MATHEMATICS FOR TTC

Tutor's Guide

YEAR

1

OPTION:

SOCIAL STUDIES EDUCATION (SSE)

FOREWORD

Dear Tutor,

Rwanda Basic Education Board is honoured to present the tutor's guide for Mathematics in the option of Social Studies Education (SSE). This book serves as a guide to competence-based teaching and learning to ensure consistency and coherence in the learning of the mathematics content. The Rwandan educational philosophy is to ensure that learners achieve full potential at every level of education which will prepare them to be well integrated in society and exploit employment opportunities.

Specifically, TTC curriculum was reviewed to train quality teachers who will confidently and efficiently implement the Competence Based Curriculum in pre-primary and primary education. The rationale of the changes is to ensure that TTC leavers are qualified for job opportunities and further studies in Higher Education in different programs under education career advancement.

In line with efforts to improve the quality of education, the government of Rwanda emphasizes the importance of aligning teaching and learning materials with the syllabus to facilitate their learning process. Many factors influence what they learn, how well they learn and the competences they acquire. Those factors include the relevance of the specific content, the quality of teachers' pedagogical approaches, the assessment strategies and the instructional materials.

The ambition to develop a knowledge-based society and the growth of regional and global competition in the jobs market has necessitated the shift to a competence-based curriculum. After a successful shift from knowledge to a competence based curriculum in general education, TTC textbooks also were elaborated to align them to the new curriculum.

The book provides active teaching and learning techniques that engage student teachers to develop competences. In view of this, your role is to:

- Plan your lessons and prepare appropriate teaching materials.
- Organize group discussions for students considering the importance of social constructivism suggesting that learning occurs more effectively when the students works collaboratively with more knowledgeable and experienced people.
- Engage students through active learning methods such as inquiry methods, group discussions, research, investigative activities and group and individual work activities.

- Provide supervised opportunities for students to develop different competences by giving tasks which enhance critical thinking, problem solving, research, creativity and innovation, communication and cooperation.
- Support and facilitate the learning process by valuing students' contributions in the class activities.
- Guide students towards the harmonization of their findings.
- Encourage individual, peer and group evaluation of the work done in the classroom and use appropriate competence-based assessment approaches and methods.

To facilitate you in your teaching activities, the content of this book is self explanatory so that you can easily use it. It is divided in 3 parts:

The part I explains the structure of this book and gives you the methodological guidance;

The part II gives a sample lesson plan;

The part III details the teaching guidance for each concept given in the student book.

Even though this Teacher's guide contains the guidance on solutions for all activities given in the learner's book, you are requested to work through each question before judging student's findings.

I wish to sincerely express my appreciation to the people who contributed towards the development of this book, particularly, REB staff, UR-CE Lecturers, TTC Tutors, Teachers from general education and experts from Local and international Organizations for their technical support. A word of gratitude goes also to the Head Teachers and TTCs principals who availed their staff for various activities.

Dr. MBARUSHIMANA Nelson
Director General, REB

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Joan MURUNGI

Head of CTLR Department

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PART I. GENERAL INTRODUCTION

1.1 The structure of the guide

The tutor's guide of Mathematics is composed of three parts:

The Part I concerns general introduction that discusses methodological guidance on how best to teach and learn Mathematics, developing competences in teaching and learning, addressing cross-cutting issues in teaching and learning and Guidance on assessment.

Part II presents a sample lesson plan. This lesson plan serves to guide the teacher on how to prepare a lesson in Mathematics.

The Part III is about the structure of a unit and the structure of a lesson. This includes information related to the different components of the unit and these components are the same for all units. This part provides information and guidelines on how to facilitate student teachers while working on learning activities. More other, all application activities from the textbook have answers in this part.

1.2 Methodological guidance

1.2.1 Developing competences

Since 2015 Rwanda shifted from a knowledge based to a competency-based curriculum for pre-primary, primary, secondary education and recently the TTC curriculum. This called for changing the way of learning by shifting from teacher centred to a learner centred approach. Teachers are not only responsible for knowledge transfer but also for fostering learners' learning achievement and creating safe and supportive learning environment. It implies also that learners have to demonstrate what they are able to transfer the acquired knowledge, skills, values and attitude to new situations.

The competence-based curriculum employs an approach of teaching and learning based on discrete skills rather than dwelling on only knowledge or the cognitive domain of learning. It focuses on what learner can do rather than what learner knows. Learners develop competences through subject unit with specific learning objectives broken down into knowledge, skills and attitudes through learning activities.

In addition to the competences related to Mathematics, student teachers also develop generic competences which should promote the development of the higher order thinking skills and professional skills in Mathematics teaching. Generic competences are developed throughout all units of Mathematics as follows:

Generic competences	Ways of developing generic competences
Critical thinking	All activities that require learners to calculate, convert, interpret, analyze, compare and contrast, etc have a common factor of developing critical thinking into learners
Creativity and innovation	All activities that require learners to plot a graph of a given algebraic data, to organize and interpret statistical data collected and to apply skills in solving problems of economics have a common character of developing creativity into learners
Research and problem solving	All activities that require learners to make a research and apply their knowledge to solve problems from the real-life situation have a character of developing research and problem solving into learners.
Communication	During Mathematics class, all activities that require learners to discuss either in groups or in the whole class, present findings, debate …have a common character of developing communication skills into learners.
Co-operation, interpersonal relations and life skills	All activities that require learners to work in pairs or in groups have character of developing cooperation and life skills among learners.
Lifelong learning	All activities that are connected with research have a common character of developing into learners a curiosity of applying the knowledge learnt in a range of situations. The purpose of such kind of activities is for enabling learners to become life-long learners who can adapt to the fast-changing world and the uncertain future by taking initiative to update knowledge and skills with minimum external support.
Professional skills	Specific instructional activities and procedures that a teacher may use in the class room to facilitate, directly or indirectly, students to be engaged in learning activities. These include a range of teaching skills: the skill of questioning, reinforcement, probing, explaining, stimulus variation, introducing a lesson; illustrating with examples, using blackboard, silence and non verbal cues, using audio – visual aids, recognizing attending behaviour and the skill of achieving closure.

The generic competences help learners deepen their understanding of Mathematics and apply their knowledge in a range of situations. As students develop generic competences they also acquire the set of skills that employers look for in their employees, and so the generic competences prepare students for the world of work.

1.2.2 Addressing cross cutting issues

Among the changes brought by the competence-based curriculum is the integration of cross cutting issues as an integral part of the teaching learning process-as they relate to and must be considered within all subjects to be appropriately addressed. The eight cross cutting issues identified in the national curriculum framework are: *Comprehensive Sexuality Education, Environment and Sustainability, Financial Education, Genocide studies, Gender, Inclusive Education, Peace and Values Education, and Standardization Culture.*

Some cross-cutting issues may seem specific to particular learning areas/subjects but the teacher need to address all of them whenever an opportunity arises. In addition, learners should always be given an opportunity during the learning process to address these cross-cutting issues both within and out of the classroom.

Below are examples of how crosscutting issues can be addressed:

Cross-Cutting Issue	**Ways of addressing cross-cutting issues**
Comprehensive Sexuality Education: The primary goal of introducing Comprehensive Sexuality Education program in schools is to equip children, adolescents, and young people with knowledge, skills and values in an age appropriate and culturally gender sensitive manner so as to enable them to make responsible choices about their sexual and social relationships, explain and clarify feelings, values and attitudes, and promote and sustain risk reducing behaviour.	Using different charts and their interpretation, Mathematics tutor should lead students to discuss the following situations: “Alcohol abuse and unwanted pregnancies” and advise student teachers on how they can instil learners to fight those abuses. Some examples can be given when learning statistics, powers, logarithms and their properties.

Environment and Sustainability: Integration of Environment, Climate Change and Sustainability in the curriculum focuses on and advocates for the need to balance economic growth, society well-being and ecological systems. Learners need basic knowledge from the natural sciences, social sciences, and humanities to understand to interpret principles of sustainability.	Using Real life models or students' experience, Mathematics Tutor should lead student teachers to illustrate the situation of "population growth" and discuss its effects on the environment and sustainability.
Financial Education: The integration of Financial Education into the curriculum is aimed at a comprehensive Financial Education program as a precondition for achieving financial inclusion targets and improving the financial capability of Rwandans so that they can make appropriate financial decisions that best fit the circumstances of one's life.	Through different examples and calculations on interest rate problems, total revenue and total cost, Mathematics Tutor can lead student teachers to discuss how to make appropriate financial decisions.
Gender: At school, gender will be understood as family complementarities, gender roles and responsibilities, the need for gender equality and equity, gender stereotypes, gender sensitivity, etc.	Mathematics Tutor should address gender as cross-cutting issue through assigning leading roles in the management of groups to both girls and boys and providing equal opportunity in the lesson participation and avoid any gender stereotype in the whole teaching and learning process.
Inclusive Education: Inclusion is based on the right of all learners to a quality and equitable education that meets their basic learning needs and understands the diversity of backgrounds and abilities as a learning opportunity.	Firstly, Mathematics Tutors need to identify/recognize students with special needs. Then by using adapted teaching and learning resources while conducting a lesson and setting appropriate tasks to the level of students, they can cater for students with special education needs. They must create opportunity where student teachers can discuss how to cater for learners with special educational needs.

Peace and Values Education: Peace and Values Education (PVE) is defined as education that promotes social cohesion, positive values, including pluralism and personal responsibility, empathy, critical thinking and action in order to build a more peaceful society.	Through a given lesson, a tutor should: • Set a learning objective which is addressing positive attitudes and values, • Encourage students to develop the culture of tolerance during discussion and to be able to instil it in colleagues and cohabitants; • Encourage students to respect ideas for others.
Standardization Culture: Standardization Culture in Rwanda will be promoted through formal education and plays a vital role in terms of health improvement, economic growth, industrialization, trade and general welfare of the people through the effective implementation of Standardization, Quality Assurance, Metrology and Testing.	With different word problems related to the effective implementation of Standardization, Quality Assurance, Metrology and Testing, students can be motivated to be aware of health improvement, economic growth, industrialization, trade and general welfare of the people.

1.2.3. Guidance on how to help students with special education needs in classroom

In the classroom, students learn in different way depending to their learning pace, needs or any other special problem they might have. However, the teacher has the responsibility to know how to adopt his/her methodologies and approaches in order to meet the learning need of each student in the classroom. Also teachers need to understand that student with special needs, need to be taught differently or need some accommodations to enhance the learning environment. This will be done depending to the subject and the nature of the lesson.

In order to create a well-rounded learning atmosphere, teachers need to:

- Remember that learners learn in different ways so they have to offer a variety of activities (e.g. role-play, music and singing, word games and quizzes, and outdoor activities);
- Maintain an organized classroom and limits distraction. This will help learners with special needs to stay on track during lesson and follow instruction easily;

- Vary the pace of teaching to meet the needs of each child. Some learners process information and learn more slowly than others;
- Break down instructions into smaller, manageable tasks. Learners with special needs often have difficulty understanding long-winded or several instructions at once. It is better to use simple, concrete sentences in order to facilitate them understand what you are asking.
- Use clear consistent language to explain the meaning (and demonstrate or show pictures) if you introduce new words or concepts;
- Make full use of facial expressions, gestures and body language;
- Pair a learner who has a disability with a friend. Let them do things together and learn from each other. Make sure the friend is not over protective and does not do everything for the one with disability. Both learners will benefit from this strategy;
- Use multi-sensory strategies. As all learners learn in different ways, it is important to make every lesson as multi-sensory as possible. Learners with learning disabilities might have difficulty in one area, while they might excel in another. For example, use both visual and auditory cues.

Below are general strategies related to each main category of disabilities and how to deal with every situation that may arise in the classroom. However, the list is not exhaustive because each child is unique with different needs and that should be handled differently.

Strategy to help learners with developmental impairment:

- Use simple words and sentences when giving instructions;
- Use real objects that learners can feel and handle. Rather than just working abstractly with pen and paper;
- Break a task down into small steps or learning objectives. The learner should start with an activity that she/he can do already before moving on to something that is more difficult;
- Gradually give the learner less help;
- Let the learner with disability work in the same group with those without disability.

Strategy to help learners with visual impairment:

- Help learners to use their other senses (hearing, touch, smell and taste) and carry out activities that will promote their learning and development;
- Use simple, clear and consistent language;
- Use tactile objects to help explain a concept;
- If the learner has some sight, ask him/her what he/she can see;
- Make sure the learner has a group of friends who are helpful and who allow him/her to be as independent as possible;
- Plan activities so that learners work in pairs or groups whenever possible;

Strategy to help learners with hearing disabilities or communication difficulties

- Always get the learner's attention before you begin to speak;
- Encourage the learner to look at your face;
- Use gestures, body language and facial expressions;
- Use pictures and objects as much as possible.
- Keep background noise to a minimum.

Strategies to help learners with physical disabilities or mobility difficulties:

- Adapt activities so that learners who use wheelchairs or other mobility aids, can participate.
- Ask parents/caregivers to assist with adapting furniture e.g. the height of a table may need to be changed to make it easier for a learner to reach it or fit their legs or wheelchair under;
- Encourage peer support when needed;
- Get advice from parents or a health professional about assistive devices if the learner has one.

Adaptation of assessment strategies:

At the end of each unit, the tutor is advised to provide additional activities to help students achieve the key unit competence. These assessment activities are for remedial, consolidation and extension designed to cater for the needs of all categories of students; slow, average and gifted students respectively. Therefore, the tutor is expected to do assessment that fits individual student.

Remedial activities	After evaluation, slow students are provided with lower order thinking activities related to the concepts learnt to facilitate them in their learning. These activities can also be given to assist deepening knowledge acquired through the learning activities for slow students.
Consolidation activities	After introduction of any concept, a range number of activities can be provided to all students to enhance/ reinforce learning.
Extended activities	After evaluation, gifted and talented students can be provided with high order thinking activities related to the concepts learnt to make them think deeply and critically. These activities can be assigned to gifted and talented students to keep them working while other students are getting up to required level of knowledge through the learning activity.

1.2.4. Guidance on assessment

Assessment is an integral part of teaching and learning process. The main purpose of assessment is for improvement of learning outcomes. Assessment for learning/ Continuous/ formative assessment intends to improve students' learning and tutor's teaching whereas assessment of learning/summative assessment intends to improve the entire school's performance and education system in general.

Continuous/ formative assessment

It is an on-going process that arises during the teaching and learning process. It includes lesson evaluation and end of sub unit assessment. This formative assessment should play a big role in teaching and learning process. The teacher should encourage individual, peer and group evaluation of the work done in the classroom and uses appropriate competence-based assessment approaches and methods.

Formative assessment is used to:

- Determine the extent to which learning objectives are being achieved and competences are being acquired and to identify which students need remedial interventions, reinforcement as well as extended activities. The application activities are developed in the learner book and they are designed to be given as remedial, reinforcement, end lesson assessment, homework or assignment
- Motivate students to learn and succeed by encouraging students to read, or learn more, revise, etc.
- Check effectiveness of teaching methods in terms of variety, appropriateness, relevance, or need for new approaches and strategies. Mathematics tutors need to consider various aspects of the instructional process including appropriate language levels, meaningful examples, suitable methods and teaching aids/ materials, etc.
- Help students to take control of their own learning.

In teaching Mathematics, formative or continuous assessment should compare performance against instructional objectives. Formative assessment should measure the student's ability with respect to a criterion or standard. For this reason, it is used to determine what students can do, rather than how much they know.

Summative assessment

The assessment can serve as summative and informative depending to its purpose. The end unit assessment will be considered summative when it is done at end of unit and want to start a new one.

It will be formative assessment, when it is done in order to give information on the progress of learners and from there decide what adjustments need to be done.

The assessment done at the end of the term, end of year, is considered as summative assessment so that the teacher, school and parents are informed of the achievement of educational objective and think of improvement strategies. There is also end of level/ cycle assessment in form of national examinations.

When carrying out assessment?

Assessment should be clearly visible in lesson, unit, term and yearly plans.

- Before learning (diagnostic): At the beginning of a new unit or a section of work; assessment can be organized to find out what students already know / can do, and to check whether the students are at the same level.
- During learning (formative/continuous): When students appear to be having difficulty with some of the work, by using on-going assessment (continuous). The assessment aims at giving students support and feedback.
- After learning (summative): At the end of a section of work or a learning unit, the Mathematics Tutor has to assess after the learning. This is also known as Assessment of Learning to establish and record overall progress of students towards full achievement. Summative assessment in Rwandan schools mainly takes the form of written tests at the end of a learning unit or end of the month, and examinations at the end of a term, school year or cycle.

Instruments used in assessment.

- **Observation:** This is where the Mathematics tutor gathers information by watching students interacting, conversing, working, playing, etc. A tutor can use observations to collect data on behaviours that are difficult to assess by other methods such as attitudes, values, and generic competences and intellectual skills. It is very important because it is used before the lesson begins and throughout the lesson since the tutor has to continue observing each and every activity.

▪ **Questioning**

(a) Oral questioning: a process which requires a student to respond verbally to questions

(b) Class activities/ exercise: tasks that are given during the learning/ teaching process

(c) Short and informal questions usually asked during a lesson

(d) Homework and assignments: tasks assigned to students by their tutors to be completed outside of class.

Homework assignments, portfolio, project work, interview, debate, science fair, Mathematics projects and Mathematics competitions are also the different forms/instruments of assessment.

1.2.5. Teaching methods and techniques that promote active learning

The different learning styles for students can be catered for, if the teacher uses active learning whereby learners are really engaged in the learning process.

The main teaching methods used in mathematics are the following:

- **Dogmatic method** (the teacher tells the students what to do, What to observe, How to attempt, How to conclude)
- **Inductive-deductive method:** Inductive method is to move from specific examples to generalization and deductive method is to move from generalization to specific examples.
- **Analytic-synthetic method:** Analytic method proceeds from unknown to known, 'Analysis' means 'breaking up' of the problem in hand so that it ultimately gets connected with something obvious or already known. Synthetic method is the opposite of the analytic method. Here one proceeds from known to unknown.
- **Skills lab method:** Skills lab method is based on the maxim "learning by doing." It is a procedure for stimulating the activities of the students and to encourage them to make discoveries through practical activities.
- **Problem solving method, Project method and Seminar Method.**

The following are some active techniques to be used in Mathematics:

- Group work
- Research
- Probing questions
- Practical activities (drawing, plotting, interpreting graphs)
- Modelling
- Brainstorming
- Quiz Technique
- Discussion Technique
- Scenario building Technique

What is Active learning?

Active learning is a pedagogical approach that engages learners in doing things and thinking about the things they are doing. Learners play the key role in the active learning process. They are not empty vessels to fill but people with ideas, capacity and skills to build on for effective learning. Thus, in active learning, learners are encouraged to bring their own experience and knowledge into the learning process.

The role of the teacher in active learning	The role of learners in active learning
• The teacher engages learners through active learning methods such as inquiry methods, group discussions, research, investigative activities, group and individual work activities. • He/she encourages individual, peer and group evaluation of the work done in the classroom and uses appropriate competence-based assessment approaches and methods. • He provides supervised opportunities for learners to develop different competences by giving tasks which enhance critical thinking, problem solving, research, creativity and innovation, communication and cooperation. • Teacher supports and facilitates the learning process by valuing learners' contributions in the class activities.	A learner engaged in active learning: • Communicates and shares relevant information with peers through presentations, discussions, group work and other learner-centred activities (role play, case studies, project work, research and investigation); • Actively participates and takes responsibility for his/her own learning; • Develops knowledge and skills in active ways; • Carries out research/ investigation by consulting print/online documents and resourceful people, and presents their findings; • Ensures the effective contribution of each group member in assigned tasks through clear explanation and arguments, critical thinking, responsibility and confidence in public speaking • Draws conclusions based on the findings from the learning activities.

Main steps for a lesson in active learning approach

All the principles and characteristics of the active learning process highlighted above are reflected in steps of a lesson as displayed below. Generally, the lesson is divided into three main parts whereby each one is divided into smaller steps to make sure that learners are involved in the learning process. Below are those main part and their small steps:

1) Introduction

Introduction is a part where the teacher makes connection between the current and previous lesson through appropriate technique. The teacher opens short discussions to encourage learners to think about the previous learning experience and connect it with the current instructional objective. The teacher reviews the prior knowledge, skills and attitudes which have a link with the new concepts to create good foundation and logical sequencings.

2) Development of the new lesson

The development of a lesson that introduces a new concept will go through the following small steps: discovery activities, presentation of learners' findings, exploitation, synthesis/summary and exercises/application activities.

- **Discovery activity**

Step 1

- The teacher discusses convincingly with learners to take responsibility of their learning
- He/she distributes the task/activity and gives instructions related to the tasks (working in groups, pairs, or individual to instigate collaborative learning, to discover knowledge to be learned)

Step 2

- The teacher let learners work collaboratively on the task;
- During this period the teacher refrains to intervene directly on the knowledge;
- He/she then monitors how the learners are progressing towards the knowledge to be learned and boosts those who are still behind (but without communicating to them the knowledge).

- **Presentation of learners' findings/productions**

- In this episode, the teacher invites representatives of groups to present their productions/findings.
- After three/four or an acceptable number of presentations, the teacher decides to engage the class into exploitation of learners' productions.

- **Exploitation of learner's findings/ productions**

- The teacher asks learners to evaluate the productions: which ones are correct, incomplete or false
- Then the teacher judges the logic of the learners' products, corrects those which are false, completes those which are incomplete, and confirms those which are correct.

- **Institutionalization or harmonization (summary/ conclusion/ and examples)**

- The teacher summarizes the learned knowledge and gives examples which illustrate the learned content.

- **Application activities**

- Exercises of applying processes and products/objects related to learned unit/sub-unit
- Exercises in real life contexts
- Teacher guides learners to make the connection of what they learnt to real life situations. At this level, the role of teacher is to monitor the fixation of process and product/object being learned.

3) Assessment

In this step the teacher asks some questions to assess achievement of instructional objective. During assessment activity, learners work individually on the task/activity. The teacher avoids intervening directly. In fact, results from this assessment inform the teacher on next steps for the whole class and individuals. In some cases, the teacher can end with a homework/ assignment. Doing this will allow learners to relay their understanding on the concepts covered that day. Teacher leads them not to wait until the last minute for doing the homework as this often results in an incomplete homework set and/or an incomplete understanding of the concept.

PART II: SAMPLE LESSON

School:.......................... **Teacher's Name:..........................**

Term	Date	Subject	Class	Unit Nº	Lesson Nº	Duration	Class size
------	----/------/-----	MATHEMATICS	Year one	1	7 of 9	40minutes	40 student-teachers

Type of Special Educational Needs to be catered for in this lesson and number of students in each category	2 student-teachers with hearing impairment will be seat near the tutor and the use of gestures will be improved in the lesson.
Unit title	Arithmetic
Key Unit Competence:	Use arithmetic operations to solve simple real life problems
Title of the lesson	Powers and related problems
Instructional Objective	Through the given activities, student-teachers should be able to solve problems involving powers accurately using properties of powers which are written on flash cards.
Plan for this Class (location: in / outside)	Inside the classroom
Learning Materials (for all learners)	Flash Cards, papers, Pens, Exercise Books, other supporting teaching aids such as Chalks and Chalkboard, etc...
References	Year one Student-teacher's book and Tutor's guide of Mathematics.

Steps and Timing	**Description of teaching and learning activities**		Competences and Cross-Cutting Issues to be addressed
	Student-Teachers are organized into groups to discuss the activity 1.7 in their mathematics books and the examples, the reporter from one group, presents the findings and the Student-Teachers interact together in group. The tutor facilitates Student-Teachers to capture the key concepts of the lesson through harmonization. Finally, the Student-Teachers are assigned to individual tasks and the correction is done on the chalk board.		
	Teachers activities	**Learners activities**	
Introduction **10mins**	**Powers and related problems** Tutor distributes flash cards to student-teachers in their small group discussions and invite them to brainstorm on the activity 1.7; Tutor moves around to help those who are struggling and guides them in finding definitions and properties of powers. Tutor invites student-teachers to present their findings. Tutor harmonizes the answers from presentation.	Student-teachers receive flashcards, discuss and brainstorm on the activity 1.7. They guess the definition of power and explore properties of powers. Group representatives present findings from groups and other students participate actively in the presentation by comments or by asking questions.	Cooperation is improved through group work: team working spirit is developed through working together in small group discussions. Communication skills are developed through small group discussions.

Development of the lesson: 20mins	Tutor gives instructions, invites students to brainstorm in their small groups the examples 1.7 (question 1 and question 2) found in the student's book. Tutor moves around to each group, ask probing questions in order to help struggling students. Tutor invites student-teachers to present their findings.	In their respective groups, Student-teachers discuss and brainstorm on examples 1.7 (question 1 and question 2). Student-teachers present their findings.	**Critical thinking**, **problem solving skills** and **Finance Education** are developed through analyzing and solving real life Mathematical problem. **Cooperation** and **communication** are developed during presentations and group discussions. **Inclusive education** is addressed by providing the remediation activities and tasks to struggling student-teachers.
Conclusion 10 min	**Summary:** Tutor guides all student-teachers to highlight the main properties of powers, their usage and to summarize the lesson of the day.	Student-teachers summarize the lesson guided by the tutor.	Communication skill is developed through small discussion on the findings and the main points of the lesson.
	Assessment -Tutor asks learners to individually work out the application activity 1.7	Student-teachers work independently on the application activity 1.7	**Critical thinking** and **problem solving skills** are developed through analyzing and solving real life Mathematical problem.

	Tutor gives the homework to student-teachers.	Write homework and ask more clarification on it.	**Critical thinking** and **problem solving skills** are developed through analyzing and solving real life Mathematical problems.
Tutor self-evaluation	To be completed after receiving the feed-back from the Student-teachers.		

PART III: UNIT DEVELOPMENT

UNIT: 1

ARITHMETIC

1.1 Key unit competence:

Use arithmetic operations to solve simple real life problems

1.2 Prerequisite

Student-teachers will perform well in this unit if they have a good background to arithmetic in senior two unit 4 and unit 7 in senior three.

1.3 Cross-cutting issues to be addressed

- Financial education
- Standardization Culture
- Inclusive Education
- Environment and sustainability
- Gender

1.4 Guidance on introductory activity

- Invite student-teachers to work in groups and give them instructions on how they do the introductory activity 1 found in unit 1of student's book;
- Guide students to read and analyse the problem insisting on mathematics operations to be used;
- Guide student-teachers to find out solutions of that problem and share them to the rest of class.
- Basing on student-teachers' experience, prior knowledge and abilities shown in answering the questions for this activity, use different questions to facilitate them to give their predictions and ensure that you arouse their curiosity on what is going to be leant in this unit.

Answer for introductory activity 1

Initial sum invested: 50,000

Interest at end of year 1== $50000(0.08) = 4,000 Frw$

Total sum invested for year 2 $= 50,000 Frw + 4,000 Frw = 54,000 Frw$

Interest at end of year 2 $= 54,000 Frw(0.08) = 4,320 Frw$

Total sum invested for year 3: $= 54,000 Frw + 4,320 Frw = 58,320 Frw$

Interest at end of year 3 $= 58,320 Frw(0.08) = 4,665.6 Frw$

Total sum invested for year 4 $= 58,320 Frw + 4,665.6 Frw = 62,985.6 Frw$

Interest at end of year 4 $= 62,985.6 Frw(0.08) = 5,038.848 Frw$

Total sum invested for year 5 $= 62,985.6 Frw + 5,038.848 Frw = 68,024.448 Frw$

Interest at end of year 5 $= 68,024.448 Frw(0.08) = 5,441.95 Frw$

Final value of investment $= 68,024.448 Frw + 5,441.95 Frw = 73,466.4 Frw$

At the end of 5 years, the investor will earn 73,466 Frw.

The mathematics operations used are the combination of successive multiplication and addition.

The following lessons will show how to simplify these successive operations in one simple formula.

1.5. List of lessons/sub-headings

#	Lesson title	Learning objectives	Number of periods
0.	Introductory activity	To arouse the curiosity of student-teacher on the content of unit 1.	1
1	Operations of real numbers and their properties	Classify real numbers and discover the main properties of operation of numbers	1
2	Fractions and related problems	Use properties for operations of fractions to solve real life problems	1

3	Decimals and related problems	Use decimals in solving real life problems	1
4	Percentages and related problems	Convert a fraction to a percentage and vice-versa Determine the percentage that corresponds to a given decimal number	1
5	Negative numbers and related problems	Appreciate the importance and the use negative numbers in real life;	1
6	Absolute value	Appreciate the use of absolute value in real life	1
7	Powers and related problems	Use properties of powers to solve some problems in Economics and finance	1
8	Radicals and related problems	Appreciate the importance of radicals in solving real life problems	1
9	Decimal logarithms and related problems	Use properties of decimal logarithms to solve real life problems.	2
10	Important application of arithmetic	Apply arithmetic in solving problems in Finance and Economics: simple and compound interest in a given problem.	3
11	End unit assessment		1
	Total		**15**

Lesson 1: Operations of real numbers and their properties

a) Learning objective:

Classify real numbers and discover the main properties of operation of real numbers

b) Teaching resources:

Graph papers, manila papers, Markers, digital technology including calculators, spreadsheets, etc

c) Prerequisites/Revision/Introduction: student-teachers will perform well in this lesson if they make revision on operations on numbers learnt in unit 2 of S1.

d) Learning activities

- Provide instructions to students and invite them to work in small groups on the activity 1.1.1;
- Ask each group to share findings with another group and ask them to complete their work or comment on the work done in order to support each other.
- Invite a member of each group to present their findings in a whole class discussion;
- As a tutor, harmonize the work done on activity 1.1.1 and use different probing questions to guide students to deduce the relationship between subsets of the set of real numbers, and the illustrating figure showing that $\mathbb{N} \subset \mathbb{Z} \subset D \subset \mathbb{Q} \subset \mathbb{R}$.
- With different questions, guide students to present some numbers on a number line, and to highlight the difference between a rational numbers and irrational numbers.
- Use the application activity 1.1.1 to assess the competences achieved.

After this session,

- Invite students to go back their groups to do the activity 1.1.2 on properties of operations in the set of real numbers.
- As this is an emphasis on what was learnt in previous years, invite one group to present their findings while others follow attentively for eventual comments where necessary.
- Basing on the presentation, guide students to discover the main properties that hold for different operations of real numbers through the use of examples given in the student book. Such properties include: Closure property, Commutative property, Associative property, Identity property, Inverse property, Distributive property.
- Invite students to exercise themselves through the application activity 1.1.2 before doing the application activity 1.1 for assessment or evaluation.

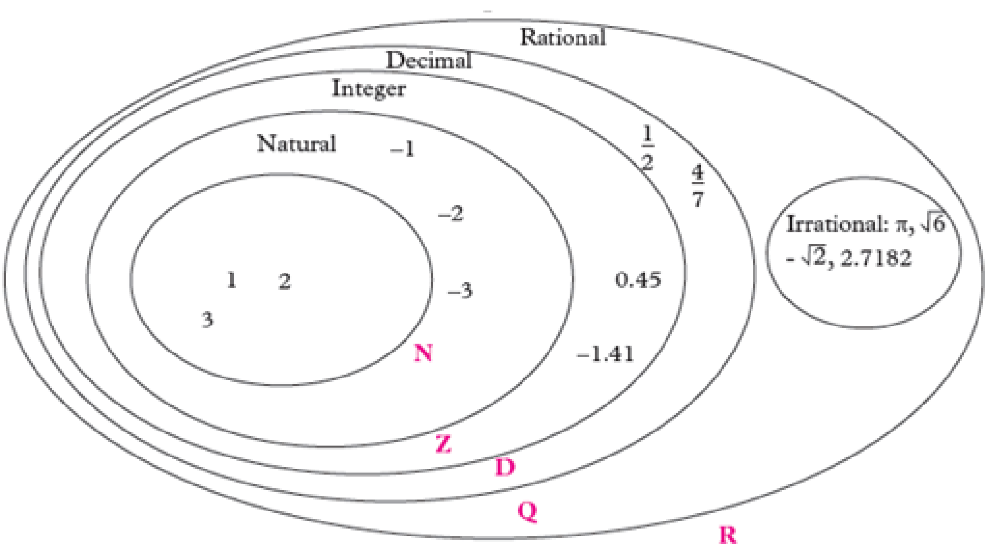
Answers for activity 1.1.1

a) Students will give different answers on the definitions requested. As a tutor, support those who eventually can provide a wrong answer.

In the list of numbers, $0;1;-5;6;\frac{3}{4};3.146;1.3333....;\pi;\sqrt{8}$, we have:

Natural numbers	Integers	Rational numbers	Irrational numbers
0, 1, 6	0, 1, -5, 6	0, 1, -5, 6 , ¾, 3.146, 1.333…	$\pi;\sqrt{8}$

b) Different answers will be given on the definition of the set $\mathbb{R}$, verify whether stipulates that the set $\mathbb{R}$ includes the union of rational and irrational numbers.



c) Given any 3 real numbers a, b *and* c of your choice,

$a+b=b+a$: with an example, they will show that addition of two real numbers is commutative

$ab=ba$, with an example, they will show that the multiplication of two real numbers is commutative

$a(bc)=(ab)c$, with an example, they will show that the multiplication of real numbers is associative.

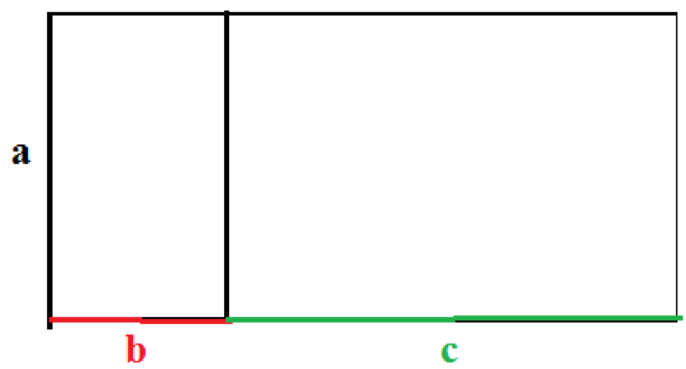
However, $\frac{a}{b}$ is a real number only when the number $b \neq 0$ because we do not divide by 0 in $\mathbb{R}$. $\frac{5}{0} \notin \mathbb{R}$

e) Expected answers on application activity1.1.1

	Number	Natural numbers	Whole numbers	Integers	Rational numbers	Irrational numbers	Real numbers
1	-17	-	-	√	√	-	√
2	-2	-	-	√	√	-	√
3	-9/37	-	-	-	√	-	√
4	0	√	√	√	√	-	√
5	-6.06	-	-	-	√	-	√
6	$4.5\bar{6}$	-	-	-	√	-	√
7	3.050050005...	-	-	-	-	√	√
8	18	√	√	√	√	-	√
9	$\frac{-43}{0}$	-	-	-	-	-	-
10	π	-	-	-	-	√	√

Expected answer on the application activity 1.1.2

Question 3: Find the area of the rectangle shown in the figure in two different ways to prove the distitributive property $a(b+c)=ab+ac$



Area of the whole rectangle is the sum S1 +S2 where S1 is the area of the small rectangle and S2 the area f the big rectangle.

ie $S = S_1 + S_2 = \quad ab + ac$

However, it is clear that the whole rectangle has the width a a and the length $b + c$. Its area is $S = a(b+c)$

As this area must be the same, we have $S = a(b+c) = S_1 + S_2 = ab + ac$

i.e $S = a(b+c) = ab + ac$

This shows that the multiplication is distributive over addition.

Lesson 2: Fractions and related problems

a) Learning objective:

Use properties for operations of fractions to solve real life problems.

b) Teaching resources:

Graph papers, manila papers, **o***ranges, sugar cane, sticks, papers* digital technology including calculators, locally made materials for learning fractions.

c) Prerequisites/Revision/Introduction

Student-teachers will perform well in this lesson if they have a good background on arithmetic learnt in senior three.

d) Learning activities

- In small group discussions, invite student-teachers to do activity 1.2 in the student's book;
- Ask student-teachers to share their answers with the neighbouring group and ask them to support each other where they become more challenged when solving problems for that activity.
- Request student-teachers to present share their findings to the whole class : only groups with different methods can be called to present while others complete the findings with new ideas.
- As a tutor, harmonizes the work done on activity 1.2 by highlighting: how to do operation of fractional expressions:

- With different probing questions, guide students to enhance how to add and subtract fractional expressions, to deal with: multiplication, division, simplification, and rationalization of fractional expressions.
- Using different examples given in the student's book, guide students to solve real life problems involving fractions; they can be given time to give their own examples of problems from their real life experience; This will help them to discover the importance of fractions.
- Invite students to the application activity 1.2 for assessing or evaluating students' abilities in dealing with fractions.

Answers for activity 1.2

1) The $\frac{2}{3}$ of teddy bears is $\frac{2}{3}\times 120 = 80$ frw

He received 80×12 Rwandan francs

2) Answers of the second question

a) $\frac{8x^2y^3}{2x^3y} = \frac{2x^2y\left(4y^2\right)}{2x^2y\left(x\right)} = \frac{4y^2}{x}$

b) $\frac{2x^2+5x^3}{2x^2+4x^3} = \frac{x^2(2+5x)}{2x^2(1+2x)} = \frac{1}{2}\left(\frac{2+5x}{1+2x}\right)$

3) $\frac{1}{x+1} - \frac{1}{2x+2} = \frac{2x+2-(x+1)}{(x+1)(2x+2)} = \frac{x+1}{(x+1)(2x+2)} = \frac{1}{(2x+2)}$

4) Refer to student's book of Mathematics, students can give different answers. As a tutor, harmonize them insisting on the role of fractions in real life.

e) Expected answers on the application activity 1.2

1) Let x be the numerator of the fraction, the denominator is $x+2$

Then the fraction is written $\frac{x}{x+2}$ and $\frac{x-3}{x+2+1} = \frac{2}{3}$

$$\frac{x-3}{x+3}=\frac{2}{3}$$

$$3(x-3)=2(x+3) \Leftrightarrow 3x-9=2x+6$$

$$3x-2x=6+9$$

$$x=15$$

The numerator x is 15.

Denominator is $x+2=15+2=17$ then, the fraction is $\frac{15}{17}$

2) $\frac{x^2+1}{x^3+4x^2+3x}$ as partial fraction,

$$\frac{x^2+1}{x(x^2+4x+3)}=\frac{A}{x}+\frac{B}{x+3}+\frac{C}{x+1}$$

Let put on the same denominator and we get

$$\frac{Ax^2+4Ax+3A+Bx^2+Bx+Cx^2+3Cx}{x^3+4x^2+3x}=\frac{x^2+1}{x^3+4x^2+3x}$$

$$\begin{cases} Ax^2+Bx^2+Cx^2=1x^2 \\ 4Ax+Bx+3Cx=0x \\ 3A=1 \end{cases}$$ Then, $A=\frac{1}{3}$ from equation (3)

And $\begin{cases} B+C=1-\frac{1}{3} \\ B+3C=-\frac{4}{3} \end{cases}$ where $B=\frac{5}{3}$ and $C=-1$

Finally, $\frac{x^2+1}{x\left(x^2+4x+3\right)}=\frac{1}{3x}+\frac{5}{3\left(x+3\right)}-\frac{1}{x+1}$ (this is written in partial fraction)

Lesson 3: Decimals and related problems

a) Learning objective

Use decimals to solve real life problems

b) Teaching resources:

Oranges, sugar cane, sticks, papers, manila papers, digital technology including calculators, pens and books.

c) Prerequisites:

Student-teachers will perform well in this unit if they have a good background on the operation of decimal numbers learnt in unit 1 of senior one Mathematics.

d) Learning activities

- Guide student-teachers in forming groups and distribute teaching materials to be used.
- Invite student-teachers to do activity 1.3 on decimals and related problems;
- Ask student-teachers to ask support to their classmates or tutor when they become more challenged in solving that activity 1.3.
- Circulate in all groups to verify students work and provide support where necessary;
- Ask student-teachers from groups with different working steps to present their answers to the whole class;
- Harmonize the work done through presentation and help student-teacher to conclude and make a summary content on decimals and related problems;
- Invite them to highlight the use of decimal numbers in real life experience: buying, selling, bank, student's marks, etc.
- Let them go through the application activity 1.3.

Answers for activity 1.3

1) a. $50:100=\frac{50}{100}=\frac{1}{2}$

b. $\frac{50}{100} = 0.5$

2. $\frac{1}{3} = 0.3333333333$ And $\frac{22}{7} = 3.1428571429$

3) Evaluate the oral expressions for students on mathematics statements given.

