MATHEMATICS BOOK FOR TTC

STUDENT'S BOOK

YEAR

1

OPTION:

SOCIAL STUDIES EDUCATION (SSE)

FOREWORD

Dear Student,

Rwanda Basic Education Board (REB) is honored to present Year one Mathematics book for Social Studies Education (SSE) Student Teachers. this book serves as a guide to competence-based teaching and learning to ensure consistency and coherence in the learning of the Mathematics content. The Rwandan educational philosophy is to ensure that you achieve full potential at every level of education which will prepare you to be well integrated in society and exploit employment opportunities.

The government of Rwanda emphasizes the importance of aligning teaching and learning materials with the syllabus to facilitate your learning process. Many factors influence what you learn, how well you learn and the competences you acquire. Those factors include the relevance of the specific content, the quality of teachers' pedagogical approaches, the assessment strategies and the instructional materials available. In this book, we paid special attention to the activities that facilitate the learning process in which you can develop your ideas and make new discoveries during concrete activities carried out individually or with peers.

In competence-based curriculum, learning is considered as a process of active building and developing knowledge and meanings by the learner where concepts are mainly introduced by an activity, situation or scenario that helps the learner to construct knowledge, develop skills and acquire positive attitudes and values.

For efficiency use of this textbook, your role is to:

- Work on given activities which lead to the development of skills;
- Share relevant information with other learners through presentations, discussions, group work and other active learning techniques such as role play, case studies, investigation and research in the library, on internet or outside;
- Participate and take responsibility for your own learning;
- Draw conclusions based on the findings from the learning activities.

To facilitate you in doing activities, the content of this book is self explanatory so that you can easily use it yourself, acquire and assess your competences. The book is made of units as presented in the syllabus. Each unit has the following structure: the key unit competence is given and it is followed by the introductory activity before the development of mathematical concepts that are connected to real world problems or to other sciences.

The development of each concept has the following points:

- It starts by a learning activity: it is a hand on well set activity to be done by students in order to generate the concept to be learnt;

- Main elements of the content to be emphasized;
- Worked examples; and
- Application activities: those are activities to be done by the user to consolidate competences or to assess the achievement of objectives.

Even though the book has some worked examples, you will succeed on the application activities depending on your ways of reading, questioning, thinking and grappling ideas of calculus not by searching for similar-looking worked out examples.

Furthermore, to succeed in Mathematics, you are asked to keep trying; sometimes you will find concepts that need to be worked at before you completely understand. The only way to really grasp such a concept is to think about it and work related problems found in other reference books.

I wish to sincerely express my appreciation to the people who contributed towards the development of this book, particularly, REB staff, UR-CE Lecturers and TTC Tutors for their technical support. A word of gratitude goes to Head Teachers and TTCs principals who availed their staff for various activities.

Any comment or contribution would be welcome to the improvement of this text book for the next edition.

Dr. MBARUSHIMANA Nelson
Director General, REB

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I wish to express my appreciation to the people who played a major role in the development of this Mathematics book for Year one student teachers in Social Studies Education (SSE).

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Joan MURUNGI
Head of CTLR Department

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UNIT: 1

ARITHMETICS

Key Unit competence: Use arithmetic operations to solve simple real life problems

1.0. Introductory Activity

The simple interest earned on an investment is $I = prt$ where I is the interest earned, P is the principal, r is the interest rate and t is the time in years. Assume that 50,000Frw is invested at annual interest rate of 8% and that the interest is added to the principal at the end of each year.

a. Discuss the amount of interest that will be earned each year for 5 years.

b. How can you find the total amount of money earned at the end of these 5 years? Classify and explain all Mathematics operations that can be used to find that money.

1.1 Operations of real numbers and their properties

1.1.1. Sets of numbers

Activity 1.1.1

a. Carry out research on sets of numbers to determine the meanings of natural numbers, integers, rational numbers and irrational numbers. Use your findings to select the natural numbers, integers, rational numbers and irrational numbers in the list of the numbers given below:

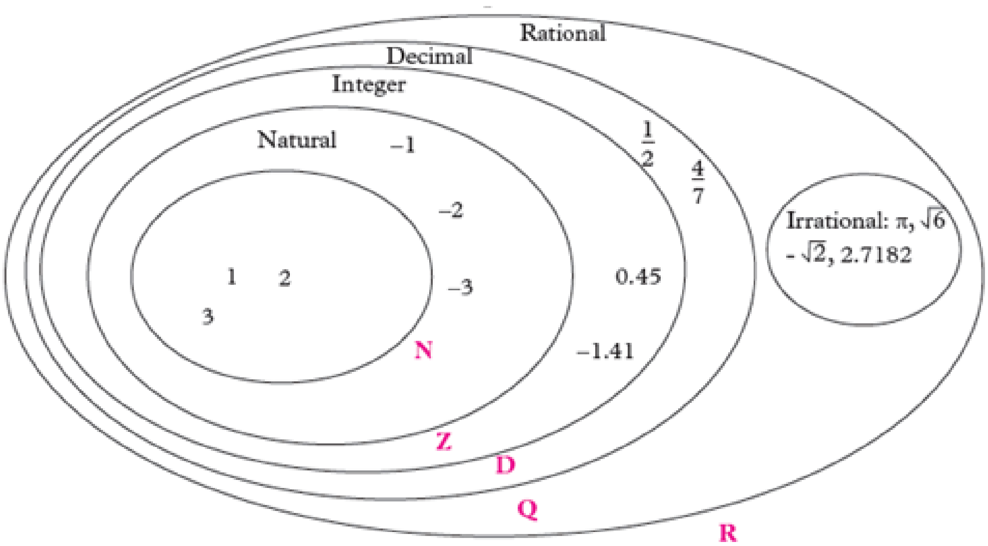
$0; 1; -5; 6; \frac{3}{4}; 3.146; 1.3333....; \pi; \sqrt{8}$.

b. From (a) deduce the definition of the set $\mathbb{R}$ of real numbers.

c. Given any 3 real numbers a, b and c of your choice, discuss the following:

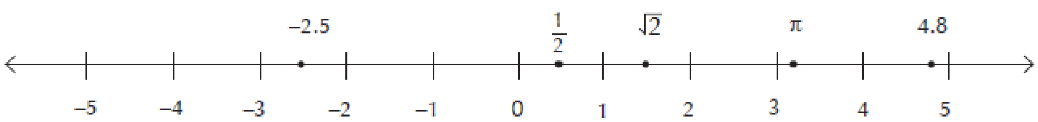
$a + b = b + a, ab = ba, a(bc) = (ab)c$. Is $\frac{a}{b}$ a real number for every value of b?

The rational and irrational numbers together make up the set of real numbers denoted by $\mathbb{R}$. The sets $\mathbb{N}$ of natural numbers, $\mathbb{Z}$ of integers, D of limited decimals and $\mathbb{Q}$ of rational numbers are all subsets of $\mathbb{R}$. In fact $\mathbb{N} \subset \mathbb{Z} \subset D \subset \mathbb{Q} \subset \mathbb{R}$.



There exits some numbers called irrational numbers which are elements of $\mathbb{R}$ but not elements of $\mathbb{Q}$; they cannot be written in the form of a fraction. The set of such number is sometimes denoted by I with $I \subset \mathbb{R}$. Example: $1.34782....; \pi; \sqrt{2}; \sqrt{5}; 3\sqrt{7}, etc$

The one-to-one correspondence between the real numbers and the points on the number line is familiar to us all. Corresponding to each real number there is exactly one point on the line: corresponding to each point on the line there is exactly one real number.



All rational numbers can be expressed as either finite or recurring decimals.

For example, $\frac{1}{2} = 0.5$ is a finite decimal and $\frac{1}{3} = 0.3333...$ is a recurring decimal.

An irrational number cannot be expressed in this way.

Example

Express the recurring decimal $0.2\overline{45}$ in the form of $\frac{a}{b}$ where a and b are integers and b is different from zero.

Solution

Let $x = 0.2\overline{45}$.

Then $100x = 24.5\overline{45}$.

By subtraction we obtain $99x = 24.3$. Hence $0.2\overline{45} = \frac{243}{990} = \frac{27}{110}$.

Application activity 1.1.1

Work in groups of five. Use a marker and manila paper to draw a larger copy of the table below.

Identify the sets to which each of the following numbers belong by marking an "X" in the appropriate boxes.

	Number	Natural numbers	Whole numbers	Integers	Rational numbers	Irrational numbers	Real numbers
1.	-17						
2.	-2						
3.	$-\frac{9}{37}$						
4.	0						
5.	-6.06						
6.	$4.5\overline{6}$						
7.	$3.050050005...$						
8.	18						
9.	$\frac{-43}{0}$						
10.	π						

1.1.2 Properties of operations in the set of real numbers

Activity 1.1.2

Work out the following operations. What do you notice for each case?

1. $3+4$ and $4+3$
2. $13-4$ and $4-13$
3. (10+345)+34 and 10+(345+34)
4. $17+0$ and $0+17$
5. 18+(-18) and -18+18
6. $9\times\frac{1}{9}$ and $\frac{1}{9}\times 9$
7. $13\times(3+2)$ and $(3+2)\times 13$

a) Closure property

The Set $\mathbb{R}$ is closed under operation *, if and only if, for any members *x* and *y* in $\mathbb{R}$, $x*y$ is always in $\mathbb{R}$.

Example:

a. Set of integers is closed under addition and multiplication since for $x, y \in \mathbb{Z},\ x+y \in \mathbb{Z}$ *and* $xy \in \mathbb{Z}$.

b. Set of natural numbers is not closed under subtraction since for any two natural numbers the difference between them is not always a natural number. For example $7 \in \mathbb{N}, 13 \in \mathbb{N}$ but $7-13=-6 \notin \mathbb{N}$.

b) Commutative property

Operation * is commutative in set $\mathbb{R}$, if and only if, for any members *x*, and *y* in $\mathbb{R}$, $x*y = y*x$.

Example:

Addition and multiplication are commutative in $\mathbb{R}$

$\forall x, y \in \mathbb{R}$; $x+y=y+x$ *and* $xy=yx$, $4+3=3+4=7$ and $4\times 3=3\times 4=12$. Given that 27 + 2.25 = 29.25 and 2.25 + 27 = 29.25, we have 27 + 2.25 = 2.25+27 which means that the order is immaterial when adding two real numbers.

Subtraction is not commutative in $\mathbb{R}$. For example $3\in\mathbb{R}, 6\in\mathbb{R}$; $3-6\neq 6-3$

c) Associative property

Operation * is associative in $\mathbb{R}$, if and only if, for any members *x*, *y* and *z* in $\mathbb{R}$, $(x*y)*z=x*(y*z)$.

Example:

Addition and multiplication are associative in $\mathbb{Z}$

$\forall x, y, z \in \mathbb{Z}$; $(x+y)+z=x+(y+z)$ *and* $(xy)z=x(yz)$

$(5+6)+2=5+(6+2)$	$(5\times 6)\times 2=5\times(6\times 2)$
$11+2=5+8$	$30\times 2=5\times 12$
$13=13$	$60=60$

Given three real numbers a, b *and* c ,we have $(a+b)+c=a+(b+c)$ is a real number. Example (2+19.4) +(- 0.61) = 2+[19.4+(-0.61)] This means that when adding 3 numbers, you can start by adding any two of those numbers.

d) Identity property

Let x be an element of set $\mathbb{R}$ and * an operation in $\mathbb{R}$. If there exists an element, say *e*, such that $x*e=x$ *and* $e*x=x$ then *e* is said to be an identity element for operation * in $\mathbb{R}$.

0 is the identity element for addition in $\mathbb{Z}$; $\forall x\in\mathbb{Z}$, $x+0=x=0+x$

The number 0 is the additive identity *in the set* $\mathbb{R}$ *; for a real number* a, $a+0=0+a=a$

Example:

97+0 = 0+97 = 97.

1 is the identity element for multiplication in $\mathbb{Z}$; $\forall x \in \mathbb{Z},\ x \cdot 1 = x = 1 \cdot x$

For a real number a we have $1.a = a.1 = a$; Example: 1.8 = 8.1= 8.

Example:

Find the identity element for operation T defined in $\mathbb{Z}$ by $xTy = x + y + 2$

Solution

Let e be the identity element. Then

$$xTe = x \Leftrightarrow x + e + 2 = x \qquad eTx = x \Leftrightarrow e + x + 2 = x$$

$$\Leftrightarrow e + 2 = x - x \qquad \Leftrightarrow e + 2 = x - x$$

$$\Leftrightarrow e = 0 - 2 \Rightarrow e = -2 \qquad \Leftrightarrow e = 0 - 2 \Rightarrow e = -2$$

Then, the identity element is -2

e) Inverse property

Let x be an element of set $\mathbb{R}$ and * an operation in $\mathbb{R}$. If there exists an element, say x', such that: $x * x' = e$ and $x' * x = e$ where e is the identity element for operation * in $\mathbb{R}$ then, x' is said to be the inverse of x under operation *.

$\forall x \in \mathbb{Z}, \exists x' : x + x' = 0$ *and* $x' + x = 0$. Here, x' is said to be the opposite of x (or additive inverse of x)

Therefore, the additive inverse of 81 is -81 because -81+81= 0

$\forall x \in \mathbb{R}, \exists x' : x \cdot x' = 1$ *and* $x' \cdot x = 1$. Here, x' is said to be the multiplicative inverse of x. Therefore, $\frac{1}{9}$ is the multiplicative inverse of 9 because $\frac{1}{9}.9 = 1$.

Example:

1. a) $5 + (-5) = 0 = (-5) + 5$. Then -5 is the additive inverse (opposite) of 5.

b) $6 \times \left(\frac{1}{6}\right) = 1$ *and* $\left(\frac{1}{6}\right) \times 6 = 1$. Then $\frac{1}{6}$ is the multiplicative inverse of 6.

Example:

Find the inverse of 1 under the operation T defined by $xTy = x + y + 3$ given that for this operation the identity element is -3

Solution

Let x be the inverse of 1. Then $xT1 = -3$ *and* $1Tx = -3$

$xT1 = -3 \Leftrightarrow x+1+3 = -3 \Rightarrow x = -7$ $\qquad$ $1Tx = -3 \Leftrightarrow 1+x+3 = -3 \Rightarrow x = -7$

The inverse of 1 is -7.

f) Distributive property

Operation * is said to be distributive over operation T in set $\mathbb{R}$ if and only if for any members x, y and z in $\mathbb{R}$,

$x*(yTz) = (x*y)T(x*z)$: * is left distributive over T

$(xTy)*z = (x*z)T(y*z)$: * is right distributive over T

Given three real numbers a, b *and* c, we have $a(b+c) = ab + ac$ and $(a+b)c = ac + bc$

Example:

$2(\sqrt{5}-5) = 2\sqrt{5} - 2(5) = 2\sqrt{5} - 10$

Multiplication is distributive over addition in $\mathbb{Z}$.

Example: Expand in the following expressions:

a) $2x(x-3y)$ b) $(3b+d)c$

Solution

a) $2x(x-3y) = 2x^2 - 6xy$ b) $(3b+d)c = 3bc + dc$

Application activity 1.1.2

1) Use the distributive property to simplify

a). $4x(2y+10)-60x$

b). $(12-3bc-c)a-2b(a+c+20)$

c). $(a+18)b-a(c-8b)$

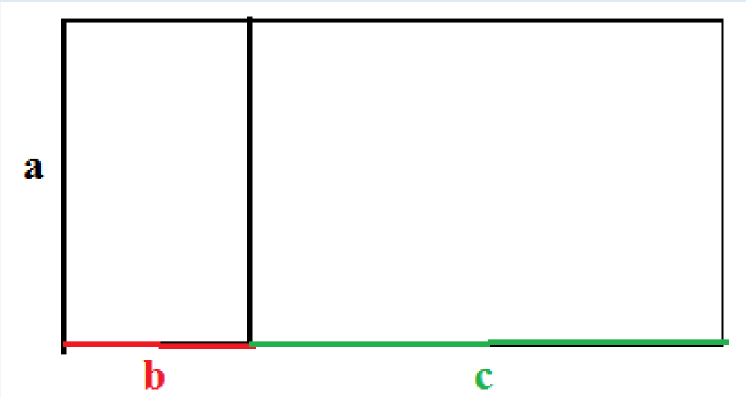
2) The binary operation is defined in the set of numbers by $aTb = ab-7$

a. Calculate

i). $(-2)T(4T1)$ ii). $(2T2)T4$

b. Is T associative and/or commutative?

3) Find the area of the rectangle shown in the figure in two different ways to prove the distitributive property $a(b+c)=ab+ac$



1.2 Fractions and related problems

Activity 1.2

1. Sam had 120 teddy bears in his toy store. He sold $\frac{2}{3}$ of them at 12 Rwandan francs each. How much did he receive?
2. Simplify the following fractions and explain the method used to simplify: a) $\frac{8x^2y^3}{2x^3y}$ b) $\frac{2x^2+5x^3}{2x^2+4x^3}$

3.Explain how to work out $\frac{1}{x+1}-\frac{1}{2x+2}$ if x is different from -1

4. Do you some times use fractions in your life? Explain your answer.

CONTENT SUMMARY

Consider the expressions $\frac{x}{4}+\frac{3y}{x},\frac{5}{x+4}$ in each of these the numerator or the denominator or both contain a variable or variables. These are examples of algebraic fractions. Since the letter used in these fractions stand for real numbers, we deal with algebraic fractions in the same way as we do with fractions in arithmetic. Fractions can be simplified and operations on fractions such as addition, subtraction, multiplication and division applied in simple arithmetic for solving related problems.

Adding fractions:

To add fractions there is a simple rule $\frac{a}{b}+\frac{c}{b}=\frac{a+c}{b}$ where $(b\neq 0)$

Example: $\frac{x}{2}+\frac{y}{5}=\frac{5x+2y}{(2)(5)}=\frac{5x+2y}{10}$

Subtracting fractions

Subtracting fractions is very similar to addition of fractions, except that the sign change.

Example: Considering that the denominator is different from zero,

$$\frac{x+2}{x}-\frac{x}{x-2}=\frac{(x+2)(x-2)-(x)(x)}{x(x-2)}=\frac{-4}{x^2-2x}$$

Multiplying fractions

Multiplying fractions is the easiest one of all, just multiply the numerators together, and the denominators together

$$\frac{a}{b}\times\frac{c}{d}=\frac{ac}{bd}, \text{ where } (b\neq 0, d\neq 0)$$

Example:

$$\frac{3x}{x-2}\times\frac{x}{3}=\frac{(3x)(x)}{3(x-2)}=\frac{3x^2}{3(x-2)}=\frac{x^2}{x-2}\ ,\ (x\neq 2).$$

Dividing fractions

To divide fractions, first flip the fraction we want to divide by, then use the same method as for multiplication:

$$\frac{a}{b}\div\frac{c}{d}=\frac{a}{b}\times\frac{d}{c}=\frac{ad}{bc}\quad \text{where}\,(b\neq 0, c\neq 0, d\neq 0).$$

Example:

$$\frac{3y^2}{x+1}\div\frac{y}{2}=\frac{3y^2}{x+1}\times\frac{2}{y}=\frac{(3y^2)(2)}{(x+1)(y)}=\frac{6y^2}{(x+1)(y)}=\frac{6y}{x+1}\ ,\ (x\neq -1).$$

As fraction is a symbol indicating the division of integers. For example: $\frac{15}{9},\frac{3}{8}$ are fractions and are called Common Fraction. The dividend (upper number) is called the numerator $N(x)$ and the divisor (lower number) is called the denominator $D(x)$ from the operations of fractions, different fractions can be found by taking Low Common Multiple (L.C.M.) and then add all the fractions.

For **example:** $\frac{1}{x-1}+\frac{2}{x+2}=\frac{3x}{(x-1)(x+2)}$ we split up a single fraction into a number of fractions whose denominators are the factors of denominator of that fraction. These fractions are called **Partial fractions** (To express a single rational fraction into the sum of two or more single rational fractions is called **Partial fraction resolution**)

Example Considering that the denominator is different from zero,

$$\frac{2x+x^2-1}{x(x^2-1)}=\frac{1}{x}+\frac{1}{x-1}-\frac{1}{x+1}$$

$\frac{2x+x^2-1}{x(x^2-1)}$ is the resultant fraction and $\frac{1}{x}, \frac{1}{x-1}$ and $\frac{1}{x+1}$ are its partial fractions.

Rational fraction

We know that $\frac{p}{q},\ q \neq 0$ is called a rational number. Similarly the quotient of two polynomials $\frac{N(x)}{D(x)}, D(\mathrm{x}) \neq 0$ with no common factors. There are two types of rational fractions such proper or improper fractions.

Proper fraction:

A rational fraction $\frac{N(x)}{D(x)}$ is called a proper fraction if the **degree of numerator is less than the degree of Denominator D(x).**

Example: $\frac{6x+27}{3x^3-9x}$ is a proper fraction

Improper Fraction:

A rational fraction $\frac{N(x)}{D(x)}$ is called improper fraction if the **degree of numerator is greater than or equal to the degree of Denominator D(x).**

Example: $\frac{6x^3-5x^2-3x-10}{x^2+1}$

An algebraic fraction exists only if the denominator is not equal to zero. The values of the variable that make the denominator zero is called a restriction on the variable(s). An algebraic fraction can have more than one restriction.

Examples:

1) Identify the restriction on the variable in the fraction $\frac{3xy}{(x+3)(x-2)}$

Solution

In the fraction $\frac{3xy}{(x+3)(x-2)}$, $(x+3)\ (x-2)$ is a denominator. We have $x+3 \neq 0$ **or** $x \neq 2$.

Therefore, if $x=-3$, $(x+3)(x-2)=0$ and if $x=2$, $(x+3)(x-2)=0$

In the fraction $\frac{3xy}{(x+3)(x-2)}$, the restrictions are $x \neq -3$ and $x \neq 2$.

2) Simplify $\frac{3x^2y}{4a^2} \div \frac{9xy}{5a}$

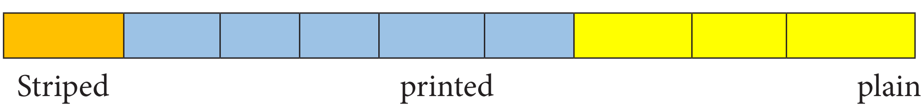
Solution: $\frac{3x^2y}{4a^2} \times \frac{5a}{9xy} = \frac{(3x^2y)(5a)}{(4a^2)(9xy)} = \frac{(x)(5)}{(4a)(3)} = \frac{5x}{12a}$

Problems related to fractions

Example

1) One ninth of the shirts sold at peter's shop are stripped. $\frac{5}{8}$ of the remainder are printed. The rest of the shirts are plain colour shirts.
If peter's shop has 81 plain colour shirts, how many more printed shirts than plain colour shirts does the shop have?

Solution



3 units equals to 81

$1 \text{ unit} = 81 \div 3 = 27$

Printed shirts have 2 parts more than plain shirts.

$2 \text{ units} = 27 \times 2 = 54$

Peter's shop has 54 more printed colour shirts than plain shirts.

2) Oscar sold 2 glasses of milk for every 5 sodas he sold. If he sold 10 glasses of milk, how many sodas did he sell?

Solution

Let x be the number of Sodas that Oscar will sell.

Set up a proportion of milk with soda as $\frac{milk}{soda}$, $\frac{2}{5} = \frac{10}{x} \Rightarrow 2x = 50$, $x = \frac{50}{2} = 25$

He will sell 25 sodas.

Application activity 1.2

1. A proper fraction is such that its numerator and denominator have a difference of 2. If one is added to the denominator and three subtracted from the numerator, the fraction becomes $\frac{2}{3}$. Find the fraction and explain your colleague how to do it.

2. What is a partial fraction? Express $\frac{x^2+1}{x^3+4x^2+3x}$ in partial fractions.

1.3 Decimals and related problems

Activity 1.3

Refer to the meaning of decimals and fractions learnt in previous years and

1. Calculate 50:100 and write it in the form of
   a) a fraction, b) a decimal number.

2. Express $\frac{1}{3}$ *and* $\frac{22}{7}$ in the form of decimal numbers . Explain the relationship between the set of fractions and the set of decimal numbers.

3. Given that the set D is a set of the limited decimal numbers, discuss the following:

(a) D is a subset of $\mathbb{Q}$; (b) $D \subset \mathbb{Q}$, (c) $\mathbb{Z} \subset D$, (d) $\mathbb{N} \subset \mathbb{Z} \subset D \subset \mathbb{Q} \subset \mathbb{R}$.

4. Do you some times use decimal numbers in your life? Explain your answer.

CONTENT SUMMARY

Decimals are just another way of expressing fractions

$$0.1 = \frac{1}{10};\ 0.01 = \frac{1}{100};\ 0.001 = \frac{1}{1000}$$

Thus 0.234 is equivalent to 234/1,000. Most of the time you will be able to perform operations involving decimals by applying what was learnt in previous years or by using a calculator.

In mathematics a decimal format is often required for a value that is usually specified as a fraction in everyday usage. For example, $\frac{62}{100} = 0.62$.

Because some fractions cannot be expressed exactly in decimals, one may need to 'round off' an answer for convenience. In many of the economic problems (of various books) there is not much point in taking answers beyond two decimal places. Where this is done then we denote '(to 2 dp)' is normally put after the answer.

For **example**: 1/7 as a decimal is 0.14 (to 2 dp).

Example:

1) $1.345 + 0.00041 = 1.34541$

2) $2.463 \times 38 = 93.954$

3) $360.54 \div 0.04 = 9{,}013.5$

Application activity 1.3

Evaluate the following:

1) $1.345 + 0.00041 + 0.20023 =$

2) $93.954 \div 2.4 =$

1.4 Percentages and related problems

Activity 1.4

Refer to the meaning of decimals and percentage learnt in previous years and

1) Calculate 60:100 and write it in the form of

a) a fraction, b) a percentage and c) a decimal number.

2) Express $\frac{1}{3}$ *and* $\frac{22}{7}$ in the form of decimal numbers . Is it possible to express this number in the form of percentage? Is the pecentage obtained an exact number?

3) 24 students in a class took English test. If 18 students passed the test, what percentage of those who did not pass?

4) Are percentages used by bank managers? Explain your answer.

CONTENT SUMMARY

Percentage is defined as the proportion, rate or ratio expressed with a denominator of 100.

For example : $\frac{3}{100}, \frac{25}{100}$ etc

Fraction can be expressed in the form of percentage as $\frac{1}{4} \times 100 = 25\%$.

Decimal format is often required for a value that is usually specified as a percentage in everyday usage. For example, interest rates are usually specified as percentages. A percentage format is really just another way of specifying a decimal fraction,

$62\% = \frac{62}{100} = 0.62$ And so, percentages can easily be converted into decimal fractions by dividing by 100. Because some fractions cannot be expressed exactly in decimals, one may need to 'round off' an answer for convenience. In many of the economic problems (of various books) there is not much point in taking answers beyond *two decimal places (2dp)*. Where this is done then we denote '(to 2 dp)' is normally putted after the answer. For example, 1/7 as a percentage is 14.29% (to 2 dp).

In solving word problems involving percentage, 3 steps can help you:

1. Make sure you understand the question
2. Sort out the information to make a basic percent problem
3. Apply the operations to find out what asked

Example:

A town council imposes different taxes on different fixed assets as follows: Commercial property 25% per year, Residential property 15% per year, Industrial property 20% per year.

An investor owns a residential building on a plot all valued at 80 000 000 Frw an industrial plot worth 75 000 000 Frw and a commercial premises worth 12 500 000 Frw. How much tax does the investor pay annually?

Solution:

Commercial: $\frac{25}{100} \times 12,500,000 Frw = 3,125,000 Frw$

Residential: $\frac{15}{100} \times 80,000,000 Frw = 12,000,000 Frw$

Industrial: $\frac{20}{100} \times 75,000,000 Frw = 15,000,000 Frw$

Total tax the investor pay annually:

$3,125,000 Frw + 12,000,000 Frw + 15,000,000 Frw = 30,125,000 Frw$.

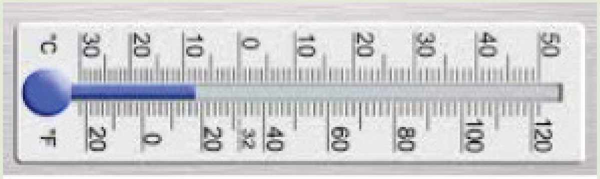
Application activity 1.4

1. In the middle of the first term, the school organize the test and the tutor of mathematics prepared 20 questions for both section A and B. peter get 80% correct. How many questions did peter missed?
2. Student earned a grade of 80% on mathematics test that has 20 questions. How many did the student answered correctly? And what percentage of that not answered correctly?
3. John took a mathematics test and got 35 correct answers and 10 incorrect answers. What was the percentage of correct answers?
4. As a future teacher, is it necessary for a student teacher to know how to determine the percentage? Explain it with supportive examples.

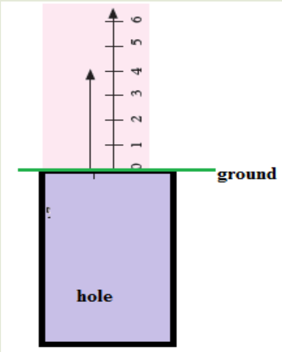
1.5 Negative numbers and related problems

Activity 1.5

1. The temperature of a juice in the bottle was $20^0 C$. They put this juice in the fridge so that its temperature decreases by $30^0 C$. What is the temperature of this juice? What can you advice the child who wishes to drink that juice?



2. Suppose that you have a long ruler fixed from the hole and graduated such that the point 0 corresponds to the ground level as illustrated on the following figure.



What is the coordinate of the point position for an insect which is at 3 units below the ground level in a hole?

CONTENT SUMMARY

There are numerous instances where one comes across negative quantities, such as temperatures below zero or bank overdrafts. For example, if you have 3500Frw in your bank account and withdraw 6000Frw with an acceptable credit, your bank balance is -2500Frw . There are instances, however, where it is not usually possible to have negative quantities. For example, a firm's production level cannot be negative.

From the above activity, you have learnt that, you can need to use **negative or a positive numbers.**

For example, when measuring temperature, the value of the temperatures of the body or surrounding can be negative or positive. The normal body temperature is about $+37^0C$ and the temperature of the freezing mercury is about -39^0C .

Example:

a) Eight students each have an overdraft (scholarship advance) of 21,000Frw. What is their total bank balance?

Solution:

The total balance in the bank is $8 \times (-21{,}000) = -168{,}000$Frw. The sign negative means that students have the credit to be paid.

b) Calculate $\frac{24}{-5} \div \frac{-32}{-10}$

Solution:

$$\frac{24}{-5} \div \frac{-32}{-10} = \frac{24}{-5} \times \frac{-10}{-32} = \frac{3}{1} \times \frac{2}{-4} = \frac{6}{-4} = -\frac{3}{2}$$

Application activity 1.5

Question1: Answer to the following questions

a) $\frac{(-10)\times(-5)\times(-6)}{(-3)\times(-2)} =$ b) $\frac{(-30)\times(+2)\times(-10)}{(-50\times(+2)} =$

c) Where negative numbers are applied in the real life? Do you think that computers of bank managers deal with negative numbers when operating loans for clients?

Question2:

A cylinder is ¼ full of water. After 60ml of water is added the cylinder is 2/3 full. Calculate the total volume of the cylinder.

Question3:

Between 1990 and 1997 the population of an Island fell by 4%. The population in 1997 was 201,600. Find the population in 1990.

1.6. Absolute value

1.6.1 Meaning of absolute value

Activity 1.6.1

1) Draw a number line and state the number of units found between

a) 0 and -8 b) 0 and 8

c) 0 and $\frac{1}{2}$ d) 4 and 17

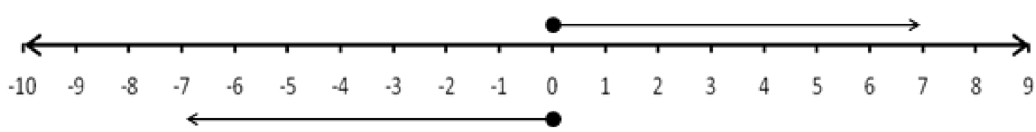
2) Do you think that a distance can be expressed by a negative number?

CONTENT SUMMARY

Absolute value of a number is the distance of that number from the origin (zero point) on a number line. The symbol $|\ |$ is used to denote the absolute value.

Example:

7 is at 7 units from zero, thus the absolute value of 7 is 7 or $|7| = 7$. Also -7 is at 7 units from zero, thus the absolute value of -7 is 7 or $|-7| = 7$. So $|-7| = |7| = 7$ since -7 and 7 are on equal distance from zero on number line.

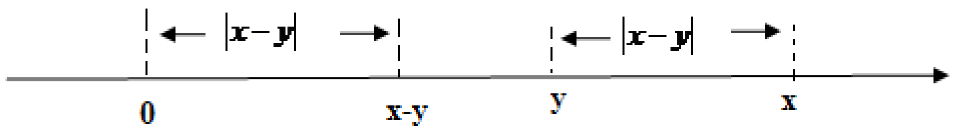


Note:

- The absolute value of zero is zero
- The absolute value of a non-zero real number is a positive real number.
- Given that $|x| = k$ where k is a positive real number or zero, then $x = -k$ *or* $x = k$.

- $|x| = \begin{cases} x, & \text{if } x \geq 0 \\ -x, & \text{if } x < 0 \end{cases}$
- Geometrically, $|x|$ represents the nonnegative distance from x to 0 on the real line. More generally, $|x-y|$ represents the nonnegative distance between the point x and y on the real line, since this distance is the same as that from the point $x-y$ to 0.(See the following figure)

$$|x-y| = \begin{cases} x-y, \text{ if } x \geq y \\ -(x-y) = y-x, \text{ if } x \leq y. \end{cases}$$



Example:

Find x in the following:

a) $|x| = 5$ b) $|x| + 5 = 1$ c) $|x-4| = 10$

Solution

a) $|x| = 5,\ x = -5 \ \ or \ \ x = 5$

b) $|x| + 5 = 1$

$\Leftrightarrow |x| = 1-5 \Rightarrow |x| = -4$ which is impossible in in the set $\mathbb{R}$.

There is no value of x since the absolute value of x must be a positive real number.

c) $|x-4| = 10$

$x - 4 = -10 \ \ or \ \ x - 4 = 10$

$x = -10 + 4 \ \ or \ \ x = 10 + 4$

$x = -6 \ \ or \ \ x = 14$

Example:

Simplify

a) $-|40-12|$ b) $|4(-3)-(2)(5)|$ c) $|-4(-2)|$

Solution

a) $-|40-12|=-|28|=-28$

b) $|4(-3)-(2)(5)|=|-12-10|=|-22|=22$

c) $|-4(-2)|=|8|=8$

1.6.2 Properties of the Absolute Value

Activity 1.6.2

Evaluate and compare the following:

1) $|3|$ and $|-3|$ 2) $|3\times 5|$ and $|3|\times|5|$ 3) $|(-8)+5|$ and $|-8|+|5|$

1. Opposite numbers have equal absolute value.

$|a|=|-a|$

Example:

$|5|=|-5|=5$

2. The absolute value of a product is equal to the product of the absolute values of the factors.

$|ab|=|a||b|$

Example:

$|4(-6)|=|4||-6|$

$$|4(-6)| = |-24| = 24$$

$$|4||-6| = 4 \times 6 = 24$$

3. The absolute value of a sum is less than or equal to the sum of the absolute values of the ends.

$$|a+b| \le |a| + |b|$$

Example:

$$|-3+2| \le |-3| + |2|$$
$$|-1| \le 3+2$$
$$1 \le 5$$

Application activity 1.6

1. Find the value(s) of x: a) $|x| = 6$ b) $|x+1| = 0$

c) $|x-3| - 4 = 2$ d) $|2x+1| = 4$ e) $|x-3| + 3 = 5$

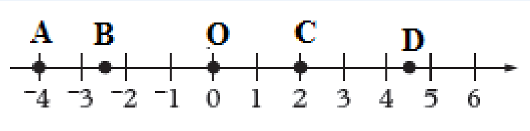
2. Simplify

(1) $|-5|$ (2) $|-4||-5|$ (3) $|-7| + |4|$ (4) $-|4 \times 6|$ (5) $-|-6+8|$

3. Let a and b be the coordinates of points A and B, respectively, on a coordinate line. The distance between A and B, denoted by d(A,B) is defined by

$d(A,B) = |b-a|$.

a) Refer to the figure below and determine the distance d(A,B), d(C,B) and d(O,A).

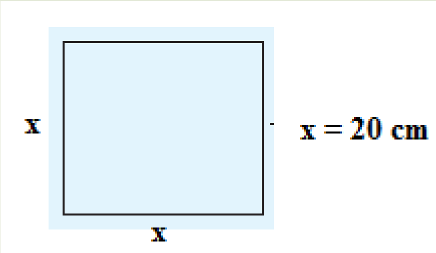


b) Dr. Makoma went from Kabgayi to Muhanga City on foot at the constant speed of 100m/min. If he used 60 minutes to go and 60 minutes to come back. Explain to your colleague the distance and the displacement covered by Dr. Makoma.

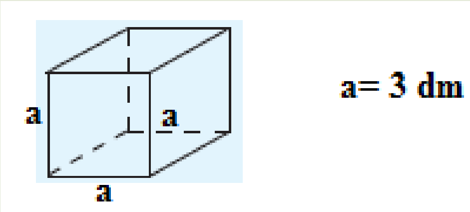
1.7. Powers and related problems

Activity 1.7

1. How can you find the area of the following paper in the form of a square?



2. Given a cube of the following form:



Determine the volume of this cube.

CONTENT SUMMARY

1.7.1 Meaning of power of a number

We call n^{th} power of a real number a that we note a^n, the product of n factors of a. that is

$$a^n = \underbrace{a \cdot a \cdot a \ldots a}_{n\ factors} \quad \begin{cases} n \text{ is an exponent} \\ a \text{ is the base} \end{cases}$$

Example 1

$$2^4 = \underbrace{2 \cdot 2 \cdot 2 \cdot 2}_{4\ factors} = 16 \qquad 3^3 = \underbrace{3 \cdot 3 \cdot 3}_{3\ factors} = 27$$

Notice

- $a^1 = a$
- $a^0 = 1, a \neq 0$
- If $a = 0, a^0$ is not defined

1.7.2 Properties of powers

Let $a, b \in \mathbb{R}$ and $m, n \in \mathbb{R}$

a) $a^m \cdot a^n = a^{m+n}$

In fact, $a^m \cdot a^n = \underbrace{a \cdot a \cdot a \cdots a}_{m\ factors} \times \underbrace{a \cdot a \cdot a \cdots a}_{n\ factors} = \underbrace{a \cdot a \cdot a \cdots a}_{m+n\ factors} = a^{m+n}$

b) $\left(a^m\right)^n = a^{mn}$

In fact, $\left(a^m\right)^n = \underbrace{\underbrace{a \cdot a \cdot a \cdots a}_{m\ factors} \times \underbrace{a \cdot a \cdot a \cdots a}_{m\ factors} \cdots \underbrace{a \cdot a \cdot a \cdots a}_{m\ factors}}_{n\ factors} = a^{mn}$

c) $\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$

In fact,

$$\left(\frac{a}{b}\right)^m = \underbrace{\frac{a}{b} \cdot \frac{a}{b} \cdot \frac{a}{b} \cdots \frac{a}{b}}_{m\ factors} = \frac{\underbrace{a \cdot a \cdot a \cdots a}_{m\ factors}}{\underbrace{b \cdot b \cdot b \cdots b}_{m\ factors}} = \frac{a^m}{b^m}$$

d) $\frac{1}{b^m} = b^{-m}$

In fact, $\frac{1}{b^m} = \frac{1}{b^m}^m = \left(\frac{1}{b}\right)^m = \left(b^{-1}\right)^m = b^{-m}$

e) $\frac{a^m}{a^n} = a^{m-n}$

In fact,

$$\frac{a^m}{a^n} = a^m \frac{1}{a^n} = a^m a^{-n} = a^{m-n}$$

f) $(ab)^m = a^m b^m$

In fact, $(ab)^m = \underbrace{ab \cdot ab \cdots ab}_{m\ factors} = \underbrace{a \cdot a \cdots a}_{m\ factors} \times \underbrace{b \cdot b \cdots b}_{m\ factors} = a^m b^m$

These properties help us to simplify some powers.

There is no general way to simplify the sum of powers, even when the powers have the same base. For instance, $2^5 + 2^3 = 32 + 8 = 40$, and 40 is not an integer power of 2. But some products or ratios of powers can be simplified using repeated multiplication models of a^n.

Example:

a) $2^4 \cdot 2^3 \cdot 4 = 2^4 \cdot 2^3 \cdot 2^2 = 2^9 = 512$

b) $a^4 \cdot b^3 \cdot a^5 \cdot b^8 = a^4 \cdot a^5 \cdot b^3 \cdot b^8 = a^9 \cdot b^{11}$

$a^9 \cdot b^{11}$ cannot be simplified further because the bases are different.

c) $\frac{y^9}{y^2} = y^{9-2} = y^7$

1.7.3 The compound interest formula

Considering the case in which P is invested in a bank for n interest periods per year at the rate r expressed as a decimal. The accumulated amount of money A after the time t (number of years P is invested) is given by:

$$A = P\left(1 + \frac{r}{n}\right)^{nt}$$

where n is the number of interest periods per year, r is the interest rate expressed as decimal, A is the amount after t years.

Note:

1) When the interest rate is compounded per year, $A = P(1+r)^n$ where r is expressed as a decimal for example $r = 9\% = 0.09$.

2) When the interest rate is compounded monthly, $A = P\left(1 + \frac{r}{12}\right)^{12.t}$ where t is the number of years and r expressed as a decimal.

This is called the compound interest formula. It is conveniently used in solving problems of compound interest especially those involving long periods of investments or payment.

In this method, the accrued compound interest is obtained by subtracting the original principal from the final amount.

Thus, Compound interest = Accumulated amount (A) – Principal (P)

Note that the principal and the interest earned increased after each interest period.

We can also deduce that $I = A - P$

Examples

Question1	Question 2
A trader deposited $63000Frw$ in a fixed deposit account with a local bank which attracted an interest of 8% per annual compound interest. Find: (a) the total amount after 4 years; (b)compound interest. **Solution:** $P = 63000Frw; n = 4; r = 8\% p.a.$ $A = P\left(1+\frac{r}{100}\right)^n$ (a) $A = 63000\left(1+\frac{8}{100}\right)^4$ $= 63000(1.08)^4 = 63000\times 1.36048896 = 85710.80Frw$ (b) $I = (85710.80 - 63000)Frw$ $= 22710.80Frw$	Find the compound interest earned on $15000Frw$ invested for 3years, at 20%p.a. compounded quarterly. **Solution:** Here, each year has 4 interest period (quarterly) i.e in 3years, there are 12 interest period $(3\times 4 = 12)$. The rate, $r\% = 20 \div 4 = 5\% p.a.$ $A = 15000\left(1+\frac{5}{100}\right)^{12}$ $A = 15000(1.05)^{12} = 26937.84Frw$ $I = 26937.84 - 15000 = 11937.84Frw$

Application activity 1.7

1) Simplify

a) x^3x^2 (b) $\left(xy^3\right)^2 + 4x^2y^6$ (c) $\frac{6xy^2}{3xy}$ (d) $\frac{ab}{a^3} - \frac{a^3b^2}{a^5b}$ (e) $\frac{yx}{4xy}$

2) Referring to your real life experience, where powers are used?

1.8 Roots (radicals) and related problems

1.8.1 Meaning of radicals

Activity 1.8.1

1) Evaluate the following powers using a calculator

a) $\sqrt{81}$ b) $(216)^{\frac{1}{3}}$ c) $(-27)^{\frac{1}{3}}$ d) $\sqrt[4]{16}$

2) Using examples, explain whether there is a difference between the square and the square root of a number. How do you calculate a square root of a given number?

CONTENT SUMMARY

The n^{th} root of a real number is $\frac{1}{n}$ power of that real number. It is noted by $\sqrt[n]{b}$, $b \in \mathbb{R}, n \in \mathbb{N} \setminus \{1\}$. $\forall a, b \in \mathbb{R}, \sqrt[n]{b} = a \Leftrightarrow b^{\frac{1}{n}} = a \Leftrightarrow b = a^n$

$$\begin{cases} n \textit{ is called the index} \\ b \textit{ is called the base or radicand} \\ \sqrt[n]{\ } \textit{ is called the radical sign} \end{cases}$$

Example

a) $\sqrt[3]{27} = (27)^{\frac{1}{3}} = (3^3)^{\frac{1}{3}} = 3^{3\times\frac{1}{3}} = 3$

b) $\sqrt[4]{16} = (16)^{\frac{1}{4}} = (2^4)^{\frac{1}{4}} = 2$

If $n = 2$, we say square root and $\sqrt[2]{b}$ is written as $\sqrt{b}$. Here b must be a positive real number or zero.

If $n = 3$, we say cube root noted $\sqrt[3]{b}$. Here b can be any real number.

If $n = 4$, we say 4th root noted $\sqrt[4]{b}$. Here b must be a positive real number or zero.

Generally, if $n = n$, we say n^{th} root noted $\sqrt[n]{b}$. Here if n is even, b must be a positive real number or zero and if n is odd b can be any real number.

Example:

$\sqrt{-9}$ is not defined in $\mathbb{R}$ the index in radical is even but $\sqrt[3]{-27}=(-27)^{\frac{1}{3}}=\left[(-3)^3\right]^{\frac{1}{3}}=-3$

1.8.2 Properties of radicals

$\forall n\in\mathbb{N}\setminus\{1\}, m\in\mathbb{R}$

a) $\sqrt[n]{a^m}=a^{\frac{m}{n}}$

In fact, $\sqrt[n]{a^m}=\left(a^m\right)^{\frac{1}{n}}=a^{m\times\frac{1}{n}}=a^{\frac{m}{n}}$

b) $\sqrt[n]{ab}=\sqrt[n]{a}\cdot\sqrt[n]{b}$

In fact, $\sqrt[n]{ab}=(ab)^{\frac{1}{n}}=a^{\frac{1}{n}}b^{\frac{1}{n}}=\sqrt[n]{a}\sqrt[n]{b}$

c) $\sqrt[n]{\frac{a}{b}}=\frac{\sqrt[n]{a}}{\sqrt[n]{b}}$

In fact, $\sqrt[n]{\frac{a}{b}}=\left(\frac{a}{b}\right)^{\frac{1}{n}}=\frac{a^{\frac{1}{n}}}{b^{\frac{1}{n}}}=\frac{\sqrt[n]{a}}{\sqrt[n]{b}}$

d) $\sqrt[n]{\sqrt[m]{a}}=\sqrt[nm]{a}=a^{\frac{1}{nm}}$

In fact,

$$\sqrt[n]{\sqrt[m]{a}}=\left(a^{\frac{1}{m}}\right)^{\frac{1}{n}}=a^{\frac{1}{m}\times\frac{1}{m}}=a^{\frac{1}{mn}}=\sqrt[mn]{a}$$

Example:

Simplify

a) $\sqrt{46656}$ b) $\sqrt[3]{\sqrt{64}}$ c) $\sqrt[3]{ab}\times\sqrt[3]{a^2b^2}$ d) $\sqrt{\frac{36}{81}}$

Solution

a) $\sqrt{46656}=\sqrt{6^6}=6^3=216$ b) $\sqrt[3]{\sqrt{64}}=\sqrt[6]{64}=\sqrt[6]{2^6}=2$

c) $\sqrt[3]{ab}\times\sqrt[3]{a^2b^2}=\sqrt[3]{a^3b^3}=\sqrt[3]{(ab)^3}=ab$ d) $\sqrt{\frac{36}{81}}=\frac{\sqrt{36}}{\sqrt{81}}=\frac{6}{9}=\frac{2}{3}$

Application activity 1.8.2

Simplify

a) $\sqrt{a}\times\sqrt{ab^3}\times\sqrt{bc^2}$ b) $\sqrt[3]{abc}\times\sqrt[3]{a^2b^2c^2}$

c) $\sqrt[3]{\frac{8}{27}}$ d) $\sqrt{\sqrt[4]{x^8}}$ e) $\sqrt{\frac{x^3y^4}{4x}}$

1.8.3 Operations on radicals

Activity 1.8.3

Simplify the following

(1) $\sqrt{18}+\sqrt{2}$ (2) $\sqrt{12}-3\sqrt{3}$ (3) $\sqrt{2}\times\sqrt{3}$ (4) $\frac{\sqrt{6}}{\sqrt{2}}$

Addition and subtraction

When adding or subtracting the radicals we may need to simplify if we have similar radicals. Similar radicals are the radicals with the same indices and same bases.

