larger and larger, $f(x)$ gets close to zero. Also, as *x* becomes more and more negative, $f(x)$ is close to zero. Thus, $\lim\limits_{x\to +\infty} f(x) = 0$ and $\lim\limits_{x\to -\infty} f(x) = 0$

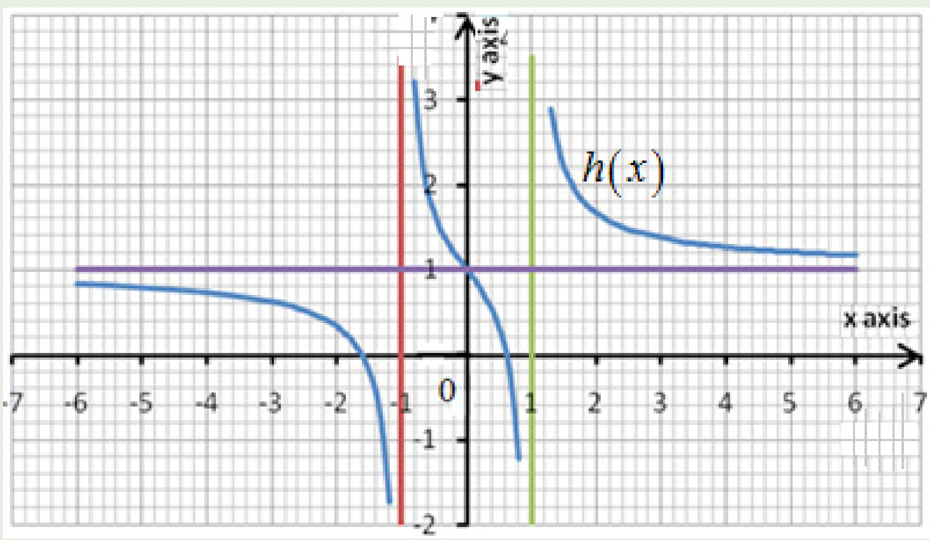
Application activity 4.1.4.

1. Find $\lim_{x \to 2} f(x)$ using the following graph of $f(x)$



2. Find the $\lim_{x \to -1} h(x)$, $\lim_{x \to +1} h(x)$, $\lim_{x \to -\infty} h(x)$, and $\lim_{x \to \infty} h(x)$;

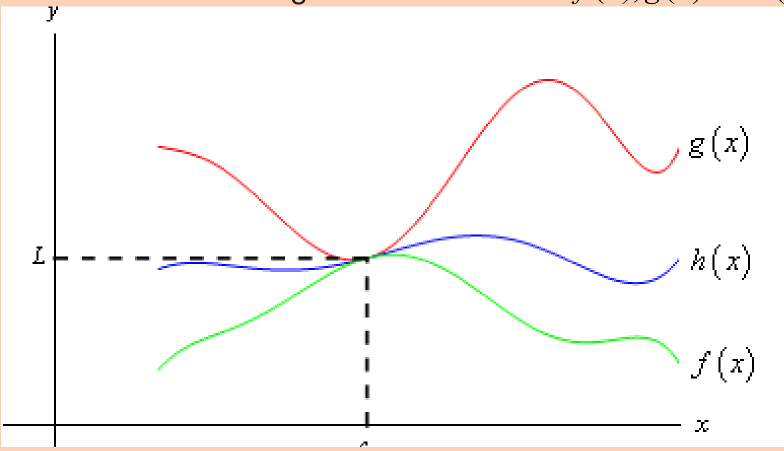
   Using the graph of $h(x) = 1 + \dfrac{x}{x^2 - 1}$ shown below:



4.1.5. The squeeze theorem /Sandwich theorem or Pinching theorem

Activity 4.1.5.

Consider the following curve of functions $f(x), g(x)$ and $h(x)$



Suppose that $f(x) \leq h(x) \leq g(x)$. If $\lim_{x\to c} f(x) = \lim_{x\to c} g(x) = L$, what do you think will be the answer for $\lim_{x\to c} h(x)$?

From the figure we can see that if the limits of $f(x)$ and $g(x)$ are equal at $x = c$ then the function values must also be equal at $x = c$.

$h(x)$ appeared to be "**squeezed**" between $f(x)$ and $g(x)$ at this point. Therefore, the limit of $h(x)$ at this point must also be the same as the limits of the other two functions.

The squeeze theorem states that: If numerical functions f, g and h are such that $f(x) \leq h(x) \leq g(x)$ for all x in some interval $]a, b[$,except possibly at point $c \in]a,b[$ and if $\lim_{x\to c} f(x) = \lim_{x\to c} g(x) = L$, then $\lim_{x\to c} h(x) = L$

Examples:

1. Given that: $1 - \frac{x^2}{4} \leq u(x) \leq 1 + \frac{x^2}{2}$. Find $\lim_{x\to 0} u(x)$

Solution: Since the $\lim_{x\to 0}\left(1 - \frac{x^2}{4}\right) = 1$ *and* $\lim_{x\to 0}\left(1 + \frac{x^2}{2}\right) = 1$, the Sandwich theorem implies that $\lim_{x\to 0} u(x) = 1$

2. Show that if $\lim_{x\to a}|f(x)|=0$ then $\lim_{x\to a} f(x)=0$

Solution: Since $-|f(x)|\le f(x)\le |f(x)|$, and $-|f(x)|$ and $|f(x)|$ both have limit 0 as x approaches a, so does $f(x)$ by the Sandwich theorem.

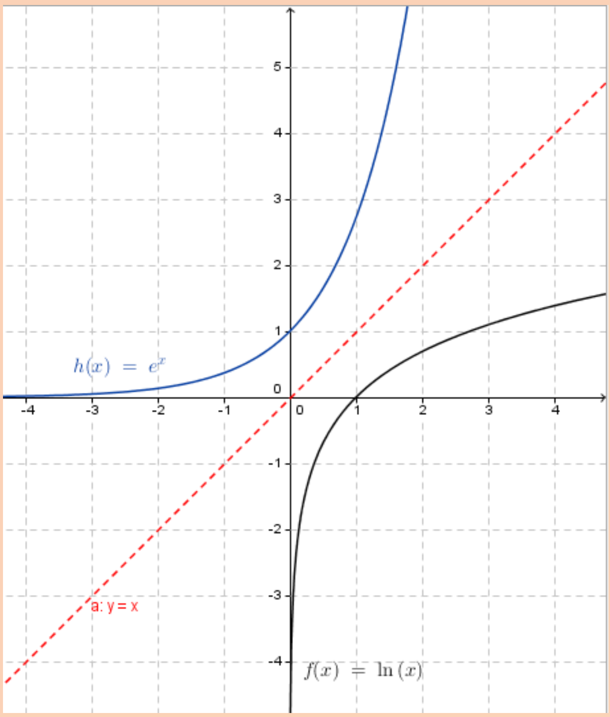
Application activity 4.1.5.

In the same Cartesian plane sketch the following curves of $f(x)=x^2+5$, $g(x)=-x^2+5$, and $h(x)=5$. What can you say about the three curves?

4.1.6. Limits of exponential functions

Activity 4.1.6.

1) Refer to the graphs of $f(x)=\ln x$ and $y=x$,



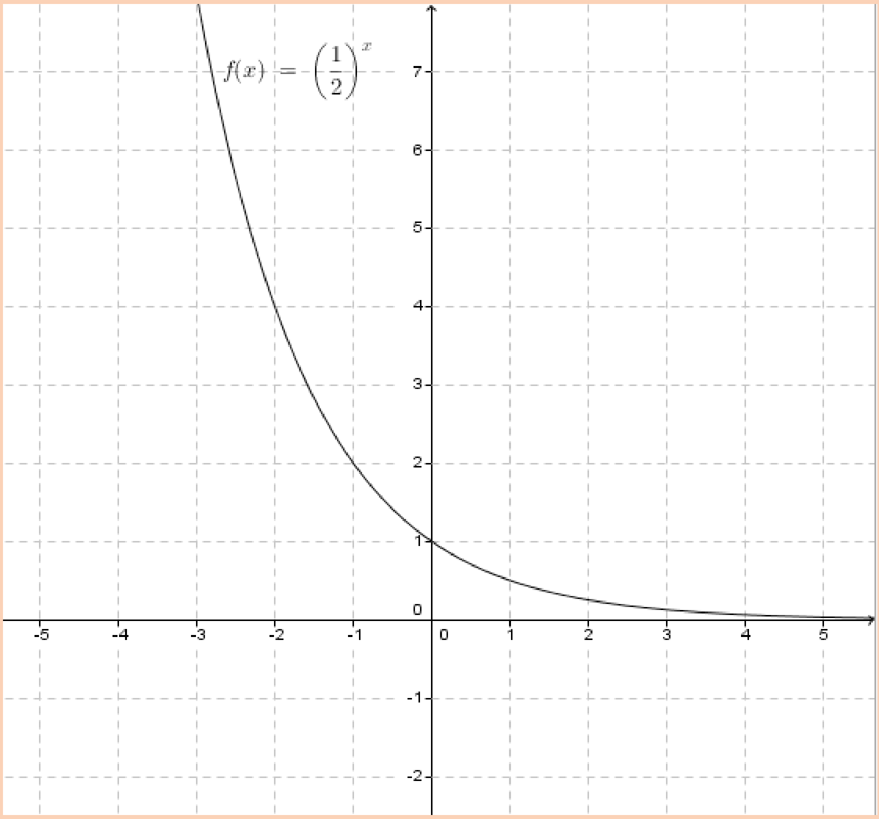
a) Explain in your own words how you can obtain the graph of its inverse $y=f^{-1}(x)=e^x$.

b) Observe the graph, and complete the following:

The function $f(x)=e^x$ approaches the line …….. when x takes ….. values and it goes to …..when x takes ……. values.

2) Observe the graph of the function $f(x)=\left(\frac{1}{2}\right)^x$ and Observe the graph, and complete the following:

The function $f(x)=\left(\frac{1}{2}\right)^x$ approaches the line …….. when x takes ….. values and it goes to …..when x takes ……. values.



From the graph of the functions $f(x)=e^x$ and $f(x)=\ln x$, It is clearly observed that: $\lim_{x\to-\infty} e^x = 0$ and $\lim_{x\to+\infty} e^x = +\infty$

In general: If $a>1$, $\lim_{x\to-\infty} a^x = 0$ *and* $\lim_{x\to+\infty} a^x = +\infty$, and If $0<a<1$, ,

$\lim_{x\to-\infty} a^x = +\infty$ *and* $\lim_{x\to+\infty} a^x = 0$

Examples

1) 1) Evaluate

a) $\lim_{x\to\infty} e^{1-4x-5x^2}$ b) $\lim_{x\to1}\left(\frac{3}{5}\right)^{\frac{1}{x-1}}$ c) $\lim_{x\to-\infty} 3^{\frac{1}{x}}$ d) $\lim_{x\to1} 3^{\frac{1}{x-1}}$

Solution

a) $\lim_{x\to\infty} e^{1-4x-5x^2}$

We know that $\lim_{x\to-\infty}\left(1-4x-5x^2\right)=-\infty$

Hence, $\lim_{x\to\infty} e^{1-4x-5x^2}=0$

b) The exponent assumes values increasing or decreasing without bound,

as x gets closer and closer to 1, that is $\lim_{x\to 1}\frac{1}{x-1}=\pm\infty$

Hence, $\lim_{x\to 1}\left(\frac{3}{5}\right)^{\frac{1}{x-1}}=\left(\frac{3}{5}\right)^{\pm\infty}=\begin{cases}0; at+\infty\\ +\infty; at-\infty\end{cases}$; the limit does not exist.

c) $\lim_{x\to-\infty} 3^{\frac{1}{x}}=3^{\lim_{x\to-\infty}\frac{1}{x}}=3^0=1$

d) $\lim_{x\to 1} 3^{\frac{1}{x-1}}=3^{\lim_{x\to 1}\frac{1}{x-1}}=3^{\frac{1}{0}}$

Study one side limit:

x	$3^{\frac{1}{x-1}}$
0	0.33
0.2	0.25
0.4	0.16
0.6	0.06
0.8	0.004
0.9	0.00001

x	$3^{\frac{1}{x-1}}$
2	3
1.8	3.948
1.6	6.24
1.4	15.59
1.2	243
1.1	59049

$\lim_{x\to 1^-} 3^{\frac{1}{x-1}}=0 \; and \; \lim_{x\to 1^-} 3^{\frac{1}{x-1}}=+\infty$, Hence, $\lim_{x\to 1} 3^{\frac{1}{x-1}}$ does not exist, as the limits from left and right hands are not the same,

Alternatively:

Since, $\lim_{x\to 1^+}\frac{1}{x-1}=+\infty$ and $\lim_{x\to 1^-}\frac{1}{x-1}=-\infty$, apply results on $\lim_{x\to\pm\infty} a^x$ for $a>1$ to have:

$\lim_{x\to 1^-} 3^{\frac{1}{x-1}}=0 \; and \; \lim_{x\to 1^-} 3^{\frac{1}{x-1}}=+\infty$

Hence, $\lim_{x\to 1} 3^{\frac{1}{x-1}}$ does not exist.

2) Consider $f(x)=\frac{e^{-3x}-2e^{8x}}{9e^{8x}-7e^{-3x}}$, evaluate each of the following:

$\lim_{x\to-\infty} f(x)$ and $\lim_{x\to+\infty} f(x)$.

Solution

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty}\frac{e^{-3x}-2e^{8x}}{9e^{8x}-7e^{-3x}}$$

$$= \lim_{x\to-\infty}\frac{e^{-3x}\left(1-2e^{11x}\right)}{e^{-3x}\left(9e^{11x}-7\right)}$$

$$= \lim_{x\to-\infty}\frac{1-2e^{11x}}{9e^{11x}-7} = -\frac{1}{7}$$

$$\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty}\frac{e^{-3x}-2e^{8x}}{9e^{8x}-7e^{-3x}}$$

$$= \lim_{x\to+\infty}\frac{e^{8x}\left(e^{-11x}-2\right)}{e^{8x}\left(9-7e^{-11x}\right)} = -\frac{2}{9}$$

Application activity 4.1.6.

For each function, evaluate limit at $+\infty$ and $-\infty$: 1. $f(x)=e^{8+2x-x^3}$

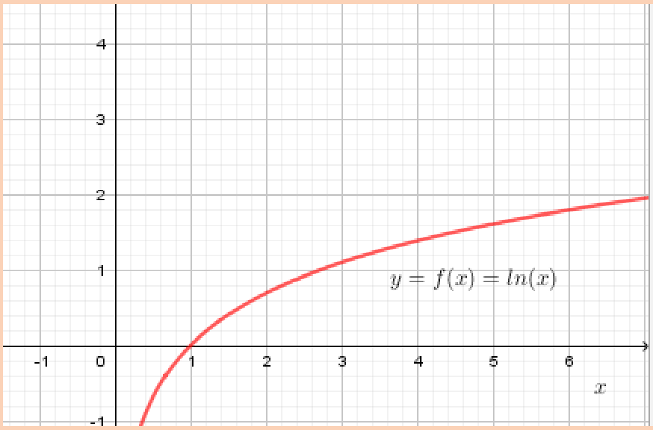
2. $f(x)=e^{\frac{6x^2+x}{5+3x}}$; 3. $f(x)=2e^{6x}-e^{-7x}-10e^{4x}$; 4. $f(x)=3e^{-x}-8e^{-5x}-e^{10x}$;

5. $f(x)=\dfrac{e^{-3x}-2e^{8x}}{9e^{8x}-7e^{-3x}}$

4.1.7. Limits of logarithmic functions

Activity 4.1.7.

The graph below represents natural logarithmic function $f(x)=\ln x$



Considering the form of this graph and the values of x in the table below,

x	0.5	0.001	0.001	0.0001	2	100	1001	10000
$\ln x$								

a) Discuss the values of $\ln x$ when x takes values closer to 0 from the right and deduce $\lim_{x\to 0^+} \ln x$.

b) Discuss the values of $\ln x$ when x take greater values and conclude about the $\lim_{x\to +\infty} \ln x$.

The limit of a logarithmic function in a base other than e can be determined in the same way as the limit of a natural logarithmic function. You can convert the logarithm to natural logarithm by using the change of base formula:

$$f(x) = \log_a u(x) = \frac{\ln(u(x))}{\ln a} \text{ if and only if } a > 0, a \neq 1$$

Example 1

Determine each of the following limit

a) $\lim_{x\to e} \ln x$; b) $\lim_{x\to 2}(1-\ln x)$; c) $\lim_{x\to +\infty} \log_3\left(\frac{x-4}{x+6}\right)$

Solution

a) $\lim_{x\to e} \ln x = 1$

b) $\lim_{x\to 2}(1-\ln x) = 1-\ln 2$

c) $\lim_{x\to +\infty} \log_3\left(\frac{x-4}{2x+6}\right) = \log_3 \frac{1}{2}$ Since $\lim_{x\to \infty}\frac{x-4}{2x+6} = \frac{1}{2}$.

$= -\log_3 2$

Alternatively, using natural logarithmic function, we have

$$\lim_{x\to +\infty} \log_3\left(\frac{x-4}{2x+6}\right) = \lim_{x\to +\infty} \frac{\ln\left(\frac{x-4}{2x+6}\right)}{\ln 3}$$

$$= \frac{1}{\ln 3} \lim_{x\to +\infty} \ln\left(\frac{x-4}{2x+6}\right)$$

$=\frac{1}{\ln 3}\times\ln\frac{1}{2}$

$=-\frac{\ln 2}{\ln 3}$

$=-\log_3 2$

$=\log_3\frac{1}{2}$

Application activity 4.1.7.

Evaluate the following limits

1) $\lim_{x\to+\infty}\ln(7x^3-x^2+1)$; 2) $\lim_{x\to1^+}\left(\ln\frac{1}{x-1}\right)$; 3) $\lim_{x\to2^-}\log_5\left(x^2-5x+6\right)$;

4) $\lim_{a\to4^+}\ln\frac{a}{\sqrt{a-4}}$; 5) $\lim_{x\to+\infty}\ln\left(x^2-4x+1\right)$; 6) $\lim_{x\to+\infty}\frac{2+4\log x}{x}$

4.1.8.Limits involving infinity, indeterminate cases

4.1.8.1. Limits involving infinity

Activity 4.1.8.1.

1. Consider the following function $f(x)=\frac{x+1}{x-1}$; Find:

   a. $f(0.97)$; b. $f(0.98)$;

   c. $f(0.99)$; d. $f(1.01)$; e. $f(1.02)$; f. $f(1.03)$

2. Evaluate the following operations: a. $-2+\infty$; b. $2-\infty$; c. $-\infty+\infty$;

   d. $-\infty(+\infty)$; e. $3(-\infty)$; f. $\frac{-\infty}{-2}$; g. $\frac{+\infty}{-\infty}$

a) Infinite limits

A function whose values grow arbitrarily large is said to have an infinite limit. Infinite limits provide a way of describing the behavior of functions that grow arbitrarily large, in absolute value

The Mathematical statement $\lim_{x \to a} f(x) = \infty$ shows that the values of $f(x)$ increases without bound while $\lim_{x \to a} f(x) = -\infty$ shows that the values of $f(x)$ decreases without bound.