4)Answers will vary depending on the group of students, assess their oral expression.

e) Answers to application activity 1.3

1) $1.345 + 0.00041 + 0.20023 = 1.54564$

2) $93.954 \div 2.4 = 39.1475$

Lesson 4: Percentages and related problems

a) Learning objective:

- Convert a fraction to a percentage and vice-versa
- Determine the percentage that corresponds to a given decimal number

b) Teaching resources:

Textbook, Manila papers, digital technology including calculators.

c) Prerequisites/Revision/Introduction:

Student-teachers will perform well in this unit if they are enough skilled in arithmetic acquired in senior one and senior two.

d) Learning activities

- In pairs, Invite student-teachers to do activity 1.4 on percentage and related problems
- Move around in the class to verify students' progress over the work.
- Ask some groups with different working steps to present their answers to the whole class.
- As a tutor, harmonize their findings insisting on how a decimal number is written in the form of percentage, fraction and vice versa.

- Ask students to brainstorm on the use of percentages in real life experience: Bank, student's marks, different exams, reports on research, statistics from local administration readers, etc.

Answers for activity 1.4

1) a) $60:100=\frac{60}{100}=\frac{6}{10}=\frac{3}{5}$

$\frac{3}{5}\times 100=60\%$

$\frac{60}{100}=\frac{3}{5}=0.6$

2) $\frac{1}{3}=0.3333333333$ and $\frac{22}{7}=3.1428571429$

Yes, we can present the above decimals in the form of percentage as

$\frac{1}{3}\times 100=33.33333\%\approx 33.33\%$

$\frac{22}{7}\times 100=314.286$

3) First, find out how many students did not pass

Let x be percentage of students who did not pass test

Students who did not pass: $24-18=6$

Then, $x\%$ of 24 is equal to 6 students

Let find $x\%$,

$x=\frac{6}{24}=0.25=\frac{25}{100}=25\%$

Then, 25% of students did not pass the test.

4) Students will give different answers. Harmonize them accordingly.

e) Answers to application activity 1.4

1. The number correct answers is 80% of 20 or $\frac{80}{100}\times 20$

$\frac{80}{100}\times 20 = 0.80\times 20 = 16$

Number of missed questions is $20-16=4$

2. The questions answered correctly= $\frac{80}{100}\times 20 = 16$

The questions that not answered correctly $20-16=4$ questions

Percentage of questions not answered correctly is $\frac{4}{20}\times 100 = 20\%$

3. Correct answers are 35 and incorrect answers are 10

   Total answers=35+10=45 questions

   Percentage of correct answers is $\frac{35}{45}\times 100 = 77.778\%$

4) Answers will vary depending on the group of students. Harmonize them accordingly.

Lesson 5: Negative numbers and related problems

a) Learning objective:

Appreciate the importance and the use negative numbers in real life

b) Teaching resources:

Rulers, sticks, Thermometer, calculators, pens and pieces of chalks, etc…

c) Prerequisite:

Student-teachers will perform well in this unit if they are enough skilled in arithmetic acquired in senior one, unit two.

d) Learning activities:

- In pairs, invite student-teachers to do activity 1.5 in student-teachers' book on negative numbers and related problems;

- Ask student-teachers to share their answers with other neighbouring pairs.
- Move around to each group and ask some probing questions leading to the objective of the lesson;
- Invite groups with different working steps to present their answers then, harmonize the presented answers.
- Invite student-teachers to perform an example from content summary in student book.
- Invite students to brainstorm on the use of negative numbers in real life, how operations of negative numbers are done: insist on multiplication and division of negative numbers.

Answers for activity 1.5

1) The temperature of a juice in the bottle was $20^0 C$.

   Temperature decreases by $30^0 C$.

   What is the temperature of this juice?

The temperature of this juice= the temperature of a juice in the bottle− temperature of decreasing $= 20^0 C - 30^0 C = -10^0 C$

Advice:

The body has to work harder to maintain its internal temperature if you are drinking water that's near the temperature of juice. During drinking water with temperature that is near of ice exercises can help keep the body from overheating and make your workout sessions more successful. This is probably because drinking cold water makes it easier for body to maintain a lower core temperature. For children, it is not good to give them very cold juice, they can get the flu as a sickness. Put the juice out of the fridge to get the allowed temperature.

e) Answers of application activity 1.5

Question1:

a. $\frac{(-10)\times(-5)\times(-6)}{(-3)\times(-2)} = \frac{-(10\times 5\times 6)}{6} = -\frac{300}{6} = -50$

b. $\frac{(-30)\times(+2)\times(-10)}{(-50)\times(+2)}=\frac{+600}{-100}=-6$

c. Answers will vary, try to harmonize them insisting that when a date of paying the loan given to you is arrived, the computer takes it as a negative number on your account.

Question 2:

The cylinder is ¼ full of water and after 60ml of water is added the cylinder is 2/3 full.

What is the total volume of the cylinder?

Let x be the quantity of the full water when the cylinder is full.

$$\frac{1}{4}x+60=\frac{2}{3}x$$

$$\frac{x+240}{4}=\frac{2}{3}x$$

$$3x+720=8x$$

$$-5x=-720$$

$$x=144$$

Then, the total volume of the cylinder is 144 cubic units.

Lesson 6: Absolute value

a) Learning objective:

Appreciate the use of absolute value in real life

b) Teaching resources:

Rulers, sticks, calculators, pens, pieces of chalks, etc…

c) Prerequisite:

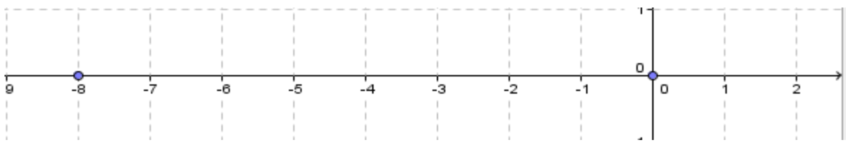
Student-teachers will perform well in this unit if they make a revision on the absolute value of a number learnt in S2 and S3.

d) Learning activities:

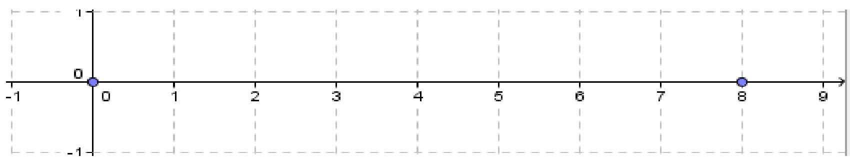
- Invite student-teachers to work in pairs the activity 1.6.1 in student-teachers' book on absolute value.
- Ask student-teachers to share their answers with neighbouring pairs.
- Move around to different groups and verify students' works.
- Invite member of group to present the findings while others follow for eventual comments.
- Together with student-teachers, invite them to go through 1st and the 2nd examples from content summary in student book. With clear examples, insist on the meaning of $|x| = \begin{cases} x, \text{ if } x \geq 0 \\ -x, \text{ if } x < 0 \end{cases}$.
- Do the same procedure above on properties of absolute value stating by the activity 1.6.2.
- Insist on properties of absolute value of a number and a variable.
- Using the example of distance, guide students to brainstorm on measurements which are not expressed with negative numbers and why it can happen.

Answers for activity 1.6.1

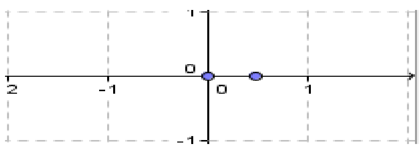
a.



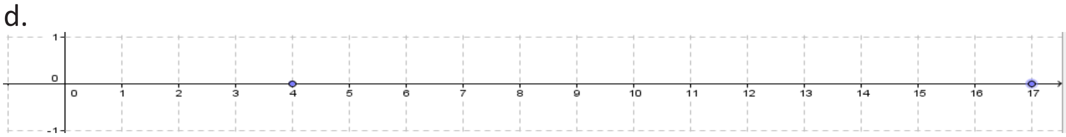
b.



c.



d.



e) Answers of application activity 1.6

1.

a) $|x| = 6$

$x = 6 \text{ or } x = -6$

$S = \{-6; 6\}$

b) $x = -1$

c)

$|x-3| - 4 = 2$
$x - 3 = 2 + 6$
$x = 9$

or

$-(x-3) - 4 = 2$
$-x + 3 - 4 = 2$
$-x = 3$
$x = -3$

Therefore, $S = \{-3; 9\}$

d) and e) can be solved in the same way.

2. a) Distance between OA is given by $d(\text{O}, \text{A}) = |A - 0|$.

Distance between OA=$|-4-0| = |-4| = 4$ lenght units

Distance between CB=$d(\text{B}, \text{C}) = |C - B|$

$d(c,b) = \left|-\frac{5}{2} - 2\right| = \left|\frac{-9}{2}\right| = \left|\frac{9}{2}\right| = \frac{9}{2} = 4.5$ lenght units

b) Distance covered by Dr. Makoma when going: (100min/sec).60min = 6000m

Distance when coming back: (100min/sec).60min = 6000m.

When considering the distance between the departure point and the arriving point, the distance is 12000m. However the displacement covered equals to ($\vec{\text{O}}$).

Lesson 7: Powers and related problems

a) Learning objective

Use properties of powers to solve some problems in Economics and finance

b) Teaching resources

Rulers, sticks, Graph papers, digital materials including calculator, manila papers, etc…

c) Prerequisite

Student-teachers will perform well in this unit if they make a short revision on powers lof a real number learnt in S2 and S3.

d) Learning activities

- Invite student-teachers to work in groups the activity 1.7 found in student-teachers' book.
- Ask student-teachers to share answers with other groups and ask support on challenging points they faced in their work.
- Move around to verify how students are working.
- Invite student-teachers to present their answers then, harmonize the presented answers.
- After doing activity 1.7, guide student-teachers on content summary by brainstorming on properties of powers referring to examples given in the students' book.
- Ask students to give examples where properties of powers are applied to solve real life problems: insist on the determination of the compound interest formula which uses the accumulated amount of money A after the time t (number of years P is invested) given by

  $A = P\left(1+\frac{r}{n}\right)^{nt}$ where n is the number of interest periods per year, r is the interest rate expressed as decimal, A is the amount after t years.

- Note that when the interest rate is compounded per year, $A = P\left(1+r\right)^{t}$ where r is expressed as a decimal for example $r = 9\% = 0.09$. When the interest rate is compounded monthly, $A = P\left(1+\frac{r}{12}\right)^{12.t}$ where t is the number of years and r expressed as a decimal.

- Compound interest = Accumulated amount (A) – Principal (P) or compound interest = End amount (A) - principal amount (P)

Answers for activity 1.7

1. **The given paper has the form of square**

$Area = x.x = x^2$

$Area = 20cm \times 20cm = (20cm)^2 = 400cm^2$

2. $Volume = a.a.a = a^3$

$volume = 3dm.3dm.3dm = (3dm)^3 = 3^3 dm^3 = 27dm^3$

e) Answers of application activity 1.7

1) simplification

a) $x^3 x^2 = x^{3+2} = x^5$

b) $(xy^3)^2 + 4x^2y^6 = x^2y^6 + 4x^2y^6 = 5x^2y^6$

c) $\frac{6xy^2}{3xy} = \frac{2y}{1} = 2y$

2) Students will provide different answers. Verify them using other reference books from general education.

Lesson 8: Radicals and related problems

a) Learning objective:

Appreciate the importance of radicals in solving real life problems.

b) Teaching resources:

Digital materials including calculator, rulers, sticks, Graph papers, manila papers, markers, etc…

c) Prerequisite:

Student-teachers will perform well in this unit if they are enough skilled in arithmetic acquired in mathematics for senior two, unit 1.

d) Learning activities:

- In group discussions, invite student-teachers to do activity 1.8.1 in student-teachers' book on radicals and related problems.
- Use gallery walk, student-teachers share their answers to others by rotating and ask support on challenging points they faced in their group.
- Move around to see student's progress in their respective groups.
- Invite groups with different working steps to present their answers then, harmonize the presented answers.

After doing activity 1.8.1, use different questions and guide student-teachers to discover properties of powers and examples.

- Invite student-teacher to do the activity 1.8.3 where one group will be invited to present and others will contribute with comments.
- Guide students to deal with operations on radicals through examples and remember to highlight different rules: simplification of radicals, rationalizing a denominator, etc.
- Ask student-teacher to brainstorm the use of radicals in real life: solving problems including square roots, cubic roots, etc.

Answers for activity 1.8.1

By the use of calculator:

1. $(81)^{\frac{1}{2}}=9$ 2. $(216)^{\frac{1}{3}}=6$ 3. $(-27)^{\frac{1}{3}}=-3$ 4. $(16)^{\frac{1}{4}}=2$.

e) Answers of application activity 1.8.2

1. $\sqrt{a}\times\sqrt{ab^3}\times\sqrt{bc^2} = ab^2c^{\frac{2}{3}}$

2. $\sqrt[3]{\frac{8}{27}}=\sqrt[3]{\frac{8}{27}}=\frac{\sqrt[3]{8}}{\sqrt[3]{27}}=\frac{8^{\frac{1}{3}}}{27^{\frac{1}{3}}}=\frac{2}{3}$

Answers for activity 1.8.3

1. $4\sqrt{2}$ **2.** $-\sqrt{3}$ 3. $\sqrt{6}$ 4. $\sqrt{3}$

Answers on application activity 1.8.3

1. $\sqrt{20}+\sqrt{5}=3\sqrt{5}$ 2. $4\sqrt{3}-\sqrt{12}=2\sqrt{3}$ 3. $3\sqrt{7}$ 4. 12

Answers on activity 1.8.4

1) $\frac{\sqrt{2}}{2}$ 2) $\frac{\sqrt{5}\left(2-\sqrt{3}\right)}{10}$ 3) $-\frac{2}{5}\left(1+\sqrt{6}\right)$

Answers of application activity 1.8

1. Distance$=1.2\sqrt{30}=\frac{12}{10}\sqrt{30}=6.57m$

2. Total surface area of cube(TSA) is given by $6x^2$

$TSA=6x^2=96$ surface area unit

$6x^2=96 \Leftrightarrow x^2=\frac{96}{6}$

$x^2=16 \Leftrightarrow x=\sqrt{16}=\pm 4$

Side of the cube is $4cm$

$volume=\left(4cm\times 4cm\times 4cm\right)=64cm^3$

3. Rationalize the denominator

a. $\frac{2\sqrt{2}}{4+3\sqrt{3}}=\frac{-2\sqrt{2}\left(4-3\sqrt{3}\right)}{11}$

b. $\frac{a-\sqrt{b}}{\sqrt{d}}=\frac{a-\sqrt{b}}{\sqrt{d}}=\frac{(a-\sqrt{b})}{\sqrt{d}}\times\frac{\sqrt{d}}{\sqrt{d}}=\frac{a\sqrt{d}-\sqrt{b}\sqrt{d}}{\sqrt{d}\sqrt{d}}=\frac{\sqrt{d}\left(a-\sqrt{b}\right)}{d}$

c. $\frac{3\sqrt{3}+2\sqrt{2}}{1+2\sqrt{2}}=\frac{3\sqrt{3}+2\sqrt{2}}{1+2\sqrt{2}}\times\frac{1-2\sqrt{2}}{1-2\sqrt{2}}=\frac{3\sqrt{3}-6\sqrt{6}+2\sqrt{2}-8}{1-8}$

4. Harmonize the findings of student-teachers from their research on library or on internet.

5. Answers may be different, try to verify their velocity and correct where necessary.

Lesson 9: Decimal logarithms and related problems

a) Learning objective:

Use properties of decimal logarithms to solve real life problems.

b) Teaching resources:

Digital materials including calculator, sticks, manila papers, markers, etc…

c) Prerequisite:

Student-teachers will perform well in this unit if they are enough skilled in properties of powers acquired in mathematics for S2 and S3 or in previous paragraphs of this unit.

d) Learning activities:

- Let students work in groups and do the activity 1.9 from the student-teachers' book;
- As students are working, circulate to each group and ask some questions which can lead to the objectives of this lesson;
- Ask groups to share their answers with other groups and allow them to share the challenging points they faced in their groups.
- Invite group representative to present their answers to the whole class;
- Harmonize students' findings through presentation
- Ask them different questions leading them to discover the meaning of decimal logarithm of a number written in the power of 10.
- After attempting different examples, help them to formulate the decimal logarithm of a number and establish how to find it. Highlight the following properties supported with examples:

$\log x$ is the same as $\log_{10} x$ and is defined for all positive real numbers only. 10 is the base of this logarithm. In general notation we do not write this base for decimal logarithm. In the notation $y = \log x$, x is said to be the antilogarithm of y.

Properties

$\forall a,b \in]0,+\infty[$

- $\log ab = \log a + \log b$
- $\log \frac{1}{b} = -\log b$
- $\log \frac{a}{b} = \log a - \log b$
- $\log a^n = n \log a$
- $\log \sqrt{a} = \log a^{\frac{1}{2}} = \frac{1}{2} \log a$
- $\log \sqrt[m]{a^n} = \log a^{\frac{n}{m}} = \frac{n}{m} \log a$
- $\text{colog}\, x = \log\left(\frac{1}{x}\right) = -\log x$
- Change of base formula: If u $(u > 0)$ and if a and b are positive real numbers different from 1, $\log_b u = \frac{\log_a u}{\log_a b}$ This means that if you

  have a logarithm in any other base, you can convert it in the decimal logarithm in the following way where $a = 10$:

  $\log_b u = \frac{\log_{10} u}{\log_{10} b} = \frac{\log u}{\log b}$.
- There is another special logarithm called natural logarithm which has the base a number $e \approx 2.71828$. This logarithm is written as: $\log_e x = \ln x$.
- Guide students to find more examples from real life that can be solved with the intervention of logarithms.
- Invite them to do application activity 1.9 to assess the competences developed.

Answers for activity 1.9

1. The number requested is the exponent of 10 in the following expressions

$1=10^0$ 2. $10=10^1$ 3. $100=10^2$ 4. $1000=10^3$ 5. $10000=10^4$
6. $100000=10^5$

2. To find the number x in $x^3=64$, we can equalize it with $x^3=64=4^3$ and deduce that $x=4$

e) Answers of Application activity 1.9

1. a. $a>b$ b. $a=b$ c. $a<b$

2. a. 2.17 b. 0.66 c. 0.30

3. a. $co\log 100=-\log 100=-2$

b. $co\log 42=-\log 42=-1.623$

c. $co\log 15=-\log 15=-1.176$

4. For the given problem, we'll use the compound Interest formula,

$F=P(1+i)^t$ where F is the final value, P the initial value of investment.

$$100000=70000\left(1+\frac{11}{100}\right)^t$$

$$10=7(1.11)^t$$

$$t=\frac{\log\left(\frac{10}{7}\right)}{\log(1.11)}=3.41$$

The time needed is 3.41 years

Lesson 10: Important applications of arithmetic

a) Learning objective:

Apply arithmetic in solving problems in Finance and Economics: simple and compound interest related problem.

b) Teaching resources:

Digital materials including calculator, sticks, manila papers, markers, student-teacher's book of mathematics, etc…

c) Prerequisite:

Student-teachers will perform well in this unit if they are enough skilled in arithmetic acquired senior two, senior three and content from previous lessons in this unit.

d) Learning activities:

- Form groups of students teachers and explain them how they are going to do the activity 1.10: to make research in the library or on internet to categorize problems of Economics and Finance that are easily solved with the use of arithmetic by focusing on Elasticity of demand, Arc of elasticity for demand, Simple interest and Compound interest, Final value of investment. Ask each group to make summary. And ask them to share their findings with other groups.
- Invite all groups to present their findings on given activity. Then, as a tutor harmonize findings from student-teachers.
- When harminizing, insist on showing students that Mathematics is needed every where and particularly when dealing with economics and finance. The main examples are given in the answer for activity 1.10.

Answers for activity 1.10

Expected answers: Price elasticity of demand is a measure of the responsiveness of demand to changes in price. It is usually defined as

$$e=(-1)\frac{\%change\,in\,quantity\,demand}{\%change\,in\,price}$$

Simple interest: Simple interest is the amount charged when one borrows money or loan from a financial institution which accrue yearly

The compound interest: Considering the case in which P is invested in a bank for n interest periods per year at the rate r expressed as a decimal. The accumulated amount A after the time t (number of years P is invested) is given by:

$A=P\left(1+\frac{r}{n}\right)^{nt}$ Where n is the number of interest periods per year, r is the interest rate expressed as decimal, A is the amount after t years.

Calculating the final value of an investment: Consider an investment at compound interest where: A is the initial sum invested, F is the final value of the investment, i is the interest rate per time period (as a decimal fraction) and n is the number of time periods. The formula is given by $F=A(1+i)^n$

Note:

When the interest rate is compounded per year, $A=P(1+r)^n$ where r is expressed as a decimal for example $r=9\%=0.09$.

When the interest rate is compounded monthly, $A=P\left(1+\frac{r}{12}\right)^{12.t}$ where t is the number of years and r expressed as a decimal.

e) Answers of application activity 1.10

1. An initial investment of £50,000

Ending amount £56,711.25 and time (t) = 2 years.

$$\frac{F}{A}=\frac{56711.25}{50000}=1.134225$$

$$i=\sqrt[n]{\left(\frac{F}{A}\right)}-1=\sqrt[2]{1.13455}-1=1.065-1=0.065$$

$$i=6.5\%$$

The interest rate has been applied is 6.5%

2. $A=P\left(1+\frac{r}{n}\right)^{nt}$

Let $k = \frac{n}{r}, \rightarrow \mathrm{n = kr}$ and $nt = krt$ and we may write the formula as

$$A = P\left(1+\frac{1}{k}\right)^{krt} = P\left[\left(1+\frac{1}{k}\right)^{k}\right]^{rt}$$

For continuously compounded interest we le n (the number of interest periods per year) increase without bound, denoted by $n \rightarrow \infty$, equivalently by $k \rightarrow \infty$, using the definition of e, we see that

$$P\left[\left(1+\frac{1}{k}\right)^{k}\right]^{rt} \rightarrow P[e]^{rt} = pe^{rt} \text{ As } k \rightarrow \infty$$

1.6. Summary of the unit

Arithmetic is used to determine the following:

Simple interest

When the principal (P), rate in percentage (R) and time in year (T) are given, then simple interest (I) for the given period is given by:

$$I = P.\frac{R}{100}.T = \frac{PRT}{100}$$

The compound interest formula

The accumulated amount of money A after the time t (number of years P is invested) is given by:

$A = P\left(1+\frac{r}{n}\right)^{nt}$ where n is the number of interest periods per year, r is the interest rate expressed as decimal, A is the amount after t years.

When the interest rate is compounded monthly, $A = P\left(1+\frac{r}{12}\right)^{12.t}$ where t is the number of years and r expressed as a decimal.

If we take $t = 1$, a general formula to solve for n can be derived as follows from the final sum formula:

$$A = P(1+r)^n, \quad \frac{A}{P} = (1+r)^n \text{ and } \quad n = \frac{\log(A/P)}{\log(1+r)}.$$

The application of logarithm respects the following properties:

$\forall a, b \in \left]0, +\infty\right[$

- $\log ab = \log a + \log b$
- $\log \frac{1}{b} = -\log b$
- $\log \frac{a}{b} = \log a - \log b$
- $\log a^n = n \log a$
- $\log \sqrt{a} = \frac{1}{2} \log a$
- $\log \sqrt[n]{a} = \frac{1}{n} \log a$
-
- $\log_b u = \frac{\log_{10} u}{\log_{10} b} = \frac{\log u}{\log b}$

1.7. Additional Information for teachers

The compound interest formula for different interest periods

To find the new amount of principal after one year if the interest is compounded quarterly, monthly, weekly, daily hourly and each minute, one can use the compound interest formula (and a calculator).

Interest period	Quarter	Month	Week	Day	Hour	Minute
n	4	12	52	365	8760	525,600

As the time is one year, this formula is $A = P\left(1 + \frac{r}{n}\right)^n$.

Table below summarize on how the formula applied in the above interest period.

Interest period	Amount after one year	Amount after t years
Quarter	$A=P\left(1+\frac{r}{4}\right)^{4}$	$A=P\left(1+\frac{r}{4}\right)^{4t}$
Month	$A=P\left(1+\frac{r}{12}\right)^{12}$	$A=P\left(1+\frac{r}{12}\right)^{12t}$
Week	$A=P\left(1+\frac{r}{52}\right)^{52}$	$A=P\left(1+\frac{r}{52}\right)^{52t}$
Day	$A=P\left(1+\frac{r}{365}\right)^{365}$	$A=P\left(1+\frac{r}{365}\right)^{365t}$
Hour	$A=P\left(1+\frac{r}{8760}\right)^{8760}$	$A=P\left(1+\frac{r}{8760}\right)^{8760t}$
Minute	$A=P\left(1+\frac{r}{525600}\right)^{525600}$	$A=P\left(1+\frac{r}{525600}\right)^{525600t}$

1.8 End unit assessment

1. The answers will be vary accordingly, the tutor harmonize the answers of student-teachers.

$$\frac{F}{A}=(1+i)^{n}\Rightarrow F=A(1+i)^{n}$$

a) $F=2000000(1+0.06)^{1}=2120000Frw$

b) $F=2000000(1+0.06)^{3}=2382032$

c) Let n be 19, then, $F=2000000(1+0.06)^{19}=6051199$

1.9 Additional activities

1.9.1 Remedial activities

1. Suppose that \$1000 is invested at an interest rate of 9% compounded monthly. Find the new amount of principal after 5 years, after 10 years, and after 15 years, calculate the amount after those periods of time.

Solutions: we find that the amount after time t is given by $A = P\left(1+\frac{r}{4}\right)^{4t}$

After 5 years: $A = \$1000\left(1+\frac{0.09}{12}\right)^{12\times 5} = \$1000(1.0075)^{60} = \$1565.68$

After 10 years, $A = \$1000\left(1+\frac{0.09}{12}\right)^{12\times 10} = \$1000(1.0075)^{120} = \$2451.36$

After 15 years, $A = \$1000\left(1+\frac{0.09}{12}\right)^{12\times 15} = \$1000(1.0075)^{180} = \$3838.04$

2. **Simplify the following**

a. $\sqrt{46656} = \sqrt{6^6} = 6^3 = 216$

b. $\sqrt[3]{ab} \times \sqrt[3]{a^2b^2} = \sqrt[3]{a^3b^3} = \sqrt[3]{(ab)^3} = ab$

c. $\sqrt{\frac{36}{81}} = \frac{\sqrt{36}}{\sqrt{81}} = \frac{6}{9} = \frac{2}{3}$

1.9.2 Consolidation activities

1. The number N of bacteria present in a culture at time t (in hours) obeys the function N (t) = $1000e^{0.01t}$.

a) Determine the number of bacteria at $t = 0$ *hour*

b) What is the growth rate of the bacteria?

c) What is the population after 4 hours?

d) When will the number of bacteria reach 1700?

e) When will the number of bacteria double?

Solution

a) $at\ t = 0$,   N (0) = 1000 bacteria's

b) Growth rate of the bacteria is 0.01

c) $N(4) = 1000e^{0.04} = 1040.8\ bacterias$

d) $1700 = 1000e^{0.01t} \Rightarrow 1.7 = e^{0.01t}$

$$\Rightarrow t = \frac{ln1.7}{0.01} = 53\ hours$$

e) $2000 = 1000e^{0.01t} \Rightarrow t = \frac{ln2}{0.01} = 69.3\ hours$

2. Rationalise $\frac{\sqrt{3}+\sqrt{7}}{4\sqrt{2}}$

Solution: $\frac{\sqrt{3}+\sqrt{7}}{4\sqrt{2}} = \frac{\left(\sqrt{3}+\sqrt{7}\right)\sqrt{2}}{\left(4\sqrt{2}\right)\sqrt{2}} = \frac{\sqrt{6}+\sqrt{14}}{8}$

1.9.3 Extended activities

1) Rationalise

a. $\frac{\sqrt{2}}{\sqrt{5}-\sqrt{3}}$

$$\frac{\sqrt{2}}{\sqrt{5}-\sqrt{3}} = \frac{\sqrt{2}\left(\sqrt{5}+\sqrt{3}\right)}{\left(\sqrt{5}-\sqrt{3}\right)\left(\sqrt{5}+\sqrt{3}\right)} = \frac{\sqrt{10}+\sqrt{6}}{5-3} = \frac{\sqrt{10}+\sqrt{6}}{2}$$

2) Solve the following into partial fractions as indicated

a) $\frac{6x^3+5x^2-7}{3x^2-2x-1} = A + \frac{1}{x-1} + \frac{B}{3x+1}$, find A and B.

b) $\frac{x^2-3x+1}{(x-1)^2(x-2)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x-2}$, find A , B and C

c) $\frac{9x-7}{(x+3)(x^2+1)} = \frac{A}{x+3} + \frac{Bx+C}{(x^2+1)}$ find A, B and C

Solution

a) $\dfrac{6x^3+5x^2-7}{3x^2-2x-1}=A+\dfrac{1}{x-1}+\dfrac{B}{3x+1}$.

The synthetic division gives

$$\frac{6x^3+5x^2-7}{3x^2-2x-1}=(2x+3)+\frac{8x-4}{3x^2-2x-1}$$

One can see that $3x^2-2x-1=(x-1)(3x+1)$.

Let $\dfrac{8x-4}{3x^2-2x-1}=\dfrac{8x-4}{(x-1)(3x+1)}=\dfrac{1}{x-1}+\dfrac{B}{3x+1}$

Therefore, $A=(2x+3)$ and we can determine B.

$$\frac{8x-4}{(x-1)(3x+1)}=\frac{1}{x-1}+\frac{B}{3x+1}$$

$$\frac{3x+1+B(x-1)}{(x-1)(3x+1)}=\frac{8x-4}{(x-1)(3x+1)}$$

$B=5$

Therefore, we see that $\dfrac{6x^3+5x^2-7}{3x^2-2x-1}=(2x+3)+\dfrac{1}{x-1}+\dfrac{5}{3x+1}$.

b) $\dfrac{x^2-3x+1}{(x-1)^2(x-2)}=\dfrac{A}{x-1}+\dfrac{B}{(x-1)^2}+\dfrac{C}{x-2}$,

Multiplying both sides by L.C.M. which is $(x-1)(x-2)$, we get

$x^2-3x+1=A(x-1)(x-2)+B(x-2)+C(x-1)^2$ is an equation.

Putting $x-1=0$, we get $x=1$ and our equation becomes

$1-3+1=-B$

$B=1$

Putting $x-2=0$, we get $x=2$ and our equation becomes

$4-6+1=C$

$C=-1$

However,

$$x^2 - 3x + 1 = A(x-1)(x-2) + B(x-2) + C(x-1)^2$$
$$x^2 - 3x + 1 = A(x^2 - 3x + 2) + B(x-2) + C(x^2 - 2x + 1)$$

Comparing the co-efficient of like powers of x on both sides, we get:

$$A + C = 1$$
$$A = 1 - C = 2$$

Hence as

$C = -1, \quad B = 1, \quad A = 2;$ the required partial fractions are

$$\frac{x^2 - 3x + 1}{(x-1)^2(x-2)} = \frac{2}{x-1} + \frac{1}{(x-1)^2} - \frac{1}{x-2}.$$

c) $\dfrac{9x-7}{(x+3)(x^2+1)} = \dfrac{A}{x+3} + \dfrac{Bx+C}{(x^2+1)}$. Find A, B and C

Multiplying both sides by L.C.M. i.e. $(x + 3)(x^2 + 1)$, we get this equation:

$$9x - 7 = A(x^2 + 1) + (Bx + C)(x + 3)$$

Putting $x + 3 = 0$, we get $x = -3$ and our equation becomes $-27 - 7 = 10A$

And $A = -\dfrac{17}{5}$

However, our equation becomes $9x - 7 = A(x^2 + 1) + B(x^2 + 3x) + C(x + 3)$

Comparing the co-efficient of like powers of x on both sides,

$A + B = 0$ (i)

$3B + C = 9$ (ii)

And putting the value of A in equation (i) we get

$$-\frac{17}{5} + B = 0 \text{ and } B = \frac{17}{5}.$$

Putting this value in equation (ii) we get $C = -\frac{6}{5}$

Hence the required partial fraction are $\frac{9x-7}{(x+3)(x^2+1)} = \frac{-17}{5(x+3)} + \frac{17x-6}{5(x^2+1)}$.

UNIT: 2

EQUATIONS AND INEQUALITIES

2.1 Key unit competence

Apply equations and inequalities to solve daily life problems.

2.2 Prerequisite

Student-teachers will perform well in this unit if they have a good background to arithmetic in senior one unit 3 and senior three unit 5.

2.3 Cross-cutting issues to be addressed

- Financial education
- Standardization Culture
- Inclusive Education
- Environment and sustainability
- Gender

2.4 Guidance on introductory activity

- Invite student-teachers to work in groups where they read and analyse the problem in introductory activity 2 found in student's book unit 2.
- During instruction, tel them that they can use a library or computer lab to search on the definition of linear equation and its application in real life.
- Ask student-teachers to complete the table found in introductory activity by using the information obtained from research.
- Invite all groups to present their findings to the whole class.
- Basing on their experience, results from their own research, prior knowledge and abilities shown in answering the questions for this activity, use different questions to facilitate student-teachers give their predictions and ensure that you arouse their curiosity on what is going to be leant in this second unit.

Answers for introductory activity 2

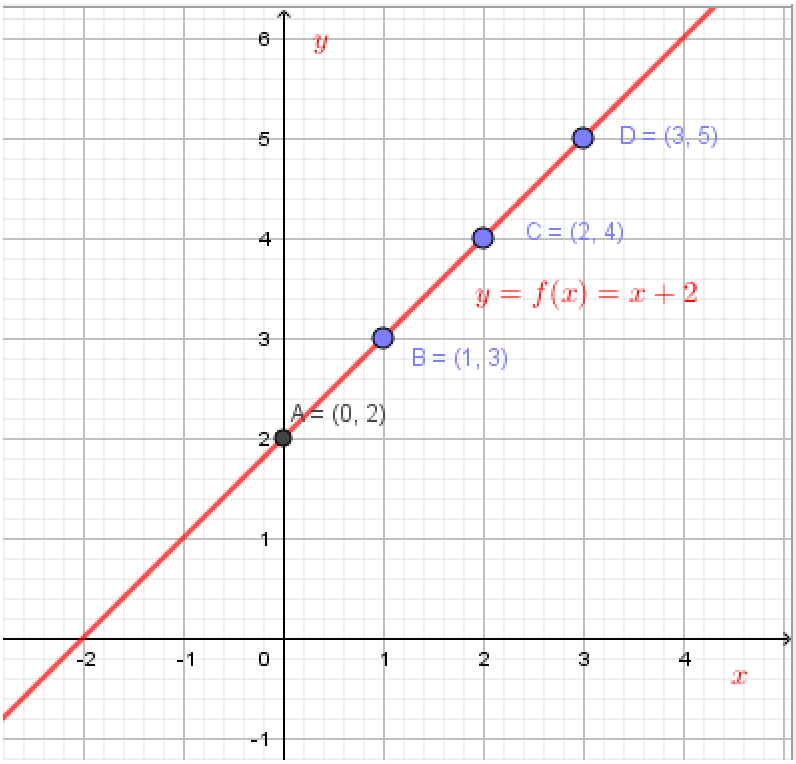
1) Refer to the student's book and verify answers for students.

2) If x is the number of pens for a learner, the teacher decides to give him/her two more pens. A learner with one pen will have $(1+2)\text{pens} = 3\text{pens}$

a) $y = f(x) = x+2$

x	-2	-1	0	1	2	3	4
$y = f(x) = x+2$	0	1	2	3	4	5	6
(x,y)	(-2;0)	(-1;1)	(0,2)	(1;3)	(2;4)	(3;5)	(4;6)

b) The graph obtained is the following:



c) The graph obtained is a line.

d) $y = x+2$ this is a linear equation because its graph is a line. Identically, $x+2 \geq 0$ is a linear inequality.

3) Students will give different examples. Verify whether the solution involves the linear equation.

2.5. List of lessons

#	Lesson title	Learning objectives	Number of periods
0.	Introductory activity	To arouse the curiosity of student-teacher on the content of unit 2.	1
1	Linear equations in one unknown and related problems	List and use the main steps in modelling a problem by linear equations and inequalities.	2
2	Linear inequalities in one unknown and related real life problems	Model a problem using linear equations or inequalities.	2
3	Solving algebraically simultaneous linear equations in two unknowns (by equating two same variables)	Use the mathematical methods to solve problems of economics and finance that involve simultaneous equations	1
4	Solving algebraically simultaneous linear equations in two unknowns (by row operations or elimination method)	Use the mathematical methods to solve problems of economics and finance that involve simultaneous equations.	1
5	Solving graphically simultaneous linear equations in two unknowns	Solve graphically and algebraically linear equations.	1
6	Solving algebraically and graphically simultaneous linear inequalities in two unknowns	Solve graphically and algebraically linear inequalities.	2
7	Solving quadratic equations by the use of factorization and discriminant	Factorize and solve quadratic equations	1
8	Applications of linear and quadratic equations in economics and finance: Problems about supply and demand (equilibrium price)	Model and solve real life problems that involve quadratic equations.	3
9	End unit assessment		1
	Total number of periods		**15**

Lesson 1: Linear equations in one unknown and related problems

a) Learning objective:

List and use the main steps in modelling a problem by linear equations and inequalities.

b) Teaching resources:

Graph papers, manila papers and calculators.

c) Prerequisites/Revision/Introduction:

Student-teachers will perform well in this unit if they are enough skills on equations and inequalities acquired in senior one and senior three.

d) Learning activities

- Invite student-teachers to work in group and answer questions for activity 2.1;
- Ask each group to share their answers with another group and ask them to support each other where they become more challenged in solving that activity.
- Request the group representative to share their findings to the whole class during a class discussion;
- As a tutor, harmonize the work done on activity 2.1 through presentation and insist on recalling the Following: increasing function, value of a function at a point, initial value for a function, solving an equation, the solution set for an equation or inequality.
- Use different questions and examples from student book and guide students on how to solve different types of equations.
- Let students go through the application activity 2.1 and evaluate whether the objectives of the lesson were achieved.

Answers for activity 2.1

1. Solve the equation $2x(3-x)=3$

$2x(3-x)=3$

$6x-2x^2=3$

$2x^2-6x+3=0$

$\Delta=\sqrt{b^2-4ac}=36-24=12$

$\sqrt{12}=2\sqrt{3}$

$$x_1=\frac{-b+\sqrt{b^2-4ac}}{2a}=\frac{6+2\sqrt{3}}{4}=\frac{3+\sqrt{3}}{2}$$

$$x_2=\frac{-b-\sqrt{b^2-4ac}}{2a}=\frac{6-2\sqrt{3}}{4}=\frac{3-\sqrt{3}}{2}$$

$$S=\left\{\frac{3+\sqrt{3}}{2},\frac{3-\sqrt{3}}{2}\right\}$$

1. $2x-1>x+3$

$x>4$

2. Let x be the first number and y be the second number

$x-y=48$

$\frac{x}{y}=\frac{7}{3}\Rightarrow 3x=7y$

$x=\frac{7}{3}y$ Replace the value of x in equation $x-y=48$

$\frac{7}{3}y-y=48$

$7y-3y=144$

$4y=144$

$y=36$

Replace the value of x in $x-y=48$ to find the value of y

$$x-36=48$$
$$x=48+36=84$$

$$S=\{84,36\}$$

e) Answers to application activities 2.1

1) a. $4x+5=20+x$

$$4x+5=20+x$$
$$4x-x=20-5$$
$$3x=15$$
$$x=5$$

b. $x-31=50-8x$

$$x+8x=50+31$$
$$9x=81$$
$$x=9$$

c. $\frac{2x+5}{x-6}=4$

$$\frac{2x+5}{x-6}=4$$
$$2x+5=4(x-6)$$
$$2x+5=4x-24$$
$$2x-4x=-24-5$$
$$-2x=-29$$
$$x=\frac{29}{2}$$

$$S=\left\{\left(\frac{29}{2}\right)\right\}$$

2) Let x be the breadth of rectangle and $2x$ be the length of rectangle

Perimeter $(p)=72m$

$p = 2(\text{L}+\text{l})$ Where L is length and l is width

$$2(x+2x) = 72$$
$$2x+4x = 72$$
$$6x = 72$$
$$x = 12m$$

The length of rectangle is $2\times 12m = 24m$

The breadth of that rectangle is $12m$

3) Let the first number be x and the second be $x+9$

$$x+x+9 = 25$$
$$2x = 16$$
$$x = 8$$

First number is 8

Second number, $8+9=17$

4) Let x be unknown number

$$\frac{x}{4}-\frac{x}{5} = 3$$

$$\frac{5x-4x}{20} = \frac{60}{20}$$

$$5x-4x = 60$$
$$x = 60$$

The unknown number is 60.