Example:

$$\sqrt{2}+\sqrt{8}=\sqrt{2}+\sqrt{2\times 4}=\sqrt{2}+\sqrt{2}\times\sqrt{4}=\sqrt{2}+2\sqrt{2}=3\sqrt{2}$$

$$\sqrt{3}-\sqrt{27}=\sqrt{3}-\sqrt{3\times 9}=\sqrt{3}-\sqrt{3}\times\sqrt{9}=\sqrt{3}-3\sqrt{5}=-2\sqrt{3}$$

Application activity 1.8.3

Simplify

1) $\sqrt{20}+\sqrt{5}$ 2) $4\sqrt{3}-\sqrt{12}$ 3) $5\sqrt{7}-\sqrt{28}$

4) $\sqrt{18}\times\sqrt{8}$ 5) $\sqrt{45}+\sqrt{80}+\sqrt{180}$ 6) $\sqrt{108}-\sqrt{48}$

1.8.4 Rationalizing radicals

Activity 1.8.4

Make the denominator of each of the following rational

1. $\frac{1}{\sqrt{2}}$ 2. $\frac{2-\sqrt{3}}{2\sqrt{5}}$ 3. $\frac{2}{1-\sqrt{6}}$ 4. $\frac{\sqrt{2}+\sqrt{3}}{\sqrt{3}+\sqrt{5}}$

Rationalizing is to convert a fraction with an irrational denominator to a fraction with rational denominator. To do this,

1) if the denominator involves radicals we multiply the numerator and denominator by the conjugate of the denominator.

Some examples are: the conjugate of $a \pm \sqrt{b}$ is $a \mp \sqrt{b}$.

2) When the denominator is made by one term, we multiply the numerator and denominator by the same radical. For example, use $\sqrt{a}$ if the denominator is $\sqrt{a}$

or if you have $a\sqrt{b}$, use $\sqrt{b}$

Remember that $(a+b)(a-b) = a^2 - b^2$

Example:

a) $$\frac{1}{1+\sqrt{2}} = \frac{1-\sqrt{2}}{\left(1+\sqrt{2}\right)\left(1-\sqrt{2}\right)} = \frac{1-\sqrt{2}}{1-2} = \frac{1-\sqrt{2}}{-1} = \sqrt{2}-1$$

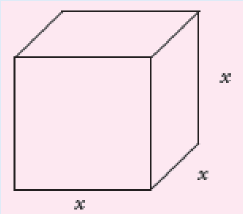
b) $$\frac{\sqrt{2}}{\sqrt{5}-\sqrt{3}} = \frac{\sqrt{2}\left(\sqrt{5}+\sqrt{3}\right)}{\left(\sqrt{5}-\sqrt{3}\right)\left(\sqrt{5}+\sqrt{3}\right)} = \frac{\sqrt{10}+\sqrt{6}}{5-3} = \frac{\sqrt{10}+\sqrt{6}}{2}$$

c) $$\frac{\sqrt{3}+\sqrt{7}}{4\sqrt{2}} = \frac{\left(\sqrt{3}+\sqrt{7}\right)\sqrt{2}}{\left(4\sqrt{2}\right)\sqrt{2}} = \frac{\sqrt{6}+\sqrt{14}}{8}$$

Application activity 1.8

1. On a clear day, the distance d (in meters) that can be seen from the top of a tall building of height h (in meter) can be approximated by $d=1.2\sqrt{h}$. Approximate the distance that can be seen from Kigali Tower which is 30m tall.

2. A cube has a total surface area of 96 square cm. Find the volume of that cube.



3. Rationalize the denominator

a) $\frac{2\sqrt{2}}{4+3\sqrt{3}}$ b). $\frac{a-\sqrt{b}}{\sqrt{d}}$ c) $\frac{3\sqrt{3}+2\sqrt{2}}{1+2\sqrt{2}}$

4. Carry out a research in the library or on internet and discover other real life problems in which you can apply the powers and radicals. Discuss them with your classmates.

5. Discuss orally how to determine the square of a square root of a number. Did you ever need to use a square root in your real life experience? Explain the answer.

1.9 Decimal logarithms and related problems

Activity 1.9

1) What is the real number at which 10 must be raised to obtain:

a) 1 b) 10 c) 100 d) 1000 e) 10000 f) 100000

2) Explain your classmate how you can find the number x if $x^3=64$.

CONTENT SUMMARY

The decimal logarithm of a positive real number x is defined to be a real number y for which 10 must be raised to obtain x. We write $\forall x > 0, \quad y = \log x$ as $x = 10^{y}$

$\log x$ is the same as $\log_{10} x$ and is defined for all positive real numbers only. 10 is the base of this logarithm. In general notation we do not write this base for decimal logarithm.

In the notation $y = \log x$, x is said to be the **antilogarithm of** y.

Example:

log (100) = ?

We are required to find the power to which 10 must be raised to obtain 100

So log(100) = 2 as $100 = 10^2$

$y = \log x$ means $10^{y} = x$

Be careful! $\log 2x + 1 \neq \log(2x+1)$

$$\log 2x + 1 = (\log 2x) + 1$$

Since logs are defined using exponentials, any "log x" has an equivalent "exponent" form, and vice-versa.

Example: (1) $\log 10^5 = 5$ (2) $\log(0.01) = \log(10^{-2}) = -2$

Properties

$\forall a, b \in \left]0, +\infty\right[$

a) $\log ab = \log a + \log b$

b) $\log \frac{1}{b} = -\log b$

c) $\log \frac{a}{b} = \log a - \log b$

d) $\log a^n = n \log a$

e) $\log \sqrt{a} = \frac{1}{2} \log a$

f) $\log \sqrt[n]{a} = \frac{1}{n} \log a$

g) If $a > b, \log a > \log b$

h) If $a = b, \log a = \log b$

i) $\log \sqrt[n]{a^m} = \frac{m}{n} \log a$

Examples:

1) Calculate in function of $\log a, \log b$ *and* $\log c$

a) $\log a^2b^2$

b) $\log \frac{ab}{c}$

c) $\log \frac{ab}{\sqrt{c}}$

Solution

a) $\log a^2b^2 = \log(ab)^2$

$$= 2\log ab$$
$$= 2(\log a + \log b)$$

b) $\log \frac{ab}{c} = \log ab - \log c$

$$= \log a + \log b - \log c$$

c) $\log \frac{ab}{\sqrt{c}} = \log ab - \log \sqrt{c}$

$$= \log a + \log b - \log (c)^{\frac{1}{2}}$$
$$= \log a + \log b - \frac{1}{2}\log c$$

2) Given that $\log 2 = 0.30$, $\log 3 = 0.48$ and $\log 5 = 0.7$. Calculate

a) $\log 6$

b) $\log 0.9$

Solution

a) $\log 6 = \log(2 \times 3)$

$$= \log 2 + \log 3$$
$$= 0.30 + 0.48$$
$$= 0.78$$

b) $\log 0.9 = \log\frac{9}{10}$

$= \log 9 - \log 10$

$= \log 3^2 - \log(2\times 5)$

$= 2\log 3 - \log 2 - \log 5$

$= 2(0.48) - 0.30 - 0.7$

$= -0.04$

Co-logarithm

Co-logarithm, sometimes shortened to **colog**, of a number is the logarithm of the reciprocal of that number, equal to the negative of the logarithm of the number itself,

$$\operatorname{colog} x = \log\left(\frac{1}{x}\right) = -\log x$$

Example:

$\operatorname{colog}\ 200 = -\log\ 200 = -2.3010$

Change of base formula

If u $(u > 0)$ and if a and b are positive real numbers different from 1, $\log_b u = \dfrac{\log_a u}{\log_a b}$

This means that if you have a logarithm in any other base, you can convert it in the decimal logarithm in the following way where $a = 10$:

$$\log_b u = \frac{\log_{10} u}{\log_{10} b} = \frac{\log u}{\log b}.$$

This is for example: $\log_2 5 = \dfrac{\log_{10} 5}{\log_{10} 2} \approx 2.322$

There is another special logarithm called natural logarithm which has the base a number $e \approx 2.71828$. This logarithm is written as : $\log_e x = \ln x$.

Example:

How long will it take10,000Frw to double it on account earning 2% compounded quarterly?

Solution

For this problem, we'll use the compound Interest formula,

$F = P(1+i)^n$ where F is the final value, P the initial value of investment.

Since we want to know how long it will take, let t represent the time in years. The number of compounding periods is four times the time or $n = 4t$. The original amount is $P = 10,000$ and the future value is the double or F = 20,000. The interest rate per period is $i = \frac{0.02}{4} = 0.005$

When these values are substituted into the compound interest formula, we get the exponential equation

$$20000 = 10000(1.005)^{4t}$$

To solve this equation for t, isolate the exponential factor by dividing both sides by 10,000 to give

$$2 = (1.005)^{4t}$$

Convert this exponential form to logarithm form and divide by 4

$$4t = \log_{1.005}(2)$$

$$t = \frac{\log_{1.005}(2)}{4}$$

To find an approximate value, use the Change of Base Formula to convert to a natural logarithm (or a common logarithm)

$$t = \frac{\frac{\ln(2)}{\ln(1.005)}}{4} \approx 34.7\,years \quad \text{or} \quad t = \frac{\frac{\log(2)}{\log(1.005)}}{4} \approx 34.7\,\text{years}.$$

It is interesting to note that the starting amount is irrelevant when doubling. If we started with P dollars and wanted to accumulate 2P at the same interest's rate and compounding periods, we would need to solve

$$2P = P(1.005)^{4t}.$$

This reduces to the same equation as above $2 = (1.005)^{4t}$ when both sides are divided by P. This means it takes about 37.4 years to double any amount of money at an interest rate of 2% compounded quarterly.

Application activity 1.9

1. Without using calculator, compare the numbers *a* and *b*.

a) $a = 3\log 2$ and $b = \log 7$

b) $a = \log 2 + \log 40$ and $b = 4\log 2 + \log 5$

c) $a = 2\log 2$ and $b = \log 16 - \log 3$

2. Given that $\log 2 = 0.30$, $\log 3 = 0.48$ and $\log 5 = 0.69$. Calculate

a) $\log 150$ b) $\log \frac{9}{2}$ c) $\log 0.2 + \log 10$

3. Find co-logarithm of

a) 100 b) 42 c)15

4. How long will 70,000Frw take to accumulate to 100,000Frw if it is invested at 11%?

5. Mr. Mateso was astonished when he was told that his money \$200 in the Bank was raised to a certain power and became \$8,000,000 after 3 years. Explain how he can discover that power *n*.

1.10 Important applications of arithmetic

Activity 1.10

Make a research in the library or on internet and categorize problems of Economics and Finance that are easily solved with the use of arithmetic.

Focus on the following: Elasticity of demand, Arc of elasticity for demand, Simple interest and compound interest, Final value of investment.

CONTENT SUMMARY

1. Elasticity' of demand and Arc of elasticity for demand

Elasticity' of demand:

Price elasticity of demand is a measure of the responsiveness of demand to changes in price. It is

usually defined as

$$e = (-1)\frac{\%change\,in\,quantity\,demand}{\%change\,in\,price}$$

Arc of elasticity for demand

The (−1) in this definition ensures a positive value for elasticity as either the change in price or the change in quantity will be negative. When there are relatively large changes in price and quantity it is best to use the concept of 'arc elasticity' to measure elasticity along a section of a demand schedule.

This takes the changes in quantity and price as percentages of the averages of their values before and after the change.

Thus arc elasticity is usually defined as

$$arc\,e = (-1)\frac{\dfrac{change\,in\,quantity}{0.5\,(1st\,quantity + 2nd\,quantity)}.100}{\dfrac{change\,in\,price}{0.5\,(1st\,price + 2nd\,price)}.100}$$

Although a positive price change usually corresponds to a negative quantity change, and vice versa, it is easier to treat the changes in both price and quantity as positive quantities. This allows the (−1) to be dropped from the formula. The 0.5 and the 100 will always cancel top and bottom in arc elasticity calculations.

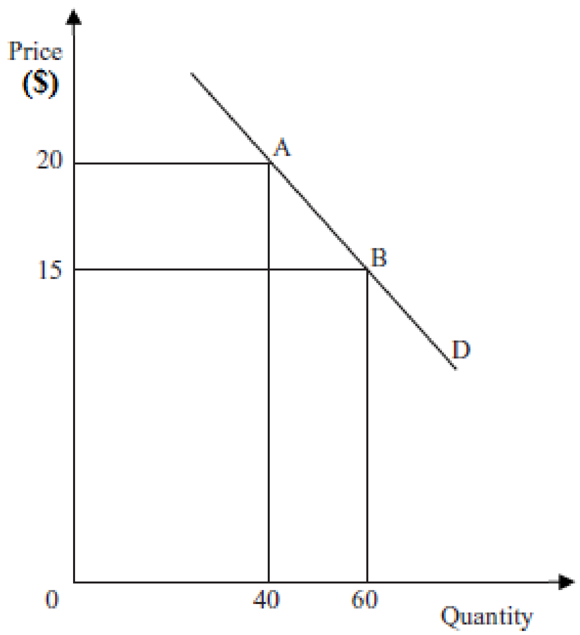
Thus we are left with $$arc\,e = (-1)\frac{\dfrac{change\,in\,quantity}{0.5\,(1st\,quantity + 2nd\,quantity)}}{\dfrac{change\,in\,price}{0.5\,(1st\,price + 2nd\,price)}}$$

as the formula actually used for calculating price arc elasticity of demand

Example:

Calculate the arc elasticity of demand between points A and B on the demand schedule shown

In the following figure:



Solution:

Between points A and B price falls by 5 from 20 to 15 and quantity rises by 20 from 40 to 60. Using the formula defined above:

$$arce = \frac{\frac{20}{40+60}}{\frac{5}{20+15}} = \frac{7}{5}$$

2. Simple interest

Simple interest is the amount charged when one borrows money or loan from a financial institution which accrue yearly.

This interest is a fixed percentage charged on money/loan that is not yet paid.

This interest is calculated based on the original principal or loan and is paid at regular intervals

From, above example, we see that when the principal (P), rate in percentage (R) and time in year (T) are given, then simple interest (I) for the given period is given by

$$I = P.\frac{R}{100}.T = \frac{PRT}{100}$$

The total amount (A) paid back by the borrower or the financial institution on the expiry of the interest period is the sum of the principal and the interest earned (I).

Thus, $A = P + I$

Example:

1. Find the simple interest earned from $3400 Frw$ borrowed for 3years at the rate of 10%p.a.

Solution: $Interst, I = \frac{PRT}{100} = \frac{3400 \times 10 \times 3 Frw}{100} = 1020 Frw$

2. Gatete borrowed 32 000Frw from a lending institution to start a business. If the institution charged interest at a rate of 8%p.a., calculate the simple interest and the total amount she eventually paid back after 4years.

Solution: $Interest, I = \frac{PRT}{100} = \frac{32000 \times 8 \times 4 Frw}{100} = 10240 Frw$

$Amount, A = (32000 + 10240) Frw = 42\,240 Frw$

3. The compound interest *(developed above)*

Considering the case in which P is invested in a bank for *n* interest periods per year at the rate r expressed as a decimal. The accumulated amount A after the time t (number of years P is invested) is given by:

$$A = P\left(1 + \frac{r}{n}\right)^{nt}$$

where *n* is the number of interest periods per year, r is the interest rate expressed as decimal, A is the amount after t years.

Note: When the interest rate is compounded per year, $A = P(1+r)^n$ where r is expressed as a decimal for example $r = 9\% = 0.09$. When the interest rate is compounded monthly, $A = P\left(1 + \frac{r}{12}\right)^{12.t}$ where t is the number of years and r expressed as a decimal.

4. Calculating the final value of an investment

Consider an investment at compound interest where:

P is the initial sum invested, A is the final value of the investment, r is the interest rate per time period (as a decimal fraction) and n is the number of time periods.

The value of the investment at the end of each year will be $1+r$ times the sum invested at the start of the year.

Thus, for any investment,

The value after one year $= P(1+r)$

Value after 2 years $= P(1+r)(1+\mathrm{r}) = \mathrm{P}(1+\mathrm{r})^2$

Value after 3 years $= P(1+r)(1+\mathrm{r})(1+r) = \mathrm{P}(1+r)^3$ *etc*

We can see that each value is multiplied by $(1+r)$ to the power of number of years that the sum is invested. Thus, after n years the initial sum A is multiplied by $(1+r)^n$.

The formula for the initial value A. Thus, after an investment of P money, for n time periods at interest rate r is therefore $A = P(1+r)^n$.

Examples:

1) If \$600 is invested for 3 years at 8% then the known values for the formula will be

$P = \$600; \quad n = 3; \quad r = 8\% = 0.08$. Thus the final sum will be

$A = P(1+r)^n = 600(1.08)^3 = \$755.83$

2) If \$4,000 is invested for 10 years at an interest rate of 11% per annum what will the final value of the investment be?

Solution:

P = \$4,000, n = 10   r = 11% = 0.11

$F = A(1 + i)n = 4{,}000(1.11)10$

$A = P(1+r)^n = 4000(1.11)^{10} = 11{,}253.68$

The final value of the investment be $\$11{,}253.68$

5. Time periods, initial amounts and interest rates

The formula for the final sum of an investment contains the four variables F, A, i and n.

So far we have only calculated F for given values of A, i and n. However, if the values of any three of the variables in this equation are given then one can usually calculate the fourth.

Initial amount

A formula to calculate A, when values for F, i and n are given, can be derived as follows.

Since the final sum formula is

$$A = P(1+r)^n,$$

then, dividing through by $(1+r)^n$ (we get the initial sum formula

$$P = \frac{A}{(1+r)^n} \quad or\ P = A(1+r)^{-n}$$

Example:

How much money needs to be invested now in order to accumulate a final sum of \$12,000 in 4 years' time at an annual rate of interest of 10%?

Solution:

Using the formula derived above, the initial amount is

$$P = A(1+r)^{-n} = 12{,}000(1.1)^{-4} = \$8{,}196.16$$

What we have actually done in the above example is find the sum of money that is equivalent to \$12,000 in 4 years' time if interest rates are 10%. An investor would therefore be indifferent between (a) \$8,196.16 now and (b) \$12,000 in 4 years' time. The \$8,196.16 is therefore known as the 'present value' (PV) of the \$12,000 in 4 years' time. We shall come back to this concept in the next few sections when methods of appraising different types of investment project are explained.

Time period

Calculating the time period is rather more tricky than the calculation of the initial amount.

From the final sum formula.

$A = P(1+r)^n$, then $\frac{A}{P} = (1+r)^n$

If the values of A, P and r are given and one is trying to find n this means that one has to work out to what power (1 + r) has to be raised to equal $\frac{A}{P}$. One way of doing this is via logarithms.

Example

For how many years must $1,000 be invested at 10% in order to accumulate $1,600?

Solution

P= $1,000, *A* = $1,600, r = 10% = 0.1

Substituting these values into the formula

$\frac{A}{P} = (1+r)^n$ then, $\frac{1600}{1000} = (1+0.1)^n$

We get $1.6 = (1.1)^n$

Since to find the *n*th power of a number its logarithm must be multiplied by *n*. Finding logs, this means that our equation becomes

$\log 1.6 = n \log(1.1)$

And $n = \frac{\log(1.6)}{\log(1.1)} = 4.93$. Given that 4,93 years is approximately 5years,

If investments must be made for whole years then the answer is 5 years.

This answer can be checked using the final sum formula

$\mathrm{A} = \mathrm{P}(1+r)^n = 1000(1.1)^5 = 1{,}610.51 \approx 1600$

If the $1,000 is invested for a full 5 years then it accumulates to just over $1,600, which checks out with the answer above.

A general formula to solve for *n* can be derived as follows from the final sum formula:

$A = P(1+r)^n$, $\frac{A}{P} = (1+r)^n$ and $n = \frac{\log(A/P)}{\log(1+r)}$

An alternative approach is to use the iterative method and plot different values on a spreadsheet. To find the value of *n* for which $1.6 = (1.1)^n$.

this entails setting up a formula to calculate the function $y=(1.1)^n$ and then computing it for different values of n until the answer 1.6 is reached. Although some students who find it difficult to use logarithms will prefer to use a spreadsheet, logarithms are used in the other examples in this section. Logarithms are needed to analyze other concepts related to investment and so you really need to understand how to use them.

Example:

1) How many years will \$2,000 invested at 5% take to accumulate to \$3,000?

Solution:

P = 2,000; A = 3,000; r= 5% = 0.05

Using these given values in the time period formula derived above gives

$$n=\frac{\log(A/P)}{\log(1+r)}=\frac{\log 1.5}{\log 1.05}=8.34$$

This money will need to be invested in 8.34 years.

2) How long will any sum of money take to double its value if it is invested at 12.5%?

Solution

Let the initial sum be A. Therefore the final sum is

$A=2P$ and $r=12.5\%=0.125$

Substituting these value for A and r into the final sum formula $A=P(1+r)^n$, we find

$2A=A(1.125)^n$

Or $2=(1.125)^n$ which gives $n=\dfrac{\log 2}{\log 1.125}=5.9$

For any sum of money, it takes 5.9 years to double its value if it is invested at 12.5%.

Interest rates

A method of calculating the interest rate on an investment is explained in the following example.

If \$4,000 invested for 10 years is projected to accumulate to \$6,000, what interest rate is used to derive this forecast?

Solution

$P = 4{,}000$ A = 6,000 and $n = 10$

Substituting these values into the final sum formula

$A = P(1+r)^n$ gives $6000 = 4000(1+r)^{10}$

$1.5 = (1+r)^{10}$

$1+r = \sqrt[10]{1.5}$

$r = 4.14\%$

A general formula for calculating the interest rate can be derived. Starting with the familiar final sum formula

$$A = P(1+r)^n \Leftrightarrow \frac{A}{P} = (1+r)^n \Leftrightarrow r = \left(\frac{A}{P}\right)^{1/n} - 1$$

Therefore, $r = \left(\frac{A}{P}\right)^{1/n} - 1$

Example:

1) At what interest rate will 3000Frw accumulate to 10,000Frw after 15 years?

Solution:

$$r = \left(\frac{A}{P}\right)^{1/n} - 1 = \left(\frac{10000}{3000}\right)^{1/15} - 1 = 0.083574 = 8.36\%$$

2) An initial investment of 50,000Frw increases to 56,711.25Frw after 2 years. What interest rate has been applied?

Solution

Given that P = 50,000 A = 56,711.25 n = 2 and $r = \sqrt[n]{\frac{A}{P}} - 1$

We have $r = \sqrt[2]{\frac{56{,}711.25}{50{,}000}} - 1 = 0.065 = 6.5\%$

Application activity 1.8.2

1. An initial investment of £50,000 increases to £56,711.25 after 2 years. What interest rate has been applied?

2. Consider the formula for $A = P\left(1+\frac{r}{n}\right)^{nt}$ and the case of the continuously compounded interest where n (the number of interest periods per year) increases without bound (ie $n \to \infty$) and prove that the amount after t years is $A = Pe^{rt}$ where P is the principal, r is an interest rate expressed as decimal, t the number of years P is invested.

1.11. END UNIT ASSESSMENT

1. Why is it necessary for a student teacher to study arithmetic? Explain your answer on one page and be ready to defend your arguments in a classroom discussion;

2. The price of a house was 2000000Frw in the year 2000. At the end of each year the price has increased by 6%.

a) Find the price of the house after one year

b) Find the price of that house after 3 years

c) Find the price that such a house should have in this year.

UNIT 2:

EQUATIONS AND INEQUALITIES

Key Unit competence: **Model and solve daily life problems using linear, quadratic equations or inequalities**

2.0. Introductory Activity

1) By the use of library and computer lab, do the research and explain the linear equation.
2) If x is the number of pens for a learner, the teacher decides to give him/her two more pens. What is the number of pens will have a learner with one pen?
   a) Complete the following table called table of value to indicate the number $y = f(x) = x + 2$ of pens for a learn who had x pens for $x \geq 0$.

x	-2	-1	0	1	2	3	4
$y = f(x) = x + 2$			2				
(x,y)			(0,2)				

   b) Use the coordinates of points obtained in the table and complete them in the Cartesian plan.
   c) Join all points obtained. What is the form of the graph obtained?
   d) Suppose that instead of writing $f(x) = x + 2$ you write the equation $y = x + 2$. Is this equation a linear equation or a quadratic equation? What is the type of the inequality "$x + 2 \geq 0$ "?
3) Find out an example of problem from the real life situation that can be solved by the use of linear equation in one unknown.

2.1 Linear equations in one unknown and related problems

Activity 2.1

1) Assume that in a competitive market, the supply schedule is $p = 60 + 0.4q$ where p is a price function and q is the quantity supplied. Is the price increasing or decreasing.
   a) Find the price for q = 600 units.
   b) What is the value of p for $q = 0$. What does it mean for an industry which is producing a certain good and has just fixed a price p_0 at the beginning ($q = 0$). Can you justify this price p_0.
2) Solve the equation $3 - x = 3$
3) Solve the following inequality: $2x - 1 > x + 3$. Express the solution set in terms of interval and present solution on graph.

CONTENT SUMMARY

An equation is a mathematical statement expressing the equality of two quantities or expressions. Equations are used in every field that uses real numbers. A linear equation is an equation of a straight line.

Consider the statement $x - 1 = 0$, this statement is equation as there is two quantities to be equal and is true for the value $x = 1$. The value $x = 1$ is called the solution of the statement $x - 1 = 0$ the number 1 is called the root of the equation. Thus, to find a solution to the given equation is to find the value that satisfies that equation.

To do this, rearrange the given equation such that variables will be in the same side and constants in the other side and then find the value of the variable.

Example:

Solve in set of real numbers the following equations

a) $x + 2 = 10$

b) $3 + x = 9 - 2x$

c) $x = 16 - x$

Solutions:

a) $x + 2 = 10$

$x = 10 - 2$

$x = 8$

b) $3 + x = 9 - 2x$

$x + 2x = 9 - 3$

$3x = 6$

$x = \frac{6}{3}$

$x = 2$

$S = \{2\}$

c) $x = 16 - x$

$x + x = 16$

$2x = 16$

$x = \frac{16}{2}$

$x = 8$

$S = \{8\}$

Real life problems involving linear equations

Problems which are expressed in words are known as problems or applied problems. A word or applied problem involving unknown number or quantity can be translated into linear equation consisting of one unknown number or quantity. The equation is formed by using conditions of the problem. By solving the resulting equation, the unknown quantity can be found.

In solving problem by using linear equation in one unknown the following steps can be used:

i) Read the statement of the word problems

ii) Represent the unknown quantity by a variable

iii) Use conditions given in the problem to form an equation in the unknown variable

iv) Verify if the value of the unknown variable satisfies the conditions of the problem.

Examples

1) The sum of two numbers is 80. The greater number exceeds the smaller number by twice the smaller number. Find the numbers.

Solution

Let the smaller number be x

Therefore the greater number be $80 - x$

According to the problem,

$(80 - x) - x = 2x$

$80 - x - x = 2x$

$80 - 2x = 2x$

$80 = 2x + 2x$

$80 = 4x$

$4x = 80$

$x = \frac{80}{4}$

$x = 20$

Now substitute the value of $x = 20$ in $80 - x$ we get $80 - 20 = 60$

Therefore, the smaller number is 20 and the greater number is 60

2) A boat covers a certain distance downstream in 2 hours and it covers the same distance upstream in 3 hours. If the speed of the stream is $2km/_{hr}$ Find the speed of the boat.

Solution

Let the speed of the boat be $xkm/_{hr}$

Speed of the stream=$2km/hr$

Speed of the boat downstream = $(x + 2)\, km/hr$

Speed of the boat upstream = $(x - 2)\, km/hr$

Distance covered in both the cases in same.

$2(x + 2) = 3(x - 2)$

$2x + 4 = 3x - 6$

$2x - 3x = -6 - 4$

$-x = -10$

$x = 10$

Therefore, the speed of the boat is $10km/hr$

Product equation

The equation in the form $(ax + b)(cx + d) = 0$ is product equation since the product of factors is null (zero) either one of them is zero. To solve this we proceed as follows:

$(ax + b)(cx + d) = 0$
$ax + b = 0$ or $cx + d = 0$

$x = \frac{-b}{a}$ or $x = \frac{-d}{c}$

Example

Solve the equation $(2x + 4)(x - 1) = 0$

Solution:

$2x + 4 = 0$ or $x - 1 = 0$

$x = -2$ or $x = 1$

Fractional equation of the first degree

The general form $\frac{ax+b}{cx+d} = 0$ to find solution of this equation we need to have the condition of existence $cx + d \neq 0$ and $\frac{ax+b}{cx+d} = 0 \Rightarrow ax + b = 0$

We solve $ax + b$ and we take the value(s) which verify the condition of existence

Example

Solve $\frac{2x-6}{x+1} = 0$

Solution:

The existence condition is $x+1 \neq 0$

$2x - 6 = 0$

$2x = 6$

$x = 3$

$S = \{3\}$

Application activity 2.1

1) Solve in set of real numbers:
   a) $4x + 5 = 20 + x$
   b) $x - 31 = 50 - 8x$
   c) $\frac{2x+5}{x-6} = 4$
2) Mr. Peter wants to fence his garden of rectangle form where the length of that rectangle is twice its breadth. If the perimeter is 72 metres, help Peter to find the length and breadth of the rectangle.
3) The sum of two numbers is 25. One of the numbers exceeds the other by 9. Explain how you can determine these numbers.
4) Find the number whose one fifth is less than the one fourth by 3.

2.2 Linear inequalities in one unknown and related real life problems

Activity 2.2

1) Find the value(s) of x such that the following statements are true
   a) $x < 5$ b) $x > 0$ c) $-4 < x < 12$
2) Use library and computer lab to do the following:
   a) Discuss the difference between linear equations and linear inequalities
   b) Find out example of linear inequalities and try to solve them.
   c) Examples on how linear inequalities are applied when solving real word problems

CONTENT SUMMARY

2.2.1 Meaning of an inequality

The statement $x+3=10$ is true only when $x=7$. If x is replaced by 5, we have a statement $5+3=10$ which is **false**. To be true we may say that 5 + 3 is less than 10 or in symbol $5+3<10$. If x is replaced by 8, the statement 8+3=10 is also **false**. In those two cases we no longer have equality but inequality.

Suppose that we have the inequality $x+3<10$, in this case we have an inequality with one unknown. Here the real value of x satisfies this inequality is not unique. For example 1 is a solution but 3 is also a solution. In general all real numbers less than 7 are solutions. In this case we will have many solutions combined in an interval.

Now, the solution set of $x+3<10$ is an open interval containing all real numbers less than 7 whereby 7 is excluded. How?

We solve this inequality as follow

$x+3<10$

$\Leftrightarrow x<10-3$

$\Leftrightarrow x<7$

And then $S=\left]-\infty,7\right[$

Note that:

- When the same real number is added or subtracted from each side of inequality the direction of inequality is not **changed**.
- The direction of the inequality is **not changed** if both sides are multiplied or divided by the same **positive real number** and is **reversed** if both sides are multiplied or divided by the **same negative real** number.

2.2.2 Intervals

A subset of real line is called an interval if it contains at least two numbers and also contains all real numbers between any two of its elements. For example, the set of real numbers x such that $x>6$ is an interval, but the set of real numbers y such that $y\neq 0$ is not an interval.

If a and b are real numbers and $a<b$, we often refer to:

i) The open interval from a to b, denoted by (a,b) or $\left]a,b\right[$, consisting of all real numbers x satisfying $a<x<b$

ii) The closed interval from a to b, denoted by $[a, b]$, consisting of all real numbers x satisfying $a \leq x \leq b$

iii) The half-open interval $[a, b[$, consisting of all real numbers x satisfying the inequalities $a \leq x < b$

iv) Half-open interval $]a, b]$, consisting of all real numbers x satisfying the inequalities $a < x \leq b$

Examples

Solve the following inequalities and express the solution in terms of intervals.

a) $2x - 1 > x + 3$ b) $-\frac{x}{3} \geq 2x - 1$

c) $2(x+5) > 2x-8$ d) $2x+5 \leq 2x+4$

Solution:

a) $2x - 1 > x + 3$

$2x > x + 3 + 1$

$2x - x > 4$

$x > 4$

The solution set is the interval $]4, \infty[$

b) $-\frac{x}{3} \geq 2x - 1$

$-x \geq 3(2x - 1)$

$x \leq -6x + 3$

$7x \leq 3$

$x \leq \frac{3}{7}$

The solution set is the interval $\left]-\infty, \frac{3}{7}\right]$

c) $2(x+5) > 2x-8$

$2x+10 > 2x-8$

$0x > -18$ which is impossible. The solution set is $S = \varnothing$

d) $2x+5\le 2x+4$

$2x-2x\le 4-5$

$0x\le -1$

Since any real number times zero is zero and zero is not less or equal to -1 then the solution set is the empty set. $S=\varnothing$

2.2.3 Inequalities products / quotients

Activity 2.2.3

Explain the method you can use to solve the following inequalities:

1) $(x+1)(x-1)<0$

2) $\dfrac{2x-3}{x}<0$

Suppose that we need to solve the inequality of the form $(ax+b)(cx+d)<0$ For this inequality we need the set of all real numbers that make the left hand side to be negative. Suppose also that we need to solve the inequality of the form $(ax+b)(cx+d)>0$. For this inequality we need the set of all real numbers that make the left hand side to be positive.
We follow the following steps:

a) First we solve for $(ax+b)(cx+d)=0$

b) We construct the table called **sign table**, find the sign of each factor and then the sign of the product or quotient if we are given a quotient.

For the quotient the value that makes the denominator to be zero is always excluded in the solution. For that value we use the symbol || in the row of quotient sign.

c) Write the interval considering the given inequality sign.

Example

Solve in set of real numbers the following inequalities

a) $(3x+7)(x-2)<0$

b) $\dfrac{x+4}{2x-1}\ge 0$

Solution

a) $(3x+7)(x-2)<0$

Start by solving $(3x+7)(x-2)=0$

$3x+7=0$

$\Leftrightarrow x=-\dfrac{7}{3}$ or $x-2=0$

$\Leftrightarrow x=2$

Then, we find the sign table.

x	$-\infty$	$-\frac{7}{3}$		2	$+\infty$
$3x+7$	-	0	+		+
$x-2$	-		-	0	+
$(3x+7)(x-2)$	+	0	-	0	+

Since the inequality is $(3x+7)(x-2)<0$ we will take the interval where the product is negative. Thus, $S=\left]-\dfrac{7}{3},2\right[$

b) $\dfrac{x+4}{2x-1}\geq 0$

$x+4=0 \Rightarrow x=-4$

$2x-1=0 \Rightarrow x=\dfrac{1}{2}$

x	$-\infty$		-4		$\frac{1}{2}$		$+\infty$
$x+4$		-	0	+		+	
$2x-1$		-		-	0	+	
$\dfrac{x+4}{2x-1}$		+	0	-	||	+	

$S=]-\infty,-4]\cup\left]\dfrac{1}{2},+\infty\right[$

2.2.4 Inequalities involving absolute value

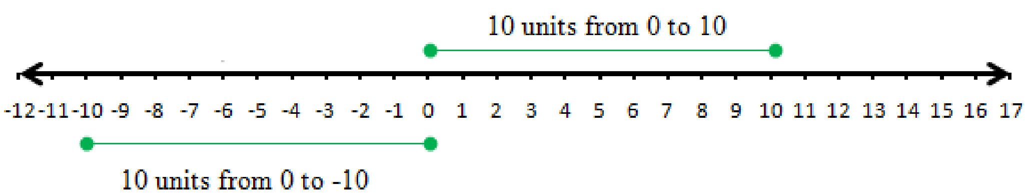
Activity 2.2.4

State the set of all real numbers whose number of units from zero, on a number line, are

1) greater than 4
2) less than 6

The inequality $|x-a|<k$ says that the distance from x to a is less than k, so x must lie between $a-k$ and $a+k$ or equivalently a must lie between $x-k$ and $x+k$ if k is a positive number.

Recall that absolute value of a number is the number of units from zero to a number line. That is, $|x|=k$ means k units from zero (k is a positive real number or zero).



For all real number x and $k \geq 0$

a) $|x|<k \Leftrightarrow -k<x<k$

b) $|x-a|<k \Leftrightarrow a-k<x<a+k$

c) $|x|>k \Leftrightarrow x>k \text{ or } x<-k$

d) $|x-a|>k \Leftrightarrow x>a+k \text{ or } x<a-k$

Example

1) Solve $|3x-2| \leq 1$

Solution:

$-1 \leq 3x-2 \leq 1$ we solve this pair of inequalities

$-1 \leq 3x-2$ and $3x-2 \leq 1$

$-1+2 \leq 3x$ $\qquad$ $3x \leq 1+2$

$^1/_3 \leq x$ $\qquad\qquad$ $x \leq 1$

Thus the solution lies in the interval $[^1/_3, 1]$

2) Find the solution set of the inequality $|3x-15|<3$

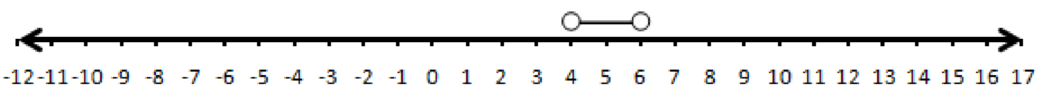
Solution

$|3x-15|<3 \Leftrightarrow -3<3x-15<3$

$-3+15<3x<3+15 \Leftrightarrow 12<3x<18$ or $\frac{12}{3}<x<\frac{18}{3} \Leftrightarrow 4<x<6$

The solution set is $S=\{x \in IR : 4<x<6\}$

Number line:



2.2.5 Real life problems involving linear inequalities

Activity 2.2.5

Sam and Alex play in the same team at their school. Last Saturday their team played with another team from other school in the same district, Alex scored 3 more goals than Sam. But together they scored less than 9 goals.
What are the possible number of goals Alex scored?

Inequalities can be used to model a number of real life situations. When converting such word problems into inequalities, begin by identifying how the quantities are relate to each other, and then pick the inequality symbol that is appropriate for that situation. When solving these problems, the solution will be a range of possibilities. Absolute value inequalities can be used to model situations where margin of error is a concern.

Examples

1) The width of a rectangle is 20 meters. What must the length be if the perimeter is at least 180 meters?

Solution:

Let x be length of rectangle

$perimeter = 2length + 2width$

$2x + 2(20) \geq 180$

$2x \geq 180 - 40$

$x \geq 70$

The length must be at least 70 meters.

2) John has 1 260 000 Rwandan Francs in an account with his bank. If he deposits 30 000 Rwanda Francs each week into the account, how many weeks will he need to have more than 1 820 000 Rwandan Francs on his account?

Solution:

Let x be the number of weeks

We have total amount of deposits to be made **+** the current balance is greater to the total amount wanted.

That is $30000x + 1260000 > 1820000$

$$30000x > 1820000 - 1260000$$

$$30000x > 560000$$

$$x > \frac{560000}{30000} \approx 19$$

Thus, John needs at least 19 weeks to have more than 1 820 000 Rwandan Francs on his account.

Application activity 2.2

1) Joe enters a race where he has to cycle and run. He cycles a distance of 25 km, and then runs for 20 km. His average running speed is half of his average cycling speed. Joe completes the race in less than 2½ hours, what can we say about his average speeds?
2) Explain your colleague whether or not a solution set for an inequality can have one element.

2.3 Simultaneous linear equations in two unknowns (Solving by equating two same variables)

Activity 2.3

In each of the following systems find the value of one variable from one equation and equalize it with the same value of another variable from second equation. Calculate the values of those variables.

1) $\begin{cases} 3x - 2y = -6 \\ x + y = -2 \end{cases}$ 2) $\begin{cases} x + 2y = 10 \\ -3x + 2y = 12 \end{cases}$ 3) $\begin{cases} x + 4y = 8 \\ y - x = 2 \end{cases}$

CONTENT SUMMARY

To find the value of unknown from simultaneous equation by equating the same variable in terms of another, we do the following steps:

i) Find out the value of one variable in first equation,
ii) Find out the value of that variable in the second equation,
iii) Equate the same values obtained from the two equations,
iv) Solve the equation obtained to find out the unknown variables.

Example

1) Algebraically, solve the simultaneous linear equation by equating the same variables.

$$\begin{cases} 4x + 5y = 2 \\ x + 2y = -1 \end{cases}$$

Solution:

$$\begin{cases} 4x + 5y = 2 \\ x + 2y = -1 \end{cases}$$

From equation (1) $4x + 5y = 2 \Rightarrow x = \frac{2-5y}{4}$, from equation (2) $x + 2y = -1 \Rightarrow x = -1 - 2y$

Equalize the values of x from equation (1) and (2)

$$\frac{2-5y}{4} = -1 - 2y$$

$$2 - 5y = 4(-1 - 2y)$$

$$2 - 5y = -4 - 8y$$

$$-5y + 8y = -4 - 2$$

$$y = -2$$

$x = \frac{2-5y}{4}$, $x = \frac{2-5(-2)}{4} = \frac{12}{4} = 3$ then, $s = \{(3, -2)\}$

2) Solve algebraically the following system by equating two same variables

$$\begin{cases} x + y = 1 \\ 2x + 3y = 2 \end{cases}$$

Solution

From equation $x + y = 1 \Rightarrow y = 1 - x$

From equation $2x + 3y = 2 \Rightarrow y = \frac{2-2x}{3}$

By equating those values of y, $1 - x = \frac{2-2x}{3}$

$$3(1 - x) = 2 - 2x$$

$$3 - 3x = 2 - 2x$$

$$-3x + 2x = 2 - 3$$

$$x = 1,\ y = 1 - x \Rightarrow y = 1 - 1 = 0$$

$$s = \{(1,0)\}$$

3) Solve the following simultaneous equation

$$\begin{cases} x + 4y = 8 \\ y - x = 2 \end{cases}$$

Solution

From equation (1), $x + 4y = 8 \Rightarrow x = 8 - 4y$

From equation (2), $y - x = 2 \Rightarrow x = -2 + y$

By equating values from (1) and (2)

$$8-4y=-2+y$$
$$-4y-y=-2-8$$
$$-5y=-10$$
$$y=2,\quad x=-2+2=0 \quad \text{then, } s=\{(0,2)\}.$$

Application activity 2.3

Solve the simultaneous linear equations by equating two same variables

1) $\begin{cases} x-y=3 \\ 2x-2y=6 \end{cases}$ 2) $\begin{cases} x+y=2 \\ 2x+2y=4 \end{cases}$ 3) $\begin{cases} 3x-5y=10 \\ 2x+y=12 \end{cases}$

4) $\begin{cases} x-4y=1 \\ x-y=2 \end{cases}$ 5) $\begin{cases} -x+y=0 \\ x+2y=3 \end{cases}$ 6) $\begin{cases} 5y+3x=2 \\ 10x+6y=0 \end{cases}$

2.4 Simultaneous linear equations in two unknowns (solving by row operations or elimination method)

Activity 2.4

For each of the following, find two numbers to be multiplied to the equations such that one variable will be eliminated when making the addition or subtraction of the two equations.

$$\begin{cases} -2x+5y=-7 \\ 7x-3y=-19 \end{cases}$$

CONTENT SUMMARY

To eliminate one of the variables from either of equations to obtain an equation in just one unknown, make one pair of coefficients of the same variable in both equations negatives of one another by multiplying both sides of an equation by the same number. Upon adding the equations, that unknown will be eliminated.

Example

1) Solve the system of equations using elimination method.

$$\begin{cases} x+y=1 \\ 2x+3y=2 \end{cases}$$

Solution

$$\begin{cases} x+y=1 \\ 2x+3y=2 \end{cases}\left|\begin{matrix} -2 \\ 1 \end{matrix}\right. \Leftrightarrow \begin{cases} -2x-2y=-2 \\ 2x+3y=2 \end{cases}$$

$$\overline{\quad y=0 \quad}$$

$x+y=1 \Leftrightarrow x=1-y=1$

$S=\{(1,0)\}$

2) Solve the system of equation by using elimination method

$$\begin{cases} -2x+5y=-7 \\ 7x-3y=-19 \end{cases}$$

Solution

$$\begin{cases} -2x+5y=-7 \\ 7x-3y=-19 \end{cases} \Leftrightarrow \begin{matrix} 7(-2x+5y)=7(-7) \\ 2(7x-3y)=2(-19) \end{matrix}$$

$$\begin{matrix} -14x+35y=-49 \\ 14x-6y=-38 \end{matrix} \Leftrightarrow \begin{array}{r} -14x+35y=-49 \\ 14x-6y=-38 \\ \hline 29y=-87 \end{array}$$

$$x=\frac{-7-5y}{-2}=\frac{-7-5(-3)}{-2}=-4$$

Then, $s=\{(-4,-3)\}$

Application activity 2.4

1) Solve the following system of equation by using elimination method.