Example:

1. Describe the behavior of the function $f(x) = \frac{1}{x^2}$ near $x = 0$.

Solution

As x approaches 0 from either side, the values of $f(x)$ are positive and grow larger and larger, We write

$\lim_{x \to 0^-} f(x) = \lim_{x \to 0} \frac{1}{x^2} = +\infty$ and $\lim_{x \to 0^+} f(x) = \lim_{x \to 0} \frac{1}{x^2} = +\infty$

2. Describe the behavior of the function $f(x) = \frac{1}{x}$ near $x = 0$.

Solution

Let x successively assume values $x = 1, \frac{1}{10}, \frac{1}{100}, \ldots,$ then $\frac{1}{x} = 1, 10, 100, \ldots$ successively. As x approaches 0 from the right the value of $\frac{1}{x}$ gets larger and larger without bound, then $\lim_{x \to 0^+} \frac{1}{x} = +\infty$

Let x successively assume values $x = -1, -\frac{1}{10}, -\frac{1}{100}, \ldots,$ then $\frac{1}{x} = -1, -10, -100, \ldots$ successively. As x approaches 0 from the left the value of $\frac{1}{x}$ decreases and becomes more and more negative without bound, then $\lim_{x \to 0^-} \frac{1}{x} = -\infty$

Another way to find this, is to construct the **sign table**:

x	$-\infty$			0			$+\infty$
1		+	+	+	+	+	
x				0	+	+	
$\frac{1}{x}$				‖ ∞	+	+	

Considering the last row, we see that for $x=0$ the value of $\frac{1}{x}$ does not exist (∞). At the left there is a negative sign, thus $\lim\limits_{x\to 0^-}\frac{1}{x}=-\infty$. At the right there is a positive sign, thus $\lim\limits_{x\to 0^+}\frac{1}{x}=+\infty$.

It follows that $\lim\limits_{x\to 0}\frac{1}{x}$ does not exist because the one-sided limits as *x* approaches zero do not exist.

3. Describe the behavior of the function $\lim\limits_{x\to 4}\frac{2-x}{x^2-2x-8}$

Solution

As *x* approaches 4 from the right, the numerator is negative quantity approaching -2 and the denominator a positive quantity approaching 0. Consequently, the ratio is a negative quantity that decreases without bound. That is

$$\lim_{x\to 4^+}\frac{2-x}{x^2-2x-8}=-\infty$$

As *x* approaches 4 from the left, the numerator is eventually a negative quantity approaching -2 and the denominator a negative quantity approaching 0. Consequently, the ratio is a positive quantity that increases without bound. That is

$$\lim_{x\to 4^-}\frac{2-x}{x^2-2x-8}=+\infty$$

Another way to see this is to construct the **sign table**:

x	$-\infty$		-2		2		4		$+\infty$
$2-x$		+	+		0			-	
x^2-2x-8		+	0		-		0	+	
$\frac{2-x}{x^2-2x-8}$		+	$\|\|$ ∞	-	0	+		-	

We find that $\lim\limits_{x\to 4^+}\frac{2-x}{x^2-2x-8}=-\infty$ and $\lim\limits_{x\to 4^-}\frac{2-x}{x^2-2x-8}=+\infty$ from the last row of this table.

b) Operations with infinity and Limits at infinity.

i) Addition

For any real number c**,** we have:

$(+\infty)+c=c+(+\infty)=+\infty;$

$(-\infty)+c=c+(-\infty)=-\infty$

One can convince himself/herself by considering the quantity of water in a lake as infinity, compared to the quantity of water in a spoon.If you add a spoon of water to the water of a lake, there is no change.

$\begin{matrix}(+\infty)+(+\infty)=+\infty; \\ (-\infty)+(-\infty)=-\infty\end{matrix}$: adding two infinities of the same sign yields to infinity of that common sign

$(+\infty)+(-\infty)=(-\infty)+(+\infty)$: adding two infinities of different signs yields to an indeterminate case(or indeterminate form): we need to remove the indetermination to discover the true value hidden by the indetermination.

ii) Subtraction

We have:

$(+\infty)-(-\infty)=+\infty;$

$(-\infty)-(+\infty)=-\infty$

$(+\infty)-(+\infty);(-\infty)-(-\infty)$:subtracting infinity from infinity of the same sign yields to an indeterminate case

iii) Multiplication

We have:

$(+\infty)(+\infty)=+\infty;$

$(-\infty)(-\infty)=+\infty;$

$(+\infty)(-\infty)=(-\infty)(+\infty)=-\infty$

$a\cdot(+\infty)=\begin{cases}+\infty; a>0 \\ -\infty; a<0\end{cases}; a\cdot(-\infty)=\begin{cases}-\infty; a>0 \\ +\infty; a<0\end{cases}$

But $\begin{matrix}0\cdot(+\infty)=(+\infty)\cdot 0; \\ 0\cdot(-\infty)=(-\infty)\cdot 0\end{matrix}$ are indeterminate cases

iv) Division

We have:

For any real number a, $\frac{\infty}{a}=\infty$: the sign is to be determined, note that ∞ stands for $+\infty$ or $-\infty$; in particular, $\frac{\infty}{0}=\infty$;

For any real number a, $\frac{a}{\infty}=0$; in particular, $\frac{0}{\infty}=0$

But $\frac{\infty}{\infty}$ is an indeterminate case.

Notice: The following $\infty-\infty;\ 0\times\infty;\ \frac{\infty}{\infty},\frac{0}{0};0^0;1^\infty;\infty^0$ are **indeterminate cases** in limits calculation.

i) Limit of polynomial functions

Let $f(x)=a_nx^n+a_{n-1}x^{n-1}+...+a_0$ and $g(x)=b_mx^m+b_{m-1}x^{m-1}+...+b_0$

$$\lim_{x\to\infty} f(x)=\lim_{x\to\infty}\left(a_nx^n+a_{n-1}x^{n-1}+...+a_0\right)$$

$$=\lim_{x\to\infty} a_nx^n\left(1+\frac{a_{n-1}}{a_nx}+\frac{a_{n-2}}{a_nx^2}+...+\frac{a_0}{a_nx^n}\right)$$

$$=\lim_{x\to\infty} a_nx^n$$

$$\lim_{x\to\infty} g(x)=\lim_{x\to\infty}\left(b_mx^m+a_{m-1}x^{m-1}+...+a_0\right)$$

$$=\lim_{x\to\infty} b_mx^m\left(1+\frac{b_{m-1}}{b_mx}+\frac{b_{m-2}}{b_mx^2}+...+\frac{b_0}{b_mx^m}\right)$$

$$=\lim_{x\to\infty} b_mx^m$$

And then $\lim_{x\to\infty}\frac{f(x)}{g(x)}=\lim_{x\to\infty}\frac{a_nx^n}{b_mx^m}$.

We have three cases

a) If $m=n$, $\lim_{x\to\infty}\frac{f(x)}{g(x)}=\lim_{x\to\infty}\frac{a_nx^n}{b_mx^m}=\frac{a_n}{b_m}$

b) If $n>m$, $\lim_{x\to\infty}\frac{f(x)}{g(x)}=\lim_{x\to\infty}\frac{a_nx^{n-m}}{b_m}=\infty$, the sign of infinity is to be determined.

c) If $n<m$, $\lim_{x\to\infty}\frac{f(x)}{g(x)}=\lim_{x\to\infty}\frac{a_n}{b_mx^{m-n}}=0$

1) $\lim_{x\to+\infty}(-6)=-6$

2) $\lim_{x\to-\infty}\left(3x^2+5x-3\right)=\lim_{x\to-\infty}3x^2$

$=+\infty$

3) *Find the limit* $\lim\limits_{x\to-\infty}\frac{1}{x}$ *and* $\lim\limits_{x\to+\infty}\frac{1}{x}$

$\lim\limits_{x\to-\infty}\frac{1}{x}=0$ and $\lim\limits_{x\to+\infty}\frac{1}{x}=0$. We can write $\lim\limits_{x\to\pm\infty}\frac{1}{x}=0$

4)

$$\lim_{x\to\infty}\frac{2x^4+3x^2+1}{3x^4+5x^2+3}=\lim_{x\to\infty}\frac{2x^4}{3x^4}$$
$$=\lim_{x\to\infty}\frac{2}{3}$$
$$=\frac{2}{3}$$

5)

$$\lim_{x\to-\infty}\frac{4x^3+5x-3}{x^2+3x+1}=\lim_{x\to-\infty}\frac{4x^3}{x^2}$$
$$=\lim_{x\to-\infty}4x$$
$$=-\infty$$

6)

$$\lim_{x\to\infty}\frac{5x+2}{3x^2+1}=\lim_{x\to\infty}\frac{5x}{3x^2}$$
$$=\lim_{x\to\infty}\frac{5}{3x}$$
$$=0$$

Application activity 4.1.8.1.

Evaluate the following limit

a. $\lim\limits_{x\to\infty}\frac{2x^2-3x+5}{x-2}$

b. $\lim\limits_{x\to\infty}\frac{2x^2+6}{3x^2-4x+2}$

c. $\lim\limits_{x\to-\infty}\frac{3x^2+4x+3}{x^3+x+14}$

4.1.8.2. Limits of functions involving indeterminate cases

Activity 4.1.8.2.

Given the function f(x), find a common factor for numerator and denominator and then find the value of f(x) for $x=1;2\ and\ \infty$ on each function What do you notice for each case?

a. $f(x)=\dfrac{x^2-1}{x-1}$

b. $f(x)=\dfrac{x^3+x^2-5x-2}{x^2-4}$

The indeterminate forms may occur in the following ways:

- Suppose that $\lim\limits_{x\to a} f(x)=0$ and $\lim\limits_{x\to a} g(x)=\infty$.

  The limit of the product $f(x)g(x)$ has the indeterminate form, $0\times\infty$ at $x=a$.

  To evaluate this limit we change the limit into one of the form $\dfrac{0}{0}$ or $\dfrac{\infty}{\infty}$ in this way: $f(x)g(x)=\dfrac{f(x)}{\dfrac{1}{g(x)}}=\dfrac{g(x)}{\dfrac{1}{f(x)}}$

- If $\lim\limits_{x\to a} f(x)=\lim\limits_{x\to a} g(x)=+\infty$, then $\lim\limits_{x\to a}\left[f(x)-g(x)\right]$ has the indeterminate form $\infty-\infty$. To evaluate this limit, we perform the algebraic manipulations by converting the limit into a form of $\dfrac{0}{0}$ or $\dfrac{\infty}{\infty}$.

  If $f(x)$ or $g(x)$ is expressed as a fraction, we find the common denominator.

1. Limits of rational functions involving indeterminate forms

Examples:

1) $\lim\limits_{x\to 2}\dfrac{x^2-4}{x-2}=\dfrac{0}{0}$, this is the indeterminate form (I.F)

By factoring the numerator and cancelling, we move out this I.F

$$\lim_{x\to 2}\frac{x^2-4}{x-2}=\lim_{x\to 2}\frac{(x+2)(x-2)}{x-2}$$
$$=\lim_{x\to 2}(x+2)$$
$$=4$$

2) $\lim_{x\to 1}\frac{x^2-1}{x-1}=\frac{1-1}{1-1}=\frac{0}{0}I.F$

$$\lim_{x\to 1}\frac{x^2-1}{x-1}=\lim_{x\to 1}\frac{(x+1)(x-1)}{x-1}$$
$$=\lim_{x\to 1}(x+1)$$
$$=2$$

3) $\lim_{x\to+\infty}\frac{x^2+4x+5}{4x^2+7x+9}=\frac{\infty}{\infty}I.F$

Factor out x^2, to move out this I.F; then we have

$$\lim_{x\to+\infty}\frac{x^2\left(1+\frac{4}{x}+\frac{5}{x^2}\right)}{x^2\left(4+\frac{7}{x}+\frac{9}{x^2}\right)}=\lim_{x\to+\infty}\frac{1+\frac{4}{x}+\frac{5}{x^2}}{4+\frac{7}{x}+\frac{9}{x^2}}$$
$$=\frac{1+0+0}{4+0+0}$$
$$=\frac{1}{4}$$

Or

Since we have a rational function and degree of numerator is equal to the degree of denominator, to find the limit a**s *x* tends to infinity** we need to divide the coefficients of the highest degree for numerator and denominator. Then the limit of this function becomes $\frac{1}{4}$

4) $\lim_{x\to-\infty}(3x^2+5x-3)=3(-\infty)^2+5(-\infty)-3$

$$=+\infty-\infty\ IF$$

Factor out x^2, to move this I.F

$$\lim_{x\to-\infty}\left(3x^2+5x-3\right)=\lim_{x\to-\infty}x^2\left(3+\frac{5}{x}-\frac{3}{x^2}\right)$$
$$=+\infty\left(3+0-0\right)$$
$$=+\infty$$

2. Limits of irrational functions involving indeterminate forms

When we are computing the limits of irrational functions, in case of indeterminate form, we need to know the conjugate of the irrational expression in that function. We may need to find the domain of the given function.

Examples

1) Let us evaluate $\lim_{x\to4}\frac{\sqrt{x-3}-1}{x-4}$.

$$\lim_{x\to4}\frac{\sqrt{x-3}-1}{x-4}=\frac{1-1}{4-4}=\frac{0}{0}I.F$$

To remove this I.F, we multiply the numerator and denominator by the conjugate of $\sqrt{x-3}-1$ which is $\sqrt{x-3}+1$, then

$$\lim_{x\to4}\frac{\sqrt{x-3}-1}{x-4}=\lim_{x\to4}\frac{\left(\sqrt{x-3}-1\right)\left(\sqrt{x-3}+1\right)}{\left(x-4\right)\left(\sqrt{x-3}+1\right)}$$
$$=\lim_{x\to4}\frac{x-3-1}{\left(x-4\right)\left(\sqrt{x-3}+1\right)}$$
$$=\lim_{x\to4}\frac{1}{\sqrt{x-3}+1}$$
$$=\frac{1}{2}$$

2) $\lim_{x\to+\infty}\left(\sqrt{4x^2+2}-2x\right)=+\infty-\infty\ I.F$

To remove this I.F, we multiply and divide by the conjugate of $\sqrt{4x^2+2}-2x$.

$$\lim_{x\to+\infty}\left(\sqrt{4x^2+2}-2x\right)=\lim_{x\to+\infty}\frac{\left(\sqrt{4x^2+2}-2x\right)\left(\sqrt{4x^2+2}+2x\right)}{\sqrt{4x^2+2}+2x}$$

$$= \lim_{x\to+\infty} \frac{4x^2+2-4x^2}{\sqrt{4x^2+2}+2x}$$
$$= \lim_{x\to+\infty} \frac{2}{\sqrt{4x^2+2}+2x}$$
$$= \frac{2}{+\infty}$$
$$= 0$$

Examples:

1) $\lim_{x\to+\infty} \frac{\sqrt{4x^2-11x-3}}{x} = \frac{\sqrt{\infty-\infty}}{\infty} I.F$

To move out this I.F, we perform the algebraic manipulations such that the denominator will be cancelled.

$$\lim_{x\to+\infty} \frac{\sqrt{4x^2-11x-3}}{x} = \lim_{x\to+\infty} \frac{\sqrt{4x^2\left(1-\frac{11}{4x}-\frac{3}{4x^2}\right)}}{x} \qquad = \lim_{x\to+\infty} \frac{\left(\sqrt{4x^2}\right)\sqrt{1-\frac{11}{4x}-\frac{3}{4x^2}}}{x}$$
$$= \lim_{x\to+\infty} \frac{\sqrt{4x^2}}{x} \lim_{x\to+\infty} \sqrt{1-\frac{11}{4x}-\frac{3}{4x^2}}$$
$$= \left(\lim_{x\to+\infty} \frac{\sqrt{4x^2}}{x}\right) \times 1$$
$$= \lim_{x\to+\infty} \frac{\sqrt{4x^2}}{x}$$
$$= \lim_{x\to+\infty} \frac{2\sqrt{x^2}}{x}$$

Recall that $\sqrt{x^2} = \begin{cases} x & if \ x \geq 0 \\ -x & if \ x < 0 \end{cases}$

We find the domain of the given function: $Domf = \left]-\infty, -\frac{1}{4}\right] \cup [3, +\infty[$.

As x tends to $+\infty$, $x \in [3, +\infty[$ and then $\sqrt{x^2} = x$.

Thus,

$$\lim_{x\to+\infty} \frac{\sqrt{4x^2-11x-3}}{x} = \lim_{x\to+\infty} \frac{2\sqrt{x^2}}{x}$$
$$= \lim_{x\to+\infty} \frac{2x}{x}$$
$$= 2$$

2) $\lim_{x\to 1}\frac{\sqrt[3]{x}-1}{x-1}=\frac{0}{0}\ IF$

To move out this I.F we use the conjugate of the numerator

$$\lim_{x\to 1}\frac{\sqrt[3]{x}-1}{x-1}=\lim_{x\to 1}\frac{\left(\sqrt[3]{x}-1\right)\left(\sqrt[3]{x^2}+\sqrt[3]{x}+1\right)}{(x-1)\left(\sqrt[3]{x^2}+\sqrt[3]{x}+1\right)}=\lim_{x\to 1}\frac{(x-1)}{(x-1)\left(\sqrt[3]{x^2}+\sqrt[3]{x}+1\right)}$$

$$=\lim_{x\to 1}\frac{1}{\left(\sqrt[3]{x^2}+\sqrt[3]{x}+1\right)}$$

$$=\frac{1}{3}$$

3) $\lim_{x\to 2^+}\frac{\sqrt{x-2}}{\sqrt[3]{x-2}}=\frac{0}{0}$

$$\lim_{x\to 2^+}\frac{\sqrt{x-2}}{\sqrt[3]{x-2}}=\lim_{x\to 2^+}(x-2)^{\frac{1}{2}}\cdot(x-2)^{-\frac{1}{3}}=\lim_{x\to 2^+}(x-2)^{\frac{1}{6}}=\lim_{x\to 2^+}\sqrt[6]{x-2}=0$$

Application activity 4.1.8.2.

Evaluate the following limits

a) $\lim_{x\to\infty}\frac{3x^3+2x^2-1}{6x^3+x+4}$	b) $\lim_{x\to-\infty}\frac{(x+3)^2}{x^3+4x^2-8x-4}$	c) $\lim_{x\to\infty}\frac{4x^3+x^2-1}{x^2-x+4}$
d) $\lim_{x\to-4}\frac{x+1}{x+4}$	e) $\lim_{x\to 3}\frac{x^2+2x+1}{x-3}$	f) $\lim_{x\to\infty}\left(x^2-2x+5\right)$
g) $\lim_{x\to-\infty}\left(4x^3+3x^2-6\right)$	h) $\lim_{x\to 4}\frac{x^4-16}{x^2-4}$	i) $\lim_{x\to 4}\frac{\sqrt{x^2-6}-10}{x-4}$
j) $\lim_{x\to-\infty}\frac{3x}{\sqrt{9x^2-3x+6}}$		

4.2. Applications of Limits functions to continuity and asymptotes

4.2.1. Continuity of a function

Activity 4.2.1.