Lesson 2: Linear inequalities in one unknown and related real life problems

a) Learning objective:

List and use the main steps in modelling a problem by linear equations and inequalities.

b) Teaching resources:

Graph papers, manila papers, calculators, markers, pens, graph editors such as Geogebra (where possible).

c) Prerequisites/Revision/Introduction:

Student-teachers will perform well in this unit if they are enough skilled on linear inequalities in one unknown studied in S1, S2 and S3.

d) Learning activities

- Invite student-teachers to work in groups on questions of activity 2.2;
- Ask student-teachers to share their answers with another group and ask them to support each other where they become more challenged.
- With the use of different questions and examples given in the student's book, guide students to establish a good method of solving different types of inequalities: involving simple expression, product, quotient and absolute values. Insist on different ways of presenting the solution set of inequality.
- Guide students to brainstorm on real life problems that involve inequalities and invite them to explore the examples given in the students' book.
- After the lesson development, invite students to do the application activity 2.2 and verify if the objectives of the lesson were achieved.

Answers for activity 2.2.1

1. a) $x < 5$ represents all numbers less than 5, for example 4, 3, 2, 1, 0, -1, -2, etc.

   b) $x > 0$ represents all numbers greater than 0, for example 1, 2, 3, 4, 5, 6 etc

   c) $-4 < x < 12$ includes for example 6, 7, 8, 9,10, etc.

2. Response on this question will vary form a group to another, as a tutor, try to verify their veracity and correct them accordingly.

Answers for Activity 2.2.3

In each case, first construct the sign table. The solution will be given by interval showing negative values for $<$:

1) $(x+1)(x-1)<0$

x	$-\infty$ -1 1 $+\infty$
$(x+1)$	- - 0 + + + +
$(x-1)$	- - - - 0 + + + +
$(x+1)(x-1)$	+ + 0 - - 0 + + +

The table shows that $\frac{2x-3}{x}<0$ for $x\in\left]0,\frac{3}{2}\right[$

Then $S=\left]0,\frac{3}{2}\right[$

Answers for activity 2.2.4

1) The set of all real numbers whose number of units from zero, on a number line, are greater than 4 is $S=\{x\in\mathbb{R}:|x|>4\}$.

2) The set of all real numbers whose number of units from zero, on a number line, are less than 6 is $S=\{x\in\mathbb{R}:|x|<6\}$

Answers for activity 2.2.5

Let A be number of goals Alex scored and S be number of goals Sam scored

$A=S+3$

$S+A<9$

$S+S+3<9$

$2S<9-3$

$S<3$

Sam could score goals which are less than to 3. Therefore could score 0,1 or 2 goals.

If Sam score 0, Alex score 0+3=3, if Sam score 1 goal, Alex score 1+3=4 and if Sam score 2 goals Alex score 2+3=5 goals. Then, Alex could score 3, 4 or 5 goals

e) Answers of Application activity 2.2

1)Let S= average running speed then the average cycling speed = 2S

Speed=$\frac{Distance}{time}$,   Total time< $2^{1/2}$

$$\frac{25}{2s}+\frac{20}{s}<2^{1/2}$$

$$25+40<2^{1/2}$$

$$s>13$$

So, his average running is greater than $13km/h$ and his average speed cycling is greater than $26km/h$

2. The answers vary according to the answers given by student-teachers, tutor harmonize the answers.

Lesson 3: Solving algebraically simultaneous linear equations in two unknowns (by equating two same variables)

a) Learning objective:

Use mathematical methods to solve problems of economics and finance that involve simultaneous linear equations.

b) Teaching resources:

Manila papers, markers, pens and calculators.

c) Prerequisites/Revision/Introduction:

Student-teachers will perform well in this unit if they have good background on solving equations as it was learnt in senior two unit 3 and senior three unit 4.

c) Learning activities

- Invite student-teachers to work in group discussions and do activity 2.3 found in their Mathematics books.
- Move around in the class for facilitating various groups during their work.

- Invite group representative to present their answers to the whole class;
- Harmonize the findings highlighting how to solve simultaneous equations by equating values of two same variables;
- Guide students to explore examples given in the students' book to enhance the methods and invite them to give their own examples from real life situations.
- After doing this activity, assign students to do the application activity 2.3 and verify whether the lesson's objective was achieved.

Answers for activity 2.3

1. $\begin{cases} 3x-2y=-6 \\ x+y=-2 \end{cases}$

   $x=-2$ and $y=0$

2. $\begin{cases} x+2y=10 \\ -3x+2y=12 \end{cases}$

   $x=\frac{-1}{2}$ and $y=\frac{21}{4}$

3. $\begin{cases} x+4y=8 \\ y-x=2 \end{cases}$

   $x=0$ and $y=2$

e) Answers for application activity 2.3

1. $\begin{cases} x-y=3 \\ 2x-2y=6 \end{cases}$

It has an Infinity of solutions, $S=\{(3+y,y):y\in\mathbb{R}\}$

2. $\begin{cases} x+y=2 \\ 2x+2y=4 \end{cases}$

It has an Infinity of solutions $S=\{(2-y,y):y\in\mathbb{R}\}$

3. $\begin{cases} 3x - 5y = 10 \\ 2x + y = 12 \end{cases}$

$x = \frac{70}{13}$ and $y = \frac{16}{13}$ and $S = \left\{\left(\frac{70}{13}, \frac{16}{13}\right)\right\}$

4. $\begin{cases} x - 4y = 1 \\ x - y = 2 \end{cases}$

$x = \frac{7}{3}$ and $y = \frac{1}{3}$ and $S = \left\{\left(\frac{7}{3}, \frac{1}{3}\right)\right\}$

5. $\begin{cases} -x + y = 0 \\ x + 2y = 3 \end{cases}$

$x = 1$ and $y = 1$ and $S = \{(1,1)\}$

Lesson 4: Solving algebraically simultaneous linear equations in two unknowns (by row operations or elimination method)

a) Learning objective:

Use the mathematics methods of raw operations to solve real life problems that involve simultaneous linear equations.

b) Teaching resources:

Manila papers, calculators, markers and pens.

c) Prerequisites/Revision/Introduction:

Student-teachers will perform well in this unit if they have good background in senior two unit 3, senior three unit 4 and in previous lessons of this unit.

c) Learning activities

- Invite student-teachers to work in group discussions and do activity 2.4 found in their Mathematics books.

- Move around in the class for facilitating various groups during their work.
- Invite group representative to present their answers to the whole class;
- Harmonize the findings highlighting how to solve simultaneous equations by multiplying one equations by a number such that when making the addition or subtraction, one variable is eliminated to obtain an equation in just one unknown,
- Guide students to explore examples given in the students' book to enhance the methods and invite them to give their own examples from real life situations.
- After doing this activity, assign students to do the application activity 2.4 and verify whether the lesson's objective was achieved.

Answers for activity 2.4

Given the following simultaneous equations:

$$\begin{cases} -2x + 5y = -7 \\ 7x - 3y = -19 \end{cases}$$

We can multiply the first equation by 7 on and the second equation by 2.

$$\begin{cases} -14x + 35y = -49 \\ 14x - 6y = -38 \end{cases}$$

The addition of the results will eliminate the variable x and get

$$29y = -87$$

This equation gives $y = -3$. Replacing this value in one equation we get $x = -4$.

The solution set $S = \{(-4, -3)\}$

e) Answers of Application activity 2.4

1) a. $x = -1$ and $y = -1$, Then $S = \{(-1, -1)\}$

b. $x = \frac{7}{3}$ and $y = \frac{1}{3}$. Then, $S = \left\{(\frac{7}{3}, \frac{1}{3})\right\}$

c. $x = -2$ and $y = 0$. Then, $S = \{(-2, 0)\}$

d. $x = \frac{14}{5}$ and $x = \frac{4}{5}$. Then, $S = \left\{ (\frac{14}{5}, \frac{1}{5}) \right\}$

2) Answers vary with the group depending on how student-teachers explained how to solve algebraically simultaneous linear equations. Try to orient them accordingly.

Lesson 5: Solving graphically simultaneous linear equations in two unknowns

a) Learning objective:

Solve graphically linear equations.

b) Teaching resources:

Manila papers, markers, rulers, graph papers, calculators and mathematics software such as Geogebra where possible.

c) Prerequisites/Revision/Introduction:

Student-teachers will perform well in this unit if they have good background in solving equations as it was learnt in senior two unit 3, senior three unit 4 and previous lessons of this unit.

c) Learning activities

- Invite student-teachers to work in group discussions and do activity 2.5 found in their Mathematics books.
- Move around in the class for facilitating various groups during their work and verify their working steps.
- Invite one member from groups with different working steps to present their answers to the whole class;
- Harmonize the findings highlighting how to solve graphically simultaneous equations. Guide them to enhance how to draw the lies using their equations and highlight that the solution is the intercept of two lines; its coordinates must be shown.
- Guide students to explore examples given in the students' book to enhance the methods and invite them to give their own examples from real life situations.
- After doing this activity, assign students to do the application activity 2.5 and verify whether the lesson's objective was achieved.

The following steps can be applied in solving system of linear equation graphically:

1. Find at least two points for each equation.
2. Plot the obtained points in XY plane and join these points to obtain the lines. Two points for each equation give one line.
3. The point of intersection for two lines is the solution for the given system

Answer for activity 2.5

1. To find the coordinate of the point intercept of two lines is found by equalizing the two equations. The point is $(3,2)$

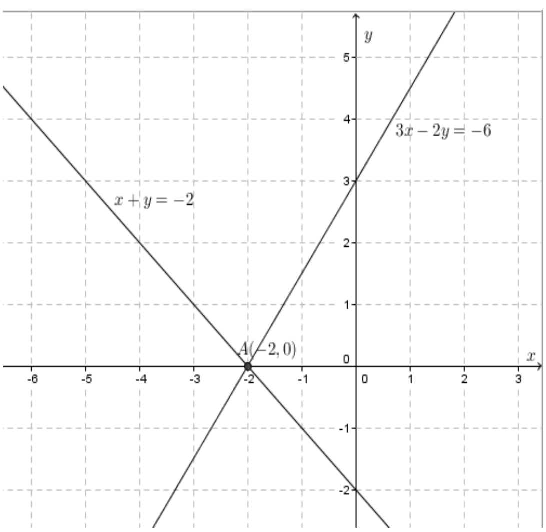
2.a. for equation 1: $y=\dfrac{6+3x}{2}$

x	-2	0	2
y	0	3	6

For equation 2: $y=-2-x$

x	0	1
y	-2	-3

b. The graph of two lines

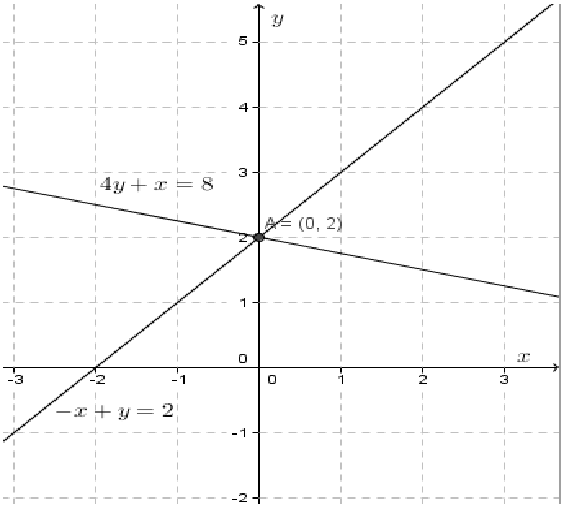


c. The point of intersection is point $A(-2,0)$

d. $S=\{(-2,0)\}$

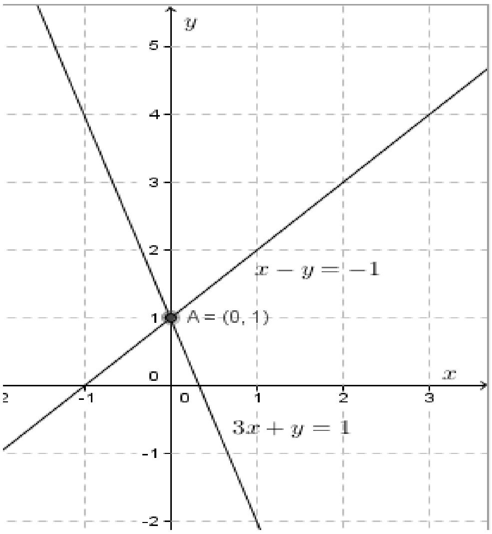
e) Answers of application activity 2.5

1. $\begin{cases}4y+x=8\\-x+y=2\end{cases}$



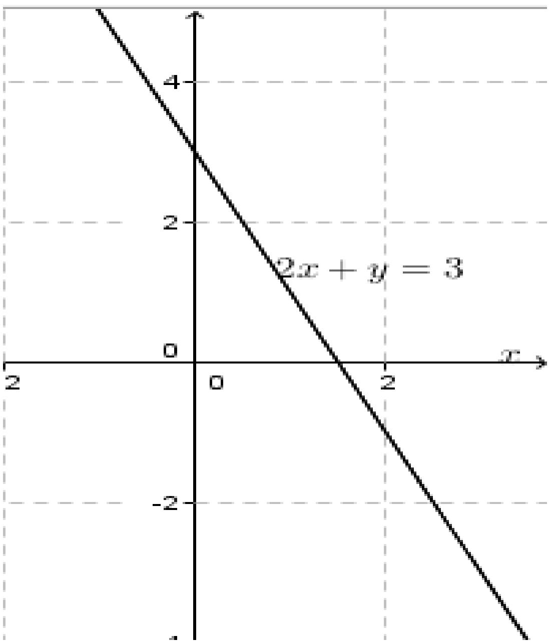
Solution of the system is $S=\{(0,2)\}$

2. $\begin{cases}3x+y=1\\x-y=-1\end{cases}$



The solution $S=\{(0,1)\}$

3. $\begin{cases} 2x+y=3 \\ 6x+3y=9 \end{cases}$



Since the two lines are coinciding the system has infinite solutions. The solution is made of the entire line. $S=\{(x,y)\in\mathbb{R}^2 : 2x+y=3\}$.

Lesson 6: Solving algebraically and graphically simultaneous linear inequalities in two unknowns

a) Learning objective:

Solve algebraically and graphically simultaneous linear inequalities.

b) Teaching resources:

Manila papers, markers, rulers, graph papers, calculators, mathematics software such as Geogebra where it is possible.

c) Prerequisites/Revision/Introduction:

Student-teachers will perform well in this unit if they revise the content related to inequalities learnt in senior two unit 3, senior three unit 4 and previous lessons of this unit.

d) Learning activities

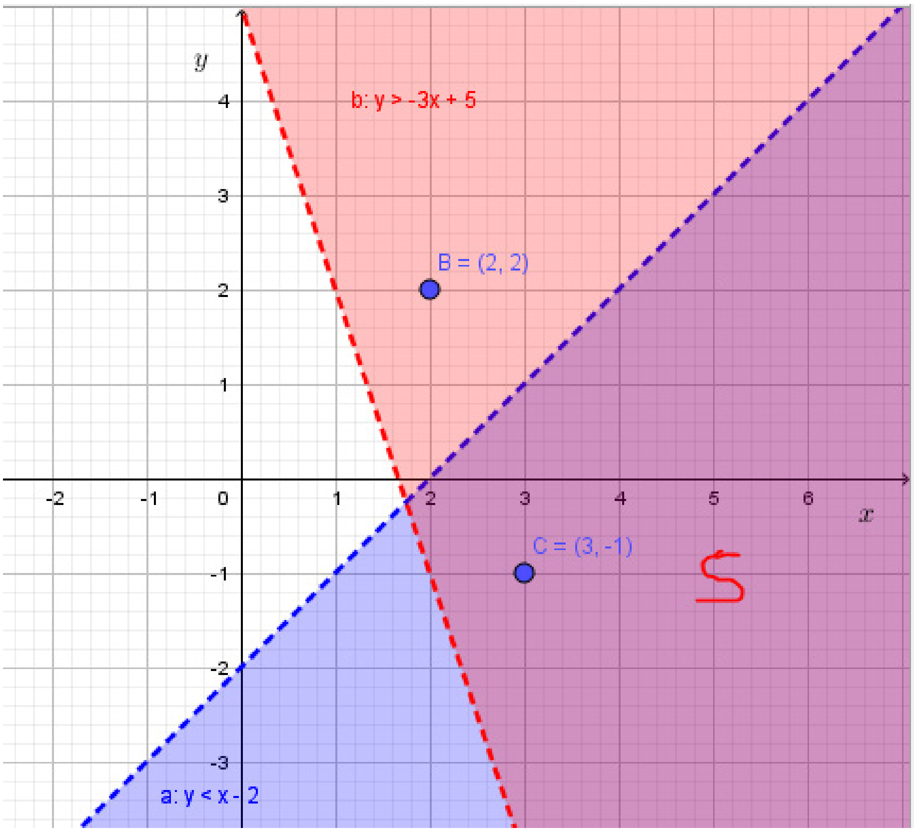
- Invite student-teachers to work in group discussions and do activity 2.6 found in their Mathematics books.
- Move around in the class for facilitating various groups during their work, verify their working steps and intervene with probing questions for orienting students where necessary. For example, this orientation concerns to take one point $A(x, y)$ out of the line drawn (using equations given) and verify if the value found verifies the inequalities.
- Ask the neighbouring groups of student-teachers to share their answers and compare them where necessary.
- Invite one member from groups with different working steps to present their answers to the whole class;
- Harmonize the findings highlighting how to solve algebraically and graphically the inequalities.
- When solving graphically, use different questions leading to identify whether a selected point in the plan is located in the region that verify the inequality. Decide which region will be shaded between the solution region or the other region.
- Guide students to explore examples given in the students' book to enhance their methods and invite them to give their own examples from real life situations.

- After doing this activity, assign students to do the application activity 2.6 and verify whether the lesson's objective was achieved.

Answers for activity 2.6

The points which verify the system of inequalities $\begin{cases} y < x-2 \\ y > -3x+5 \end{cases}$ are $B(2,2)$ and $C(3,-1)$

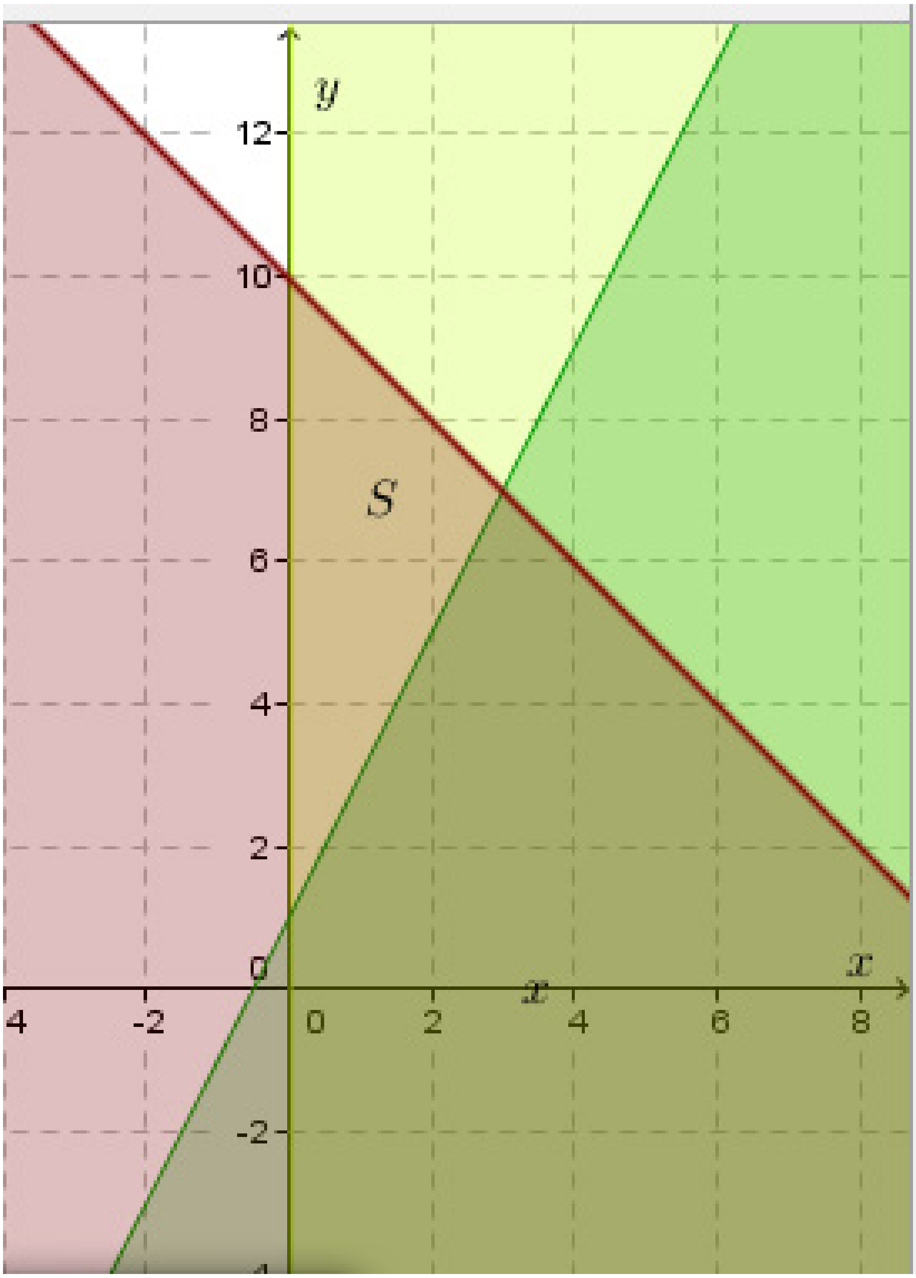
In the following graph, the shaded region is the one which is a solution.



It is clear that the point B (2,2) is in the region solution of the first inequality but it is not in the region solution of the second inequality. The intersection of the shaded regions is the solution region S. This is the reason why the point C (3,-1) verify all inequalities.

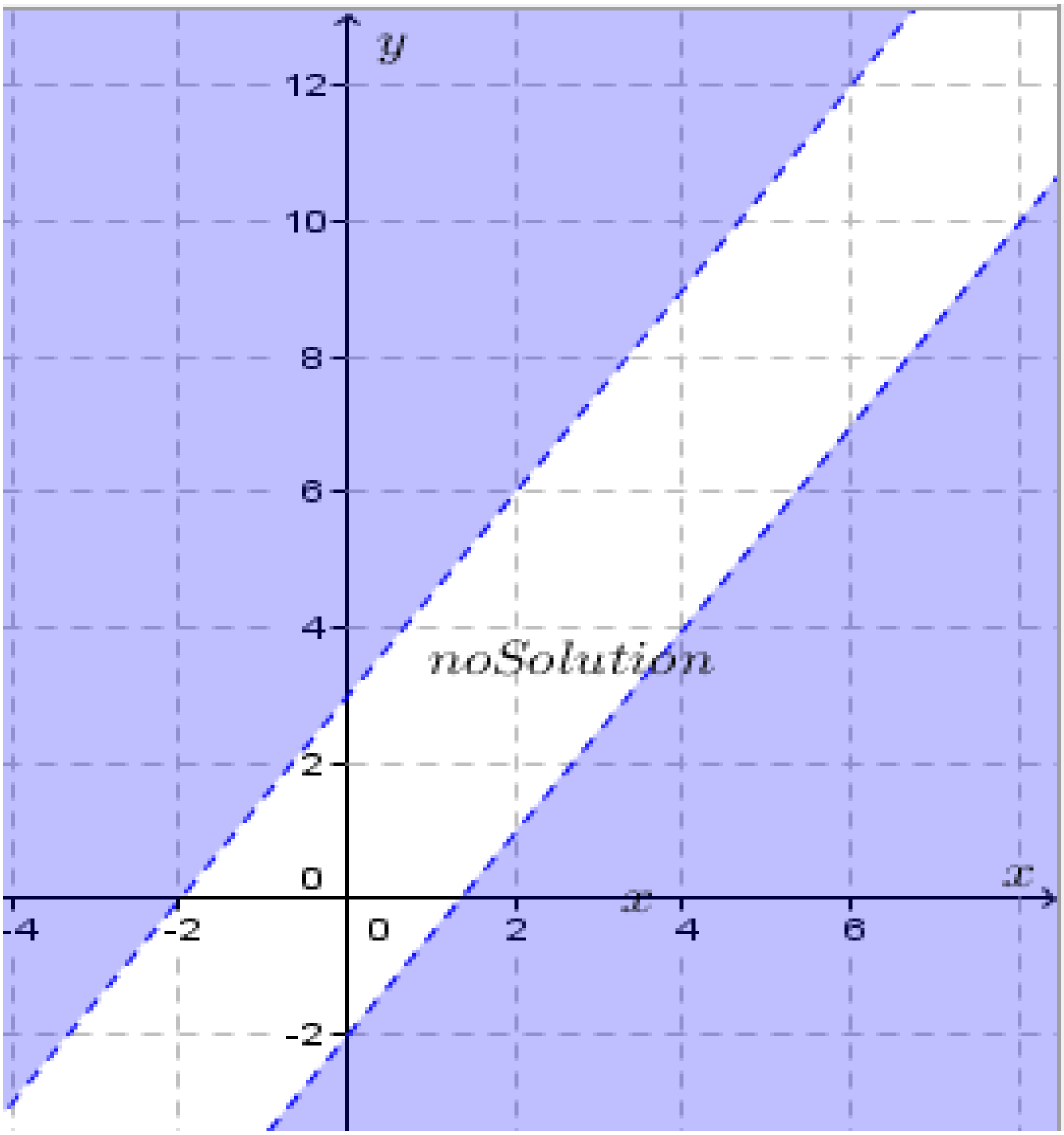
e) Answers for application activity 2.6

1.a. $\begin{cases} y - 2x \le 1 \\ x + y \le 10 \\ x \ge 0 \end{cases}$

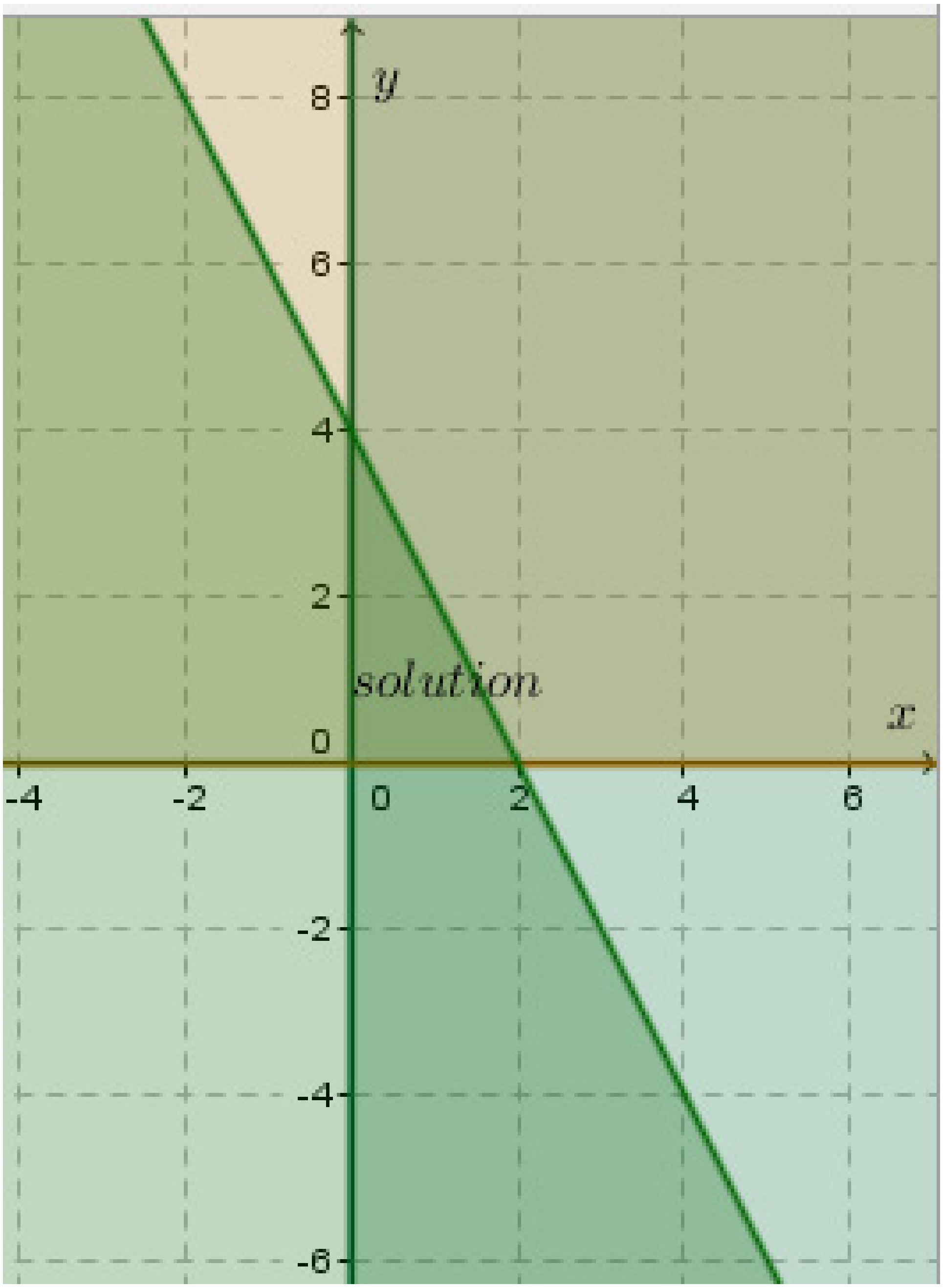


Where S is the solution

$$b.\begin{cases}-3x + 2y > 6\\ 6x - 4y > 8\end{cases}$$



c. $\begin{cases} x \geq 0 \\ y \geq 0 \\ 2x + y \leq 4 \end{cases}$



2) Answers will vary according to student-teachers observations. As a tutor, harmonize the answers referring to the steps for finding solutions when solving graphically the simultaneous inequalities.

Lesson 7: Solving quadratic equations by the use of factorization and discriminant

a) Learning objective:

Factorize and solve quadratic equation

b) Teaching resources:

Manila papers, markers, rulers, graph papers, calculators, pencils, pens, mathematics software such as Geogebra.

c) Prerequisites/Revision/Introduction:

Student-teachers will perform well in this unit if they revise the content related quadratic equations learnt in senior three unit 5.

d) Learning activities

- Invite student-teachers to work in group discussions and do activity 2.7 found in their Mathematics books.
- Move around in the class for facilitating various groups during their work, verify their working steps and intervene with probing questions for orienting students where necessary. For example, you can ask them to recall what the equation product AB = 0 means leading them to guess towards the factorization of the quadratic equation before solving;
- Ask the neighbouring groups of student-teachers to share their answers and compare them where necessary.
- Invite one member from groups with different working steps to present their answers to the whole class;
- Harmonize the findings highlighting how to solve algebraically the quadratic equations;
- Use different types of equations and some examples given in the student's book and guide students to identify a type of equation and discuss how they can solve them: Use of factorization, use of discriminant, ploting graphs, etc. In each case, help them to discover 3 main cases: equation with two real roots, equation with one double root and equation without root in the set of real numbers:

The quadratic equation has the form $ax^2+bx+c=0$. The discriminant is given by $\Delta=b^2-4ac$

The two values of unknown x are generated as follows:

$x_1 = \frac{-b+\sqrt{b^2-4ac}}{2a}$ And $x_2 = \frac{-b-\sqrt{b^2-4ac}}{2a}$

Note that, when the $\Delta = 0$ two equal roots are generated as $x_1 = x_2 = \frac{-b}{2a}$.

When $\Delta < 0$ the equation has no root in the set of real numbers.

- Assign students to do the application activity 2.7 and verify whether the lesson's objective was achieved.

Answers for activity 2.7

a) $y = -16t^2 + 1600$, for $y = 1000$, we have $1000 = -16t^2 + 1600$.

Solve this equation to find the time requested. $t = \frac{\sqrt{600}}{4} \approx 6.1$

The jumper is in free fall for about 6 seconds.

b)Table of value:

t	0	1	2	3	4	5	6
y	1600	1616	1664	1744	1856	2000	2176

e) Answers for application activity 2.7

a) 1. $x_1 = -2$ and $x_2 = \frac{5}{3}$

2). No solution

3). $x_1 = 1$ and $x_2 = 11$

4) $x_1 = x_2 = 4$

b) The area of rectangle is given by $w.\mathrm{L} = w(w+7) = 30$

$w^2 + 7w - 30 = 0$ By solving we get

$w = 3$

$w = -10$

Consider the width $w = 3$ meters

The length L = (3+7) m.

So, the width of the garden is 3 m, and the length is 10 m.

c) Help student-teachers to decide referring to the 3 cases observed different examples.

Lesson 8: Applications of linear and quadratic equations in economics and finance

a) Learning objective:

Model and solve real life problems involving linear and quadratic equations.

b) Teaching resources:

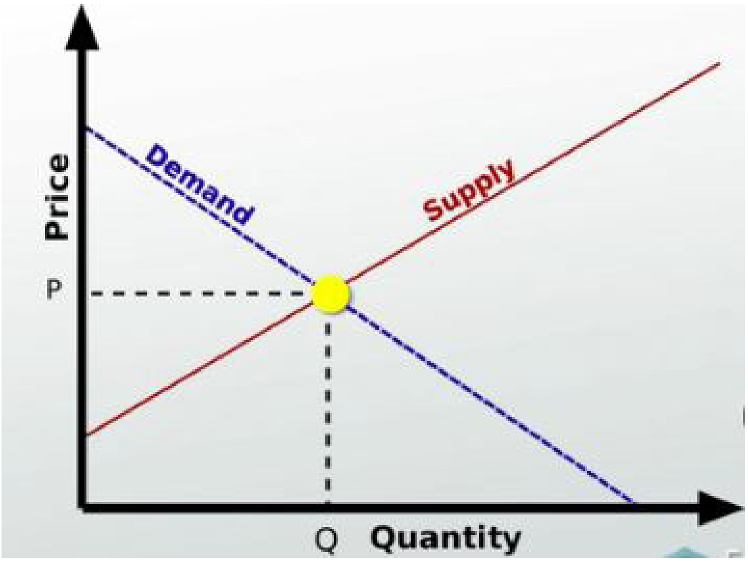
Manila papers, markers, rulers, graph papers, calculators and where possible use the mathematics software for graphing such as Geogebra.

c) Prerequisites/Revision/Introduction:

Student-teachers will perform well in this unit if they learnt well the content for previous lessons in this unit.

d) Learning activities

- Invite student-teachers to work in group and do the activity 2.8 found in their Mathematics books.
- Move around in the class for facilitating where necessary and give more clarification on eventual challenges they may face during their work; Identify groups which have different working steps.
- Invite each groups with different working steps to present their answers in a whole class discussion;
- As a tutor, harmonize the findings from presentation of student-teachers and ask them to give other examples of problems from real life which involve the use of linear or quadratic equations especially in Economics and finance.
- Guide them to explore examples given in the student's book: equilibrium values of p and q in a competitive market:



Answers for activity 2.8

From the information in the question we can work out that this firm faces the total revenue function TR = 180q and the total cost function

TC = 2400 + 140q, where q is output.

$$TC = 2400 + 140q$$
$$TR = 180q$$

The break-even point is where $TR = TC$ Then,

$$180q = 2400 + 140q$$
$$180q - 140q = 2400$$
$$40q = 2400$$
$$q = 60$$

Therefore the output required to break even is 60 units.

e) Answers of Application activity 2.8

1) Let y equal the money the house worker has at after t days.

After one day the house worker has $3000 + (400).1$

After t days, the house worker has $3000 + (400).\text{t}$; Meaning that $y = 3000 + 400\,\text{t}$

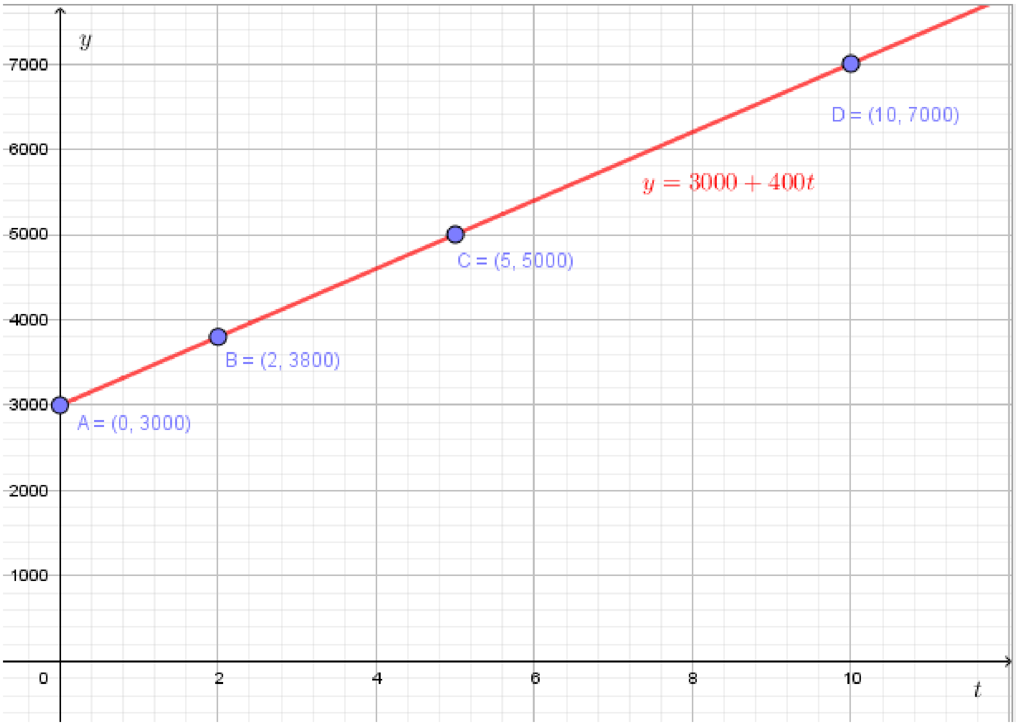
a) After the second day he will have $(3000+400(2))\,\text{Frw}=3800\,Frw$

After 5 days, he will have $(3000+400(5))\,\text{Frw}=5000\,Frw$

After 10 days he will have 7000Frw

b) The function $y=3000+400\text{t}$ shows that as days increases, the money for the house workers is increasing.

c) The graph is below



d) After two months (60 days), the house worker will have
$(3000+400(60))\,\text{Frw}=27{,}000\,Frw$.

e) Yes a house work can save money with the aim of getting enough money for solving a specified problem he has.

2) $\begin{cases} Y=C+I \\ C=40+0.5Y \\ I=200 \end{cases}$

$Y = 40 + 0.5Y + 200$

$0.5Y = 240$

$Y = 480$

The national income is 480 (units of money).

2.6. Summary of the unit

Linear equation

A linear equation is an equation of a straight line, it has the form of $y = ax + b, a \neq 0 \quad and\ a, b \in \mathbb{R}$.

Solve inequality

To solve inequality of the form $(ax+b)(cx+d) < 0$, we follow the following steps:

a) First we solve for $(ax+b)(cx+d) = 0$

b) We construct the table called **sign table**, find the sign of each factor and then the sign of the product or quotient if we are given a quotient.

For the quotient the value that makes the denominator to be zero is always excluded in the solution. For that value we use the symbol **||** in the row of quotient sign.

c) Write the interval considering the given inequality sign.

Solving graphically a system of linear equation

The following steps can be applied in solving system of linear equation graphically:

1. Find at least two points for each equation.
2. Plot the obtained points in XY plane and join these points to obtain the lines. Two points for each equation give one line.
3. The point of intersection for two lines is the solution for the given system

Solving graphically a system of linear inequalities

The following steps used to find the solution of simultaneous inequalities graphically:

1. Rearrange the equation so "y" is on the left and everything else on the right.

2. Plot the y line (make it a solid line for $y \leq$ or $y \geq$ and a dashed line for $y <$ or $y >$

3. Shade above the line for a greater than $y >$ or $y \geq$ or below the line for a less than $y <$ or $y \leq$

4. The intersection will define the set of simultaneous ordered pair solutions.

Quadratic equation

It has the form $ax^2 + bx + c = 0 \ (a \neq 0)$

Solve a quadratic function by

a) Use of factorization or finding square roots

$(Ax+B)(\mathrm{Cx}+\mathrm{D})=0$ when $(Ax+B)=0$ or $(\mathrm{Cx}+\mathrm{D})=0$

b) Use of discriminant

Let $\Delta = b^2 - 4ac$ be discriminant

The equation has two real roots if $\Delta > 0$. These are:

$$x_1 = \frac{-b+\sqrt{\Delta}}{2a} \text{ and } x_2 = \frac{-b-\sqrt{\Delta}}{2a}.$$

It has a double root if $\Delta = 0$. These are $x_1 = x_2 = \frac{-b}{2a}$;

It has no real root if $\Delta < 0$. This means $x \notin \mathbb{R}$ *and* $S = \phi$.