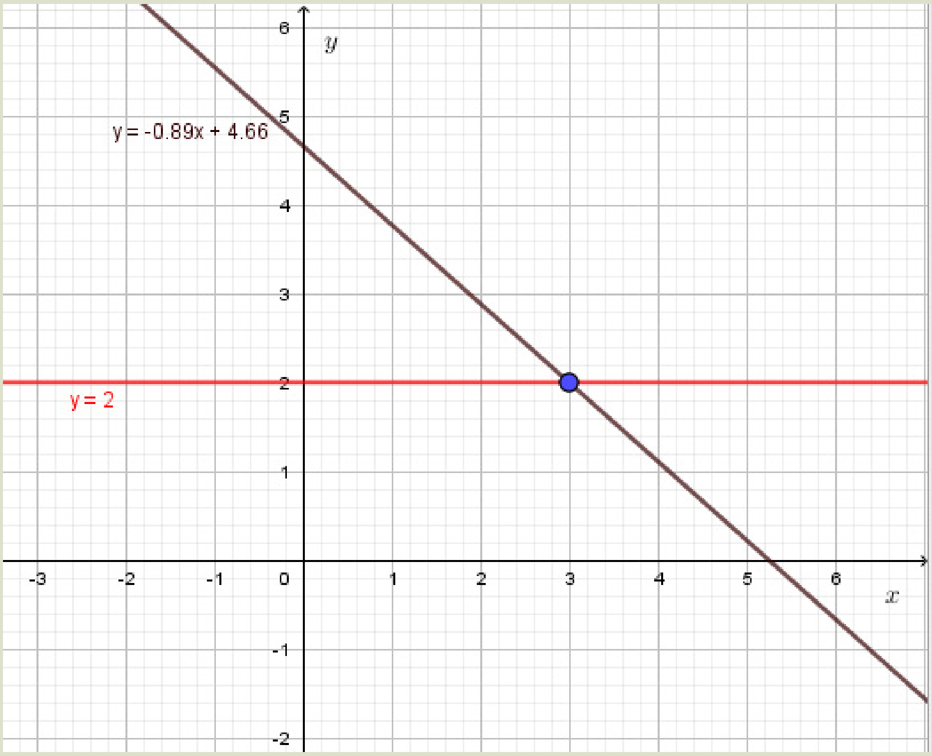
a) $\begin{cases} 3x-4y=1 \\ x-3y=2 \end{cases}$ b) $\begin{cases} x-4y=1 \\ x-y=2 \end{cases}$ c) $\begin{cases} 3x-2y=-6 \\ x+y=-2 \end{cases}$ d) $\begin{cases} 2x+3y=8 \\ x-y=2 \end{cases}$

2) Use your own words to explain how to solve algebraically simultaneous linear equations.

2.5 Solving graphically simultaneous linear equations in two unknowns

Activity 2.5

1) Discuss how you can find the coordinate of the point intercept of two lines whose equations are known.



2) Given the system of equations

$$\begin{cases} 3x - 2y = -6 \\ x + y = -2 \end{cases}$$

a) For each equation from the system, choose any two values of x and use them to find values of y . this gives you two points in the form

b) Plot the obtained points in XY plane and join these points to obtain the lines.

c) What is the point of intersection for two lines?

d) Write the obtained point as solution of the system.

CONTENT SUMMARY

One way to solve a system of linear equations is by graphing. The intersection of the graphs represents the point at which the equations have the same x-value and the same y-value. Thus, this ordered pair represents the solution common to both equations. This ordered pair is called the solution to the system of equations.

The following steps can be applied in solving system of linear equation graphically:

1) Find at least two points for each equation.
2) Plot the obtained points in XY plane and join these points to obtain the lines. Two points for each equation give one line.
3) The point of intersection for two lines is the solution for the given system

Examples

1) Solve the following system by graphical method.

$$\begin{cases} x+y=4 \\ x-y=2 \end{cases}$$

Solution

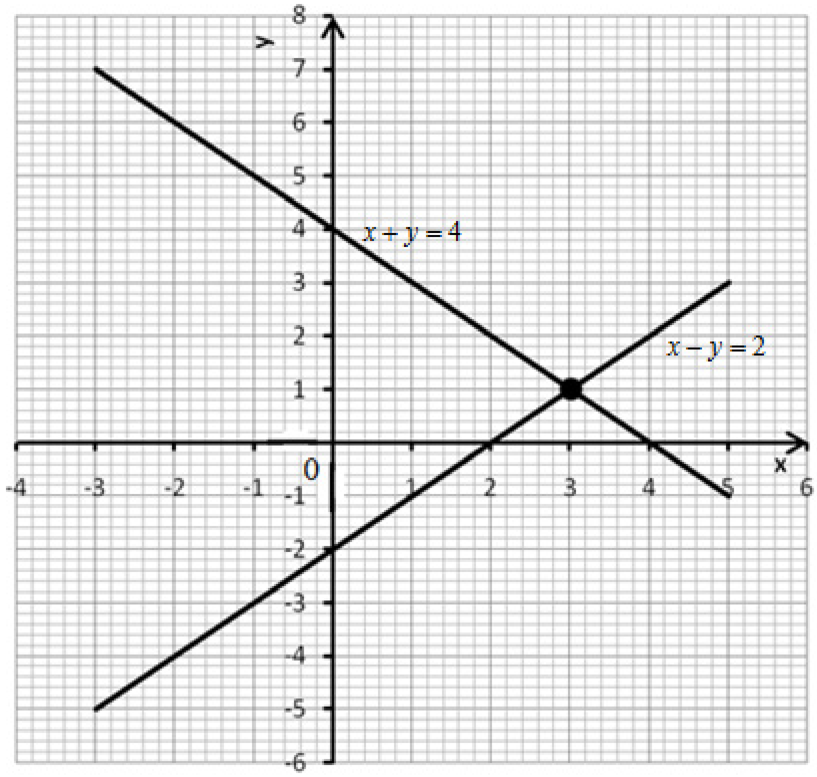
For $x+y=4$

x	-3	5
y	7	-1

For $x-y=2$

x	-3	5
y	-5	3

Graph



The two lines intersect at point $(3,1)$. Therefore the solution set is $S=\{(3,1)\}$.

2) Solve graphically the following system of linear equations

Solve the following equations graphically

$$\begin{cases} x+y=2 \\ 2y=4-2x \end{cases}$$

Solution

For $x+y=2$

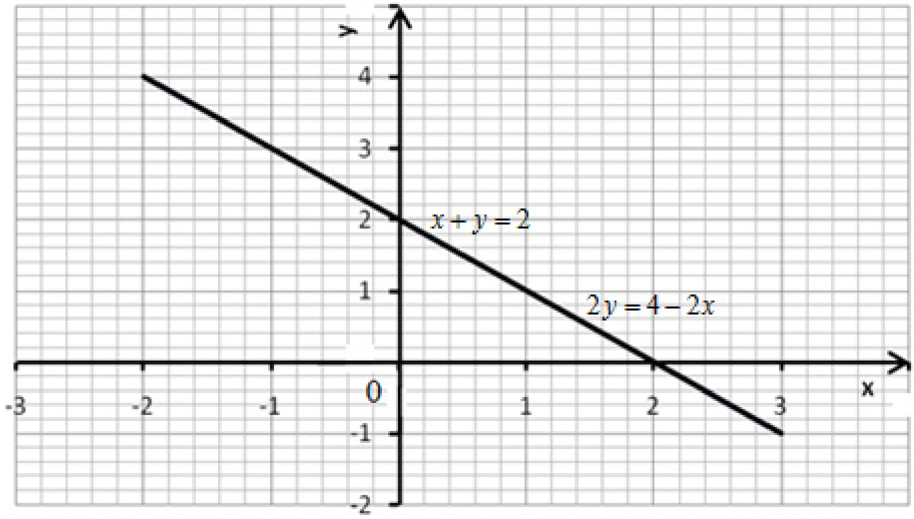
x	-2	3
y	4	-1

For $2y=4-2x$

x	-2	3

y	4	-1

Graph



We see that the two lines coincide as a single line. In such case there is infinite number of solutions.

Application activity 2.5

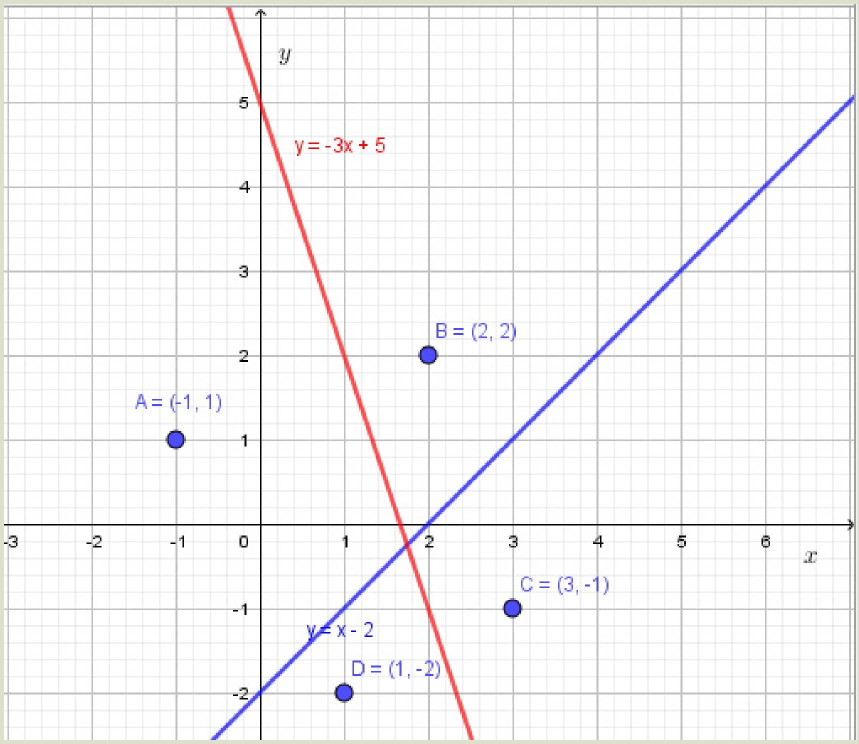
1) Solve the graphically the following system of linear equations.

a) $\begin{cases} 4y + x = 8 \\ -x + y = 2 \end{cases}$ b). $\begin{cases} 3x + y = 1 \\ x - y = -1 \end{cases}$ c) $\begin{cases} 2x + y = 3 \\ 6x + 3y = 9 \end{cases}$

2.6 Solving algebraically and graphically simultaneous linear inequalities in two unknowns

Activity 2.6

The following graph illustrates two lines and their equations



For each point A, B, C, and D, replace its coordinate in the two inequalities to verify which one satisfies the following system:

$$\begin{cases} y < x - 2 \\ y > -3x + 5 \end{cases}$$

CONTENT SUMMARY

A system of inequalities consits of a set of two inequalities with the same variables. The inequalities define the conditions that are to be considered simultaneously.

Each inequality in the set contains infinetely many ordered pair solutions defined by a region in rectangular coordinate plane. When considering two of these inequalities together, the intersection of these sets will define the set of simultaneous ordered pair solutions.

Linear inequalities with two unknowns are solved to find a range of values of the two unknowns which make the inequalities true at the same time. The solution is represented graphically by a region.

In finding solution, first, graph the "equals" line, then shade in the correct area.

The following steps can be used to find the solution of simultaneous inequalities graphically:

1) Rearrange the equation so that "y" is on the left and everything else on the right.
2) Plot the y line (make it a solid line for $y \leq$ or $y \geq$ and a dashed line for $y <$ or $y >$
3) Shade above the line for a greater than $y >$ or $y \geq$ or below the line for a less than $y <$ or $y \leq$
4) The intersection will define the set of ordered pair solutions.

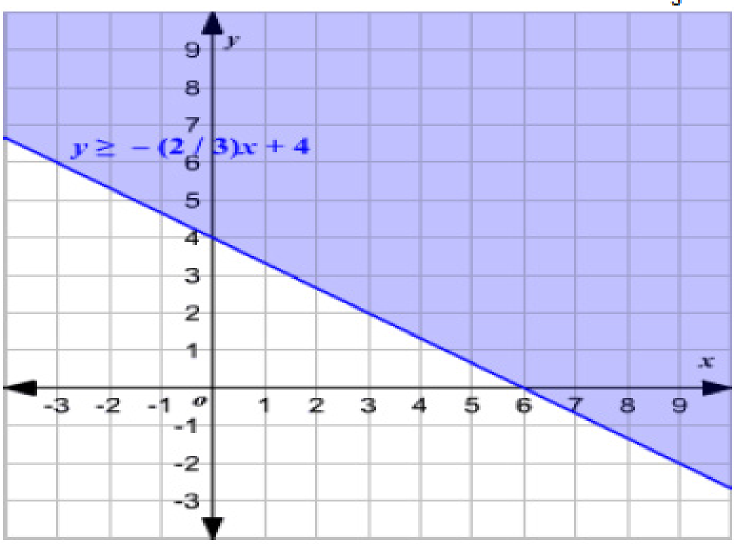
Example

Solve the system of inequalities by graphing:

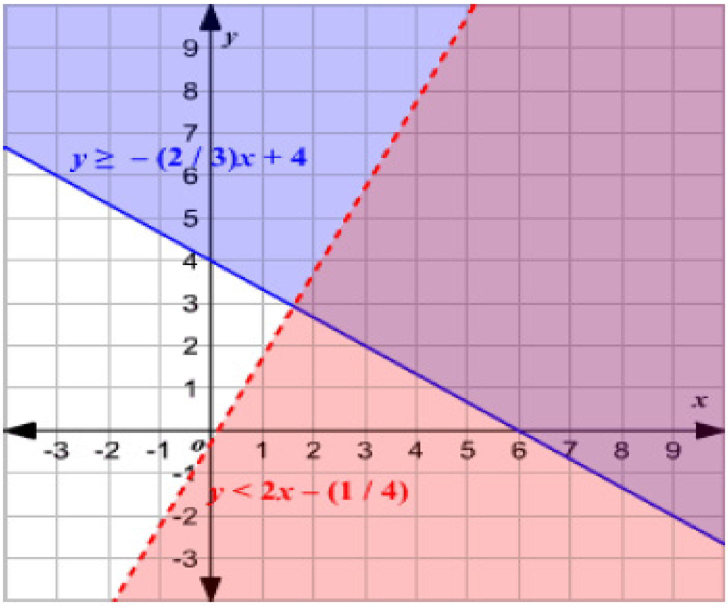
$$\begin{cases} 2x + 3y \geq 12 \\ 8x - 4y > 1 \\ x < 4 \end{cases}$$

Solution

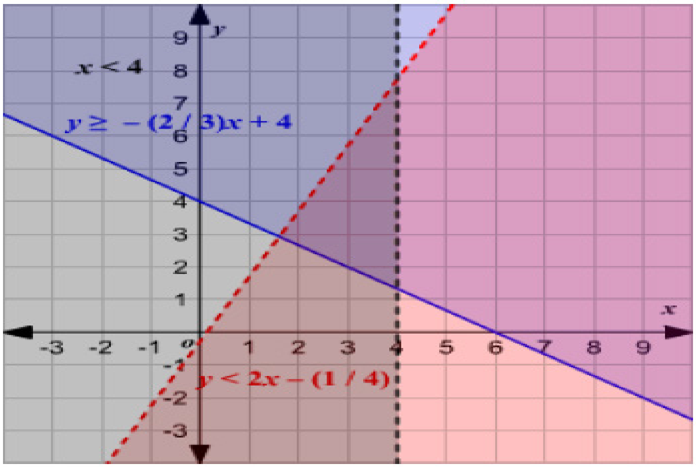
From Each inequality from the system we have $2x + 3y \geq 12$ which implies $y \geq -\frac{2}{3}x + 4$, the related equation to this is $y = -\frac{2}{3}x + 4$ since the inequality is $\geq$, not a strict one, the border line is solid. Graph the line $y = -\frac{2}{3}x + 4$



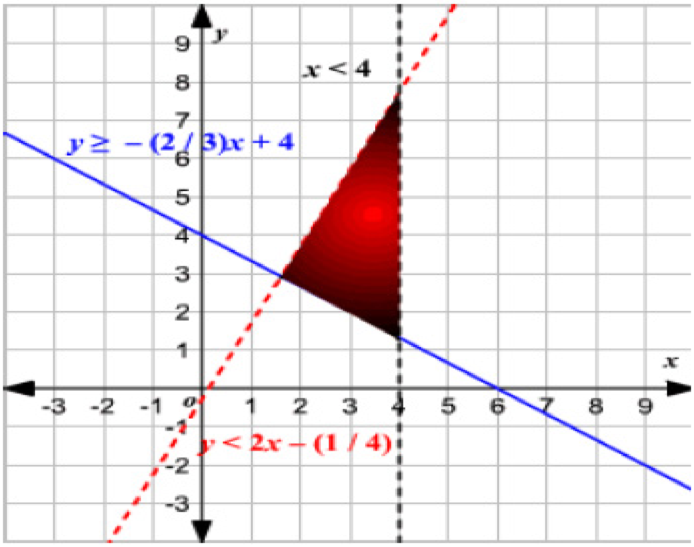
Similarly, draw a dashed line of related equation of the second inequality $y < 2x - \frac{1}{4}$ which has a strict inequality.



Draw the dashed vertical line $x = 4$ which is the related to the equation of the third inequality.



The solution of the system of inequalities is the intersection region of the solutions of the three inequalities as it is done in the following figure.



Application activity 2.6

1) Algebraically and graphically, solve the following simultaneous inequalities.

2) a) $\begin{cases} y - 2x \leq 1 \\ x + y \leq 10 \\ x \geq 0 \end{cases}$ b) $\begin{cases} -3x + 2y > 6 \\ 6x - 4y > 8 \end{cases}$ c) $\begin{cases} x \geq 0 \\ y \geq 0 \\ 2x + y \leq 4 \end{cases}$

3) Use your own words to explain how to solve graphically the simultaneous inequalities.

2.7 Solving quadratic equations by the use of factorization and discriminant

Activity 2.7

Smoke jumpers are fire fighters who parachute into areas near forest fires. Jumpers are in free fall from the time they jump from a plane until they open their parachutes. The function $y = -16t^2 + 1600$ gives a jumper's height y in metre after t seconds for a jump from $1600m$.

a) How long is free fall if the parachute opens at $1000m$?

b) Complete a table of values for t = 0, 1, 2, 3, 4, 5 and 6.

CONTENT SUMMARY

Equations which are written in the form of $ax^2+bx+c=0$ $(a\neq 0)$ are called quadratic equations. To find solution of this equation the two main ways can be used in solving such equation

a) Use of factorization or finding square roots

Grouping terms or decomposition can be used to factorize the quadratic equations; and later help us to find the solution of equation. By having the product of ac and the sum of those two integers which gives b, it helps you to decompose into a product of factors.

Example

1) Solve in $\mathbb{R}$: $6y^2+5y-25=0$

Solution:

To obtain a common factor, terms that have a common factor are grouped together, and the common factor of each group is divided as follows

$$6y^2+5y-25=0$$

Guess two integers whose sum is 5 and product is -150

$$6y^2+15y-10y-25=0$$

$$(6y^2+15y)-(10y+25)=0$$

$$3y(2y+5)-5(2y+5)=0$$

$$(2y+5)(3y-5)=0$$

$$(2y+5)=0 \quad \text{or} \quad (3y-5)=0$$

$$y=\frac{-5}{2} \quad \text{or} \quad y=\frac{5}{3}$$

2) Solve the equation $x^2-4y-45=0$

Solution:

$$x^2-4x-45=0$$

$$(x-9)(x+5)=0$$

$$x=9 \quad or \quad x=-5$$

Then, the solution set is $S=\{-5,9\}$

3)Find the solution set of $(3x-1)(2x+3)=-5$

Solution:

We expand the product as the equation is different from zero

$$6x^2+7x-3=-5$$

$$6x^2+7x-3+5=0$$

$$(3x+2)(2x+1)=0$$

$$3x+2=0 \quad or \quad 2x+1=0$$

$x=-\frac{2}{3}$ or $x=-\frac{1}{2}$ Thus, the solution set is $\left\{-\frac{2}{3},-\frac{1}{2}\right\}$

b) Use of discriminant

In quadratic equation: $ax^2+bx+c=0$, Let $\Delta=b^2-4ac$ be discriminant

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

(There are two values of x which can help us to write down factor form of a quadratic expression).

Examples

1) Solve in $\mathbb{R}$: $6x^2+5x-25=0$

Solution:

$$\boldsymbol{a=6\,,\ b=5,\ c=-25}$$

$$\Delta=b^2-4ac\,,$$

$$\Delta=(5)^2-4(6)(-25)=625\,,\ \sqrt{\Delta}=\sqrt{625=\pm25}$$

$$x_1=\frac{-5+25}{12}=\frac{20}{12}=\frac{5}{3} \qquad x_2=\frac{-5-25}{12}=\frac{-30}{12}=\frac{-5}{2}$$

Then, the solution set is $S=\left\{-\frac{5}{2},\frac{5}{3}\right\}$.

2) For what value of k will the equation : $x^2+2x+k=0$ have one double roots? Find that root.

Solution:

For one double root $\Delta=0$.

$$\Delta=4-4k$$

$$4-4k=0 \Rightarrow k=1$$

Thus, the value of k is 1.

That root is $x=-\frac{2}{2}=-1$. The solution set is $S=\{-1\}$.

Application activity 2.7

a) Use factorization and discriminant to solve the following equations

1) $3x^2=10-x$
2) $x^2-3x=-11$
3) $x^2-12x+11=0$
4) $x^2+16=8x$

b) The area of a rectangular garden is 30 meter square. If the length is 7 meter longer than the width, find the dimensions of the rectangle.

c) Does a quadratic equation have more than one solution? Explain your answer.

2.8 Applications of linear and quadratic equations in economics and finance: Problems about supply and demand (equilibrium price)

Activity 2.8

Assume that a firm can sell as many units of its product as it can manufacture in a month at

180 Rwandan francs each. It has to pay out 2400 Rwandan francs fixed costs plus a marginal cost of 140 Rwanda francs for each unit produced. How much does it need to produce to break even (where total revenue equals to total cost)?

CONTENT SUMMARY

When only two or single variables and equations are involved, a simultaneous equation system can be related to familiar graphical solutions, such as supply and demand analysis.

For **example,** assume that in a competitive market the demand schedule is given by

$p = 420 - 0.2q$ *and* the supply schedule is given by $p = 60 + 0.4q$,

If this market is in equilibrium, the equilibrium price and quantity will be where the demand and supply schedules intersect. This requires you to solve the system formed by the two simultaneous equations. Its solution will correspond to a point which is on both the demand schedule and the supply schedule. Therefore, the equilibrium values of p and q will be such that both equations (1) and (2) hold.

Example

1) In a competitive market the demand schedule is given by $p = 420 - 0.2q$ *and* the supply schedule is given by $p = 60 + 0.4q$, Solve for p and q *the* simultaneous equation and determine the point at which the market is in equilibrium.

Solution:

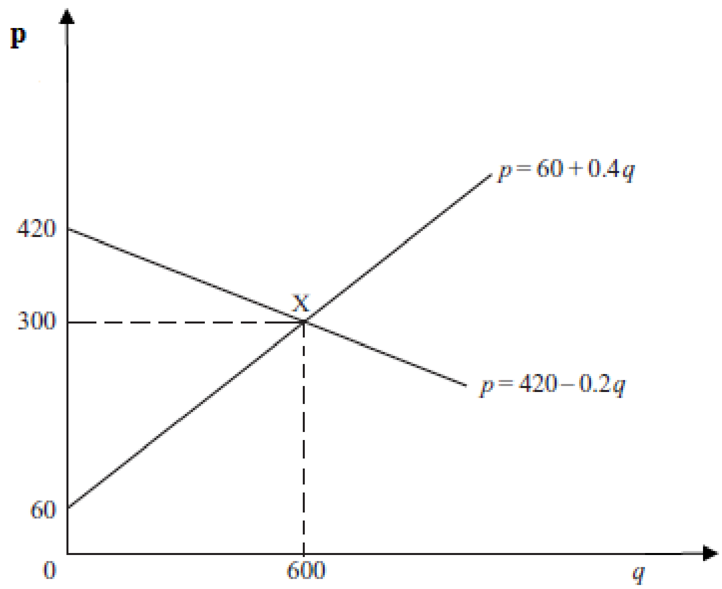
Let us solve the system

$$\begin{cases} p = 420 - 0.2q \\ P = 60 + 0.4q \end{cases}$$

Equalizing the value of p, we find $420 - 0.2q = 60 + 0.4q$.

Which gives $q = 600$. Replacing this value in the given two equations, we find $p = 300$. The market is in equilibrium at the point $q = 600$.

These two functional relationships are plotted in the figure and both hold at the intersection point $X(600, 300)$.



2) Calculate the equilibrium values of p and q in a competitive market where the demand schedule is $p = 200q^{-1}$ and the supply is $p = 30 + 2q$

Solution

In equilibrium, demand price equals supply price. Therefore

$$200q^{-1} = 30 + 2q$$

Multiplying through by q, $200 = 30q + 2q^2$

$$0 = 30q + 2q^2 - 200$$

$$0 = (2q - 10)(q + 20)$$

Therefore $2q - 10 = 0$ and $q + 20 = 0$

$$q = 5 \quad \text{or} \quad q = -20$$

We can ignore the second solution as negative quantities cannot exist. Thus the equilibrium quantity is 5.

Substituting this value into the supply function gives equilibrium price

$$p = 30 + 2 \times 5 = 40$$

3) A firm makes two goods A and B which require two inputs K and L. One unit of A requires 6 units of K plus 3 units of L and one unit of B requires 4 units of K plus 5 units of L. The firm has 420 units of K and

300 units of L at its disposal. How much of A and B should it produce if it wishes to exhaust its supplies of K and L totally?

Solution:

This question requires you to use the economic information given to set up a mathematical problem in a format that can be used to derive the desired solution.

The total requirements of input K are 6 for every unit of A and 4 for each unit of B, which

Can be written as $K = 6A + 4B$

Similarly, the total requirements of input L can be specified as $L = 3A + 5B$

As we know that $K = 420$ and $L = 300$ because all resources are used up

Then, $420 = 6A + 4B$ and $300 = 3A + 5B$

Solve the system of linear equations to find values of A and B

$$\begin{cases} 420 = 6A + 4B \\ 300 = 3A + 5B \end{cases}$$

From first equation $420 = 6A + 4B \Rightarrow A = \frac{420-4B}{6}$

$$420 = 6A + 4B \Rightarrow A = \frac{420-4B}{6}$$

From second equation $300 = 3A + 5B \Rightarrow A = \frac{300-5B}{3}$

$$300 = 3A + 5B \Rightarrow A = \frac{300-5B}{3}$$

$$\frac{420-4B}{6} = \frac{300-5B}{3}$$

$$1260 - 12B = 1800 - 30B$$

$$-12B + 30B = 1800 - 1260$$

$$18B = 540$$

$B = 30$ or $A = \frac{420-4(30)}{6} = 50$

The firm should therefore produce 50 units of A and 30 units of B.

4) Two student teachers were driving at constant speeds to the same direction during their holidays. The first travelling at 40 km/hr left

Kigali at 8:30 a.m. The second travelling at 60 km/hr followed him after 1 hour.

a) What is the distance between the two divers one hour after the departure of the first driver?

b) If t is the same time used by the two drivers after the departure of the second driver and y the entire distance covered,

i) Establish the functions $y = f(t)$ of the first;

ii) Establish the functions $y = g(t)$ of the second driver

iii) When did the second bus over take the first bus?

iv) Illustrate the two functions on the Cartesian plan from $t = 0$ to $t = 5\,hrs$ and show the point where the second met the first driver.

Solution:

The speed of the first driver is 40 km/hour. $v_1 = 40km/hr$

The second driver has a speed of 60km/hr and left after the departure of the first. $v_2 = 60km/hr$.

a) Distance covered by the first driver is $y_0 = v_1.t = (40km/hr).1hr = 40km$

b) i) The distance covered by the first driver is the initial distance plus the distance covered after the departure of the second driver.

This is $y = y_0 + v_1.t = 40 + 40.t$.

Then $y = f(t) = 40 + 40.t$

ii) The distance y covered by the second driver is $y = v_2.t = 60.t$.

Therefore, $y = g(t) = 60.t$

iii) After the departure of the second driver, the two drivers will be using the same time t and when this second driver will meet the first, the two will be covered the same distance

$$d = y = g(t) = f(t)$$

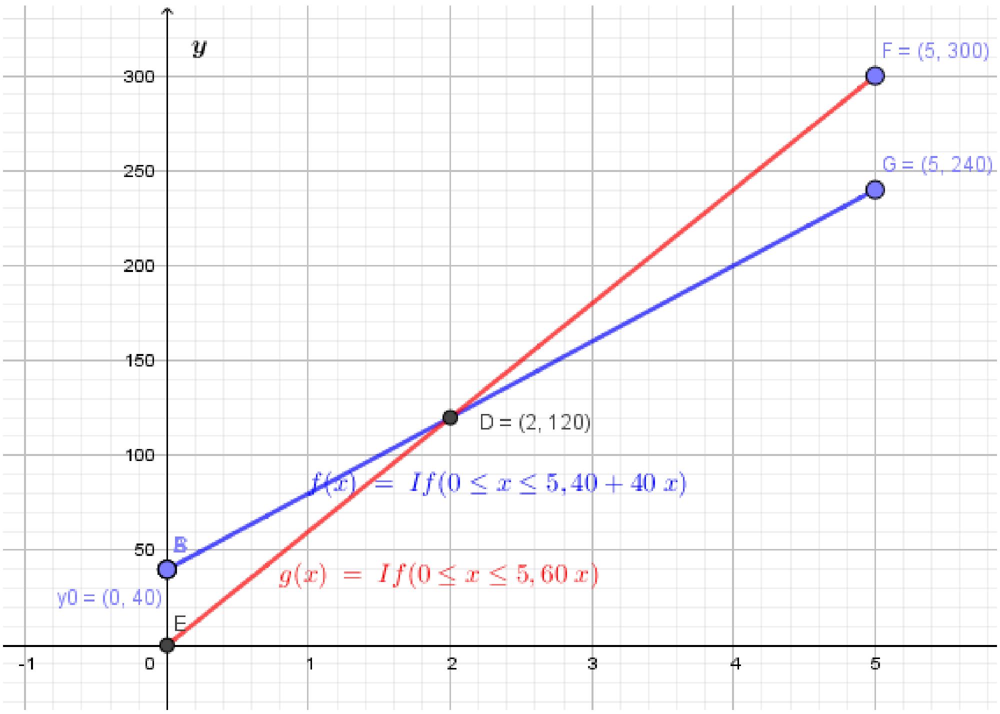
This means: $40 + 40.t = 60.t$

$$40+40.t=60.t$$
$$60t-40t=40$$
$$20t=40$$
$$t=2$$

The second driver will meet the first after 2 hours. It will be at 11h30 because 9h30+2hs =11h30.

iv) The table of values can help to draw the graphs of the function f and g.

T	0	1	2	3	4	5
$y=f(t)=40+40.t$ **(in km)**	40	80	120	160	200	240
$y=g(t)=60.t$ **(in km)**	0	60	120	180	240	300



Application activity 2.8

1) Today a house-worker has 3000 Frw. His boss pays him/her 400Frw per day and he saves all the money received.

   a) What is the money the house-worker will have after 2 days, 5 days and after 10 days ?

   b) Discuss the function $y = f(t)$ for the money saved by the house-worker where t represents the time in days.

   c) Illustrate the function $y = f(t)$ in the Cartesian plan on a manila paper;

   d) Discuss the money the house-worker will have after 2 months.

   e) Is it possible for a house-worker to save the money? Explain your answer.

2) The national income in the basic Keynesian macroeconomic model is given by $Y = C + I$. If C is given by $40 + 0.5Y$ and $I = 200$, calculate the national income in the basic Keynesian macroeconomic model.

2.9. END UNIT ASSESSMENT

1) Solve by factorization the quadratic equation $2x^2-6x-20=0$.
2) Solve the inequalities $-4x-3>-2x-11$
3) Algebraically and graphically, solve the simultaneous inequalities.

$$\begin{cases} x\geq 0 \\ y\geq 0 \\ 2x+y\leq 4 \end{cases}$$

4) The length of a rectangular garden is $5cm$ more than its width and the area is $50cm^2$. Calculate the length and width of the garden.
5) A ball is thrown upwards from a rooftop, 80m above the ground. It will reach a maximum vertical height and then fall back to the ground. The height of the ball from the ground at time t is h, which is given by, $h=-16t^2+64t+80$
    a) What is the height reached by the ball after 1 second?
    b) What is the maximum height reached by the ball?
    c) How long will it take before hitting the ground?
6) Two cyclists move away from a town along two perpendicular paths at $20km/hr$ and $40km/hr$ respectively. The second cyclist starts the journey an hour later than the first one. Find the time taken for them to be $100km$ apart.

UNIT: 3

GRAPHS AND FUNCTIONS

Key unit Competence: Apply graphical representation of function in economics models

3.0. Introductory Activity 3

Suppose that average weekly household expenditure on food C depends on average net household weekly income Y according to the relationship $C = 12 + 0.3Y$.

a) Can you find a value of Y for which C is not a real number?

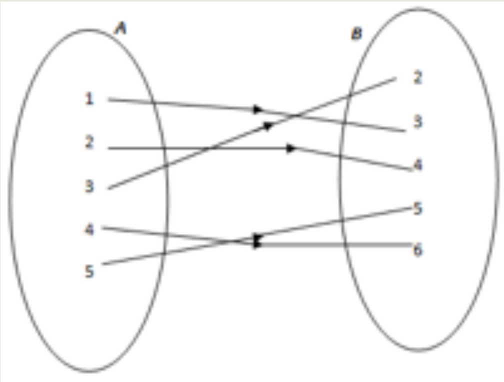
b) Complete a table of value from Y= 0 to Y= 10 and use it to draw the graph of $C = 12 + 0.3Y$

c) If Y=90, what is the value of C?

3.1 Generalities on numerical functions

Activity 3.1

In following arrow diagram, for each element of set A state which element of B is mapped to it.



a) What is the set of elements of A which have images in B?

b) Determine the set of elements in B which have antecedent in A.

c) Is there any element of A which has more than one image?

A function is a rule that assigns to each element in a set A one and only one element in set B. We can even define a function as any relationship which takes one element of one set and assigns to it one and only one element of second set.

If x is an element in the domain of a function f, then the element that f associates with x is denoted by the symbol $f(x)$ (read f of x) and is called the image of x under f or the value of f at x.

The set of all possible values of $f(x)$ as x varies over the domain is called the range of f and it is denoted $R(f)$.

The set of all values of A which have images in B is called Domain of f and denoted $Domf$.

We shall write $f(x)$ to represent the image of x under the function f. The letters commonly used for this purpose are f, g and h.

Examples

1. Given that $f(x)=x^2$*, find the values of* $f(0), f(2), f(3), f(4)$ *and* $f(5)$

Solution

$f(0)=0^2=0$ $\qquad$ $f(2)=2^2=4$

$f(3)=3^2=9$ $\qquad$ $f(4)=4^2=16$

$f(5)=5^2=25$

Note:

$f(x)=x^2$ can also be written as $f: x \to x^2$ which is read as

"f is a function which maps x onto x^2"

2. Draw arrow diagrams for the functions. Use the domain $\{1,2,3\}$

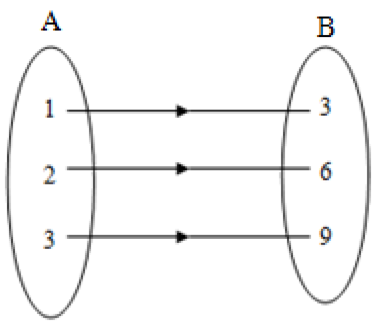
a. $f: x \to 3x$

b. $h: x \to x^2+1$

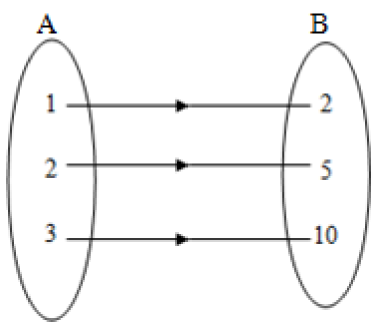
c. $g: x \to 2x+1$

Solution

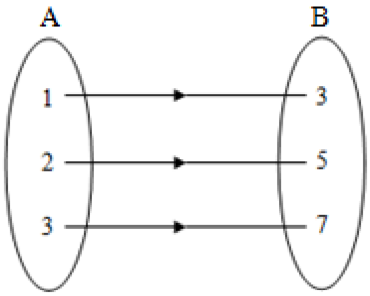
a. $f : x \rightarrow 3x$



b. $h : x \rightarrow x^2 + 1$



c. $g : x \rightarrow 2x + 1$



3. The functions f and g are given as $f(x) = x + 3 \;\; for \;\; x \geq 0$ and

$g(x) = x^2 \;\; for \;\; -2 \leq x \leq 3$

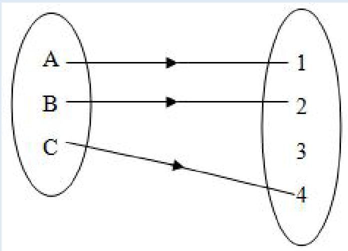
State the range of each of these functions.

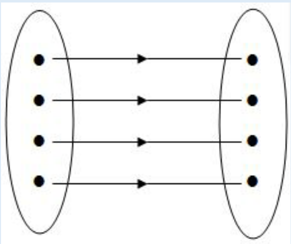
Solution

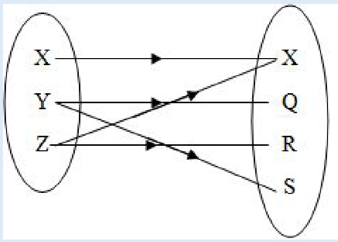
If $x \geq 0$, then $x+3 \geq 3$. Thus, the range of f will be $f(x) \geq 3$. If $-2 \leq x \leq 3$, then $0 \leq x^2 \leq 9$. Thus, the range of g will bethe set of all x such that $0 \leq g(x) \leq 9$

Application activity 3.1

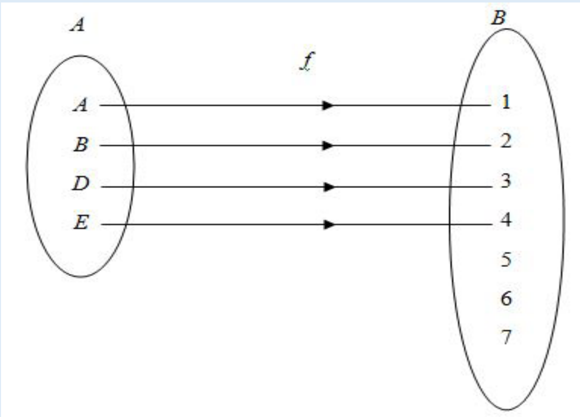
1. State which of the following relations shows a function







2. In the following arrow diagram, state the domain, co-domain and range



3. If $f(x) = 2x+4$, find

a. $f(2)$

b. $f(-2)$

c. $f(d)$

d. The value of a if $f(a) = a$

3.2 Types of functions

Activity 3.2

Differentiate rational from irrational numbers. Guess which of the following functions is a polynomial, rational or irrational function

1. $f(x)=(x+1)^2$
2. $h(x)=\dfrac{x^3+2x+1}{x-4}$
3. $f(x)=\sqrt{x^2+x-2}$

a) Constant function

A function that assigns the same value to every member of its domain is called **a constant function**. This is $f(x)=c$ where c is a given real number.

Example: $f(x)=4$

The function f given by $f(x)=3$ *is constant.*

Remark

The constant function that assigns the value *c* to each real number is sometimes called **the constant function *c***.

b) Identity: The identity function is of the form $f(x)=x$

c) Monomial

A function of the form cx^n, where c is constant and n a nonnegative integer is called a **monomial in *x***.

Example: $f(x)=0.3x$ is a monomial in x.

$2x^3$; πx^7; $4x^0$; $-6x$ *and* x^{17} *are monomials*

The functions $4x^{\frac{1}{2}}$ and x^{-3} are not monomials because the powers of x are not nonnegative integers.

d) Polynomial

A function that is expressible as the sum of finitely many monomials in x is called **polynomial in x.**

Example: x^3+4x+7; $17-\frac{2}{3}x$; y and x^5 are polynomials. Also $(x-2)^3$ is a polynomial in x because it is expressible as a sum of monomials.

In general, f is a polynomial in x if it is expressible in the form

$f(x)=a_0+a_1x+a_2x^2+......+a_nx^n$ where n is a nonnegative integer and $a_0, a_1, ..., a_n$ are real constants.

A polynomial is called

- **linear** if it has the form $a_0+a_1x,\ a_1\neq 0$, with degree 1
- **quadratic** if it has the form $a_0+a_1x+a_2x^2,\ a_2\neq 0$, with degree 2
- **cubic** if it has the form $a_0+a_1x+a_2x^2+a_3x^3,\ a_3\neq 0$, with degree 3
- **nth degree** polynomial if it has the form $a_0+a_1x^1+a_2x^2+a_3x^n+...+a_nx^n; a_n\neq 0$, with degree n

e) Rational function

A function that is expressible as ratio of two polynomials is called **rational function.**

It has the form f(X)=$\frac{a_0+a_1x+a_2x^2+......+a_nx^n}{b_0+b_1x+b_2x^2+......+b_mx^m}$.

Example:

$f(x)=\frac{x^2+4}{x-1},\ g(x)=\frac{1}{3x-5}$ are rational functions

f) Irrational function

A function that is expressed as root extractions is called irrational function. It has the form $\sqrt[n]{f(x)}$, where $f(x)$ is a polynomial and n is positive integer greater or equal to 2.

Example:

$f(x)=\frac{\sqrt{x^2+4}}{\sqrt[3]{x-1}},\ g(x)=\sqrt{\frac{x}{x-5}}$ are irrational functions

Application activity 3.2

What is the type of the following function:

1. $f(x)=x^3+2x^2-2$
2. $g(x)=-2$
3. $h(x)=\dfrac{x^3+2x^2-2}{x-5}$
4. $f(x)=\dfrac{x^2-2}{x^2-8x+15}$

3.3 Domain of definition for a numerical function

Activity 3.3.1

For which value(s) of x the following functions are not defined

1. $f(x)=x^3+2x+1$
2. $f(x)=\dfrac{1}{x}$
3. $g(x)=\dfrac{x+2}{x-1}$

Activity 3.3.2

Find the domain of definition for each of the following functions

1. $f(x)=x^3+2x^2-2$
2. $g(x)=-2$
3. $h(x)=\dfrac{x^3+2x^2-2}{x-5}$
4. $f(x)=\dfrac{x^2-2}{x^2-8x+15}$

Activity 3.3.3

For each of the following functions, give a range of values of the variable x for which the function is not defined

1. $f(x)=\sqrt{2x+1}$
2. $f(x)=\sqrt[3]{x^2+x-2}$
3. $g(x)=\sqrt{\dfrac{x-2}{x+1}}$

Case 1: The given function is a polynomial

Given that $f(x)$ is polynomial, then the domain of definition is the set of real numbers. That is $Domf = \mathbb{R}$

Case 2: The given function is a rational function

Given that $f(x)=\dfrac{g(x)}{h(x)}$ where $g(x)$ and $h(x)$ are polynomials, then the domain of definition is the set of real numbers excluding all values where the denominator is zero. That is $Domf = \{x \in \mathbb{R} : h(x) \neq 0\}$.

Case 3: The given function is an irrational function

Given that $f(x)=\sqrt[n]{g(x)}$ where $g(x)$ is a polynomial, there are two cases

a) If n is odd number, then the domain is the set of real numbers. That is $Domf = \mathbb{R}$

b) If n is even number, then the domain is the set of all values of x such $g(x)$ is positive or zero. That is $Domf = \{x \in \mathbb{R}: g(x) \geq 0\}$

Examples:

The domain of the function $f(x)=3x^5+2x^4+4x+6$ is $\mathbb{R}$ since it is a polynomial.

1. Given $f(x)=\dfrac{x+1}{3x+6}$, find the domain of definition.

Solution

Condition: $3x+6 \neq 0$

$3x+6=0 \Rightarrow x=-2$

Then, $Domf = \mathbb{R} \setminus \{-2\}$ or $Domf =]-\infty -2[\cup]-2,+\infty[$

2. Given $f(x)=\sqrt{x^2-1}$, find domain of definition

Solution: Condition: $x^2-1 \geq 0$.