Given the function $f(x)=\begin{cases}\dfrac{x^2-4}{x-2}, x\neq 2\\ 4, \qquad x=2\end{cases}$, find

a. $f(2)$

b. $\lim\limits_{x\to 2} f(x)$

c. What can you say about $f(2)$ and $\lim\limits_{x\to 2} f(x)$?

a) Continuity of a function at a point or on interval *I*

A function $f(x)$ is said to be **continuous at point *c*** if the following conditions are satisfied:

a) $f(c)$ is defined

b) $\lim\limits_{x\to c} f(x)$ exists

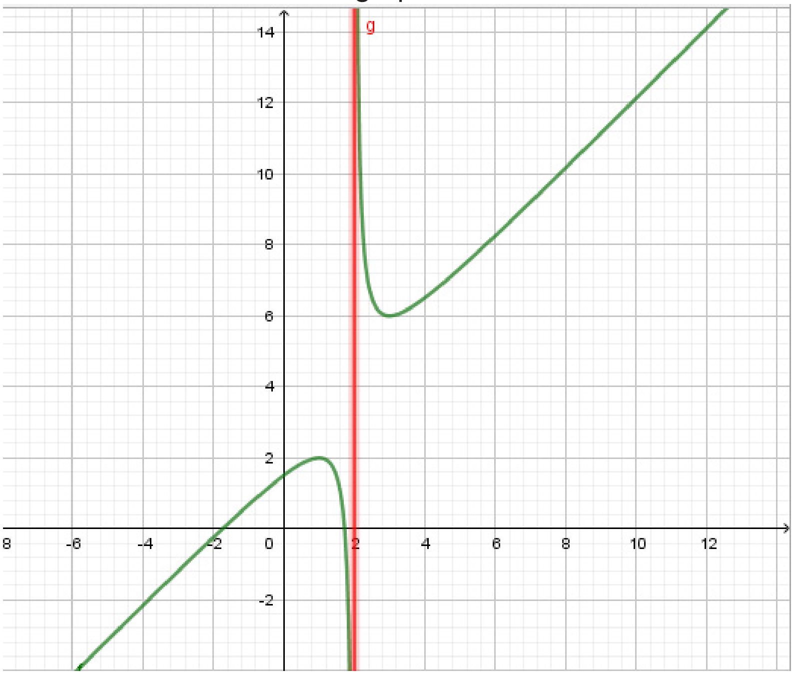
c) $\lim\limits_{x\to c} f(x)=f(c)$

If one or more conditions in this definition fails to hold, then *f* is said to be **discontinuous at point *c***, and *c* is called **a point of discontinuity** of *f*. If *f* is continuous at all point of an open interval $]a,b[$, then *f* is said to be continuous on $]a,b[$.

A function that is continuous on $]a,b[$ is said to be **continuous everywhere** or simply **continuous**.

Examples:

1) The function $f(x)=\dfrac{x^2-3}{x-2}$ is discontinuous at 2 because $f(2)$ is undefined, see the graph below.



2) The function $g(x)=\begin{cases}\dfrac{x^2-9}{x-3} & if\ \ x\neq 3\\ 6 & if\ \ x=3\end{cases}$ is continuous at 3 because $g(3)=6$ and $\lim\limits_{x\to 3^+} g(x)=\lim\limits_{x\to 3^-} g(x)=\lim\limits_{x\to 3}\dfrac{x^2-9}{x-3}=\lim\limits_{x\to 3}(x+3)=6$ *so that* $\lim\limits_{x\to 3} g(x)=g(3)$.

b) Continuity at the left side and continuity at the right side of a point

A function *f* is continuous at the left of point *c* if the following conditions are satisfied:

a) $f(c)$ is defined

b) $\lim\limits_{x\to c^-} f(x)$ exists

c) $\lim\limits_{x\to c^-} f(x)=f(c)$

A function *f* is continuous at the right of a point *c* if the following conditions are satisfied:

a) $f(c)$ is defined

b) $\lim_{x \to c^+} f(x)$ exists

c) $\lim_{x \to c^+} f(x) = f(c)$

Example ,Function $f(x) = \begin{cases} 1; x < 2 \\ 2; x \geq 2 \end{cases}$ is continuous at the right of point 2.In fact, $f(x)$ is defined at 2, $f(2) = 2$, $\lim_{x \to 2^+} f(x)$ exists(=2), and $\lim_{x \to 2^+} f(x) = f(2)$

c) Continuity on an interval

We say that *f* is **continuous on the interval *I*** if it is continuous at each point of *I*.

Examples

1) The function $f(x) = \sqrt{x}$ is a continuous function on its domain. Its domain is $[0, +\infty[$.

2) The function $f(x) = \frac{x}{\sqrt{1-x^2}}$ is continuous on its domain. Its domain is $]-1,1[$.

Properties

a) Polynomials are continuous functions

b) If the functions *f* and *g* are continuous at *c*, then

i. $f + g$ is continuous at *c*

ii. $f - g$ is continuous at *c*

iii. $f \cdot g$ is continuous at *c*

iv. $\frac{f}{g}$ is continuous at *c* if $g(c) \neq 0$, and is discontinuous at *c* if $g(c) = 0$.

c) A rational function is continuous everywhere except at the point where the denominator is zero.

Examples

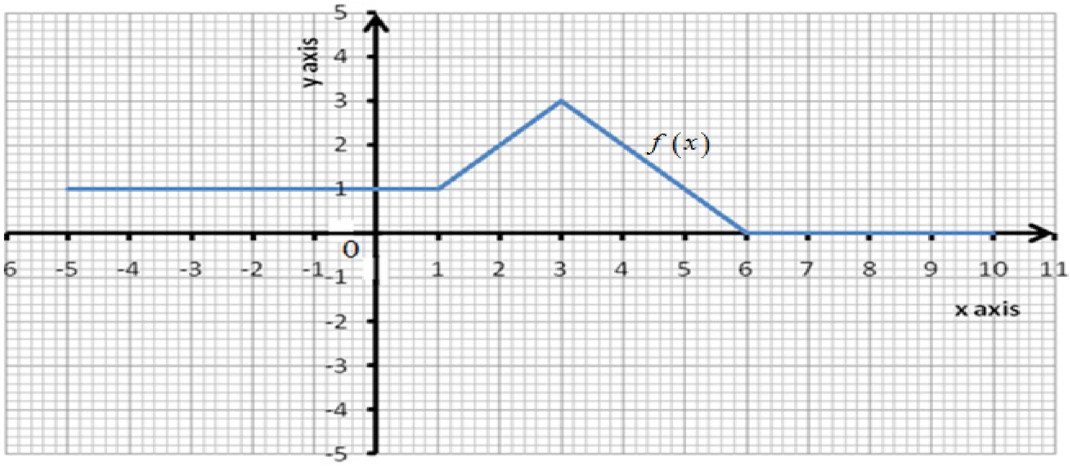
1) The function $h(x) = \frac{x^2 - 9}{x^2 - 5x + 6}$ is continuous everywhere except at point 2 and 3 because the numerator and denominator are polynomials and denominator is zero at points 2 and 3.

2) The function $f(x)$ which is defined by:

$$f(x)=\begin{cases} 1 & if\ x\le 1 \\ x & if\ 1<x\le 3 \\ -x+6 & if\ 3<x\le 6 \\ 0 & if\ x>6 \end{cases}$$

is continuous on $\mathbb{R}$,

The following is the graph of f(x)



Note:

If $f(x)$ is continuous at $x=a$, and $g(x)$ is continuous at $f(a)$, then

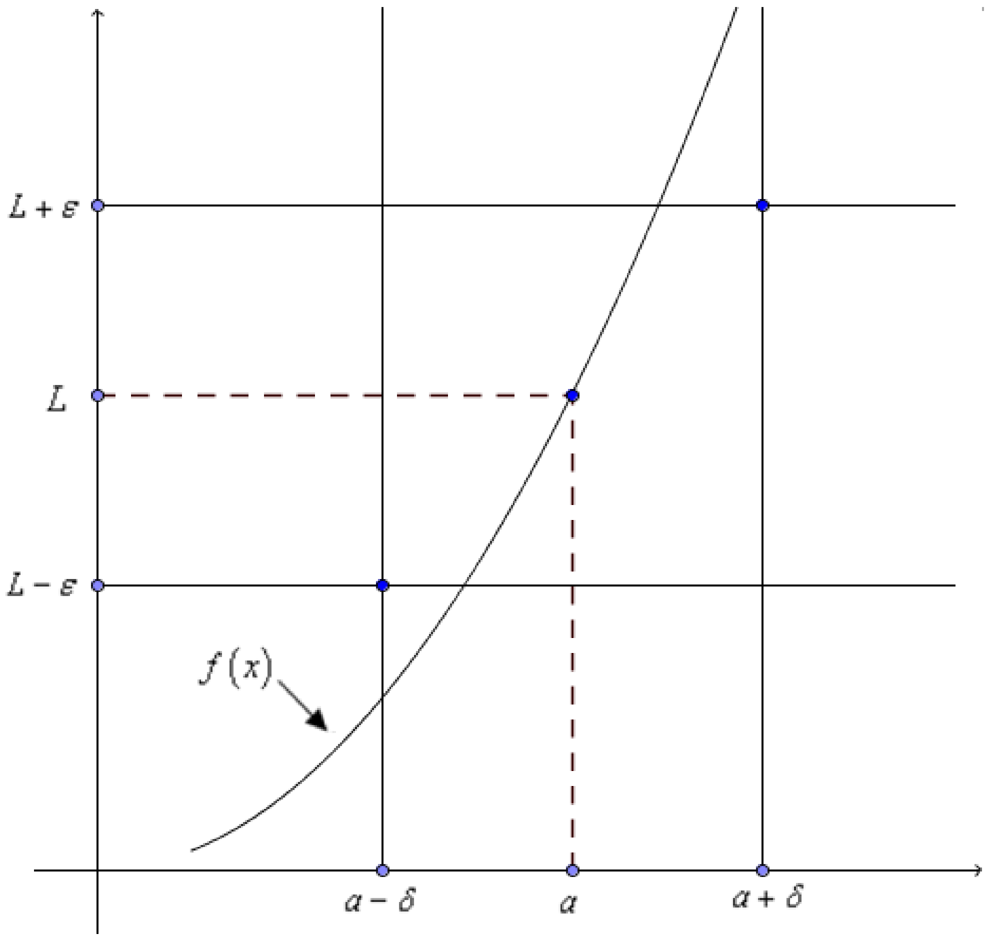
$\lim\limits_{x\to a} g[f(x)] = g\left[\lim\limits_{x\to a} f(x)\right]$: the limit of the image equals the image of the limit,

we have: $\lim\limits_{x\to a} f(x) = f(a)$; and if

$$\lim_{x\to f(a)} g(x) = g[f(a)] = g\left[\lim_{x\to a} f(x)\right] = \lim_{x\to a} g[f(x)],$$

Formal definition:

Let $f(x)$ be a function on an interval that contains $x=a$, except possibly at $x=a$. Then we say that: $\lim\limits_{x\to a} g(x) = L$ if for every number $\varepsilon>0$ there is some number $\delta>0$ such that $|f(x)-L|<\varepsilon$ whenever $0<|x-a|<\delta$.



What the definition is telling us is that for **any** open interval, with center at L and radius ε , $\varepsilon > 0$, that is for any open interval $]L-\varepsilon, L+\varepsilon[$,on the y-axis, there exists an open interval ,with center at a and radius δ, that is $]a-\delta, a+\delta[$ on the x-axis, such that $f(]a-\delta, a+\delta[) \subset]\mathrm{L}-\varepsilon, \mathrm{L}+\varepsilon[$, δ is to be determined in terms of ε

Application activity 4.2.1

1. Determine where the function $f(x)=\dfrac{4x+10}{x^2-2x-15}$ is not continuous for $x=-3 \; or \; x=5$

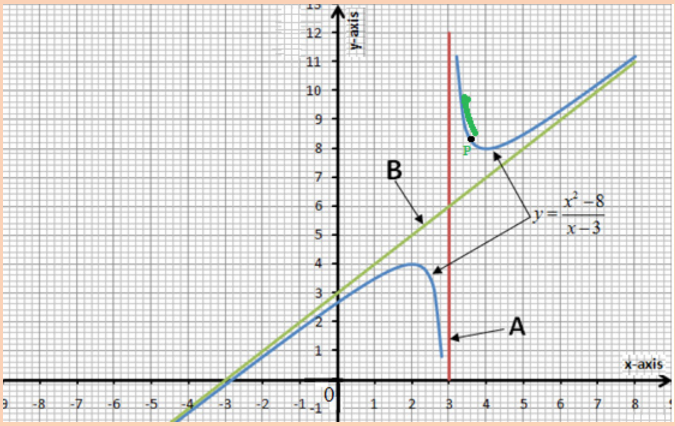
2. Given the function: $f(x)=\begin{cases}\dfrac{x^2-9}{x-3}, x \neq 3\\ k, x=3\end{cases}$

   Determine the value of ***k*** for which the function is continuous at *x=3*.

4.2.2.Asymptotes to curve or graph of a function

Activity 4.2.2.

Consider the following curve of function y



What can you say about the curve of y and the lines A and B?

Consider point P moving on the graph in the direction shown by the arrow in green. We observe that:

The distance from point P to the line in red approaches 0, the x-coordinate of point P approaches 3, and the y-coordinate of P increases without bound, that is the y-coordinate approaches $+\infty$.The straight line $x=3$ is said to be asymptote of the curve

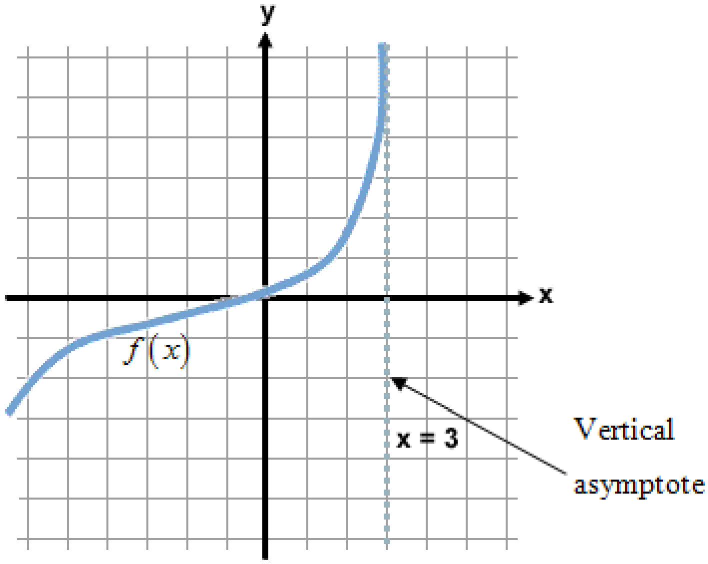
More generally, a straight line is said to be **asymptote** of the graph of a function if its distance, from a point moving on the graph such that at least one of its coordinate approaches infinity, approaches zero

Types of asymptotes

There are three types of asymptotes: Vertical asymptote, Horizontal asymptote, and Oblique asymptote.

a) Vertical asymptote

A line with equation $x=x_0$ $(D \equiv x=x_0)$ is called a **vertical asymptote** for the graph of a function $f(x)$ if $\lim_{x\to x_0} f(x)=\pm\infty$ Or $V.A \equiv x=x_0$, *where* $\lim_{x\to x_0} f(x)=\pm\infty$



Example: *Let* $f(x)=\dfrac{2x^2+7x-1}{x+1}$; $Domf=\left]-\infty,-1\right[\cup\left]-1,+\infty\right[$;

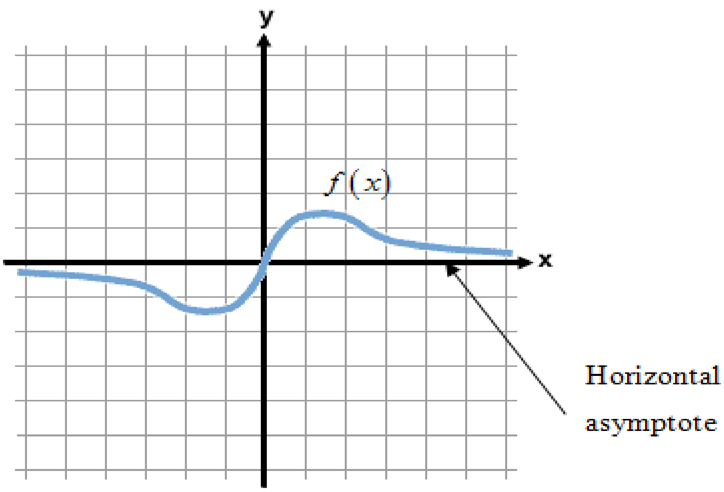
$\lim\limits_{x\to-1} f(x)=\lim\limits_{x\to-1}\dfrac{2x^2+7x-1}{x+1}$

$=\infty$

Hence, $VA \equiv x=-1$ is vertical asymptote for $f(x)=\dfrac{2x^2+7x-1}{x+1}$.

b) Horizontal asymptote

A line with equation $y=L$ $(D\equiv y=L)$ is called a **horizontal asymptote** for the graph of a function $f(x)$ if $\lim\limits_{x\to\pm\infty} f(x)=L$ or $H.A\equiv y=L;\ where\ \lim\limits_{x\to\pm\infty} f(x)=L$



Example

Let $f(x)=\dfrac{3x^2+4x-9}{2x^2+1}$, $Domf=]-\infty,+\infty[$

$$\lim_{x\to\pm\infty} f(x)=\lim_{x\to\pm\infty}\frac{3x^2}{2x^2}$$

$$=\frac{3}{2}$$

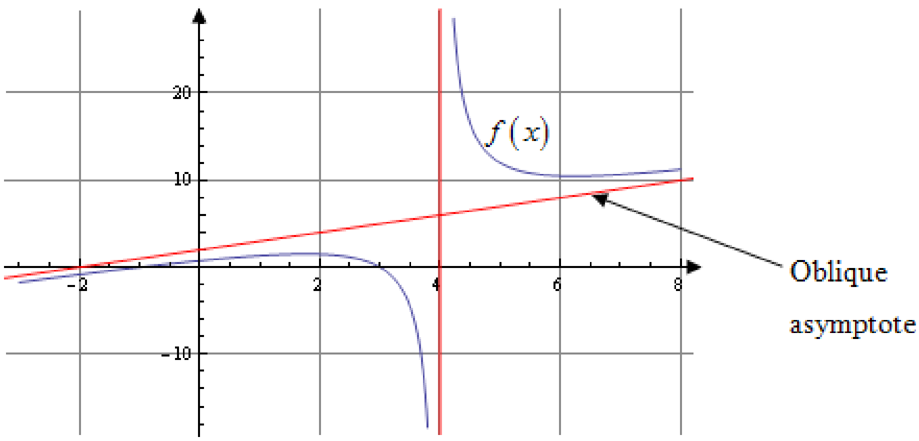
Thus, $y=\dfrac{3}{2}$ is a horizontal asymptote for $f(x)=\dfrac{3x^2+4x-9}{2x^2+1}$.