2.7. Additional Information for teachers

The quadratic equation is given by$ax^2 + bx + c = 0ax^2 + bx + c = 0$,

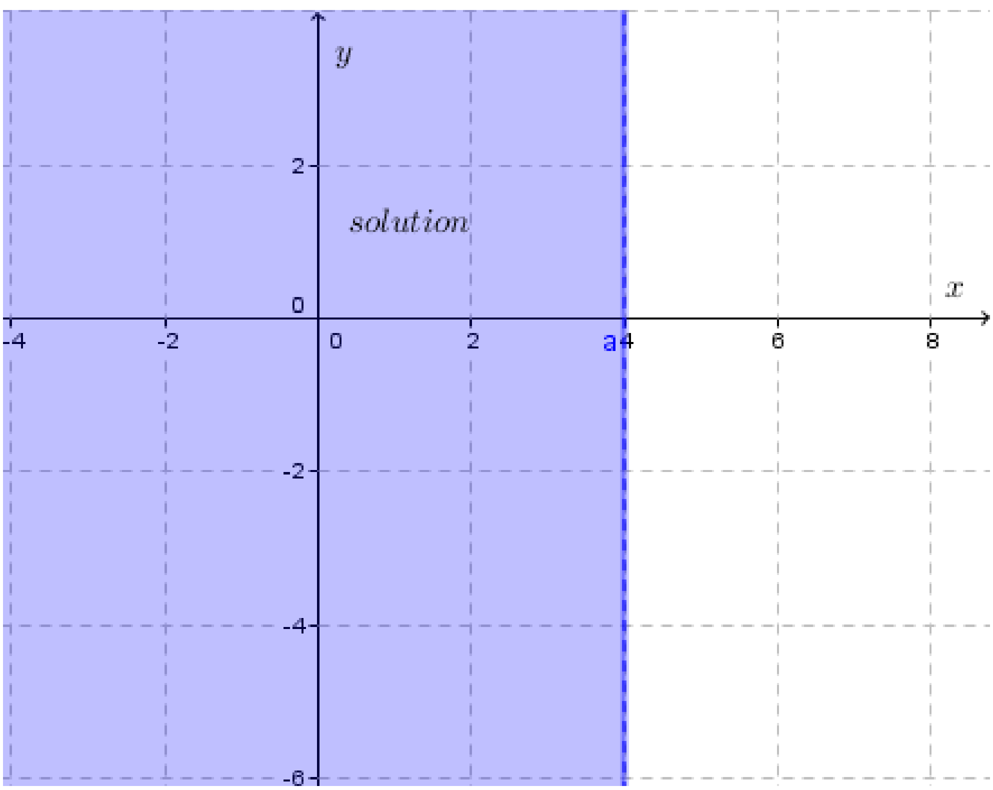
The discriminant $\Delta = b^2 - 4ac$

- if $\Delta = 0$ then, $x_1 = x_2 = \frac{-b}{2a}$ there are double roots
- if $\Delta > 0$ then, $x_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$, $x_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$ there are two real distinct roots
- if $\Delta < 0$ then, there is no real roots

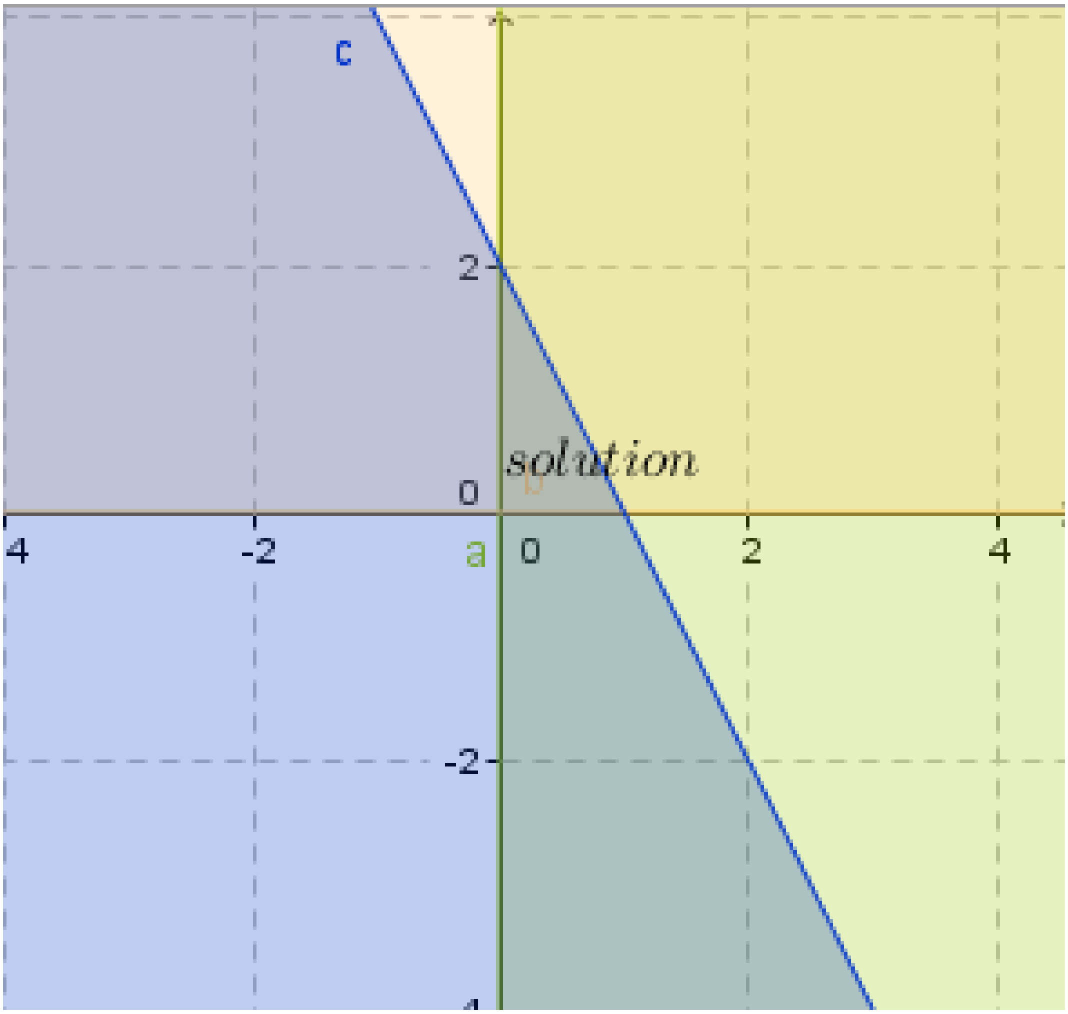
2.8 End unit assessment

1. $x = -2$ or $x = 5$

2. $x < 4$



3. $\begin{cases} x \geq 0 \\ y \geq 0 \\ 2x + y \leq 4 \end{cases}$



4. Let the width be x. Then the length $x+5$

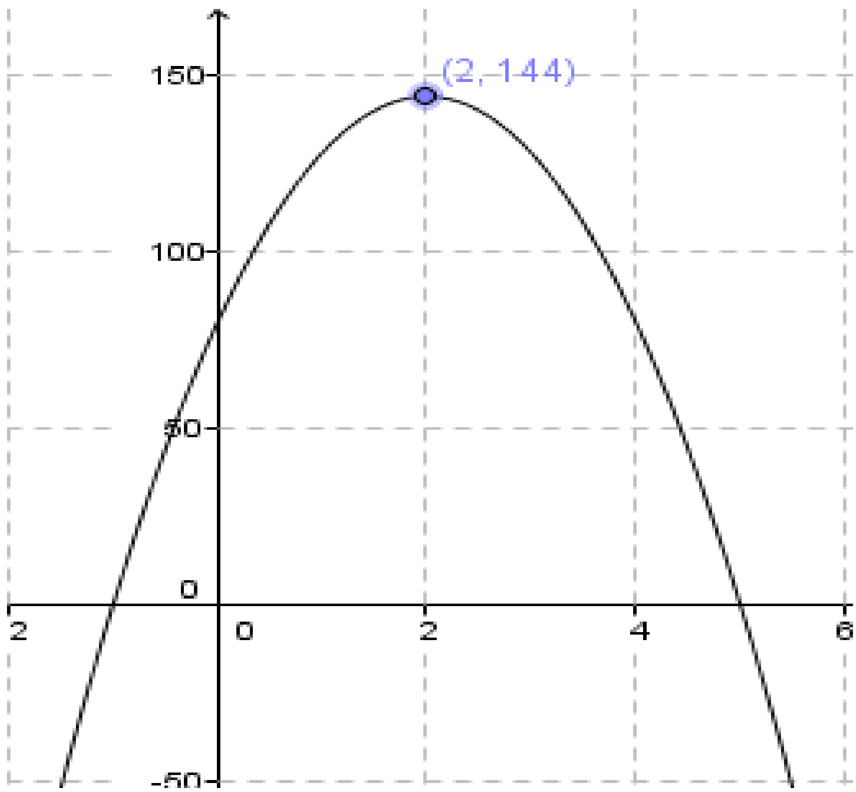
$x(x+5)=50$, $x^2+5x=50$

$x^2+5x-50=0$, $(x+10)(x-5)=0$

$x=-10$ or $x=5$ Width is $5cm$ and the length is $10cm$

The perimeter is $2(10+5)cm=30cm$

5.



$h = -t^2 + 64t + 80$

$h = -16 + 64 + 80 = 128m$

$h = -16(t^2 - 4t - 5)$

When the height is maximum, t = 2; therefore, maximum height = 144m.

When the ball hits the ground, h=0; i.e 0; $0 = -16t^2 + 64t + 80 \Rightarrow t^2 - 4t + -5 = 0$

Solve the quadratic equation then, $t = 5$ or $t = -1$

The time cannot be negative; so, the time = 5 seconds

6. Suppose the time taken by the first cyclist is t;
   Then the time taken by the other cyclist = (t-1)
   The distances travelled by them are 20t and 40(t-1) respectively.

Using Pythagoras Theorem,

$$(20t)^2 + \left[40(t-1)\right]^2 = (100)^2$$

$$400t^2 + 1600(t-1)^2 = 10000$$

$$5t^2 - 8t - 21 = 0$$

$$(5t+7)(t-3) = 0$$

$$t = 3$$

$$t = -1.4$$

Since time cannot be negative, t = 3hrs.

2.9 Additional activities

2.9.1 Remedial activities

A Company produced a product with 18000 frw as fixed costs. The variable cost is estimated to be 30% of the total revenue when it is sold at a rate of 20 frw per unit. Find the total revenue, total cost and profit functions.

Solution:

$R(x) = 20x$ .Where x is the number of units sold.

$$C(x) = 18000 + \frac{30}{100}R(x)$$

$$= 18000 + \frac{30}{100} \times 20x$$

$$= 18000 + 60x$$

$$P(x) = R(x) - C(x)$$

$$= 20x - (18000 + 6x)$$

$$= 14x - 18000$$

2.9.2. Consolidation activities

1) A small toy rocket is launched from a 4-foot pedestal. The height (*h,* in feet) of the rocket t seconds after taking off is given by the formula $h = -2t^2 + 7t + 4$ .How long will it take the rocket to hit the ground?

Solution:

$$0=-2t^2+7t+4$$
$$0=-2t^2+8t-t+4$$
$$0=-(2t+1)(t-4)$$
$$t=\frac{-1}{2}$$
$$t=4$$

The rocket will hit the ground 4 seconds after being launched

2) Solve the following equation $3x^2-2x-7=0$

Solution: $\Delta=b^2-4ac$

$$x_1=\frac{-b+\sqrt{b^2-4ac}}{2a}=\frac{2+\sqrt{88}}{6}$$
$$x_2=\frac{-b-\sqrt{b^2-4ac}}{2a}=\frac{2-\sqrt{88}}{6}$$

Then the solution set $S=\left\{\left(\frac{2-\sqrt{88}}{6},\frac{2+\sqrt{88}}{6}\right)\right\}$

2.9.3. Extended activities

1) A 3 hour river cruise goes 15 km upstream and then back again. The river has a current of 2 km an hour. What is the boat's speed and how long was the upstream journey?

Solution:

There are two speeds to think about: the speed boat makes in the water, and the speed relative to the land:

Let $x=$ the boat's speed in the water (km/h)

Let v=the speed relative to the land (km/h)

Because the river flows downstream at $2km/h$

When going up, $v=x-2$ (its speed is reduced by $2km/h$

When going downstream, $v = x + 2$ (its speed is increased by $2km/h$

$$Time = \frac{distance}{speed}$$

Total time = time upstream + time downstream = 3 hours

$$\text{Total time} = \frac{15}{(x-2)} + \frac{15}{(x+2)} = 3h$$

$$\frac{15}{(x-2)} + \frac{15}{(x+2)} = 3h$$

$$3(x-2)(x+2) = 15(x+2) + 15(x-2)$$

$$3x^2 - 30x - 12 = 0$$

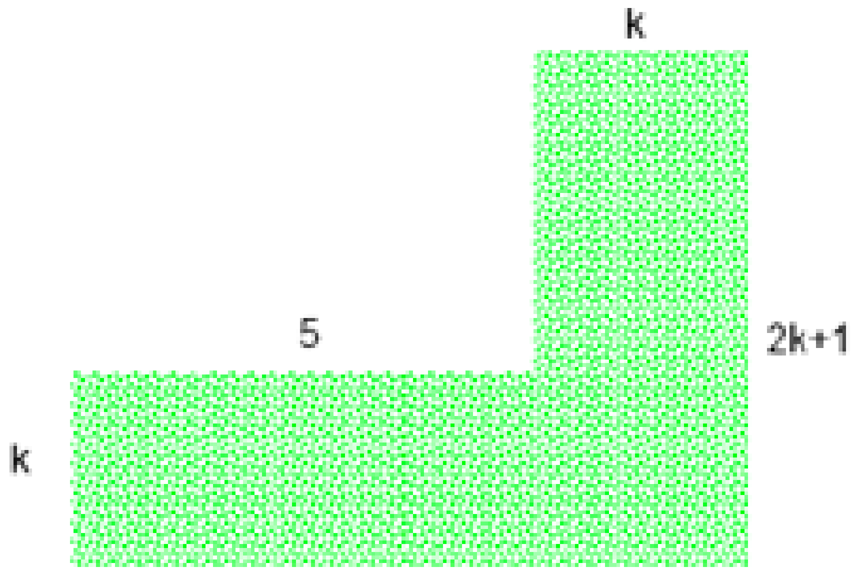
$x = -0.39$ or $x = 10.39$

Consider positive $x = 10.39$ as is more perfect then, Boat's speed is 10.39 km/h

$$\text{The upstream journey} = \frac{15}{(10.39-2)} = 1.79 hours = 1 hours 47 \min$$

$$\text{The downstream journey} = \frac{15}{(10.39+2)} = 1.21 hours = 1 hours 13 \min$$

2) The following picture shows the shape of grass patch prepared by Mary at home referring to the figures studied in primary. If the area of that patch is $80m^2$ find the value of k on that patch.



Solution:

The total area of the patch is

$$5k + k(2k+1)$$
$$= 5k + 2k^2 + k$$
$$= 2k^2 + 6k$$

Since the area is $80m^2$

$$2k^2 + 6k = 80$$
$$2k^2 + 6k - 80 = 0$$
$$(2k-10)(k+8) = 0$$
$$k = 5, k = -8$$

Since the length ca not be negative $k = 5$

UNIT: 3

GRAPHS AND FUNCTIONS

3.1 Key unit competence

Apply graphical representation of function in economics models.

3.2 Prerequisite

In this lesson, Student-teachers must be skilled in linear functions learnt in Unit 6 of S3 and unit 2 of Year 1.

3.3 Cross-cutting issues to be addressed

- Inclusive education (promote education for all while teaching);
- Peace and value Education (respect others' view and thoughts during class discussions);
- Gender (equal opportunity of boys and girls in the lesson participation);

3.4 Guidance on introductory activity

Note to the tutor: this unit should take 6 periods to gain time for teaching other units.

In groups, facilitate student-teachers to read and do the introductory activity 3 from Student -teacher's book;

- Facilitate discussions to avoid noise or other unnecessary conversation,
- Move around in the classroom to get aware of straggling Student –teachers;
- Invite group representatives to present their findings and promote gender into presentation;
- Through class discussions, let student-teachers think on different ways of getting solutions.
- Through question-answer, arouse the curiosity of student teacher on the content of unit 3.

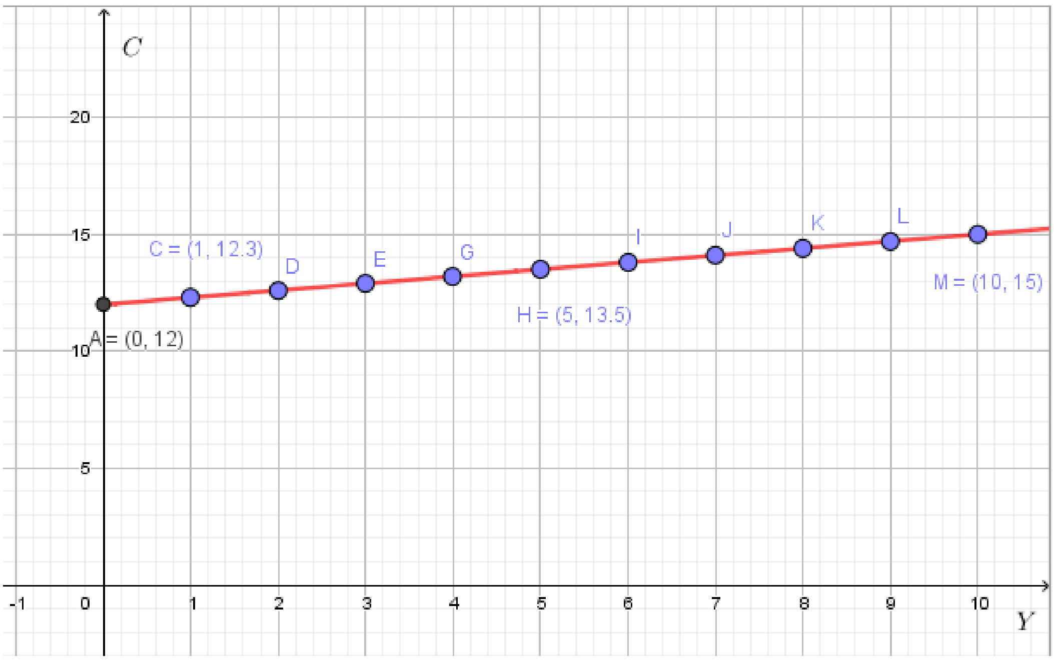
Answers for introductory activity 3

Suppose that average weekly household expenditure on food C depends on average net household weekly income Y according to the relationship $C = 12 + 0.3Y$.

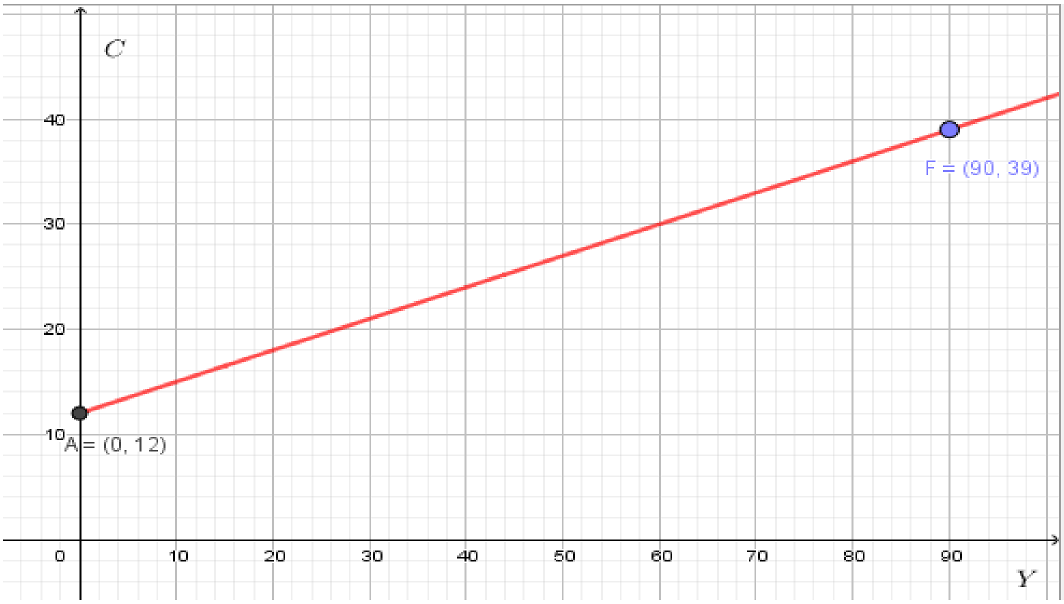
a) For every value of Y, C is a real number. This means that $\forall y \in \mathbb{R}, C \in \mathbb{R}$. The domain of $C(Y)$ is $\mathbb{R}$.

b) Table of value from Y= 0 to Y= 10 and use it to draw the graph of $C = 12 + 0.3Y$

Y	0	1	2	3	4	5	6	7	8	9	10
$C(Y)$	12	12.3	12.6	12.9	13.2	13.5	13.8	14.1	14.4	14.7	15



c) If Y=90, the value of C is $C(90) = 12 + 0.3(90) = 39$



3.5 List of lessons/sub-heading

#	Lesson title	Learning objectives	Number of periods
0	Introductory activity	To arouse the curiosity of student teacher on the content of unit 3	1
1	Generalities on numerical functions	Identify a function and recognize rules that are not functions	
2	Types of numerical functions	Differentiate the types of functions.	1
3	Domains of definition of numerical functions	Determine the domains of definition of different numerical functions	1
4	Parity of a function (odd or even).	Differentiate even functions from odd functions	1
5	Operations on functions	Perform operations on functions and use them to determine the composite functions and inverse of a function.	1
6	Graphical representation and interpretation of economics functions.	Use the properties of functions to explain different concepts of Economics and finance.	1
7	End unit Assessment		
Total number of periods			**6**

Lesson 1: Generalities on numerical functions

a) Learning objective

Identify a function and recognize rules that are not functions

b) Teaching resources

Student-teacher's book and other Reference textbooks to facilitate research, Mathematical set, calculator, manila paper, markers, pens, pencils, etc.

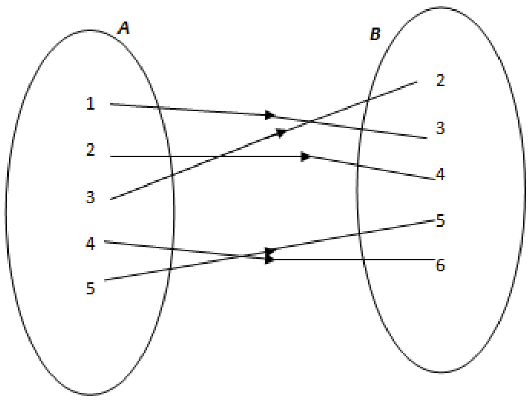
c) Prerequisites/Revision/Introduction

In this lesson, Student-teachers must be skilled in Unit 1 &3 of S1, Unit 2 of S2 and Unit 6 of S3.

d) Learning activities

- Invite student-teachers to work in group and do the activity 3.1 found in their Mathematics books;
- Move around in the class for facilitating where necessary and give more clarification on eventual challenges they may face during their work;
- Invite one member from each group with different working steps to present their work;
- As a tutor, harmonize the findings from presentation of student-teachers and guide them to specify the set of elements which have images, the set of all elements which have antecedents.
- Help them to check whether there exist an element which has more than one image.
- Guide them to explore the content and examples given in the student's book where they will be able to differentiate a function from relations and determine image for a point per a given function.
- After the lesson, guide students to do the application activity 3.1 and evaluate whether lesson objectives were achieved.

Answer for activity 3.1



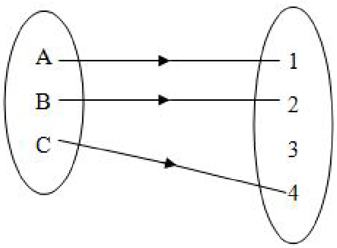
a) The set of elements of A which have images in B is $\{1;2;3;4;5\}$

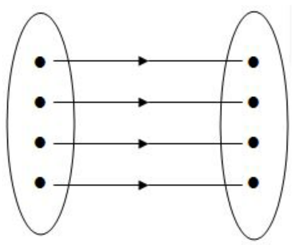
b) The set of elements in B which have antecedent in A is $\{2;3;4;5;6\}$

c) Each element of the set A has one image in the set B.

Answer of application activity 3.1

1. The 1^{st} and the 2^{nd} arrow diagrams are functions.





2. $Dom=\{a,b,c,d,e\}$ $co-domain=\{1,2,3,4,5,6\}$ $Range=\{1,2,3,4\}$

3.

$a) f(2)=8$

$b) f(-2)=0$

$c) f(d)=2d+8$

$d) f(a)=a \quad 2a+8=a$

$a=-8$

Lesson 2: Types of numerical functions

a) Learning objective

Differentiate the types of functions

b) Teaching resources

Student-teacher's book and other Reference textbooks to facilitate research, Mathematical set, calculator, Manila paper, markers, pens, pencils, etc.

c) Prerequisites/Revision/Introduction

In this lesson, Student-teachers will perform better if they revise the content on functions learnt in S2 and S3.

d) Learning activities

- Invite student-teachers to work in group and do the activity 3.2 found in their Mathematics books;
- Move around in the class for facilitating where necessary and give more clarification on eventual challenges they may face during their work;
- Invite a student from each group to present their findings;
- As a tutor, harmonize the findings from presentation and guide them to explain why they take such type of function.
- Use different probing questions and guide them to explore the content and examples given in the student's book lead them to be able to differentiate different types of functions: Constant function, Identity, Monomial, Polynomial, Rational and Irrational functions.
- Guide them to classify polynomial functions eitherby number of terms or by degrees and guide them to establish the general form of a polynomial function as $f(x)=a_nx^n+a_{n-1}x^{n-1}+a_{n-2}x^{n-2}+\ldots+a_2x^2+a_1x^1+a_0x^0$

Degree 0	Degree 1	Degree2	Degree3	Degree4	etc.
Constant polynomial function.	Linear polynomial function/1st degree polynomial function.	quadratic polynomial function/ 2nd degree polynomial function.	Cubic polynomial function/3rd degree polynomial function.	4th degree polynomials (bi-quadratic polynomial functions).	

One term	Two terms	Three terms	Four terms	Etc
Monomial function	Binomial function	Trinomial function	Qua-trinomial Function	

- After this step, guide students to do the application activity 3.2 and evaluate whether lesson objectives were achieved.

Answer for activity 3.2

Polynomial	Rational	Irrational
$f(x)=(x+1)^2$	$h(x)=\dfrac{x^3+2x+1}{x-4}$	$f(x)=\sqrt{x^2+x-2}$

d) Answers of application activity 3.2

1. $f(x)=x^3+2x^2-2$ is a polynomial function.
2. $g(x)=-2$ is a constant function.
3. $h(x)=\dfrac{x^3+2x^2-2}{x-5}$ is a rational function.
4. $f(x)=\dfrac{x^2-2}{x^2-8x+15}$ is a rational function.

Lesson 3: Domain of definition of numerical functions

a) Learning objective

Determine the domains of definition of different numerical functions.

b) Teaching resources

Student -teacher's book and other reference textbooks to facilitate research, Mathematical set, calculator, manila paper, markers, pens, pencils, graphing software such as Geogebra, etc.

c) Prerequisites/Revision/Introduction

Student-teachers should be able to explain the difference between types of functions learnt in previous lessons of year 1.

d) Learning activities

- Invite student-teachers to work in pairs and do the activity 3.3 found in their Mathematics books;
- Move around in the class for facilitating students where necessary and give more clarification on eventual challenges they may face during their work;
- Let the 2 neighbouring pairs to work together and share their works to and improve them by highlighting the set for the values obtained;
- Verify and identify groups with different working steps;
- Invite one member from each group with different working steps to present their work.
- As a tutor, harmonize the findings from presentation and guide studets to explain why they took such values.
- Use different probing questions and guide them to explore the content and examples given in the student's book and lead them to determine the domain and range for specified functions: Constant, linear, quadratic, Polynomial, Rational and Irrational functions. Note that the range will be determined only for elementary functions.
- After this step, guide students to do the application activity 3.3 and evaluate whether lesson objectives were achieved.

Answer for activity 3.3.1

1) None. It means that for all real numbers $f(x)=x^3+2x+1$ is defined.

2) $f(x)=\frac{1}{x}$ is not defined for $x = 0$. It means that for

$x=0 \Rightarrow f(0)=\frac{1}{0} \notin \mathbb{R}$ (it is impossible to divide by zero in the set of real numbers).

3) $g(x)=\frac{x+2}{x-1}$ is not defined for $x = 1$. It means that for

$x=1 \Rightarrow f(1)=\frac{3}{0} \notin \mathbb{R}$.

Answer for activity 3.3.2

1) $domf=\mathbb{R}=]-\infty,+\infty[$ in this case f(x) is defined for all real numbers.

2) $domg=\mathbb{R}=]-\infty,+\infty[$ in this case f(x) is defined for all real numbers.

3) $domh=\mathbb{R}-\{5\} or \mathbb{R}\setminus\{5\}$ in this case f(x) is defined for all real numbers except 5.

4) $domf=\mathbb{R}\setminus\{3,5\}$ in this case f(x) is defined for all real numbers except 3 and 5.

Answer of activity 3.3.3

1) $domf=\left[-\frac{1}{2},+\infty\right[$

2) $domf=\mathbb{R}=]-\infty,+\infty[$

3) $domg=]-\infty,-1[\cup[2,+\infty[$

e) Answers for application activity 3.3

1. $domf=[2,+\infty[$
2. $domg=]-\infty,-6]\cup[1,+\infty[$
3. $domh=\mathbb{R}\setminus\{-4\}$
4. $domf=]-\infty,-5[\cup]5,+\infty[$
5. $domf=\mathbb{R}=]-4,+\infty[$

Lesson 4: Parity of a function

a) Learning objective

Differentiate even functions from odd functions.

b) Teaching resources

Student-teacher's book and other Reference textbooks to facilitate research, Mathematical set, calculator, manila paper, markers, pens, pencils, gridded paper, etc.

c) Prerequisites/Revision/Introduction

In this lesson, Student-teachers must be well skilled in the content of Unit 2 of S2.

d) Learning activities

- Invite student-teachers to work in groups and do the activity 3.4 found in their Mathematics books;
- Move around in the class for facilitating students where necessary and give more clarification on eventual challenges they may face during their work;
- Verify and identify groups with different working steps;
- Invite one member from each group with different working steps to present their work.
- As a tutor, harmonize the findings from presentation and guide them to enhance the characteristics of even functions and odd functions.
- Use graphs for simple functions to illustrate the characteristics of even and odd functions: The graph of even function is symmetric about the vertical axis (the line $x = 0$ is the axis of symmetry) while the graph of odd function looks the same when rotated through half a revolution about 0 (the point $(0,0)$ is the centre of symmetry for its parts).
- Use different probing questions and guide them to explore the content and examples given in the student's book and lead them to verify the parity of different functions.
- After this step, guide students to do the application activity 3.4 and evaluate whether lesson objectives were achieved.

Answer for activity 3.4

1. $f(x)=x^2+3$

   - $f(-x)=(-x)^2+3=x^2+3$
   - $-f(x)=-(x^2+3)=-x^2-3$

   $\therefore f(-x)=f(x) \quad and \quad -f(x)\neq f(-x)$

2. $f(x)=\sqrt[3]{x^2+x}$

   $f(-x)=\sqrt[3]{(-x)^2+(-x)}=\sqrt[3]{x^2-x}$

   $-f(x)=-\sqrt[3]{x^2+x}=\sqrt[3]{-x^2-x}$

   $\therefore f(-x)\neq -f(x)$

3. $f(x)=\dfrac{x^2-3}{x^2+1}$

   $f(-x)=\dfrac{(-x)^2-3}{(-x)^2+1}=\dfrac{x^2-3}{x^2+1}$

   $-f(x)=-\dfrac{x^2-3}{x^2+1}=\dfrac{-x^2+3}{x^2+1}$

   $\therefore -f(x)\neq f(-x)$

e) Answers for application activity 3.4

1) $f(x)=2x^2+2x-3$

   - $f(-x)=2(-x)^2+2(-x)-3=2x^2-2x-3$
   - $-f(x)=-(2x^2+2x-3)=-2x^2-2x+3$

   $f(-x)\neq -f(x) and$

   $f(-x)\neq -f(x)$

$\therefore f(x)=2x^2+2x-3$ is neither odd nor even.

2) $f(x)=\dfrac{3x^3+2x^2+8}{x-5}$ is neither odd nor even.

3) $g(x)=x^3-x$

$\circ g(-x)=(-x)^3-(-x)=-x^3+x$

$\circ -g(x)=-(x^3-x)=-x^3+x$ $\quad\therefore g(x)=x^3-x$ is odd.

$g(-x)=-g(x)$

4) $h(x)=\dfrac{x^2+4}{x^2-4}$ is even function.

5) $g(x)=x(x^2+x)=x^3+x^2$

$\circ g(-x)=(-x)^3+(-x)^2=-x^3+x^2$

$\circ -g(x)=-(x^3+x^2)=-x^3-x^2$ $\quad\therefore g(x)=x(x^2+x)$ is neither odd nor even.

$g(-x)\neq -g(x)$ *and*

$g(-x)\neq -g(x)$

Lesson 5: Operations on the numerical functions

a) Learning objective

Perform operations on functions and use them to determine the composite functions and inverse of a function.

b) Teaching resources

Student-teacher's book and other Reference textbooks to facilitate research, Mathematical set, calculator, Manila paper, markers, pens, pencils, etc.

c) Prerequisites/Revision/Introduction

In this lesson, student-teachers must be skilled in unit 6 of S3.

c) Learning activities

- Invite student-teachers to work in groups and do the activity 3.5.1 found in their Mathematics books;
- Move around in the class for facilitating students where necessary and ask some challenging questions to lead them to work correctly;
- Verify and identify groups with different working steps;
- Invite one member from each group with different working steps to

present their work.

- As a tutor, harmonize the findings from presentation and guide them to enhance the the operation of functions.
- Use different probing questions and guide them to explore the content and examples given in the student's book and lead them to explore the addition, multiplication and division of functions.
- Invite students to work on the activity 3.5.2 and the activity 3.5.3.
- Move to every group and verify their working steps.
- Invite all students for a whole class discussion and guide them to establish how to determine the composite of functions and the inverse of a function;
- Use different probing questions and guide them to explore the content and examples on the composite of functions and the inverse of a function as it is given in the student's book;
- After this step, guide students to do the application activity 3.5 and evaluate whether lesson objectives were achieved.

Answer for activity 3.5.1

1) $f(x)=\frac{x+1}{2x-3}; g(x)=x+1$

$$f(x)+g(x)=\left(\frac{x+1}{2x-3}\right)+\left(\frac{x+1}{1}\right)=\frac{(x+1)+(x+1)\cdot(2x-3)}{2x-3}$$

$$=\frac{x+1+2x^2-3x+2x-3}{2x-3}=\frac{2x^2-3}{2x-3}; x\neq\frac{3}{2}$$

2) $f(x)=\frac{x+1}{2x-3}; g(x)=x+1$

$$f(x)-g(x)=\left(\frac{x+1}{2x-3}\right)-\left(\frac{x+1}{1}\right)=\frac{(x+1)-(x+1)\cdot(2x-3)}{2x-3}$$

$$=\frac{x+1-(2x^2-x-3)}{2x-3}=\frac{-2x^2+2x+4}{2x-3}; x\neq\frac{3}{2}$$

3) $f(x)=\frac{x+1}{2x-3}; g(x)=x+1$

$$f(x)\cdot g(x)=\left(\frac{x+1}{2x-3}\right)\left(\frac{x+1}{1}\right)=\frac{(x+1)\cdot(x+1)}{2x-3}=\frac{x^2+2x+1}{2x-3}; x\neq\frac{3}{2}$$

4) $f(x)=\frac{x+1}{2x-3}; g(x)=x+1$

$$\frac{f(x)}{g(x)}=\left(\frac{\frac{x+1}{2x-3}}{\frac{x+1}{1}}\right)=2x-3$$

Answer for activity 3.5.2

1) $f(x)=3x+2; g(x)=x^2-1$

$f(g(x))=3(x^2-1)+2=3x^2-3+2=3x^2-1$

2) $f(x)=3x+2; g(x)=x^2-1$

$g(f(x))=(3x+2)^2-1=9x^2+12x+3$

3) $f(g(x))\neq g(f(x))$

Answer for activity 3.5.3

1) $y=x+1 \Rightarrow x=y-1$

2) $y=3x-2 \Rightarrow x=\frac{y+2}{3}$

3) $y=\frac{-x+3}{2x-1} \Rightarrow x=\frac{3+y}{2y+1}$

e) Answers of application activity 3.5

1) $(f+g)(x)=2x^3+8x-5$

2) $f.g(x)=6x^5-13x^4+23x^3-30x^2+30x-12$

3) a) $f^{-1}(x)=\frac{x-2}{5}$

b) $f^{-1}(x)=\frac{-x-2}{7}$

c) $f^{-1}(x)=\frac{2x+1}{x+2}$

4) a) $f\circ g(x)=-3;\quad g\circ f(x)=2$

b) $f\circ g(x)=72x^2+6x-3;\quad g\circ f(x)=12x^2+6x-18$

Lesson 6: Graphical representation and interpretation of economics functions

a) Learning objective

Use the properties of functions to explain different concepts of Economics and finance.

b) Teaching resources

Student-teacher's book and other Reference textbooks to facilitate research, Mathematical set, calculator, Manila paper, markers, pens, pencils, etc.

c) Prerequisites/Revision/Introduction

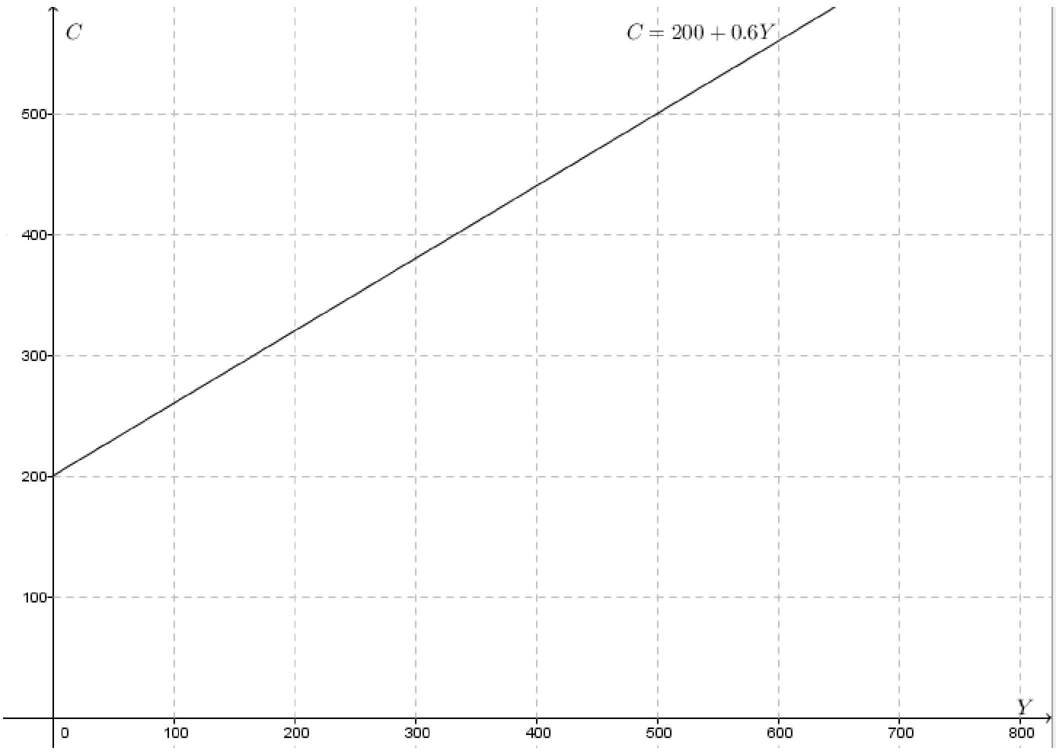
In this lesson, student-teachers must be skilled in Unit 6 of S3 and all lessons above in this unit.

d) Learning activities

- Invite student-teachers to work in group and do the activity 3.6 found in their Mathematics books;
- Move around in the class for facilitating groups where necessary and ask some guiding questions on eventual challenges they may face during their work;
- Invite groups with different working steps to present their work;
- As a tutor, harmonize the findings from presentation of student-teachers and guide them to relate the solved equation with usual equations with y and x variables.
- Invite them to brainstorm on other application of functions they have observed in economics or finance;
- Guide them to explore the content and examples given in the student's book where they will be able to explore the following: Price as function of quantity supplied, Consumption as function of income, Price as function of quantity demanded, Point elasticity of demand, The Cost Function, The Profit Function, The Marginal Cost, Marginal Revenue, and Marginal Profit and Equilibrium Price and Quantity.
- After the lesson, guide students to do the application activity 3.6 and evaluate whether lesson objectives were achieved.