We need to construct sign table to see where x^2-1 is positive

$x^2-1=0 \Rightarrow x=\pm 1$

x	$-\infty$		-1		1		$+\infty$
x^2-1		+	0	-	0	+	

Thus, $Domf =]-\infty,-1] \cup [1,+\infty[$

3. Find domain of definition of $f(x)=\sqrt[3]{x+1}$

Solution

Since the index in radical sign is odd number, then $Domf = \mathbb{R}$

4. What is the domain of definition of $g(x)$ if $g(x)=\sqrt[4]{x^2+1}$?

Solution

Condition: $x^2+1 \geq 0$

Clearly x^2+1 is always positive. Thus $Domg = \mathbb{R}$

5. Find domain of $f(x)=\dfrac{x}{\sqrt{x^3-4x^2+x+6}}$

Solution

Condition: $x^3-4x^2+x+6>0$. Here we have combined two conditions:
$x^3-4x^2+x+6 \geq 0$ and $x^3-4x^2+x+6 \neq 0$

$x^3-4x^2+x+6=(x+1)(x-2)(x-3)$

x	$-\infty$		-1		2		3		$+\infty$
$x+1$		-	0	+	+	+	+	+	
$x-2$		-	-	-	0	+	+	+	
$x-3$		-	-	-	-	-	0	+	
x^3-4x^2+x+6		-	0	+	0	-	0	+	

Then, $Domf =]-1,2[\cup]3,+\infty[$

6. The following functions map an element x of the domain onto its image y. i.e $f : x \to y$

For each of the three functions below, state

i) the domain for which the function is defined,

ii) the corresponding range of the function,

iii) whether the function is one-to-one or many-to-one.

a) $f : x \to x+3$

b) $f : x \to \sqrt{x}$

c) $f : x \to \dfrac{1}{x^2}$

Solution

a) $f : x \to x+3$

i) The function is defined for all real number, so the domain is $\mathbb{R}$,

ii) For this domain, the range will contain all real numbers, so the range is $\mathbb{R}$,

iii) Each element of the range is obtained from only one element of the domain, so the function is **one-to-one (1 to 1)**.

b) $f : x \to \sqrt{x}$

i) The function is not defined for negative x, so the domain is $\{x \in \mathbb{R} : x \geq 0\}$

, $Domf = [0, +\infty[$

ii) The range will contain all positive numbers in $\mathbb{R}$. The range is therefore

$\{y \in \mathbb{R} : y \geq 0\}$, $R(f) = [0, +\infty[$

iii) Each element of the range is obtained from only one element of the domain, so the function is one-to-one (1 to 1).

c) $f : x \to \dfrac{1}{x^2}$

i) The function is defined for all real x except for $x = 0$. We write the domain as

$\{x \in \mathbb{R} : x \neq 0\}$ or $Domf =]-\infty, 0[\cup]0, +\infty[$.

ii) For this domain, the range will contain neither zero nor any negative numbers because x^2 will be positive. The range is therefore $\{y \in \mathbb{R} : y > 0\}$ or $R(f) =]0, +\infty[$

iii) Here the element of the domain can be obtained by more than one element of the domain. For example, $f(2) = f(-2) = \frac{1}{4}$. So the function is **many-to-one.**

Application activity 3.3

Find the domain of definition for each of the following functions

1. $f(x) = \sqrt{4x-8}$
2. $g(x) = \sqrt{x^2+5x-6}$
3. $h(x) = \dfrac{x^3+2x^2-2}{\sqrt[3]{x+4}}$
4. $f(x) = \dfrac{x-2}{\sqrt[4]{x^2-25}}$
5. $f(x) = \sqrt{\dfrac{(x-1)^2}{x+4}}$

3.4. Parity of a function (odd or even)

Activity 3.4

For each of the following functions, find $f(-x)$ *and* $-f(x)$. *Compare* $f(-x)$ *and* $-f(x)$ *using* $=$ *or* $\neq$

1) $f(x) = x^2+3$ 2) $f(x) = \sqrt[3]{x^3+x}$ 3) $f(x) = \dfrac{x^2-3}{x^2+1}$

Even function

A function $f(x)$ is said to be **even** if the following conditions are satisfied

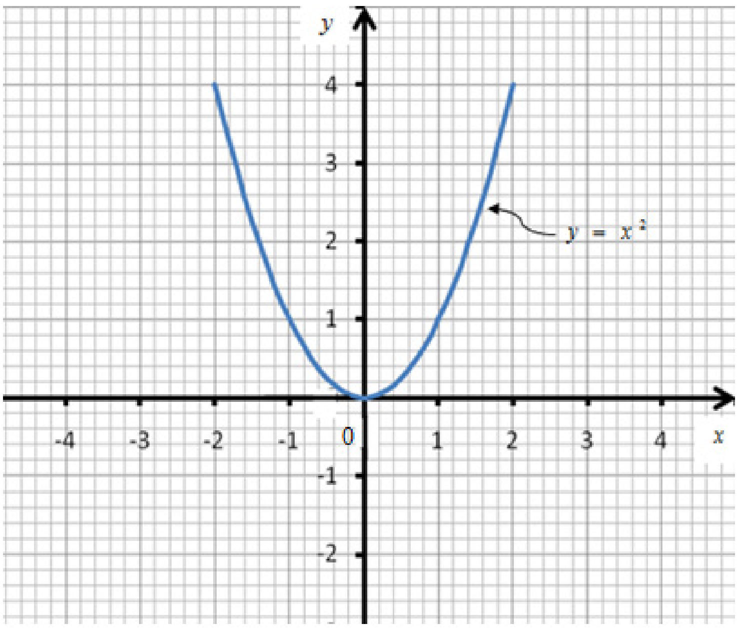
- $\forall x \in Domf, -x \in Domf$
- $f(-x) = f(x)$

The graph of such function is **symmetric about the vertical axis.** i.e $x = 0$

Example:

The function $f(x) = x^2$ is an even function since $\forall x \in Domf = \mathbb{R}, -x \in Domf = \mathbb{R}$ and $f(-x) = (-x)^2 = x^2 = f(x)$

Here is the graph of $f(x) = x^2$



Odd function

A function $f(x)$ is said to be **odd** if the following conditions are satisfied

- $\forall x \in Domf, -x \in Domf$
- $f(-x) = -f(x)$

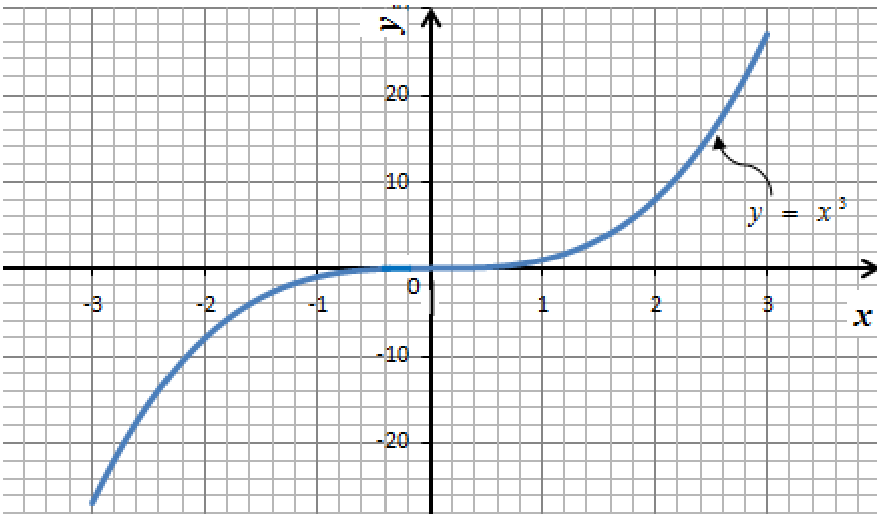
The graph of such a function looks the same when rotated through half a revolution about 0. The graph of such a function is symmetric vis-a-vis the central symmetry of centre (0,0). This is called **rotational symmetry**.

Example:

1. $f(x)=x^3$ is an odd function since $\forall x\in Domf=\mathbb{R}, -x\in Domf=\mathbb{R}$ and

$f(-x)=(-x)^3=-x^3=-f(x)$

Here is the graph of $f(x)=x^3$

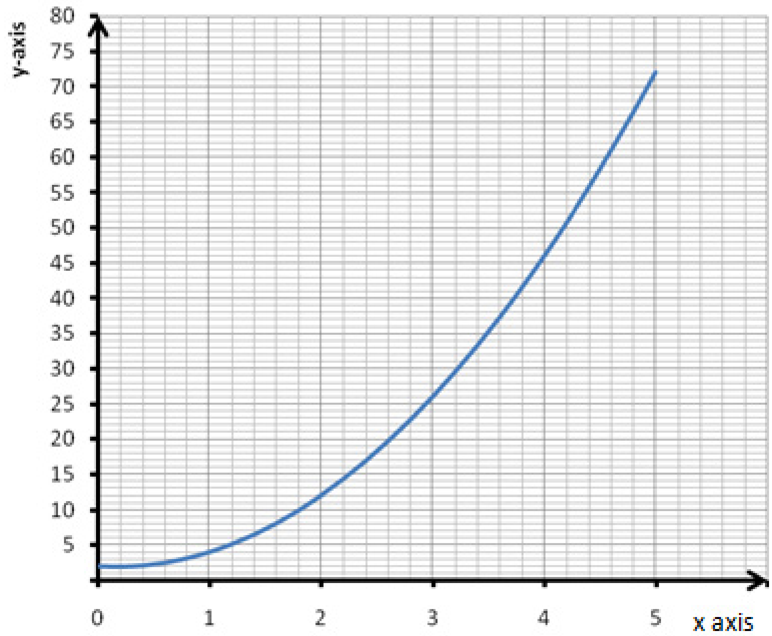


2. Consider the function $f(x)=3x^2-|x|+2$ for $x\geq 0$. Is this function odd, even or neither?

Here, the domain of the given function is restricted to $Domf=[0,+\infty[$ since $x\geq 0$.
$\forall x\in Domf, -x\notin Domf$

Thus, the given function is neither even nor odd.

Here is the graph



Application activity 3.4

Study the parity of the following functions

1. $f(x)=2x^2+2x-3$
2. $f(x)=\dfrac{3x^3+2x^2+8}{x-5}$
3. $g(x)=x^3-x$
4. $h(x)=\dfrac{x^2+4}{x^2-4}$
5. $g(x)=x(x^2+x)$

3.5 Operations on functions

3.5.1 Addition, subtraction, multiplication and division

Activity 3.5.1

Activity 3.5.1

Given the functions $f(x)=\dfrac{x+1}{2x-3}$ and $g(x)=x+1$, find 1) $f(x)+g(x)$

2) $f(x)-g(x)$ 3) $f(x)\cdot g(x)$ 4) $\dfrac{f(x)}{g(x)}$

Just as numbers can be added, subtract, multiplied and divided to produce other numbers, there is a useful way of adding, subtracting, multiplying and dividing functions to produce other functions. These operations are defined as follows: Given functions f and g, sum $f+g$, difference $f-g$, product $f\cdot g$ and quotient $\frac{f}{g}$, are defined by

- $(f+g)(x)=f(x)+g(x)$
- $(f-g)(x)=f(x)-g(x)$
- $(f\cdot g)(x)=f(x)\cdot g(x)$
- $\left(\frac{f}{g}\right)(x)=\frac{f(x)}{g(x)}$

For the functions, $f+g$, $f-g$ and $f\cdot g$, the domain is defined to be the intersection of the domains of f and g and for $\frac{f}{g}$, as we have seen it, the domain is this intersection with **the points where** $g(x)=0$ **excluded.**

Example:

1. Let f and g be the functions $f(x)=3x^4-5x^3+x-4$ and $g(x)=4x^3-3x^2+4x+3$. Find $(f+g)(x)$ and $(f-g)(x)$

Solution

$$\begin{array}{rrrrrr} & f(x)=3x^4 & -5x^3 & & +x & -4 \\ + & g(x)= & 4x^3 & -3x^2 & +4x & +3 \\ \hline (f+g)(x)= & 3x^4 & -x^3 & -3x^2 & +5x & -1 \end{array}$$

$$\begin{array}{rrrrrr} & f(x)=3x^4 & -5x^3 & & +x & -4 \\ - & g(x)= & 4x^3 & -3x^2 & +4x & +3 \\ \hline (f-g)(x)= & 3x^4 & -9x^3 & +3x^2 & -3x & -7 \end{array}$$

2. If $f(x)=\frac{9}{x+2}$ and $g(x)=x^3$. Find

a. $h(x)=f(x)+g(x)$

b. $t(x)=f(x)\cdot g(x)$

c. $k(x)=\dfrac{f(x)}{g(x)}$

Solution

$h(x)=f(x)+g(x)$ $=\dfrac{9}{x+2}+x^3$ $=\dfrac{9+x^3(x+2)}{x+2}$ $=\dfrac{x^4+2x^3+9}{x+2}$	$t(x)=f(x)\cdot g(x)$ $=\dfrac{9}{x+2}\cdot x^3$ $=\dfrac{9x^3}{x+2}$	$k(x)=\dfrac{f(x)}{g(x)}$ $=\dfrac{\frac{9}{x+2}}{x^3}$ $=\dfrac{9}{x+2}\left(\dfrac{1}{x^3}\right)$ $=\dfrac{9}{x^4+2x^3}$

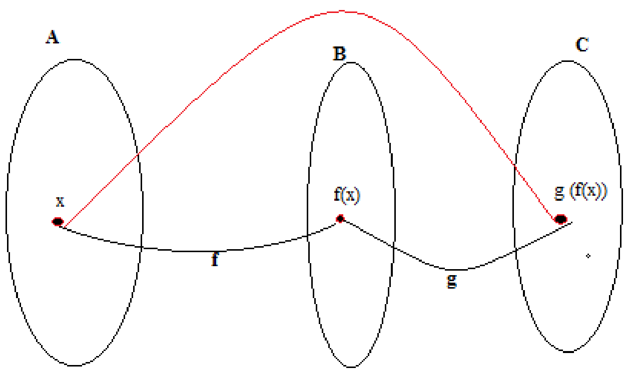
3.5.2. Composite functions

Activity 3.5.2

Consider two functions $f(x)=3x+2$ and $g(x)=x^2-1$. Find

1. $f[g(x)]$
2. $g[f(x)]$
3. Compare the two results

Consider the functions $f(x)=2x-1$ and $g(x)=x^2$.



$g_o f(x) = g(f(x))$

This combined or composite function is written $(g \circ f)(x)$ or $g[f(x)]$ or simply gf . The function f is performed first and so is written nearer to the variable x.

The set $\{1,3,5,7\}$ is the domain for the composite function and $\{1,25,81,169\}$ is the range.

Note that $(f \circ g)(x) \neq (g \circ f)(x)$

Example

1. If $f(x)=2x$ and $g(x)=3x+1$, express $g \circ f$ as a single function $h(x)$.

Solution

$f(x)=2x$ so $(g \circ f)(x)=g(2x)=3(2x)+1=6x+1$

$\therefore\ h(x)=6x+1$

2. Let $f(x)=x-1$ *and* $g(x)=\sqrt{x}$, *find* $(f \circ g)(x)$ *and* $(g \circ f)(x)$

- $g(x)=\sqrt{x}$, *so* $fg(x)=f(\sqrt{x})=\sqrt{x}-1$ $\qquad \therefore (f \circ g)(x)=\sqrt{x}-1$
- $f(x)=x-1$, *so* $gf(x)=g(x-1)=\sqrt{x-1}$ $\qquad \therefore (g \circ f)(x)=\sqrt{x-1}$

3.5.3 The inverse of a function

Activity 3.5.3

Find the value of x in function of y if

1) $y=x+1$ $\qquad$ 2) $y=3x-2$ $\qquad$ 3) $y=\dfrac{-x+3}{2x-1}$

Consider a function *f* which maps each element x of the domain x onto its image y in the range y that is $f: x \to y$ where $x \in X, y \in Y$. If this map can be reversed, i.e $f^{-1}: y \to x$ and resulting relationship is a function, it is called **the inverse of the original function**, and is denoted by f^{-1}

Only one-to-one functions can have an inverse function. To find the inverse of one-to-one functions, we change the subject of a formula.

Examples

1. Find the inverse function of $f(x)=2x+3$.

Solution

Let us make x the subject of $y=2x+3$ as follows:

$y=2x+3$ (Solve for the variable x);

Then, $y-3=2x$, thus, $x=\dfrac{y-3}{2}$

$\therefore f^{-1}(x)=\dfrac{x-3}{2}$

2. Find the inverse of the function $f(x)=3x-1$

Solution

If $f(x)=3x-1$, we require $f^{-1}(y)=x$. If $y=3x-1$ then $x=\dfrac{y+1}{3}$

So, given y, we can return to x using the expression $\dfrac{y+1}{3}$. Thus, $f^{-1}(x)=\dfrac{x+1}{3}$

Application activity 3.5

1) Given the functions $f(x)=2x^3+5x-1$ and $g(x)=3x-4$. Find $(f+g)(x)$

2) Given the functions $f(x)=3x^3-5x^2+7x-4$ and $g(x)=2x^2-x+3$. Find $(f\cdot g)(x)$

3) Find the inverse of the following functions

a) $f(x)=5x+2$ b) $g(x)=-7x-2$ c) $h(x)=\dfrac{-2x+1}{x-2}$

4) Find $(f \circ g)(x)$ and $(g \circ f)(x)$

a) If $f(x) = x^3 - 3x^2 + 1$ and $g(x) = 2$

b) If $f(x) = 2x^2 + x - 3$ and $g(x) = 6x$

c) If $f(x) = x^3 + 2x^2 + x - 4$ and $g(x) = 7x + 3$

3.6 Graphical representation and interpretation of functions in economics and finance

Activity 3.6

Considering that C is the dependent variable, measured in the vertical axis, and Y is the independent variable, measured on the horizontal axis, Draw the graph of the function

$C = 200 + 0.6Y$ Where C is consumer spending and Y is income. Note that the income cannot be negative.

Determine the point (Y, C) at which the line cuts the vertical axis.

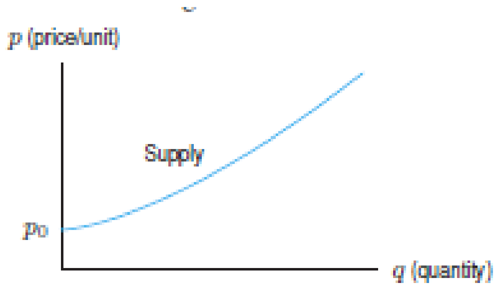
1. Price as function of quantity supplied

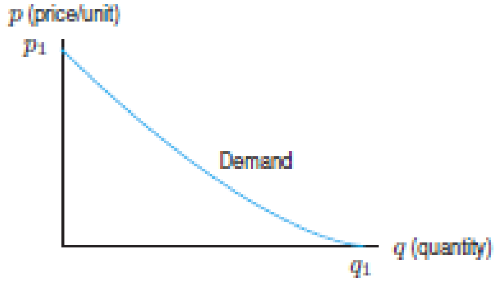
The quantity Q of an item that is manufactured and sold depends on its price P. As the price increases, manufacturers are usually willing to supply more of the product, whereas the quantity demanded by consumers falls.

The supply curve, for a given item, relates the quantity Q of the item that manufacturers are willing to make per unit time to the price P for which the item can be sold.

The demand curve relates the quantity, q, of an item demanded by consumers per unit time to the price P of the item.

Economists often think of the quantities supplied and demanded Q as functions of price P. However, for historical reasons, the economists put price (the independent variable) on the vertical axis and quantity (the dependent variable) on the horizontal axis. (The reason for this state of affairs is that economists originally took price to be the dependent variable and put it on the vertical axis.





Example : the demand function $Q = 800 - 4P$.

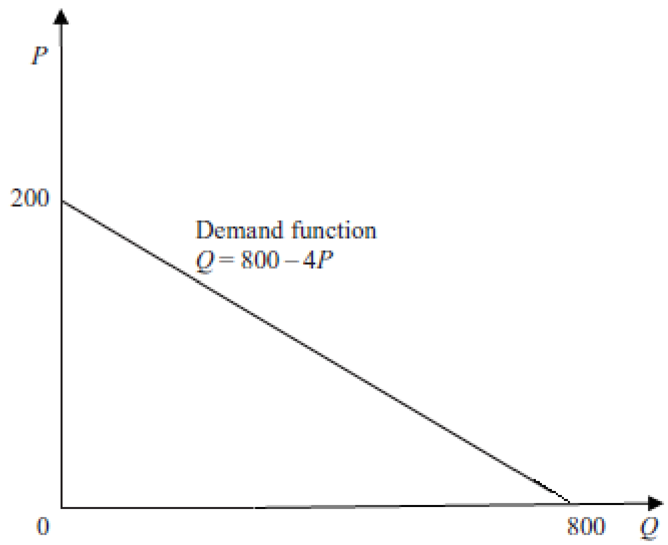
Theoretically, it does not matter which axis is used to measure which variable. However, one of the main reasons for using graphs is to make analysis clearer to understand. Therefore, if one always has to keep checking which axis measures which variable this defeats the objective of the exercise. Thus, even though it may upset some mathematical purists, the economists sometimes stick to the convention of measuring **quantity on the horizontal axis and price on the vertical axis**, even if price is the independent variable in a function.

This means that care has to be taken when performing certain operations on functions. If necessary, one can transform monotonic functions to obtain the inverse function (as already explained) if this helps the analysis.

Examples

a) The demand function $Q = 800 - 4P$ has the inverse function

$$P = \frac{800 - Q}{4} = 200 - 0.24Q$$



This figure shows that when the quantity Q is increasing, the price P reduces progressively. This can be caused by the fact that every consumer has sufficient quantity of goods and does not want to by any more.

b. Suppose that a firm faces a linear demand schedule and that 400 units of output Q are sold when price is \$40 and 500 units are sold when price is \$20. Once these two price and quantity combinations have been marked as points A and B, then the rest of the demand schedule can be drawn in. Use this data to determine the function that can help to predict quantities demanded at different prices and draw the corresponding graph.

Solution:

Accurate predictions of quantities demanded at different prices can be made if the information that is initially given is used to determine the algebraic format of the function.

A linear demand function must be in the form $P = a - bQ$, where a and b are parameters that we wish to determine the value of.

when P = 40 then Q = 400 and so 40 = a – 400b (1)

when P = 20 then Q = 500 and so 20 = a – 500b (2)

Equations (1) and (2) are what is known as simultaneous linear equations.

$$\begin{cases} 40 = a - 400b \\ 20 = 1 - 500b \end{cases}$$

We can solve this by one of the methods we used above. We find $a = 120, b = 0.02$.

Our function can now be written as P = 120 – 0.2Q

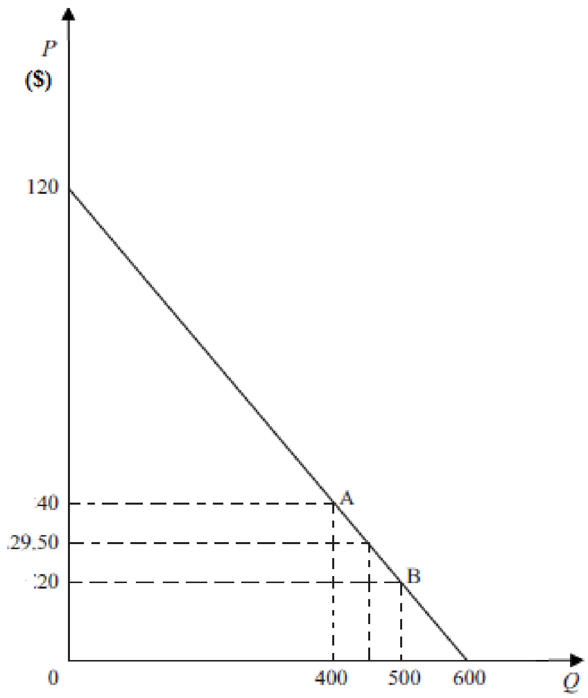
We can check that this is correct by substituting the original values of Q into the function.

If Q = 400 then P = 120 – 0.2(400) = 120 – 80 = 40

If Q = 500 then P = 120 – 0.2(500) = 120 – 100 = 20

These are the values of P originally specified and so we can are satisfied that the line that passes through points A and B is the linear function P = 120 – 0.2Q.

The inverse of this function will be Q = 600–5P. Precise values of Q can now be derived for given values of P. For example, when P = £29.50 then Q = 600 – 5(29.50) = 452.5.



2. Consumption as function of income

It is assumed that consumption C depends on income Y and that this relationship takes the form of the linear function $C = a + bY$.

Example:

When *the income* is $600, the consumption observed is $660. When *the income* is $1,000, the consumption observed is $900. Determine the equation "consumption function of income".

Solution:

To determine the required equation we can solve this system of equations

$$\begin{cases} 660 = a + 600b \\ 900 = a + 1000b \end{cases}$$

However, let us use another way as follows: We expect b to be positive, i.e. consumption increases with income, and so our function will slope upwards. As this is a linear function then equal changes in Y will cause the same changes in C.

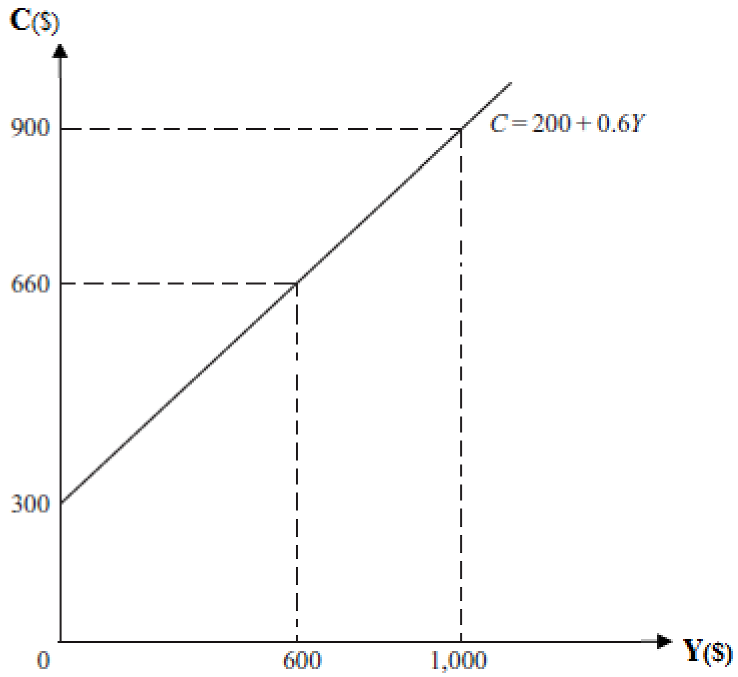
A decrease in Y of $400, from $1,000 to $600, causes C to fall by $240, from $900 to £660.

If Y is decreased by a further $600 (i.e. to zero) then the corresponding fall in C will be 1.5 times the fall caused by an income decrease of $400, since $600 = 1.5 × $400.

Therefore the fall in C is 1.5 × \$240 = \$360. This means that the value of C when Y is zero is £660 – £360 = £300. Thus a = 300.

A rise in Y of \$400 causes C to rise by \$240. Therefore a rise in Y of \$1 will cause C to rise by \$240/400 = \$0.6. Thus b = 0.6.

The function can therefore be specified as $C = 300 + 0.6Y$.



The graph shows that when the income increases, the consumption increases also.

3. Price as function of quantity demanded

The linear demand function is in the form $P = a - bQ$, where a and b are parameters, P is the price and Q is the quantity demanded.

Example:

Consider the function P = 60–0.2Q where P is price and Q is quantity demanded. Assume that P and Q cannot take negative values, determine the slop of this function and sketch its graph.

Solution:

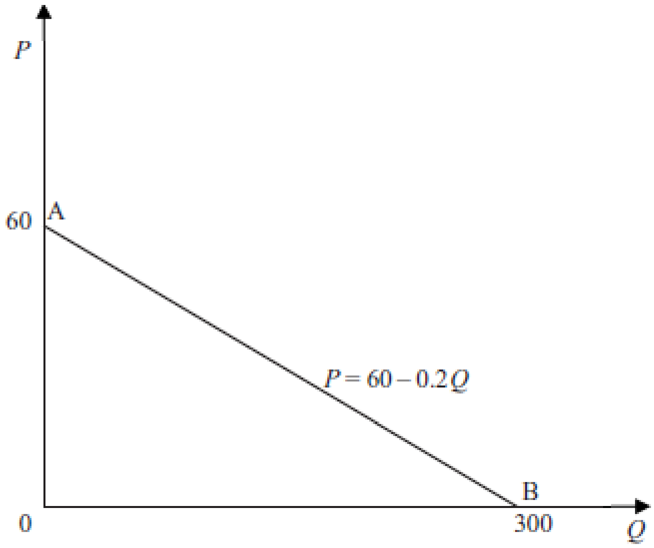
When Q = 0 then P = 60

When P = 0 then 0 = 60 – 0.2Q

0.2Q = 60

$$Q = \frac{60}{0.2} = 300$$

Using this points: $(0,60)$ (0, 60) and $(300,0)$, we can find the graph as follows:



The slope of a function which slopes down from left to right is found by applying the formula

$$slope = (-1)\frac{height}{base}$$

to the relevant right-angled triangle. Thus, using the triangle 0BA, the slope of our function

is $(-1)\dfrac{60}{300} = -0.2$

This, of course, is the same as the coefficient of Q in the function P = 60 – 0.2Q.

Remember that in economics the usual convention is to measure P on the vertical axis of a graph. If you are given a function in the format Q = f(P) then you would need to derive the inverse function to read off the slope.

Example:

What is the slope of the demand function Q = 830 – 2.5P when P is measured on the vertical axis of a graph?

Solution:

If $Q = 830 - 2.5P$; then $2.5P = 830 - Q$

$P = 332 - 0.4Q$

Therefore the slope is the coefficient of Q, which is –0.4.

4. Point elasticity of demand

Elasticity can be calculated at a specific point on a linear demand schedule. This is called '*point elasticity of demand*' and is defined as

$$e = (-1)\left(\frac{P}{Q}\right)\left(\frac{1}{slope}\right)$$

where P and Q are the price and quantity at the point in question. The slope refers to the slope of the demand schedule at this point although, of course, for a linear demand schedule the slope will be the same at all points.

Example:

Calculate the point elasticity of demand for the demand schedule P = 60 – 0.2Q where price is

(i) zero, (ii) \$20, (iii) \$40, (iv) \$60.

Solution

This is the demand schedule referred to earlier and illustrated above. Its slope must be –0.2 at all points as it is a linear function and this is the coefficient of Q.

To find the values of Q corresponding to the given prices we need to derive the inverse function. Given that

P = 60 – 0.2Q then 0.2Q = 60 – P

Q = 300 – 5P

(i) When P is zero, at point B, then Q = 300 – 5(0) = 300. The point elasticity will therefore be

$$e = (-1)\frac{P}{Q}\left(\frac{1}{slope}\right) = -1\frac{0}{30}\left(\frac{1}{-0.2}\right) = 0$$

ii) When P = 20 then Q = 300 – 5(20) = 200.

$$e = (-1)\frac{P}{Q}\left(\frac{1}{slope}\right) = -1\frac{20}{200}\left(\frac{1}{-0.2}\right) = 0.5$$

iii) When $P = 40$ then $Q = 300 - 5(40) = 100$

$$e = (-1)\frac{P}{Q}\left(\frac{1}{slope}\right) = -1\frac{40}{100}\left(\frac{1}{-0.2}\right) = 2$$

iv) When P = 60 then Q = 300 – 5(60) = 0.

If Q = 0, then $\frac{P}{Q} \to \infty$.

Therefore, $e = (-1)\frac{P}{Q}\left(\frac{1}{slope}\right) = -1\frac{60}{0}\left(\frac{1}{-0.2}\right) \to \infty$

5. The Cost Function

The **cost function**, C(q), gives the total cost of producing a quantity q of some good. Costs of production can be separated into two parts: the *fixed costs,* which are incurred even if nothing is produced, and the *variable costs*, which depend on how many units are produced.

Example:

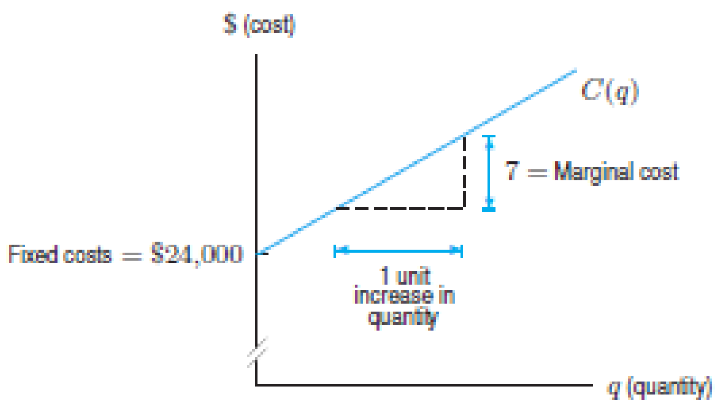
Let's consider a company that makes radios. The factory and machinery needed to begin production are fixed costs, which are incurred even if no radios are made. The costs of labor and raw materials are variable costs since these quantities depend on how many radios are made. The fixed costs for this company are \$24,000 and the variable costs are \$7 per radio.

Then, Total costs for the company = Fixed costs + Variable costs = 24,000 + 7.(Number of radios),

so, if q is the number of radios produced,

C(q) = 24,000 + 7q.

This is the equation of a line with slope 7 and vertical intercept 24,000.



If C(q) is a linear cost function,

- Fixed costs are represented by the vertical intercept.
- Marginal cost is represented by the slope.

6. The Revenue Function

The **revenue function,** R(q), gives the total revenue received by a firm from selling a quantity, q, of some good.

If the good sells for a price of p per unit, and the quantity sold is q, then Revenue = Price · Quantity,

so R = pq.

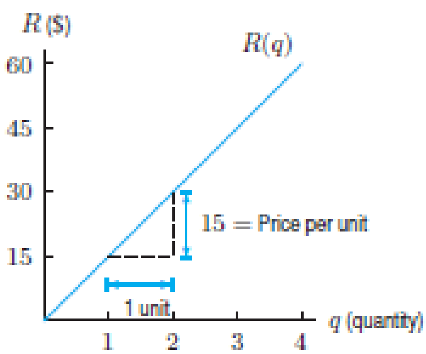
If the price does not depend on the quantity sold, so p is a constant, the graph of revenue as a function of q is a line through the origin, with slope equal to the price p.

Example:

1. If radios sell for $15 each, sketch the manufacturer's revenue function. Show the price of a radio on the graph.

Solution:

Since R(q) = pq = 15q, the revenue graph is a line through the origin with a slope of 15. See the figure. The price is the slope of the line.



2. Graph the cost function C(q) = 24,000 + 7q and the revenue function R(q) = 15q on the same axes. For what values of q does the company make money?

Solution:

The company makes money whenever revenues are greater than costs, so we find the values of q for which the graph of R(q) lies above the graph of C(q). See Figure 1.45.

We find the point at which the graphs of R(q) and C(q) cross:

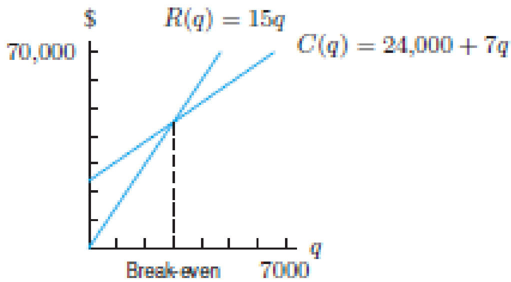
Revenue = Cost

$15q = 24{,}000 + 7q$

$8q = 24{,}000$

$q = 3000.$

The company makes a profit if it produces and sells more than 3000 radios. The company loses money if it produces and sells fewer than 3000 radios.



7. The Profit Function

Decisions are often made by considering the profit, usually written as π to distinguish it from the price, p.

We have: Profit = Revenue – Cost.

So, $\pi = R - C$

The *break-even point* for a company is the point where the profit is zero and revenue equals cost.

Example:

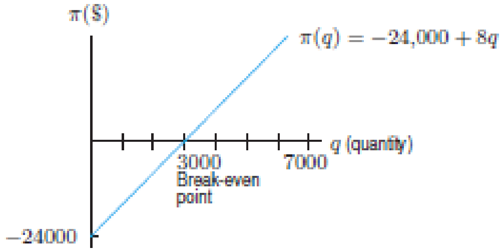
Find a formula for the profit function of the radio manufacturer. Graph it, marking the break-even point

Solution:

Since $R(q) = 15q$ and $C(q) = 24{,}000 + 7q$, we have

$$\pi(q) = R(q) - C(q)$$
$$= 15q - (24000 + 7q) = -2400 + 8q$$

Notice that the negative of the fixed costs is the vertical intercept and the break-even point is the horizontal intercept. See the figure;



8. The Marginal Cost, Marginal Revenue, and Marginal Profit

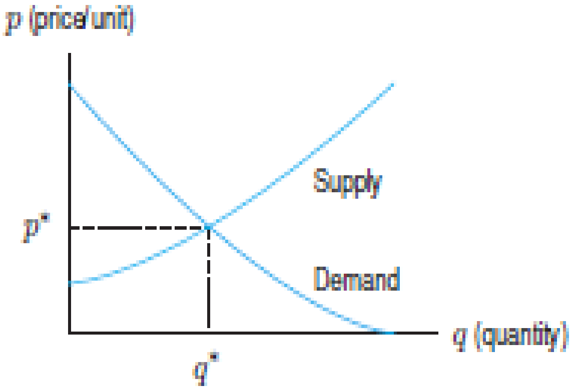
Just as we used the term marginal cost to mean the rate of change, or slope, of a linear cost function, we use the terms *marginal revenue and marginal profit* to mean the rate of change, or slope, of linear revenue and profit functions, respectively.

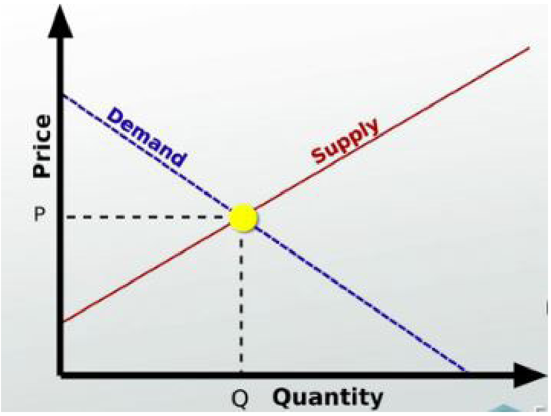
The term *marginal* is used because we are looking at how the cost, revenue, or profit change "at the margin," that is, by the addition of one more unit.

For **example,** for the radio manufacturer, the marginal cost is 7 dollars/item (the additional cost of producing one more item is $7), the marginal revenue is 15 dollars/item (the additional revenue from selling one more item is $15), and the marginal profit is 8 dollars/item (the additional profit from selling one more item is $8).

9. Equilibrium Price and Quantity

If we plot the supply and demand curves on the same axes, the graphs cross at the *equilibrium point*. The values p^* and q^* at this point are called the *equilibrium price* and *equilibrium quantity*, respectively. It is assumed that the market naturally settles to this equilibrium point.





Example:

Find the equilibrium price and quantity if Quantity supplied = 3p – 50 and

Quantity demanded = 100 – 2p.

Solution:

To find the equilibrium price and quantity, we find the point at which

Supply = Demand

3p – 50 = 100 – 2p

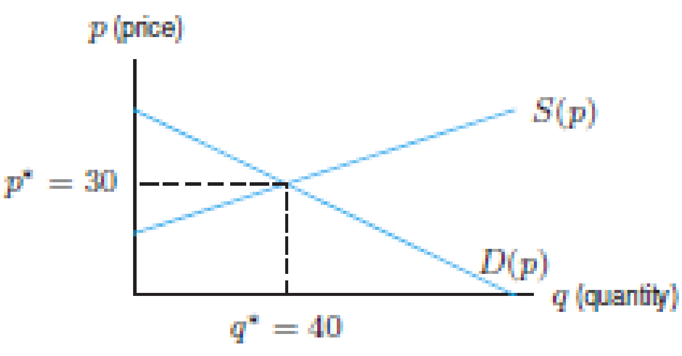
5p = 150

p = 30.

The equilibrium price is \$30. To find the equilibrium quantity, we use either the demand curve or the supply curve. At a price of \$30, the quantity produced is 100 – 2 (30) = 40 items.

The equilibrium quantity is 40 items.

In the figure, the demand and supply curves intersect at $p^* = 30$ and $q^* = 40$.



Application activity 3.6

1. Assume that consumption C depends on income Y according to the function

   $C = a + bY$, where a and b are parameters. If C is \$60 when Y is \$40 and C is \$90 when Y is \$80,

   a) What are the values of the parameters a and b?

   b) Sketch the graph of C(Y) and interpret it.

2. Suppose that $q = f(p)$ is the demand curve for a product, where p is the selling price in dollars and q is the quantity sold at that price.

   (a) What does the statement $f(12) = 60$ tell you about demand for this product?

   (b) Do you expect this function to be increasing or decreasing? Why?

3. A demand curve is given by $75p + 50q = 300$, where p is the price of the product, in dollars, and q is the quantity demanded at that price. Find p^* and q^* intercepts and interpret them in terms of consumer demand.

3.7. END UNIT ASSESSMENT

The total cost C for units produced by a company is given by $C(q) = 50000 + 7q$ where q is the number of units produced.

a) What does the number 50000 represent?

b) What does the number 8 represent?

d) Plot the graph of C and indicate the cost when $q = 5$.

e) Determine the real domain and the range of $C(q)$.

f) Is $C(q)$ an odd function?

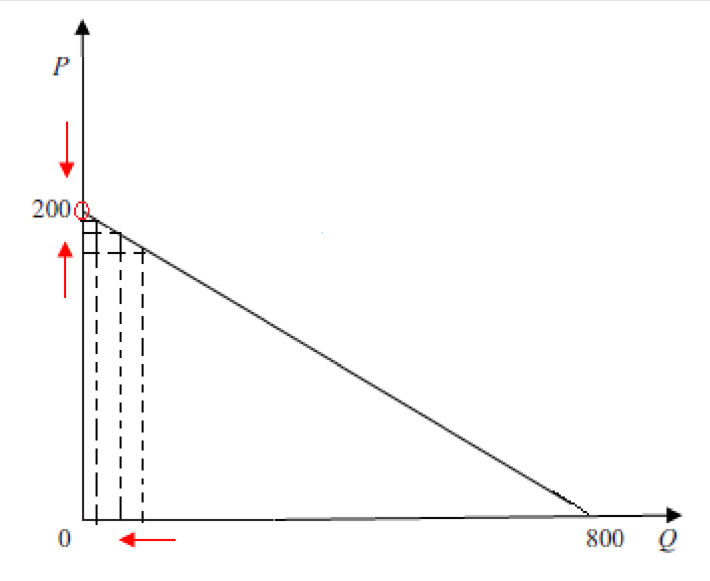
UNIT: 4

LIMITS OF FUNCTIONS

Key unit competence: Evaluate correctly limits of functions and apply them to solve related problems

4.0. Introductory Activity 4

Consider the price function of quantity Q such that $P = 200 - 0.24Q$



1) Complete the table and approximate the value of P when Q approaches to 0.

Q	1	0.5	0.1	0.01	0.001	0.0001	…	0
P								

When Q approaches 0, the price gets closer and closer to …..

2) Complete the table and approximate the value of P when Q approaches 20.

Q	19.5	19.9	19.999	20	20.1	20,2	20,5	21
P								

When Q approaches 20, the price gets "closer and closer" to ……, This can be written as $\lim\limits_{Q \to 20} P = \ldots$

4.1 Concepts of limits

4.1.1 Meaning for limit of a function

Activity 4.1.1

When finding the value of a function $f(x)$ when x approaches 2, a student used a calculator and dressed a table as follows:

x	$f(x)$	x	$f(x)$
2.5	3.4	1.5	5.0
2.1	3.857142857	1.9	4.157894737
2.01	3.985074627	1.99	4.015075377
2.001	3.998500750	1.999	4.001500750
2.0001	3.999850007	1.9999	4.000150008
2.00001	3.999985000	1.99999	4.000015000

a) Is it possible to put the values of x on a number line? Try to do it and locate the point $x = 2$

b) Write 2 possible open intervals of the number line such that their centre is

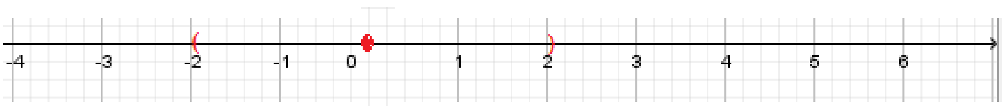
x = 2.

c) Try to approximate the value of $f(x)$ when x approaches 2.

Neighbourhood of a real number

Let A be a number (thought of as a point on the real line). A neighborhood of A is any open interval centered at A.

A neighborhood can be large or small; for example the intervals (-2;2), (-100; 100) and (-0.001;0.001) are both neighborhoods of 0, while the intervals (-2; 0) and (-1.01; -0.99) are neighborhoods of -1.



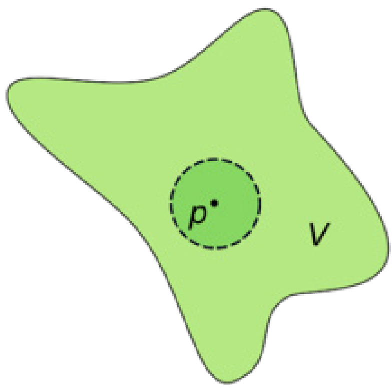
(-2;2) is neighbourhood of 0

Note that we made sure and picked values of x that were on both sides of $x = 2$ and that we moved in very close to $x = 2$ to make sure that any trends that we might be seeing are in fact correct.

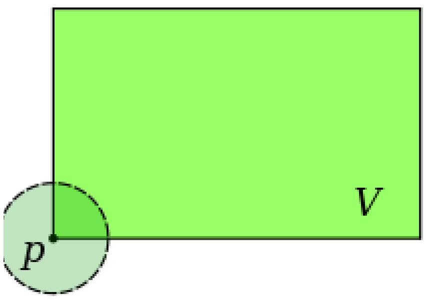
Mathematically, a set N is called a neighbourhood of point P if there exist an open interval I such that $x \in I \subset N$. The collection of all neighbourhoods of a point is called the **neighbourhood system** at the point.

Examples:

1. A set V in the plane is a neighbourhood of a point p if a small disk around p is contained in V as illustrated below.



2. A rectangle is not a neighbourhood of any of its corner.



3. The interval $(-1,1)=\{y:-1<y<1\}$ is a neighbourhood of $x=0$ in the real line, so the set $(-1,0)\cup(0,1)=(-1,1)\setminus\{0\}$ is a deleted neighbourhood of 0.

4.1.2 Limit of a function

Activity 4.1.2

Evaluate

1) $f(2)$ if $f(x)=\dfrac{x+1}{x+2}$

2) $f(1)$ if $f(x)=\dfrac{\sqrt{x+3}}{\sqrt[3]{3x-2}}$

3) $f(3)$ if $f(x)=4x^3-2x^2+3x-1$

Limits are used to describe how a function behaves as the independent variable moves towards a certain values.

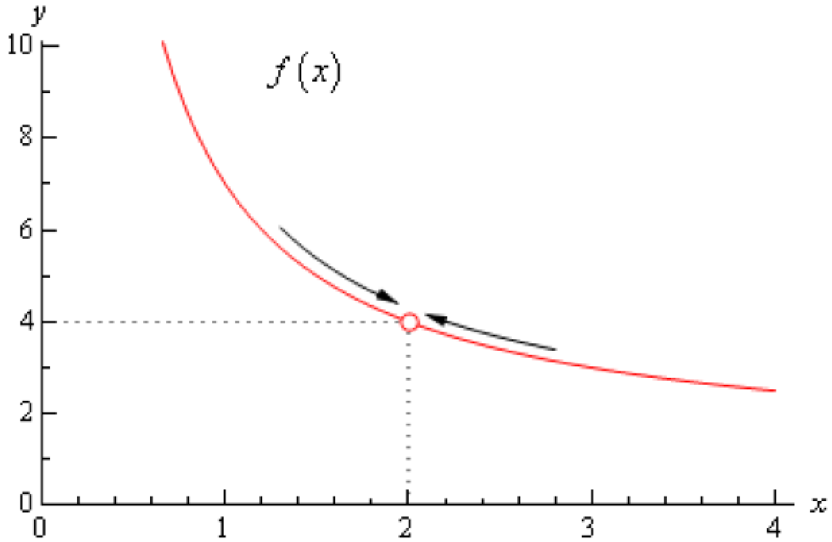
Frequently when studying function $y = f(x)$, we find ourselves interested in the function's behaviour near a particular point x_0, but not at x_0.

Example:

1. Let us estimate the $\frac{x^2+4x-12}{x^2-2x}$ as x approaches 2 in its neighbourhood.

As we were plugging in values of x into the function we are in effect moving along the graph in towards the point as $x = 2$.

This is shown in the graph by the two arrows on the graph that are moving in towards the point.



When we are computing limits the question that we are really asking is what y value is our graph approaching as we move in towards on our graph. We are **NOT** asking what y value the graph takes at the point in question. In other words, we are asking what the graph is doing around the point. In our case we can see that as x moves in towards 2 (from both sides) the function is approaching $y = 4$ even though the function itself doesn't even exist at $x = 2$.

Therefore we can say that the limit is in fact 4 and we can write $\lim_{x\to 2}\frac{x^2+4x-12}{x^2-2x}$.

So what have we learned about limits? Limits are asking what the function is doing around $x = a$ and are not concerned with what the function is actually doing at $x = a$. This is a good thing as many of the functions that we'll be looking at won't even exist at $x = a$ as we saw in our last example.