c) Oblique asymptote

If a rational function, $\dfrac{P(x)}{Q(x)}$, is such that the degree of the numerator exceeds the degree of the denominator by one, then the graph of $\dfrac{P(x)}{Q(x)}$ will have an **oblique asymptote (or a slant asymptote)**; that is, an asymptote which is neither vertical nor horizontal. We perform the division of $P(x)$ by $Q(x)$ to obtain $\dfrac{P(x)}{Q(x)}=(ax+b)+\dfrac{R(x)}{Q(x)}$, where, $ax+b$ is the quotient and $R(x)$ is the remainder. Another way to find the values of constants *a* and *b* is $a=\lim\limits_{x\to\pm\infty}\dfrac{f(x)}{x}$ $a\neq 0$, and $b=\lim\limits_{x\to\pm\infty}\left[f(x)-ax\right]$. Therefore, **we can write that** $O.A\equiv y=ax+b$**,** **where** $a=\lim\limits_{x\to\pm\infty}\dfrac{f(x)}{x}$**,** $a\neq 0$**, and** $b=\lim\limits_{x\to\pm\infty}\left[f(x)-ax\right]$

Notice

Horizontal asymptote and oblique asymptote do not exist simultaneously on the same side, that is, at infinity of the same sign



Example:

1. Find the equation of the oblique asymptote of function $f(x)=\dfrac{3x^3+4x-5}{x^2+1}$

Solution:

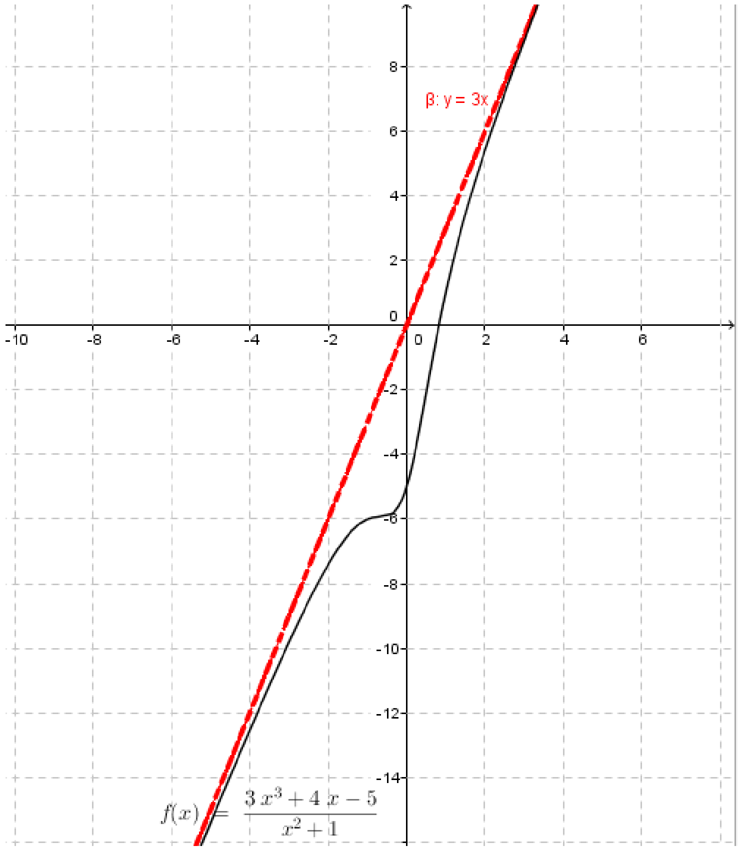
$Domf =]-\infty,+\infty[$

Let $y=ax+b$ be the oblique asymptote. Then:

$$a=\lim_{x\to\pm\infty}\frac{f(x)}{x}=\lim_{x\to\pm\infty}\frac{3x^3+4x-5}{x^3+x}$$
$$=3$$

$$b=\lim_{x\to\pm\infty}\left[f(x)-3x\right]$$
$$=\lim_{x\to\pm\infty}\frac{3x^3+4x-5-3x^3-3x}{x^2+1}$$
$$=\lim_{x\to\pm\infty}\frac{x-5}{x^2+1}$$
$$=0$$

Thus, $y=3x$ is the oblique asymptote for $f(x)=\dfrac{3x^3+4x-5}{x^2+1}$



2. Let $f(x)=\dfrac{x}{x-2}$. Find all possible asymptotes

Solution:

$Domf=]-\infty,2[\cup]2,+\infty[$ $\lim_{x\to 2^+} f(x)=+\infty$ and $\lim_{x\to 2^-} f(x)=-\infty$. Thus, there exists a vertical asymptote $\text{V.A}\equiv x=2$

$\lim_{x\to\pm\infty} f(x)=1$. Thus, there exists a horizontal asymptote $\text{H.A}\equiv y=1$

Note that there is no oblique asymptote.

3. Let $f(x)=\dfrac{x^2+2x-3}{x}$. Find all possible asymptotes

Solution:

$Domf=]-\infty,0[\cup]0,+\infty[$

$\lim_{x\to 0^+} f(x)=-\infty$ and $\lim_{x\to 0^-} f(x)=+\infty$. Thus, there is a vertical asymptote $\text{V.A}\equiv x=0$

$\lim_{x\to\pm\infty} f(x)=\pm\infty$. Thus, the horizontal asymptote does not exist.

To find oblique asymptote, let $y=ax+b$ be the oblique asymptote.

$$a=\lim_{x\to\pm\infty}\frac{f(x)}{x}=\lim_{x\to\pm\infty}\frac{x^2+2x-3}{x^2}=1$$

Since $1\neq 0$, let us find b.

$$b=\lim_{x\to\pm\infty}\left[f(x)-ax\right]=\lim_{x\to\pm\infty}\left[\frac{x^2+2x-3}{x}-x\right]=\lim_{x\to\pm\infty}\frac{2x-3}{x}=2$$

Then, $\text{O.A}\equiv y=x+2$.

As the degree of the numerator exceeds the degree of the denominator by one, we could find oblique asymptote after performing long division

	$x+2$
x	x^2+2x-3 -(x^2) $2x-3$ -$(2x)$ -3

$f(x)=x+2-\dfrac{3}{x}$. Thus, there exists an oblique asymptote $\text{O.A}\equiv y=x+2$

Application activity 4.2.2

Find asymptotes of the following functions:

1) $f(x)=\dfrac{x^3+x^2-5x-2}{x^3-x^2-2x}$; 2) $y=\dfrac{x+3}{x^2+9}$; 3) $y=\dfrac{x^2+3x+1}{4x-9}$;

4) $y=\dfrac{x^2-x-2}{x-2}$

4.2.3. Continuity and asymptotes of logarithmic functions

Activity 4.2.3.

I. Let us consider the logarithmic function $f:\mathbb{R}^+\to\mathbb{R}, y=f(x)=\log_2(x)$

1. Use a scientific calculator to complete the following table:

$x=x_0$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4
$y=\log_2 x$					
$\lim\limits_{x\to x_0}\log_2 x$					

2. Can you conclude that $\lim\limits_{x\to x_0}\log_2 x=\log_2(x_0)$ **?** What about the continuity of $y=f(x)=\log_2(x)$?

3. By using the information from the above table, try to plot the graph of $y=\log_2(x)$

4. Give any justification that allows you to decide on the continuity of the function

5. Find $\lim\limits_{x\to 0^+}\ln x$ and deduce the equation of asymptotes of $f(x)=\ln x$ if any.

II. Observe the graph of the function $p(x)=\dfrac{\ln x}{x}$ and deduce

$$\lim_{x\to+\infty}\frac{\ln x}{x},\ \lim_{x\to0^+}\frac{\ln x}{x},\ \lim_{x\to1}\frac{\ln x}{x},\ \lim_{x\to\frac{1}{5}}\left(\frac{\ln x}{x}\right)$$

The limit $\lim_{x \to 0^+} \ln x = -\infty$ shows that the line OY with equation $x = 0$ is the vertical asymptote. This means that as the independent variable x takes values approaching 0 from the right, the graph of the function approaches the line of equation $x = 0$ without intercepting. In other words, the dependent variable y decreases without bound.

$\lim_{x \to +\infty} \ln x = +\infty$, implies that there is no horizontal asymptote.

The $\lim_{x \to 0^-} \ln x$ does not exist because values closer to 0 from the left are not included in the domain of the given function. The graph of the logarithmic function $f : \mathbb{R}^+ \to \mathbb{R}, f(x) = \log_a(x),\ a > 1$ has the following characteristics:

- The domain is $]0, +\infty[$ and $f(x)$ is continuous on this interval.
- The range is $\mathbb{R}$
- The graph intersects the $x - axis$ at $(1, 0)$
- As $x \to 0^+, y \to -\infty$, so the line of equation $x = 0$ (the y- axis) is an asymptote to the curve
- As x increases, the graph rises more steeply for $x \in [0,1]$ and is flatter for $x \in [1, +\infty[$
- The logarithmic function is increasing and takes its values (range) from negative infinity to positive infinity.

Example: Let us consider the logarithmic function $y = \log_2(x - 3)$

a) What is the equation of the asymptote line?

b) State the domain and range

c) Find the $x-$ intercept.

d) Determine another point through which the graph passes

e) Sketch the graph

Solution

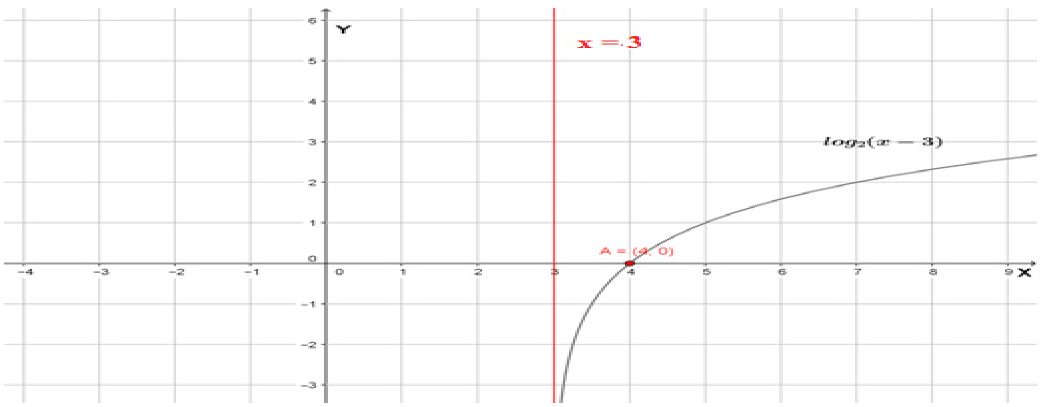
a) The basic graph of $y = \log_2 x$ has been translated 3 units to the right, so the line $L \equiv x = 3$ is the vertical asymptote.

b) The function $y = \log_2(x-3)$ is defined for $x-3>0$, So, the domain is $]3,+\infty[$ .The range is $\mathbb{R}$

c) The x-intercept is (4, 0) since $\log_2(x-3)=0 \Leftrightarrow x=4$

d) Another point through which the graph passes can be found by allocating an arbitrary value to x in the domain then compute y. For example, when $x=4, y=\log_2(4-2)=\log_2 2=1$ which gives the point $(4,1)$.

Note that the graph does not intercept *y*-axis because the value 0 for *x* does not belong to the domain of the function. **The graph of** $y=f(x)=\log_2(x-3)$



Application activity 4.2.3.

1. Given the logarithmic function $y=-1+\ln(x+1)$,

   i) Find equation of asymptote lines (if any)?;

   ii) State the domain and range;

   iii) Find the $x-$ intercept;

   iv) Find the $y-$ intercept;

   v) Determine another point belonging to the graph; (vi) Sketch the graph.

2. Sketch the graph of the logarithmic function of base a with $0<a<1$ Precise the characteristics of the graph.

4.2.4. Continuity and asymptotes of exponential functions

Activity 4.2.4.

Given the function $f(x) = 2^{(x-2)}$,

a. Find the domain and range of $f(x)$.

b. Determine $\lim_{x\to-\infty} f(x)$ and deduce the equation of horizontal asymptote for the graph.

c. Evaluate the value of $f(x)$ *for* $x=0$ and deduce y- intercept

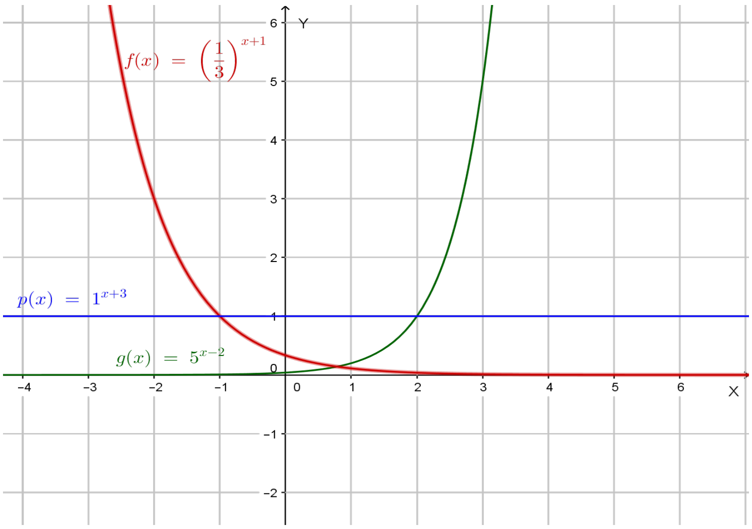
d. Determine $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} \frac{f(x)}{x}$

e. Evaluate $\lim_{x\to0^+} f(x)$ and $\lim_{x\to0^-} f(x)$. Discuss the continuity of this function at $x=0$.

f. Sketch the graph of $f(x)$

For $a>0, a\neq1$, the exponential function $f(x) = a^x$ is continuous on $\mathbb{R}$ and takes always nonnegative values. Its graph admits the line of equation y = 0 as horizontal asymptote and intercepts y-axis at (0, 1). The function f is increasing from 0 to $+\infty$ if a is greater than 1 and decreasing from $+\infty$ to 0 if a is smaller than 1. The function is the constant 1 if $a=1$ and its graph is the horizontal line of equation $y=1$.

Graphs of $g(x)=5^{x-2}$ **,** $f(x)=\left(\frac{1}{3}\right)^{x+1}$ and $p(x)=1^{x+3}$ are shown on the diagram below:



Example: Let $f(x)=3^{x+1}-1$.

Find the domain, range and equation of the horizontal asymptote of the graph of f. Precise intercepts (if any) of the graph with axes.

Solution:

The domain of f is the set of all real numbers since the expression $x+1$ is defined for all real values. To find the range of f, we start with the fact that $3^{(x+1)}$ > 0 as exponential function. Then, subtract 1 to both sides to get $3^{x+1}-1>-1$. Therefore, for any value of *x*, $f(x)>-1$. In other words, the range of *f* is $]-1,\infty[$; As x decreases without bound, $f(x)=3^{x+1}-1$ approaches -1, in other words $\lim_{x\to-\infty} f(x)=-1$. Thus, the graph of f has horizontal asymptote the line of equation $y=-1$. To find the x- intercept we need to solve the equation $f(x) = 0$.

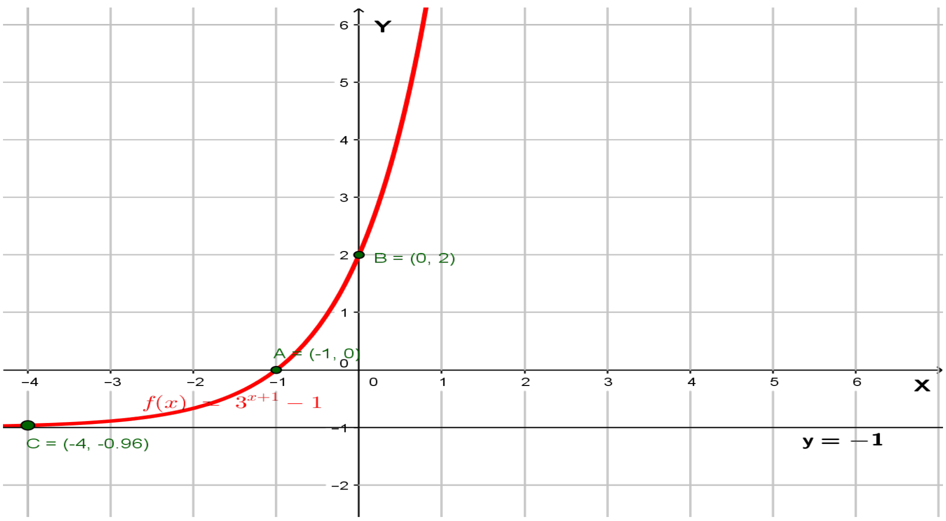
This is $3^{(x+1)}-1=0$. Solving yields to $x=-1$. The x- intercept is the point (-1, 0).

The y- intercept is given by (0, f (0)) = (0, $3^{(0+1)}$ - 1) = (0, 2).

Extra points: (-2, f(-2)) = (-2, $3^{(-2+1)}$ - 1) = (2, -4/3) , and

(-4, f(-4))= (-4, $3^{(-4+1)}$ - 1) = (-4, -26/27)

We can now use all the above information to plot $f(x)=3^{(x+1)}-1$:



Application activity 4.2.4.

Find out the asymptotes to the following functions

a. $y = 2^{x-4} + 3$

b. $y = 2e^{x+1} + 2$

c. $y = \frac{1}{e^x - 1}$

d. $y = e^{-\frac{1}{x^2}}$

4.3. End unit Assessment

End of unit assessment

1. Given the function $f(x) = \frac{x^2 + 3x + 1}{4x - 9}$; Find the limits:

   (a) $\lim_{x \to \frac{9}{5}} f(x)$;

   (b) $\lim_{x \to \pm\infty} f(x)$,

   (c) Verify if the graph has asymptotes and determine their equations.

2. Evaluate the following limit $\lim_{x \to \infty} \frac{e^x + 1}{e^x - 2}$;

3. Find the value for a such that $\lim_{x \to -2} \frac{3x^2 + ax + 3}{x^2 + x - 2}$ has the indeterminate form of $\frac{0}{0}$;

4. Given the following functions: (a) $f(x) = \frac{x^2 - x - 2}{x - 2}$;

   (b) $f(x) = \begin{cases} \frac{1}{x^2}, x \neq 0 \\ 1, x = 0 \end{cases}$; (c) $f(x) = \begin{cases} x^2 - x - 2, x \neq 0 \\ 1, x = 2 \end{cases}$;

   Does (a),(b), and (c) exist and continuous functions ?

UNIT 5 FINANCIAL MATHEMATICS

Key unit competence: Use Financial Mathematics techniques in solving production, financial and economical related problems such as simple and compound interests, annuity and sinking funds.