Answer for activity 3.6

The function is $C = 200 + 0.6Y$



The point at which the line cuts the vertical axis $(0, 200)$.

e) Answers for Application activity 3.6

1. a) $C(Y) = a + bY$

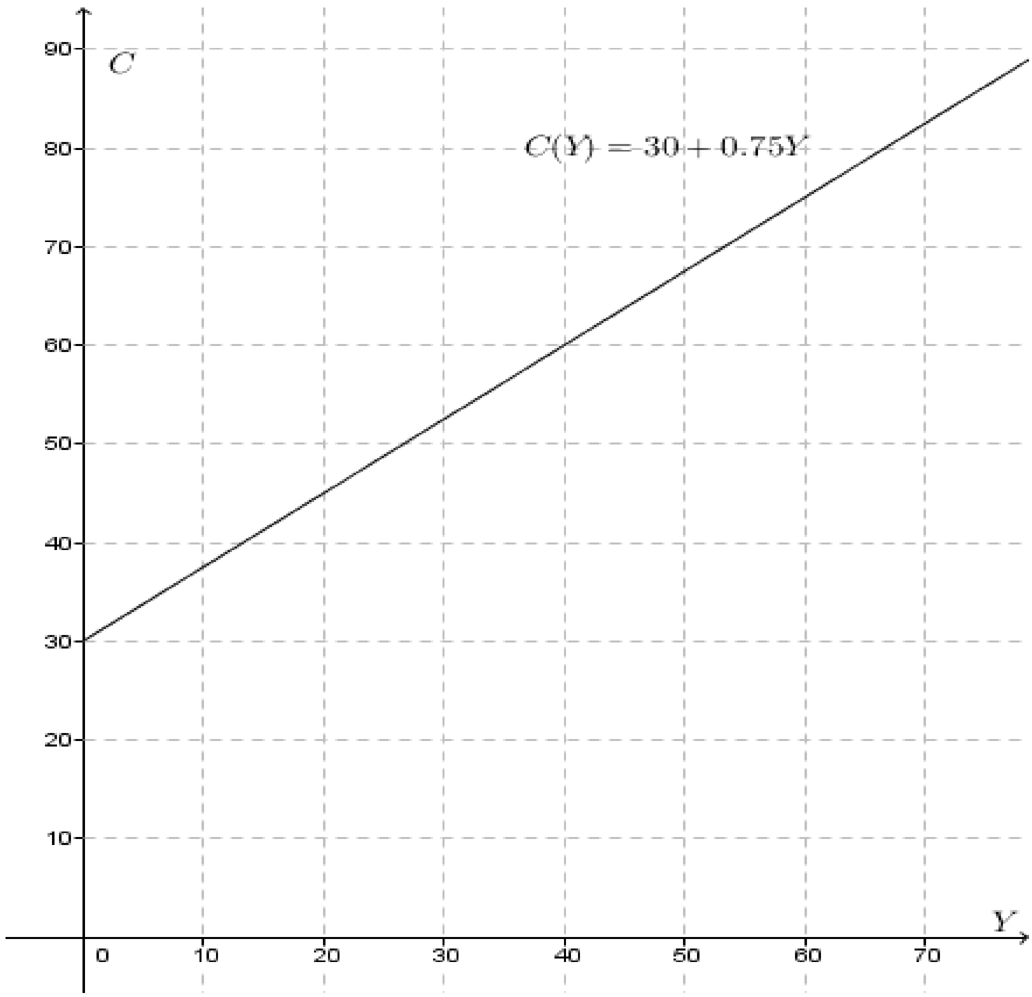
$$\begin{cases} 60 = a + 40b(\times -1) \\ 90 = a + 80b(\times 1) \end{cases}$$

$$\begin{cases} -60 = -a - 40b \\ 90 = a + 80b \end{cases}$$

$$30 = 40b \Rightarrow b = \frac{3}{4} = 0.75$$

$$a = 30$$

b)

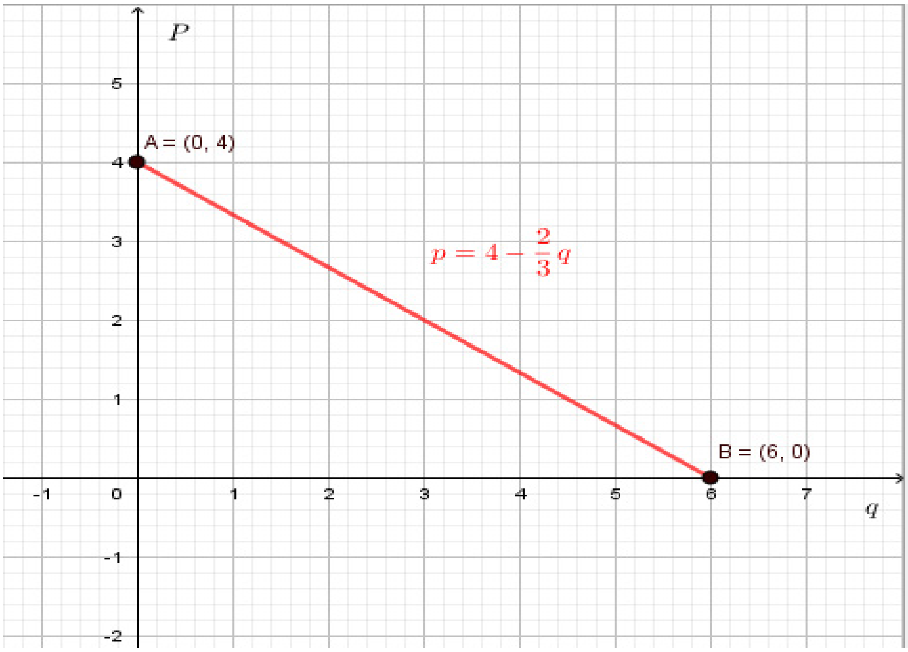


2) a) $f(12) = 60$ means that when the product is 12 units, the price is 60 units of money.

b) Normally the price increases when the quantity is reducing.

3) $75p + 50q = 300$

$$p = 4 - \frac{2}{3}q$$



It is clear that the Vertical intercept is $p = 4$ dollars and the horizontal intercept is $q = 6$ units.

3.6. Summary of the unit

Function

A function is a rule that assigns to each element in a set *A* one and only one element in set *B*. We can even define a function as any relationship which takes one element of one set and assigns to it one and only one element of second set

Types of functions:

There are: constant function, Identity, Monomial, Polynomial function, Rational function and Irrational function.

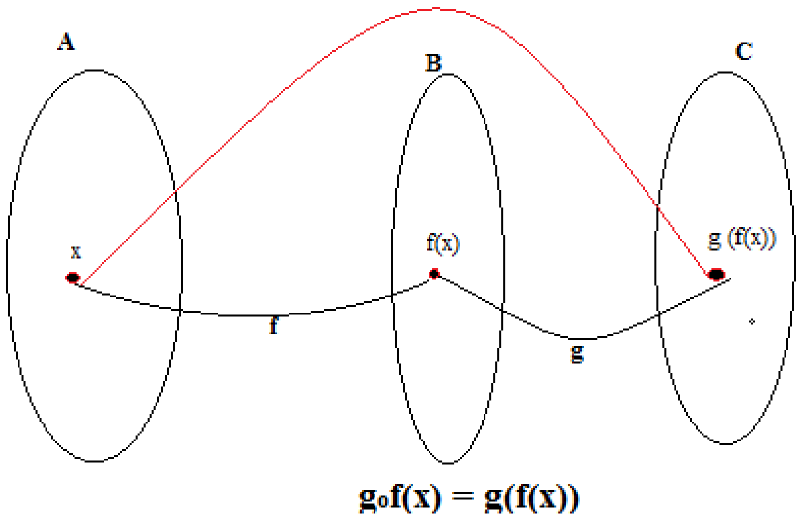
Parity of a function (odd or even

Even function: $f(-x) = f(x)$

Odd function: $f(-x) = -f(x)$

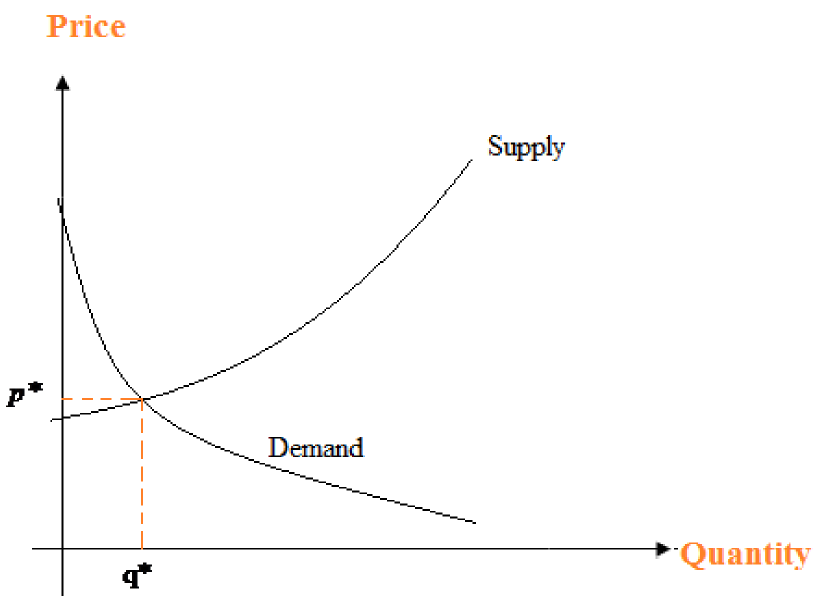
Composite function

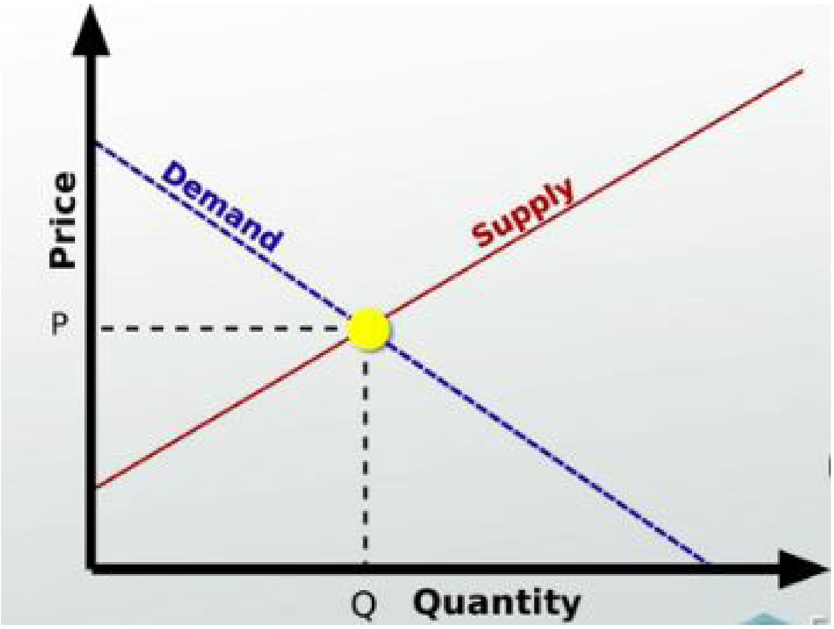
This composite function of f(x) and g(x) is written $(g \circ f)(x)$ or $g[f(x)]$



Equilibrium Price and Quantity

If we plot the supply and demand curves on the same axes, the graphs cross at the *equilibrium point*. The values p^* and q* at this point are called the *equilibrium price* and *equilibrium quantity*, respectively. It is assumed that the market naturally settles to this equilibrium point.





3.7. Additional Information for Teachers

Be careful!

- Emphasize the use of graph paper/grided paper while student-teachers draw the graphs.
- Emphasize and facilitate students to use of geometric materials to ameliorate the quality of graphs.
- Remind students to name axes (x-axis and y-axis).
- Recall them to mention/highlight the origin/intersection point of axes by 0.

3.8. Answer for end unit assessment 3

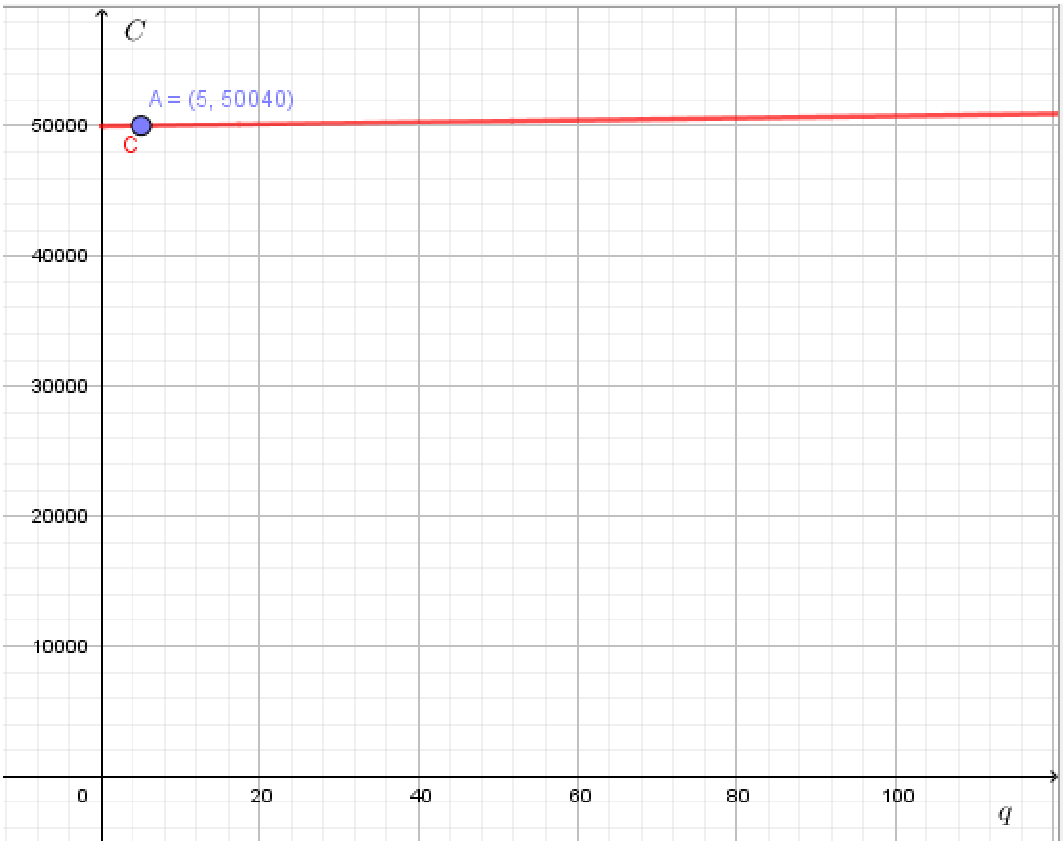
The total cost C for units produced by a company is given by $C(q) = 50000 + 7q$ where q is the number of units produced.

Solution:

a) The amount 50000 represents the fixed cost;

b) the number 8 represents the the marginal cost(cost of a unit of product);

c) The graph for $C(q) = 50000 + 7q$ is the following:



d) The real domain of C that corresponds to q which is positive is $[0;\infty[$.

The range is $[50000;\infty[$

e) C(q) is not an odd function because $C(-q) \neq -C(q)$.

3.9. Additional activities

1) A small ball is projected vertically upward from the top of a building with the initial velocity $v_0 = 144\ m/\sec$. Its distance $s(t)$ in meter above the ground after t seconds is given by the equation $s(t) = -16t^2 + 144t + 100$.

a) What is the distance $s(t)$ at the initial time when $t = 0$?

b) Make a table of variation for $s(t)$ to show the distance from the initial time t=0 to t=10 seconds.

c) Use the table to draw the graph of $s(t)$ and show the position of the ball at t = 5 seconds

d) Discuss the parity of the function $s(t)$.

Solution:

$s(t) = -16t^2 + 144t + 100$ represents the height or the vertical distance above the ground.

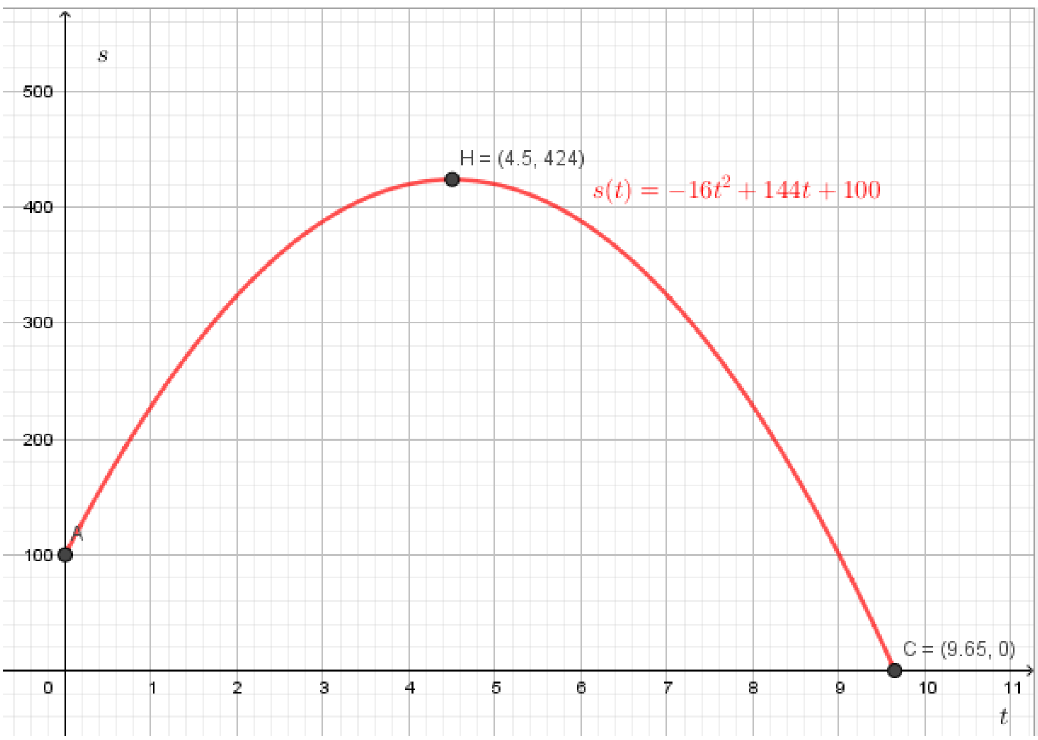
a) At the initial time $t = 0$, then the distance is

$s(0) = -16(0) + 144(0) + 100 = 100$

This distance is 100m.

b)

t	0	1	2	3	4	5	6	7	8	9	$\frac{483}{50}$
$s(t)$	100	228	324	388	420	420	388	324	328	100	0



The graph shows that the ball falls on the ground between $t = 9$ and $t = 10$.

To find this time, you must solve the equation $s(t) = -16t^2 + 144t + 100 = 0$

· $t = \frac{483}{50}$

2) Let W and Z be sets;

$W \times Z$ the Cartesian product of W and Z ;

C be the set of all correspondences (relations) from W to Z ;

F be the set of all functions from W to Z ;

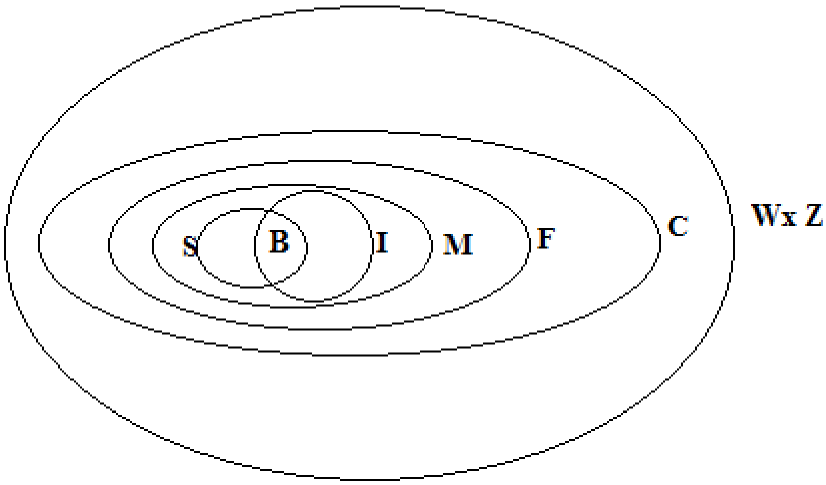
M be the set of all mappings from W to Z ;

I be the set of all one to one mappings from W to Z ;

S be the set of all onto mappings from W to Z ;

B be the set of all bijective mappings from W to Z ;

Then we have the following sequence of inclusion of sets



Using examples, explain in your own words the relationship amongst these sets decide on the following inclusion: $S \subset M \subset F \subset C \subset (W \times Z)$.

Solution:

The following inclusion is true: $S \subset M \subset F \subset C \subset (W \times Z)$.

A bijection is a type of mapping which is at the same time a surjective (S) and injective (I) mapping.

All mappings are functions f(x) for which every point x has an image.

All functions are types of correspondences for which you cannot observe a point which has more than one image.

UNIT: 4

LIMITS OF FUNCTIONS

4.1 Key unit competence

Evaluate correctly limits of functions and apply them to solve related problems

4.2 Prerequisite:

Student-teachers will perform well in this unit if they are skilled enough in the content of second unit (Equations and Inequalities) of mathematics in student-teacher's book for year one Social Studies Education.

4.3 Cross-cutting issues to be addressed:

- Inclusive education (promote education for all while teaching)
- Peace and value Education (respect others view and thoughts during class discussions)
- Gender (equal opportunity of boys and girls in the lesson participation)

4.4 Guidance on introductory activity

- In groups, facilitate student-teachers to read and do the introductory activity 4 from Student -teacher's book;
- Facilitate discussions to avoid noise or other unnecessary conversation,
- Move around in the classroom to get aware of straggling Student –teachers and provide assistance where necessary;
- Invite group representatives to present their findings and promote gender into presentation;
- Through class discussions, let student-teachers think on different ways of getting solutions.
- Through question-answer, arouse the curiosity of student teacher on the content of unit 4.

Answers for introductory activity

1) As $P = 200 - 0.24Q$, then

Q	1	0.5	0.1	0.01	0.001	0.0001	…	0
P	200	200	200	200	200	200		200

When Q approaches 0, the price gets closer and closer to 200.

2) The values of P when Q approaches 20 are given in the table below:

Q	19.5	19.9	19.9999	20	20.1	20.2	20.5	21
P	200	200	195.20024	195.2	195.176	195.152	195.08	194.96

When Q approaches 20, the price gets "closer and closer" to 195.2. This can be written as $\lim_{Q\to 20} P = \lim_{Q\to 20}(200 - 0.24Q) = 195.2$

4.5 List of lessons

#	Lesson title	Learning objectives	Number of periods
1.	Introductory activity	Arouse the curiosity of student-teacher on the content of unit 4	1
2.	Concepts of limits	Define the concept of limit for a real-valued function, the neighbourhood of a real number and the value of a function at a given point of one real variable.	1
3.	Limit of a variable (Conditions, One-sided limits)	Evaluate the limit of a function at a given point.	2
4.	Limits of functions at infinity and involving infinity, indeterminate cases	Perform operations on limits involving infinity.	2
5.	Graphical interpretation of limit of a function	Guess the limit of a function on its graphical representation	1
6.	Applications of limits in mathematics: Continuity of a function, Asymptotes	Extend the concept of limit to determine the asymptotes of a given function	2

7.	Applications of limits in real life: Solving Problems involving limits in real life	solve problems involving limits in real life	2
8	End unit assessment		1
Total			**12**

Lesson 1: Concepts of limits

Learning objective:

Define the concept of limit for a real-valued function, the neighbourhood of a real number and the value of a function at a given point of one real variable.

a) Teaching resources:

Student-teacher's book and other reference textbooks to facilitate research, Mathematical set, calculator, Manila paper, graph paper, ruler, markers, pens, pencils, etc.

b) Prerequisites/Revision/Introduction:

Student-teachers will perform well in this unit if they are skilled enough in the content of the second unit (Equations and Inequalities) of mathematics in student-teacher's book for year one social studies education.

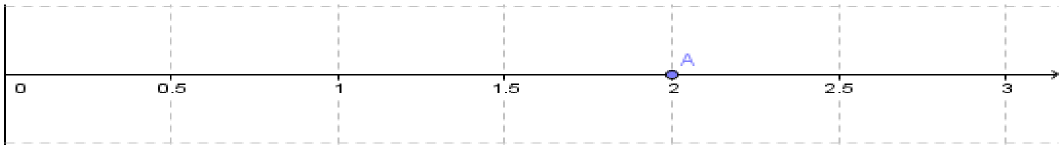
c) Learning activities

- Invite student-teachers to work in group and do the activity 4.1.1 and activity 4.1.2 found in their Mathematics books;
- Move around in the class for facilitating where necessary and give more clarification on eventual challenges they may face during their work;
- Invite one member from each group with different working steps to present their work;
- As a tutor, harmonize the findings from presentation of student-teachers and guide them to explore the content and examples given in the student's book where they will be able to differentiate the neighbourhood of a real number and the value of a function at a given point.

- After the lesson, guide students to do the application activity 4.1 and evaluate whether lesson objectives were achieved.

Answer for activity 4.1.1

a.



b. Open intervals: $]0,4[$, $]1.9,2.1[$, $]1.99,2.01[$ and $]1,3[$

c) The values given in the table show that when $x=2.00001$ or $x=1.99999$ approaches 2, $f(x)=3.99998$ or $f(x)=4.00001$ approaches the value 4 (i.e $f(x)=4$).

Answers to activity 4.1.2

1) $f(2)=\frac{3}{4}$; 2) $f(1)=2$; 3) $f(x)=80$

Answer of Application activity 4.1

1) Sutudents may give different answers, verify if they are correct.

Examples:

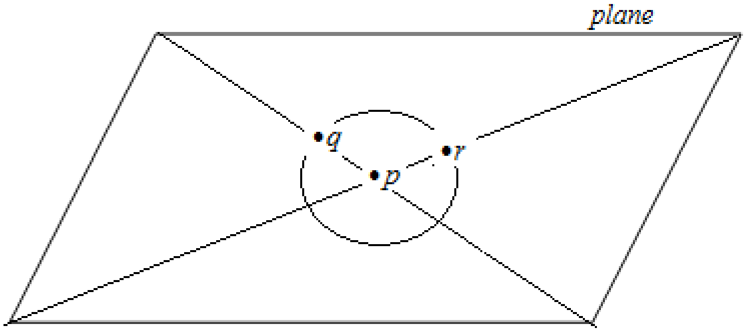
San Marino, a state surrounded by **Italy.**

Vatican City, a state forming part of **Rome,** thereby surrounded by **Italy.**

2) There are many possible answers verify that -5 is the center of the interval. For example: $(-6,-4)$, $(-7,-3)$, $(-10,0)$.

3) No, the circle is a neighborhood of it center but not for its points.

4) The following plane is a neighborhood of points p, q and r.



Lesson 2: Limit of a function and conditions of existence

a) Learning objective:

Evaluate the limit of a function at a given point.

b) Teaching resources

Student-teacher's book and other textbooks to facilitate research, Mathematical set, calculator, manila paper, markers, pens, pencils, etc.

c) Prerequisites/Revision/Introduction

In this lesson, Student-teachers will perform better if they are enough skilled in unit 3 (Graphs and functions) of mathematics for social studies education.

d) Learning activities

- Invite student-teachers to work in groups and do the activity 4.2 found in their Mathematics books;
- Move around in the class for facilitating where necessary and give more clarification on eventual challenges they may face during their work;
- Invite a student from each group to present their findings;
- As a tutor, harmonize the findings from presentation and guide them to guess the true definition of limit of a function at a given point basing on the value of the left hand limit and the right hand limit.
- Use different probing questions and guide them to explore the

content and examples given in the student's book lead them to be able to determine the limit of a function at a point when applying related properties.

- After this step, assign students to do the application activity 4.2 and evaluate whether the lesson objectives were achieved.

Answers for activity 4.2

a. $f(x)=\begin{cases} 0 \text{ for } x<0 \\ 1 \text{ for } x \geq 0 \end{cases}$ $\qquad g(x)=\begin{cases} 0 \text{ for } x \leq 1 \\ 1 \text{ for } x>1 \end{cases}$

b. If we stay to the left side, as x approaches 0, $f(x)$ gets closer to 0

c. If we stay to the right side, as x approaches 0, $f(x)$ gets closer to 1

d. If we stay to the left side, as x approaches 1, $g(x)$ gets closer to 0

e. If we stay to the right side, as x approaches 1, $g(x)$ gets closer to 1

e) Answers for application activity 4.2

1. 0

2. a) 3 b) -5832

3. a) $\infty-\infty$ is indeterminate form not zero

b) $\lim\limits_{x \to 0^+}\left(\frac{1}{x}-\frac{1}{x^2}\right)=\lim\limits_{x \to 0^+}\frac{x-1}{x^2}$

x	$-\infty$		0		1		$+\infty$
$x-1$		-		-	**0**	+	+
x^2		+	**0**	+		+	
$\frac{x-1}{x^2}$		- -	‖	- -	**0**	+	+

Hence, $\lim\limits_{x \to 0^+}\left(\frac{1}{x}-\frac{1}{x^2}\right)=-\infty$

4 a) *If* $f(x)=\begin{cases} x^2-2x+1, & x\neq 3 \\ 7 & ,x=3 \end{cases}$, $\lim_{x\to 3} f(x)$ =7.

b) *If* $h(x)=\begin{cases} x^2-x-1, & x<3 \\ 3x-5 & , x\geq 3 \end{cases}$, $\lim_{x\to 2} h(x)$ =1;

c) *If* $g(x)=\begin{cases} x, & x\leq 0 \\ x^2, & x>0 \end{cases}$; $\lim_{x\to 0} g(x)$ =0;

d) *If* $h(x)=\begin{cases} 1, & x>1 \\ 3, & x\leq 1 \end{cases}$, $\lim_{x\to 1} h(x)$ Doesn't exist

Lesson 3: Limits of functions at infinity and involving infinity, indeterminate cases

a) Learning objective:

Perform operations on limits involving infinity.

b) Teaching resources

Student-teacher's book and other reference textbooks to facilitate research, Mathematical set, calculator, manila paper, markers, pens, pencils, etc.

c) Prerequisites/Revision/Introduction

Student-teachers will perform well in this unit if they have good background in lesson two of this unit.

d) Learning activities

- Invite student-teachers to work in groups and do the activity 4.3.1 found in their Mathematics books;
- Move around in the class for facilitating students where necessary and ask some challenging questions to lead them to work correctly;
- Verify and identify groups with different working steps;
- Invite one member from each group with different working steps to present their work.
- As a tutor, harmonize the findings from presentation and guide them to enhance the limits involving infinity and the operations on limits in general.

- Use different probing questions and guide them to explore the content and examples given in the student's book and lead them to explore the addition, multiplication and division on limits.
- Invite students to work on the activity 4.3.3 and the activity 4.3.4
- Move to every group and verify their working steps.
- Invite all students for a whole class discussion and guide them to guess how to move out the indeterminate cases;
- Use different probing questions and guide them to explore the content and examples related to limits of functions at infinity and involving infinity, indeterminate cases as it is given in the student's book;
- After this step, guide students to do the application activity 4.3 and evaluate whether lesson objectives were achieved.

Answers for activity 4.3.1

1. a. -65.6 b. -99

   c. -199 d. 201

   e. 101 f. 67.6

2. a. $+\infty$ b. $-\infty$

   c. indeterminate d. $-\infty$

   e. $-\infty$ f. $+\infty$

   g. indeterminate

Answers for activity 4.3.3

a.
$$f(x)=\frac{x^2-1}{x-1}=\frac{(x-1)(x+1)}{x-1}$$

b.
$$f(x)=\frac{x^3+x^2-5x-2}{x^2-4}=\frac{x^3-2x^2+3x^2-6x+x-2}{(x-2)(x+2)};x\neq-2,x\neq2$$
$$=\frac{x^2(x-2)+3x(x-2)+(x-2)}{(x-2)(x+2)}=\frac{(x-2)(x^2+3x+1)}{(x-2)(x+2)}$$

Answers for Activity 4.3.4

a)The conjugate of $f\left(x\right)=\sqrt{x^2-2}+3$ is $f(x)=\dfrac{(\sqrt{x^2-2}+3)(\sqrt{x^2-2}-3)}{(\sqrt{x^2-2}-3)}$

b) The conjugate of $f\left(x\right)=\dfrac{\sqrt{x-2}-1}{x-3}$ is $f(x)=\dfrac{(\sqrt{x-2}-1)(\sqrt{x-2}+1)}{(x-3)(\sqrt{x-2}+1)}$

e) Answers for application activity 4.3

$$\lim_{x\to\infty}\frac{3x^3+2x^2-1}{6x^3+x+4}$$

$$\lim_{x\to-\infty}\frac{\left(x+3\right)^2}{x^3+4x^2-8x-4}$$

Lesson 4: Graphical interpretation of limit of a function

a) Learning objective:

Guess the limit of a function on its graphical representation.

b) Teaching resources:

Student-teacher's book and other reference textbooks to facilitate research, Mathematical set, calculator, Manila paper, markers, pens, pencils, etc.

c) Prerequisites/Revision/Introduction:

Student-teachers will perform well in this unit if they have good background in previous lessons of this unit.

d) Learning activities

- Invite student-teachers to work in groups and do the activity 4.4 found in their Mathematics books;
- Move around in the class for facilitating groups where necessary and ask some guiding questions on eventual challenges they may face during their work;
- Invite groups with different working steps to present their work;
- As a tutor, harmonize the findings from presentation of student-

teachers and guide them to predict the limit they observe on a graph of function.

- Guide them to explore the content and examples given in the student's book where they will be able to explore the limits of functions using their graphical representations.
- After this step, guide students to do the application activity 4.4 and evaluate whether lesson objectives were achieved.

Answers for activity 4.4

1) $\lim_{x\to 1^-} f(x)=1$

2) $\lim_{x\to 1^+} f(x)=2$

3) The limit of $\lim_{x\to 1} f(x)$ doesn't exist.

e) Answers for Application activity 4.4,

Question 1

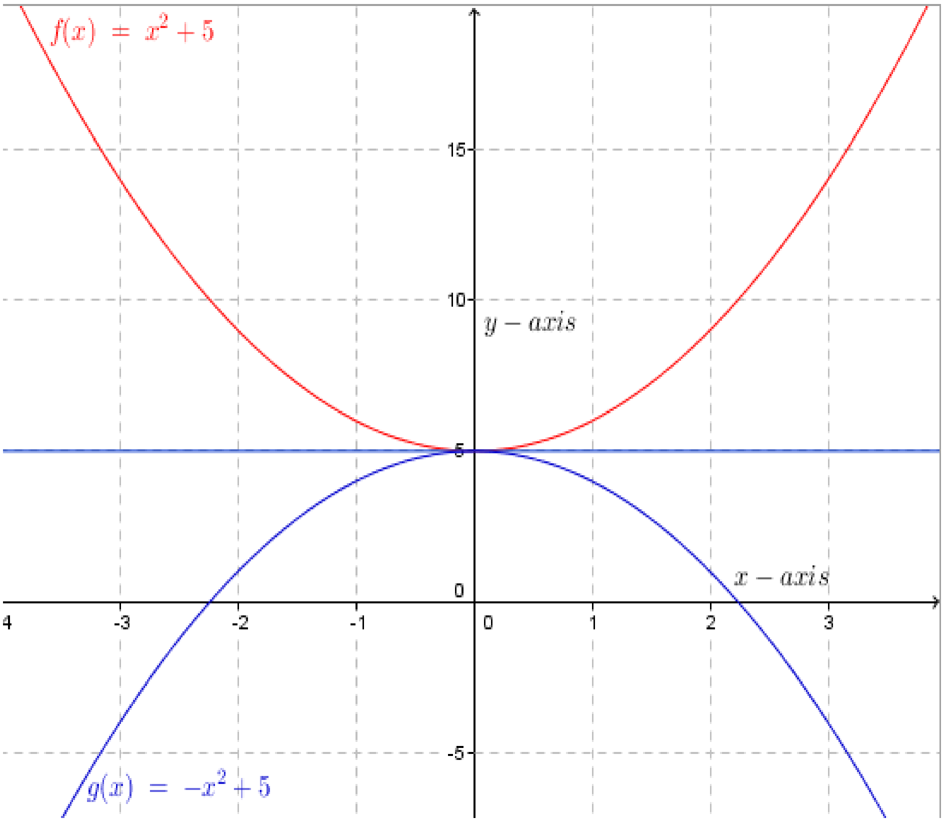
The limit $\lim_{x\to 2} f(x)$ is 4 .

Question 2:

$\lim_{x\to -1} h(x)$ doesn't exist, $\lim_{x\to 1} h(x)$ doesn't exist,

$\lim_{x\to -\infty} h(x)=1$ $\lim_{x\to \infty} h(x)=1$

2)



The curve of $h(x) = 5$ lies between other two curves and the three curves meet at the same point $(0,5)$

Therefore, $\lim\limits_{x\to 0} f(x) = \lim\limits_{x\to 0} g(x) = \lim\limits_{x\to 0} h(x) = 5$

3) (a) $\lim\limits_{x\to 0}[3(3x-1)]$ = -3; $\lim\limits_{x\to 0}[3(3x-1)]$= -3

(b) $\lim\limits_{x\to 0}(x^2)$ =0, $\lim\limits_{x\to 0}(x^2+3x-1)$ = -1; $\lim\limits_{x\to 0}(x^2+3x-1)$ =-1

(c) $\lim\limits_{x\to 1}(x^2+3x-6)$, = -2; $\lim\limits_{x\to 1}(x+4)$ =5; $\lim\limits_{x\to 1}\dfrac{x^2+3x-6}{x+4} = -\dfrac{2}{5}$

(d) $\lim\limits_{x\to 2}(x-1) = 1$; $\lim\limits_{x\to 2}(x+4) = 6$; $\lim\limits_{x\to 2}(x^2+3x-4)$=6

(e) $\lim\limits_{x\to -4}\left[(x^2+1)^2\right]$ = 289; $\left[\lim\limits_{x\to -4}(x^2+1)\right]^2$ =289

Lesson 5: Applications of limits in mathematics: Continuity of a function, Asymptotes

a) Learning objective:

Extend the concept of limit to determine the asymptotes of the given function

b) Teaching resources:

Student-teacher's book and other Reference textbooks to facilitate research, Mathematical set, calculator, manila paper, graph papers, markers, pens, pencils, etc.

c) Prerequisites/Revision/Introduction:

Student-teachers will perform well in this unit if they have good background in previous lessons of this unit.

d) Learning activities

- Invite student-teachers to work in group and do the activity 4.5 found in their Mathematics books;
- Move around in the class for facilitating groups where necessary and ask some guiding questions on eventual challenges they may face during their work;
- Invite groups with different working steps to present their work;
- Guide them to explore the content and examples given in the student's book where they will be able to evaluate the continuity of limits toward a point .
- After the lesson, guide students to do the application activity 4.5 and evaluate whether lesson objectives were achieved.