Application activity 4.1

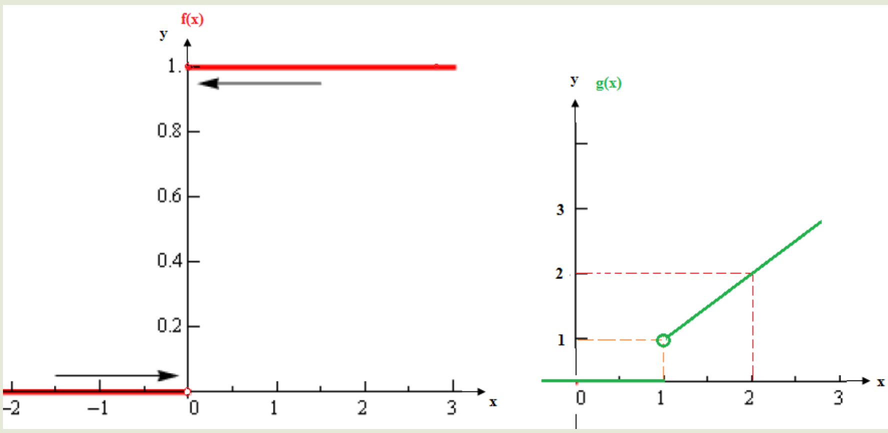
1. Apart from the Kingdom of Lesotho, give two examples of countries or Cities in the world that are surrounded by a single country or city.
2. Give three examples of intervals that are neighbourhoods of -5
3. Is a circle a neighbourhood of each of its points? explain your answer.
4. Draw any plane and show three points on that plane for which the plane is their neighbourhood.

4.2 One-sided limits, existence of limit and properties

4.2.1 One-sided limits, existence of limit

Activity 4.2

Consider the graphs of two functions f(x) and g(x) plotted below:



a) Complete the following:

$$f(x)=\begin{cases} \text{....for } x<0 \\ \text{.....for } x\geq 0 \end{cases} \text{ and } g(x)=\begin{cases} \text{....for } x\leq 1 \\ \text{.....for } x>1 \end{cases}$$

b) If we stay to the left side, as x approaches 0, $f(x)$ gets closer to .....;

c) If we stay to the right side, as x approaches 0, $f(x)$ gets closer to ......

d) If we stay to the left side, as x approaches 1, $g(x)$ gets closer to .....;

e) If we stay to the right side, as x approaches 1, $g(x)$ gets closer to .....;.

Right-handed limit

We say that $\lim_{x\to a^+} f(x)=L$ *provided we can make f(x) as close to L as we want for all x sufficiently close to a and x>a without actually letting x be a.*

Left-handed limit

We say $\lim_{x\to a^-} f(x)=L$ provided we can make f(x) as close to L as we want for all x sufficiently close to a and x<a without actually letting x be a.

For the right-handed limit we now have $x\to a^+$ (note the "+") which means that we will only look at x>a.

Likewise for the left-handed limit we have $x\to a^-$ (note the "-") which means that we will only be looking at x<a.

So when we are looking at limits it's now important to pay very close attention to see whether we are doing a normal limit or one of the one-sided limits.

Condition of existence for a limit

If the value of $f(x)$ approaches L_1 as x approaches x_0 from the right side we write

$\lim_{x\to x_0^+} f(x)=L_1$ and we read "**the limit of $f(x)$ as x approaches x_0 from the right equals L_1**.

If the value of $f(x)$ approaches L_2 as x approaches x_0 from the left side we write

$\lim_{x\to x_0^-} f(x)=L_2$ and we read "**the limit of $f(x)$ as x approaches x_0 from the left equals L_2**

If the limit from the left side is the same as the limit from the right side, say

$\lim_{x\to x_0^-} f(x)=\lim_{x\to x_0^+} f(x)=L$, then we write $\lim_{x\to x_0} f(x)$ and we read "the limit of $f(x)$ as x approaches x_0 equals L.

Note that $\lim_{x\to x_0} f(x)$ exists if and only if $\lim_{x\to x_0^-} f(x)=\lim_{x\to x_0^+} f(x)$

Always recall that the value of a limit does not actually depend upon the value of the function at the point in question. The value of a limit only depends on the values of the function around the point in question. Therefore, even though the function doesn't exist at this point the limit can still have a value.

Examples:

1. If $f(x)=\dfrac{|x-2|}{x^2+x-6}$. *Find* $\lim\limits_{x\to 2^-} f(x)$ *and* $\lim\limits_{x\to 2^+} f(x)$

Solution

Observe that $|x-2|=x-2 \text{ if } x>2$ and $|x-2|=-(x-2) \text{ if } x<2$

Therefore,

$$\lim_{x\to 2^+} f(x)=\lim_{x\to 2^+}\frac{x-2}{x^2+x-6}$$
$$=\lim_{x\to 2^+}\frac{x-2}{(x-2)(x+3)}$$
$$=\lim_{x\to 2^+}\frac{1}{x+3}$$
$$=\frac{1}{5}$$

$$\lim_{x\to 2^-} f(x)=\lim_{x\to 2^-}\frac{-(x-2)}{x^2+x-6}$$
$$=\lim_{x\to 2^-}\frac{-(x-2)}{(x-2)(x+3)}$$
$$=\lim_{x\to 2^-}\frac{-1}{x+3}$$
$$=\frac{-1}{5}$$

Since $\lim\limits_{x\to 2^-} f(x)\neq \lim\limits_{x\to 2^+} f(x)$, then $\lim\limits_{x\to 2} f(x)$ does not exist.

2. Find $\lim\limits_{x\to 3} f(x)$ *for* $f(x)=\begin{cases} x^2-5 & \text{if } x\le 3 \\ \sqrt{x+13} & \text{if } x>3 \end{cases}$

Solution

As x approaches 3 from the left, the formula for f is $f(x)=x^2-5$. So that

$$\lim_{x\to 3^-} f(x)=\lim_{x\to 3}\left(x^2-5\right)=4$$

As x approaches 3 from the right, the formula for f is $f(x)=\sqrt{x+13}$. So that

$$\lim_{x\to 3^+} f(x)=\lim_{x\to 3}\sqrt{x+13}$$
$$=\sqrt{\lim_{x\to 3}(x+13)}$$
$$=\sqrt{16}$$
$$=4$$

Since the one-sided limits are equals $\lim\limits_{x\to 3} f(x)=4$.

3. Let us explore numerically how the function $f(x)=\frac{x^2-9}{x-3}$ behaves near $x=3$.

Note that $f(x)=\frac{x^2-9}{x-3}$ is defined for all real numbers x except for $x=3$. For any $x=3$ we can simplify the expression for $f(x)$ by factoring the numerator and cancelling common factors:

$$f(x)=\frac{(x+3)(x-3)}{x-3}=x+3 \quad for \;\; x\neq 3$$

Even though $f(3)$ is not defined, it is clear that we can make the value of $f(x)$ as close as we want to 6 by choosing x close enough to 3. Therefore, we say that $f(x)$ approaches arbitrarily close to 6 as x approaches 3, or, more simply, $f(x)$ approaches the limit 6 as x approaches 3. We write this as

$$\lim_{x\to 3} f(x)=6 \;\text{ or }\; \lim_{x\to 3}\frac{x^2-9}{x-3}=6$$

Informally,

If $f(x)$ is defined for all x near a, except possibly at a itself, and if we can ensure the $f(x)$ is as close as we want to L by taking x close enough to a, but not equal to a, we say that the function f approaches the ***limit L*** as x approaches a, and we write $\lim_{x\to a} f(x)=L$.

To find limit of a function $f(x)$ as x approaches a, first we need to substitute that value a in the function and see what happen. The limit can exist or not.

Example:

1. $\lim_{x\to 2}(2x+1)=2(2)+1=5$

2. Since a constant function $f(x)=k$ has the same value k everywhere, it follows that at each point $\lim_{x\to a} k=k$. For example $\lim_{x\to 4} 5=5$

3. The limit $\lim_{x\to a} x=a$ is self-evident. For example, $\lim_{x\to -5} x=-5$, $\lim_{x\to 0} x=0$

4. $\lim_{x\to 2}\sqrt{x^2-2x+1}=\sqrt{4-4+1}=1$

5) $\lim_{x\to 3}\frac{\sqrt{2x+1}}{\sqrt[3]{3x-1}}=\frac{\sqrt{7}}{\sqrt[3]{8}}=\frac{\sqrt{7}}{2}$

For a rational function $f(x)=\dfrac{g(x)}{h(x)}$

If x approaches $a\in\mathbb{R}$, we have three cases:

a) $h(x)\neq 0$ for $x=a\Rightarrow \lim\limits_{x\to a} f(x)=f(a)$

b) $g(x)\neq 0$, $h(x)=0$ for $x=a\Rightarrow \lim\limits_{x\to a} f(x)=\infty$

c) $g(x)=0$, $h(x)=0$ for $x=a\Rightarrow \lim\limits_{x\to a} f(x)=\dfrac{0}{0}$ (Indeterminate form)

Later, we will see how to remove the indeterminate forms

Example:

1) $\lim\limits_{x\to 2}\dfrac{x+4}{2+x}=\dfrac{2+4}{2+2}=\dfrac{6}{4}=\dfrac{3}{2}$

2) $\lim\limits_{x\to 0}\dfrac{x^2-2x-3}{x+6}=\dfrac{0-0-3}{0+6}=-\dfrac{1}{2}$

4.2.2 Properties of limits

Let lim stands for one of the limits $\lim\limits_{x\to a}$, $\lim\limits_{x\to a^-}$, $\lim\limits_{x\to a^+}$, $\lim\limits_{x\to -\infty}$ or $\lim\limits_{x\to +\infty}$. If $\lim f(x)$ and $\lim g(x)$ both exist, say $\lim f(x)=L_1$ and $\lim g(x)=L_2$, then

a) A constant factor can be moved through a limit sign. That is, if k is a constant, then $\lim\left[kf(x)\right]=k\lim f(x)$

b) $\lim\left[f(x)+g(x)\right]=\lim f(x)+\lim g(x)=L_1+L_2$

c) $\lim\left[f(x)-g(x)\right]=\lim f(x)-\lim g(x)=L_1-L_2$

d) $\lim\left[f(x)\bullet g(x)\right]=\lim f(x)\bullet\lim g(x)=L_1\bullet L_2$

e) $\lim\left[\dfrac{f(x)}{g(x)}\right]=\dfrac{\lim f(x)}{\lim g(x)}=\dfrac{L_1}{L_2}$ if $L_2\neq 0$

f) If n and m are positive integers, then $\lim\left[f(x)\right]^{\frac{m}{n}}=L_1^{\frac{m}{n}}$ provided that $L_1\geq 0$ if n is even.

Examples

1. $\lim_{x\to 3} x^4 = \left[\lim_{x\to 3} x\right]^4 = 3^4 = 81$

2. Find $\lim_{x\to 5}\left(x^4 - 4x + 3\right)$

Solution

$$\begin{aligned}\lim_{x\to 5}\left(x^2 - 4x + 3\right) &= \lim_{x\to 5} x^2 - \lim_{x\to 5} 4x + \lim_{x\to 5} 3\\ &= \lim_{x\to 5} x^2 - 4\lim_{x\to 5} x + \lim_{x\to 5} 3\\ &= 5^2 - 4\lim_{x\to 5} x + \lim_{x\to 5} 3\\ &= 25 - 4(5) + 3\\ &= 8\end{aligned}$$

2. Find $\lim_{x\to 2}\frac{f(x)}{g(x)}$ *if* $f(x) = 5x^3 + 4$ *and* $g(x) = x - 3$

Solution

$$\lim_{x\to 2}\frac{f(x)}{g(x)} = \lim_{x\to 2}\frac{5x^3 + 4}{x - 3}$$

$$\begin{aligned}\lim_{x\to 2}\frac{5x^3 + 4}{x - 3} &= \frac{\lim_{x\to 2}\left(5x^3 + 4\right)}{\lim_{x\to 2}(x - 3)}\\ &= \frac{5(2)^3 + 4}{2 - 3}\\ &= -44\end{aligned}$$

2. Find $\lim_{x\to 0} f(x)g(x)$ *if* $f(x) = 6x^2 + 2$ *and* $g(x) = x + 2$

Solution

$$\lim_{x\to 0} f(x)g(x) = \lim_{x\to 0}\left(6x^2 + 2\right)\lim_{x\to 0}(x + 2) = 2\times 2 = 4$$

Application activities 4.2

1. Given that $-x^2 \le g(x) \le x^2$. Find $\lim_{x \to 0} g(x)$

2. If $\lim_{x \to 3} f(x) = 6$ and $\lim_{x \to 3} g(x) = -3$. Find

a. $\lim_{x \to 3}\left[f(x) + g(x)\right]$

b. $\lim_{x \to 3}\left[f(x) g(x)\right]^3$

3. Explain why the following calculation is incorrect

a) $\lim_{x \to 0^+}\left(\frac{1}{x} - \frac{1}{x^2}\right) = \lim_{x \to 0^+}\frac{1}{x} - \lim_{x \to 0^+}\frac{1}{x^2} = \infty - \infty = 0$

b. Show that $\lim_{x \to 0^+}\left(\frac{1}{x} - \frac{1}{x^2}\right) = -\infty$

4) Evaluate the following limits:

a) $\lim_{x \to 3} f(x)$ *if* $f(x) = \begin{cases} x^2 - 2x + 1, & x \ne 3 \\ 7 & , x = 3 \end{cases}$

b) $\lim_{x \to 2} h(x)$ *if* $h(x) = \begin{cases} x^2 - x - 1, & x < 3 \\ 3x - 5 & , \quad x \ge 3 \end{cases}$

c) $\lim_{x \to 0} g(x)$ *if* $g(x) = \begin{cases} x, & x \le 0 \\ x^2, & x > 0 \end{cases}$

d) $\lim_{x \to 1} h(x)$ *if* $h(x) = \begin{cases} 1, & x > 1 \\ 3, & x \le 1 \end{cases}$

4.3 Limits of functions at infinity and involving infinity, indeterminate cases

4.3.1 Infinite limits

Activity 4.3.1

1. Consider the following function $f(x)=\dfrac{x+1}{x-1}$ Find:

a) $f(0.97)$ b) $f(0.98)$

c) $f(0.99)$ d) $f(1.01)$

e) $f(1.02)$ f) $f(1.03)$

2. Evaluate the following operations

a) $-2+\infty$ b) $2-\infty$

c) $-\infty+\infty$ d) $-\infty\times\infty$

e) $3(-\infty)$ f) $\dfrac{-\infty}{-2}$ g) $\dfrac{\infty}{-\infty}$

A function whose values grow arbitrarily large can sometimes be said to have an infinite limit. Since infinity is not a number, infinite limits are not really limits at all, but they provide a way of describing the behaviour of functions that grow arbitrarily large positive or negative.

Example:

1. Describe the behaviour of the function $f(x)=\dfrac{1}{x^2}$ *near* $x=0$.

Solution

As x approaches 0 from either side, the values of $f(x)$ are positive and grow larger and larger, so the limit of $f(x)$ as x approaches 0 does not exist. It is nevertheless convenient to describe the behaviour of f near 0 by saying that $f(x)$ approaches ∞ as x approaches zero. We write

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0}\frac{1}{x^2} = +\infty \quad \text{and} \quad \lim_{x\to 0^+} f(x) = \lim_{x\to 0}\frac{1}{x^2} = +\infty$$

2. *Describe the behaviour of the function* $f(x)=\dfrac{1}{x}$ *near* $x=0$.

Solution

Let x successively assumes values $x=1,\frac{1}{10},\frac{1}{100},\ldots$, then $\frac{1}{x}=1,10,100,\ldots$

successively. As x approaches 0 from the right the value of $\frac{1}{x}$ gets larger and larger

without bound, then $\lim\limits_{x\to 0^+}\frac{1}{x}=+\infty$

Let x successively assumes values $x=-1,-\frac{1}{10},-\frac{1}{100},\ldots$, then $\frac{1}{x}=-1,-10,-100,\ldots$

successively. As x approaches 0 from the left the value of $\frac{1}{x}$ decreases and becomes

more and more negative without bound, then $\lim\limits_{x\to 0^-}\frac{1}{x}=-\infty$

Another way to find this is to construct the **sign table**:

x	$-\infty$	0	$+\infty$
1	+ + +	+	+ +
x	- -	0	+ +
$\frac{1}{x}$	- -	$\Vert$ ∞	+ +

Considering the last row, we see that for $x=0$ the value of $\frac{1}{x}$ does not exist (∞).

At the left there is a negative sign, thus $\lim\limits_{x\to 0^-}\frac{1}{x}=-\infty$. At the right there is a positive

sign, thus $\lim\limits_{x\to 0^+}\frac{1}{x}=+\infty$.

It follows that $\lim_{x\to 0}\frac{1}{x}$ does not exist because the one sided limits as x approaches zero do not exist.

4. Describe the behaviour of the function $\lim_{x\to 4}\frac{2-x}{x^2-2x-8}$

Solution

As x approaches 4 from the right, the numerator is negative quantity approaching -2 and the denominator a positive quantity approaching 0. Consequently the ratio is a negative quantity that decreases without bound. That is

$$\lim_{x\to 4^+}\frac{2-x}{x^2-2x-8}=-\infty$$

As x approaches 4 from the left, the numerator is eventually a negative quantity approaching -2 and the denominator a negative quantity approaching 0. Consequently the ratio is a positive quantity that increases without bound. That is

$$\lim_{x\to 4^-}\frac{2-x}{x^2-2x-8}=+\infty$$

Another way to see this is to construct the **sign table**:

x	$-\infty$		-2		2		4		$+\infty$
$2-x$			+		0			-	
x^2-2x-8		+	0		-		0	+	
$\frac{2-x}{x^2-2x-8}$		+	$\|\|$ ∞	-	0	+	$\|\|$ ∞	-	

From the last row of the above table, we find that

$$\lim_{x\to 4^+}\frac{2-x}{x^2-2x-8}=-\infty \text{ and } \lim_{x\to 4^-}\frac{2-x}{x^2-2x-8}=+\infty$$

4.3.2 Limits at infinity

Let us start by looking about **operations with infinity**

a) Addition

When you add two non-zero numbers you get a new number. For example $3+8=11$. But with infinity this is not true. A really large positive number plus any positive number, regardless of size, is still a really large positive number.

With infinity you have the following:

$$\infty + c = \infty \text{ with } c \neq -\infty$$

$$\infty + \infty = \infty$$

b) Subtraction

A really large negative number minus any positive number, regardless of size, is still a really large negative number. In the case of subtraction we have

$$-\infty - c = -\infty \quad \text{with} \quad c \neq -\infty$$

$$-\infty - \infty = -\infty$$

c) Multiplication

A really large (positive or negative) number times any number, regardless of size, is still a really large number and we have to be careful with signs. In the case of multiplication we have

$$(a)(\infty) = \infty \text{ with } a > 0$$

$$(a)(\infty) = -\infty \text{ with } a < 0$$

$$(\infty)(\infty) = \infty$$

$$(-\infty)(-\infty) = \infty$$

$$(-\infty)(\infty) = -\infty$$

d) Division

A really large (positive or negative) number dived by any number that is not too large, is still a really large number and we have to be careful with signs.

$$\frac{\infty}{a} = \infty \text{ if } a > 0 \text{ and } a \neq \infty$$

$$\frac{\infty}{a} = -\infty \text{ if } a < 0 \text{ and } a \neq -\infty$$

$$\frac{-\infty}{a} = -\infty \text{ if } a > 0 \text{ and } a \neq \infty$$

$$\frac{-\infty}{a} = \infty \text{ if } a < 0 \text{ and } a \neq -\infty$$

$$\frac{a}{\infty} = 0$$

$$\frac{a}{-\infty} = 0$$

Beware!

The following form $\infty-\infty;\ 0\times\infty;\ \frac{\infty}{\infty}$ are some cases of the indeterminate form in limits calculation.

a) Limit of polynomials

Let $f(x)=a_nx^n+a_{n-1}x^{n-1}+...+a_0$ and $g(x)=b_mx^m+b_{m-1}x^{m-1}+...+b_0$

$$\lim_{x\to\infty} f(x)=\lim_{x\to\infty}\left(a_nx^n+a_{n-1}x^{n-1}+...+a_0\right)$$

$$=\lim_{x\to\infty} a_nx^n\left(1+\frac{a_{n-1}}{a_nx}+\frac{a_{n-2}}{a_nx^2}+...+\frac{a_0}{a_nx^n}\right)$$

$$=\lim_{x\to\infty} a_nx^n$$

$$\lim_{x\to\infty} g(x)=\lim_{x\to\infty}\left(b_mx^m+a_{m-1}x^{m-1}+...+a_0\right)$$

$$=\lim_{x\to\infty} b_mx^m\left(1+\frac{b_{m-1}}{b_mx}+\frac{b_{m-2}}{b_mx^2}+...+\frac{b_0}{b_mx^m}\right)$$

$$=\lim_{x\to\infty} b_mx^m$$

And then $\lim_{x\to\infty}\frac{f(x)}{g(x)}=\lim_{x\to\infty}\frac{a_nx^n}{b_mx^m}$.

We have three cases

a) If $m=n$, $\lim_{x\to\infty}\frac{f(x)}{g(x)}=\lim_{x\to\infty}\frac{a_nx^n}{b_mx^m}=\frac{a_n}{b_m}$

b) If $n>m$, $\lim_{x\to\infty}\frac{f(x)}{g(x)}=\lim_{x\to\infty}\frac{a_nx^{n-m}}{b_m}=\infty$

c) If $n<m$, $\lim_{x\to\infty}\frac{f(x)}{g(x)}=\lim_{x\to\infty}\frac{a_n}{b_mx^{m-n}}=0$

1) $\lim_{x\to+\infty}(-6)=-6$

2) $\lim_{x\to-\infty}\left(3x^2+5x-3\right)=\lim_{x\to-\infty}3x^2$

$=+\infty$

3) *Find the limit* $\lim_{x\to-\infty}\frac{1}{x}$ *and* $\lim_{x\to+\infty}\frac{1}{x}$

$\lim_{x\to-\infty}\frac{1}{x}=0$ and $\lim_{x\to+\infty}\frac{1}{x}=0$. We can write $\lim_{x\to\pm\infty}\frac{1}{x}=0$

4) $\lim_{x\to\infty}\frac{2x^4+3x^2+1}{3x^4+5x^2+3}=\lim_{x\to\infty}\frac{2x^4}{3x^4}$

$=\lim_{x\to\infty}\frac{2}{3}$

$=\frac{2}{3}$

5) $\lim_{x\to-\infty}\frac{4x^3+5x-3}{x^2+3x+1}=\lim_{x\to-\infty}\frac{4x^3}{x^2}$

$=\lim_{x\to-\infty}4x$

$=-\infty$

6) $\lim_{x\to\infty}\frac{5x+2}{3x^2+1}=\lim_{x\to\infty}\frac{5x}{3x^2}$

$=\lim_{x\to\infty}\frac{5}{3x}$

$=0$

4.3.3 Indeterminate cases

Activity 4.3.3

Find a common factor for numerator and denominator.

a) $f(x)=\frac{x^2-1}{x-1}$

b) $f(x)=\frac{x^3+x^2-5x-2}{x^2-4}$

An **indeterminate form** is a certain type of expression with a limit that is not evident by inspection. There are several types of indeterminate forms such as

$\frac{0}{0}, \frac{\infty}{\infty}, 0\cdot\infty, \infty-\infty, 0^0, 1^\infty$ in this section we will study the forms $\frac{0}{0}, \frac{\infty}{\infty}, 0\cdot\infty, \infty-\infty$.

The indeterminate forms may be produced in the following ways:

- Suppose that $\lim_{x\to a} f(x)=0$ and $\lim_{x\to a} g(x)=\infty$.

The limit of the product $f(x)g(x)$ has the indeterminate form, $0\times\infty$ at $x=a$.

To evaluate this limit we change the limit into one of the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$ in this way:

$$f(x)g(x)=\frac{f(x)}{\frac{1}{g(x)}}=\frac{g(x)}{\frac{1}{f(x)}}$$

- If $\lim_{x\to a} f(x)=\lim_{x\to a} g(x)=+\infty$, then $\lim_{x\to a}\left[f(x)-g(x)\right]$ has the indeterminate form $\infty-\infty$. To evaluate this limit, we perform the algebraic manipulations by converting the limit into a form of $\frac{0}{0}$ or $\frac{\infty}{\infty}$.

If $f(x)$ or $g(x)$ is expressed as a fraction, we find the common denominator.

Examples:

1) $\lim_{x\to 2}\frac{x^2-4}{x-2}=\frac{0}{0}$, this is the indeterminate form (I.F)

Solution

By factoring the numerator and cancelling, we move out this I.F

$$\begin{aligned}\lim_{x\to 2}\frac{x^2-4}{x-2}&=\lim_{x\to 2}\frac{(x+2)(x-2)}{x-2}\\&=\lim_{x\to 2}(x+2)\\&=4\end{aligned}$$

2) $\lim_{x\to 1}\frac{x^2-1}{x-1}=\frac{1-1}{1-1}=\frac{0}{0}I.F$

$$\lim_{x\to 1}\frac{x^2-1}{x-1}=\lim_{x\to 1}\frac{(x+1)(x-1)}{x-1}$$
$$=\lim_{x\to 1}(x+1)$$
$$=2$$

3) $\lim\limits_{x\to+\infty}\dfrac{x^2+4x+5}{4x^2+7x+9}=\dfrac{\infty}{\infty}I.F$

Factor out x^2, to move out this I.F; then we have

$$\lim_{x\to+\infty}\frac{x^2\left(1+\frac{4}{x}+\frac{5}{x^2}\right)}{x^2\left(4+\frac{7}{x}+\frac{9}{x^2}\right)}=\lim_{x\to+\infty}\frac{1+\frac{4}{x}+\frac{5}{x^2}}{4+\frac{7}{x}+\frac{9}{x^2}}$$
$$=\frac{1+0+0}{4+0+0}$$
$$=\frac{1}{4}$$

Or

Since we have a rational function and degree of numerator is equal to the degree of denominator, to find the limit a**s x tends to infinity** we need to divide the coefficients of the highest degree for numerator and denominator. Then the limit of this function becomes

$\frac{1}{4}$

4)

$$\lim_{x\to-\infty}(3x^2+5x-3)=3(-\infty)^2+5(-\infty)-3$$
$$=+\infty-\infty\ IF$$

Factor out x^2, to move this I.F

$$\lim_{x\to-\infty}(3x^2+5x-3)=\lim_{x\to-\infty}x^2\left(3+\frac{5}{x}-\frac{3}{x^2}\right)$$
$$=+\infty(3+0-0)$$
$$=+\infty$$

4.3.4 Indeterminate forms in irrational functions

Activity 4.3.4

What is the conjugate of the irrational expression in each of the following functions?

a) $f(x)=\sqrt{x^2-2}+3$ b) $f(x)=\dfrac{\sqrt{x-2}-1}{x-3}$

When we are computing the limits of irrational functions, in case of indeterminate form, we need to know the conjugate of the irrational expression in that function. We may need to find the domain of the given function.

Examples

1) Let us evaluate $\lim\limits_{x\to 4}\dfrac{\sqrt{x-3}-1}{x-4}$.

$$\lim_{x\to 4}\frac{\sqrt{x-3}-1}{x-4}=\frac{1-1}{4-4}=\frac{0}{0}\,I.F$$

To move out this I.F, we multiply the numerator and denominator by the conjugate of $\sqrt{x-3}-1$ which is $\sqrt{x-3}+1$, then

$$\lim_{x\to 4}\frac{\sqrt{x-3}-1}{x-4}=\lim_{x\to 4}\frac{\left(\sqrt{x-3}-1\right)\left(\sqrt{x-3}+1\right)}{(x-4)\left(\sqrt{x-3}+1\right)}$$

$$=\lim_{x\to 4}\frac{x-3-1}{(x-4)\left(\sqrt{x-3}+1\right)}$$

$$=\lim_{x\to 4}\frac{1}{\sqrt{x-3}+1}$$

$$=\frac{1}{2}$$

2) $\lim\limits_{x\to +\infty}\left(\sqrt{4x^2+2}-2x\right)=+\infty-\infty\ \ I.F$

To move out this I.F , we multiply and divide by the conjugate of $\sqrt{4x^2+2}-2x$.

$$\lim_{x\to+\infty}\left(\sqrt{4x^2+2}-2x\right)=\lim_{x\to+\infty}\frac{\left(\sqrt{4x^2+2}-2x\right)\left(\sqrt{4x^2+2}+2x\right)}{\sqrt{4x^2+2}+2x}$$
$$=\lim_{x\to+\infty}\frac{4x^2+2-4x^2}{\sqrt{4x^2+2}+2x}$$
$$=\lim_{x\to+\infty}\frac{2}{\sqrt{4x^2+2}+2x}$$
$$=\frac{2}{+\infty}$$
$$=0$$

Examples:

1) $\lim\limits_{x\to+\infty}\frac{\sqrt{4x^2-11x-3}}{x}=\frac{\sqrt{\infty-\infty}}{\infty}\ I.F$

To move out this I.F, we perform the algebraic manipulations such that the denominator will be cancelled.

$$\lim_{x\to+\infty}\frac{\sqrt{4x^2-11x-3}}{x}=\lim_{x\to+\infty}\frac{\sqrt{4x^2\left(1-\frac{11}{4x}-\frac{3}{4x^2}\right)}}{x}$$

$$=\lim_{x\to+\infty}\frac{\left(\sqrt{4x^2}\right)\sqrt{1-\frac{11}{4x}-\frac{3}{4x^2}}}{x} \qquad =\lim_{x\to+\infty}\frac{\sqrt{4x^2}}{x}\lim_{x\to+\infty}\sqrt{1-\frac{11}{4x}-\frac{3}{4x^2}}$$

$$=\left(\lim_{x\to+\infty}\frac{\sqrt{4x^2}}{x}\right)\times 1$$
$$=\lim_{x\to+\infty}\frac{\sqrt{4x^2}}{x}$$
$$=\lim_{x\to+\infty}\frac{2\sqrt{x^2}}{x}$$

Recall that $\sqrt{x^2}=\begin{cases}x & if\ \ x\geq 0\\ -x & if\ \ x<0\end{cases}$

We find the domain of the given function: $Domf=\left]-\infty,-\frac{1}{4}\right]\cup[3,+\infty[$.

As x tends to $+\infty$, $x \in [3, +\infty[$ and then $\sqrt{x^2} = x$.

Thus,

$$\lim_{x\to+\infty} \frac{\sqrt{4x^2 - 11x - 3}}{x} = \lim_{x\to+\infty} \frac{2\sqrt{x^2}}{x}$$
$$= \lim_{x\to+\infty} \frac{2x}{x}$$
$$= 2$$

2) $\lim_{x\to1} \frac{\sqrt[3]{x} - 1}{x - 1} = \frac{0}{0}$ *IF*

To move out this I.F we use the conjugate of the numerator

$$\lim_{x\to1} \frac{\sqrt[3]{x} - 1}{x - 1} = \lim_{x\to1} \frac{\left(\sqrt[3]{x} - 1\right)\left(\sqrt[3]{x^2} + \sqrt[3]{x} + 1\right)}{(x-1)\left(\sqrt[3]{x^2} + \sqrt[3]{x} + 1\right)} \qquad = \lim_{x\to1} \frac{(x-1)}{(x-1)\left(\sqrt[3]{x^2} + \sqrt[3]{x} + 1\right)}$$

$$= \lim_{x\to1} \frac{1}{\left(\sqrt[3]{x^2} + \sqrt[3]{x} + 1\right)}$$

$$= \frac{1}{3}$$

3) $\lim_{x\to2} \frac{\sqrt{x-2}}{\sqrt[3]{x-2}} = \frac{0}{0}$ *IF*

$$\lim_{x\to2} \frac{\sqrt{x-2}}{\sqrt[3]{x-2}} = \lim_{x\to2} \frac{(x-2)^{\frac{1}{2}}}{(x-2)^{\frac{1}{3}}} \qquad = \lim_{x\to2} (x-2)^{\frac{1}{2}} (x-2)^{-\frac{1}{3}}$$

$$= \lim_{x\to2} (x-2)^{\frac{1}{2}-\frac{1}{3}} \qquad = \lim_{x\to2} (x-2)^{\frac{1}{6}}$$

$$= \lim_{x\to2} \sqrt[6]{x-2}$$
$$= 0$$

Note that the limits involving indeterminate form $\frac{0}{0}$ or $\frac{\infty}{\infty}$ can be evaluated by successive derivatives of numerator and denominator. This method is called L'Hôpital rule. It can be applied after studying the fifth unit.

Application activity 4.3

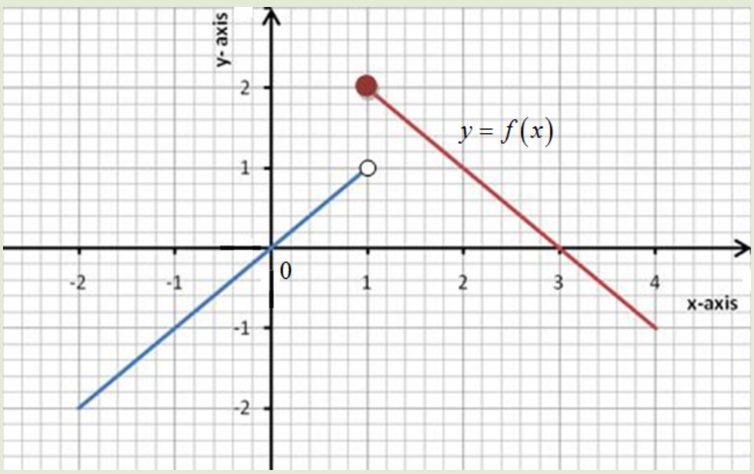
Evaluate the following limits

a) $\lim\limits_{x\to\infty}\dfrac{3x^3+2x^2-1}{6x^3+x+4}$	b) $\lim\limits_{x\to-\infty}\dfrac{(x+3)^2}{x^3+4x^2-8x-4}$	c) $\lim\limits_{x\to\infty}\dfrac{4x^3+x^2-1}{x^2-x+4}$
d) $\lim\limits_{x\to-4}\dfrac{x+1}{x+4}$	e) $\lim\limits_{x\to3}\dfrac{x^2+2x+1}{x-3}$	f) $\lim\limits_{x\to\infty}(x^2-2x+5)$
g) $\lim\limits_{x\to-\infty}(4x^3+3x^2-6)$		
h) $\lim\limits_{x\to4}\dfrac{x^4-16}{x^2-4}$	i) $\lim\limits_{x\to4}\dfrac{\sqrt{x^2-6}-10}{x-4}$	j) $\lim\limits_{x\to-\infty}\dfrac{3x}{\sqrt{9x^2-3x+6}}$

4.4 Graphical interpretation of limit of a function

Activity 4.4

Consider the following curve of function $f(x)$



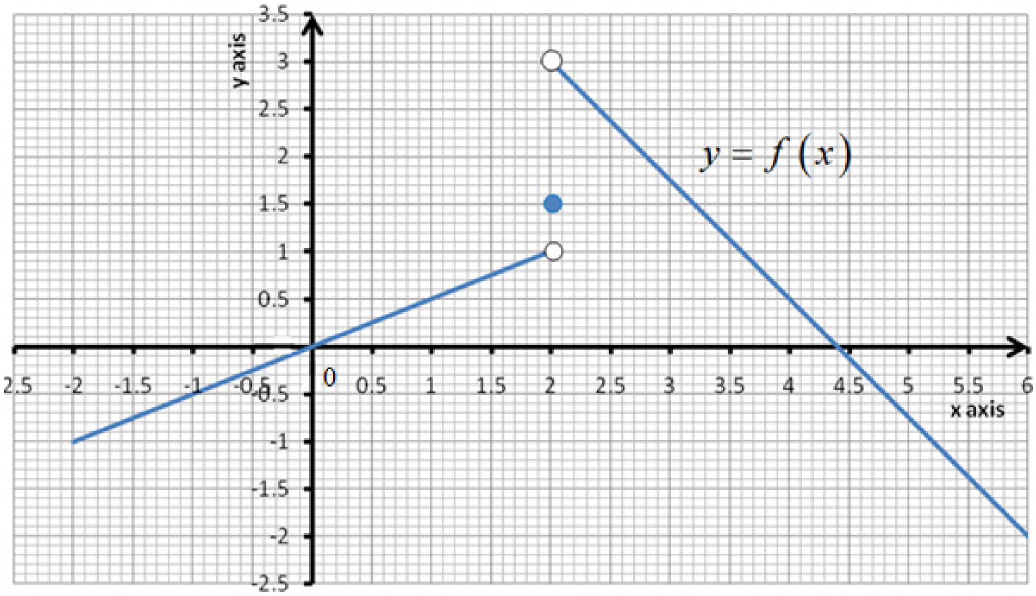
Use this graph to find

1. $\lim\limits_{x\to1^-} f(x)$
2. $\lim\limits_{x\to1^+} f(x)$
3. What can you say about $\lim\limits_{x\to1} f(x)$?

To find a limit graphically, we must understand each component of the limit to insure the graph is used properly to evaluate the limit.

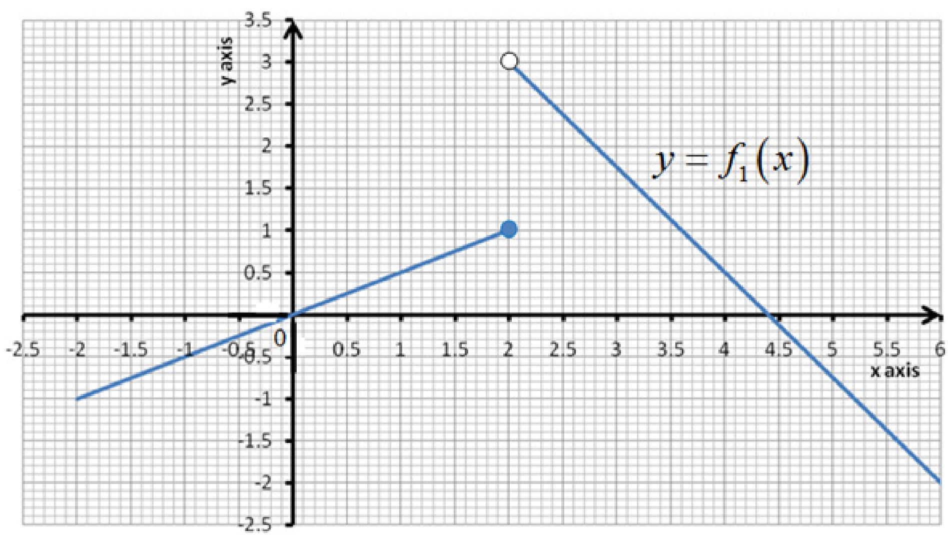
Example:

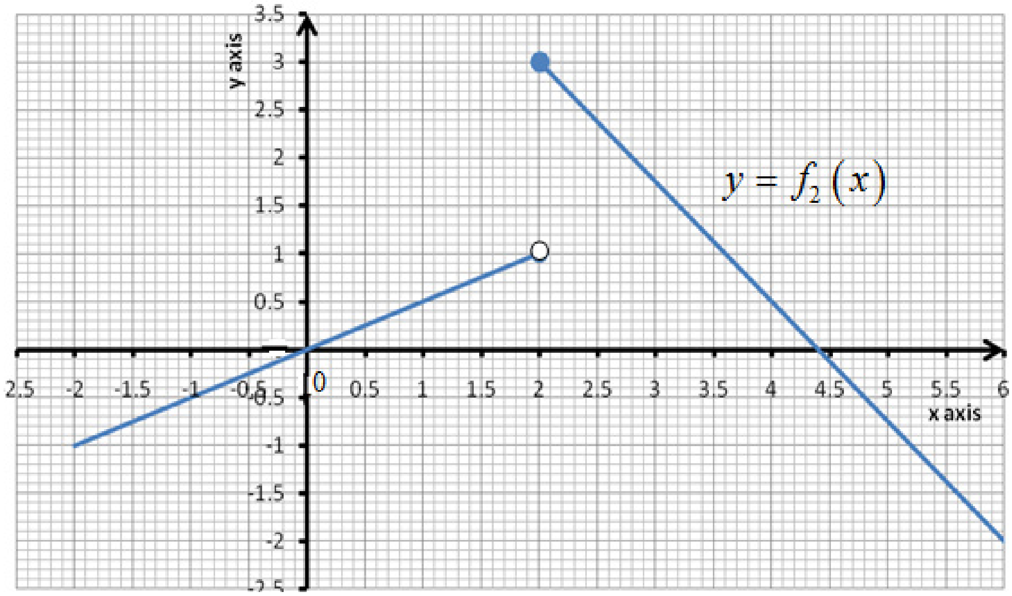
1. Let f be the function whose graph is shown below



As x approaches 2 from the left, $f(x)$ approaches 1, so $\lim\limits_{x \to 2^-} f(x) = 1$. As x approaches 2 from the left, $f(x)$ approaches 3, so $\lim\limits_{x \to 2^+} f(x) = 3$ but $f(2) = 1.5$

Therefore, the value of a function at a point, and the left and right hand limits at the point can all be different.



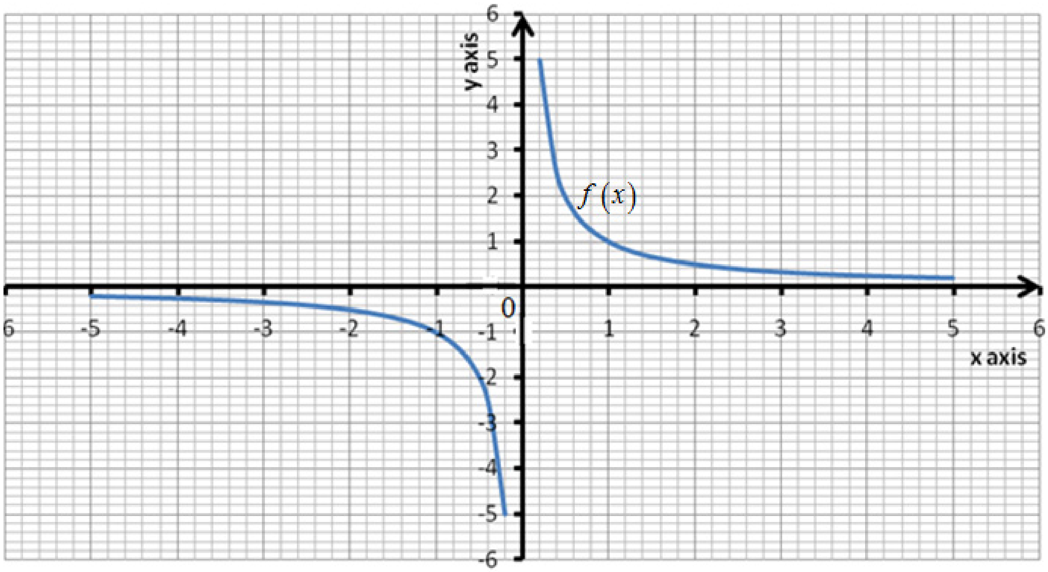


If we compare f_1, f_2 and f, we find that $f(2)=1.5$ while $f_1(2)=1$ but

$\lim\limits_{x\to 2^-} f(x) = \lim\limits_{x\to 2^-} f_1(x) = \lim\limits_{x\to 2^-} f_2(x) = 1$ and

$\lim\limits_{x\to 2^+} f(x) = \lim\limits_{x\to 2^+} f_1(x) = \lim\limits_{x\to 2^+} f_2(x) = 3$.

2. Let f be the function whose graph is



As x approaches 0 from the right side, $f(x)$ gets larger and larger without bound and consequently approaches no fixed value. In this case, we have $\lim_{x\to 0^+} f(x) = +\infty$ to indicate that the real limit fails to exist because $f(x)$ is increasing without bound.

As x approaches 0 from the left side, $f(x)$ becomes more and more negative without bound and consequently approaches no fixed real value. In this case, we have $\lim_{x\to 0^-} f(x) = -\infty$ to indicate that the real limit fails to exist because $f(x)$ is decreasing without bound. Therefore, as $\lim_{x\to 0^-} f(x) \neq \lim_{x\to 0^+} f(x)$,

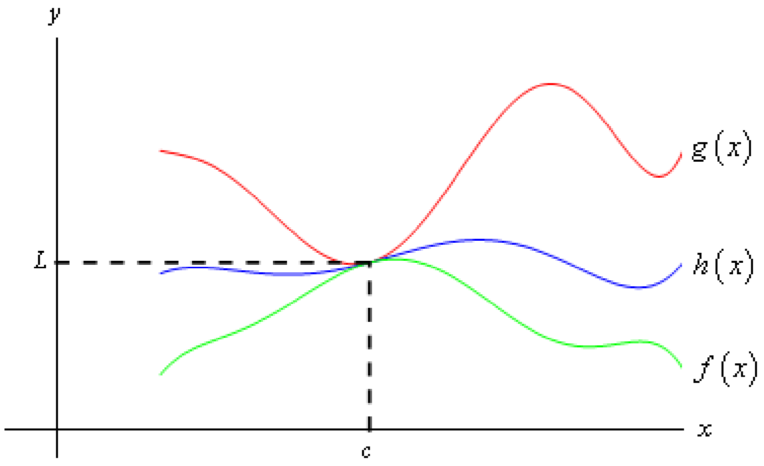
we conclude that $\lim_{x\to 0} f(x)$ does not exist.

As x gets larger and larger, $f(x)$ gets close to zero. Also as x becomes more and more negative, $f(x)$ is close to zero. Thus, $\lim_{x\to +\infty} f(x) = 0$ and $\lim_{x\to -\infty} f(x) = 0$

The squeeze theorem (or Sandwich theorem or Pinching theorem)

Suppose that $f(x) \le h(x) \le g(x)$. If $\lim_{x\to c} f(x) = \lim_{x\to c} g(x) = L$, then $\lim_{x\to c} h(x) = L$

The following figure illustrates what is happening in this theorem



From the figure we can see that if the limits of $f(x)$ and $g(x)$ are equal at $x = c$ then the function values must also be equal at $x = c$. However, because $h(x)$ is "squeezed" between $f(x)$ and $g(x)$ at this point then $h(x)$ must have the same value. Therefore, the limit of $h(x)$ at this point must also be the same. Similar statements hold for left and right limits.