5.0. Introductory activity

Introductory activity 5

In simple interest earned on an investment, $I = p \times r \times t$, I represents interest earned, p represents principal, r represents interest rate, and t represents time in years. Suppose 50 000 Frw is invested at an annual interest rate of 8% and the interest is added to the principal at the end of each year.

1) Classify and explain any financial concepts that can be used to find the total amounts of money over those five years.
2) Apart from the scenario above, (i) what are your observations when you need to make money in a very short time? (ii) Is it the same if you make money in the long term?
3) Assuming a future amount triples in 5 years, does this relationship indicate an investment opportunity? Is this investment profitable, if so, who can handle it? Why?

5.1. Financial Mathematics concepts

5.1.1. Interest and interest rates

Activity 5.1.1.

Suppose you want to borrow money from a bank for your project.

a) List and explain the main concepts you will discuss about with the bank manager.

b) Attempt to differentiate between "interest" and "interest rate"

- **Interest:** is the compensation one gets for lending a certain asset. **For instance,** suppose that you put some money in a bank account for a year. Then, the bank can do whatever it wants with that money for one year. To reward you for that, it pays you your money plus an extra amount. The extra amount is called **interest**. The asset being lent out is called the **capital** or the **principal**.
- The amount of interest due, expressed as proportion of the principal, or as percentage, is called the **interest rate.**
- Interest rates are broadly divided into types that comprise:
  1. **Nominal interest rate:** the stated rate on which the interest is calculated.
  2. **Effective interest rate:** The actual interest rate (as opposed to the stated or nominal interest rate) obtained after accounting for the effects of variation along the period of investment.

Note: Usually, both principal and interest are expressed in money. However, this is not required.

For example**,** a farmer can lend his tractor to a neighbor and receive 10% of the grain harvested.

The capital is always expressed in money.

Application activity 5.1.1

A parent gets, from a bank, a loan of $1\,000\,000$ Frw to pay the school fees of his children. The bank charges an interest rate of 5% over one year.

a) Express the interest rate as an irreducible fraction

b) Find the interest the bank will get at the end of one-year period.

c) Is this interest rate nominal or effective? Explain.

5.1.2. Simple interest and Compound interest

Activity 5.1.2.

Suppose that $50\ 000$ Frw is invested at a bank at annual interest rate of 8%. Find the total amount(principal and interest) at the end of three years in each of the following cases:

a) The interest is charged only on the money initially invested.

b) The interest is added to the principal at the end of each year to make a new principal for the following year.

c) What are your observations as future Accountant Technicians?

- **Simple interest**

Calculating interest using simple interest method consists in charging the interest only on the original principal. The quantities involved when using this method are **the interest rate(i), the principal or capital (P)** and **the term(n).**thus, the simple interest(SI) yield on capital *P* lent at interest rate *i* over a period *n* is: $n \times i \times P$

Example 1:

How much interest do you get if you put 10 000 Frw in a savings account that pays simple interest at 9% per annum for two years?

What is the interest if you only invest the money in the account for half a year?

Answer:

If you invest for two years, you get $\left(2 \times 0.09 \times 10\ 000\right) Frw = 1\ 800 Frw$ as interest.

If you invest for only half a year, you get $\left(0.5 \times 0.09 \times 10\ 000\right) Frw = 450 Frw$ as interest.

Note: in practice, the application of simple interest is limited.

- **Compound interest**

Calculating interest using compound interest method consists in charging the interest on the principal plus any accumulated interest.

Example 2:

How much interest do you get if you put $10\ 000$ Frw in a savings account that pays compound interest at 9% per annum for two years?

Answer:

- The first year you would earn $(1 \times 0.09 \times 10\,000)\,Frw = 900Frw$ as interest, so after one year you have $10\,900$ Frw

- The second year you earn FRW $(1 \times 0.09 \times 10\,900)\,Frw = 981Frw$ as interest

  At the end of the two years, you have $10\,900Frw + 981Frw = 11\,881Frw$

Now compare example of simple interest and example of compound interest, we observe the following:

- The first example shows that if you invest 10 000 Frw for two years, simple interest, the capital grows to $11\,800$ Frw
- The second example shows that if you invest 10 000 Frw for two years, compound interest, the capital grows to $11\,881$ Frw. The investor put $10\,000$ Frw into an account at 9% for two years at a compound interest and in the first year, the investor receives $900Frw$ interest (9% of $10\,000Frw$). These would be credited to his / her account, so he/she now has $10\,900$ Frw. In the second year he/she would get $981Frw$ interest (9% of $10\,900Frw$), so he /she ended up with $11\,881Frw$.

More generally, the interest over a year is iP, where i is the interest rate and P is the principal at the beginning of the year.

Thus, at the end of the year, the capital has grown to P + iP = (1 + i) P.

In the second year, the principal is (1 + i) P and interest is calculated over this amount, so the interest is i(1 + i)P and the capital has grown to $(1+i)^2 P$. In the third year the interest has increased, and the capital has grown to $(1+i)^3 P$,

In general, this can lead to, in the nth year the capital grows to $(1+i)^n P$

Example 3:

How much do you have after two years of investing 1 000 000 Frw in a savings account that pays compound interest at 9% per annum?

How much do you have if you invest the money in the account only for half a year?

Answer:

If you leave the money in the account for two years, then at the end you have $\left[(1+0.09)^2 \times 1\,000\,000\right] Frw = 1188\,100 Frw$, as we computed above.

If you leave the money in the account for only half a year, then at the end you have $\left[(1+0.09)^{\frac{1}{2}} \times 1\,000\,000\right] Frw = \left[\sqrt{1+0.09} \times 1\,000\,000\right] Frw = 1\,044\,030 Frw$.

Example 4:

Suppose 500 000 Frw principal earns 150 000 Frw in interest over 6 years. If compound interest is used, what was the interest rate? What if simple interest is used?

Answer:

The principal has accumulated to 650 000 Frw, so in the case of compound interest we need to solve the interest rate i from the equation $500000(1+i)^6 = 650000$. Thus, the interest rate is 4.47% rounded to the nearest basis point (one basis point equals 0.01%). Note that the calculation is the same regardless of the currency used.

In the case of simple interest, the equation to solve is $6 \times i \times 500,000 = 150,000$, so $i = \frac{150,000}{6 \times 500,000} = 0.05$, therefore, the interest rate is 5%.

- **Comparing simple interest and compound interest**

	Simple interest	**Compound interest**
Total amount after n years, **P: initial principal**	$(1+ni)P$	$(1+i)^n P$

To compare simple interest and compound interest, it is sufficient to compare $1+ni$ and $(1+i)^n$

We have, for $n \geq 2$, $(1+i)^n > 1+ni$,showing that the compound interest is more beneficial than simple interest.

Example 5:

How long does it take to double your capital if you deposit it into an account with 7.5% compound interest? What if the account pays simple interest?

Solution:

The question is, for what value of n does a capital P accumulate to 2P when i = 0.075. So we have to solve for n the equation $(1.075)^n P = 2P$. The first step is to divide both sides by P to get $1.075^n = 2$ then take the logarithm: $\log(1.075)^n = \log 2 \Leftrightarrow n\log 1.075 = \log 2$ so, $n = \frac{\log 2}{\log 1.075} = 9.58$ So it takes 9.58 years to double your capital. Note that it doesn't matter how much you start with. It takes as long for a pound to grow into two pounds as it does for a million RWF into two million RWF. With simple interest, the calculation is simpler. We need to solve the equation $(1+n0.075)\,\mathrm{P} = 2P$, so $n = \frac{1}{0.075} = 13.3$, so with simple interest it takes 13.3 years to double your capital.

- **Discrete and continuous compound interest rates**

If you put money in a savings account then the bank will pay you interest (a percentage of your account balance) at the end of each time of period, typically one day, one week, one month, quarterly, semiannually, or annually.

The method whereby the interest is calculated and added to the principal at specific periods is called **discrete compounding.**

For example, if the time of period is one month this process is called **Monthly compounding.** The term compounding refers to the fact that interest is added to your account each month. If it is one day is called **daily compounding**. The exponential model that describes this situation is called discrete compounding interest formula; it is given by:

$A = P_0\left(1+\frac{r}{n}\right)^{nt}$ Where A is total amount at the end of periods of time, P_0 is the principal amount, n is the number of times that the interest is compounded, r is the interest rate per period, t is the time.

Example 6:

1. If the principal money is \$100 the annual interest rate is 5% and the interest is compounded daily. What will be the balance after ten years?

2. An amount of 500 000 Frw is invested at a bank that pays an interest rate of 12% compounded annually.

   a) How much will the owner have at the end of 10 years, in each of the following cases? The interest rate is compounded: (i). once a year, (ii). Twice a year

   b) What type of interest rate among the two would the client prefer? Explain why.

Solution:

1. Let $P_0 = 100, r = 5\% = 0.05, n = 1 year = 365 days\ and\ t = 10$ **;**

$P(t) = 100\left(1+\frac{0.05}{365}\right)^{365\times 10} = 164.87$ After 10 years balance will be $164.87

2. a)The interest rate is calculated as follows:

   i) For once a year, at the end of 10 years the owner will have

   $A = P(1+r)^t = 500\,000(1+0.12)^{10} = 500\,000(1.12)^{10} = 1\,552 \in 924.10 Frw$

   ii) For twice a year, at the end of 10 years the owner will have

   $A = P(1+\frac{r}{2})^{2t} = 500\,000(1+\frac{0.12}{2})^{2(10)} = 500\,000(1.06)^{20} = 1\,603\,567 Frw$

   b) Since $1\,603\,567 > 1\,552\,924.10$, the client will prefer compounding many times per year as it results in more money.

Continuous compound interest

If we start with discrete compound interest formula and let the number of times compounded per year approaches ∞, then we end up with what is known as **continuous compounding** then the balance at time t years is given by $A = P_0 e^{rt}$ where P_0 is the principal amount, r is annual interest and time t years.

Example7

If the principal money is $10 000 the annual interest rate is 5% and the interest is compounded continuously. What will be the balance after 40 years?

Solution:

$P_0 = 10\,000, r = 5\% = 0.05, t = 40$

$P(40) = 10\,000 e^{0.05\times 40} = 73\,890.56$

The balance after 40 years is $73890.56.

Example 8:

Find the accumulated amount after 3 years if $1 000 is invested at 8% per year compounded (a) daily, (b) continuously.

Solution:

a) Using the compound interest formula with P=1 000, r =0.08, m= 365, and t = 3, we find $A = C(1+\frac{i}{m})^{mt} = 1\,000(1+\frac{0.08}{365})^{365} = 1\,271.22$

b) Using the continuous compound interest formula with P = 1 000, r = 0.08, and t = 3, we find $A = Pe^{rt}$= 1 000 $e^{(0.08)(3)} \approx$ 1 271.25, Note that the two solutions are very close to each other.

Keep in mind that with simple interest, you could increase the interest you earn by withdrawing your money halfway from the account. Compound interest has the desirable property that it makes no difference. Suppose you put your money in one account for m years and then in another account for n years, and both accounts pay compound interest at rate i. Then, after the first m years, your capital has grown to $(1+i)^m P$. You withdraw it and put it in another account for n years, after which your capital has grown to $(1+i)^n(1+i)^m P$.

This is the same as what you would get if you had kept the principal m + n years in the same account, because $(1+i)^n(1+i)^m P = (1+i)^{n+m} P$ this is why so much compound interest is used in practice. Unless otherwise stated, interest always refers to compound interest.

In summary,

Type of compounding	**Formula**
Discrete compounding	$A = P_0\left(1+\frac{r}{n}\right)^{nt}$
Continuous compounding	$A = P_0 e^{rt}$

Where**:** P_0 is the principal or capital at the beginning of a certain period.

$r\%$ is the constant interest rate per period,

t: number of *years*

n: number of periods per year

- **Effective Interest Rate:**

In section **5.1.1,** we saw that the **Effective interest rate**, as opposed to the stated or nominal interest rate, is the actual interest rate, obtained after accounting for the effects of variation along the period of investment. Now, let us see how it is applied to compound interest:

The actual rate of return on an investment depends on the frequency of compounding. To clarify the interest rate comparison, we can use the **effective interest** rate (also called annual return). We want to derive a relationship between the nominal compound interest and the effective interest rate.

$(1+i_e)^t P = P(1+\frac{i}{m})^{mt}$, dividing both sides by C and considering the t^{th} root of each side,

$1+i_e = \left(1+\frac{i}{m}\right)^m$. therefore, $i_e = \left(1+\frac{i}{m}\right)^m - 1$

The cumulative amount after one year at a simple interest rate i per year, when the interest is compounded m times per year, is $r_{eff} = (1+\frac{i}{m})^m - 1$; this is called the **Annual Equivalent Rate,** denoted by **AER.** Thus, $AER = \left(1+\frac{i}{m}\right)^m - 1$, where i is the nominal interest rate, and m is the number of periods. The **AER** is also called the **effective annual rate** of interest.

For continuous compounding, it can be shown that $AER = e^r - 1$

Example 9

Find the effective annual rate of interest for a nominal interest of 10% when compounded for two years:

a) Semiannually

b) Continuously

Solution:

a) Semiannually, $AER = \left(1+\frac{i}{m}\right)^m - 1 = \left(1+\frac{0.1}{2}\right)^2 - 1 = 0.1025$, that is the $AER = 10.25\%$

b) Continuously, $AER = e^r - 1 = e^{0.10} - 1 = 0.1052$, that is, $AER = 10.52\%$

Note:

- The relationship between the monthly interest rate i_m on a deposit account and the AER can be formulated as $AER = (1+i_m)^{12} - 1$.
- The relationship between the daily interest rate i_d on a deposit account and the AER can be formulated as $AER = (1+i_d)^{365} - 1$.
- For loan repayments the annual equivalent rate is usually referred to as the **Annual Percentage Rate**, this is denoted by APR. If you take out a bank loan you will usually be quoted an APR even though you will be asked to make monthly repayments.

Example10

If the monthly rate of interest on a loan is 1.75% ,what is the corresponding Annual Percentage Rate (APR)?

Solution:

$i_m = 1.75\% = 0.0175$

$APR = (1+i_m)^{12} - 1 = (1.0175)^{12} - 1 = 0.02314393$

Thus, the Annual Percentage Rate is 23.14%.

Application activity 5.1.2.

1. Your aunt would like to invest 300 000 Frw at a bank. The Bank I pays an interest rate of 10% compounded once annually. The Bank II pays an interest rate of 9.8% compounded continuously. Your aunt will withdraw the money plus interest after 10 years.
2. At which bank do you advice your Aunt to invest her money so as to get much money at the end of 10 years?
3. 2.The interest rate on a given bank deposit account is A per annum. Find the accumulation of 5 000 after seven years on this account.
4. 3. Find the *effective* rate of interest corresponding to a nominal rate of 8% per year compounded: Annually, Semiannually, Quarterly, Monthly, and Daily
5. 4.How long does it take for 900 to accumulate to 1 000 at an interest rate of 4% per annum?
6. 5. Suppose that you save £1 000 in an account that pays 2% interest every quarter. How much do you have in one year, if the interest is paid in the same account?

5.1.3.Present value and future value

Activity 5.1.3.

Assume that your aunt would like to invest money in an account today to make sure that she has enough money in 10 years to buy a car. She estimated that in 10 years, a car will be costing $10\,000$ US Dollars. The bank offers a compound interest rate of 5% per year. Let A be the amount your aunt should invest now.

a) Compare A to 10 000 US Dollars.

b) Suggest a name for the amount A and a name for 10 000 US Dollars

c) Suggest a name for the process of finding the amount A from 10 000 US Dollars under annual compounding.

- Suppose you want to determine the current value of the ultimate returns on an investment. This question could be rephrased as follows: What is the present value of my investment maturing in t years at an interest rate of i percent?

To solve this problem, you need to know the future value of your investment, how many years it will take for the investment to mature, and the interest or discount rate on your investment. The result of the equation is a unit amount that is less than the future principal and interest you will earn; it is the amount the investment is worth at the present time. The equation for present value (PV) is as follows: $PV = \frac{FV}{(1+i)^t}$.

Where, **FV =** the **future value** of the investment at the end of t years; **t =** the number of years in the future; **i =** the interest rate or annual interest rate and **PV =** the current or **present value** of a sum of money that you have invested or intend to invest today. So that these inputs help each to apply formulae in solving financial related problems.

Vice versa, the future value is found from the formula $FV = PV(1+i)^t$

Let's say you want to determine what an investment will be worth at some point in the future, **i.e.** What will my investment be worth in N years if my interest rate is 1 percent? You need to know how many years **t** it will take to have the investment, the interest rate **i**, and the amount of the investment (the present value of the investment; PV). **The future value (FV)** is then given by the equation $FV = PV(1+i)^t$

Example1

a) Calculate the future value (15 years from now) of \$5 000, which yields 10 percent; assume an annual compounding period.

b) Calculate the future value (15 years from now) of \$5 000 earning 10 percent; accept simple interest (no interest is paid on the interest earned).

c) How much did interest earn on interest in the first problem (i)?

Solution:

a) The future value is given by

$$FV = PV(1+i)^t = 5\,000(1+0.1)^{15} = USD\ 20\,886.24$$

b) The future value is given by

$$FV = 5\,000(1+15\times 0.1) = 5\,000(2.5) = USD\ 12\,500$$

c) The difference between USD 20 886 and USD 12 500 is USD 8 386, which is the amount of interest your interest earned.

This concept is the key to financial success. Earn interest on your interest.

- **Discounting:**

Note that the same amount received today and, in the future, have different values. The process of finding the present value of a future amount of money is called **discounting.** The discount rate d is the interest paid at the beginning of a period divided by the principal at the end of the period. The discount factor v is the amount of money one has to invest to get a unit of capital after a unit of time.

Example2:

How much do you have to invest now to get 2 000 000 Frw after five years at an interest rate of 4% ?

Solution:

One Rwandan Franc will accumulate to $(1+0.04)^5 = 1.216652924$ Frw in five years, so you need to invest $\frac{2\,000\,000}{(1+0.04)^5} = 1\,643\,854 Frw$.

We say that $1\,643\,854$ Frw will be equal to 2 000 000 Frw in five years at a rate of 4%. We call 1 643 854 Frw the present value and 2 000 000 Frw the future value.

When you defer a payment, it accumulates; moving it backwards will discount it. This shows that money has a time value: the value of money depends on time. 2 000 will be worth more than 2 000 in five years.

In financial mathematics, all payments must be dated. Let us generally assume that the interest rate is i. How much do you have to invest to get a capital C after a unit of time?

The answer is $\frac{C}{1+i}$. The factor $v=\frac{1}{1+i}$ is called the **discount factor**. It's the factor by which you have to multiply a payment to push it back a year. If the interest rate is 0.25% then discount factor is $\frac{1}{1+0.0425}=0.95923$.

Unless the interest rate is too high, the discount factor is close to one. Therefore, the discount rate $d=\frac{1}{v}$, usually expressed as a percentage, is often used (compare how the interest rate i is used instead of the accumulation factor 1+i. In the example above, the discount rate is 0.0425 or 4.25%

Example3:

Suppose the interest rate is 7%. What is the present value of a payment of 70 euros in one year?