Answer for activity 4.5.1

1. a) 4

   b) 4

   c) $f(2)$ and $\lim_{x\to 2} f(x)$ exist and are equal, $f(2)=\lim_{x\to 2} f(x)$

e) Answers of application activity 4.5.1

1. The function is not continuous at $x=-3$ and $x=5$

2. $k=6$ **,** the value of ***k*** is 6 for which the function is continuous at $x = 3$.

3. $a=-1, b=1$

Answers of activity 4.5.2

As x increases or decreases the curve comes closer and closer to the line B. As x approaches 3 from the right or from the left, the curve comes closer and closer to the line A.

Answers for application activity 4.5.2

1. Horizontal asymptote: $HA \equiv y=1$

   Vertical asymptotes: $VA \equiv x=-1$ and $VA \equiv x=0$

2. Horizontal asymptote: $HA \equiv y=0$

3. Vertical asymptote: $VA \equiv x=\frac{9}{4}$

Oblique asymptote: $OA \equiv y=\frac{1}{4}x+\frac{21}{16}$

Lesson 6: Applications of limits in real life: Solving Problems involving limits

a) Learning objective

Use the properties of limits to Solve real life problems involving limits.

b) Teaching resources:

Student-teacher's book and other reference textbooks to facilitate research, Mathematical set, calculator, Manila paper, markers, pens, pencils, etc.

c) Prerequisites/Revision/Introduction:

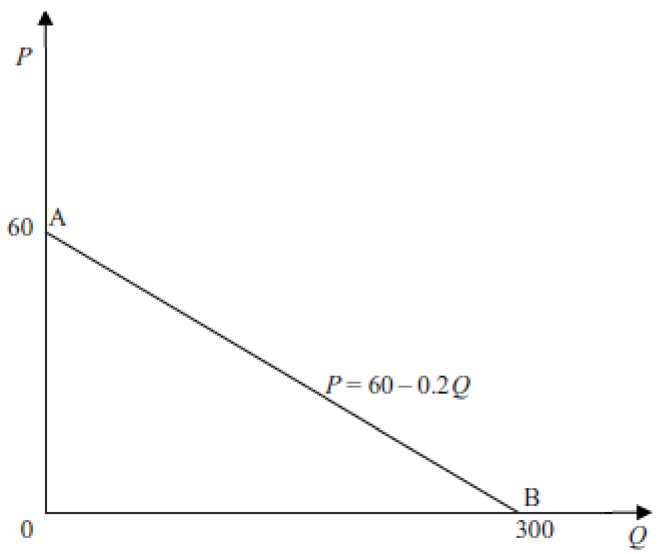
Student-teachers will perform well in this unit if they have good background in unit 2& 3 of this book and in previous lessons of this unit.

d) Learning activities

- Invite student-teachers to work in group and do the activity 4.6 found in their Mathematics books;
- Move around in the class for facilitating groups where necessary and ask some guiding questions on eventual challenges they may face during their work;
- Invite groups with different working steps to present their work;
- As a tutor, harmonize the findings from presentation of student-teachers and guide them to evaluate the rate of change, instantaneous velocity and instantaneous acceleration of a moving body.
- Guide them to explore the content and examples given in the student's book where they will be able to solve real problems involving limits.
- After the lesson, guide students-Teacher to do the application activity 4.6 and evaluate whether lesson objectives were achieved.

Answers for activity 4.6

$$\lim_{Q\to 300}(60-0.2Q)=\lim_{Q\to 300}(60-60)=0$$



e) Answers of application activity 4.6

$$a=\frac{\Delta v}{\Delta t}=\frac{v(2)-v(1)}{2-1}=\frac{(2^2-2)-(1^2-1)}{1}=2$$

4.6. Summary of the unit

Neighborhood of a real number

Definition: Mathematically, a set N is called a neighbourhood of point P if there exist an open interval I such that $x\in I\subset N$. The collection of all neighbourhoods of a point is called the neighborhood system at the point.

Condition of existence for a limit

If the limit from the left side is the same as the limit from the right side, say $\lim_{x\to x_0^-} f(x)=\lim_{x\to x_0^+} f(x)=L$, then we write $\lim_{x\to x_0} f(x)$ and we read "the limit of $f(x)$ as x approaches x_0 equals L.

Note that $\lim_{x\to x_0} f(x)$ exists if and only if $\lim_{x\to x_0^-} f(x)=\lim_{x\to x_0^+} f(x)$

Properties of limits

Let lim stands for one of the limits $\lim_{x\to a}$, $\lim_{x\to a^-}$, $\lim_{x\to a^+}$, $\lim_{x\to -\infty}$ or $\lim_{x\to +\infty}$. If $\lim f(x)$ and $\lim g(x)$ both exist, say $\lim f(x)=L_1$ and $\lim g(x)=L_2$, then

a) A constant factor can be moved through a limit sign. That is, if k is a constant, then $\lim\left[kf(x)\right]=k\lim f(x)$

b) $\lim\left[f(x)+g(x)\right]=\lim f(x)+\lim g(x)=L_1+L_2$

c) $\lim\left[f(x)-g(x)\right]=\lim f(x)-\lim g(x)=L_1-L_2$

d) $\lim\left[f(x)\bullet g(x)\right]=\lim f(x)\bullet\lim g(x)=L_1\bullet L_2$

e) $\lim\left[\frac{f(x)}{g(x)}\right]=\frac{\lim f(x)}{\lim g(x)}=\frac{L_1}{L_2}$ if $L_2\neq 0$

If n and m are positive integers, then $\lim\left[f(x)\right]^{\frac{m}{n}}=L_1^{\frac{m}{n}}$ provided that $L_1\geq 0$ if n is even

Limits are used to determine Asymptotes to curve of a function

There are three types of asymptotes:

- Vertical asymptote,
- Horizontal asymptote
- Oblique asymptote

a) Vertical asymptote

A line with equation $x = x_0$ $(D \equiv x = x_0)$ is called a vertical asymptote for the graph of a function $f(x)$ if $\lim_{x \to x_0} f(x) = \pm\infty$ 0r $V.A \equiv x = x_0$, *where* $\lim_{x \to x_0} f(x) = \pm\infty$

b) Horizontal asymptote

A line with equation $y = L$ $(D \equiv y = L)$ is called a horizontal asymptote for the graph of a function $f(x)$ if $\lim_{x \to \pm\infty} f(x) = L$ or $H.A \equiv y = L$; *where* $\lim_{x \to \pm\infty} f(x) = L$

c) Oblique asymptote

If a rational function, $\frac{P(x)}{Q(x)}$, is such that the degree of the numerator exceeds the degree of the denominator. Then the graph of $\frac{P(x)}{Q(x)}$ will have an oblique asymptote (or a slant asymptote); that is, an asymptote which is neither vertical nor horizontal.

We perform the division of $P(x)$ by $Q(x)$ to obtain $\frac{P(x)}{Q(x)} = (ax+b) + \frac{R(x)}{Q(x)}$ Where, $ax+b$ is the quotient and $R(x)$ is the remainder.

Another way to find the values of constants a and b is $a = \lim_{x \to \pm\infty} \frac{f(x)}{x}$ $a \neq 0$ and $b = \lim_{x \to \pm\infty} [f(x) - ax]$

We can write that $O.A \equiv y = ax + b$ where $a = \lim_{x \to \pm\infty} \frac{f(x)}{x}$, $a \neq 0$ and

$b = \lim_{x \to \pm\infty} [f(x) - ax]$.

4.7. Additional Information for teachers

Emphasize on the following:

a) Polynomials are continuous functions

b) If the functions f and g are continuous at c, then

i) $f+g$ is continuous at c

ii) $f-g$ is continuous at c

iii) $f.g$ is continuous at c

iv) $\frac{f}{g}$ is continuous at c if $g(c)\neq 0$, and is discontinuous at c if $g(c)=0$

c) A rational function is continuous everywhere except at the point where the denominator is zero.

d) Piecewise functions (functions that are defined on a sequence of intervals) are continuous if every function is in its interval of definition, and if the functions match their side limits at the points of separation of their intervals.

If $f(x)$ is continuous at $x=a$, then

$$\lim_{x\to a} f(x)=f(a);\quad \lim_{x\to a^-} f(x)=f(a);\ \lim_{x\to a^+} f(x)=f(a)$$

and if $f(x)$ is continuous at $x=b$, and $\lim_{x\to a} g(x)=b$, then

$$\lim_{x\to a} f(g(x))=f\left(\lim_{x\to a} g(x)\right)=f(b)$$

Horizontal asymptote and oblique asymptote do not exist on the same side. That means if $f(x)\to L$ as $x\to +\infty$, there is no oblique asymptote on the right side since there is horizontal asymptote and if $f(x)\to L$ as $x\to -\infty$, there is no oblique asymptote on the left side since there is horizontal asymptote.

4.8 End unit assessment

1. $y = 2t - 1$

2. a. $-\frac{241}{45}$

   b. $\infty, \quad -\infty$

The function has asymptotes: $VA \equiv x = \frac{9}{4}$ and $OA \equiv \frac{1}{4}x + \frac{21}{16}$

4.9 Additional activities

4.9.1 Remedial activities

For the given function

$$\begin{cases} x+2, x<1 \\ x^2, 1 \le x \le 2 \\ 5x-6, x \ge 2 \end{cases}$$

i) identify the discontinuity,

ii) where $f(x)$ is discontinuous or continuous

Answers

$\lim_{x \to 1^-} f(x) = 3$

$\lim_{x \to 1^+} f(x) = 1$

$\lim_{x \to 2^-} f(x) = 4$

$\lim_{x \to 2^+} f(x) = 4$

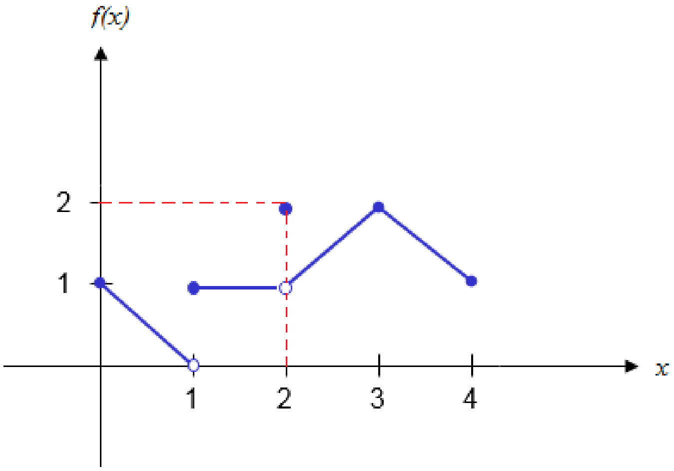
Conclusion,

$\lim_{x \to 1^-} f(x) \neq \lim_{x \to 1^+} f(x)$ therefore $f(x)$ is discontinuous at $x = 1$

$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^+} f(x)$ Therefore $f(x)$ is continuous at $x = 2$

4.9.2 Consolidation activities

1) Observe the graph bellow, analyse it and interpret it then find out if $\lim_{x\to 3} f(x)$ exist.



Solution:

$\lim_{x\to 3^-} f(x) = 2$

$\lim_{x\to 3^+} f(x) = 2$

$f(3) = 2$

Hence, $\lim_{x\to 3^-} f(x) = \lim_{x\to 3^+} f(x)$ then $f(x)$ exist

Let W and Z be sets;

4.9.3 Extended activities

Evaluate $\lim_{x\to +\infty} \frac{\sqrt{4x^2 - 11x - 3}}{x}$

Solution:

$$\lim_{x\to +\infty} \frac{\sqrt{4x^2 - 11x - 3}}{x} = \frac{\sqrt{\infty - \infty}}{\infty} I.F$$

To evaluate this limit, we try the algebraic manipulations such that the denominator will be cancelled.

$$\lim_{x\to+\infty}\frac{\sqrt{4x^2-11x-3}}{x}=\lim_{x\to+\infty}\frac{\sqrt{4x^2\left(1-\frac{11}{4x}-\frac{3}{4x^2}\right)}}{x}$$

$$=\lim_{x\to+\infty}\frac{\left(\sqrt{4x^2}\right)\sqrt{1-\frac{11}{4x}-\frac{3}{4x^2}}}{x}$$

$$=\lim_{x\to+\infty}\frac{\sqrt{4x^2}}{x}\lim_{x\to+\infty}\sqrt{1-\frac{11}{4x}-\frac{3}{4x^2}}$$

$$=\left(\lim_{x\to+\infty}\frac{\sqrt{4x^2}}{x}\right)\times 1$$

$$=\lim_{x\to+\infty}\frac{\sqrt{4x^2}}{x}$$

$$=\lim_{x\to+\infty}\frac{2\sqrt{x^2}}{x}$$

Recall that $\sqrt{x^2}=\begin{cases}x & if\ x\geq 0\\ -x & if\ x<0\end{cases}$

We need to find the domain of the given function: $Domf=\left]-\infty,-\frac{1}{4}\right]\cup\left[3,+\infty\right[$

As x tends to $+\infty$, $x\in\left[3,+\infty\right[$ and then $\sqrt{x^2}=x$.

Thus,

$$\lim_{x\to+\infty}\frac{\sqrt{4x^2-11x-3}}{x}=\lim_{x\to+\infty}\frac{2\sqrt{x^2}}{x}$$

$$=\lim_{x\to+\infty}\frac{2x}{x}$$

$$=2$$

UNIT: 5

DERIVATIVE OF FUNCTIONS AND THEIR APPLICATIONS

5.1 Key unit competence:

Use the concepts of derivative to solve and interpret related problems in various contexts

5.2 Prerequisite:

Student-teachers will perform well in this unit if they are skilled enough in the second, third and fourth units of mathematics for year one social studies education.

5.3 Cross-cutting issues to be addressed

- Inclusive education (promote education for all while teaching)
- Peace and value Education (respect others view and thoughts during class discussions)
- Gender (equal opportunity of boys and girls in the lesson participation)
- Environment sustainability

5.4 Guidance on introductory activity

- In groups, facilitate student-teachers to do the introductory activity 5 from Student -teacher's book;
- Facilitate discussions to avoid noise or other unnecessary conversation,
- Move around in the classroom to get aware of straggling Student –teachers;
- Invite group representatives to present their findings and promote gender into presentation.
- Basing on students' experience, results from their own research, prior knowledge and abilities shown in answering the questions for this activity, use different questions to facilitate student-teachers give their predictions and ensure that you arouse their curiosity on

what is going to be leant in this fifth unit

Answers for introductory activity 5

1) a) The slop of $f(x)$ in the point for $x_0 = 1$ is $\lim_{h\to 0}\frac{f(x_0+h)-f(x_0)}{h} = 2x_0 = 2\times 1 = 2$

b) The derivative of $f(x) = x^2+1$ by definition is

$$\begin{aligned} f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} \\ &= \lim_{h\to 0}\frac{(x+h)^2+1-x^2-1}{h} \\ &= \lim_{h\to 0}\frac{x^2+2hx+h^2+1-x^2-1}{h} \\ &= \lim_{h\to 0}\frac{2hx+h^2}{h} \\ &= \lim_{h\to 0}(2x+h) \\ &= 2x \end{aligned}$$

The value of $f'(x_0)$ at $x_0 = 1$ is 2

Thus, $m_p = f(x_0) = 2 \quad at\ x_0 = 1$

1) a.The derivative of a function $f(x)$ with respect to x is denoted by $f'(x)$ or $\frac{d}{dx}f(x)$ and defined as $f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$ provided that the limit exists b. The derivative is used in differents domain such as in Economics and finance, chemistry, physics, etc.

5.5 List of lessons

#	Lesson title	Learning objectives	Number of periods
1.	Introductory activity	To arouse the curiosity of student-teachers on the content of unit 5	1
2.	Concepts of derivative of a function	Define concept of derivative and evaluate derivatives of functions using the definition and properties of derivative.	2

	Rules of differentiation	Use differentiation rules to find out derivatives of first order for a function and apply them for variations of functions.	2
4.	High order derivatives	Use differentiation rules to find out derivatives of an order greater than one and apply them on variation of functions	2
5	Derivative and the variation of a function	Evaluate differentiation from first principles to find out the gradient at a point.	2
6	Applications of differentiation in Economics and finance	Apply differentiation in economics and other social sciences	5
7	End unit assessment		1
Total			15

Lesson 1: Concepts of derivative of a function

a) Learning objective:

Define the concept of derivative and Evaluate derivatives of functions using the definition of derivative.

b) Teaching resources:

Student-teacher's book and other Reference textbooks to facilitate research, Mathematical set, calculator, Manila paper, graph paper, ruler, markers, pens, pencils, etc.

c) Prerequisites/Revision/Introduction:

Student-teachers will perform well in this unit if they are enough skilled in unit 4 of mathematics in year one social studies education.

d) Learning activities

- Invite student-teachers to work in group and do the activity 5.1 found in their Mathematics books;
- Move around in the class for facilitating where necessary and give more clarification on eventual challenges they may face during their work;

- Invite one member from each group with different working steps to present their work;
- As a tutor, harmonize the findings from presentation of student-teachers and Guide them to explore the content and examples given in the student's book where they will be able to find the definition of derivative.
- After the lesson, guide students to do the application activity 5.1 and evaluate whether lesson objectives were achieved.

Answers to activity 5.1

1. The answers on activity 5.1 are:

a. $\frac{y_1 - y_0}{x_1 - x_0} = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$

b. Q will coincide with P

c. $m_{\tan} = \lim_{x \to x_0} \frac{f(x_1) - f(x_0)}{x_1 - x_0}$

d. $m_{\tan} = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$

e) Application activity 5.1

$$1\ m_p = \lim_{h \to 0} \frac{f(3+h) - f(3)}{h} = \lim_{h \to 0} \frac{(3+h)^2 - 3^2}{h} = \lim_{h \to 0} \frac{9 + 6h + h^2 - 9}{h}$$

$$= \lim_{h \to 0} \frac{h(6+h)}{h} = 6 + 0 = 6$$

2) By the use of definition

a. The derivative of $f(x)=x^2+1$ is

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$
$$=\lim_{h\to 0}\frac{(x+h)^2+1-x^2-1}{h}$$
$$=\lim_{h\to 0}\frac{x^2+2hx+h^2+1-x^2-1}{h}$$
$$=\lim_{h\to 0}\frac{2hx+h^2}{h}$$
$$=\lim_{h\to 0}(2x+h)$$
$$=2x$$

b.

$f(x)=x^2+2x-1$

$$\lim_{h\to 0}\frac{(x+h)^2+2(x+h)-x^2-2x+1}{h}=\lim_{h\to 0}\frac{\cancel{x^2}+2xh+h^2+\cancel{2x}+2h-\cancel{1}-\cancel{x^2}-\cancel{2x}+\cancel{1}}{h}$$

$$\lim_{h\to 0}\frac{\cancel{h}(2x+h+2)}{\cancel{h}}=\lim_{h\to 0}2x+2=2x+2$$

3)The graph is turning down

a. If a curve has equation $y=f(x)$, considering a nearby point $Q(x,f(x)),\quad x\neq a$ and the slop $PQ=m_{PQ}=\dfrac{f(x)-f(a)}{x-a}$

The tangent line to the curve $y=f(x)$ at the point $P(a,f(a))$ is the line p with slope (m): $m=\lim\limits_{x\to a}\dfrac{f(x)-f(a)}{x-a}$

b. By definition, $f'(x)=\lim\limits_{h\to 0}\dfrac{f(x+h)-f(x)}{h}$

c. $f(x)=x+1$

$$f'(x)=\lim_{h\to 0}\frac{x+h+1-x-1}{h}$$

$$f'(x)=\lim_{h\to 0}\frac{h}{h}=1$$

4) $f'(x)=\lim\limits_{h\to 0}\dfrac{f(x+h)-f(x)}{h}=\lim\limits_{h\to 0}\dfrac{(x+h)^2-8(x+h)+9-x^2+8x-9}{h}=2x-8$

$f'(a)=2a-8$

Lesson 2: Rules of differentiation

a) Learning objective:

Use differentiation rules to find out derivatives of first order of function and apply them for variations of functions.

b) Teaching resources:

Student-teacher's book and other Reference textbooks to facilitate research, Mathematical set, calculator, manilla paper, graph paper, ruler, markers, pens, pencils, etc.

c) Prerequisites/Revision/Introduction:

Student-teachers will perform well in this unit if they are enough skilled in mathematics on Limits (unit 4) of functions for year one social studies education

d) Learning activities

- Invite student-teachers to work in group and do the activity 5.2 found in their Mathematics books;
- Move around in the class for facilitating where necessary and give more clarification on eventual challenges they may face during their work;
- Invite a student from each group to present their findings;
- Use different probing questions and guide them to explore the content and examples given in the student's book lead them to be able to differentiate different types of rules of differentiation of the first order.
- As a tutor, guide students-Teacher to do the application activity 5.2 and evaluate whether lesson objectives were achieved

Answers for activity 5.2

1) a) $g'(x) = 2x$; b) $h'(x) = 3$; c) $g'(x) = 2x + 3$

2) $f'(t) = \frac{1' \times t - 1 \times t'}{t^2} = \frac{-1}{t^2}$

e) Answers of application activity 5.2

1. a. $2x + 5$

   b. $2x + 5$

   c. $2x + 5$

Note that $(f[g(x)]' = f'[g(x)].g'(x)$

2. $100(3x^2)(x^3 - 1)^{99}$
   $300x^2(x^3 - 1)^{99}$

3. $x(t) = t^3 - 3t$ $V(t) = 3t^2 - 3$ $a(t) = 6t$

Lesson 3: High order derivatives

a) Learning objective:

Use differentiation rules to find out derivatives of order greater than one of function and apply them in the variation of functions.

b) Teaching resources:

Student-teacher's book and other Reference textbooks to facilitate research, Mathematical set, calculator, manila paper, markers, pens, pencils

c) Prerequisites/Revision/Introduction:

Student-teachers will perform well in this unit if they have good background in lesson two of this unit.

d) Learning activities

- Invite student-teachers to work in group and do the activity 5.3 found in their Mathematics books;

- Move around in the class for facilitating where necessary and give more clarification on eventual challenges they may face during their work;
- Invite a student from each group to present their findings;
- Use different probing questions and guide them to explore the content and examples given in the student's book lead them to be able to perform the differentiation of order which is greater than one of a given function.
- Tutor guides students-Teacher to do the application activity 5.3 and evaluate whether lesson objectives were achieved

Answers for activity 5.3

1. a. $v(t)=3t^2+6t-9$

   At $t=1$, $v(1)=0m/s$ the body is at rest

   At $t=2$, $v(2)=15m/s$

   At $t=3$, $v(3)=36m/s$

b. Acceleration $a(t)=6t+6$

   At t=1, $a=12m/s^2$

   At t=2, $a=18m/s^2$

2. $f''(x)=6x+4$.

e) Answers of application activity 5.3

1. a. $f''(x)=24x^2-6$

   b. $f''(x)=-6x$

2. a. $v(t)=3t^2+6t-9$

   b. Acceleration is $18m/s^2$

Lesson 4: Derivative and the variation of a function

a) Learning objective:

Evaluate differentiation from first principles to find out the gradient at a point.

b) Teaching resources:

Student-teacher's book and other Reference textbooks to facilitate research, Mathematical set, calculator, manila paper, graph paper, ruler, markers, pens, pencils, etc.

c) Prerequisites/Revision/Introduction:

Student-teachers will perform well in this unit if they have good background in previous lesson of this unit and the unit 4 (limits of function)

d) Learning activities

- Invite student-teachers to work in group and do the activity 5.4 found in their Mathematics books;
- Move around in the class for facilitating where necessary and give more clarification on eventual challenges they may face during their work;
- Invite a student from each group to present their findings;
- Use different probing questions and guide them to explore the content and examples given in the student's book lead them to be able to evaluate the variation of the function towards a point using differentiation.
- Tutor guides students-Teacher to do the application activity 5.4 and evaluate whether lesson objectives were achieved

Answers for activity 5.4

1) a. The given function is differentiable

b) $y' = 3x^2 - 12$

c) Function $y = x^3 - 12x - 5$ is increasing on $]-\infty, -2[\cup]2, +\infty[$

Function $y = x^3 - 12x - 5$ is decreasing on $]-2, 2[$

d) Function $y = x^3 - 12x - 5$ is down on $]-\infty, 0[$

Function $y = x^3 - 12x - 5$ is concave up on $]0, +\infty[$

2) The function increase on $]-\infty, 0[\cup]2, +\infty[$

Concave down $]-\infty, 1[$

Concave up on $]1, +\infty[$

e) Answers of Application activity 5.4

a. $(2, -21)$, $(-2, 11)$

b. Function $y = x^3 - 12x - 5$ is increasing on $]-\infty, -2[\cup]2, +\infty[$

Function $y = x^3 - 12x - 5$ is decreasing on $]-2, 2[$

c. Function $y = x^3 - 12x - 5$ is down on $]-\infty, 0[$

Function $y = x^3 - 12x - 5$ is concave up on $]0, +\infty[$

Lesson 5: Applications of differentiation in Economics and finance

a) Learning objective:

Apply differentiation in economics and other social sciences

b) Teaching resources:

Student-teacher's book and other Reference textbooks to facilitate research, Mathematical set, calculator, manila paper, graph paper, ruler, markers, pens, pencils, etc.

c) Prerequisites/Revision/Introduction:

Student-teachers will perform well in this unit if they have good background in previous lessons of this unit.

d) Learning activities

- In group discussions, invite student-teachers to conduct research on the internet or in library about the application of differentiation in economics and finance.
- Invite student-teachers to work in group and do the second question of activity 5.5 found in their Mathematics books;
- Move around in the class for facilitating groups where necessary and ask some guiding questions on eventual challenges they may face during their work;
- Invite groups with different working steps to present their work;
- As a tutor, harmonize the findings from presentation of student-teachers and guide them to explore the content and examples given in the student's book where they will be able to solve real life problem of economics and finance by using differentiation.
- After the lesson, guide students-Teacher to do the application activity 5.5 and evaluate whether lesson objectives were achieved.

Answer for activity 5.5

1. Answers vary accordingly, facilitate student-teachers to present their findings and harmonize their answers from their own research.

2. $MC = \frac{dTC}{dq}$

$TC = 6 + 4q^2$

$MC = \frac{dTC}{dq} = \frac{d(6+4q^2)}{dq} = 8q$

The function $MC = 8q$

e) Answers of application activity 5.5

Question: A market faces the demand schedule $p=58-\frac{q}{2}$ and the cost schedule

$$TC=97q-17\frac{q^2}{2}+\frac{q^3}{3}$$

How much should it sell to maximize profit and what will be this maximum profit? (All costs and prices are in Rwandan Francs)

$TC=97q-17\frac{q^2}{2}+\frac{q^3}{3}$ as $TR=p.q=58q-\frac{q^2}{2}$

$$MC=\frac{dTC}{dq}=97-17q+q^2$$
$$MR=\frac{dTR}{dq}=58-q$$

To stabilize the profit, $MC=MR$. Equating the two equations, we find:

$$97-17q+q^2=58-q$$
$$(3-q)(13-q)=0$$
$$q=3 \quad ,q=13$$

The profit will be:

$$TR-TC=58q-q^2-\left(97q-17\frac{q^2}{2}+\frac{q^3}{3}\right)=58q-q^2-97q+17\frac{q^2}{2}-\frac{q^3}{3}$$
$$=58q-97q-q^2+17\frac{q^2}{2}-\frac{q^3}{3}=-39q+7.5q^2-\frac{q^3}{3}$$

The profit: $TR-TC=-39q+7.5q^2-\frac{q^3}{3}$

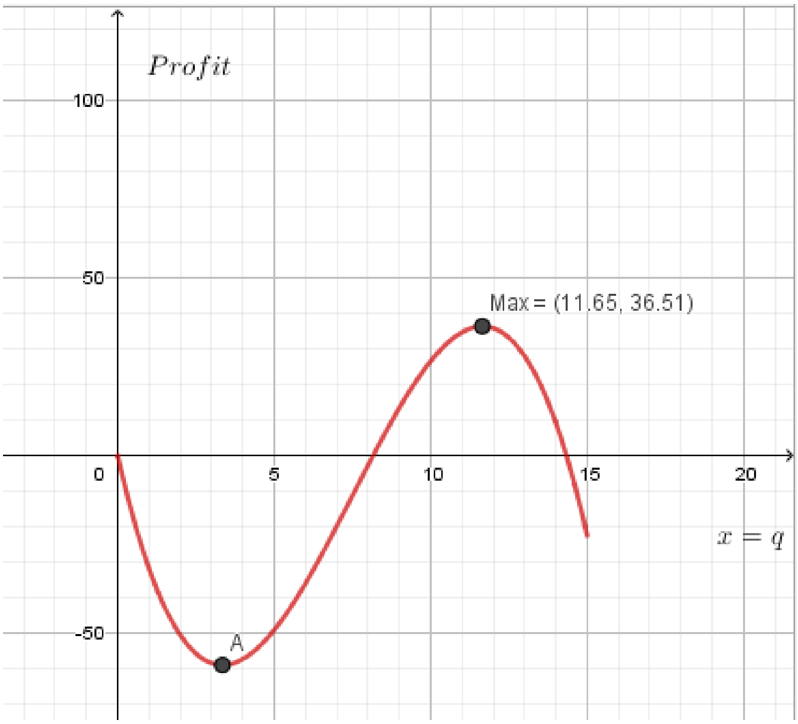
When we consider the values $q=3$ *and* $q=13$, we find that for $q=3$, there is no profit. This is to be rejected.

For $q=13$, there is stabilized market, the corresponding profit for which $MC=MR$. Therefore, this required profit is

$TR - TC = -39q + 7.5q^2 - \frac{q^3}{3}; q = 13$. This is 478.16 Units of money.

To maximize the profit, we must study the variation of $\frac{d(TR - TC)}{dq}$

This is the value that can be observed on the following figure:



Let us study the variation of $\frac{d(TR - TC)}{dq}$

$\frac{d(TR - TC)}{dq} = -39 + 15q - q^2$

q	0		3.35		$\frac{22.2-\sqrt{38.25}}{2}$		11.65		$\frac{22.2+\sqrt{38.25}}{2}$
$-39 + 15q - q^2$	-		0	+	0	+	0	-	0
$-39q + 7.5q^2 - \frac{q^3}{3}$	0	↘		↗	0	↗	36.51	↘	0

The profit is maximum when $\frac{d(TR-TC)}{dq} = -39+15q-q^2 = 0$

This means for $q = 11.65 Frw$

It is minimum for $q = 3.35$ on the point A of the graph given above.

The maximum profit MaxP is $Max(TR-TC) = -39q+7.5q^2 - \frac{q^3}{3}$, for $q = 11.65 Frw$

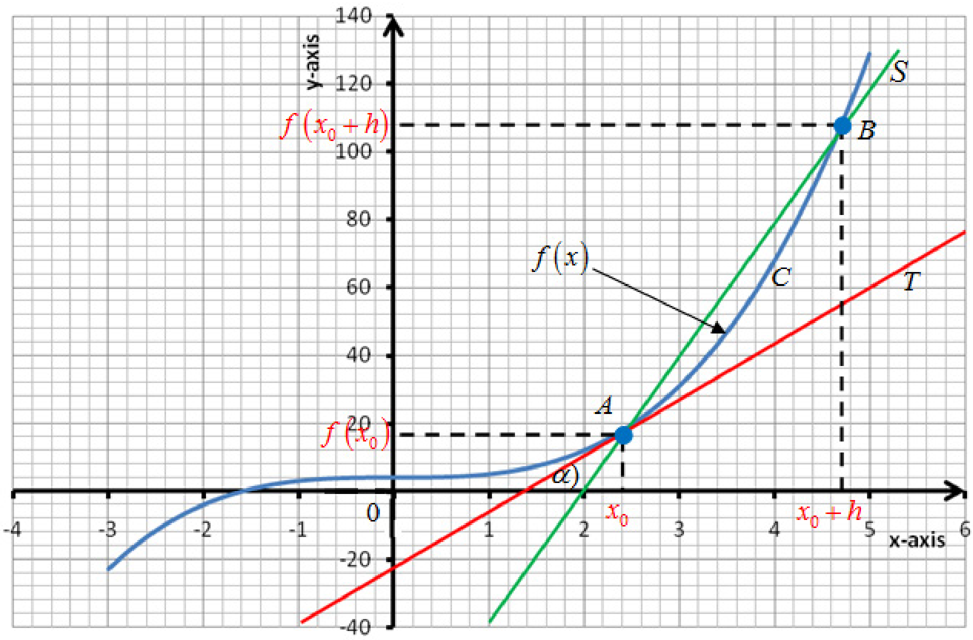
We find that MaxP = 36.51 Frw.

5.6. Summary of the unit

Definition of derivative of a function

The derivative of a function $f(x)$ with respect to x is denoted by $f'(x)$ or $\frac{d}{dx}f(x)$ and defined as $f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$ provided that the limit exists.

Graphical interpretation of derivative and the slope of a function using differentiation



Some rules of differentiation

Derivative of a constant function:

If f is a constant function, $f(x) = c$, for all x then $\frac{df}{dx} = \frac{d}{dx}(c) = 0$

Derivative of identity function:

The derivative of the identity function is the constant function 1, that is

$$if \quad f(x) = x, \frac{df}{dx} = \frac{dx}{dx} = 1$$

Multiplication by a scalar:

If f is a differentiable function of x, and c is a constant, then

$$\frac{d}{dx}\left(cf\left(x\right)\right) = c\frac{d}{dx}f\left(x\right)$$

Derivative of a power:

If n is any real number,

then $\frac{d}{dx}x^n = nx^{n-1}$ for all x where the powers x^n and x^{n-1} are defined.
This holds for any function with power. Thus, if $f \in (D, I)$ for positive and negative, and fractional value of n, $\left[f^n(x)\right]' = nf^{n-1}(x)f'(x)$

Sum rule

$$\frac{d}{dx}(u+v) = \frac{du}{dx} + \frac{dv}{dx}$$

Product rule

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$$

Notice that this is not just the product of two derivatives.

Quotient rule

$\frac{d}{dx}\left(\frac{u}{v}\right)=\frac{v\frac{du}{dx}-u\frac{dv}{dx}}{v^2}$ or can be written as $d\left(\frac{u}{v}\right)=\frac{v\,du-u\,dv}{v^2}$

Derivative of the reciprocal function

Let $D(I,\mathbb{R})$ be the set of functions differentiable on I. If $f\in D(I,\mathbb{R})$, then $\frac{1}{f}\in D(I,\mathbb{R})$ $f(x)\neq 0$. Moreover $\frac{d}{dx}\left(\frac{1}{f}\right)=-\frac{\frac{df}{dx}}{f^2}$ or $\frac{d}{dx}\left(\frac{1}{f}\right)=-\frac{f'}{f^2}$

Derivative of a composite function: Chain rule

If f and g are both differentiable and F is the composite function defined by $F(x)=f(g(x))$, then F is differentiable and F' is given by the product $F'(x)=f'(g(x))\cdot g'(x)$

In Leibniz notation, if $y=f(u)$ and $u=g(x)$ are both differentiable

5.7. Additional Information for teachers

Emphasize on the following:

- Marginal Cost $=\frac{dC}{dq}$
- Marginal Revenue $=\frac{dR}{dq}$

Denote P (q) to be the **profit** of producing and selling q units of the product, that is,

$P(q)=R(q)-C(q)$ Thus P is a function of q and it is called the profit function in order to have maximum profit, we need $\frac{dP}{dq}=0$ Or eventually, $\frac{dC}{dq}=\frac{dR}{dq}$

5.8 End unit assessment

1. $f'(x)=1$

2. a. $p(x)=19-\frac{1}{3000}x$

b. Ticket prices be set to maximize revenue is 9.50 dollars

3. $f'(x)=6x^2-10x+4=3x^2-5x+2 \qquad f''(x)=12x-10$

These points are $(1,3)$ $,(\frac{2}{3},\frac{82}{27})$.

$f'(x)=6x^2-10x+4=3x^2-5x+2$

$f''(x)=12x-10=6x-5$

The interval of increasing: $]-\infty,1[\cup]0.667,+\infty[$

The interval of decreasing: $]1,0.667[$

5.9 Additional activities

5.9.1 Remedial activities

Calculate the derivative of $f(x)=(2x+1)^6$

Solution:

$f(x)=(2x+1)^6$

$$
\begin{aligned}
f(x)&=(2x+1)^6\\
f'(x)&=6(2x+1)'(2\text{x}+1)^5\\
f'(x)&=6(2)(2x+1)^5\\
f'(x)&=12(2x+1)^5
\end{aligned}
$$

5.9.2 Consolidation activities

a. Find the derivative of $f(x)=\frac{3}{2}\sqrt[3]{x}$

$$\frac{d}{dx}(f(x))=\frac{d}{dx}\left(\frac{3}{2}\sqrt[3]{x}\right)=\frac{3}{2}\frac{d}{dx}\left(x^{\frac{1}{3}}\right)=\frac{3}{2}\cdot\frac{1}{3}x^{\frac{1}{3}-1}=\frac{1}{2}\sqrt[3]{x^{-2}}$$

b. A firm was assumed to have the total cost function $TC = 18q$ and the total revenue function $TR = 240 + 14q$ calculate the profit-maximizing output at $q = 1$

Solution

The profit function will be $TR - TC$

$$= TR - TC = 240 + 14q - 18q$$
$$= 240 - 4q$$

The profit-maximizing output $= 240 - 4 = 236$

5.9.3 Extended activities

A firm faces the demand schedule $p = 200 - 2q$ and the total cost function given by the function $TC = \frac{2}{3}q^3 - 14q^2 + 222q + 50$ .Derive expressions for the following functions and find out whether they have maximum or minimum points.

If they do, say what value of q this occurs at and calculate the actual value of the function at this output.

(a) Marginal cost (b) Average variable cost (c) Average fixed cost

(d) Total revenue (e) Marginal revenue (f) Profit

UNIT: 6

DESCRIPTIVE STATISTICS

6.1 Key unit competence

Analyse and interpret statistical data from daily life situations

6.2 Prerequisite

Student-teachers will perform well in this unit if they have a good background to arithmetic in senior one, senior two and senior three

6.3 Cross-cutting issues to be addressed

- Inclusive education (promote education for all while teaching)
- Peace and value Education (respect others view and thoughts during class discussions)
- Gender (equal opportunity of boys and girls in the lesson participation)
- Financial education

6.4 Guidance on introductory activity

- Invite student-teachers to work in groups and give them instructions on how they can do the introductory activity 6 found in unit 6 of student's book;
- Guide students to read and analyze the questions insisting on the analysis of data, the more repeated value and how they can find the mean;
- Basing on student-teachers' experience, prior knowledge and abilities shown in answering the questions for this activity, use different questions to facilitate them to give their predictions and ensure that you arouse their curiosity on what is going to be learnt in this unit.

Answers for introductory activity 6

1) The total amount of money she got is:

$1000Frw + 1200Frw + 1250Frw + 1000Frw + 1300Frw = 5,750Frw$

To get the same total amount of money 5750 Frw, if the sales are equally distributed per day

She could get $\frac{5750Frw}{5} = 1,150Frw$ per day.

2) Ten students have got the following marks: 3, 5,6,3,8,7,8,4,8 and 6

a) Their mean is $\frac{3+5+6+3+8+7+8+4+8+6}{10} = 5.8$

b) The mark that was obtained by many students is 8 as it was obtained by 3 students;

c) As the mean is 5.8, we see that 4 students have a mark which is below the mean(3;3;4;5) while 6 students have a mark which is above the mean(6;6;7;8;8;8).

6.5 List of lessons/sub-headings

#	Lesson title	Learning objectives	Number of periods
0.	Introductory activity	To arouse the curiosity of student-teacher on the content of unit 6	1
1	Definition and type of data	Differentiate the types of data	1
2	Data presentation or organization	Represent statistical information using: histogram, polygon, frequency distribution table and pie chart.	2
3	Graph interpretation and Interpretation of statistical data	Read and interpret a diagram of statistical data.	2
4	Measures of central tendencies for ungrouped data	Determine the mode, mean, median and range of statistical data	2
5	Measures of central tendencies for grouped data: mode, mean and median	Determine the mean, mode, median for a grouped statistical data.	2

6	Measures of dispersion for ungrouped data and for grouped data	Use the measures of dispersion in solving real life word problem.	3
7	Practical activity in statistics	Collect, organize and represent statistical information using histogram, polygon, frequency distribution table and pie chart and then analyse that data.	2
8	End unit assessment		1
	Total		15

Lesson 1: Definition and type of data

a) **Learning objective:**

Differentiate the types of data

b) **Teaching resources:**

Manila papers, Graph Papers, ruler, calculators.

c) **Prerequisites/Revision/Introduction:**

Student-teachers will perform well in this unit if they are enough skilled on the content of statistics learnt from senior one up to senior three.

d) **Learning activities**

- In small group discussions, invite student-teachers to carry out research on statistics to determine the meanings of statistics and types of data
- Ask student-teachers to share their findings with the other groups
- Harmonize the results presented by the student-teachers and facilitate them to use the results obtained in selecting qualitative and quantitative data from a given list of data.
- Use different questions and guide students to explore examples and the content to enhance the ability of differentiating and giving different types of data.
- After this step, guide students to do the application activity 6.1 and evaluate whether lesson objectives were achieved.