Examples:

1) Given that $1-\frac{x^2}{4} \le u(x) \le 1+\frac{x^2}{2}$. Find $\lim_{x\to 0} u(x)$

Solution: Since $\lim_{x\to 0}\left(1-\frac{x^2}{4}\right)=1$ *and* $\lim_{x\to 0}\left(1+\frac{x^2}{2}\right)=1$, the Sandwich theorem implies that $\lim_{x\to 0} u(x)=1$

2) Show that if $\lim_{x\to a}|f(x)|=0$ then $\lim_{x\to a} f(x)=0$

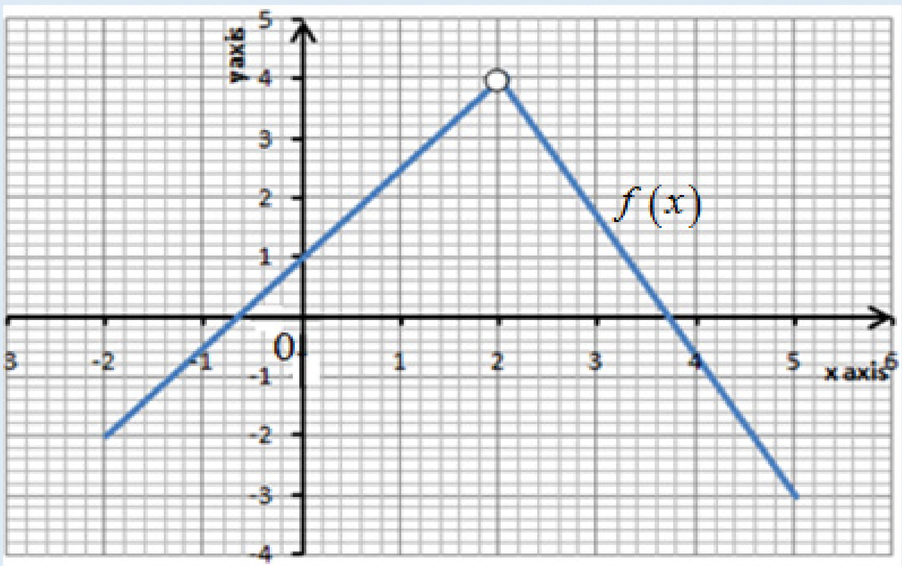
Solution

Since $-|f(x)| \le f(x) \le |f(x)|$, and $-|f(x)|$ and $|f(x)|$ both have limit 0 as x approaches a, so does $f(x)$ by the Sandwich theorem.

Application activity 4.4

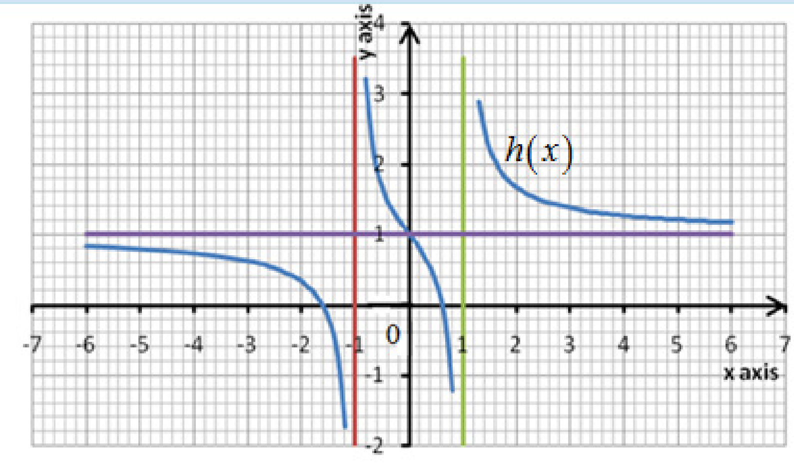
Question1:

Find $\lim_{x\to 2} f(x)$ using the following graph of $f(x)$



Question 2

1. Find $\lim_{x \to -1} h(x)$, $\lim_{x \to 1} h(x)$, $\lim_{x \to -\infty} h(x)$, $\lim_{x \to \infty} h(x)$ using the following graph of $h(x)$



2. In the same Cartesian plane sketch the curves of $f(x) = x^2 + 5$, $g(x) = -x^2 + 5$ and $h(x) = 5$. What can you say about the three curves?

3. Evaluate the following limits and compare 2 of each sub question.

a. $\lim_{x \to 0} [3(3x-1)]$, $3\left[\lim_{x \to 0}(3x-1)\right]$

b. $\lim_{x \to 0}(x^2)$, $\lim_{x \to 0}(3x-1)$, $\lim_{x \to 0}(x^2+3x-1)$

c. $\lim_{x \to 1}(x^2+3x-6)$, $\lim_{x \to 1}(x+4)$, $\lim_{x \to 1} \dfrac{x^2+3x-6}{x+4}$

d. $\lim_{x \to 2}(x-1)$, $\lim_{x \to 2}(x+4)$, $\lim_{x \to 2}(x^2+3x-4)$

e. $\lim_{x \to -4}\left[(x^2+1)^2\right]$, $\left[\lim_{x \to -4}(x^2+1)\right]^2$

4.5 Applications of limits in mathematics

Activity 4.5.1

1. Given the function $f(x)=\begin{cases}\dfrac{x^2-4}{x-2}, & x\neq 2\\ 4, & x=2\end{cases}$,

find: a) $f(2)$

b) $\lim\limits_{x\to 2} f(x)$

c) What can you say about $f(2)$ and $\lim\limits_{x\to 2} f(x)$?

4.5.1 Continuity of a function

a) Continuity of a function at a point or on interval *I*

A function $f(x)$ is said to be **continuous at point *c*** if the following conditions are satisfied:

a) $f(c)$ is defined

b) $\lim\limits_{x\to c} f(x)$ exists

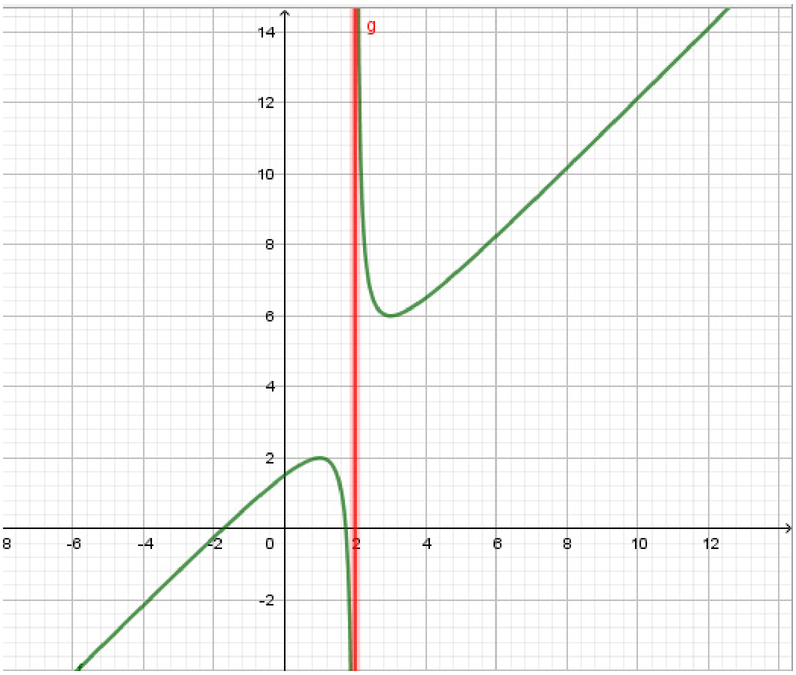
c) $\lim\limits_{x\to c} f(x)=f(c)$

If one or more conditions in this definition fails to hold, then f is said to be **discontinuous at point *c***, and *c* is called **a point of discontinuity** of *f*. If *f* is continuous at all point of an open interval (a,b), then *f* is said to be continuous on (a,b).

A function that is continuous on $(-\infty,+\infty)$ is said to be **continuous everywhere** or simply **continuous**.

Examples:

1) The function $f(x)=\dfrac{x^2-3}{x-2}$ is discontinuous at 2 because $f(2)$ is undefined, see the graph below.



2) The function $g(x)=\begin{cases}\dfrac{x^2-9}{x-3} & \text{if } x\neq 3\\ 6 & \text{if } x=3\end{cases}$ is continuous at 3 because $g(3)=6$ and

$$\lim_{x\to 3^+} g(x)=\lim_{x\to 3^-} g(x)=\lim_{x\to 3}\frac{x^2-9}{x-3}=\lim_{x\to 3}(x+3)=6 \text{ so that } \lim_{x\to 3} g(x)=g(3).$$

b) Continuity at the left and continuity at the right of a point

A function *f* is continuous at the left of point *c* if the following conditions are satisfied:

a) $f(c)$ is defined b) $\lim_{x\to c^-} f(x)$ exists c) $\lim_{x\to c^-} f(x)=f(c)$

A function *f* is continuous at the right at of point *c* if the following conditions are satisfied:

a) $f(c)$ is defined b) $\lim_{x\to c^+} f(x)$ exists c) $\lim_{x\to c^+} f(x)=f(c)$

*From **Example**,* $\lim_{x\to 2^+}[2]=2=[2]$. *Therefore* $[x]$ *is continuous at the right of point 2.*

c) Continuity on an interval

We say that f is **continuous on the interval *I*** if it is continuous at each point of *I*. in particular, we will say that f is a continuous function if f is continuous at every point of its domain.

Examples

1) The function $f(x)=\sqrt{x}$ *is a continuous function on its domain. Its domain is* $[0,+\infty)$. *It is continuous at the left endpoint 0 because it is right continuous there.*

Also, f is continuous at every number $c>0$ *since* $\lim\limits_{x\to c}\sqrt{x}=\sqrt{c}$.

2) The function $f(x)=\dfrac{x}{\sqrt{1-x^2}}$ *is a continuous function on its domain.*

Its domain is $]-1,1[$.

It is continuous at the left endpoint -1 because it is right continuous there. It is continuous at the right endpoint 1 because it is left continuous there. Also, f is

continuous at every number $c\in]-1,1[$ since $\lim\limits_{x\to c}\dfrac{x}{\sqrt{1-x^2}}=\dfrac{c}{\sqrt{1-c^2}}$.

Theorem 1

a) Polynomials are continuous functions

b) If the functions f and g are continuous at c, then

i) $f+g$ is continuous at c

ii) $f-g$ is continuous at c

iii) $f\cdot g$ is continuous at c

iv) $\dfrac{f}{g}$ is continuous at c if $g(c)\neq 0$, and is discontinuous at c if $g(c)=0$.

c) A rational function is continuous everywhere except at the point where the denominator is zero.

d) Piecewise functions (functions that are defined on a sequence of intervals) are continuous if every function is in its interval of definition, and if the functions match their side limits at the points of separation of their intervals.

Examples

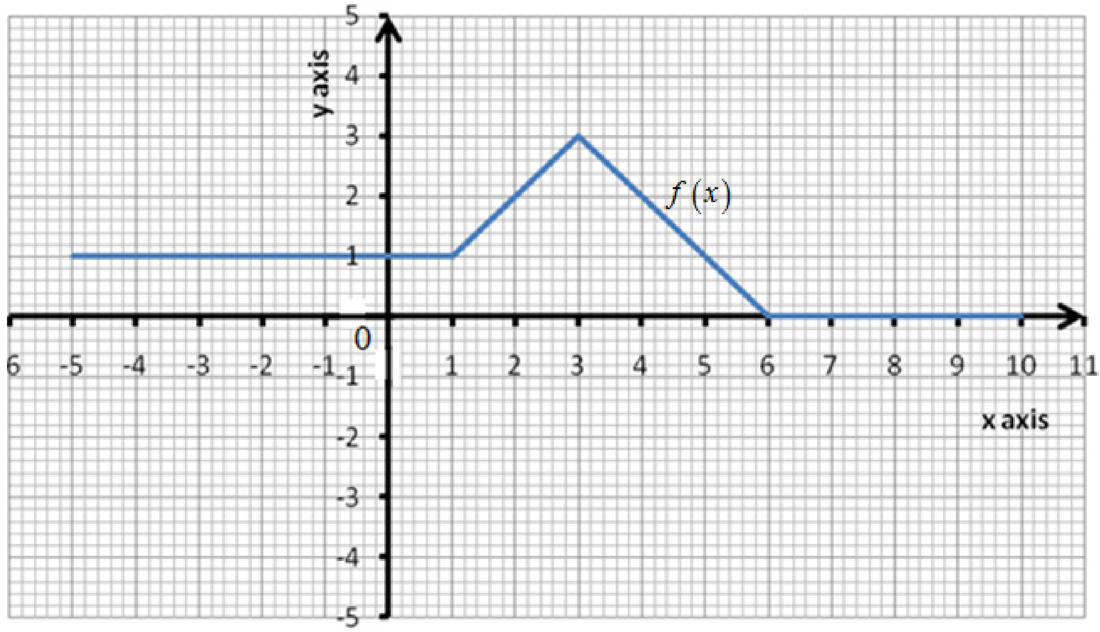
1) The function $h(x)=\dfrac{x^2-9}{x^2-5x+6}$ is continuous everywhere except at point 2 and 3 because the numerator and denominator are polynomials and denominator is zero at points 2 and 3.

2) The function

$$f(x)=\begin{cases} 1 & \text{if } x\le 1 \\ x & \text{if } 1<x\le 3 \\ -x+6 & \text{if } 3<x\le 6 \\ 0 & \text{if } x>6 \end{cases}$$

is continuous on $\mathbb{R}$, because its constituent functions are polynomials and the side limits at the points of division coincide.

Here is the graph of f(x)



Conclusion:

If $f(x)$ is continuous at $x=a$, then

$\lim\limits_{x\to a} f(x)=f(a)$; $\lim\limits_{x\to a^-} f(x)=f(a)$; $\lim\limits_{x\to a^+} f(x)=f(a)$

and if $f(x)$ is continuous at $x=b$, and $\lim\limits_{x\to a} g(x)=b$, then

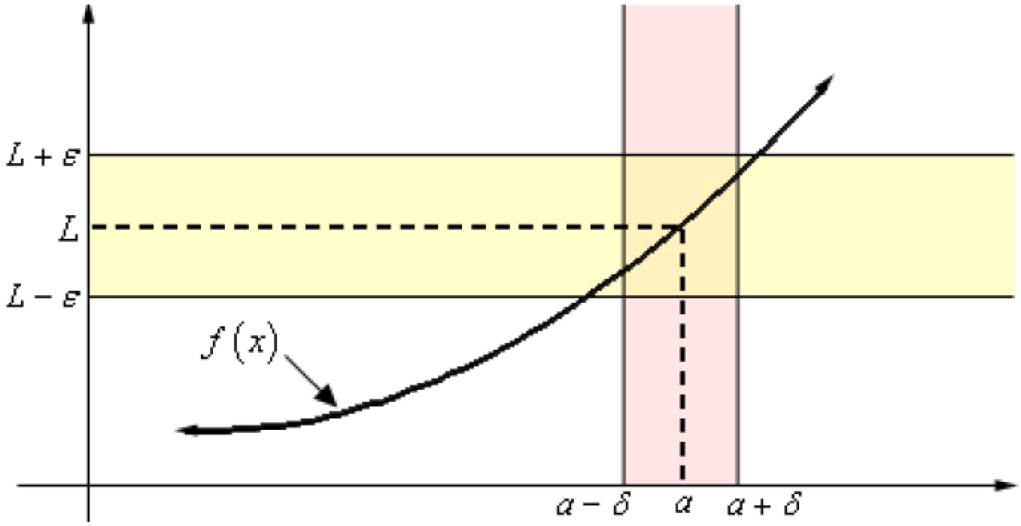
$$\lim_{x\to a} f\left(g\left(x\right)\right) = f\left(\lim_{x\to a} g\left(x\right)\right) = f\left(b\right)$$

Definition:

Let $f(x)$ be a function on an interval that contains $x = a$, except possibly at $x = a$. Then we say that:

$\lim_{x\to a} g\left(x\right) = L$ if for every number $\varepsilon > 0$ there is some number $\delta > 0$ such that

$\left|f\left(x\right) - L\right| < \varepsilon$ whenever $0 < \left|x - a\right| < \delta$.



What the definition is telling us is that for any number $\varepsilon > 0$ that we pick we can go to our graph and sketch two horizontal lines at $L + \varepsilon$ and $L - \varepsilon$ as shown on the graph above. Then somewhere out there in the world is another number $\delta > 0$, which we will need to determine, that will allow us to add in two vertical lines to our graph at $a + \delta$ and $a - \delta$.

Application activity 4.5.1

Determine where the function below is not continuous.

$$f\left(x\right) = \frac{4x + 10}{x^2 - 2x - 15}$$

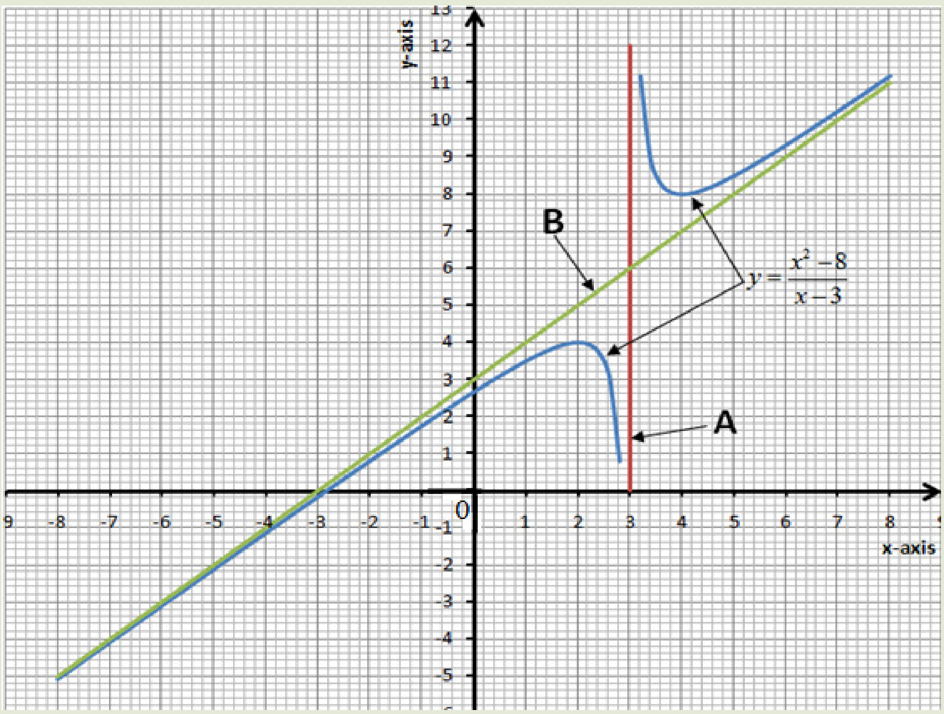
Given the function: $f\left(x\right) = \begin{cases} \dfrac{x^2 - 9}{x - 3}, & x \neq 3 \\ k, & x = 3 \end{cases}$

Determine the value of **k** for which the function is continuous at x = 3.

4.5.2 Asymptotes to curve of a function

Activity 4.5.2

Consider the following curve of function $y = f(x) = \dfrac{x^2 - 8}{x - 3}$



What can you say about the curve of y and the lines A and B?

Recall that if $P(x)$ and $Q(x)$ are polynomials,

then their ratio $f(x) = \dfrac{P(x)}{Q(x)}$ is called a rational function of x.

The discontinuity occurs at points where $Q(x) = 0$.

An asymptote on the curve is a straight line that is closely approached by that curve so that the perpendicular distance between them decreases to zero.

To find any asymptote of the function first we need to determine its domain of definition and evaluate the limits at the boundaries of the domain.

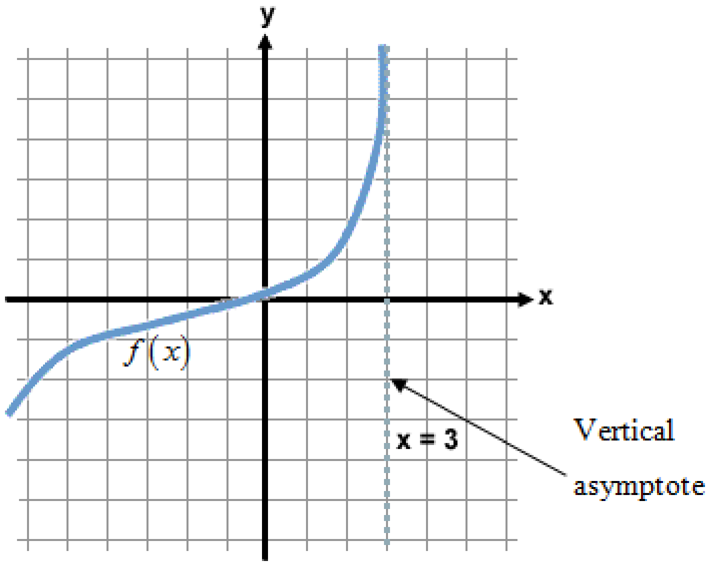
Types of asymptotes

There are three types of asymptotes:

- Vertical asymptote,
- Horizontal asymptote and
- Oblique asymptote.

a) Vertical asymptote

A line with equation $x = x_0$ ($D \equiv x = x_0$) is called a **vertical asymptote** for the graph of a function $f(x)$ if $\lim\limits_{x\to x_0} f(x) = \pm\infty$ 0r $V.A \equiv x = x_0$, *where* $\lim\limits_{x\to x_0} f(x) = \pm\infty$



Example: *Let* $f(x) = \dfrac{2x^2 + 7x - 1}{x+1}$

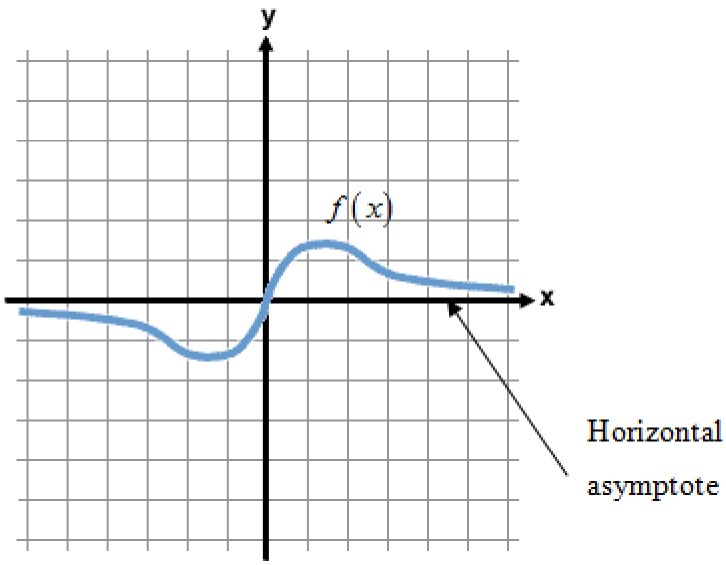
$Domf =]-\infty,-1[\cup]-1,+\infty[$

$$\lim_{x \to -1} f(x) = \lim_{x \to -1} \frac{2x^2+7x-1}{x+1}$$
$$= \infty$$

Hence, the line D with equation $x = -1$ is a vertical asymptote for $f(x) = \frac{2x^2+7x-1}{x+1}$.

b) Horizontal asymptote

A line with equation $y = L$ ($D \equiv y = L$) is called a **horizontal asymptote** for the graph of a function $f(x)$ if $\lim_{x \to \pm\infty} f(x) = L$ or $H.A \equiv y = L;\ where\ \lim_{x \to \pm\infty} f(x) = L$



Example

$Let\ f(x) = \frac{3x^2+4x-9}{2x^2+1}$

$Domf =]-\infty, +\infty[$

$$\lim_{x \to \pm\infty} f(x) = \lim_{x \to \pm\infty} \frac{3x^2}{2x^2}$$
$$= \frac{3}{2}$$

Thus, $y = \frac{3}{2}$ is a horizontal asymptote for $f(x) = \frac{3x^2 + 4x - 9}{2x^2 + 1}$.

c) Oblique asymptote

If a rational function, $\frac{P(x)}{Q(x)}$, is such that the degree of the numerator exceeds the degree of the denominator by one, then the graph of $\frac{P(x)}{Q(x)}$ will have an **oblique asymptote (or a slant asymptote)**; that is, an asymptote which is neither vertical nor horizontal.

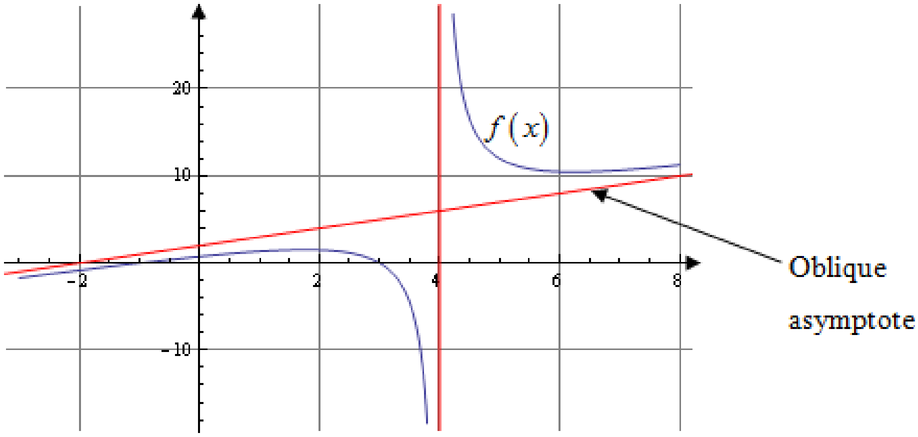
We perform the division of $P(x)$ by $Q(x)$ to obtain $\frac{P(x)}{Q(x)} = (ax + b) + \frac{R(x)}{Q(x)}$
Where, $ax + b$ is the quotient and $R(x)$ is the remainder.

Another way to find the values of constants a and b is $a = \lim_{x \to \pm\infty} \frac{f(x)}{x}$ $a \neq 0$ and $b = \lim_{x \to \pm\infty} \left[f(x) - ax \right]$

We can write that $O.A \equiv y = ax + b$ *where* $a = \lim_{x \to \pm\infty} \frac{f(x)}{x}$,
$a \neq 0$ *and* $b = \lim_{x \to \pm\infty} \left[f(x) - ax \right]$

Notice

Horizontal asymptote and oblique asymptote do not exist on the same side. That means if $f(x) \to L$ as $x \to +\infty$, there is no oblique asymptote on the right side since there is horizontal asymptote and if $f(x) \to L$ as $x \to -\infty$, there is no oblique asymptote on the left side since there is horizontal asymptote.



Example:

Let $f(x)=\dfrac{3x^3+4x-5}{x^2+1}$

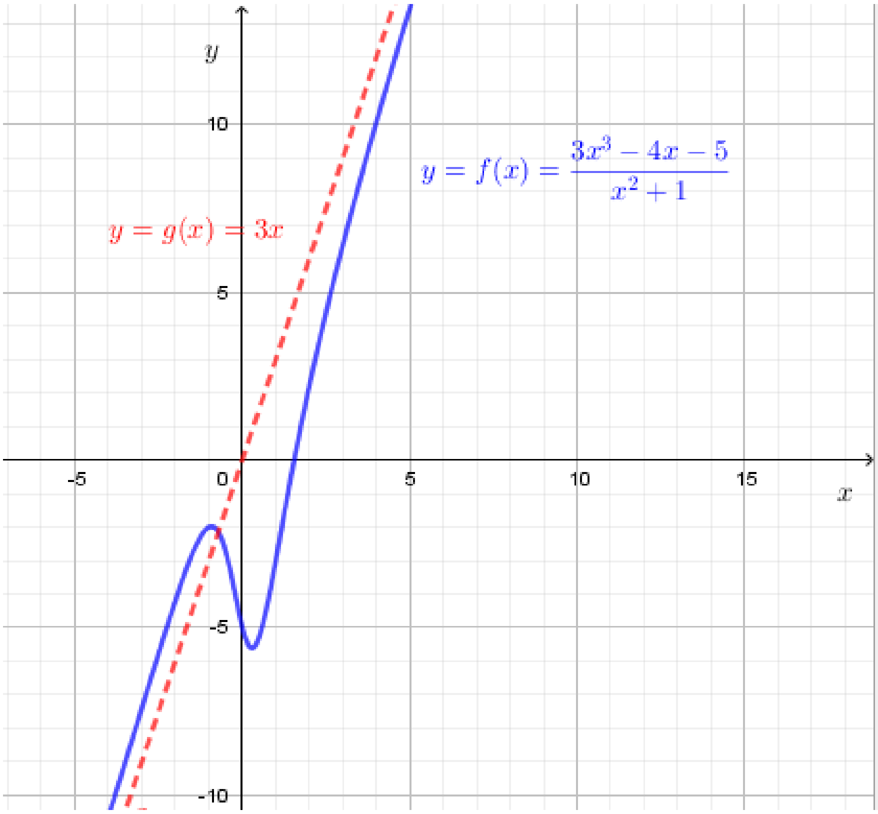
$Domf=\left]-\infty,+\infty\right[$

Let $y=ax+b$ *be the oblique asymptote.*

$$a=\lim_{x\to\pm\infty}\frac{f(x)}{x}=\lim_{x\to\pm\infty}\frac{3x^3+4x-5}{x^3+x}$$
$$=3$$

$$b=\lim_{x\to\pm\infty}\left[f(x)-3x\right]$$
$$=\lim_{x\to\pm\infty}\frac{3x^3+4x-5-3x^3-3x}{x^2+1}$$
$$=\lim_{x\to\pm\infty}\frac{x-5}{x^2+1}$$
$$=0$$

Thus, $y=3x$ is the oblique asymptote for $f(x)=\dfrac{3x^3+4x-5}{x^2+1}$



2. *Let* $f(x) = \dfrac{x}{x-2}$. *Find relative asymptotes*

$Domf = (-\infty, 2) \cup (2, +\infty)$

$\lim\limits_{x \to 2^+} f(x) = +\infty$ and $\lim\limits_{x \to 2^-} f(x) = -\infty$. Thus, there exists a vertical asymptote V.A $\equiv x = 2$

$\lim\limits_{x \to \pm\infty} f(x) = 1$. Thus, there exists a horizontal asymptote H.A $\equiv y = 1$

Note that there is no oblique asymptote.

3. *Let* $f(x) = \dfrac{x^2 + 2x - 3}{x}$. *Find relative asymptotes*

$Domf = (-\infty, 0) \cup (0, +\infty)$

$\lim\limits_{x \to 0^+} f(x) = -\infty$ and $\lim\limits_{x \to 0^-} f(x) = +\infty$. Thus, there is a vertical asymptote V.A $\equiv x = 0$

$\lim\limits_{x \to \pm\infty} f(x) = \pm\infty$. Thus, the horizontal asymptote does not exist.

To find oblique asymptote, let $y = ax + b$ be the oblique asymptote.

$a = \lim_{x\to\pm\infty} \frac{f(x)}{x}$ $= \lim_{x\to\pm\infty} \frac{x^2+2x-3}{x^2}$ $=1$ Since $1 \neq 0$, let us find b.	$b = \lim_{x\to\pm\infty} \left[f(x) - ax\right]$ $= \lim_{x\to\pm\infty} \left[\frac{x^2+2x-3}{x} - x\right]$ $= \lim_{x\to\pm\infty} \frac{2x-3}{x}$ $=2$

Then, $\text{O.A} \equiv y = x+2$.

As the degree of the numerator exceeds the degree of the denominator by one, we could find oblique asymptote after performing long division

	$x+2$
x	x^2+2x-3 $-(x^2)$ $2x-3$ $-(2x)$ -3

$$f(x) = x+2-\frac{3}{x}.$$

Thus, there exists an oblique asymptote $\text{O.A} \equiv y = x+2$

Application activity 4.5.2

Find relative asymptotes of

1) $f(x) = \frac{x^3+x^2-5x-2}{x^3-x^2-2x}$ 2) $y = \frac{x+3}{x^2+9}$

3) $y = \frac{x^2+3x+1}{4x-9}$ 4) $y = \frac{x^2-x-2}{x-2}$

4.6 Applications of limits in real life: Solving Problems involving limits

Activity 4.6

Consider the function price in function of supplied quantity P = 60−0.2Q where P is price and Q is quantity. Assume that P and Q cannot take negative values, determine the limit of the price when the quantity approaches 300 units.

Plot the graph of $P = f(Q)$.

1. Instantaneous rate of change of a function

The **instantaneous rate of change** of f at a, also called the **rate of change** of f at a, is defined to be the limit of the average rates of change of f over shorter and shorter intervals around a.

Since the average rate of change is a difference quotient of the form $\frac{\Delta y}{\Delta t}$, the instantaneous rate of change is a limit of difference quotients. In practice, we often approximate a rate of change by one of these difference quotients.

Example:

1. The quantity (in mg) of a drug in the blood at time t (in minutes) is given by $Q == 25(0.8)^t$. Estimate the rate of change of the quantity at t = 3 and interpret your answer.

Solution:

We estimate the rate of change at t = 3 by computing the average rate of change over intervals near t = 3. We can make our estimate as accurate as we like by choosing our intervals small enough.

Let's look at the average rate of change over the interval 3 ≤ t ≤ 3.01:

$$\text{Average rate of change} = \frac{\Delta Q}{\Delta t} = \frac{25\left(0.8\right)^{3.01} - 25\left(0.8\right)^{3}}{3.01 - 3.00} = -2.85$$

A reasonable estimate for the rate of change of the quantity at t = 3 is −2.85. Since Q is in mg and t in minutes, the units of $\frac{\Delta Q}{\Delta t}$ are mg/minute. Since the rate of change is negative, the quantity of the drug is decreasing. After 3 minutes, the quantity of the drug in the body is decreasing at 2.85 mg/minute.

2. Instantaneous velocity

Instantaneous velocity of a moving body is the limit of average velocity over an infinitesimal interval of time.

$$v = \lim_{t \to 0} \frac{\Delta s}{\Delta t}.$$

3. Instantaneous acceleration

Instantaneous acceleration for a moving body is the limit of average acceleration over an infinitesimal interval of time.

$$a = \lim_{\Delta t \to 0} \frac{\Delta v}{\Delta t}$$

(Here and elsewhere, if motion is in a straight line, vector quantities can be substituted by scalars in the equations.)

Application activity 4.6

A particle moves along a line so that its velocity at time t is $v(t) = t^2 - t$ (measured in meters per second). Find the instantaneous acceleration of the particle during the time period $1 \le t \le 2$.

4.7. END UNIT ASSESSMENT

1. Use limits to find the slope of the tangent line to the graph of $s = t^2$ at the point $(1,1)$.

2. Given the function f(x)= $y = \dfrac{x^2 + 3x + 1}{4x - 9}$

Find the limits: a) $\lim_{x \to \frac{9}{5}} f(x)$

b) $\lim_{x \to \pm\infty} f(x)$

Verify if the graph of this function has asymptotes and determine their equations.

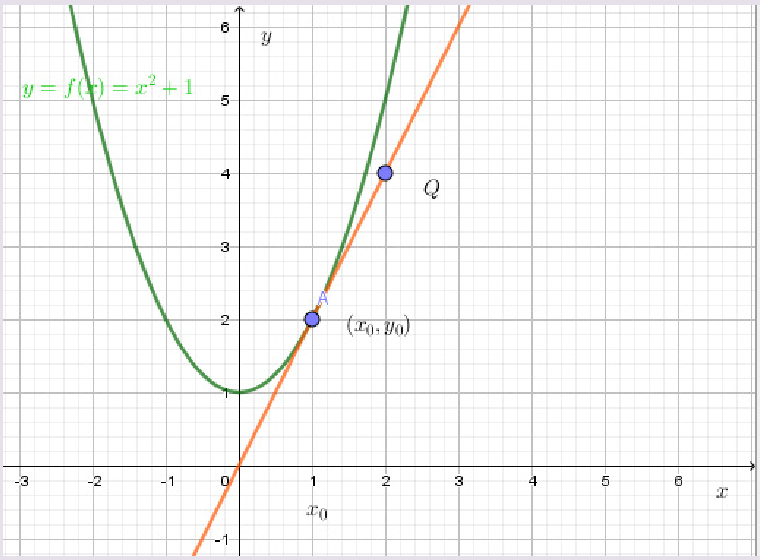
UNIT: 5

DERIVATIVE OF FUNCTIONS AND THEIR APPLICATIONS

Key Unit competence: Use the concepts of derivative to solve and interpret related problems in various contexts

5.0. Introductory Activity

1. Consider the function $f(x)=x^2+1$ illustrated on the following graph;



It is defined that the slop m_P of the tangent of the curve of $f(x)$ in a point $P(x_0, y_0)$

is obtained by $\lim\limits_{h\to 0}\dfrac{f(x_0+h)-f(x_0)}{h}$,

a) Determine the slop of $f(x)$ in the point for which $x_0=1$.

b) Deduce the value of the function $f'(x)=\lim\limits_{h\to 0}\dfrac{f(x+h)-f(x)}{h}$ for $f(x)=x^2+1$

and compare the slop m_P and $f'(x_0)$ for $x_0=1$.

2) Use books or internet and do research on the following:

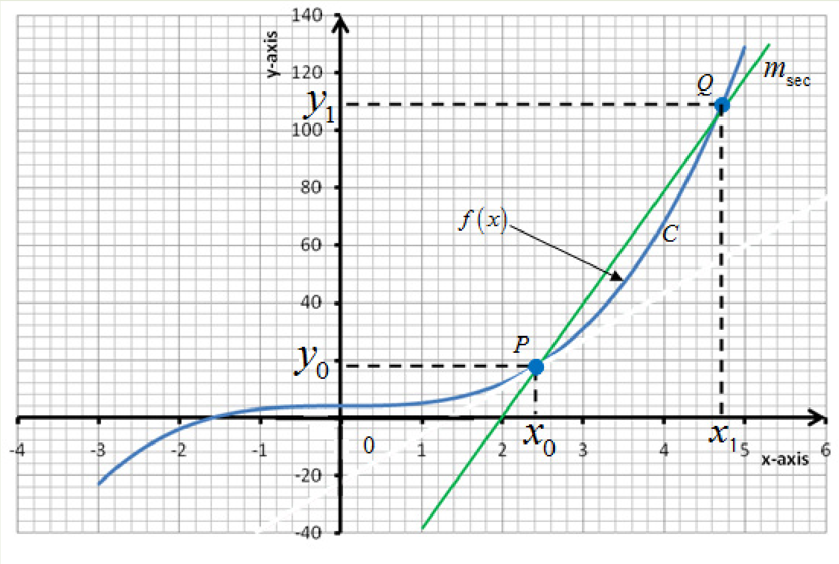
a) The meaning for the derivative of a function

b) Different applications of derivatives in solving problems.

5.1 Concepts of derivative of a function

Activity 5.1

Consider the figure below, analyse it and answer the questions that follow:



a. If $P(x_0, y_0)$ and $Q(x_1, y_1)$ are two points on the graph of a function f, refer to what you learned in S3 and find the slope of secant line (m_{sec}) passing through P and Q. Since $y_0 = f(x_0)$ and $y_1 = f(x_1)$, express the slope in terms of $f(x_0)$ and $f(x_1)$.

b. If we let x_1 approach x_0, how can you conclude about position of Q to P?

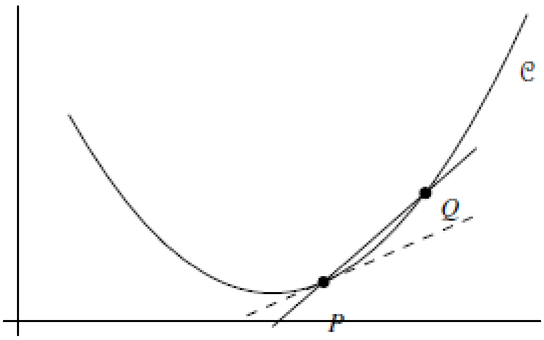
c. Let $m_{tan} = \lim\limits_{x_1 \to x_0} m_{sec}$, write down expression of m_{tan} in terms of $f(x_0)$ and $f(x_1)$.

d. After letting $h = x_1 - x_0$, rewrite m_{tan} in terms of $f(x_0)$ and $f(x_0 + h)$.

CONTENT SUMMARY

Slop of a function at a point

To define the slope of the curve C at P, take a point Q on the curve different from P. The line PQ is called a secant line at P. Its slope, denoted by m_{PQ}, can be found using the coordinates of P and Q. If we let Q move along the curve, the slope m_{PQ} changes.



Suppose that as Q approaches P, the number m_{PQ} approaches a fixed value and the increment $h = x_Q - x_P$ of x approaches 0. This value, denoted simply by m_P if the curve is understood, is called the slope of C at P; and the line with slope m_P and passing through P is called the tangent line to the curve C at P.

In view of the concept "*limit of a function at a point*", the slope m_{PQ} of the secant line *PQ* is

$$m_{PQ} = \frac{f(x_0+h)-f(x_0)}{(x_0+h)-x_0}$$

$$= \frac{f(x_0+h)-f(x_0)}{h}$$

Note that as Q approaches *P*, the number *h* approaches 0. From these, we see that the slope of C at *P* (denoted by m_P) is

$$m_P = \lim_{h\to 0}\frac{f(x_0+h)-f(x_0)}{h}$$

If this limit exists, the number m_P is called ***slope of tangent line*** to the graph of f at P or at $x = x_0$

Definition of derivative of a function

The slope m_P has a special notation, we denote it by $f^{'}(x_0)=\lim\limits_{h\to 0}\dfrac{f(x_0+h)-f(x_0)}{h}$ and $f^{'}(x_0)$ is read $f^{'}$ prime of x_0.

Dropping the subscript on x_0 in notation of m_P, we get one of the most important in mathematics, the ***derivative of function f at*** $x=x_0$

The derivative of a function $f(x)$ with respect to *x* is denoted by $f'(x)$ or $\dfrac{d}{dx}f(x)$ and defined as $f'(x)=\lim\limits_{h\to 0}\dfrac{f(x+h)-f(x)}{h}$ provided that the limit exists.

Examples:

1) Let $f(x)=x^2+1$

The derivative of $f(x)$ is

$$
\begin{aligned}
f'(x)&=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}\\
&=\lim_{h\to 0}\frac{(x+h)^2+1-x^2-1}{h}\\
&=\lim_{h\to 0}\frac{x^2+2hx+h^2+1-x^2-1}{h}\\
&=\lim_{h\to 0}\frac{2hx+h^2}{h}\\
&=\lim_{h\to 0}(2x+h)\\
&=2x
\end{aligned}
$$

Thus, $f'(x)=2x$

2) By using definition, calculate the first derivative for $y = x^2 - 3$

Solution:

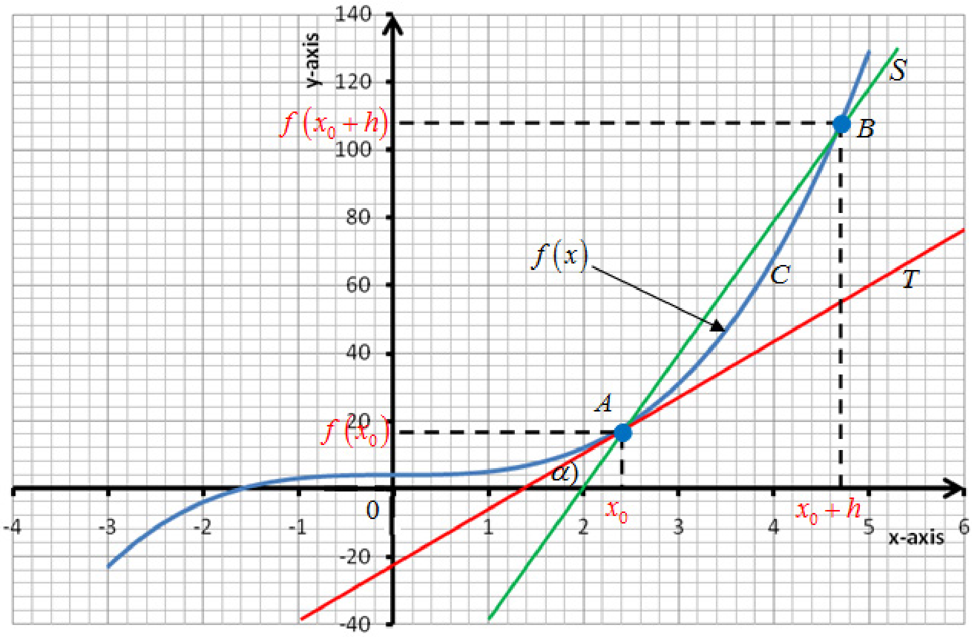
$$f'(x_0) = \lim_{h\to 0}\frac{f(x_0+h)-f(x_0)}{h}, \qquad y' = \lim_{h\to 0}\frac{(x+h)^2-3-(x^2-3)}{h}$$

$$y' = \lim_{h\to 0}\frac{x^2+2xh+h^2-x^2}{h} \quad = \lim_{h\to 0} 2x+h \quad = 2x$$

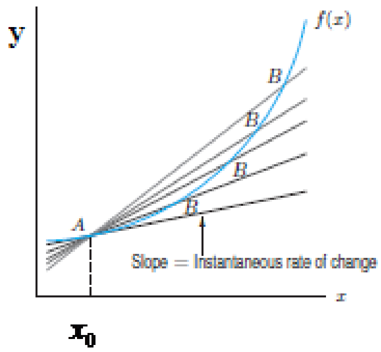
Remarks

- If $t \to f(t)$ represents the law of a moving object, then the derivative number of f represents the instantaneous speed of that moving object at instant t.
- The process of finding derivative of a function is called *differentiation* of that function.

Graphical interpretation of derivative and the slope of a function using differentiation



The graph shows the **average rate of change of a function** represented by the slope of the secant line joining points A and B.



The derivative is found by taking the average rate of change over smaller and smaller intervals. In the figure, as point B moves toward point A, the secant line becomes the tangent line at point A. Thus, **the derivative is represented by the slope of the tangent line to the graph at the point**.

The derivative of a function at the point A is equal to:

- The slope of the graph of the function at A.
- The slope of the line tangent to the curve at A.

Then AB has slope of $\frac{f(x_0+h)-f(x_0)}{(x_0+h)-x_0}=\frac{f(x_0+h)-f(x_0)}{h}$. When h approaches to zero, the point B approaches point A .

Then by definition $f^{/}(x_0)=\lim\limits_{h\to 0}\frac{f(x_o+h)-f(x_o)}{h}$ for $y=f(x)$ at point $(x_0, f(x_0))$
.

Alternative notations

The derivative: $f'(x)=y'=\frac{dy}{dx}=\frac{df}{dx}=\frac{d}{dx}f(x)$ where $\frac{dy}{dx}$ *and* $\frac{d}{dx}$ are called differentiation operators but $\frac{dy}{dx}$ should not be regarded as a ratio.

If x = t is time and y = $s(t)$ is the displacement function of a moving object then $s'(\mathrm{t}_0)$ or $\left.\frac{ds}{dt}\right|_{t=t_0}$ is the rate of change of displacement with respect to time when $t = t_0$, that is, the (instantaneous) velocity at $t = t_0$.

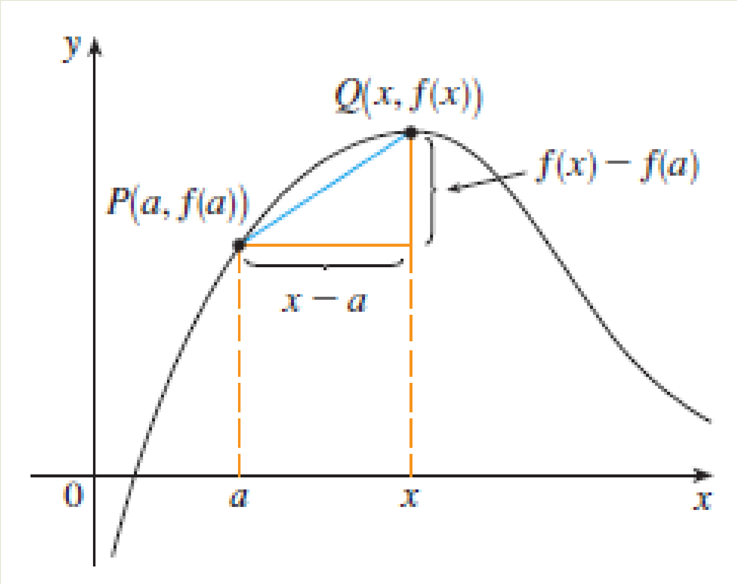
Application activity 5.1

1. Find the slope of the curve given by $f(x)=x^2$ at the point P(3; 9).

2. By the use of definition, find the derivative of:

   a. $f(x)=x^2+1$ b. $f(x)=x^2+2x-1$

3. Observe the graph and answer the questions that follow:



   a. From your observation, interpret the graph by indicating the rate of change given that the variable x changes from a to x

   b. Deduce the formula of derivative;

   c. Use your interpretation and the formula deduced to find out the derivative of $f(x)=x+1$.