Solution:

The discount factor is v = 1/1.07 = 0.934579, so the present value is $0.934579\times 70=65.42$ euros (to the nearest cent). Interest is usually paid in arrears. If you borrow money for a year, you have to pay it back at the end of the year plus interest. However, there are also some situations where the interest is paid in advance. The discount rate is useful in these situations, as shown in the example below.

Example 4

Suppose the interest rate is 7%. If you borrow $1 000 for a year and have to pay interest at the beginning of the year, how much do you have to pay? Translate the final amount in Rwandan francs as per today's exchange rate.

Solution:

If interest were to be paid later, you would have to pay $0.07\times 1\,000=70$ euros at the end of the year. However, you have to pay the interest a year earlier. As we have seen in the example, the equivalent value is $v\times 70=65.42$ euros. There is another way to get to the answer. At the beginning of the year, you get $1 000 from the lender, but you have to pay interest immediately, so you get less from the lender. At the end of the year you pay back $1 000.

The amount you should get at the start of the year should be equivalent to the 1 000 euros you pay at the end of the year. The discount factor is

v = 1/1.07 = 0.934579, so the present value of the 1 000 euros at the end of the year is 934.58 euros. Thus, the interest you have to pay is 1 000 euros – 934.58 euros = 65.42 euros. In terms of the interest rate i = 0.07 and the capital P = 1 000, the first method calculates ivC and the second method calculates P – vP = (1 – v)P = dP. Both methods yield the same answer, so we arrive at the important relation $d = iv$.

Example 5

Sales of a new product tend to decrease over time. Sales levels are described by the function $S(t) = 1\,500 + 750e^{-0.2t}$, where t is measured in years. Find out the discount of sales by evaluating the sales levels for $t = 1, 2, 4\, and\, 10\, years$

Solution:

Simply substitute the respective value of time in the sales function,

For the beginning $S(0) = \$1\,500 + \$750 = \$2\,250$

For 1 year, $t = 1, S(1) = \$1\,500 + 750e^{-0.2} \approx \$2\,114$

For 4 years, $t = 4, S(4) = \$1\,500 + 750e^{-0.8} \approx \$1\,837$

For 10 years, $t = 10, S(10) = \$1\,500 + 750e^{-2} \approx \$1\,601$

We see that the sales to the firm get smaller every single year, as the product matures.

Example 6

The number of credits in the bank increases according to the following provided equation of the type $N(t) = N_0 e^{kt}$, where N_0 is the initial value.

a) Given that the number of loans triples in 2years, solve the equation to find the value of k.

b) How long would it take for the number of loans to be 5 times the initial number?

Solution

a) $N(2) = 3N_0 \Leftrightarrow 3N_0 = N_0 e^{k(2)} \Leftrightarrow e^{2k} = 3 \Rightarrow 2k = \ln 3 \Leftrightarrow k = \dfrac{\ln 3}{2} = 0.5493$

b) $5N_0 = N_0 e^{0.5493t} \Leftrightarrow e^{0.5493t} = 5 \Rightarrow t = \dfrac{\ln 5}{0.5493} \approx 2.93.$

It will take 2.93 years for the number of loans to be 5 times the initial number.

Application activity 5.1.3.

1. How many days does it take for £1 450 to accumulate to £1 500 under an interest rate of 4% p.a. compounded monthly?
2. Compute the nominal interest rate per annum payable monthly that is equivalent to the simple interest rate of 7% p.a. over a period of three months.

5.2.Sequences

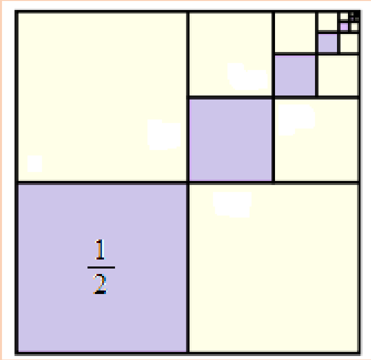
5.2.1. Introduction to sequences

Activity 5.2.1.

1. Suppose that your investments are growing in such a way that each new return per year is 2 times as large as the previous year. If there are 126 000 000 Frw in your investments, use a piece of paper (measured in million) and write down the number of investments that will be there in second, third, fourth, …n^{th} years. How can you classify this kind of investments financially and economically?
2. Fold once a square paper, what is the fraction that represents the colored part?

   Fold it twice, what is the fraction that represents the colored part?

- What is the fraction that represents the colored part if you fold it ten times?
- What is the fraction that represents the colored part if you fold it n times?
- Write a list of the fractions obtained starting from the first until the n^{th} fraction.



A sequence is a function whose domain is a subset of the set of natural numbers. The terms of a sequence are the range elements of the function. A sequence is denoted by $\{u_n\}$ or (u_n).

We can also write $\{u_1, u_2, u_3, ..., u_{n-1}, u_n\}$ for a finite sequence whose first term is u_1. The numbers $u_1, u_2, u_3, ..., u_{n-1}, u_n$ in a sequence are called **terms of the sequence**.

The natural number *n* is called **term number** and value u_n is called the **general term** of the sequence. If a sequence continues indefinitely, it can be denoted as $\{u_n\}_{n=1}^{\infty}$. The number of terms of a sequence (possibly infinite) is called the **length of the sequence**.

Note:

- Sometimes, the term number, *n*, starts from 0. In this case the terms of the sequence are $u_0, u_1, u_2, ..., u_{n-1}, u_n, ...$ and this sequence is denoted by $\{u_n\}_{n=0}^{+\infty}$, the initial term is u_0.The term number n may start from 1, in the sequence $u_1, u_2, u_3, ..., u_{n-1}, u_n$ the term u_1 is the **initial term** or **the first term**
- A sequence can be finite, like the sequence $2, 4, 8, 16, ..., 256$. A finite sequence can be empty. This particular case will not be considered in our study of sequences.

Let us consider the following list of numbers: $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, ... \frac{1}{n}, ...$. The terms of this list are compared to the images of the function $f(x) = \frac{1}{x}$. The list never ends, as the ellipsis indicates. The list is called a sequence, and the numbers in this ordered list are called the **terms** of the sequence. In dealing with sequences, we usually use subscripted letters, such as u_1 to represent the first term, u_2 for the second term, u_3 for the third term, and so on such as in the sequence $f(n) = u_n = \frac{1}{n}$. However, in the sequence such as $\{u_n\}: u_n = \sqrt{n-3}$, the first term is u_3 as the previous are not possible, in the sequence $\{u_n\}: u_n = 2n-5$, the first term is u_0.

Usually, a numerical sequence is given by some formula $u_n = f(n)$, permitting to find any term of the sequence in terms of its number *n*; this formula is called a **general term formula**.

A second way of defining a sequence is to assign a value to the first (or the first few) term(s) and specify the *n*th term by a formula or equation that involves one or more of the terms preceding it. Sequences defined this way are said to be defined **recursively,** and the rule or formula is called a **recursive formula.**

Example: The sequence $(u_n):\begin{cases} u_1 = 1 \\ u_n = nu_{n-1} \end{cases}$

Infinite and finite sequences

Consider the sequence of odd numbers less than 11: This is 1, 3,5,7,9. This is a finite sequence as the list is limited and countable. However, the sequence made by all odd numbers is:

$1,3,5,7,9,...2n+1,...$This suggests that an i**nfinite sequence** is a sequence whose terms are infinite. Note that it is not always possible to give the numerical sequence by a general term formula; sometimes a sequence is given by description of its terms.

Examples:

1) Numerical sequences:

$0,1,2,3,4,5,...$ is a sequence of natural numbers.

0, 2, 4, 6, 8, 10, … is a sequence of even numbers.

$1.4,1.41,1.414,1.4142,...$ is a numerical sequence of approximates, defined more precisely values of $\sqrt{2}$. For this numerical sequence it is impossible to give a general term formula, nevertheless this sequence is described completely.

2) List the first five terms of the sequence $\left\{2^n\right\}_{n=1}^{+\infty}$

Solution

Here, we substitute $n=1,2,3,4,5,...$ into the formula 2^n. This gives $2^1,2^2,2^3,2^4,2^5,...$

Or, equivalently, $2,4,8,16,32,...$

3) Express the following sequences in general notation

a) $\frac{1}{2},\frac{2}{3},\frac{3}{4},\frac{4}{5},...$; b) $\frac{1}{2},\frac{1}{4},\frac{1}{8},\frac{1}{16},...$

Solution

a) $\frac{1}{2},\frac{2}{3},\frac{3}{4},\frac{4}{5},...$ Beginning by comparing terms and term numbers:

Term number	1	2	3	4	…
Term	$\frac{1}{2}$	$\frac{2}{3}$	$\frac{3}{4}$	$\frac{4}{5}$	…

In each term, the numerator is the same as the term number, and the denominator is one greater the term number. Thus, the n^{th} term is $\frac{n}{n+1}$ and the sequence may be written as $\left\{\frac{n}{n+1}\right\}_{n=1}^{+\infty}$.

b) $\frac{1}{2},\frac{1}{4},\frac{1}{8},\frac{1}{16},...$ Beginning by comparing terms and term numbers:

Term number	1	2	3	4	...
Term	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{16}$	...

Or

Term number	1	2	3	4	...
Term	$\frac{1}{2^1}$	$\frac{1}{2^2}$	$\frac{1}{2^3}$	$\frac{1}{2^4}$	...

In each term, the denominator is equal to the power of 2, where the exponent is the term number. We observe that the n^{th} term is $\frac{1}{2^n}$ and the sequence may be written as $\left\{\frac{1}{2^n}\right\}_{n=1}^{+\infty}$.

4) A sequence is defined by $\{u_n\}:\begin{cases} u_0 = 1 \\ u_{n+1} = 3u_n + 2 \end{cases}$; Determine $u_1,\ u_2\ and\ u_3$

Solution: Since $u_0 = 1$ and $u_{n+1} = 3u_n + 2$, replace *n* by $0,1,2$ to obtain u_1, u_2, u_3 respectively.

$$n = 0,\quad u_{0+1} = u_1 = 3u_0 + 2 = 3\times 1 + 2 = 5$$

$$n = 1,\quad u_{1+1} = u_2 = 3u_1 + 2 = 3\times 5 + 2 = 17$$

$$n = 2,\quad u_{2+1} = u_3 = 3u_2 + 2 = 3\times 17 + 2 = 53$$

Thus, $\begin{cases} u_1 = 5 \\ u_2 = 17 \\ u_3 = 53 \end{cases}$

Application activity 5.2.1.

1. A sequence is given by $\{u_n\}:\begin{cases} u_0 = 1 \\ u_n = \dfrac{2n^2}{n^2+1} \end{cases}$; Determine u_1, u_2 and u_3

2. List the first five terms of the sequence $\left\{\sqrt{n+1}-\sqrt{n}\right\}_{n=1}^{+\infty}$

3. Express the following sequence in general notation: 1, 3, 5, 7, 9, 11, …

5.2.2.Arithmetic sequences

Activity 5.2.2.

1. In each of the following sequences, each term can be found by adding a constant number to the previous. Guess that constant number:

   a) $(w_n): 20;18;16;14;...$

   b) $(t_n): t_n = 16-6n$

2. Consider the following sequence: $2;5;8;11;14;...$

   a) What is the first term?

   b) Guess the constant number added to a term to obtain the next term

   c) determine the sum of the first 6 terms and try to derive a formula for finding the sum of the first n terms.

Let u_1 be an initial term of a sequence. If we add d successively to the initial term to find other terms, the difference between successive terms of a sequence is always the same number and the sequence is called **arithmetic.** This sequence has the following term $u_1, u_2 = u_1 + d$,

$$u_3 = u_2 + d = u_1 + 2d\ ,\ u_4 = u_3 + d = u_1 + 3d\ ,\ ....u_n = u_{n-1} + d = u_1 + (n-1)d\ ,...$$

An **arithmetic sequence** may be defined recursively as $u_n = u_1 + (n-1)d$ where u_1 and d are real numbers. The number u_1 is the first term, and the number d is called the **common difference.**

Examples:

The following sequences are arithmetic sequences:

Sequence $\{u_n\}$: $5, 8, 11, 14, 17, \ldots$;

Sequence $\{v_n\}$: $26, 31, 36, 41, 46, \ldots$;

Sequence $\{w_n\}$: $20, 18, 16, 14, 12, \ldots$

Common difference

The fixed number that binds the terms of the sequence together is called the **common difference**; it may be represented by letter d. The common difference can be calculated by subtracting any term from the immediately following term. That is $d = u_{n+1} - u_n$ or $d = u_n - u_{n-1}$.

Note:

In an arithmetic sequence, three consecutive terms are such that, the double of the middle term is equal to the sum of extreme terms. That is for an arithmetic sequence u_{n-1}, u_n, u_{n+1}, we have $2u_n = u_{n-1} + u_{n+1}$.

Proof:

If u_{n-1}, u_n, u_{n+1} form an arithmetic sequence, then $u_{n+1} = u_n + d$ and $u_{n-1} = u_n - d$

Adding two equations, you get $u_{n+1} + u_{n-1} = 2u_n$

General term of an arithmetic sequence

The n^{th} term, u_n of an arithmetic sequence $\{u_n\}$ with common difference d and initial term u_1 is given by $u_n = u_1 + (n-1)d$ which is **the general term of an arithmetic sequence.**

If the initial term is u_0, then the general term of an arithmetic sequence becomes $u_n = u_0 + nd$. Generally, if u_p is any p^{th} term of a sequence then the n^{th} term is given by $u_n = u_p + (n-p)d$

Examples

1) $4, 6, 8$ Are three consecutive terms of an arithmetic sequence because $2 \times 6 = 4 + 8 \Leftrightarrow 12 = 12$

2) If the first and tenth terms of an arithmetic sequence are 3 and 30, respectively, find the fiftieth term of the sequence.

Solution: $u_1 = 3$ and $u_{10} = 30$. But $u_n = u_1 + (n-1)d$, $u_{10} = u_1 + (10-1)d$, then

$30 = 3 + (10-1)d \Leftrightarrow 30 = 3 + 9d \Rightarrow d = 3$, now,

$u_{50} = u_1 + (50-1)d = 3 + 49 \times 3 = 150$

Thus, the fiftieth term of the sequence is 150.

3) If the 3rd term and the 8th term of an arithmetic sequence are 5 and 15 respectively, find the common difference.

Solution: $u_3 = 5,\ u_8 = 15$, using the general formula: $u_n = u_p + (n-p)d$

$u_3 = u_8 + (3-8)d$

$5 = 15 - 5d$

$\Leftrightarrow 5d = 15 - 5$

$\Rightarrow 5d = 10 \Rightarrow d = 2$

The common difference is 2.

4) Consider the sequence $5,\ 8,\ 11,\ 14,\ 17,\ ...,\ 47$. Find the number of terms in this sequence

Solution: We see that $u_1 = 5,\ u_n = 47$ and $d = 3$.

But we know that $u_n = u_1 + (n-1)d$. This gives

$47 = 5 + (n-1)3$

$\Leftrightarrow 42 = 3n - 3 \Rightarrow n = 15$

This means that there are 15 terms in the sequence.

5) Consider the sequence which went from 20 to -26 with -2 as common difference. Find the number of terms.

Solution: We have

$-26 = 20 + (n-1)(-2)$

$\Leftrightarrow -46 = -2n + 2 \Rightarrow n = 24$

This means that there are 24 terms in the sequence.

6) Show that the following sequence is arithmetic. Find the first term and the common difference. $\{s_n\} = \{3n+5\}$

Solution: The first term is $s_1 = 8$. The *n*th term and the $(n-1)^{st}$ term of the sequence $\{s_n\}$ are $s_n = 3n+5$ *and* $s_{n-1} = 3(n-1)+5 = 3n+2$, The first term is $s_1 = 8$, Their difference *d* is $s_n - s_{n-1} = (3n+5)-(3n+2) = 3$. Since the difference of any two successive terms is the constant 3, the sequence $\{s_n\}$ is arithmetic and the common difference is 3.

1. Arithmetic Means of an arithmetic sequence

If three or more than three numbers form an arithmetic sequence, then all terms lying between the first and the last numbers are called **arithmetic means.** If B is arithmetic mean between A and C, then $B=\frac{A+C}{2}$.

Let us see how to insert *k* terms between two terms u_1 and u_n to form an arithmetic sequence:

u_1	...	...	...	...	u_n

The terms to be inserted are called **arithmetic means** between two terms u_1 and u_n . This requires to form an arithmetic sequence of $n=k+2$ terms, whose first term is u_1 and last term is u_n. While there are several methods, we will use the n^{th} term formula $u_n=u_1+(n-1)d$. As u_1 and u_n are known, we need to find the common difference *d* taking $n=k+2$ where *k* is the number of terms to be inserted and 2 stands for the number of extreme values (the first and last terms).

Examples

1) Insert three arithmetic means between 7 and 23.

Solution:

Here $k=3$ and then $n=k+2=5$, $u_1=7$ and $u_n=u_5=23$.

Then

$$u_5=u_1+(5-1)d$$
$$\Leftrightarrow 23=7+4d \Rightarrow d=4$$

Now, insert the terms using $d=4$, the sequence is $7,11,15,19,23$.

2) Insert five arithmetic means between 2 and 20.

Solution: Here $k=5$ and then $n=k+2=7$, $u_1=2$ and $u_n=u_7=20$.

Then $u_7=u_1+(7-1)d$
$$\Leftrightarrow 20=2+6d \Rightarrow d=3$$

Now, insert the terms using $d=3$, the sequence is $2,5,8,11,14,17,20$.

2. Nth terms and arithmetic sum

For finite arithmetic sequence $\{u_n\} = u_1, u_2, u_3, ...u_n$, the sum

$\sum_{r=1}^{n} u_r = u_1 + u_2 + u_3 + ... + u_n$ is called an **arithmetic serie.** We denote the sum of the first *n* terms of the sequence by S_n.