Answers for activity 6.1

Qualitative data: Male, female, tall,

Quantitative data**:** 20 sticks, 45 student-teachers, 20 meters, 4 piece of chalks

Lead them to know definitions of some terms those we are familial with in statistics such as:

Statistics: the branch of mathematics that deals with the collection, presentation, interpretation and analysis of data.

Qualitative data

Qualitative data is a categorical measurement expressed not in terms of numbers, but rather by means of a natural language description.

Example of qualitative data	Possible categories variable
• Marital status	• Single, married, divorced
• Gender	• Male, Female
• Pain level	• None, moderated, severe
• Color	• Red, black, green, yellow

Quantitative data

Quantitative data is a numerical measurement expressed not by means of a natural language description, but rather in terms of numbers.

Discrete data represent items that can be counted; they take on possible values that can be listed out.

Continuous data represent measurements; their possible values cannot be counted and can only be described using intervals on the real number line.

Descriptive statistics consists of the collection, organization, summarization, and presentation of data.

Raw data

Data which have been collected in original form, they are called raw data

e) **Answers for application activity 6.1**

1) Qualitative data: basketball team classification, Product rating,

Quantitative data: number of student-teachers in the classroom, weight, age, number of rooms in a house, number of tutors in school.

2) The variable is the fasting blood sugar reading for a patient. The observations are 125, 175, 160, and 110.

The second patient has the high level of fasting blood sugar, the fourth patient has the low level of fasting blood sugar.

Lesson 2: Data presentation or organization

a) Learning objective:

Represent statistical information using: histogram, polygon, frequency distribution table and pie chart.

b) Teaching resources:

Graph papers, manila papers, calculators, markers, pens, mathematical set.

c) Prerequisites/Revision/Introduction:

Student-teachers will perform well in this unit if they are enough skilled in the content of statistics learnt in to senior three.

d) Learning activities

- Invite student-teachers to work in groups and do the activity 6.2 found in their Mathematics books;
- Move around in the class for facilitating students where necessary and give more clarification on eventual challenges they may face during their work;
- Verify and identify groups with different working steps;
- Invite one member from each group with different working steps to present their work where they must explain the working steps;
- As a tutor, harmonize the findings from presentation;.
- Use different probing questions and guide them to explore the

content and examples given in the student's book and lead them to discover the different ways of presenting simple statistical data: raw data, frequency distribution, Cumulative frequency, Histograms, Frequency Polygons, and ogives, Pie chart, Stem and Leaf Plots.

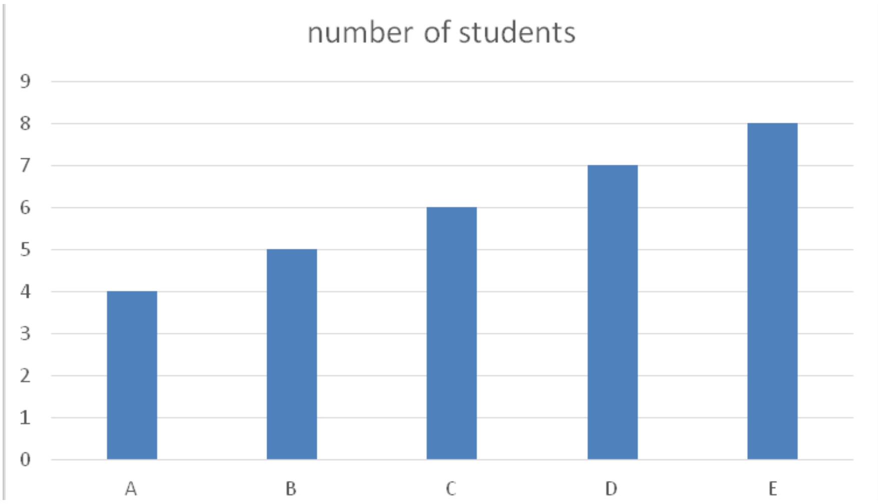
- Retake the presentation for statistical grouped dada where different methods can intervene: group works, whole class discussion, question answer, etc;
- After this step, guide students to do the application activity 6.2 and evaluate whether lesson objectives were achieved.

Answers for activity 6.2

1) The table below shows the number of student-teachers who attended the school in 5 classrooms on first day.

Number of student-teachers	4	5	6	7	8
Classroom	A	B	C	D	E

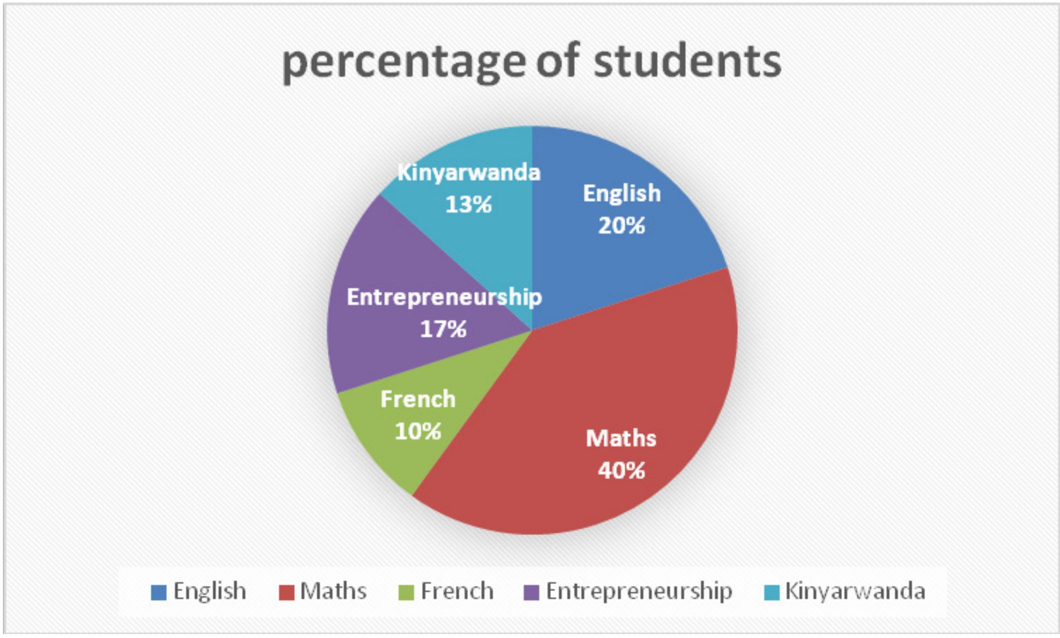
This data on the bar chart:



2) Here is the number of participants of the survey.

Subject	**Number of students**
English	12
Mathematics	24
French	6
Entrepreneurship	10
Kinyarwanda	8

Pie chart representing the percentage of students for each subject:



Definitions related to grouped frequency distribution

- **Class limits:** The class limits are the lower and upper values of the class
- **Lower class limit:** Lower class limit represents the smallest data value that can be included in the class.
- **Upper class limit:** Upper class limit represents the largest data value that can be included in the class.
- **Class mark or class midpoint:**

$$class\ midpoint = \frac{lower\ class\ limit + upper\ class\ limt}{2}$$

e) Answers of application Activity 3.2

1) a.

x	f
3	4
4	1
5	1
6	3
7	1
8	5
9	2
10	4

b. Relative frequency table and calculate percentage for each, n=21

x	f	Relative frequency	percentage
3	4	4/21=0.1904	19.04%
4	1	1/21=0.0476	4.76%
5	1	1/21=0.0476	4.76%
6	3	3/21=0.1428	14.28%
7	1	1/21=0.0476	4.76%
8	5	5/21=0.2380	23.8%
9	2	2/21=0.0952	9.52%
10	4	4/21=0.1904	19.04%

c. Cumulative frequency

x	f	cuf
3	4	4
4	1	5
5	1	6
6	3	9
7	1	10
8	5	15
9	2	17
10	4	21

There are 21 students

2) Present the results using stem and leaf:

Stem	**Leaf**						
2	4	5	7	8			
3	0	2					
4	2	3	3	5	6	7	9
5	0	4	4	4	5		
6	1	2	3				

Lesson 3: Graph Interpretation of statistical data

a) Learning objective:

Read and interpret a diagram of statistical data.

b) Teaching resources:

Manila papers, calculators, markers, student's book, pens, notebooks.

c) Prerequisites/Revision/Introduction:

Student-teachers will perform well in this unit if they make a good revision on the content of statistics learnt in senior three and in previous lessons of this unit.

d) Learning activities

- Invite student-teachers to work in groups and do the activity 6.3 found in their Mathematics books;
- Move around in the class for facilitating students where necessary and give more clarification on eventual challenges they may face during their work;
- Verify and identify groups with different working steps;
- Invite one member from each group with different working steps to present their work where they must explain the working steps;
- As a tutor, harmonize the findings from presentation highlighting elements to be verified on a graph of data;
- Use different probing questions and guide them to explore examples given in the student's book and lead them to discover the different

ways of interpreting statistical data given graphically in reports or newspapers.

- After this step, guide students to do the application activity 6.3 and evaluate whether lesson objectives were achieved.

Answers for activity 2.3

a. There are 5 students with small size (S);

b. There are 13 students with medium size, 8 Students with large size and 4 students with Extra-large size.

e) Answers of application activity 3.3

a. There are 240 bags of cement produced in 8 minutes;

There are 96 bags of cement produced in 3min 12 seconds

There are 150 bags of cement produced in 5 minutes;

There 210 bags of cement produced in 7 minutes;

b. It will take 2 minutes 48 seconds to produce 78 bags of cement.

c.

No bags	96	150	210	240
Time	3min12sec	5	7	8

Lesson 4: Measures of central tendencies for ungrouped data

a) Learning objective:
Determine the mode, mean, median and range of statistical data.

b) Teaching resources:
Manila papers, calculators, markers;

c) Prerequisites/Revision/Introduction:
Student-teachers will perform well in this unit if they have good background in senior two unit 3, senior three unit 4 and in previous two lessons of this unit.

d) Learning activities

- In group discussions, invite student-teachers to conduct a research in the library or on the internet and explain measures of central tendency, their types and provide related examples as instructed in the activity 6.4;
- Organize a whole class discussion where representative of groups present and explain their findings from research.
- As a tutor harmonize the findings from presentation of student-teachers and guide them to highlight the meaning of Mean, Mode, Median and their role when interpreting statistic data;
- Use different probing questions and guide them to explore examples given in the student's book and lead them to discover the different ways finding measures of central tendency for ungrouped data;
- After this step, guide students to do the application activity 6.4 and evaluate whether lesson objectives were achieved.

Answers of activity 6.4

Use reference books or the student's book to verify answers for students.

The mean

The *mean,* also known as the *arithmetic average,* is found by adding the values of the data and dividing by the total number of values.

Is given by the formula $\bar{x} = \frac{1}{n}\sum xfi$

The median:

If the data is well arranged in an order from the smallest to the largest, the median is the middle number or the central number of the range.

$$Me \to \left(\frac{n+1}{2}\right)^{th} \text{ or } Me = x_{\frac{n+1}{2}}$$

When n is even, the median is given by

$$Me = \frac{1}{2}\left[\left(\frac{n}{2}\right)^{th} + \left(\frac{n}{2}+1\right)^{th}\right] \text{ or } Median = \frac{x_{\frac{n}{2}} + x_{\frac{n}{2}+1}}{2}$$

The median for grouped data is given by

$$Median = L_m + c\left(\frac{\frac{n}{2} - CF_b}{f_m}\right)$$

The mode:

The mode is the number that appears the most often from the set of data. It represents the value which appears more frequently in the data.

The mode for grouped data is given by

$$Mode = L_m + c\left(\frac{\Delta_1}{\Delta_1 + \Delta_2}\right)$$

e) Answers of Application activity 6.4

1. a) by the use of $\bar{x} = \frac{1}{n}\sum xfi$ we get $\bar{X} = 3.38$

   b) by the use of $Me = \frac{1}{2}\left[\left(\frac{n}{2}\right)^{th} + \left(\frac{n}{2}+1\right)^{th}\right]$ we get $me = 3.5$

   c.) The mode is 4

2. a) By the use of $\bar{x} = \frac{1}{n}\sum xfi$ we get $\bar{X} = 8.2$

   b) By the use of $Me = \frac{1}{2}\left[\left(\frac{n}{2}\right)^{th} + \left(\frac{n}{2}+1\right)^{th}\right]$ we get $me = 8$

Lesson 5: Measures of central tendency for grouped data

Learning objective:

Determine the mean, mode, median for a grouped statistical data.

b) Teaching resources:

Manila papers, calculators, markers, rulers, graph papers, pens.

c) Prerequisites/Revision/Introduction:

Student-teachers will perform well in this unit if they make a short revision on the content for previous lesson of this unit.

d) Learning activities

- Invite students invite student-teachers to conduct a research in the library or on the internet and explain measures of central tendency for grouped data, their types and provide related examples as instructed in the activity 6.5;
- Organize a whole class discussion where representative of groups present and explain their findings from research.
- As a tutor harmonize the findings from presentation of student-teachers and guide them to highlight different ways of determining the Mean, Mode, Median and midrange for grouped data. Insist on the explanation of the role of these measures when interpreting data ; use the question such as "what does the mean show in this data?"
- Use different probing questions and guide them to explore examples given in the student's book and lead them to discover the different ways finding measures of central tendency for grouped data;
- After this step, guide students to do the application activity 6.5 and evaluate whether lesson objectives were achieved.

Answer for activity 6.5

The mean

The process of finding the mean is the same as the one applied in the ungrouped data with the exeption that the midpoints x_m of each class in grouped data plays the role of x_i used in ungrouped data. $\overline{X} = \frac{\sum f.X_m}{n}$

The mode

The mode for grouped data is the modal class. The **modal class** is the class with the largest frequency.

Median of grouped data

$$Median = l_1 + \frac{\frac{n}{2} - cufi}{fi}(l_2 - l_1)$$

Where l_1 is lower limit of the median class, l_2 is upper limit of the median class, fi is the frequency and $cufi$ is the cumulative frequency of the class preceding the median class.

2) The mean for the data is $\overline{X}=\frac{\sum f.X_m}{n}$ and $\overline{X}=\frac{490}{20}=24.5$

The Modal class is $20.5-25.5$ because it has the high frequency.

e) Answers for application activity 6.5

The data below shows the marks scored by a group of students in a mathematics out of 100: 72; 63; 51; 25; 31; 49; 51; 27; 46; 42; 25; 39; 38; 39; 55; 38; 35; 64;67;37.

The lowest value is L = 25, the highest value is H =72.

Find the range: R = highest value-lowest value =72 -25 = 47

The number of classes desired is 5.

- Find the class width by dividing the range by the number of classes

$$width=\frac{R}{Number\ of\ classes}=\frac{47}{5}=9.4$$

Class	Frequency f	Mid point x_m	$f.x_m$
25-34.4	4	29.7	118.8
34.5-43.9	7	39.2	274.4
44-53.4	4	48.7	194.8
53.5-62.9	1	58.2	58.2
63-72.4	4	67.7	270.8
Total	$\sum f=20$		$\sum f.x_m=917$

a) The mean mark

$$\bar{x}=\frac{\sum fx_m}{\sum f}=\frac{917}{20}=45.85$$

b) The median: …

c) The modal class is *34.5-43.9*, its frequency is *7*.

e) The range is *72 - 25 = 47*.

Lesson 6: Measures of dispersion for ungrouped data and for grouped data

a) Learning objective:

Use the measures of dispersion in solving real life word problem.

b) Teaching resources:

Manila papers, calculators, markers, rulers, graph papers.

c) Prerequisites/Revision/Introduction:

Student-teachers will perform well in this unit if they revise the content on measures of dispersion learnt in senior three unit 4 and the previous lessons of this unit.

d) Learning activities

- Invite student-teachers in small group discussions, to use calculators and do activity 6.6 in their Mathematics books related to measures of dispersion for ungrouped data and for grouped data;
- Move around in the class for facilitating and give more clarification on eventual challenges they may face during their practices on activity 6.6;
- Ask neighboring groups of student-teachers to share their answers for improvement;
- Invite each group to present their working steps to the whole class discussion;
- As a tutor, harmonize the findings from presentation highlighting how to avoid misconception when completing the table;
- Through the use of probing questions and examples given in the student's book, guide students to use data from the completed table to determine the variance, the standard deviation for ungrouped data;

- From the table completed above, ask students to deduce how to complete the table to be used when calculating the variance and the standard deviation for the grouped data;
- Invite students to brainstorm other measures of dispersion and guide them to discover the role of each measure when interpreting statistical data.
- After this step, guide students to do the application activity 6.6 and evaluate whether lesson objectives were achieved.

Answers for activity 6.6

1)

x	f	$x-\bar{x}$	$(x-\bar{x})^2$	$f(x-\bar{x})^2$
12	4	-4.875	23.765	95.0625
13	2	-3.875	15.0156	30.0312
15	1	-1.875	3.5156	3.5156
19	4	2.125	4.5162	18.0625
21	5	4.125	17.015	85.0781
	$\sum f = 16$			$\sum f(x-\bar{x})^2 =$ 231.75

$\sum f\left(x-\bar{x}\right)^2$ is given by the product of frequency to difference of data and mean squared

The answers vary according to the student-teachers observation, tutor harmonize the answers given by student-teachers.

- quartiles are calculated as follows:

$Q_1 \to \frac{1}{4}(n+1)^{th}$ or $Q_1 = x_{\frac{n+1}{4}}$. $Q_2 \to \frac{1}{2}(n+1)^{th}$ or $Q_2 = x_{\frac{n+1}{2}} = Me$

$Q_3 \to \frac{3}{4}(n+1)^{th}$ or $Q_3 = x_{\frac{3(n+1)}{4}}$

The inter-quartile range is given by the difference between third quartile and the first quartile.

- The variance is denoted and defined by

$$\sigma^2 = \frac{\sum_{i=1}^{n}\left(x_i - \bar{x}\right)^2}{n}$$

Developing this formula, we have

$$\sigma^2 = \frac{\sum_{i=1}^{n}\left(x_i^2 - 2x_i\bar{x} + \left(\bar{x}\right)^2\right)}{n}$$

$$= \frac{1}{n}\sum_{i=1}^{n} x_i^2 - \frac{1}{n}2\bar{x}\sum_{i=1}^{n} x_i + \frac{1}{n}\left(\bar{x}\right)^2\sum_{i=1}^{n} 1$$

$$= \frac{1}{n}\sum_{i=1}^{n} x_i^2 - 2\bar{x}\bar{x} + \left(\bar{x}\right)^2$$

$$= \frac{1}{n}\sum_{i=1}^{n} x_i^2 - \left(\bar{x}\right)^2$$

Thus, the variance is also defined by:

$$\sigma^2 = \frac{1}{n}\sum_{i=1}^{n} x_i^2 - \left(\bar{x}\right)^2$$

Recall that the mean of the set of n values $x_1, x_2, x_3, ..., x_n$ is denoted and defined by

$$\bar{x} = \frac{x_1 + x_2 + x_3 + ... + x_n}{n}$$

$$= \sum_{i=1}^{n} \frac{x_i}{n}$$

$$= \frac{1}{n}\sum_{i=1}^{n} x_i$$

- The standard deviation is denoted and defined by

$$\sigma = \sqrt{\frac{\sum_{i=1}^{n}\left(x_i - \bar{x}\right)^2}{n}} \text{ or } \sigma = \sqrt{\frac{1}{n}\sum_{i=1}^{n} x_i^2 - \left(\bar{x}\right)^2}$$

Coefficient of variation

The coefficient of variation is a calculation built on other calculations: the standard deviation and the mean as follows:

$$Cv = \frac{\sigma}{\bar{x}} \times 100$$

Where:

- σ is the standard deviation
- $\bar{x}$ is the mean.

Range

In the case of ungrouped data, the range of a set of observations is the difference in values between the largest and the smallest observations in the set.

In the case of grouped data, the range is defined as the difference between the upper limit of the highest class and the lower limit of the smallest class.

e) Answers of application activity 6.6

1. $Me = \frac{64+49}{2} = 56.5$

2. For $\overline{X} = \frac{54+55+55+56+57+58+59}{7} = 56.285$

x	f	$x-\bar{x}$	$(x-\bar{x})^2$	$f(x-\bar{x})^2$
54	1	−2.285	5.221225	5.221225
55	2	−1.285	1.651225	3.30245
56	1	−0.285	0.081225	0.081225
57	1	0.715	0.511225	0.511225
58	1	1.715	2.941255	2.941255
59	1	2.715	7.371225	7.371225
	$\sum f = 7$			$\sum f(x-\bar{x})^2 = 19.428575$

Therefore, standard deviation

$$\sigma = \sqrt{\frac{\sum_{i=1}^{n}\left(x_i - \overline{x}\right)^2}{n}}$$

$$\sigma = \sqrt{\frac{19.428575}{7}} \simeq 1.67$$

Question 3

a) $\overline{X} = 5.8$ $\quad Mode = 5$ $\quad Median = 5.5$

b) $Q_1 = 5$ $\quad Q_2 = 5.5$ $\quad Q_3 = 8.25$

c) $Variance = 3.76$ $\quad \sigma = 1.94$ $\quad C.v = 33.45$

Lesson 7: Practical activity in statistics

a) Learning objective:

Collect, organize and represent statistical information using histogram, polygon, frequency distribution table and pie chart and then analyze that data.

b) Teaching resources:

Manila papers, digital technology including calculators, markers, rulers, graph papers,

c) Prerequisites/Revision/Introduction:

Student-teachers will perform well in this unit if they have good background in senior three unit 5 and the previous lessons in this unit.

c) Learning activities

- Invite student-teachers to work in group discussions and do activity 6.7 found in their Mathematics books: set a limited number of groups as every group will present its work.
- Move around in the class for facilitating and give more clarification on the eventual challenges they may face during their work on activity 6.7

- Invite all groups to present their findings for the whole class discussion;
- As a tutor, harmonize the findings from presentation and guide students to enhance how to interpret the data using measures for centre tendency and measures of dispersion;
- After this step, guide students to do the application activity 6.7 and evaluate whether lesson objectives were achieved.
- Organize another session in which each group will present the findings realized when answering the question 2 for the application activity 6.7.
- Guide student-teachers discover many examples of real life experience where statistics is applied: marks for learners, reports from local leaders, national economy, development of the population, annual agricultural production, etc.

Answers for activity 6.7

Question 1

Modal class: 17.5-22.5

Apply the given formulae in content summary to get average, median and standard deviation.

The points of advice provided by student-teachers are referred to answers on average, standard deviation and the median.

e) Answers of application activity 6.7

Solution

1. Martha had a total of 144 fruits remaining,

a) Number of mangoes: $\frac{100}{360}\times 144 = 40$ mangoes

Number of paw paws: $\frac{40}{360}\times 144 = 16$ paw paws

Cost of mangoes = 40 × 30 Frw= 1 200 FRW

Cost of paw paws = 16 × 160Frw= 2 560 FRW

Total cost = 1 200 + 2 560= 3 760 FRW

b) Apples remained the most unsold: $\frac{150}{360} \times 144 = 60$ apples

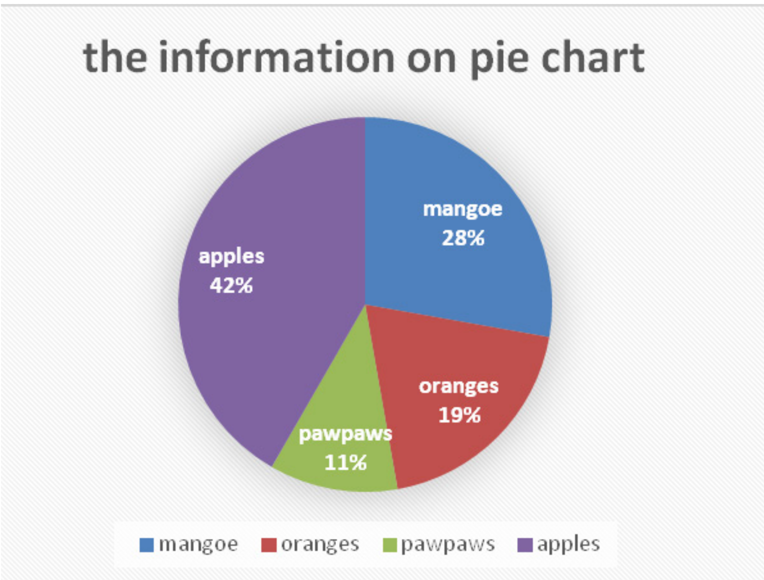
c) Median number of remaining fruit

Mangoes = 40 ; Paw paws = 16 .; Apples = 60; Oranges=28

Median=34fruits

Frequency table

Types of fruits	Frequency(number remaining)
Mangoes	40
Oranges	28
Paw paws	16
Apples	60
Total	144



2) Answers will vary from group to another. Try to organize a session where every group will have time to present its findings and others will ask questions and provide constructive feedback for learning purpose. .

6.6. Summary of the unit

Two types of data:

Qualitative and quantitative data

Data presentation

To present data, one can use:

- Frequency distribution
- Cumulative frequency
- Histograms, Frequency Polygons, and Ogives
- Pie chart
- Stem and leaf plots

Measures of centre tendency for ungrouped data

The Mean $\bar{x} = \frac{1}{n}\sum xfi$

Mode: The more repeated value

Median: $Me = \left(\frac{n}{2}\right)^{th} + \left(\frac{n}{2}+1\right)^{th}$

Measures of centre tendency for grouped data:

The Mean $\bar{X} = \frac{\sum f.X_m}{n}$

Modal class: The class with more values

Median: $Mediam = l_1 + \frac{\frac{n}{2} - cufi}{fi}(l_2 - l_1)$

where l_1 is lower limit of the median class, l_2 is upper limit of the median class, fi is the frequency and $cufi$ is the cumulative frequency of the class preceding the median class.

The Midrange

$$MR = \frac{Lowest\ value + highest\ value}{2}$$

Weighted mean

$$\bar{X} = \frac{w_1X_1 + w_2X_2 + w_3X_3 + \ldots w_n X_n}{w_1 + w_2 + w_3 + \ldots + w_n} = \frac{\sum wX}{\sum w}$$

Where $w_1, w_2, \ldots w_n$ are the weights and $X_1, X_2, \ldots X_n$ are the values.

Measures of dispersion

Quartiles:

$Q_1 = \frac{1}{4}(n+1)^{th}$ observation $Q_2 = \frac{1}{2}(n+1)^{th}$ observation $Q_3 = \frac{3}{4}(n+1)^{th}$ observation

The inter-quartile range is given by the difference between third quartile and the first quartile $Q_3 - Q_1$.

Variance

For ungrouped data

$$\sigma^2 = \frac{1}{n}\sum_{i=1}^{n} x_i^2 - \left(\bar{x}\right)^2$$

For grouped data

$$\delta^2 = \frac{\sum\left\{f\left(x - \bar{x}\right)^2\right\}}{\sum f}$$

Standard deviation

$$\sigma = \sqrt{\frac{\sum_{i=1}^{n}\left(x_i - \bar{x}\right)^2}{n}} \quad \text{or} \quad \sigma = \sqrt{\frac{1}{n}\sum_{i=1}^{n} x_i^2 - \left(\bar{x}\right)^2}$$

For grouped data

$$\delta = \sqrt{\frac{\sum\left\{f\left(x - \bar{x}\right)^2\right\}}{\sum f}}$$

Coefficient of variation

$$Cv = \frac{\sigma}{\overline{x}} \times 100$$

Range: It is the difference in values between the largest and the smallest observations in the set of data.

6.7 Additional Information for teachers

- Emphasize on the following results from the definitions of mean and standard deviation:

a) When all the data values are multiplied by a constant a, the new mean and new standard deviation are equal to a times the original mean and standard deviation. That is, the mean of $ax_1, ax_2, ax_3, ..., ax_n$ is $a\left(\overline{x}\right)$ and the standard deviation is $a\sigma$.

b) When a constant value, b,is added to all data values, then new mean is increased by b. However standard deviation does not change. That is, the mean of $ax_1, ax_2, ax_3, ..., ax_n$ is $\overline{x} + b$ and the standard deviation is σ .

- Emphasize on calculating median of ungrouped data. We need to clarify formula

for $n - odd$ and for $n - even$ as highlighted in the student-teachers book and paying attention on notation.

a) When n is odd, the median is given by

$$Me \to \left(\frac{n+1}{2}\right)^{th} \text{ or } Me = x_{\frac{n+1}{2}} \text{ we don't write } Me = \left(\frac{n+1}{2}\right)^{th}$$

b) When n is even, the median is given by

$$Me = \frac{1}{2}\left[\left(\frac{n}{2}\right)^{th} + \left(\frac{n}{2}+1\right)^{th}\right] \text{ or } Median = \frac{x_{\frac{n}{2}} + x_{\frac{n}{2}+1}}{2}$$

- Emphasize on calculating mode and median of grouped data.

$$Mode = L_m + c\left(\frac{\Delta_1}{\Delta_1 + \Delta_2}\right)$$

$$Median = L_m + c\left(\frac{\frac{n}{2} - CF_b}{f_m}\right)$$

Be specific on this notation symbols used in above formulae

L_m this is lower boundary of modal class.

C this is class width: the difference between upper and lower boundary of modal class ($C = U_m - L_m$)

f_m : frequency of modal class

$n = \sum f$: the sum of frequencies of data.

CF_b: cumulative frequency proceeded by cumulative frequency of modal class

(cumulative frequency before modal class)

$\Delta_1 = f_m - f_b$

$\Delta_2 = f_m - f_a$

f_b is frequency followed by f_m and f_a is frequency follows f_m.

- We need to clarify formulae of quartiles for $n - odd$ and for $n - even$

$Q_1 \to \frac{1}{4}(n+1)^{th}$ or $Q_1 = x_{\frac{n+1}{4}}$. Don't write $Q_1 = \frac{1}{4}(n+1)^{th}$

$Q_2 \to \frac{1}{2}(n+1)^{th}$ or $Q_2 = x_{\frac{n+1}{2}} = Me$ Don't write $Q_2 = \frac{1}{2}(n+1)^{th}$

$Q_3 \to \frac{3}{4}(n+1)^{th}$ or $Q_3 = x_{\frac{3(n+1)}{4}}$ Don't write $Q_3 = \frac{3}{4}(n+1)^{th}$

The Five-Number Summary and Boxplots

A **boxplot** can be used to graphically represent the data set. These plots involve five specific values:

1. The lowest value of the data set (i.e., minimum)

2. $Q1$

3. The median

4. $Q3$

5. The highest value of the data set (i.e., maximum).

These values are called a **five-number summary** of the data set.

A **boxplot** is a graph of a data set obtained by drawing a horizontal line from the minimum data value to $Q1$, drawing a horizontal line from $Q3$ to the maximum data value, and drawing a box whose vertical sides pass through $Q1$ and $Q3$ with a vertical line inside the box passing through the median or $Q2$.

Procedure for constructing a boxplot

1. Find the five-number summary for the data values, that is, the maximum and minimum data values, $Q1$ and $Q3$, and the median.

2. Draw a horizontal axis with a scale such that it includes the maximum and minimum data values.

3. Draw a box whose vertical sides go through $Q1$ and $Q3$, and draw a vertical line though the median.

4. Draw a line from the minimum data value to the left side of the box and a line from the maximum data value to the right side of the box.

Example:

The number of meteorites found in 10 states of the United States is 89, 47, 164, 296, 30,

215, 138, 78, 48, 39. Construct a boxplot for the data.

Solution

Step 1: Arrange the data in order: 30, 39, 47, 48, 78, 89, 138, 164, 215, 296

Step 2: Find the median.

30, 39, 47, 48, 78, 89, 138, 164, 215, 296

Median

$$= \frac{78 + 89}{2} = 83.5$$

Step 3: Find $Q1$: 30, 39, 47, 48, 78

Q1 is 47.

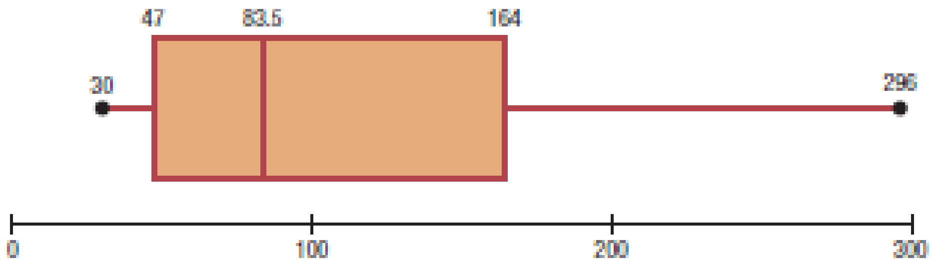
Step 4: Find $Q3$: 89, 138, 164, 215, 296

Q3 is 164.

Step 5: Draw a scale for the data on the x axis.

Step 6: Located the lowest value, $Q1$, median, $Q3$, and the highest value on the scale.

Step 7: Draw a box around $Q1$ and $Q3$, draw a vertical line through the median, and connect the upper value and the lower value to the box.



This figure indicates that the distribution is slightly positively skewed.

If the box plots for two or more data sets are graphed on the same axis, the distributions can be compared. To compare the averages, use the location of the medians. To compare the variability, use the inter quartile range, i.e., the length of the boxes.

6.8 End unit assessment

1) a) Mode is 70,000 Frw

a) Range is 60,000 Frw

b)

Monthly wage (Frw)	40,000	50,000	60,000	70,000	80,000	90,000	100,000
Number of workers	4	10	12	24	12	4	18

2) a) $Mean = 6.6$ $Q_1 = 6$ $Q_2 = 6.5$ $Q_3 = 8$

$interquartile\ range = 2$

c) $Variance = 2.04$

$Standard\ deviation \approx 1.43$

d) $Coeff.of\ variance \approx 21.7$

6.9 Additional activities

6.9.1 Remedial activities

1) Calculate the mean, variance and standard deviation for the following data

2, 4, 5, 6, 8, 17

Solution

Mean is 7

Variance is 23.33

Standard deviation is 4.83

3) The measurements in mm of the diameters of the heads of 107 screws gave the following frequency table

Diameter in mm	No. of screws
33-35	17
36-38	19
39-41	23
42-44	21
45-47	27

Calculate the standard deviation

Solution

Variance is 17.7876

Standard deviation is 4.22mm.

6.9.2 Consolidation activities

The six runners in a 200 meters race clocked times (in seconds) of 24.2, 23.7, 25.0, 23.7, 24.0, 4.6;

a) Find the mean and standard deviation of these times.

b) These readings were found to be 10% too low due to faulty timekeeping. Write down the new mean and standard deviation.

c) Draw bar graph of the above information

Solution

a) $\bar{x} = \frac{24.2 + 23.7 + 25.0 + 23.7 + 24.0 + 24.6}{6} = 24.2 \text{ seconds}$

$$\sigma = \sqrt{\frac{(24.2-24.2)^2 + (23.7-24.2)^2 + (25.0-24.2)^2 + (23.7-24.2)^2 + (24.0-24.2)^2 + (24.6-24.2)^2}{6}}$$

$= 0.473 \text{ seconds}$

b) We must divide each term 0.9 to find the correct time. The new mean is $\bar{x} = \frac{24.2}{0.9} = 26.9 \text{ sec}$. The new standard deviation is

$$\sigma = \frac{0.4726}{0.9} = 0.525 \text{ sec}$$

6.9.3 Extended activities

Hospital records indicated that maternity patients stayed in the hospital for the number of days shown in the distribution:

Number of days stayed	Frequency
3	15
4	32
5	56
6	19
7	5
Total	127

Construct

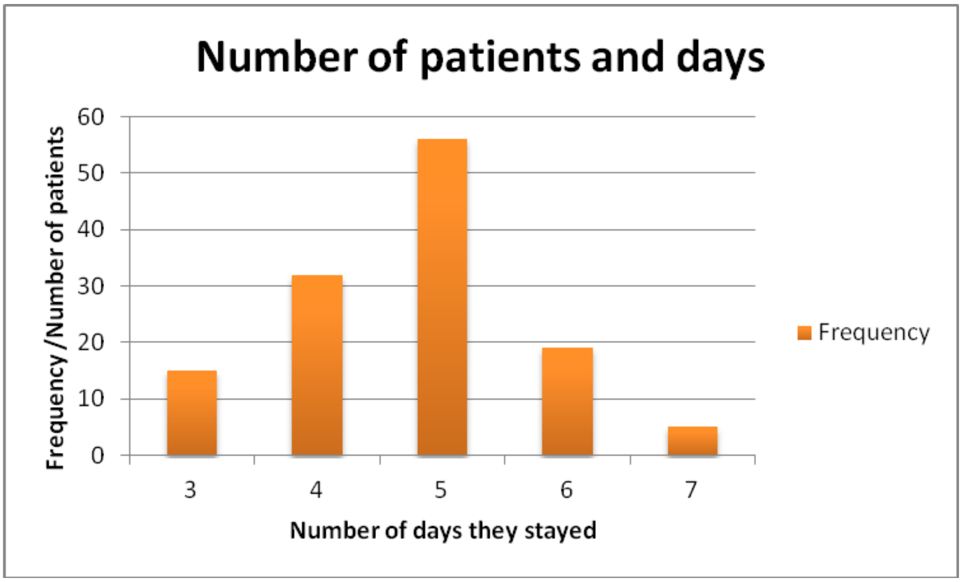
i) Bar graph

ii) Pie Chart

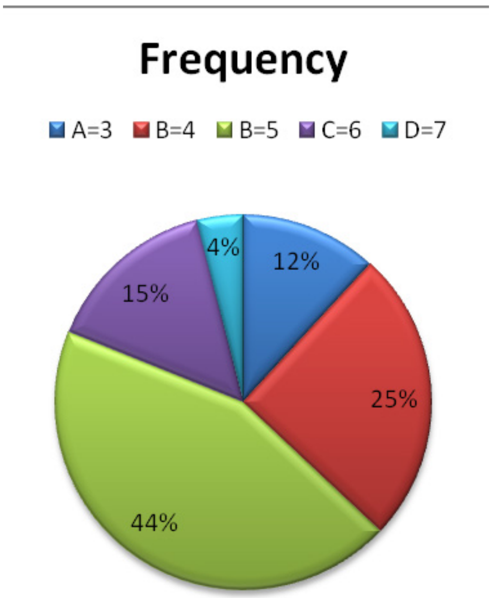
iii) Polygon

Solution

i) Bar graph



ii) Pie chart



UNIT: 7

ELEMENTARY PROBABILITY

7.1 Key unit competence

Use combinations and permutations to determine probabilities of occurrence of an event.

7.2 Prerequisite

Student-teachers will perform well in this unit if they make a short revision on the elementary probability learnt in S2 unit 11.

7.3 Cross-cutting issues to be addressed

- Inclusive education (promote education for all while teaching)
- Peace and value Education (respect others view and thoughts during class discussions)
- Gender (equal opportunity of boys and girls in the lesson participation and making groups)

7.4 Guidance on introductory activity 7

- Form groups of students and invite student-teachers to work on questions for introductory activity 7 found in student's book unit 7;
- Guide students to read and analyse the problem related different cases of the gender that 3children can have: they have to write all those cases on a sheet of paper;
- Guide student-teachers to find out many possibilities as they can;
- Invite students with different working steps to present their findings to the whole class discussion;
- Basing on their experience, prior knowledge and abilities shown in answering the questions for this activity , use different questions to guide student-teachers to give their predictions and ensure that you arouse their curiosity on what is going to be leant in this unit.

Answer for introductory activity 7:

Three children who can be born can have the following gender:If F=female or Girl and M= male or Boy,

$\Omega = \{GGG, GGB, GBG, GBB, BGG, BGB, BBG, BBB\}$; There are 8 possibilities.

Therefore, there is one case under which the woman can have a girl at the first and the second delivery with a boy at the last delivery. This is $\Omega' = \{GGB\}$ which means that she has one chance among 8 possible cases.