4. Find the derivative of the function $f(x)=x^2-8x+9$ at the number a.

5.2 Rules of differentiation

Activity 5.2

1) Find the derivative of

a) $f(x)=x^2$ b) $h(x)=3x-1$ c) $g(x)=x^2+3x-1$

Deduce the derivative of $S(x)=g(x)+h(x)$

2) Calculate the derivative of $f(t)=\frac{1}{t},\ t\neq 0$

CONTENT SUMMARY

Suppose that f is differentiable at every point belonging to an open interval (a, b). Then we say that f is differentiable on interval (a, b).

Derivative of a constant function

If f is a constant function, $f(x)=c$, for all x then $\frac{df}{dx}=\frac{d}{dx}(c)=0$

Example: calculate the derivative of $f(x)=8$

Solution: $\frac{df}{dx}=\frac{d}{dx}(8)=0$

Derivative of identity function

The derivative of the identity function is the constant function 1, that is

$$if\ \ f(x)=x, \frac{df}{dx}=\frac{dx}{dx}=1$$

Multiplication by a scalar

If f is a differentiable function of x, and c is a constant, then

$$\frac{d}{dx}\left(cf(x)\right)=c\frac{d}{dx}f(x)$$

Example: find the derivative of $f(x)=3x$

Solution: $f'(x)=3(x)'=3x^{1-1}=3$

Derivative of a power

If n is any real number,

then $\frac{d}{dx}x^n = nx^{n-1}$ for all x where the powers x^n and x^{n-1} are defined.

This holds for any function with power. Thus, if $f\in(D,I)$ for positive and negative,

and fractional value of n, $\left[f^n(x)\right]' = nf^{n-1}(x)f'(x)$

Example: Differentiate $f(x)=(2x+1)^4$ respecting the value of x.

Solution:

$$f'(x)=\left((2x+1)^4\right)' = 4(2x+1)'(2x+1)^{4-1}$$

$$f'(x)=4(2)(2x+1)^3$$

$$f'(x)=8(6x^3+12x^2+6x+1)$$

$$f'(x)=64x^3+96x^2+48x+8$$

Sum rule

$$\frac{d}{dx}(u+v)=\frac{du}{dx}+\frac{dv}{dx}$$

Example: calculate the derivative of $y=x^4+12x$ respecting x

Solution: $y'=4x^{4-1}+12x^{1-1}$

$y'=4x^3+12$

The Difference Rule

$$\frac{d}{dx}(u-v)=\frac{du}{dx}-\frac{dv}{dx}$$

Example

$y=x^4-2x^2+2$ $\qquad\qquad$ $\frac{dy}{dx}=4x^3-4x$

Product rule

$$\frac{d}{dx}(uv)=u\frac{dv}{dx}+v\frac{du}{dx}$$

Notice that this is not just the product of two derivatives

Example

$$\frac{d}{dx}\left[\left(x^2+3\right)\left(2x^3+5x\right)\right]=\left(x^2+3\right)\left(6x^2+5\right)+\left(2x^3+5x\right)\left(2x\right)$$

$$=6x^4+5x^2+18x^2+15+4x^4+10x^2$$

$$=10x^4+33x^2+15$$

Quotient rule

$$\frac{d}{dx}\left(\frac{u}{v}\right)=\frac{v\frac{du}{dx}-u\frac{dv}{dx}}{v^2}$$ or can be written as $d\left(\frac{u}{v}\right)=\frac{v\,du-u\,dv}{v^2}$

Examples:

1. Find out the derivative for $f(x)=\frac{2x^3+5x}{x^2+3}$

Solution:

$$f'(x)=\frac{(x^2+5)(2x^3+5x)'-(2x^3+5x)(x^2+3)}{(x^2+3)^2}$$

$$f'(x)=\frac{\left(x^2+3\right)\left(6x^2+5\right)-\left(2x^3+5x\right)\left(2x\right)}{\left(x^2+3\right)^2}$$

2. Find the derivative of $f(x)=\dfrac{2x^2+3x}{4x^3+x+1}$

$$f'(x)=\left(\frac{2x^2+3x}{4x^3+x+1}\right)'$$
$$=\frac{(2x^2+3x)'(4x^3+x+1)-(2x^2+3x)(4x^3+x+1)'}{(4x^3+x+1)^2}$$
$$=\frac{(2x^2+3x)'(4x^3+x+1)-(2x^2+3x)(4x^3+x+1)'}{(4x^3+x+1)^2}$$
$$=\frac{(4x+3)(4x^3+x+1)-(2x^2+3x)(12x^2+1)}{(4x^3+x+1)^2}$$
$$=\frac{16x^4+4x^2+4x+12x^3+3x+3-24x^4-2x^2-36x^3-3x}{(4x^3+x+1)^2}$$
$$=\frac{-8x^4-24x^3+2x^2+4x+3}{(4x^3+x+1)^2}$$

$$f'(x)=\left(\frac{2x^2+3x}{4x^3+x+1}\right)'$$
$$=\frac{(2x^2+3x)'(4x^3+x+1)-(2x^2+3x)(4x^3+x+1)'}{(4x^3+x+1)^2}$$
$$=\frac{(2x^2+3x)'(4x^3+x+1)-(2x^2+3x)(4x^3+x+1)'}{(4x^3+x+1)^2}$$
$$=\frac{(4x+3)(4x^3+x+1)-(2x^2+3x)(12x^2+1)}{(4x^3+x+1)^2}$$
$$=\frac{16x^4+4x^2+4x+12x^3+3x+3-24x^4-2x^2-36x^3-3x}{(4x^3+x+1)^2}$$
$$=\frac{-8x^4-24x^3+2x^2+4x+3}{(4x^3+x+1)^2}$$

Derivative of polynomial

Let $y=a_nx^n+a_{n-1}x^{n-1}+\ldots+a_2x^2+a_1x^1+a_0$ be a polynomial. Then, we have

$$\frac{dy}{dx}=na_nx^{n-1}+(n-1)a_{n-1}x^{n-2}+\ldots+2a_2x^1+a_1$$

Derivative of the reciprocal function

Let $D(I,\mathbb{R})$ be the set of functions differentiable on I. If $f\in D(I,\mathbb{R})$, then

$\dfrac{1}{f}\in D(I,\mathbb{R})$ $f(x)\neq 0$. Moreover $\dfrac{d}{dx}\left(\dfrac{1}{f}\right)=-\dfrac{\frac{df}{dx}}{f^2}$ or $\dfrac{d}{dx}\left(\dfrac{1}{f}\right)=-\dfrac{f'}{f^2}$

Example: Calculate the derivative of $f(x)=\frac{1}{x}$ respecting the value of x

Solution: $\left(\frac{1}{x}\right)'=-\frac{(x)'}{x^2}=-\frac{1}{x^2}$

Derivative of a composite function: Chain rule

If f and g are both differentiable and F is the composite function defined by $F(x)=f(g(x))$, then F is differentiable and F' is given by the product $F'(x)=f'(g(x))\cdot g'(x)$

In **Leibniz notation**, if $y=f(u)$ and $u=g(x)$ are both differentiable functions, then $\frac{dy}{dx}=\frac{dy}{du}\frac{du}{dx}$

Example: Find the derivative of $f\circ g$ if $f(x)=x^2+3x+3$ and $g(x)=\frac{2x+1}{x}$

Solution:

$$
\begin{aligned}
(f\circ g)'(x)=f'[g(x)]g'(x)&=\left[2\left(\frac{2x+1}{x}\right)+3\right]\left(\frac{2x+1}{x}\right)'\\
&=\left(\frac{4x+2+3x}{x}\right)\left(\frac{2x-2x-1}{x^2}\right)=\left(\frac{7x+2}{x}\right)\left(\frac{-1}{x^2}\right)\\
&=\frac{-7x-2}{x^3}
\end{aligned}
$$

Derivative of the square root function:

$\frac{d}{dx}\left(\sqrt{x}\right)=\frac{1}{2\sqrt{x}}$ and $\frac{d}{dx}\left(\sqrt{u(x)}\right)=\frac{u'}{2\sqrt{u}}$

Application activity 5.2

1) Given the function $f(x)=x^2+3x-4$ and $g(x)=x+1$. Find

$\left(f\left[g(x)\right]\right)'$ **b.** $f'\left[g(x)\right]$ **c.** $f'\left[g(x)\right]\cdot g'(x)$

2) Differentiate $y=\left(x^3-1\right)^{100}$

3) A body is moving such that its displacement is given by

$x(t)=t^3-3t$. What will the acceleration $a(t)$ of the body given that

$a(t)=\frac{dv}{dt}$ *and* $v(t)=\frac{dx}{dt}$.

5.3 High order derivatives

Activity 5.3

1. A body moves along such that at time t seconds the displacement is $X(t)=t^3+3t^2-9t$

Find: a. the position and velocity of the body at $t=1$, 2 and 3

b. Acceleration at $t=1$, 2 and 3.

2. Calculate $f''(x)$ for $f(x)=x^3+2x^2+1$ respecting x

CONTENT SUMMARY

We have seen that the derivative of $y=f(x)$ is in general also a function of x. This new function can be also differentiable, in which case the derivative of the first derivative is called the second derivative of the original function. It is written in several ways:

$$f''(x)=\frac{d^2y}{dx^2}=\frac{d}{dx}\left(\frac{dy}{dx}\right)=y''=f''(x)=D^2(f(x))$$

Meaning of Second Derivative

- The graph of $y = f(x)$ is the curve. Note that $f^{'}(x) = \frac{dy}{dx}$ is the slope function; it is the rate of change of y with respect to x. Since, $f^{"}(x) = \frac{d^2y}{dx^2}$ is the derivative of the slope function, it is the rate of change of slope and is related to a concept called **convexity** (bending) of a curve.
- If $x = t$ is time and if $y = s(t)$ is the displacement function of a moving object, then $s^{'}(t) = \frac{ds}{dt}$ is the velocity function. The derivative of velocity is $s^{"}(t)$ or $\frac{d^2s}{dt^2}$; it is the rate of change of the velocity (function), that is , the acceleration (function).

The symbol D^2 means the operation of differentiation is performed twice, similarly, the derivative of the second derivative is called the **third derivative** and so on.

Thus, if for example $y = 3x^4$ then

$$\frac{dy}{dx} = 12x^3, \frac{d}{dx}\left(\frac{dy}{dx}\right) = 36x^2, \frac{d}{dx}\left[\frac{d}{dx}\left(\frac{dy}{dx}\right)\right] = 72x \text{ and so on.}$$

The successive derivatives of a function f are **higher order derivatives** of the same function.

We denote higher order derivatives of the same function as follows:

The second derivative is:

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2} = f''(x) = y''$$

The third derivative is:

$$\frac{d}{dx}\left(\frac{d^2y}{dx^2}\right) = \frac{d^3y}{dx^3} = f'''(x) = y'''$$

And the n^{th} derivative is:

$$\frac{d}{dx}\left(\frac{d^{n-1}y}{dx^{n-1}}\right) = \frac{d^ny}{dx^n} = f^{(n)}(x) = y^{(n)}$$

Examples

1) Successive derivatives of $y = x^n \quad n \in \mathbb{N}$

$y' = nx^{n-1}$

$y'' = n(n-1)x^{n-2}$

$y''' = n(n-1)(n-2)x^{n-3}$

$\vdots$

$y^{(n)} = n(n-1)(n-2)...x^{n-n} = n(n-1)(n-2)....1 = n!$

Thus, if $y = x^n \quad n \in \mathbb{N}$, $y^{(n)} = n!$

2) Given $y = x^4 - 3x + 4$. Let us find $\frac{d^5 y}{dx^5}$,

$y' = 4x^3 - 3$

$y'' = 12x^2$

$y''' = 24x$

$y^{(4)} = 24$

$y^{(5)} = 0$

Thus, $y^{(5)} = 0$

3) Calculate $f''(x)$ if $f(x) = x^3 - 3x^2$

Solution: $f''(x) = (f'(x))' = ((x^3 - 3x^2)')'$, $f''(x) = (3x^2 - 6x)'$; Then,

$f''(x) = 6x - 6$

Application activity 5.3

1. Calculate the second derivative of the following functions

   a. $f(x) = 2x^4 - 3x^2 + 1$

   b. $f(x) = -x^3 - 3x^1 - 9$

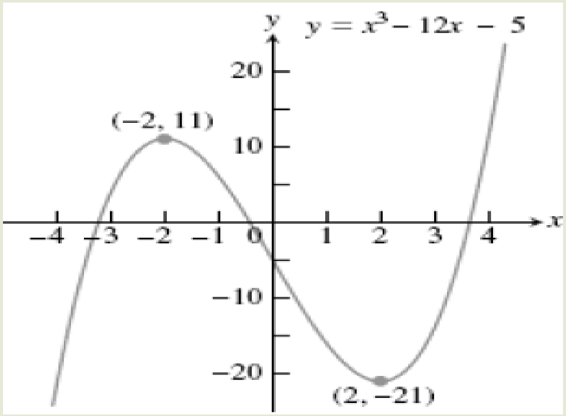
2. A body moves in a way such that at time t seconds the displacement is $X(t) = t^3 + 3t^2 - 9t$

   a. Calculate the velocity of the body

   b. Calculate the acceleration used by the body at time t of 2 seconds.

Activity 5.4

1. Observe the graph bellow and answer the questions that follows



a. Is it the given function $y = x^3 - 12x - 5$ differentiable? Justify your answer.

b. Find y' by respecting x

c. Show the increasing and decreasing interval of $y = x^3 - 12x - 5$

2. Find where the function $f(x) = x^3 - 3x^2$ is concave up or where it is concave down.

CONTENT SUMMARY

Increasing and decreasing of a function

Theorem: Let f be a function differentiable on an interval $]a,b[$

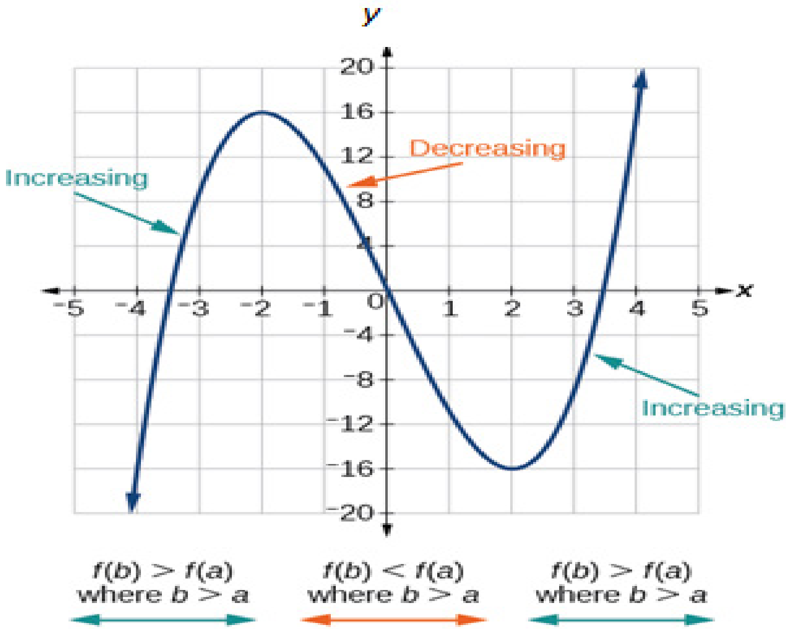
a) If $f'(x) > 0$ on each point x of $]a,b[$, then f is increasing on $]a,b[$

b) If $f'(x) < 0$ on each point x of $]a,b[$, then f is decreasing on $]a,b[$

c) If $f'(x) = 0$ for all $x \in]a,b[$, then f is constant on this interval, that is $f(x_1) = f(x_2)$ for all $x_1, x_2 \in]a,b[$, or equivalently, there exists a real c such that $f(x) = c$ for all $x \in]a,b[$.

Given x_1 and x_2 from an interval I with $x_1 < x_2$ if $f(x_1) < f(x_2)$ then $f(x)$ is increasing on I

Given any x_1 and x_2 from an interval I with $x_1 < x_2$ if $f(x_1) > f(x_2)$ then $f(x)$ is decreasing on I .



If the derivative of a function is positive at a point then the function is increasing at that point and if the derivative is negative at a point then the function is decreasing at that point. Also, the fact that the derivative of a function is zero at a point then the function is not changing at that point. These ideas used previously to identify the intervals in which a function is increasing and decreasing. This can be summarized in the following fact.

- If $f'(x) > 0$ for every x on some interval I , then $f(x)$ is increasing on the interval that interval
- If $f'(x) < 0$ for every x on some interval I , then $f(x)$ is decreasing on the interval.
- If $f'(x) = 0$ for every x on some interval I , then $f(x)$ is constant on the interval.

Example:

1) Let us determine all intervals where the following function is increasing or decreasing. $f(x) = -x^5 + \frac{5}{2}x^4 + \frac{40}{3}x^3 + 5$

To determine if the function is increasing or decreasing we will need the derivative.

$$f'(x) = -5x^4 + 10x^3 + 40x^2$$
$$= -5x^2(x^2 - 2x - 8)$$
$$= -5x^2(x-4)(x+2)$$

From the factored form of the derivative we see that we have three critical points: $x=-2$, $x=0$, and $x=4$. We will need these in a bit.

We now need to determine where the derivative is positive and where it's negative. We will build sign table of $f'(x)$, graph the critical points and pick test points from each region to see if the derivative is positive or negative in each region.

x	$-\infty$		-2		0		4		$+\infty$
$f'(x)$		-	0	+	0	+	0	-	
$f(x)$	$+\infty$	$\searrow$	$f(-2)$	$\nearrow$	$f(0)$	$\nearrow$	$f(4)$	$\searrow$	$-\infty$

Increase is symbolized by the arrow $\nearrow$ on $-2 < x < 0$ and $0 < x < 4$

Decrease is symbolized by the arrow $\searrow$ on $-\infty < x < -2$ and $4 < x < +\infty$

2) Let $f : \mathbb{R} \to \mathbb{R}$ be the function given by $f(x) = 27\mathrm{x} - \mathrm{x}^3$

Solution:

Differentiating $f(x)$, we get $f'(x) = 27 - 3x^2 = 3(3+x)(3-x)$.

	$(-\infty,-3)$	$(-3,3)$	$(3,\infty)$
3	$+$	$+$	$+$
$3+x$	$-$	$+$	$+$
$3-x$	$+$	$+$	$-$
f'	$-$	$+$	$-$
f	$\searrow$	$\nearrow$	$\searrow$

From the table, we see that the function $f(x)$ is increasing on the interval $\left]-3,3\right]$.

$f(x)$ is decreasing on the interval $\left]-\infty,3\right]$ and $\left[3,\infty\right[$

3) Let $f:\mathbb{R}\to\mathbb{R}$ be the function given by $f(x)=x^4-4x^3+5$.

Find the intervals where the function $f(x)$ is increasing or decreasing.

Solution:

Differentiating $f(x)$, we get $f'(x)=4x^3-12x^2=4x^2(x-3)$

Table of variation:

	$(-\infty,0)$	$(0,3)$	$(3,\infty)$
4	$+$	$+$	$+$
x^2	$+$	$+$	$+$
$x-3$	$-$	$-$	$+$
f'	$-$	$-$	$+$
f	$\searrow$	$\searrow$	$\nearrow$

From the table, we see that the function $f(x)$ is increasing on the interval $\left[3,\infty\right[$ and

$f(x)$ is decreasing on the interval $\left]-\infty,3\right]$

First derivative test for local extrema

Knowing where a function increases and decreases also tells us how to test for the nature of **local extreme values**.

Theorem

Suppose that c is a critical point of a continuous function f, and that f is differentiable at every point in some interval containing c except possibly at c itself. Moving across c from left to right,

1. If f' changes from negative to positive at c, then f has a **local minimum** at c

2. If f' changes from positive to negative at c, then f has a **local maximum** at c

3. If f' does not change sign at c (that is, f' is positive on both sides of c or negative on both sides), then f has no local extreme.

It means that, the relative extrema of a continuous function occur at those critical points where the first derivative changes sign.

Definition: Let f be a function and let x_0 be a real number such that f is defined on an open interval containing x_0.

- If $f'(x_0)=0$, then we say that x_0 is a stationary number of f.

Explanation: If x is the time and $y=f(t)$ is the displacement (function) of a moving object, then $\frac{dy}{dt}=f'(t)$ is the velocity (function). Thus $f'(t_0)=0$ means that the velocity at time t_0 is 0, that is, the object is stationary at that moment.

Definition:

Let f be a function and let x_0 be a real number such that f is defined on an open interval containing x_0. We say that

- f has a relative maximum at $x=x_0$ if $f(x_0)\geq f(x)$ for all x sufficiently close to x_0.

- f has a relative minimum at $x = x_0$ if $f(x_0) \le f(x)$ for all x sufficiently close to x_0.

Example

Locate the relative extreme points of $f(x) = 3x^{5/3} - 15x^{2/3}$

Solution:

$$f'(x) = 5x^{2/3} - 10x^{-1/3} = 5x^{-1/3}(x-2)$$

If $x = 0$ then, $f'(x)$ does not exist at $x = 0$ but $f(0)$ exists and $f'(x) = 0$ if $x = 2$, the critical points are $(0,0)$ and $(2,0)$

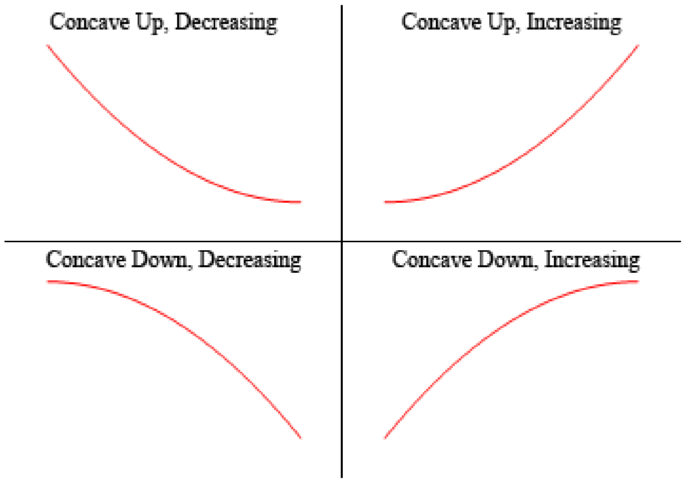
Sign table of $f'(x)$

x		0		2	
$x-2$	- - -	- - -	- - -	0 + + +	+ + +
$x^{\frac{-1}{3}}$	- - -	0 + + +	+ + +	+ + +	+ + +
$f'(x)$	+ + +	$\|\|$ - - - ∞	- - -	0 + + +	+ + +

There is a relative maximum at 0 and a relative minimum at 2.

Concavity of a function

The second derivative of a function can give us information about the graph of a function such as concave up or concave down. The following figure gives us the idea of concavity



Function is **turned up (concavity up)** if it "opens" up and the function is **turned down** (**concavity down)** if it "opens" down. Notice as well that concavity has nothing to do with increasing or decreasing. A function can be concave up and either increasing or decreasing. Similarly, a function can be concave down and either increasing or decreasing.

Given the function $f(x)$ then

- $f(x)$ is **concave up** on an interval I if all of the tangents to the curve on I are below the graph of $f(x)$.
- $f(x)$ is **concave down** on an interval I if all of the tangents to the curve on I are above the graph of $f(x)$

Therefore,

Theorem:

Let f be a function that is defined and is twice differentiable on an open interval $]a,b[$

1) If $f''(x)>0$ *for all* $x\in]a,b[$, *then* f *is convex on* $]a,b[$

2) If $f''(x)<0$ *for all* $x\in]a,b[$, *then* f *is concave on* $]a,b[$

Notice that this fact tells us that a list of possible inflection points will be those points where the second derivative is zero or doesn't exist. Be careful however to not make the assumption that just because the second derivative is zero or doesn't exist that the point will be an inflection point.

We will only know that it is an inflection point once we determine the concavity on both sides of it. It will only be an inflection point if the concavity is different on both sides of the point.

Example:

1) Let us find where the function $f(x) = x^3 - 3x^2$ is concave up or down.

We need the second derivative so that we will find where concave is up or down

$f'(x) = 3x^2 - 6x$,

$f''(x) = 6x - 6$

$f''(x) = 0, x = 1$

Make a table of sign for second derivative

Sign of $f''(x)$

x	1
$f''(x)$ $f(x)$	- - - - - 0 + + + + + + ∩ -2 ∪

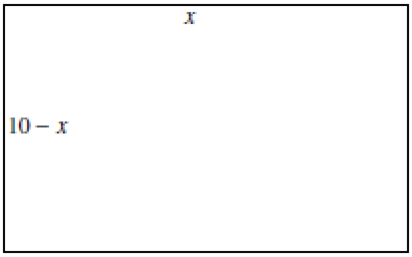
Thus, $f(x)$ is concave up if $x > 1$ or written as $]1, +\infty[$

$f(x)$ is concave down if $x < 1$ or written as $]-\infty, 1[$

2) Find the dimensions of the rectangle that has maximum area if its perimeter is 20 cm.

Solution :

Let x be the length, the width w is such that 2(x + w) = 20 i.e, w = 10-x



we will consider where the area A is increasing or decreasing where $A = x(10-x)$, 0<x<10.

We want to find the value of x at which A attains its maximum. Differentiating $A(x)$, we get $A'(x) = \frac{d}{dx}(10x - x^2) = 10 - 2x \quad (0 < x < 10)$

Solving $A'(x) = 0$, we obtain the critical number of $A: x_1 = 5$.

Since A is increasing on $(0,5)$ and decreasing on $(5,10)$, it follows that A attains its absolute maximum at $x_1 = 5$.The dimensions of the largest rectangle is $5cm \times 5cm$

	(0,5)	(5,10)
A'	+	-
A	$\nearrow$	$\searrow$

Application activity 5.4

1. Let f be the function $f(x) = x^3 - 12x - 5$

   a. Calculate extreme points

   b. Show the interval of increasing and decreasing

   c. Show where the function is concave up or down

5.5 Applications of differentiation in Economics and finance

Activity 5.4

1) Go in library or computer lab, do research on application of differentiation in economics and finance.

2) The marginal cost (MC) is the rate of change of the total cost (TC) function. In fact, in nearly all situations where one is dealing with the concept of a marginal increase, the marginal function is equal to the rate of change of the original function. This means that $MC = \frac{dTC}{dq}$, determine the function MC when $TC = 6 + 4q^2$.

CONTENT SUMMARY

Marginal cost

Suppose a manufacturer produces and sells a product. Denote $C(q)$ to be the total cost for producing and marketing q units of the product. Thus C is a function of q and it is called the *(total) cost function*. The rate of change of C with respect to q is called the *marginal cost*, that is,

$$\text{Marginal Cost } = \frac{dC}{dq}$$

Marginal revenue

Denote $R(q)$ to be the total amount received for selling q units of the product. Thus R is a function of q and it is called the *revenue function*. The rate of change of R with respect to q is called the *marginal revenue*, that is,

$$\text{Marginal Revenue } = \frac{dR}{dq}$$

Denote $P(q)$ to be the **profit** of producing and selling q units of the product, that is,

$$P(q) = R(q) - C(q)$$

Thus P is a function of q and it is called the profit function.

In introductory economics texts, marginal revenue (MR) is sometimes defined as the increase in total revenue (TR) received from sales caused by an increase in output by 1 unit.

This is not a precise definition though. It only gives an approximate value for marginal revenue and it will vary if the units that output is measured in are changed. A more precise definition of *marginal revenue is that it is the rate of change of total revenue relative to increases in output*.

Denote q_{max} to be the largest number of units of the product that the manufacturer can produce. Assuming that q can take any value between 0 and q_{max}. Then for each of the functions C, R and P, the domain is $[0; q_{max}]$. Suppose that the cost function and the revenue function are differentiable on $(0; q\text{max})$ and suppose that producing 0 or q_{max} units of the product will not give maximum profit. Then in order to have maximum profit, we need

$$\frac{dP}{dq} = 0$$

Or eventually,

$$\frac{dC}{dq} = \frac{dR}{dq},$$

that is, marginal cost = marginal revenue.

To have a maximum profit, marginal cost = marginal revenue

Example:

1) Given that $\mathrm{TR(q)} = 80q - 2q^2$,

a) Find the function of the marginal revenue MR.

b) Find the exact value of the output at which TR is a maximum.

Solution:

a) $MR = \dfrac{dTR}{dq} = 80 - 4q$

Given that the $\mathrm{TR(q)} = P.q$, we have the price $\mathrm{P(q)} = 80 - 2q$

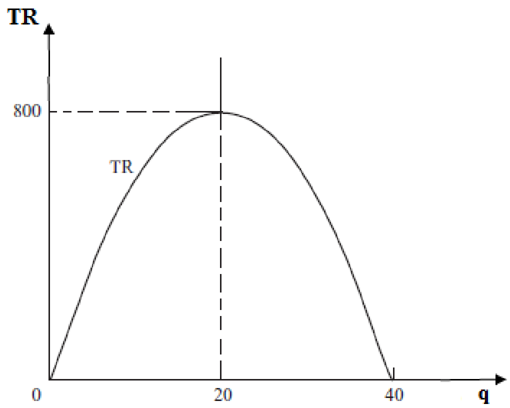
b) When TR is at its maximum, $\dfrac{dTR(q)}{dq} = MR = 0$

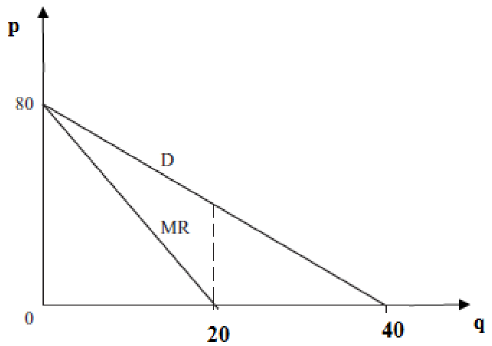
Thus,

$MR = 80 - 4q = 0$

$80 = 4q$

$20 = q$





2) For the total revenue function $TR = 500q - 2q^2$, find the value of MR when $q = 80$

Solution

$MR = \dfrac{dTR}{dq} = 500 - 4q$. Thus, when $q = 80$, $MR = 500 - 4.80 = 180$

3) Profit maximization

A monopoly faces the demand schedule $p = 460 - 2q$ and the cost schedule TC = 20 + $0.5q^2$

How much should it sell to maximize profit and what will this maximum profit be? (All costs and prices are in $.)

Solution

To maximize profits, the rule MC = MR must be realized.

To find the output where $MC = MR$ we first need to derive the MC and MR functions.

Given $TC = 20 + 0.5q^2$, then $MC = \dfrac{dTC}{dq} = q$

As $TR = pq = (460 - 2\text{q}).\text{q} = 460\text{q} - 2\text{q}^2$

Then $MR = \frac{dTR}{dq} = 460 - 4q$

To maximize profit $MR = MC$. Therefore, equating the two equations, we find:

$460 - 4q = q$ which implies that $q = 92$

The actual maximum profit when the output is 92 will be:

$TR - TC = (460q - 4q^2) - (20 + 0.5q^2)$

$= 460q - 2.5q^2 - 20 = 460.(92) - 2.5(8,464) - 20 = 21,140$

The actual maximum profit when the output is 92 will be $21,140

4) A firm faces the demand schedule $p = 184 - 4q$ and the TC function $TC = q^3 - 21q^2 + 160q + 40$.

What output will maximize profit?

Solution

Given that $TR = pq = (184 - 4q)q = 184q - 4q^2$, we have $MR = \frac{dTR}{dq} = 184 - 8q$

And $MC = \frac{dTC}{dq} = 3q^2 - 42q + 160$.

To maximize the profit, MC=MR. Therefore,

$3q^2 - 42q + 160 = 184 - 8q$

$(q - 12)(3q + 2) = 0$

$q - 12 = 0 \; or \; 3q + 2 = 0$

$q = 12 \; or \; q = -\frac{2}{3}$

As one cannot produce a negative quantity, the firm must produce 12 units of output in order to maximize profits.

3) Suppose that a company has estimated that the cost (in Rwandan francs) of producing x items is $C(x) = 10000 + 5x + 0.01x^2$. What is the marginal cost at the production level of 500 items?

Solution: Then the marginal cost function is $c'(x) = 5 + 0.02x$

The marginal cost at the production level of 500 items is $c'(500) = 5 + 0.02(500) = 15$ Rwandan Francs per item.

2) The demand equation for a certain product is $q - 90 + 2p = 0, \quad 0 \le q \le 90$ where q is the number of units and p is the price per unit, and the average cost function is

$C_{av} = q^2 - 8q + 57 + \frac{2}{q}, \quad 0 \le q \le 90$. At what value of q will there be maximum profit? What is the maximum profit?

Solution:

Although the average cost function is undefined at $q = 0$, we may include 0 in the domain of the cost function. The cost function and the revenue function are differentiable on (0; 90). However, we do not know whether maximum profit would be attained in (0; 90) or at an endpoint. So we use the method for finding absolute extrema for functions on closed and bounded intervals.

The cost function C is given by

$C(q) = q.C_{av} = q^3 - 8q^2 + 57q + 2, \quad 0 \le q \le 90)$ and the total revenue function R is given by

$$R(q) = p.q = \frac{90-q}{2}.q; \quad 0 \le q \le 90)$$

Therefore the profit function P is given by

$$p(q) = R(q) - C(q) = (45\text{q} - \frac{q^2}{2}) - (q^3 - 8q^2 + 57q + 2)$$

$$= -q^3 + \frac{15}{2}q^2 - 12q - 2, \quad (0 \le q \le 90)$$

Differentiating $P(q)$, we get

$$p'(q) = \frac{d\left(-q^3 + \frac{15}{2}q^2 - 12q - 2\right)}{dq} = -3q^2 + 15\text{q} - 12, \quad (0 \le q \le 90)$$

Solving $P'(q) = 0$, that is,

$$-3q^2 + 15\text{q} - 12 = 0$$

$$-3(q-1)(q-4) = 0; \quad (0 < q < 90)$$

We get the critical number of P where $q_1 = 1$ and $q_2 = 4$.

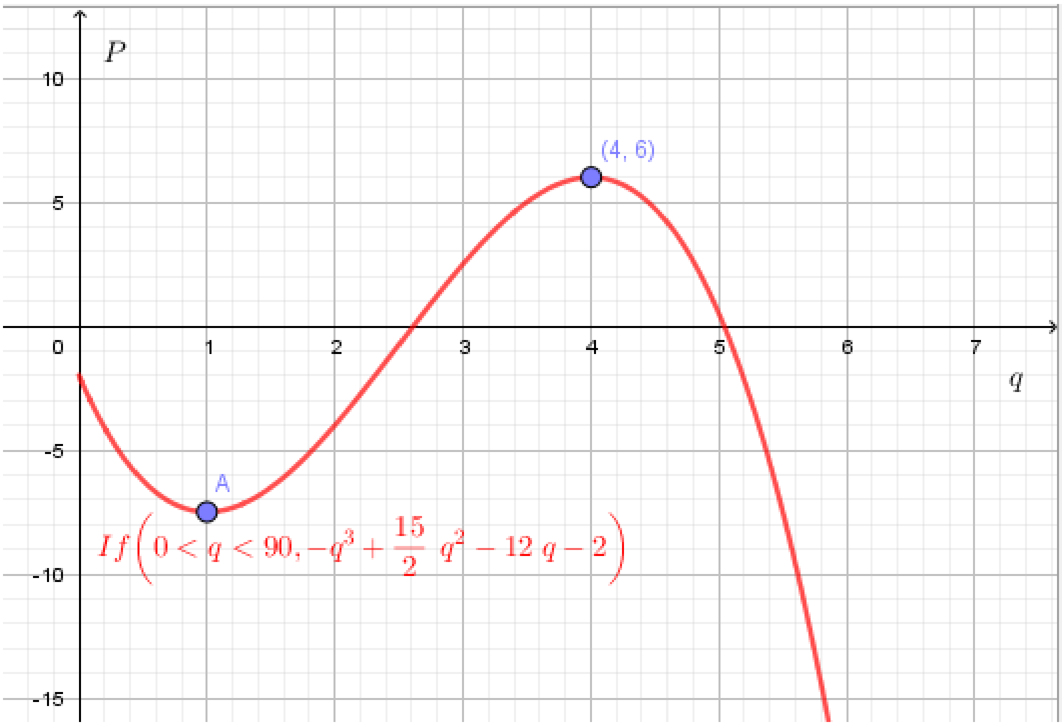
Comparing the values of P at the critical numbers as well as that at the endpoints:

q	0	1	4	90
P(q)	-2	$-\frac{15}{2}$	6	-669332

We see that maximum profit is attained at $q_2 = 4$ and the maximum profit is 6 (units of money).

Remarks:

If we know that maximum profit is not attained at the end points, we can simply compare the values of P at $q_1 = 1$ and $q_2 = 4$.



4) A store has been selling 200 DVD burners a week at 350 dollars each. A market survey indicates that for each 10 dollars rebate offered to buyers, the number of units sold will increase by 20 a week. Find the demand function and the revenue function. How large a rebate should the store offer to maximize its revenue?

Solution

If x is the number of DVD burners sold per week, then the weekly increase in sales is $x-200$. For each increase of 20 units sold, the price is decreased by 10 dollars. So for each additional unit sold, the decrease in price will be $\frac{1}{20}\times 10$

The demand function is $p(x)=350-\frac{10}{20}(x-200)=450-\frac{2}{2}x$

The revenue function is $R(x)=xp(x)=450x-\frac{1}{2}x^2$

Since $R'(x)=450-x$, we see that $R'(x)=0$ when $x=450$

This value of x gives an absolute maximum by the First Derivative Test

$p(450)=450-\frac{1}{2}(450)=225$

The rebate is 350-225=125, Therefore, to maximize revenue, the store should offer a rebate of 125 Rwandan francs.

4) Tax yield

Elementary supply and demand analysis tells us that the effect of a per-unit tax t on a good sold in a competitive market will effectively shift up the supply schedule vertically by the amount of the tax. This will cause the price paid by consumers to rise and the quantity bought to fall. The change in total revenue spent by consumers will depend on the price elasticity of demand.

Example

A market has the demand schedule p = 92−2q and the supply schedule p = 12+3q. What per-unit tax will raise the maximum tax revenue for the government? (All prices are in $.).

Solution:

Let the per-unit tax be t. This changes the supply schedule to $p=12+t+3q$

i.e the intercept on the price axis shifts vertically upwards by the amount t.

we now need to derive a function for q in terms of the tax t. In equilibrium, supply price equals demand price. Therefore,

$$12+3q+t=92-2q$$
$$q=16-0.2t$$

The tax yield is (amount sold) x(per unit tax). Therefore,

$$TY=q.t=(16-0.2t).t=16t-0.2t^2$$

And so the rate of change of TY with respect to t is

$$\frac{dTY}{dt}=16-0.4t$$

If $\frac{dTY}{dt}>0$, an increase in t will increase TY. However, from the formula for $\frac{dTY}{dt}$ derived above, one can see that as the amount of tax t is increased the value of $\frac{dTY}{dt}$ falls. Therefore in order to maximize TY, t should be increased until $\frac{dTY}{dt}=0$.Any further increases in t would cause $\frac{dTY}{dt}$ to become negative and cause TY to start to fall.

Thus, $\frac{dTY}{dt}=16-0.4t=0$

$t=40$

Therefore a per-unit tax of \$40 will maximize the tax yield.

Application activity 5.5

A market faces the demand schedule $p=58-\frac{q}{2}$ and the cost schedule $TC=97q-17\frac{q^2}{2}+\frac{q^3}{3}$

Which price should it adopt to maximize profit and what will be this maximum profit? (All costs and prices are in Rwandan Francs).

5.6 END UNIT ASSESSMENT

1. By definition calculate the derivative of $f(x) = x - 2$

2. A baseball team plays in a stadium that holds 55,000 spectators. With ticket prices at 10 dollars, the average attendance had been 27,000. When ticket prices were lowered to 8 dollars, the average attendance rose to 33,000.

   (a) Find the demand function, assuming that it is linear.

   (b) How should ticket prices be set to maximize revenue?

3. The given $f(x) = 2x^3 - 5x^2 + 4x + 2$

   a) Calculate extreme points;

   b) Calculate first and second derivative;

   c) Show the interval of increasing and decreasing of the given function;

   d) Show where concave is up or down.

UNIT: 6

DESCRIPTIVE STATISTICS

Key Unit competence: **Analyse and interpret statistical data from daily life situations**

6.0. Introductory Activity

1) 1. At the market a fruit-seller has the following daily sales Rwandan francs for five consecutive days: 1000Frw, 1200Frw, 125Frw, 1000Frw, and 1300Frw. Help her to determine the money she could get if the sales are equally distributed per day to get the same total amount of money.

2) During the welcome test of Mathematics for the first term 10 student-teachers of year one language education scored the following marks out of 10 : 3, 5,6,3,8,7,8,4,8 and 6.

   a) What is the mean mark of the class?

   b) Chose the mark that was obtained by many students.

   c) Compare the differences between the mean of the group and the mark for every student teacher.

6.1 Definition and type of data

Activity 6.1

Carry out research on statistics to determine the meanings of statistics and types of data. Use your findings to select qualitative and quantitative data from this list: Male, female, tall, age, 20 sticks, 45 student-teachers, and 20 meters, 4 pieces of chalk.

CONTENT SUMMARY

Statistics is the branch of mathematics that **deals with** data collection, data organization, summarization, analysis and draws conclusions from data.

The use of graphs, charts, and tables and the calculation of various statistical measures to organize and summarize information is called descriptive statistics. Descriptive statistics helps to reduce our information to a manageable size and put it into focus.

Every day, we come across a wide variety of information in form of facts, numerical figures or table groups. A **variable** is a characteristic or attribute that can assume different values. **Data** are the values (measurements or observations) that the variables can assume.

Variables whose values are determined by chance are called **random variables.** A collection of data values forms a **data set.** Each value in the data set is called a **data value** or a **datum.**

For example information related to profit/ loss of the school, attendance of students and tutors, used materials, school expenditure in term or year, etc. These facts or figure which is numerical or otherwise, collected with a definite purpose is called data. This is the word derived from Latin word Datum which means pieces of information.

Qualitative variable

The qualitative variables are variables that cannot be expressed using a number. A qualitative data is determined when the description of the characteristic of interest results is a non-numerical value. A qualitative variable may be classified into two or more categories. Data obtained by observing values of a qualitative variable are referred to as **qualitative data**.

Examples

Qualitative variable	Possible categories of data
Marital status	Single, married, divorced
Gender	Male, Female
Pain level	None, moderated, severe
Colour	Red, black, green, yellow

The possible categories for qualitative variables are often coded for the purpose of performing computerized statistical analysis so that we have qualitative data.

Quantitative variable

Quantitative variables are variables that are expressed in numerical terms, counted or compared on a scale. A quantitative data is determined when the description of the characteristic of interest results in numerical value. When measurement is required to describe the characteristic of interest or when it is necessary to perform a count to describe the characteristic, a quantitative variable is defined.

Discrete data is a quantitative data whose values are countable. Discrete data usually result from counting.

Continuous data is a quantitative data that can assume any numerical value over an interval or over several intervals. Continuous data usually results from making a measurement of some type. Data obtained by observing values of a quantitative variable are referred to as **quantitative data**. Quantitative data obtained from a discrete variable are also referred to as discrete data and quantitative data obtained from a continuous variable are called continuous data.

Examples

Discrete variable	Possible values of data
The number of defective needles in boxes of 100 diabetic syringes	0,1,2,.................100
The number of individuals in groups of 30 with a type **A** personality	0,1,2,3,...............30
Number of student-teachers in classroom	0,1,2,..................45

Continuous variable	Possible values of data
The household income for households with incomes less than or equal to 200,000 Rwandan francs	All the real numbers between A and 200,000 where A is the smallest household income in the population
Height	All values for the length in length measurements.

Data can be used in different ways. The body of knowledge called statistics is sometimes divided into two main areas, depending on how data are used. The two areas are descriptive statistics and inferential statistics.

Descriptive statistics consists of the collection, organization, summarization, and presentation of data.

Inferential statistics consists of generalizing from samples to populations, performing estimations and hypothesis tests, determining relationships among variables, and making predictions.

A **population** consists of all subjects (human or otherwise) that are being studied.

A **sample** is a group of subjects selected from a population.

Application activity 6.1

1) Select qualitative and quantitative data from the list below:

   Product rating, basketball team classification, number of student-teachers in the classroom, weight, age, number of rooms in a house, number of tutors in school.

2) The table below gives the fasting blood sugar reading for five patients at a small medical clinic. What is the variable? Are they continuous? Compare this data and give your observations.

Patient	Fasting blood sugar level
1st Patient	125
2nd patient	175
3rd patient	160
4th patient	110

6.2 Data presentation or organization

Activity 6.2

1) At the beginning of the school year student-teachers came to school with all requirements materials and others don't attend the school on first day. The table below shows the number of student-teachers who attended the school in 5 classrooms on first day.

Number of student-teachers	4	5	6	7	8
Classroom	A	B	C	D	E

Present the above data on bar chart and discuss what you can say when interpreting the chart.

2) One tutor of TTC wants to check the level of how his/her student-teachers like different subjects taught in Language Education option. The survey is done on 60 student-teachers. Here is the number of participants of the survey.

Subject	Number of students
English	12
Mathematics	24
French	6
Entrepreneurship	10
Kinyarwanda	8

Present these data on a pie chart. Using that chart, explain the data.

CONTENT SUMMARY

After the collection of data, the researcher needs to organize and present them in order to help those who will benefit from the research and lead him or her to the conclusion. When the data are in original form, they are called **raw data** and are listed next.

Raw data

After the collection of data, one can present them in the raw form presentation.

Example

The following are numbers of notebooks for 54 student-teachers.

3 4 5 6 6 7 3 2 5 4 5 7 4 3 2 3 6 5

8 5 6 4 2 6 5 3 4 7 9 8 6 7 4 5 4 6

3 4 3 6 7 8 2 7 6 5 4 6 4 7 8 9 9 5

Since little information can be obtained from looking at raw data, the researcher organizes the data into what is called a *frequency distribution.* A frequency distribution consists of the number of times each values appears in the raw data and sometimes the corresponding percentage vis-à-vis the total number in the sample.