Thus, $S_n = u_1 + u_2 + u_3 + ... + u_n = \sum_{r=1}^{n} u_r$

The sum of first *n* terms of a finite arithmetic sequence with initial term u_1 is given by

$$S_n = u_1 + u_2 + u_3 + ... + u_{n-2} + u_{n-1} + u_n;$$

$$S_n = u_n + u_{n-1} + u_{n-2} + ... + u_3 + u_2 + u_1$$

Then $2S_n = \underbrace{(u_1 + u_n) + (u_2 + u_{n-1}) + ... + (u_{n-1} + u_2) + (u_n + u_1)}_{nbracketsterms}$

$$2S_n = n(u_1 + u_n)$$

$S_n = \frac{n(u_1 + u_n)}{2}$, where n is the number of the terms of the arithmetic sequence

Alternatively,

$$s_n = u_1 + u_2 + u_3 + ... + u_n$$

$$= u_1 + (u_1 + d) + (u_1 + 2d) + ... + (u_1 + (n-1)d)$$

$$= \underbrace{(u_1 + u_1 + ...u_1)}_{n\ terms} + (d + 2d + ... + (n-1)d)$$

$$= nu_1 + d[1 + 2 + 3 + ... + (n-1)] = nu_1 + d[\frac{n(n-1)}{2}]$$

$$s_n = nu_1 + \frac{n}{2}(n-1)d = \frac{n}{2}[2u_1 + (n-1)d]$$

$$= \frac{n}{2}[u_1 + u_1 + (n-1)d]$$

$$= \frac{n}{2}(u_1 + u_n)$$

$$s_n = \frac{n}{2}(u_1 + u_n)$$

If the initial term is u_0, the formula becomes $S_n = \frac{n+1}{2}(u_0 + u_n)$

Examples

1) Calculate the sum of first 100 terms of the sequence $2, 4, 6, 8, \ldots$

Solution

We see that the common difference is 2 and the initial term is $u_1 = 2$. We need to find $u_n = u_{100}$.

$$u_{100} = 2 + (100 - 1)2$$
$$= 2 + 198$$
$$= 200$$

Now,

$$S_{100} = \frac{100}{2}(u_1 + u_{100})$$
$$= 50(2 + 200)$$
$$= 10100$$

2) Find the sum of first k positive even integers $(k \neq 0)$.

Solution

$u_1 = 2$ and $d = 2$

$$u_n = u_k$$
$$u_k = 2 + (k - 1)2$$
$$u_k = 2k$$

$$S_n = S_k$$
$$S_k = \frac{k}{2}(2 + 2k)$$
$$S_k = k(k + 1)$$

3) Find the sum: 60 + 64 + 68 + 72 + … + 120

Solution:

This is the sum s_n of an arithmetic sequence u_n whose first term is $u_1 = 60$ and whose common difference is $d = 4$. The *n*th term is u_n.

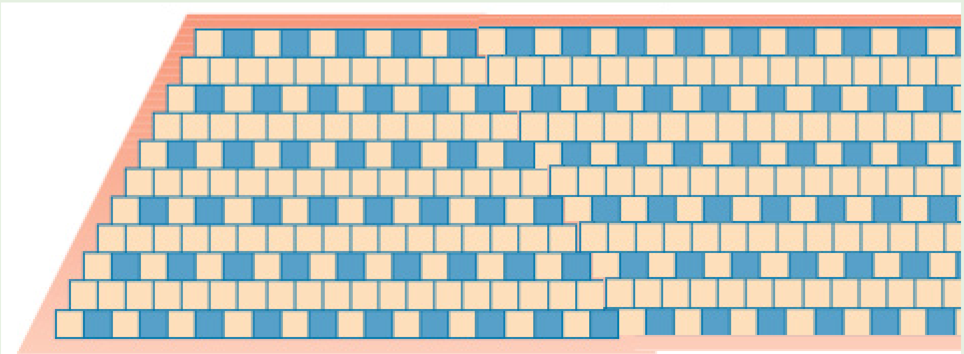
We have $u_n = u_1 + (n - 1)d$ and

$$120 = 60 + (n - 1)4 \Leftrightarrow 60 = 4(n - 1)$$
$$n = 16$$

Now, the sum is $u_{16} = 60 + 64 + 68 + ... + 120 = \frac{16}{2}(60 + 120) = 1\,440$

Application activity 5.2.2.

1. Insert 4 arithmetic means between -3 and 7
2. Insert 9 arithmetic means between 2 and 32
3. Between 3 and 54, n terms have been inserted in such a way that the ratio of 8th mean and $(n-2)$ th mean is $\frac{3}{5}$. Find the value of n.
4. Form an arithmetic progression of three numbers such that the sum of its terms is 30 and the sum of the squares of its terms is 332.
5. Consider the arithmetic sequence 8, 12, 16, 20, … Find the expression for S_n
6. Sum the first twenty terms of the sequence 5, 9, 13, …
7. The sum of the terms in the sequence 1, 8, 15, … is 396. How many terms does the sequence contain?
8. A ceramic tile floor is designed in the shape of a trapezoid 10m wide at the base and 5m wide at the top as illustrated on the figure bellow.



The tiles, 10cm by 10cm, are to be placed so that each successive row contains one less tile than the preceding row. How many tiles will be required?

5.2.3. Geometric sequences

Activity 5.2.3.

During a competition of students at the district level, 5 first winners were paid an amount of money in the way that the first got 100 000Frw , the second earned the half of this money, the third got the half of the second's money, and so on until the fifth who got the half of the fourth's money.

a) Discuss and calculate the money earned by each student from the second to the fifth.;
b) Determine the total amount of money for all the 5 student teachers;
c) Compare the money for the first and the fifth student and discuss the importance of winning at the best place;
d) Try to generalize the situation and guess the money for the student who passed at the n^{th} place if more students were paid. In this case, evaluate the total amount of money for n students.

Sequences of numbers that follow a pattern of multiplying a term by a fixed number ***r***, from one term u_1 to the next, are called **geometric sequences**.

The following are examples of geometric sequences: Sequence $\{u_n\}$: $5,10,20,40,80,...$

Sequence $\{v_n\}$: $2,1,\frac{1}{2},\frac{1}{4},\frac{1}{8},...$; Sequence $\{w_n\}$: $1,-2,4,-8,16,...$

Common ratio

The fixed number that binds all the terms of a geometric sequence is called the **common ratio** of the geometric sequence; it can be denoted by the letter ***r***. This means if u_1 the first term is, $u_2 = ru_1; u_3 = r^2u_1; u_4 = r^3u_1;...u_n = r^{n-1}u_1;...$

The n^{th} term or the general term of a geometric sequence becomes $u_n = r^{n-1}u_1$.

If the first term is u_0, then the general term of an geometric sequence becomes $u_n = r^n u_0$.

The common ration *r* can be calculated by dividing any two consecutive terms in a geometric sequence.

That is $r = \frac{u_{n+1}}{u_n}$ or $r = \frac{u_n}{u_{n-1}}$ or $u_n = ru_{n-1}$.

Generally, If u_p is the p^{th} term of the sequence, then the n^{th} term is given by $u_n = u_p r^{n-p}$.

Note that if three terms are consecutive terms of a geometric sequence, the square of the middle term is equal to the product of extreme terms. That is for a geometric sequence u_{n-1}, u_n, u_{n+1}, we have $u_n^2 = u_{n-1} \cdot u_{n+1}$.

Examples

1) $6, 12, 24$ are consecutive terms of a geometric sequence because

$$(12)^2 = 6\times 24 \Leftrightarrow 144 = 144$$

Find b such that $8, b, 18$ will be in geometric sequence.

Solution: $b^2 = 8\times 18 = 144$; $b = \pm\sqrt{144} = \pm 12$; and thus, $8, 12, 18$ or $8, -12, 18$ are in geometric sequence.

2) The product of three consecutive numbers in geometric progression is 27. The sum of the first two and nine times the third is -79. Find the numbers.

Solution: Let the three terms be $\frac{x}{a}, x, ax$. The product of the numbers is 27. So $\frac{x}{a}xax = 27 \Rightarrow x^3 = 27 \Rightarrow x = 3$; the sum of the first two and nine times the third is -79:

$$\frac{x}{a} + x + 9ax = -79 \Rightarrow \frac{3}{a} + 3 + 27a = -79$$

$$27a^2 + 82a + 3 = 0 \Rightarrow a = -3 \;\; or \;\; a = -\frac{1}{27}$$

The numbers are: -1, 3, -9 or -81, 3, $-\frac{1}{9}$.

3) If the first and tenth terms of a geometric sequence are 1 and 4, respectively, find the nineteenth term.

Solution: $u_1 = 1$ and $u_{10} = 4$; but $u_n = u_1 r^{n-1}$,

then $4 = 1r^9 \Leftrightarrow r = \sqrt[9]{4} \;\; or \;\; r = 4^{\frac{1}{9}}$

Now,

$$\begin{aligned} u_{19} &= u_1 r^{19-1} \\ &= 1\left(4^{\frac{1}{9}}\right)^{18} \\ &= 16 \end{aligned}$$

Thus, the nineteenth term of the sequence is 16.

4) If the 2nd term and the 9th term of a geometric sequence are 2 and $-\frac{1}{64}$ respectively, find the common ratio.

Solution: $u_2 = 2,\ u_9 = -\frac{1}{64}$; using the general formula: $u_n = u_p\, r^{n-p}$

$u_2 = u_9\, r^{2-9}$

$2 = -\frac{1}{64} r^{-7}$

$\Leftrightarrow 128 = -\frac{1}{r^7}$

$\Leftrightarrow r^7 = -\frac{1}{128}$

$\Leftrightarrow r = \sqrt[7]{-\frac{1}{128}} \Rightarrow r = -\frac{1}{2}$

The common ratio is $r = -\frac{1}{2}$.

5) Find the number of terms in sequence $2, 4, 8, 16, ..., 256$.

Solution:

This sequence is geometric with common ratio 2, $u_1 = 2$,

and $u_n = 256$; but $u_n = u_1 r^{n-1}$,

then $256 = 2 \times 2^{n-1} \Leftrightarrow 256 = 2^n$ or $2^8 = 2^n \Rightarrow n = 8$.

Thus, the number of terms in the given sequence is 8.

Some problems about geometric sequences

A. Insert *k* terms, called geometric means, between two terms u_1 and u_n.

Solution: We have to form a geometric sequence of $n = k + 2$ terms whose the first term is u_1 and the last term is u_n.

While there are several methods, we will use the n[th] term formula $u_n = u_1 r^{n-1}$.

As u_1 and u_n are known, we need to find the common ratio *r* taking $n = k + 2$ where *k* is the number of terms to be inserted.

Example:

1) Insert three geometric means between 3 and 48.

Solution:

Here $k=3$, then $n=5, u_1=3$ and $u_n=u_5=48$; $u_5=u_1r^{n-1} \Leftrightarrow 48=3r^4 \Rightarrow r=2$

Inserting three terms using common ratio $r=2$ gives $3,6,12,24,48$

2) Insert 6 geometric means between 1 and $-\frac{1}{128}$.

Solution:

Here $k=6$, then $n=8, u_1=1$ and $u_n=u_8=-\frac{1}{128}$

$u_8=u_1r^{n-1}$

$\Leftrightarrow -\frac{1}{128}=1r^7$

$\Leftrightarrow r^7=-\frac{1}{128}$

$\Leftrightarrow r^7=-\frac{1}{(2)^7}$

$\Leftrightarrow r=\left[-\frac{1}{(2)^7}\right]^{\frac{1}{7}}=-\frac{1}{2}$

Inserting 6 terms using common ratio $r=-\frac{1}{2}$ gives

$1,-\frac{1}{2},\frac{1}{4},-\frac{1}{8},\frac{1}{16},-\frac{1}{32},\frac{1}{64},-\frac{1}{128}$.

B. Find the sum of the first n terms of a geometric sequence

Solution:

A is an infinite sum $\sum_{n=1}^{\infty}u_n=u_1+u_1r+u_1r^2+....+u_1r^{n-1}+...$ of the terms a geometric sequence. If we have **a finite geometric sequence** $\{u_n\}=u_1,u_2,u_3,...u_n$, the sum is $S_n=\sum_{r=1}^{n}u_r=u_1+u_1r+u_1r^2+....+u_1r^{n-1}$.

The sum S_n of the first *n* terms of a geometric sequence $\{u_n\}=\{u_1r^{n-1}\}$ is

$S_n=u_1+ru_1+....+r^{n-1}u_1$ (1)

Multiply each side by *r* to obtain $rS_n=ru_1+r^2u_1+....+r^nu_1$ (2)

Subtracting equation (2) from equation (1) we obtain

$$S_n - rS_n = u_1 - u_1 r^n$$

$$(1-r)S_n = u_1(1-r^n)$$

If $r \neq 1$, then we can solve for S_n and find $S_n = u_1 \frac{(1-r^n)}{(1-r)}$

If the initial term u_1 and common ratio *r* are given, then the sum is $s_n = \frac{u_1(1-r^n)}{1-r}$, with $r \neq 1$.

If the initial term is u_0, then the formula is $s_n = \frac{u_0(1-r^{n+1})}{1-r}$ with $r \neq 1$;in this case, there are $n+1$ terms in the geometric sequence.

If $r=1$, $s_n = nu_1$; Also, the product of first *n* terms of a geometric sequence with initial term u_1 and common ratio *r* is given by $P_n = (u_1)^n r^{\frac{n(n-1)}{2}}$

If the initial term is u_0 then $P_n = (u_0)^{n+1} r^{\frac{n}{2}(n+1)}$

Examples

1) Find the sum of the first 20 terms of a geometric sequence if the first term is 1 and common ratio is 2.

Solution:

Here $u_1 = 1, r = 2, n = 20$; then, $s_{20} = \frac{1(1-2^{20})}{1-2} = \frac{1-2^{20}}{-1} = 1048575$

2) Consider the sequence $\{u_n\}$ defined by $u_0 = 0$ and $u_{n+1} = u_n + \frac{1}{2^n}$.
Consider another sequence $\{v_n\}$ defined by $v_n = u_{n+1} - u_n$.

a) Show that $\{v_n\}$ is a geometric sequence and find its first term and common ratio.

b) Express $\{v_n\}$ in term of n.

Solution:

a) $u_0 = 0,\ v_0 = u_1 - u_0 = 1$

$$u_1 = u_0 + \frac{1}{2^0} = 1,\ u_2 = u_1 + \frac{1}{2^1} = \frac{3}{2};$$

$$v_1 = u_2 - u_1 = \frac{1}{2},\ v_2 = u_3 - u_2 = \frac{1}{4}$$

$\{v_n\}$ is a geometric sequence if $v_1^2 = v_0 \cdot v_2$.

$v_1^2 = \frac{1}{4}$ and $v_0 \cdot v_2 = \frac{1}{4}$.

Thus, $\{v_n\}$ is a geometric sequence.

First term is $v_0 = 1$

Common ratio is $r = \frac{v_1}{v_0} = \frac{1}{2}$

b) General term

$v_n = v_0 r^n$

$= \frac{1}{2^n}$ Thus, $\{v_n\}$ is defined by $v_n = \frac{1}{2^n}$

C. Find the product of the first n terms of a geometric sequence:

Solution:

It can be shown that the product is $P_n = (u_1)^n r^{\frac{n(n-1)}{2}}$,

Example:

Find the product of the first 20 terms of a geometric sequence if the first term is 1 and common ratio is 2.

Solution:

Here $u_1 = 1, r = 2, n = 20$, then, $P_{20} = (1)^{20} 2^{\frac{20(19)}{2}} = 2^{190}$

Application activity 5.2.3.

1. Insert 5 geometric means between $\frac{1}{4}$ and $\frac{1}{256}$.
2. Insert 5 geometric means between 2 and $\frac{2}{729}$
3. Find the geometric mean of:

   a) 2 and 98;

   b) $\frac{3}{2}$ and $\frac{27}{2}$
4. Suppose that $4; 36; 324; ...$ are in geometric progression. Write down the next two terms of the geometric sequence.

5. The arithmetic mean of two numbers is 34 and their geometric mean is 16. Find the numbers.

6. Find the sum of the first 8 terms of the geometric sequence 32, -16, 8, …

7. Find the sum of the terms of the geometric sequence with the first term 0.99 and the common ratio is equal to the first term.

8. Find the first term and the common ratio of the geometric sequence for which $S_n = \frac{5^n - 4^n}{4^{n-1}}$

5.3. Applications of Financial Mathematics

5.3.1. Annuities

Activity 5.3.1.

Consider the following problems to fill the table that follows:

a) In 18 years your uncle would like to have $50\,000$ US Dollars saved for the college education of his new born- child. The bank offers 6% annual interest rate compounded monthly. What monthly deposit must be made to accomplish this goal?

b) A man borrows a loan of $20\,000$ US Dollars to purchase a car at annual rate interest of 6%.He will pay back the loan through monthly installments over 5 years, with the first installment to be made one month after the release of the loan. What is the monthly installment he needs to pay?

c) A four- year lease agreement requires anticipatory payments of 12 000 US Dollars every year. The interest rate is 6% compounded monthly. What is the cash Value of the lease?

	The payment is made at the		Payment frequency and Compounding frequency are	
	End of the period	Beginning of the period	equal	Different
a)				
b)				
c)				

Definition and characteristics

An **annuity** is a sequence of continuous payments of equal amounts, at fixed intervals, for a fixed period. Fundamentally an annuity is a sequence of equal payments made over equally spaced time intervals. The four characteristics of an annuity are the following:

- **Continuous**: payments with no interruption from the beginning through the end of the annuity.
- **Equal** amounts: The same amount is paid each period;
- **Periodic**: the payments are done at fixed intervals, that is, the amount of time between two consecutive payments is constant
- **Specific** period: the payments are not endless.

An annuity for which the payments are not endless is said to be a **certain annuity** (this is the object of our concern). Otherwise, the annuity is said to be contingent (we are not concerned with this).

Types of annuity

Depending on whether the annuity payment is made at the beginning of a period or at the end of a period, we have:

- **Ordinary annuity:** the payment is made at the end of a period;
- **Annuity Due:** the payment is made at the beginning of a period.
- Depending on whether the payment frequency is the same, or not, as the compounding frequency, we have:
- **Simple annuity:** the payment frequency is the same as the compounding frequency
- **General annuity:** the payment frequency is not the same as the compounding frequency

Taking into account these two factors simultaneously, there are four types of annuities:

- **Simple annuity Due;**
- **Simple ordinary annuity;**
- **General annuity Due;**
- **General ordinary annuity**

Note:

Calculation of annuities

Since annuities involve payments over a period, we are interested in the future value of annuities.

From the concepts of present value (PV) and future value (FV), we have:

Type of annuity	Future value(FV)	Present value(PV)
Simple ordinary General ordinary	$FV = C\left[\frac{(1+i)^n - 1}{i}\right]$	$PV = C\left[\frac{1-(1+i)^{-n}}{i}\right]$
Simple due General due	$FV = C\left[\frac{(1+i)^n - 1}{i}\right](1+i)$	$PV = C\left[\frac{1-(1+i)^{-n}}{i}\right](1+i)$

Where:

C: cash flow: the constant amount deposited once at an interval time;

i: interest rate;

n: number of payments;

FV: future value;

PV: present value

Deferred annuities

A deferred annuity is an Annuity in which payback does not start until a specified time in future, such as after a certain number of years from the annuity contract date or at a certain age of a beneficiary. For example, a student loan is paid in deferred annuities

Example1:

Machine A costs \$8 000 and can work for 7 years. Machine B costs \$6 000 and can work for 5 years. A Company can invest money in either machine and receive 10% per annum. Which of the two machines will generate more annual savings to the company?