7.5. List of lessons/sub-headings

No	Lesson title	Learning objectives	Number of periods
0	Introductory activity	Arouse the curiosity of stent teacher.	1
1.	Concepts of probability: Sample space and Events	Explain probability as a measure of chance and define terms related to probability.	1
2.	Simple counting techniques	Use different counting techniques (Venn diagram, tree diagram and tables) to determine the number of possibilities or occurrence outcomes for an event.	1
3.	Permutation and Arrangements	Define factorial notation and determine the number of different permutations of n different objects (unlike objects) in a row.	1
4.	Permutations of indistinguishable objects	Determine the number of different permutations of n indistinguishable objects with some alike objects.	2
5.	Circular arrangements	Determine the number of arrangements of n unlike things in a circle.	1
6.	Mutually exclusive situation	Distinguish between mutually exclusive and non-exclusive events and determine the number of outcomes for event one or event two given that the two events have different outcomes m and n.	1

7	Distinguishable permutations	Determine the number of different permutations (ways) of r unlike objects selected from n different objects.	2
8	Combinations	Determine the number of different groups of r items that could be formed from a set of n distinct objects where the order of selections is being ignored.	2
9.	Binomial expansion and Pascal's triangles	Apply Pascal's triangle to complete a binomial expansion	2
10.	Determination of Probability of an event, properties and formulas	Determine the probability of an events in real life as a measure of chance	4
11.	Examples of Events in real life and determination of related probability	Appreciate the importance of probability in social sciences.	2
12.	End assessment		1
Total number of periods			21

Lesson 1: Concepts of probability: Sample space and Events

a) Learning objective

Explain probability as a measure of chance and define terms related to probability.

b) Teaching resources

Playing cards, graph papers, manila papers, calculators, coin, dice.

c) Prerequisites/Revision/Introduction:

Student-teachers will perform well in this lesson if they make a short revision on the content learnt as introduction to probability in S2.

d) Learning activities

- Form small groups of student-teachers and give them instructions on how to work on the activity 7.1;

- Walk around to each group and ask probing questions leading them to consider the total number of cards and the number of specified cards;
- Ask each group to share their answers with neighbouring group and ask them to support each other for improvement;
- Invite groups with different working steps to present their findings to the whole class for discussion;
- As a tutor, harmonize their answers highlighting that there are a number of chance of choosing a card.
- Use different probing questions and guide them to explore examples given in the student's book and lead them to explain the main concepts of probability accompanied with examples: events and their types, outcome, sample space, etc.
- After this step, guide students to do the application activity 7.1 and evaluate whether lesson objectives were achieved.

Answers for activity 7.1

1. a) There are 52 cards that can be chosen;

   b) There are 4 kings that can be can be chosen in 52 cards;

   c) There is 1 ace of hearts that can be chosen.

2. Other example of event:

Students can give many examples. As a tutor, verify if they are correct events. Example: selecting a black card, selecting a diamond, etc.

e) Answer for application activity 7.1

Two dice are thrown simultaneously, on has $\{1,2,3,4,5,6\}$ and the second $\{1,2,3,4,5,6\}$. The sample space made by the sum of points is noted which is: $\Omega=\{2,3,4,5,6,7,8,9,10,11,12\}$.

Lesson 2: Simple counting techniques

a) Learning objective

Use different counting techniques (Venn diagram, tree diagram and tables) to determine the number of possibilities or occurrence outcomes for an event.

b) Teaching resources:

Graph papers, manila papers, calculators, coins, dice.

c) Prerequisites/Revision/Introduction:

Student will perform well in this lesson if they make a good revision on the introduction to probability learnt in S2 and the previous lesson.

d) Learning activities

- Help students to form small groups of student-teachers and give them instructions on how to work on the activity 7.2.1;
- Walk around to each group and ask probing questions leading them to determine the total number of roads from A to C via B;
- Ask each group to share their answers with neighbouring group and ask them to support each other for improvement;
- Invite groups with different working steps to present their findings to the whole class for discussion;
- As a tutor, harmonize their answers highlighting that there is a technique of finding the total number of outcomes for a given random experiment;
- Use different probing questions and guide them to explore examples given in the student's book and lead them to determine total number of outcomes for a given random experiment using: venn diagram, tree diagram or a table.
- Guide them to discover that if a sequence of n events in which the first one has n_1 possibilities, the second with n_2 possibilities the third with n_3 possibilities, and so forth until n_k, the total number of possibilities of the sequence will be $= n_1.n_2.n_3 \ldots n_k$
- After this step, guide students to do the application activity 7.2.1 and evaluate whether lesson objectives were achieved.

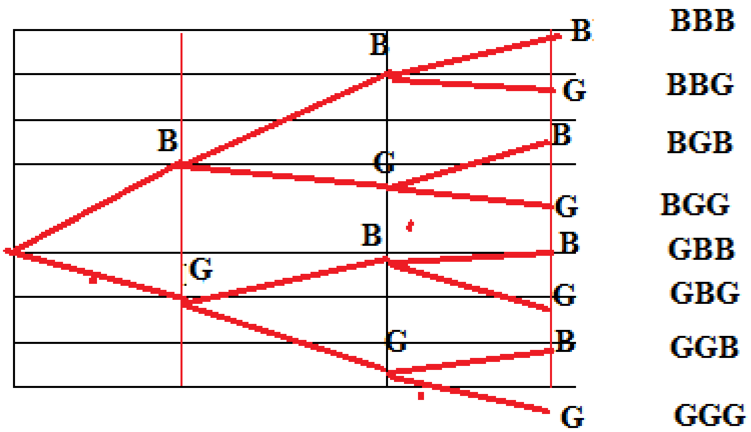
Answers for activity 7.2.1

To find all possible roads, students can use allows to join points or a try and fail method.

$\Omega = \{AB_1C_1, AB_1C_2, AB_1C_3, AB_2C_1, AB_2C_2, AB_2C_3\}$ so they are 6.

e) Answers for application activity 7.2.1

1) Using the tree diagram, one can find:



$S = \{BBB, BBG, BGB, BGG, GBB, GBG, GGB, GGG\}$

2) The coin can land either head up or tails up.

	1	**2**	**3**	**4**	**5**	**6**
H	H1	H2	H3	H4	H5	H6
T	T1	T2	T4	T4	T5	T6

There are 2.6 = 12 possibilities.

Lesson 3: Arrangementof n unlike objects in a row

a) Learning objective:

Define factorial notation and determine the number of different permutations of n different objects (unlike objects) in a row.

b) Teaching resources:

Manila papers, cards with letters, calculators, coins, dice, a bench, etc.

c) Prerequisites/Revision/Introduction:

Student-teachers will perform well in this lesson if they refer to the content of the previous lesson and the properties for multiplication learnt in unit 1 for this level (year 1 SSE).

d) Learning activities

- Form small groups of student-teachers and give them instructions on how to work on the activity 7.2.2: give each group the letter cards to be used and ask them to make all possible arrangements and permutations of those letters (for example: letter R, E and B);
- Walk around to each group and ask probing questions leading them to determine the total number of ways starting by the number of ways to chose the first letter, the second letter and the third letter;
- Ask each group to share their answers with neighbouring group and ask them to support each other for improvement;
- Invite groups with different working steps to present their findings to the whole class for discussion;
- As a tutor, harmonize their answers highlighting that the arrangement of letters is the same as ways of sitting different people on the same bench and that a permutation is an arrangement of n objects in a specific order.
- Use different probing questions and guide them to explore examples given in the student's book and lead them to discover the formula which gives the number of different permutations of *n* different objects (unlike objects) in a row.
- Guide them to discover that this number corresponds to $n!$(read *n* factorial) and explore the related properties.
- After this step, guide students to do the application activity 7.2.2 and evaluate whether lesson objectives were achieved.

Answers for activity 7.2.2

Possible arrangements for three letters R, E and B are $\{REB, \mathrm{R}\,BE, ERB, EBR, BER, BRE\}$;

The possible arrangement for these three letters is 6. This can be found by: $3! = 3.2.1 = 6$

e) Answers for application activity 7.2.2

1) a) $\frac{5!}{2!}=\frac{5\times4\times3\times2\times1}{2\times1}=60$;

b) $\frac{10!}{6!7!}=\frac{10\times9\times8\times7!}{6\times5\times4\times3\times2\times1\times7!}=\frac{5\times2\times3\times3\times4\times2}{3\times2\times5\times4\times3\times2\times1}=1$

2) Four different English books, five different Biology books and ten other books are to be arranged on a bookshelf:

a) $(4+5+10)! = 19! = 1.216451004\times10^{17}$;

b) Since the 3 biology books have to be together, consider these bound together as one book, there are now $(16+1)!=17!$ books to be arranged and these can be calculated using a calculator and find 2.134124569×10^{15}

Lesson 4: Arrangement of indistinguishable objects (Permutations with repetition)

a) Learning objective:

Determine the number of different permutations of n indistinguishable objects with some alike objects.

b) Teaching resources:

Bench, shelves of books, manila papers, calculators, coins, dice.

c) Prerequisites/Revision/Introduction:

Students will perform well in this lesson if they learnt well the content of the previous lesson in this unit;

d) Learning activities

- Form groups of student-teachers and give them instructions on how to work on the activity 7.2.3: give each group the letter cards to be used and ask them to make all possible permutations of those letters in which some letters are the same (for example: letter of the word BOOM, the two O are not separable);
- Walk around to each group and ask probing questions leading them to determine the total number of permutations considering that it

is not possible to distinguish the two letters "O";

- Ask each group to share their answers with neighbouring group and ask them to support each other for improvement;
- Invite groups with different working steps to present their findings to the whole class for discussion;
- As a tutor, harmonize their answers highlighting that the number of permutations of will reduce: the total number of permutations 4! Will be divided by the number of permutations of identical letters which is 2! and find $\frac{4!}{2!}$.
- Use different probing questions and guide students to explore examples given in the student's book and lead them to discover the formula which gives the number of different permutations of n i**ndistinguishable** objects with n_1 alike, n_2 alike, ..., which is $\frac{n!}{n_1!n_2!...}$.
- After this step, guide students to do the application activity 7.2.3 and evaluate whether lesson objectives were achieved.

Answers for activity 7.2.3

Let us take numbers from 1,2,3,and 4.

1) $\Omega = \begin{Bmatrix} 1234, 1243, 1324, 1342, 1423, 1432 \\ 2134, 2143, 2314, 2341, 2413, 2431 \\ 3124, 3142, 3214, 3241, 3412, 3421 \\ 4123, 4132, 4213, 4231, 4312, 4321 \end{Bmatrix}$ the possible number is 4!=24;

2) Students will try to make possible arrangements but some of them will be the same.

The number of all possible arrangements when writing once the identical arrangement is $\frac{4!}{2!}$.

e) Answers for application activity 7.2.3

1) a) Arrangements that can be made from the letters of the word ENGLISH are 7!=5,040;

b) Arrangements that can be made from the letters of the word MATHEMATICS are

$\frac{11!}{(2!)(2!)(2!)}$ because there are 2M, 2A and 2T which are indistinguishable.

2) Alphabet in English=26!= 4.032914611×10^{26};

3) $\Omega=\frac{9!}{4!3!2!}=1,260$.

Lesson 5: Circular arrangements

a) Learning objective:

Determine the number of arrangements of *n* unlike things in a circle

b) Teaching resources:

Manila papers, calculators, coins, dice.

c) Prerequisites/Revision/Introduction:

Student-teachers will perform well in this lesson if they learnt well the content of previous lesson.

d) Learning activities

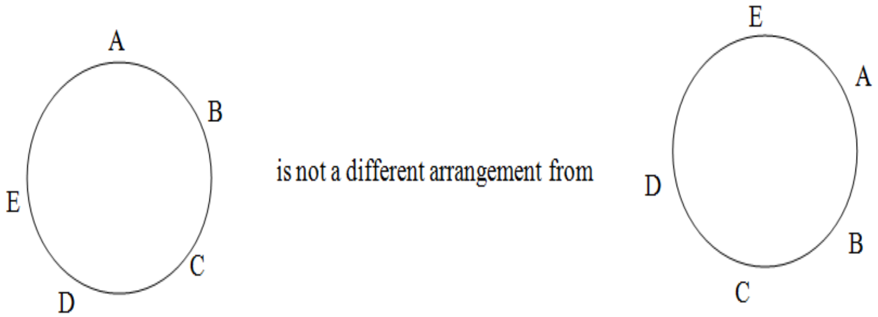
- Form groups of student-teachers and give them instructions on how to work on the activity 7.2.4: each group may have a circular table and objects to be arranged on that table;
- Walk around to each group and ask probing questions leading them to determine the total number of arrangements when one item is fixed and the remaining items arranged around it;
- Invite groups with different working steps to present their findings to the whole class for discussion;
- As a tutor, harmonize their answers and guide them to discover that the number of arrangements of *n* unlike things in a circle will

therefore be $(n-1)!$. Guide students to note that where clockwise and anticlockwise arrangements are not considered to be different, this reduces to $\frac{1}{2}(n-1)!$.

- After this step, guide students to do the application activity 7.2.4 and evaluate whether lesson objectives were achieved.

Answers for activity 7.2.4:

As one notebook will be fixed, for example A, must be: $(n-1)! = (5-1)! = 24$



e) Answers for application activity 7.2.4

1. Five men will seat on a circular table in $(5-1)!\text{ways} = 24\text{ ways}$.
2. Eleven different books will be placed on a circular table in $(11-1)!\text{ways} = 3,268,800\text{ ways}$.

Lesson 6: Basic sum principle of counting for Mutually exclusive situations

a) Learning objective:

Distinguish between mutually exclusive and non-exclusive events and determine the number of outcomes for event one or event two given that the two events have different outcomes m and n.

b) Teaching resources:

Manila papers, calculators, coins, dice.

c) Prerequisites/Revision/Introduction:

Students will perform well in this lesson if they learnt well introduction to probability found in unit 11of S2.

d) Learning activities

- Invite students to discuss in pairs the activity 7.2.5:
- Walk around to each pair and ask probing questions leading them to determine the total number of outcomes given that the number of outcomes for event one and the outcomes for event 2 are known;
- Invite two neighbouring pairs to work together, exchange ideas and improve their work;
- Visit each new formed group and identify groups with different working steps;
- Invite representatives from groups with different working steps to present their work for a whole class discussion;
- As a tutor, harmonize their answers and guide students discover that If Experiment 1 has **m** possible outcomes and if experiment 2 has **n** possible outcomes, then an experiment which might be experiment 1 or experiment 2, called **experiment 1 or 2**, has $m+n$ possible outcomes;
- After this step, guide students to do the application activity 7.2.4 and evaluate whether lesson objectives were achieved.

Answers for activity 7.2.5

1) Answers will vary from group to another. But help them to conclude that one has chances of picking either the soup or the juice but not all together. One is allowed two chances.

2) We need the numbers that end in 3: the last digit can be chosen in one way, as it must be a 3, the first digit can then be chosen in 3 ways, the second in 2 ways and the third in 1 way. Thus, there are $1\times3\times2\times1=6$ numbers that end in a 3.

Similarly, there are $1\times3\times2\times1=6$ numbers that end in a 4.

The number that ends in a 3 cannot also end in a 4, so these are mutually exclusive situations.

Thus, there are $6+6=12$ numbers end either in a 3 or in a 4.

Alternatively, this can be solved as follows:

The last digit can be chosen in 2 ways (3 or 4); the first digit can be chosen in 3 ways, the second in 2 ways and the third in 1 way, i.e, $2\times3\times2\times1=12$ numbers end either in a 3 or in a 4.

e) Answer for application activity 7.2.5:

1) 4

2) 336

Lesson 7: Permutation of distinguishable objects

a) Learning objective:

Determine the number of different permutations (ways) of r unlike objects selected from n different objects.

b) Teaching resources:

Manila papers, calculators, coins, dice.

c) Prerequisites/Revision/Introduction:

Students will perform well if they learnt well the previous lessons of this unit.

d) Learning activities

- Form groups of student-teachers and give them instructions on how to work on the activity 7.2.5: give each group the letter cards to be used and ask them to make all possible ways of selecting 3 letters from 7 given letters of the word *PRODUCT*.
- Walk around to each group and ask probing questions leading them to determine the total number of ways of selecting the first letter, the second letter and the third letter;
- Invite groups with different working steps to present their findings to the whole class for discussion;
- As a tutor, harmonize their answers highlighting that the number of ways for selecting 3 letters from 7 is the number of permutations of 3 objects taken from 7 written as ${}^7P_3$. Guide them to enhance that the first letter can be selected in 7 ways, the second can be selected in 6 ways and the third can be selected in 5 ways;
- Ask all students to guess how they can write the product $7.6.5$ using the factorial notation which lead them to guess

$$7.6.5 = \frac{7.6..5.4.3.2.1}{4.3.2.1} = \frac{7!}{(7-3)!};$$

- Use different probing questions and guide students to explore examples given in the student's book and lead them to discover the formula which gives the number of different permutations (ways) of r unlike objects selected from n different objects given by $^{n}P_{r} = \frac{n!}{(n-r)!}$ which can also be written as $P(n,r) = \frac{n!}{(n-r)!}$;
- After this step, guide students to do the application activity 7.2.6 and evaluate whether lesson objectives were achieved.

Answers for activity 7.2.6

Students will work in different ways; as a tutor, verify whether all possible ways are given:

Selection: PRO: permutations: PRO,POR, OPR, ORP, RPO, ROP

Selection: ROD: permutations: ROD, RDO, ORD, ODR, DRO, DOR

Selection: ODU: permutations: ..., ..., ..., ..., ..., ...

Selection: DUC: Permutations: ..., ..., ..., ..., ..., ...

Selection: UCT: Permutations: ..., ..., ..., ..., ..., ...

.... : .....: ....

There are 35 lines with 6 permutations which gives the number $7.6.5 = 210$ permutations.

e) Answer for the application activity 7.2.6

1) Number of permutations with 4 letters chosen from letters of the word ENGLISH: $^{7}P_{4} = 840$

2) Number of permutations with 2 letters chosen from letters of the word PACIFIC: 13

3) Number of permutations with 5 letters chosen from letters A, B, C, D, E, F, and G is

$^{7}P_{5}$.

4) Number of permutations with 10 letters chosen from English alphabet is $^{26}P_{10}$.

Lesson 8: Combinations

a) Learning objective:

Determine the number of different groups of *r* items that could be formed from a set of *n* distinct objects where the order of selections is being ignored.

b) Teaching resources:

Manila papers, calculators, coins, dice.

c) Prerequisites/Revision/Introduction:

Students will perform well if they learnt well the content for the previous lessons for this level.

d) Learning activities

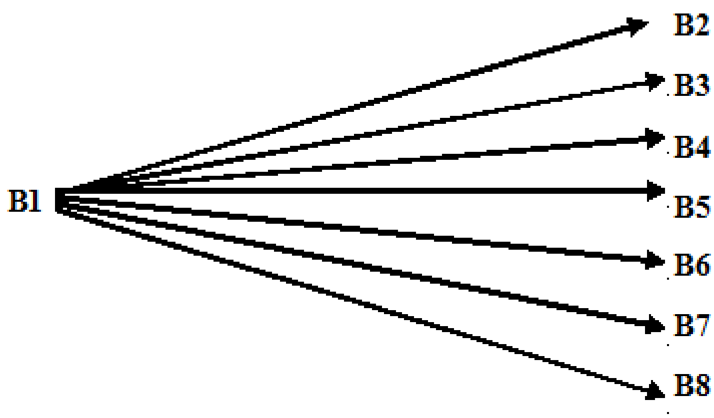
- Form groups of student-teachers and give them instructions on how to work on the activity 7.2.7;
- Walk around to each group and ask probing questions leading them to determine the total number of groups each containing 2 mathematics books from 8 mathematics books;
- Invite groups with different working steps to present their findings to the whole class for discussion;
- As a tutor, harmonize their answers highlighting that in this case, the order in which books are placed is not important (the group of B1B2 is the same as the group B2B1) which is contrary to the permutation of *r* unlike objects selected from *n* different objects where the order in which those objects are placed is important.
- Lead students to see that in this case, we must divide by the 2! (or generally by the arrangement r!) as the order is not important; we get $\frac{7.8}{2}=\frac{8!}{(8-2)!2!}$.
- Use different probing questions and guide students to explore examples given in the student's book and lead them to discover the formula which gives the number of different groups of r items that

could be formed from a set of n distinct objects with the order of selections being ignored is $^{n}C_r = \frac{n!}{(n-r)!r!}$

- After this step, guide students to do the application activity 7.2.6 and evaluate whether lesson objectives were achieved.

Answers for activity 7.2.6:

The book B1 can be participating in 7 different groups as follows:



Idem, every book Bi can participate in 7 different groups. This means that we have $8.7 = 56$ groups. However, as for example the group B2B3 and the group B3B2 make a same group, we have to divide by 2.

Which gives $\frac{7.8}{2}$ groups.

By the use of factorial notation we have: $\frac{7.8}{2} = \frac{8!}{(8-2)!2!} = 28$

e) Application activity 7.2.7

1) Four men can be selected from 10 men, i.e $^{10}C_4 = \frac{10!}{(10-4)!4!}$ ways

Two women can be selected from 12women, i.e $^{12}C_2 = \frac{12!}{(12-2)!2!}$ ways

By the basic product principle of counting, there are $\left(^{10}C_4\right)\left(^{12}C_2\right)$ ways of selecting the committee.

2) In the same ways, groups containing 4 Mathematics books and 5 Physics books is formed from 9 Mathematics books and 10 Physics books in ${}^{9}C_4 \times {}^{10}C_5$ ways.

Lesson 9: Binomial expansion and Pascal's triangles

a) Learning objective:

Apply Pascal's triangle to complete a binomial expansion in mathematics expressions.

b) Teaching resources:

Manila papers, calculators, notebooks and pens.

c) Prerequisites/Revision/Introduction:

Students will perform well if they learnt well the binomial expansion and properties of powers learnt in S2.

d) Learning activities

- In small group discussions, invite student-teachers to answer the questions in activity 7.3
- Ask student-teachers to share their answers with another group and ask them to support each other where they became more challenged in solving that activity.
- Request student-teachers to present their findings in a whole class discussion.
- As a tutor, harmonize answers presented by students and guide theme to determine the coefficients of powers in a binomial expansion.
- Use different probing questions and guide students to explore examples given in the student's book and lead them to discover the coefficient of $a^{n-r}b^r$ in the expansion of $(a+b)^n$ is given by ${}^{n}C_r = \dfrac{n!}{(n-r)!r!}$.

Answers for activity 7.3

Use these expansions and complete the table

$(a+b)^2 = a^2 + 2ab + b^2$

$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$

$(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$

$(a+b)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$

Power	Coefficient of powers of a and b					Binomial expression
0	1					$(a+b)^0$
1	1	1				$(a+b)^1$
2	1	2	1			$(a+b)^2$
3	1	3	3	1		$(a+b)^3$
4	1	4	6	4	1	$(a+b)^4$

It is clear that the coefficients of $a^{n-r}b^r$ in the expansion of $(a+b)^n$ are given by ${}^nC_r = \frac{n!}{(n-r)!r!}$

e) Answers for Application activity 7.3:

1) The coefficient of x^2 in the expansion of $(4x+1)^6$ is 240

2) The coefficient of x^3 in the expansion of $\left(x+\frac{1}{x}\right)^4$ is 0

3) The coefficient of x^6 in the expansion of $(9x-3)^{10}$ is 9039811410

Lesson 10: Determination of Probability of an event, properties and formulas

a) Learning objective:

Determine the probability for outcomes of an event in real life as a measure of chance.

b) Teaching resources:

Manila papers, calculators, coins, dice, etc.

c) Prerequisites/Revision/Introduction:

Student-teachers will perform well if they learnt well all previous lessons for this unity and the introduction to probability learnt in S1 and S2.

d) Learning activities

- Let students work in groups and do the activity 7.4;
- Go around to each group and ask probing questions to guide students to work towards the correct answer;
- Ask neighbouring groups to share their answers and ask them to support each other where they become more challenged in solving that activity.
- Request student-teachers to present their findings in a whole class discussion;
- As a tutor, harmonize answers for students and highlight how to determine the probability of an event using the classical probability.
- Use different probing questions and guide students to explore examples given in the student's book and lead them to establish and use properties of probability, determine probability of different events: certain event, impossible event, probability of complementary event, mutually exclusive or incompatible or events.
- After this step, guide students to do the application activity 7.4 and evaluate whether lesson objectives were achieved.

Answers for activity 7.4

a) There are 25 black cards in an ordinary deck of 52 cards.

b) $P(\text{A}) = \dfrac{n}{number\ of\ allcards} = \dfrac{26}{52} = 0.5$

c) $P(A) = \dfrac{Number\ of\ outcome\text{s in E}}{Total\ \text{nu}mber\ of\ outcome\text{s in}\ the\ sample\ space} = \dfrac{n(\text{E})}{n(\Omega)}$

e) Answers for application activity 7.4:

1) $P(A \cup B) = \frac{1}{5} + \frac{1}{3} = \frac{8}{15}$

2) $P(A \cup B) = \frac{1}{3} + x = \frac{7}{10} \Rightarrow x = \frac{11}{30}$

3) a) $\frac{3}{8}$; b) $\frac{5}{8}$; c) $\frac{1}{32}$

Lesson 11: Examples of Events in real life and determination of related probability

a) Learning objective:

Appreciate the importance of probability in social sciences

b) Teaching resources:

Manila papers, calculators, coins, dice, etc.

c) Prerequisites/Revision/Introduction:

Students will perform well in this lessons if they try to relate the content for all previous lessons for this units to real life problems involving events of chance.

d) Learning activities

- Invite students to work in small groups, discuss the betting explained in the activity 7.5 and answer to related questions;
- Ask neighbouring groups to share their answers and support each other by exchanging arguments on that activity;
- Invite group representatives to resent the findings in a whole class discussions;
- Tutor harmonizes answers for students on activity 7.5 and guide students to brainstorm their worries about the betting without a good prediction of probability for winning.
- Guide students to discuss other application of probability in real life and take decisions on eventual risks in betting and other probability related games.

- Use different probing questions and guide students to explore examples given in the student's book and lead them to realize that probability is applicable in the real life.
- After this step, guide students to do the application activity 7.5 and evaluate whether lesson objectives were achieved.

Answers for activity 7.5

1) (a) Let A stands for APR and R stands for RAYON SPORTS;
$\Omega = \{RRR, RRA, RAR, RAA, ARR, ARA, AAR, AAA\}$.

Matayo said that APR will gain the first Match only and Rayon Sport will gain the second and the third, it means that =
$E = \{ARR\}$ *and* $P(E) = \frac{1}{8}$

Manasseh said that APR will gain at least two matches, it means that:
$F = \{RAA, ARA, AAR, AAA\}$ *and* $P(F) = \frac{4}{8} = \frac{1}{2}$

From these results, we see that Manasseh has more chances of winning that money than Matayo.

b) Normally betting is a game of chance, it is not good to bet much money without a good and clear prediction of the probability for winning. When you bet without such clear prediction, you are wasting your money. We can advise the young people not to spend much money in such games which do not have clear rules which can help the player to predict the probability of winning.

2) Students may come up with different applications of probability in real life, analyse them and organize a session for feedback in which they can discuss their strengths and weaknesses.

e) Answers for Application activity 7.5:

1) Let E: a student own a car, $P(\text{E}) = 0.65$, F: a student owns a computer; $P(\text{F}) = 0.82$

We have $P(\text{E} \cap \text{F}) = 0.55$

Question: what is the probability that a given student owns neither a car nor a computer?

i.e $1-P(\mathrm{E}\cup\mathrm{F})=?$

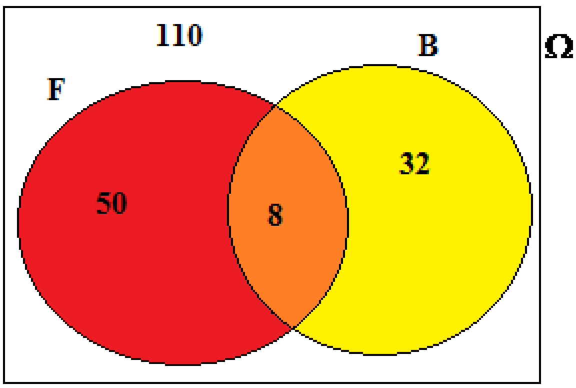
We have:

$$P(\mathrm{E}\cup\mathrm{F})=P(E)+P(E)-P(E\cup F)$$
$$=0.65+0.82-0.55$$
$$=0.92$$

Therefore, $1-P(\mathrm{E}\cup\mathrm{F})=1-0.92=0.08$

2) Using a Venn diagram, one can represent the number of students:

Let Ω be the sample space formed by all students, F the set of students who play Football and B the set of students who play Basket Ball. The number of students can be given in the following sets.



$n(\Omega)=200$, $n(\mathrm{F})=58$, $n(\mathrm{B})=40$, and $n(\mathrm{F}\cap\mathrm{B})=8$.

The number of students who plays Foot Ball or basket Ball is $n(\mathrm{F}\cup\mathrm{B})=90$

The number of student who play neither sport is

$$n(\mathrm{F}\cup\mathrm{B})'=n(\Omega)-n\left(F\cup B\right)$$
$$=200-90=110$$

7.6. Summary of the unit

Sample space

The totality of all possible outcome (or sample points) of a random experiment constitutes the sample space which is denoted by Ω.

Complementary events

If E is an event, then E' is the event which occurs when E does not occur. Events E and E' are said to be complementary events

Mutually exclusive Events

When $A \cap B = \phi$, the two events A and B are said to be mutually exclusive. This means that they cannot occur at the same time, they do not have outcomes in common.

Counting techniques

- Use of Venn diagram,
- Use of tree diagrams,
- Use of a table,
- The number of different permutations of n different objects (unlike objects) in a row is

  $n \times (n-1) \times (n-2) \times (n-3) \times \ldots \times 2 \times 1$

- *The number of different permutations of n i**ndistinguishable***

  objects with n_1 alike, n_2 alike, …, is $\frac{n!}{n_1!n_2!\ldots}$.
- The number of different permutations (ways) of r unlike objects

  selected from n different objects is $^nP_r = \frac{n!}{(n-r)!}$ or we can use the

  denotation $P_r^n = \frac{n!}{(n-r)!}$ or $P(n,r) = \frac{n!}{(n-r)!}$
- Circular arrangements

The *number of arrangements of n unlike things in a circle will therefore be* $(n-1)!$. *In those cases where clockwise and anticlockwise arrangements are not consid*ered to be different, this reduces to $\frac{1}{2}(n-1)!$.

- **Mutually exclusive situations Mutually exclusive situations**

"*If Experiment* 1 *has* **m** *possible outcomes*
and if experiment 2 *has* **n** *possible outcomes*,
then an experiment which might be experiment 1 *or experiment* 2,
called **experiment 1 or 2**, *has* $(m+n)$ *possible outcomes*."

- Combination

The number of different groups of r items that could be formed from a set of n distinct objects with the order of selections being ignored is

$$^{n}C_{r}=\frac{n!}{(n-r)!r!}.$$

Probability of an event

- The probability of an event $A\subset\Omega$, is a real number obtained by applying to A the function P defined by

$$P(A)=\frac{Number\ of\ outcomes\ in\ A}{Total\ number\ of\ outcomes\ in\ the\ sample\ space}=\frac{n(A)}{n(\Omega)}$$

- When E and E' are complementary events, $P(E)=1-P(E')$.
- *When two events A and B are not mutually exclusive,* $A\cap B=\phi$ *the probability that A or B occurs is given by:*

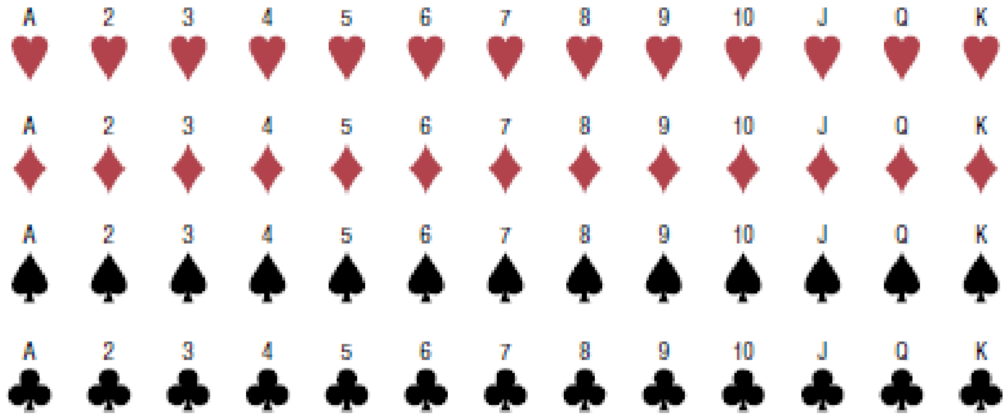
  $P(A\cup B)=\text{P(A)}+\text{P(B)}-\text{P}(A\cap B)$

Note:

We have to think big before taking decision regarding our engagement in the games of chance. Such games are for example: betting on card games, slot machine (ikiryabarezi), lotteries, and weather forecasting. In such games, predictions are based on probability and hypotheses are tested by using probability.

7.7 Additional Information for teachers

7.7.1 Components of an ordinary deck of cards:



7.7.2 Law of Large Numbers

When a coin is tossed one time, it is common knowledge that the probability of getting a head is $\frac{1}{2}$. But what happens when the coin is tossed 50 times? Will it come up heads 25 times? Not all the time. You should expect about 25 heads if the coin is fair. But due to chance variation, 25 heads will not occur most of the time.

If the empirical probability of getting a head is computed by using a small number of trials, it is usually not exactly $\frac{1}{2}$. However, as the number of trials increases, the empirical probability of getting a head will approach the theoretical probability of $\frac{1}{2}$, if in fact the coin is fair (i.e., balanced). This phenomenon is an example of the law of large numbers.

7.7.3 Independent events

Two events A and B are **independent events** if the fact that A occurs does not affect the probability of B occurring.

When two events are independent, the probability of both occurring is

$$\text{P(A and B)} = P(A).P(B)$$

Example:

A coin is flipped and a die is rolled. Find the probability of getting a head on the coin and a 4 on the die.

Solution:

P(Head and 4)= P(head).P(4)= $\frac{1}{2}.\frac{1}{4}=\frac{1}{8}$

7.8 End unit assessment

This question is related to a Lottery that can be played at school to show that it is not good to be blindly engaged in some games of chance as the participant can lose his/her money.

An urn contains 20 lottery tickets numbered from 1 to 20.

To buy a ticket, each one is selected at random and replaced before the next selection. The organizer of the lottery decided to pay 1000Frw to the one who will select a number divisible by 4 and 3 at the same time. He will pay also 500Frw to the one who will select a number which is divisible by 5 and 2 at the same time.

1) a) The game is not fair because the organizer decided him/her self the number of money to be collected;

b) The organize will receive 200Frw x 20 = 4000Frw

c) Let the following events: *A: selecting a number divisible by 4;*

B: selecting the number divisible by 3;

C: selecting the number divisible by 5, and

D: Selecting the number divisible by 2.

E= selecting a number divisible by 3 and 4 =$\{12\}$, P(E)= $\frac{1}{20}$. This means that only one participant will win and get 1000F.

d) F= Selecting a number divisible by 2 and 5=$\{10;20\}$; P(F) = $\frac{2}{20}$. This means that only two participants will win and each one will get 500Frw

e) The organizer will pay: $1000Frw + 2(500Frw) = 2000Frw$.

Therefore, the organizer will make money.
This money equals to $4000Frw - 2000Frw = 2000Frw$

2. The parents of your friend Anne Marie gave her 200Frw for buying two

pens, however, she wants to participate in the lottery to get more money before buying pens. What can you advise her?

Answers from students will vary, however, guide them to conclude that this is a game of chance where there is a high probability of loosing. It is clear that the organizer planned to get money without investing anything. It is clear that only 3 participants will win. Therefore, there is a high probability for Anne Marie to lose the money which and miss a pen.

7.9 Additional activities

7.9.1 Remedial activity

1) A card is drawn at random from an ordinary pack of 52 playing cards. Find the probability that the card drawn is a club or a diamond.

Solution

There are 13 clubs, then $P(\text{club}) = \frac{13}{52}$

There are 13 diamonds, then $P(\text{diamond}) = \frac{13}{52}$

Since a card cannot be both a club and a diamond, $P(\text{club} \cap \text{diamond}) = 0$

Therefore, P(a club or a diamond) = P(club) + P(diamond)

$$P(\text{a club or a diamond}) = P(\text{club}) + P(\text{diamond})$$

$$= \frac{13}{52} + \frac{13}{52}$$

$$= \frac{26}{52}$$

$$= \frac{1}{2}$$

7.9.2 Consolidation activity

1) In a group of 20 adults, 4 out of the 7 women and 2 out of the 13 men wear glasses. What is the probability that a person chosen at random

from the group is a woman or someone who wears glasses?

Solution

Let A be the event: "the person chosen is a woman".

B be the event: "the person chosen wears glasses".

Now, there are 7 women, then $P(A)=\frac{7}{20}$

There are 6 persons who wear glasses, then $P(B)=\frac{6}{20}$

There are 4 women who wear glasses, then $P(A\cap B)=\frac{4}{20}$

The probability that a person chosen at random from the group is a woman or someone who wears glasses is given by $P(A\, or\, B)$ which is

$$P(A\cup B)=P(A)+P(B)-P(A\cap B)$$

$$=\frac{7}{20}+\frac{6}{20}-\frac{4}{20}$$

$$=\frac{9}{20}$$

On the other hand:

There are 7 women and 6 persons who wear glasses but also 4 women who wear glasses. Then

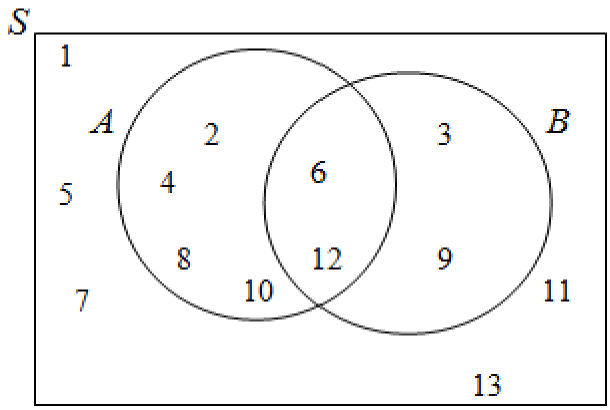
$A\cup B=9$ and $P(A\cup B)=\frac{9}{20}$.

7.9.3 Extended activity

1) An integer is chosen at random from the set $S=\{x:x\in\mathbb{Z}^{+},x<14\}$. Let A be the event of choosing a multiple of 2 and let B be the event of choosing a multiple of 3.

Find $P(A\cup B)$, $P(A\cap B)$ and $P(A-B)$.

Solution



From the diagram, $\#S = 13$

$A \cup B = \{2,3,4,6,8,9,10,12\} \Rightarrow \#(A \cup B) = 8$, thus $P(A \cup B) = \frac{8}{13}$

$A \cap B = \{6,12\} \Rightarrow \#(A \cap B) = 2$, thus $P(A \cap B) = \frac{2}{13}$

$A - B = \{2,4,8,10\} \Rightarrow \#(A - B) = 4$, thus $P(A - B) = \frac{4}{13}$

2) Suppose, that a researcher in RAB asked 50 staff members how they go home.

The results can be categorized in a frequency distribution as shown in the table below

Method	**Frequency**
drive	20
Fly	6
Bus	24

Determine:

a) The probability of selecting a person who goes home by driving;

b) Probability of selecting a person who goes home in an air plane;

c) The probability of selecting a person who goes home in a bus.

d) The sum of the probability.

3) In a sample of 50 people, 21 had type O blood, 22 had type A blood, 5 had

type B blood, and 2 had type AB blood. Set up a frequency distribution and find the following probabilities.

a) A person has type O blood.

b) A person has type A or type B blood.

c) A person has neither type A nor type O blood.

d) A person does not have type AB blood.

Solution

Type	**Frequency**
A	22
B	5
AB	2
O	21
Total	50

They are mutually exclusive.

a) $P(O) = \frac{f}{n} = \frac{21}{50}$

b) $P(\text{A or B}) = \frac{22}{50} + \frac{5}{50} = \frac{27}{50}$

c) Neither A nor O means that a person has either type B or type AB blood.)

$$P(\text{neither A nor O}) = \frac{5}{50} + \frac{2}{50} = \frac{7}{50}.$$

d) Find the probability of not AB by subtracting the probability of type AB

from 1.

$$P(\text{not AB}) = 1 - P(AB) = 1 - \frac{2}{50} = \frac{48}{50} = \frac{24}{25}$$

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