Solution:

For machine A:

From the formula $PV = C\left[\frac{1-(1+i)^{-n}}{i}\right]$,making C the subject of the formula, we have $C = \frac{iPV}{1-(1+i)^{-n}}$,where $PV = 8000; i = 0.10; n = 7$

Thus, the annual payment required is $C = \frac{iPV}{1-(1+i)^{-n}} = \frac{0.10 \times 8\,000}{1-(1+0.10)^{-7}} = 1\,643.24$

The annual cost of Machine A to the company is \$1 643.24

For machine B:

From the formula $PV = C\left[\frac{1-(1+i)^{-n}}{i}\right]$,making C the subject of the formula, we have $C = \frac{iPV}{1-(1+i)^{-n}}$,where $PV = 6\,000; i = 0.10; n = 5$

Thus, the annual payment required is $C = \frac{iPV}{1-(1+i)^{-n}} = \frac{0.10 \times 6\,000}{1-(1+0.10)^{-5}} = 1\,582.78$

The annual cost of Machine B is \$1 582.78.

Machine A will generate more annual savings to the company than Machine B. Machine A is preferable to Machine B.

Note: If we take into account the labour savings and the cost for each machine:

	A	B
Annual labour savings	\$2 000.00	\$1 800.00
Annual cost	\$1 643.24	\$1 582.78
Net Savings	\$356.76	\$217.12

Example2

If you deposit \$5 000 into an account paying 6% annual interest compounded monthly, how long will it take until there is \$8000 in the account?

Solution:

Plug in the formula the given $FV = 8\,000, P = 5\,000, r = 0.06\, and\; n = 12$

$8\,000 = 5\,000\left(1+\frac{0.06}{12}\right)^{12t}$, simplify by using operations to get $1.6 = (1.005)^{12t}$,

Take the logarithm of each side. Then use Property to rewrite the problem as multiplication,

$\log(1.6) = \log\left(1.005^{(12t)}\right) \Rightarrow \log 1.6 = 12t \log 1.005$

$\frac{\log 1.6}{\log 1.005} = 12t \;\Rightarrow 94.23553232 \approx 12t$. Therefore, $t = 7.9$

It will take approximately 7.9 years for the account to go from $5 000 to $8 000.

Note: Working with annuities is just like working with future and present values. The only difference is that we make a succession of deposits or a succession of payments.

Corresponding to future value, we consider how much money will be accrued, including interest, if we make regular deposits into a bank.

Corresponding to present value, we ask how much money we need to have in the bank now, taking into account interest, in order to make a sequence of regular payments in the future.

Example3:

Suppose I invest $100 at the end of every year for 5 years. If the interest rate is 8%, how much will my investment be worth after 5 years?

Solution:

In order to see what is happening, let us draw a simple diagram.

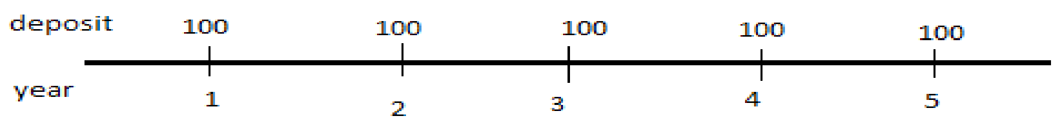


Figure above shows the payments and when they are deposited. Now consider the first payment. It remains in the bank for 4 years. The accumulated amount (i.e. its future value) is therefore $100\times(1+0.08)^4$

The next deposit is in the bank for 3 years and is worth $100\times(1+0.08)^3$

Continuing in this way, the other 3 deposits are worth $100\times(1+0.08)^2$, $100\times(1+0.08)^1$ and. 100

Notice that the last payment accrues no interest, as it is made at the end of the fifth year. The total is therefore $100(1.08)^4+100(1.08)^3+100(1.08)^2+100(1.08)+100$

$$=100(1+1.08+1.08^2+1.08^3+1.08^4)$$

This is a geometric series of 5 terms, first term 100 and common ratio 1.08,

The sum is $S_5=100\left(\frac{1-1.08^5}{1-1.08}\right)=586.66$ $.

Remember that:

- The **payment interval** is the time between two consecutive payments (in the case above, 1 year),

- The **term** is the length of time from the beginning to the end (in the case above, 5 years), the **annual rate** is the amount paid per year (in the case above \$ 100),
- An annuity is an **ordinary annuity** if payments are made at the end of each time period, as is the case above.
- An annuity is a **due annuity** if payments are made at the beginning of each time period.
- The amount is the total of all the payments made plus their interest, that is, the future value of all deposits.

Alternative way of calculating the value of an Annuity Due:

An annuity due is calculated in reference to an ordinary annuity. In other words, to calculate either the present value (**PV**) or future value (**FV**) of an annuity-due, we simply calculate the value of the comparable ordinary annuity and multiply the result by a factor of (1 + i) as shown below...

$$\text{Annuity}_{\text{Due}} = \text{Annuity}_{\text{Ordinary}} \times (1 + i)$$

This makes sense because if we go back to our earlier definitions, we see that the difference between the ordinary annuity and the annuity due is one compounding period.

Present Value of an Annuity

The present value P of an annuity consisting of n payments of C dollars each, paid (cash) at the end of each investment period into an account that earns interest at the rate of i per period, is

$$PV = C\left(\frac{1-(1+i)^{-n}}{i}\right);$$

Example4

a) Calculate the present value of an ordinary annuity of \$50 per year over 3 years at 7%

b) Calculate the present value of an **annuity due** under the same terms

Solution:

a) $PV_{ord} = C\frac{1-(1+i)^{-n}}{i} = 50\frac{1-(1+0.07)^{-3}}{0.07} = 131.22;$

b) $PV_{due} = PV_{ord}(1+i) = 131.22(1+0.07) = 140.40$

Future Value of an Annuity

The future value of an annuity of n payments of C dollars each, paid at the end of each investment period into an account that earns interest at the rate of i per period, is

$$FV = C[\frac{(1+i)^n - 1}{i}]$$

Example5

a) Calculate the future value of an ordinary annuity of $25 per year over 3 years at 9%

b) Calculate the future value annuity due under the same terms.

Solution:

a) $FV_{ord} = C\left[\frac{(1+i)^n - 1}{i}\right] = 25\left[\frac{(1+0.09)^3 - 1}{0.09}\right] = 81.95$

b) The future value of annuity due under the same terms is

$$FV_{due} = FV_{ord}(1+i) = 81.95(1.09) = 89.33$$

- The periodic payment C on a loan of P dollars to be paid over n periods with interest charged at the rate of i per period is $C = \frac{Pi}{1-(1+i)^{-n}}$.

Example6

The Baker borrowed 12 000 000Frw from a bank to help finance the purchase of a house for that business. The bank charges interest at a rate of 9% per year on the unpaid balance, with interest computations made at the end of each month. The Baker has agreed to repay the loan in equal monthly installments over 30 years. How much should each payment be if the loan is to be paid at the end of term?

Solution:

Here, P = 12,000,000, $i = \frac{0.09}{12} = 0.0075$, and n = (30) (12) = 360. Using the payment formula we find that the size of each monthly installment required is given by

$$C = \frac{Pi}{1-(1+i)^{-n}} = \frac{12\,000\,000 \times 0.075}{1-(1+0.0075)^{-360}} = 96\,555$$

Application activity 5.3.1.

1. At 3% annual interest compounded monthly, how long will it take to double your money?
2. How much money would you need to deposit today at 9% annual interest compounded monthly to have $12 000 in the account after 6 years?
3. How long will it take our money to triple in a bank account with an annual interest rate of 8.45% compounded annually?

5.3.2. Mortgage

Activity 5.3.2.

Read the following problem, and answer the questions that follow:

Your elder brother is newly employed at a company and earns 500 000 Frw per month. He would like to know if he can afford monthly payments on a mortgage of 20 000 000 Frw with an interest rate of 6% that runs for 20 years.

a) What can a mortgage mean?

b) Discuss about the amount of the loan, the time of repayments and how the repayments are made

A **mortgage** is a loan, generally of huge amount of money, to be repaid in regular periodic installments, generally for a long period of time, with a compounded interest on the amount not yet paid.

The process of paying off the debt over time in equal installments is called **amortization**

Formula mortgage payments

Geometric series are used to figure out the mortgage payments

We obtain the formula $M=\frac{P\times i\times(1+i)^n}{(1+i)^n-1}$: the monthly amount to pay;

where P: principal amount of the loan,

i: monthly interest rate and

n: number of months required to repay the total amount of the loan.

Alternatively, $M=\frac{P\times i\times(1+i)^n}{(1+i)^n-1}=\frac{P\times(\frac{r}{n})}{(1+i)^{-n}[(1+i)^n-1]}=\frac{P\times(\frac{r}{n})}{1-(1+i)^{-n}}=\frac{P\times(\frac{r}{n})}{1-(1+\frac{r}{n})^{-nt}}$;

where $i=\frac{r}{n}$

Example1

Your elder brother is newly employed at a company and earns 500 000 Frw per month. He would like to know if he can afford monthly payments on a mortgage of 20 000 000 Frw with an interest rate of 6% that runs for 20 years.

Show to your brother that he can afford the monthly payments by determining the following:

a) The monthly payment that will be retained at the bank;
b) The balance that your brother can withdraw each month from the bank;
c) How much interest your brother will pay to the bank by the end of 20 years.
d) The interest the bank will realize at the end of all payments

Solution:

$$M=\frac{\frac{rP}{n}}{1-\left(1+\frac{r}{n}\right)^{-nt}}=\frac{\frac{(0,06)(20\,000\,000)}{12}}{1-\left(1+\frac{0.06}{12}\right)^{(-12)(20)}}=143\,286.2$$

a) The amount to pay per month is 143 287 Frw,
b) the balance the brother will withdraw each month is:

500 000Frw-143 287 Frw =356 713 Frw

c) At the end of 20 years, your brother would have paid $143\,287\times12\times20=34\,388\,880$ Frw,
d) The interest the bank will realize is :

34 388 880 Frw -20 000 000 Frw=14 338 880Frw

Note:

When a person gets a loan (mortgage) from the bank, the mortgage amount ***P***, the number of payments or the number ***t*** of years to cover the mortgage, the amount of the payment ***M***, how often the payment is made or the number ***n*** of payments per year, and the interest rate ***r***, it is proved that all the 5 components are related by the following formula:

$$M = \frac{\frac{rP}{n}}{1-\left(1+\frac{r}{n}\right)^{-nt}}$$

The payment M required to pay off a loan of P Francs borrowed for n payment periods at a rate of interest i per payment period is $M = P\left[\frac{i}{1-(1+i)^{-n}}\right]$; where $i = \frac{r}{n}$.

Example2

A businesswoman wants to apply for a mortgage of 7 500 000 Frw with an interest of 8% that runs for 20 years. How much interest she will pay over the 20 years?

Solution:

Substituting for *M=7 500 000, r =0.08, t=20, n=12* in the equation

$$P = \frac{\frac{rM}{n}}{1-\left(1+\frac{r}{n}\right)^{-nt}}, \text{ we have } P = \frac{\frac{(0.08)(7\,500\,000)}{12}}{1-\left(1+\frac{0.08}{12}\right)^{-(12)(20)}} = 62\,733.$$

Each month she will be paying FRW 62,733. The total amount she will pay is $62\,733 Frw \times 12 \times 20 = 15\,055\,920 Frw$;

The interest will be $(15\,055\,920 - 7\,500\,000)Frw = 7\,555\,920 Frw$.

Example3

What monthly payment is necessary to pay off a loan of \$800 at 10% per annum?

a) In 2 years? b) In 3 years? c) What total amount is paid out for each loan?

Solution

a) For the 2-year loan, $P = \$800, \quad n = 24, \quad and\ i = \dfrac{0.10}{12}$. The monthly payment *P* is

$$M = P\left[\frac{i}{1-(1+i)^{-n}}\right] = 800\left[\frac{\frac{0.10}{12}}{1-\left(1+\frac{0.10}{12}\right)^{-24}}\right] = 36.92.$$

b) For the 3-year loan, $M = \$800, \quad n = 36, \quad and\ i = \dfrac{0.10}{12}$.

The monthly payment *P* is $P = M\left[\dfrac{i}{1-(1+i)^{-n}}\right] = 800\left[\dfrac{\frac{0.10}{12}}{1-\left(1+\frac{0.10}{12}\right)^{-36}}\right] = 25.81$

c) For the 2-year loan, the total amount paid out is $(36.92)(24) = \$886.08$;

For the3-year loan, the total amount paid out is $(25.81)(36) = \$929.16$.

Example4

A certain family has just purchased a $300 000 house and has made a down payment of $60 000. It can amortize the balance $(\$3000\,000 - \$60\,000)$ at 6% for 30 years.

a) What are the monthly payments?

b) What is their total interest payment?

c) After 20 years, what equity does it have in the house (that is, what is the sum of the down payment and the amount paid on the loan)?

Solution

a) The monthly payment *P* needed to pay off the loan of $240 000 at 6% for 30 years (360 months) is

$$M = P\left[\frac{i}{1-(1+i)^{-n}}\right] = \$240,000\left[\frac{\frac{0.06}{12}}{1-\left(1+\frac{0.06}{12}\right)^{-360}}\right] = \$1438.92$$

b) The total paid out for the loan is $(\$1\,438.92)(360) = \$518\,011.20$. The interest on this loan amounts to $\$518\,011.20 - \$240\,000 = \$278\,011.20$.

After 20 years (240 months) there remains 10 years (or 120 months) of payments. The present value of the loan is the present value of a monthly payment of $1 438.92 for 120 months at 6%, namely,

$$P = M\left[\frac{1-(1+i)^{-n}}{i}\right] = \$1\,438.92\left[\frac{1-\left(1+\frac{0.06}{12}\right)^{-120}}{1-\frac{0.06}{12}}\right] = \$129\,608.49 .$$

The amount paid on the loan is $(Original\ loan\ amount)-(Present\ value)$

$= \$240\,000 - \$129\,608.49 = \$110\,391.51$

The equity after 20 years is $(Down\ payment)+(Amount\ paid\ on\ loan)$

$= \$60\,000 + \$110\,391.51 = \$170\,391.51$

Note:

This equity does not include any appreciation in the value of the house over the 20 year period. The table below gives a partial schedule of payments for the loan. It is interesting to observe how slowly the amount paid on the loan increases early in the payment schedule, with very little of the payment used to reduce principal, and how quickly the amount paid on the loan increases during the last 5 years.

Payment number	Monthly payment	Principal	Interest	Amount paid on loan
1	$ 1 438.92	$ 238.92	$ 1 200.00	$238.92
60	$ 1 438.93	$ 320.67	$ 1 118.26	$ 1 6 669.54
120	$ 1 438.94	$ 432.53	$ 1 006.39	$ 39 154.26
180	$ 1 438.95	$ 583.42	$ 855.50	$ 69 482.77
240	$ 1 438.96	$ 786.94	$ 651.98	$ 110 391.39
300	$ 1 438.97	$ 1 061.47	$ 377.45	$ 165 570.99
360	$ 1 438.98	$ 1 431.76	$ 7.16	$ 240 000.00

Application activity 5.3.2.

A bank can offer a mortgage at 10 % interest rate to be paid back with monthly payments for 20 years. After analysis, a potential borrower finds that she can afford monthly payment of 200 000Frw. How much of mortgage can she ask for?

5.3.3. Sinking funds

Activity 5.3.3.

Consider the following problems:

a) Suppose I invest $10\,000$ Frw, at the end of every year, at a bank that pays an annual interest rate of 8% compounded annually for 3 years. By the end of three years, will I be able to buy, using the total money I have at the bank, an item that costs $32\,400 Frw$?

b) You get a loan of $324\,000 Frw$, from a bank that charges you an annual interest rate of 8% compounded annually for 3 years. How much will you owe the bank at the end of the three years.

Explain the difference between the two problems in terms of saving or paying a debt

Sinking funds is a special type of annuity; it consists in saving periodically a constant amount with a compound interest, to meet a future financial need. The formula for future value of an ordinary annuity applies for sinking funds

$$FV = C\left[\frac{(1+i)^n - 1}{i}\right]$$

Where:

C: cash flow: the constant amount deposited once at an interval time;

i: interest rate;

n: number of payments

Example

I invest $100\,000$ Frw, at the end of every year, at a bank that pays an annual interest rate of 8% compounded annually for 3 years. By the end of three years, will I be able to buy, using the total money I have at the bank, an item that costs $324\,000 Frw$?

Solution:

The future value of $100\,000 Frw$ is

$$FV = C\left[\frac{(1+i)^n - 1}{i}\right] = 100\,000\left[\frac{(1+0.08)^3 - 1}{0.08}\right] \approx 324\,640$$

Since $324\,640 > 324\,000$, I will be able to buy the item. Yet the money invested was $10\,000 \times 3 = 300\,000 Frw$

Note that for sinking funds, the money invested is always less than the amount we aim to obtain, but for a loan, the money owed is always greater than the money received.

Application activity 5.3.3.

Assume that you save $1\,800$ US Dollars in a sinking fund for two years. The account pays 6% compounded quarterly and you will also make payments quarterly. What should be your quarterly payment?

5.3.4. Financial Risk management

Activity 5.3.4.

Consider the following scenarios:

1. A bank offers loans at various interest rates
2. The currency of a given country fluctuates against the currency of another country
3. Customers are not able to pay their loans
4. A bank's cash for offering loans has decreased

Explain the problem faced in each case and suggest how the problem can be solved

- **Financial risks** are risks faced by a business in terms of handling its finances, such as defaulting on loans, debt load, and so on.

  A risk related to variable interest rates is called **interest rate risk**
- **Financial risk management**

Different ways of managing financial risks include:

- Requesting cash payments;
- Well defining the terms of a credit
- Rewarding best performers in terms of paying their loans, etc.

Example

At a particular period, it is observed that money has lost its value due to inflation. A bond on sale was costing 100 000 000 Frw before inflation. It costs 100 000 000 Frw during inflation period, state whether the value of the bond is standby, the bond lost value, or it gains value.

Solution

The bond lost value since the future value of the bond is greater than 100 000 000 Frw

Application activity 5.3.4.

Your uncle, who is a Primary teacher, fifty years old, would like to get a 20-year mortgage of 20 000 000 Frw at an annual interest rate of 4% compounded monthly.

As a future Accountant, which advice can you give to your uncle?

5.4. End unit assessment

End unit assessment 5

1. Calculate the effective annual interest rate on 100 000Frw at 6% compounded:

   a) Semiannually;

   b) Continuously

2. An investor can put money into one of the following two projects:

   Project A costs 200 000Frw and pays back 400 000 Frw in 6 years

   Project B costs 300 000Frw now and pays back 480 000 Frw in 5 years.

   The current interest rate is 10%. Which project should be chosen?

3. A firm expects its sales to grow by 12% per month. If its January sales are 100 000 Frw , what will its expected total annual sales be?

4. What are the monthly payments of a mortgage of 6 000 000 Frw taken out for 25 years if the monthly rate is 0.75%?

5. A father promised to reward his daughter whatever she wishes. The girl, with a mathematical mind, requested her father to reward her in the following way: day one she will receive 100 FRW, day two, she will receive 200FRW, the third day she will receive 400 FRW, the fourth day she will receive 800FRW, the fifth day she will receive 1600 FRW, and so on (the amount keeps doubling from one day to the next day), until the end of the month. What is the total amount the girl requested for one month of 30 days? Can the father honor his promise?

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