1. Frequency distribution

Data can be presented in various forms depending on the type of data collected. A **frequency distribution** is the organization of raw data in the table form by the use of frequencies.

Example

The following data represent marks obtained by 12 student-teachers out of 20 in mathematics test of a certain TTC.

13	10	15	17	17	18
17	17	11	10	17	10

The set of outcomes is displayed in a frequency table, as illustrated below:

Marks	Tallies	Frequencies (fi)
10	||	2
11	|	1
13	|	1
15	||	2
17	~~||||~~	5
18	|	1
Total		**12**

The total frequency is the number of items in the population. In the survey about the marks of student-teachers above, the total frequency is the number of all student-teachers. We find this by adding up the numbers from the third column of the table: $2+1+1+2+5+1$ which is equal to 12.

Grouped data

When the number of data is too large, a simple distribution is not appropriate. In this case we come up with a grouped frequency distribution. A grouped frequency distribution organizes data into **groups or classes.**

Definitions related to grouped frequency distribution

a) **Class limits:** The class limits are the lower and upper values of the class

b) **Lower class limit:** Lower class limit represents the smallest data value that can be included in the class.

c) **Upper class limit:** Upper class limit represents the largest data value that can be included in the class.

d) **Class mark or class midpoint:**

$$class\,mid\;po\,\text{int} = \frac{Lower\ class + upper\ class}{2}$$

Example:

The following data represent the marks obtained by 40 students in Mathematics test. Organize the data in the frequency table; grouping the values into classes, stating from 41-50:

54 83 67 71 80 65 70 73 45 60 72 82 79 78 65 54 67 64 54 76 45 63 49 52 60 70 81 67 45 58 v69 53 65 43 55 68 49 61 75 52.

Solution:

Classes	Class midpoint x	Frequency f
41-50	45.5	6
51-60	55.5	10
61-70	65.5	13
71-80	75.5	8
81-90	85.5	3

The classes: 41-50, 51-60, 71-80, 81-90

Lower class limits: 41, 51, 61, 71, and 81

Upper class limits: 50, 60, 70, 80, and 90

$$Class\ midpoint\ for\ the\ first\ class = \frac{\mathbf{41+50}}{\mathbf{2}} = \mathbf{45.5}$$

Class boundaries

Class boundaries are the midpoints between the **upper class limit** of a class and **the lower class limit of the next class.**

Therefore each class has a lower and an upper class boundary.

Example

Classes	Class midpoint	Frequency
[5,10[	7.5	2
[10,15[	12.5	6
[15,20[	17.5	4
[2,25[	22.5	3
[25,30[	27.5	4
[30,35[	32.5	1

For [10,15[the lower class boundary is 10, The upper class boundary is 15

$$\textit{Class width} = 15 - 10 = 5$$

We have **Categorical Frequency Distributions and Grouped Frequency Distributions**

The **categorical frequency distribution** is used for data that can be placed in specific categories, such as nominal- or ordinal-level data.

Example:

Data such as political affiliation, religious affiliation, or major field of study would use categorical frequency distributions.

Example:

Twenty-five army inductees were given a blood test to determine their blood type.

The data set is:

A	B	B	AB	O
O	O	B	AB	B
B	B	O	A	O
A	O	O	O	AB
AB	A	O	B	A

A Class	B Tally	C Frequency	D Percent
A	$\cancel{||||}$	5	20
B	$\cancel{||||}$ //	7	28
O	$\cancel{||||}$ ////	9	36
AB	////	4	16
		Total 25	100

Grouped Frequency Distributions

When the range of the data is large, the data must be grouped into classes that are more than one unit in width, in what is called a **grouped frequency distribution.** For example, a distribution of the number of hours that boat batteries lasted is the following

Class limits	Class boundaries	Tally	Frequency
24–30	23.5–30.5	///	3
31–37	30.5–37.5	/	1
38–44	37.5–44.5	~~////~~	5
45–51	44.5–51.5	~~////~~ ////	9
52–58	51.5–58.5	~~////~~ /	6
59–65	58.5–65.5	/	1
			25

The procedure for constructing a grouped frequency distribution, i.e., when the classes contain more than one data value

These data represent the record high temperatures in degrees Fahrenheit (*F) for each of the 50 states. Construct a grouped frequency distribution for the data using 7 classes.

112	100	127	120	134	118	105	110	109	112
110	118	117	116	118	122	114	114	105	109
107	112	114	115	118	117	118	122	106	110
116	108	110	121	113	120	119	111	104	111
120	113	120	117	105	110	118	112	114	114

Source: *The World Almanac and Book of Facts.*

1) Determine the classes
   - Find the highest value and lowest value: H =134 and L=100.
   - Find the range: R = highest value-lowest value H -L,

     so R=134 -100 = 34
   - Select the number of classes desired (usually between 5 and 20). In this case, 7 is arbitrarily chosen.
   - Find the class width by dividing the range by the number of classes

$$width = \frac{R}{Number\ of\ classes} = \frac{34}{7} = 4.9$$

   - Select a starting point for the lowest class limit. This can be the smallest data value or any convenient number less than the smallest data value. In this case, 100 is used. Add the width to the lowest score taken as the starting point to get the lower limit of the next class. Keep adding until there are 7 classes, as shown, 100, 105, 110, etc.

- Subtract one unit from the lower limit of the second class to get the upper limit of the first class. Then add the width to each upper limit to get all the upper limits. 105-1=104.

  The first class is 100–104, the second class is 105–109, etc.

- Find the class boundaries by subtracting 0.5 from each lower class limit and adding 0.5 to each upper class limit:

  99.5–104.5, 104.5–109.5, etc.

2) Tally the data

3) Find the numerical frequencies from the tallies.

   The completed frequency distribution is

Class limits	Class boundaries	Tally	Frequency
100–104	99.5–104.5	//	2
105–109	104.5–109.5	~~////~~ ///	8
110–114	109.5–114.5	~~////~~ ~~////~~ ~~////~~ ///	18
115–119	114.5–119.5	~~////~~ ~~////~~ ///	13
120–124	119.5–124.5	~~////~~ //	7
125–129	124.5–129.5	/	1
130–134	129.5–134.5	/	1
			$n = \Sigma f = 50$

2. Cumulative frequency

The cumulative frequency corresponding to a particular value is the sum of all frequencies up to the last value including the first value. Cumulative frequency can also defined as the sum of all previous frequencies up to the current point.

Example:

The set of data below shows marks obtained by student-teachers in Mathematics. Draw a cumulative table for the data.

11	15	18	15	10	16	11	10	17
13	17	11	17	16	17	15	13	16

Solution:

The cumulative frequency at a certain point is found by adding the frequency at the present point to the cumulative frequency of the previous point.

Marks	Frequencies (fi)	Cumulative frequencies (cufi)
10	2	2
11	3	$2+3=5$
13	2	$5+2=7$
15	3	$7+3=10$
16	3	$10+3=13$
17	4	$13+4=17$
18	1	$17+1=18$

3. Relative frequency and percentage

The relative frequency is obtained by dividing the frequency for every data by the sum of all the frequencies. The percentage for a category is obtained by multiplying the relative frequency for that category by 100.

Example:

Marks	Frequencies (fi)	Relative frequency	Percentage
10	2	$2/18=0.111$	11.1
11	3	$3/18=0.167$	16.7
13	2	$2/18=0.111$	11.1
15	3	$3/18=0.167$	16.7
16	3	$3/18=0.167$	16.7
17	4	$4/18=0.222$	22.2
18	1	$1/18=0.55$	5.5

4. Histograms, Frequency Polygons, and Ogives

After you have organized the data into a frequency distribution, you can present them in graphical form. The purpose of graphs in statistics is to convey the data to the viewers in pictorial form. It is easier for most people to comprehend the meaning of data presented graphically than data presented numerically in tables or frequency distributions.

The three most commonly used graphs in research are

1) The histogram.
2) The frequency polygon.
3) The cumulative frequency graph or ogive (pronounced o-jive).

a) The Histogram

The **histogram** is a graph that displays the data by using contiguous vertical bars (unless the frequency of a class is 0) of various heights to represent the frequencies of the classes.

Example:

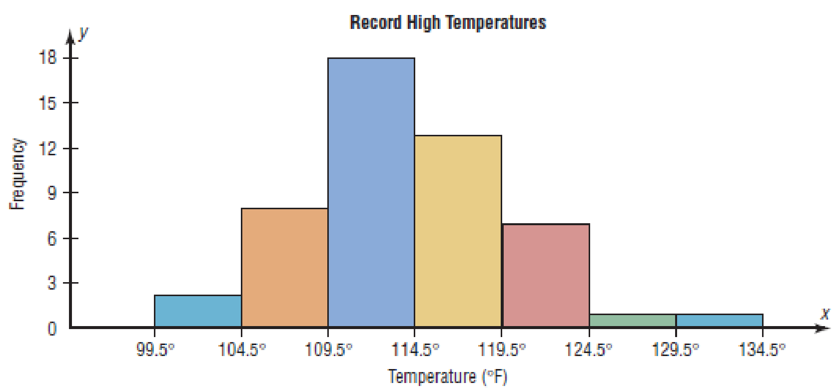
Construct a histogram to represent the data shown for the record high temperatures for each of the 50 states.

Class boundaries	Frequency
99.5–104.5	2
104.5–109.5	8
109.5–114.5	18
114.5–119.5	13
119.5–124.5	7
124.5–129.5	1
129.5–134.5	1

Step 1: Draw and label the x and y axes. The x axis is always the horizontal axis, and the y axis is always the vertical axis.

Step 2: Represent the frequency on the y axis and the class boundaries on the x axis.

Step 3: Using the frequencies as the heights, draw vertical bars for each class. See Figure below



b)The Frequency Polygon

The **frequency polygon** is a graph that displays the data by using lines that connect points plotted for the frequencies at the midpoints of the classes. The frequencies are represented by the heights of the points.

Example:

Using the frequency distribution given in Example 2–4, construct a frequency polygon

Step 1 Find the midpoints of each class. Recall that midpoints are found by adding the upper and lower boundaries and dividing by 2:

$$\frac{99.5+104.5}{2}=102, \quad \frac{104.5+109.5}{2}=107$$

and so on.

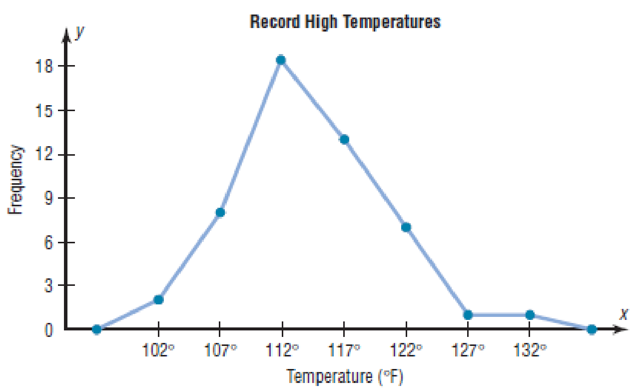
The midpoints are:

Class boundaries	Midpoints	Frequency
99.5–104.5	102	2
104.5–109.5	107	8
109.5–114.5	112	18
114.5–119.5	117	13
119.5–124.5	122	7
124.5–129.5	127	1
129.5–134.5	132	1

Step 2 Draw the x and y axes. Label the x axis with the midpoint of each class, and then use a suitable scale on the y axis for the frequencies.

Step 3 Using the midpoints for the x values and the frequencies as the y values, plot the points.

Step 4 Connect adjacent points with line segments. Draw a line back to the x axis at the beginning and end of the graph, at the same distance that the previous and next midpoints would be located, as shown in Figure 2–3.



The frequency polygon and the histogram are two different ways to represent the same data set. The choice of which one to use is left to the discretion of the researcher.

c) The Ogive

The **ogive** is a graph that represents the cumulative frequencies for the classes in a frequency distribution.

Step 1: Find the cumulative frequency for each class.

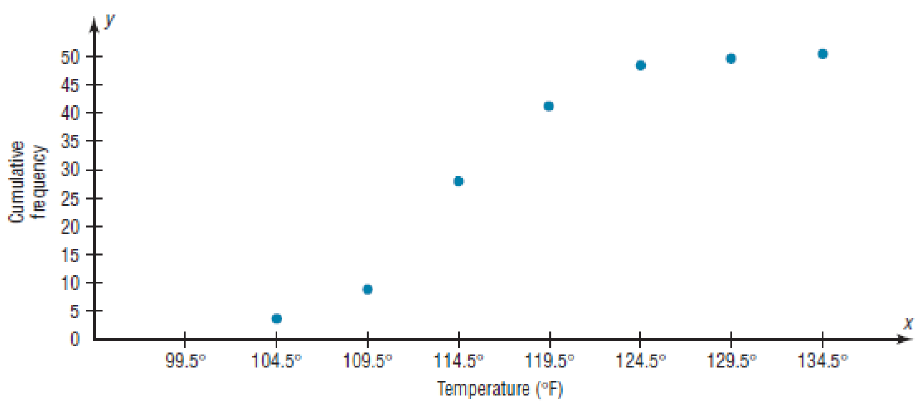
	Cumulative frequency
Less than 99.5	0
Less than 104.5	2
Less than 109.5	10
Less than 114.5	28
Less than 119.5	41
Less than 124.5	48
Less than 129.5	49
Less than 134.5	50

Step 2: Draw the *x* and *y* axes. Label the *x* axis with the class boundaries. Use an appropriate scale for the *y* axis to represent the cumulative frequencies.

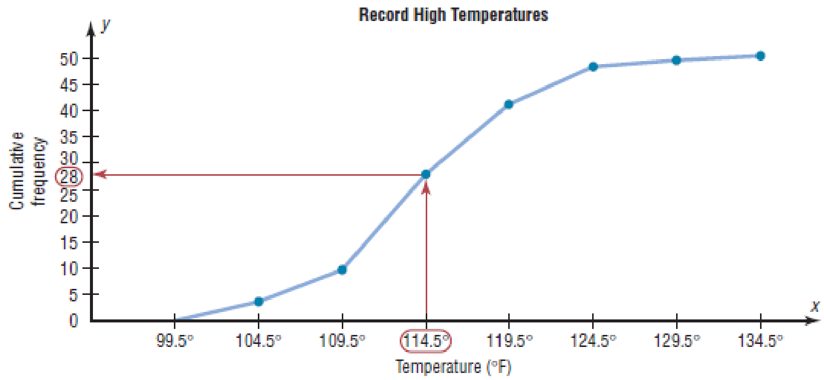
(Depending on the numbers in the cumulative frequency columns, scales such as 0, 1, 2, 3, . . . , or 5, 10, 15, 20, . . . , or 1000, 2000, 3000, . . . can be used.

Do *not* label the *y* axis with the numbers in the cumulative frequency column.) In this example, a scale of 0, 5, 10, 15, . . . will be used.

Step 3 Plot the cumulative frequency at each upper class boundary, as shown in Figure below. Upper boundaries are used since the cumulative frequencies represent the number of data values accumulated up to the upper boundary of each class.



Step 4 Starting with the first upper class boundary, 104.5, connect adjacent points with line segments, as shown in the figure. Then extend the graph to the first lower class boundary, 99.5, on the *x* axis.



Cumulative frequency graphs are used to visually represent how many values are below a certain upper class boundary. For example, to find out how many record high temperatures are less than 114.5_F, locate 114.5_F on the *x* axis, draw a vertical line up until it intersects the graph, and then draw a horizontal line at that point to the *y* axis. The *y* axis value is 28, as shown in the figure.

5. Relative Frequency Graphs

The histogram, the frequency polygon, and the ogive shown previously were constructed by using frequencies in terms of the raw data. These distributions can be converted to distributions using *proportions* instead of raw data as frequencies. These types of graphs are called **relative frequency graphs.**

Example:

Construct a histogram, frequency polygon, and ogive using relative frequencies for the distribution (shown here) of the kilometers that 20 randomly selected runners ran during a given week.

Class boundaries	Frequency
5.5–10.5	1
10.5–15.5	2
15.5–20.5	3
20.5–25.5	5
25.5–30.5	4
30.5–35.5	3
35.5–40.5	2
	20

Solution:

Step 1: Convert each frequency to a proportion or relative frequency by dividing the frequency for each class by the total number of observations.

For class 5.5–10.5, the relative frequency is $\frac{1}{20} = 0.05$; for class 10.5–15.5, the relative frequency is $\frac{2}{20} = 0.10$; for class 15.5–20.5, the relative frequency is $\frac{3}{20} = 0.15$; and so on.

Place these values in the column labeled Relative frequency.

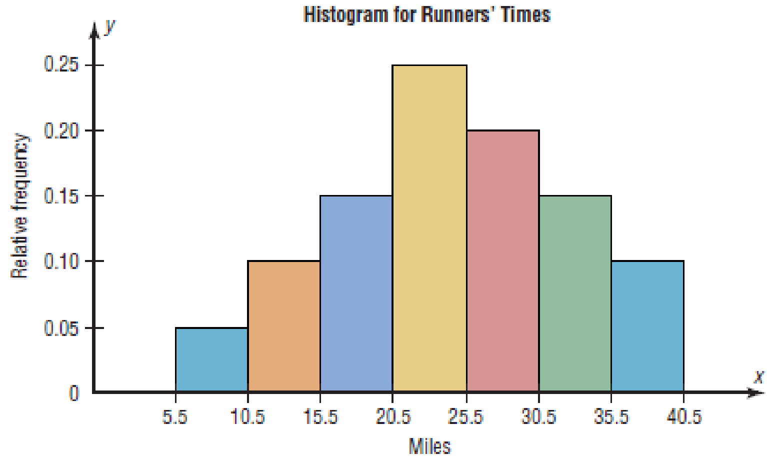
Class boundaries	Midpoints	Relative frequency
5.5–10.5	8	0.05
10.5–15.5	13	0.10
15.5–20.5	18	0.15
20.5–25.5	23	0.25
25.5–30.5	28	0.20
30.5–35.5	33	0.15
35.5–40.5	38	0.10
		1.00

Step 2: Find the cumulative relative frequencies. To do this, add the frequency in each class to the total frequency of the preceding class. In this case, 0 +0.05 = 0.05, 0.05 + 0.10 = 0.15, 0.15 +0.15 =0.30 , 0.30 + 0.25 = 0.55, etc. Place these values in the column labeled Cumulative relative frequency.

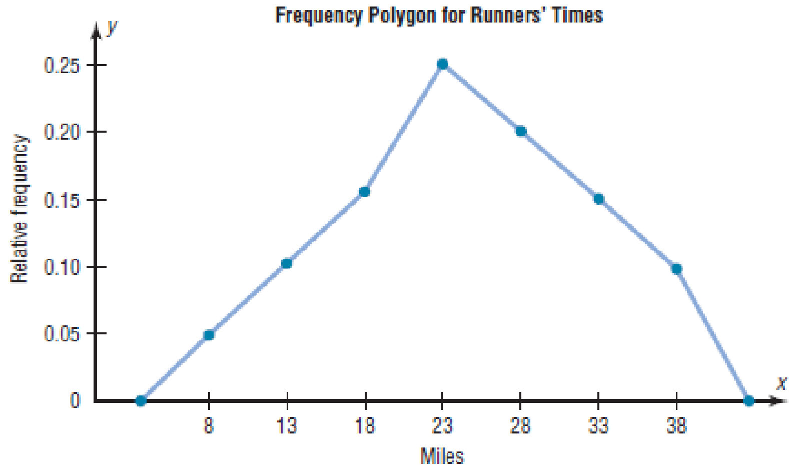
An alternative method would be to find the cumulative frequencies and then convert each one to a relative frequency.

	Cumulative frequency	Cumulative relative frequency
Less than 5.5	0	0.00
Less than 10.5	1	0.05
Less than 15.5	3	0.15
Less than 20.5	6	0.30
Less than 25.5	11	0.55
Less than 30.5	15	0.75
Less than 35.5	18	0.90
Less than 40.5	20	1.00

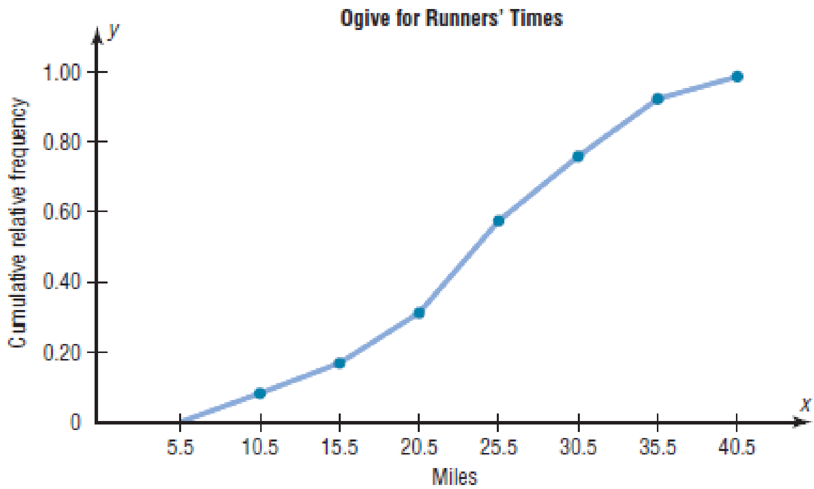
Step 3: Draw each graph as shown in Figure 2–7. For the histogram and ogive, use the class boundaries along the *x* axis. For the frequency polygon, use the midpoints on the *x* axis. The scale on the *y* axis uses proportions.



(a) Histogram



(b) Frequency polygon



(c) Ogive

6. Pie chart

A pie chart is used to display a set of categorical data. It is a circle that is divided into sections or wedges according to the percentage of frequencies in each category of the distribution.

$$Angle\ for\ sector\ S = \frac{Frequency\ of\ S . 360^0}{Total\ frequency}$$

Since there are 360^0 in a circle, the frequency for each class must be converted into a proportional part of the circle. This conversion is done by using the formula

$$Degrees = \frac{f}{n}.360^0$$

where f is frequency for each class and n is the sum of the frequencies.

Hence, the following conversions are obtained. The degrees should sum to 360.

Each frequency must also be converted to a percentage by using the formula

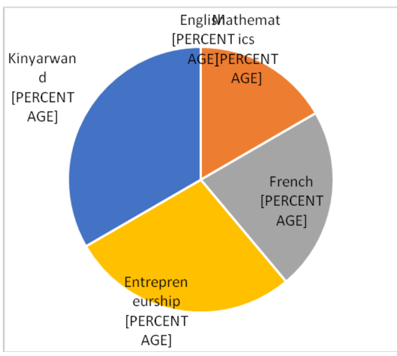
$$\% = \frac{f}{n}.100\%$$

Next, using a protractor and a compass, draw the graph using the appropriate degree measures found in step 1, and label each section with the name and percentages.

Example:

1) One tutor in TTC want to check the level of how his/her student-teachers like different subject taught in Language Education option. The survey done on 60 student-teachers in English, Mathematics, French, Entrepreneurship and Kinyarwanda.

   Here is the results he/she obtained respectively after making survey 12, 24, 6, 10, and 8. Present the data on pie chart.



2) Construct a pie graph showing the blood types of the army inductees described above. The frequency distribution is repeated here.

Class	Frequency	Percent
A	5	20
B	7	28
O	9	36
AB	4	16
	25	100

Solution:

Step 1: Find the number of degrees for each class, using the formula.

$$Degrees = \frac{f}{n}.360^0$$

For each class then the following results are obtained:

$A: \frac{5}{25}.360^0 = 72^0;$ B: $\frac{7}{25}.360^0 = 100.8^0;$

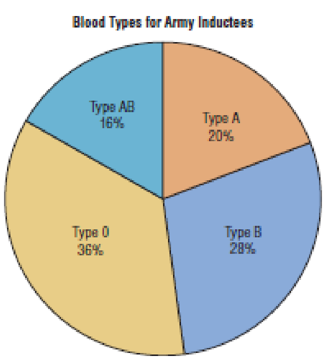
O: $\frac{9}{25}.360^0 = 129.6^0;$ $AB: \frac{4}{25}.360^0 = 57.6^0$

Step 2: Find the percentages.

A: $\frac{5}{25} \times 100\% = 20\%$ **B:** $\frac{7}{25} \times 100\% = 28\%$

O: $\frac{9}{25} \times 100\% = 36\%$ **AB:** $\frac{4}{25} \times 100\% = 16\%$

Step 3: Using a protractor, graph each section and write its name and corresponding percentage, as shown in the Figure



7. Stem and Leaf Plots

The stem and leaf plot is a method of organizing data and is a combination of sorting and graphing. It uses part of the data value as the stem and part of the data value as the leaf to form groups or classes. It has the advantage over a grouped frequency distribution of retaining the actual data while showing them in graphical form.

Examples:

1) At an outpatient testing center, the number of cardiograms performed each day for 20 days is shown. Construct a stem and leaf plot for the data.

25	31	20	32	13
14	43	02	57	23
36	32	33	32	44
32	52	44	51	45

Step 1: Arrange the data in order: 02, 13, 14, 20, 23, 25, 31, 32, 32, 32, 32, 33, 36, 43, 44, 44, 45, 51, 52, 57

***Note**:* Arranging the data in order is not essential and can be cumbersome when the data set is large; however, it is helpful in constructing a stem and leaf plot. The leaves in the final stem and leaf plot should be arranged in order.

Step 2 Separate the data according to the first digit, as shown.

02 13, 14 20, 23, 25 31, 32, 32, 32, 32, 33, 36, 43, 44, 44, 45 51, 52, 57

Step 3: A display can be made by using the leading digit as the *stem* and the trailing digit as the *leaf*.

For **example**, for the value 32, the leading digit, 3, is the stem and the trailing digit, 2, is the leaf. For the value 14, the 1 is the stem and the 4 is the leaf. Now a plot can be constructed as follows:

Leading digit (stem)	Trailing digit (leaf)
0	2
1	3 4
2	0 3 5
3	1 2 2 2 2 3 6
4	3 4 4 5
5	1 2 7

It shows that the distribution peaks in the center and that there are no gaps in the data. For 7 of the 20 days, the number of patients receiving cardiograms was between 31 and 36. The plot also shows that the testing center treated from a minimum of 2 patients to a maximum of 57 patients in any one day.

If there are no data values in a class, you should write the stem number and leave the leaf row blank. Do not put a zero in the leaf row.

2) The mathematical competence scores of 10 student-teachers participating in mathematics competition are as follows: 15, 16, 21, 23, 23, 26, 26, 30, 32, 41. Construct a stem and leaf display for these data by using 2**,** 3, and 4 as your stems.

Solution

Stem	Leaf
1	5 6
2	1 3 3 6 6
3	0 2
4	1

This means that data are concentrated in twenties.

3) The following are results obtained by student-teachers in French out of 50.

37, 33, 33, 32, 29, 28, 28, 23, 22, 22, 22, 21, 21, 21, 20,

20, 19, 19, 18, 18, 18, 18, 16, 15, 14, 14, 14, 12, 12, 9, 6

Use stem and leaf to display data

Solution:

Numbers 3, 2, 1, and 0, arranged as a stems to the left of the bars. The other numbers come in the leaf part.

Stem	Leaf
3	2 3 3 7
2	0 0 1 1 1 2 2 2 3 8 8 9
1	2 2 4 4 4 5 6 8 8 8 8 9 9
0	6 9

From the table, we see that data are concentrated in tens and twenties.

Application activity 6.2

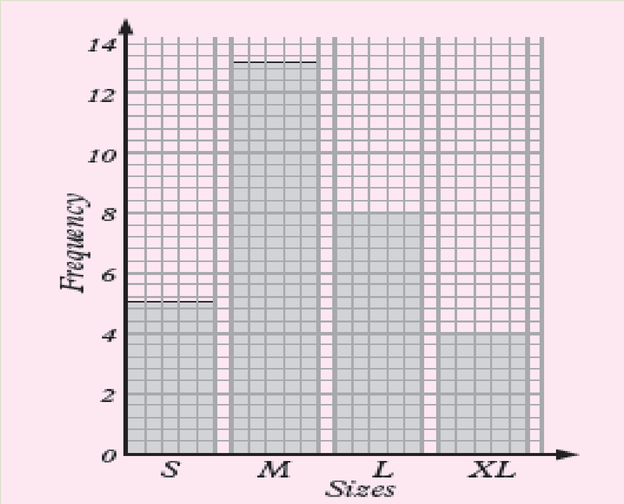
1) Suppose that a tutor conducted a test for student-teachers and the marks out of 10 were as follows: 3 3 3 5 6 4 6 7 8 3 8 8 8 10 9 10 9 10 8 10 6
   a) Draw a frequency table;
   b) Draw a relative frequency table and calculate percentage for each;
   c) Present data in cumulative frequency table, hence show the number of student-teachers who did the test.
2) During the examination of English student-teacher got the following results out of 80: 54, 42, 61, 47, 24, 43, 55, 62, 30, 27, 28, 43, 54, 46, 25, 32, 49, 73, 50, 45.

   Present the results using stem and leaf.
3) A firm making artificial sand sold its products in four cities: 5% was sold in Huye, 15% in Musanze, 15% in Kayonza and 65% was sold in Rwamagana.
   a) What would be the angles on pie chart?
   b) Draw a pie chart to represent this information.
   c) Use the pie chart to comment on these findings.

6.3 Graph interpretation and Interpretation of statistical data

Activity 6.3

The graph below shows the sizes of sweaters worn by 30 year 1 students in a certain school. Observe it and interpret it by answering the questions below it:



a) How many students are with small size?

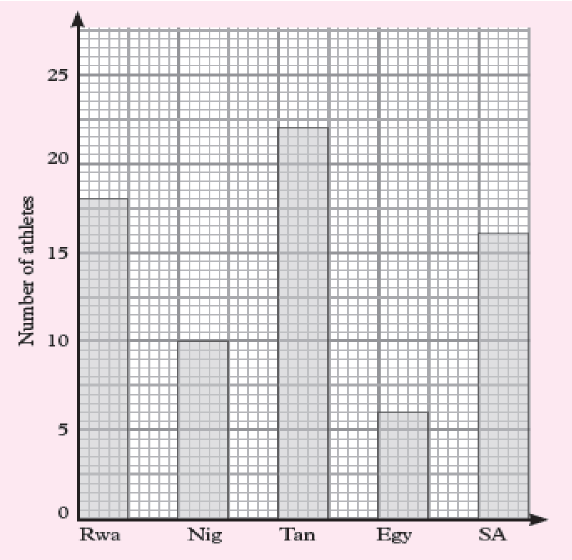
b) How many students with medium size, large size and extra large size are there?

CONTENT SUMMARY

Once data has been collected, they may be presented or displayed in various ways including graphs. Such displays make it easier to interpret and compare the data.

Examples

1) The bar graph shows the number of athletes who represented five African countries in an international championship.



a) What was the total number of athletes representing the five countries?

b) What was the smallest number of athletes representing one country?

c) What was the most number of athletes representing a country?

d) Represent the information on the graph on a frequency table.

Solution:

We read the data on the graph:

a) Total number of athletes are: 18 + 10 + 22 + 6 + 16 = 72 athletes

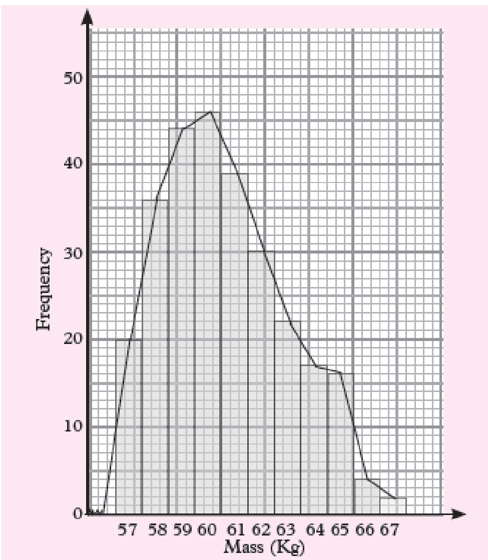
b) 6 athletes

c) 22 athletes

d) Representation of the given information on the graph on a frequency table.

Country	**Number of athletes**
Rwanda	18
Nigeria	10
Tanzania	22
Egypt	6
South Africa	16
Total	**72**

2) Use a scale vertical scale 2cm: 10 students and Horizontal scale 2cm: 10 represented on histogram below to answers the questions that follows



a) Estimate the mode

b) Calculate the range

Solution:

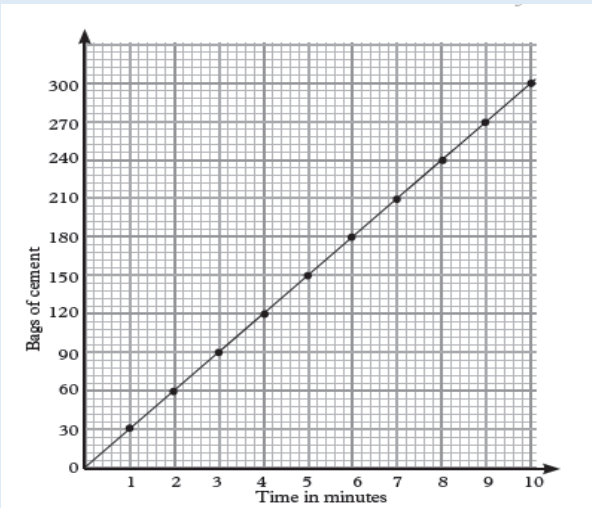
a) To estimate the mode graphically, we identify the bar that represents the highest frequency. The mass with the highest frequency is 60 kg. It represents the mode.

b) The highest mass = 67 kg and the lowest mass = 57 kg

Then, The range=highest mass-lowest mass=$67kg - 57kg = 10kg$

Application activity 6.3

The line graph below shows bags of cement produced by CIMERWA industry cement factory in a minute.



a) Find how many bags of cement will be produced in: 8 minutes, 3 minutes12 seconds, 5 minutes and 7 minutes.

b) Calculate how long it will take to produce: 78 bags of cement.

c) Draw a frequency table to show the number of bags produced and the time taken.

6.4 Measures of central tendencies for ungrouped data

Activity 6.4

Conduct a research in the library or on the internet and explain measures of central tendency, their types and provide examples.

Insist on explaining how to determine the Mean, Mode, Median and their role when interpreting statistic data.

CONTENT SUMMARY

Measures of central tendency were studies in S1 and S2.

1. The mean

The *mean,* also known as the *arithmetic average,* is found by adding the values of the data and dividing by the total number of values.

Suppose that a fruit seller earned the flowing money from Monday to Friday respectively: 300, 200, 600, 500, and 400 Rwandan francs. The mean of this money explains the same daily amount of money that she should earn to totalize the same amount in 5 days.

$$\bar{X} = \frac{X_1 + X_2 + X_3 + \dots X_n}{n} = \frac{\sum X}{n} = \frac{300+200+600+500+400}{5} = 400$$

Or $\bar{x} = \frac{1}{n}\sum xfi$

This is called the mean and is equivalent to sharing out all data evenly.

2. The median:

If the data is well arranged in an order from the smallest to the largest, the median is the middle number or the central number of the range.

When total observation ($\sum fi = n$) is odd the median is given

by $\left(\frac{n+1}{2}\right)^{th}$ number which is located on this position. On the other side

when n is even, $\frac{1}{2}\left[\left(\frac{n}{2}\right)^{th} + \left(\frac{n}{2}+1\right)^{th}\right]$, then the median is a half of the sum

of number located on those two positions.

Examples

1) Calculate the median of the following numbers: 4, 5, 7,2, 1

Solution:

Arrange data from lowest to highest number as 1, 2, 4, 5, 7

$$Me = \left(\frac{n+1}{2}\right)^{th}$$

$$Me = \left(\frac{5+1}{2}\right)^{th} = 3^{rd} \ position, Then\ Me = 4$$

2) Calculate the median of the following numbers: 4, 5, 7, 2, 1 and 8

Solution:

Arrange numbers in ascending order: 1, 2, 4, 5, 7, 8. Total observation $(n) = 6$ since the total observation is even then, the position of

$$Me = \left(\frac{n}{2}\right)^{th} + \left(\frac{n}{2}+1\right)^{th}.$$

The position of $Me = \left(\frac{6}{2}\right)^{th} + \left(\frac{6}{2}+1\right)^{th} = (3)^{rd} + (4)^{th}$

$$Me = \frac{4+5}{2} = 4.5$$

3. The mode:

The mode is the number that appears the most often from the set of data. It represents the value which appears more frequently in the data.

Examples:

Calculate the mean and mode of the following set of numbers: 3, 4, 4, 6, 8, 5, 4, and 8

Solution:

$$\bar{x} = \frac{1}{n}\sum xfi$$

x	fi	xfi
3	1	3
4	3	12
5	1	5
6	1	6
8	2	16
	$\sum fi = n = 8$	$\sum xfi = 42$

The mean is given by $\bar{x} = \frac{1}{n}\sum xfi$, $\bar{x} = \frac{42}{8} = 5.25$

The median: Arrange data first 3, 4, 4, 4, 5, 6, 8,8. Total observation is 8,

Mode is 4

Application activity 6.4

1) A group of student-teachers from language education were asked how many books they had read in previous year, the results are shown in the frequency table below. Calculate the mean, median and mode of the number books read.

Number of books	0	1	2	3	4	5	6	7	8
Frequency (number of student teachers)	5	5	6	9	11	7	4	2	1

2) During oral presentation of internship report for year three student-teachers the first 10 student-teachers scored the following marks out of 10:

   8, 7, 9, 10, 8, 9, 8, 6, 7 and 10

   Calculate the mean and the median of the group.

6.5 Measures of central tendencies for grouped data: mode, mean, median and midrange

Activity 6.5

1) Conduct a research in the library or on the internet and explain measures of central tendency for grouped data and provide examples.

   Insist on explaining how to determine the Mean, Mode, Median and their role when interpreting statistic data.

2) Using the frequency distribution given below, find the mean. The data represent the number of kilometers run during one week for a sample of 20 runners.

A Class	B Frequency f	C Midpoint X_m	D $f \cdot X_m$
5.5–10.5	1		
10.5–15.5	2		
15.5–20.5	3		
20.5–25.5	5		
25.5–30.5	4		
30.5–35.5	3		
35.5–40.5	2		
	$n = 20$		

   What does this mean represent considering the class in which it is located in the data?

1. The mean

The process of finding the mean is the same as the one applied in the ungrouped data with the exeption that the midpoints x_m of each class in grouped data plays the role of x_i used in ungrouped data.

$$\bar{X} = \frac{\sum f.X_m}{n}$$

2. The mode

The mode for grouped data is the modal class. The **modal class** is the class with the largest frequency. The mode can be determined using the following formula:

$$Mode = L + \left(\frac{f_m - f_1}{(2f_m - f_1 - f_2)}\right)w$$

Where:

L: the lower limit of the modal class

f_m: the modal frequency

f_1: the frequency of the immediate class below the modal class

f_2: the frequency of the immediate class above the modal class

w: modal class width.

Example:

Find the modal class for the frequency distribution of kilometers that 20 runners ran in one week.

Class	Frequency
5.5–10.5	1
10.5–15.5	2
15.5–20.5	3
20.5–25.5	5 ← Modal class
25.5–30.5	4
30.5–35.5	3
35.5–40.5	2

The modal class is 20.5–25.5, since it has the largest frequency. Sometimes the midpoint of the class is used rather than the boundaries; hence, the mode could also be given as 23 km per week.

3. Median of grouped data

In the case of continuous frequency of distribution, we first locate the median by cumulating the frequency until $(\frac{n}{2})^{th}$ point is reached. Finally, the median is determined within this class by using formula. The procedures thus involve the following steps:

1) Compute cumulative frequencies
2) Find the size of $\left(\frac{n}{2}\right)^{th}$ item, see that $\left(\frac{n+1}{2}\right)^{th}$ is not used in this case. N is the total frequency.
3) Locate the median class in cumulative frequency column where the size of $\left(\frac{n}{2}\right)^{th}$ item falls.

4) Obtain the median value by applying the formula

$$Mediam = l_1 + \frac{\frac{n}{2} - cufi}{fi}(l_2 - l_1)$$

where l_1 is lower limit of the median class, l_2 is upper limit of the median class, fi is the frequency and $cufi$ is the cumulative frequency of the class preceding the median class.

Example

The following table shows the weekly consumption of electricity of 56 families

weekly consumption	0-10	10-20	20-30	30-40	40-50
Number of families	16	12	18	6	4

Calculate the median weekly consumption.

Solution

Weekly consumption	**Frequency**	**Cumulative frequency**
0-10	16	16
10-20	12	28
20-30	18	46
30-40	6	52
40-50	4	56
	Summation=56	

Median is the value of $\frac{n}{2} = \frac{56}{2} = 28^{th}$ position, which lies in the class 10-20.

Thus 10-20 is the median class

$$Mediam = l_1 + \frac{\frac{n}{2} - cufi}{fi}(l_2 - l_1)$$

$$Median = 10 + \frac{28 - 16}{12}(20 - 10)$$

$$Me = 10 + \frac{12}{12} * 10 = 10 + 10 = 20$$

Hence the median is 20 units.

Question

1) Calculate the median for the following distribution

Class interval	130-140	140-150	150-160	160-170	170-180	180-190	190-200
Frequency	5	9	17	28	24	10	7

4. The Midrange

The *midrange* is a rough estimate of the middle. It is found by adding the lowest and highest values in the data set and dividing by 2. It is a very rough estimate of the average and can be affected by one extremely high or low value.

$$MR = \frac{Lowest\ value + highest\ value}{2}$$

5. Weighted mean

It is a mean of a data set in which not all values are equally represented. Find the weighted mean of a variable X by multiplying each value by its corresponding weight and dividing the sum of the products by the sum of the weights.

$$\bar{X} = \frac{w_1X_1 + w_2X_2 + w_3X_3 + \ldots \mathrm{w}_n X_n}{w_1 + w_2 + w_3 + \ldots . + w_n} = \frac{\sum wX}{\sum w}$$

Where $w_1, w_2, \ldots w_n$ are the weights and $X_1, \mathrm{X}_2, \ldots \mathrm{X}_n$ are the values.

Example

A student received an A in English Composition I (3 credits), a C in Introduction to Psychology (3 credits), a B in Biology I (4 credits), and a D in Physical Education (2 credits). Assuming A=4 grade points, B=3 grade points, C =2 grade points, D =1 grade point, and F=0 grade points, find the student's grade point average.

Solution

Course	Credits (w)	Grade (X)
English Composition I	3	A (4 points)
Introduction to Psychology	3	C (2 points)
Biology I	4	B (3 points)
Physical Education	2	D (1 point)

$$\overline{X} = \frac{\Sigma wX}{\Sigma w} = \frac{3 \cdot 4 + 3 \cdot 2 + 4 \cdot 3 + 2 \cdot 1}{3 + 3 + 4 + 2} = \frac{32}{12} = 2.7$$

$$\bar{X} = \frac{\sum wX}{\sum w} = \frac{3.4+3.2+4.3+2.1}{3+3+4+2} = \frac{32}{12} = 2.7$$

Application activity 6.5

The data below shows the marks scored by a group of students in a mathematics out of 100: 72; 63; 51; 25; 31; 49; 51; 27; 46; 42; 25; 39; 38; 39; 55; 38; 35; 64;67;37.

Use a grouped data of 5 intervals and determine:

a) The mean mark

b) The median

c) The modal class

d) The range.

6.6 Measures of dispersion for ungrouped data and for grouped data

Activity 6.6

Before starting the third tern, tutor calculated the mean mark of five student-teachers got in second term in Mathematics and he/she obtained that the mean mark is $\bar{x} = 16.875$. Use this mean to complete the table below:

x	f	$x-\bar{x}$	$(x-\bar{x})^2$	$f(x-\bar{x})^2$
12	4			
13	2			
15	1			
19	4			
21	5			
	$\sum f =$			$\sum f(x-\bar{x})^2 =$

Explain the expression $\sum f\left(x-\bar{x}\right)^2$ in your own words.

CONTENT SUMMARY

The word dispersion has a technical meaning in statistics. The average measures the center of the data. It is one aspect of observations. Another feature of the observations is how the observations are spread about the center. The observation may be close to the center or they may be spread away from the center. If the observation are close to the center (usually the arithmetic mean or median), we say that dispersion, scatter or variation is small. If the observations are spread away from the center, we say that dispersion is large.

The study of dispersion is very important in statistical data. If in a certain factory there is consistence in the wages of workers, the workers will be satisfied. But if some workers have high wages and some have low wages, there will be unrest among the low paid workers and they might go on strikes and arrange demonstrations. If in a certain country some people are very poor and some are very high rich, we say there is economic disparity. It means that dispersion is large.

The extent or degree in which data tend to spread around an average is also called the dispersion or variation. Measures of dispersion help us in studying the extent to which observations are scattered around the average or central value. Such measures are helpful in comparing two or more sets of data with regard to their variability.

Properties of a good measure of dispersion

i) It should be simple to calculate and easy to understand

ii) It should be rigidly defined

iii) Its computation be based on all the observations

iv) It should be amenable to further algebraic treatment

Some measures of dispersion are Quartiles, variance, Range, standard deviation, coefficient of variation.

1. Quartile

A measure which divides any array into four equal parts is known as quartile. Each proportion contains equal number of items. The fist and the second and the third points are termed as first quartile. The first quartile has 25% of the items of the distribution below it and 75% of the items are greater than it. The second quartile which is the median has 50% of the observations above it and 50% of the observations below it. For the arranged data in ascending order and quartiles are calculated as follows:

$Q_1 = \frac{1}{4}(n+1)^{th}$ observation $Q_2 = \frac{1}{2}(n+1)^{th}$ observation $Q_3 = \frac{3}{4}(n+1)^{th}$ observation

The inter-quartile range is given by the difference between third quartile and the first quartile $Q_3 - Q_1$.

Examples

1) Find the first and the second quartiles of the data set: 1, 3, 4, 5, 5, 6, 9, 14, 21.

Solution:

$$Q_1 = \frac{1}{4}(n+1)^{th} = \frac{1}{4}(9+1)^{th} = (2.5)^{th}, \qquad Q_1 = 4$$

$$Q_2 = Me, \; Q_2 = \frac{1}{2}(n+1)^{th} = \frac{1}{2}(9+1)^{th} = 5^{th}, \; Q_2 = 5$$

2) In the series given below, calculate the first, the third quartile and inter-quartile range

   2, 4, 6, 8, 10, 12, 14, 16, 18, 20, 22

Solution: $Q_1 = \frac{1}{4}(n+1)^{th} = \left(\frac{11+1}{4}\right)^{th} = 3^{rd}$ observation, then the first quartile is $Q_1 = 6$

$$Q_3 = \frac{3}{4}(n+1)^{th} = \left(\frac{3}{4}(11+1\right)^{th} = 9^{th} \text{ Observation,}$$

then the third quartile is $Q_3 = 18$

Inter-quartile range $= Q_3 - Q_1 = 18 - 6 = 12$

2. Variance

Variance measures how far a set of numbers is spread out. A variance of zero indicates that all the values are identical. Variance is always non-negative: a small variance indicates that the data points tend to be very close to the mean and hence to each other, while a high variance indicates that the data points are very spread out around the mean and from each other.

The variance is denoted and defined by

$$\sigma^2 = \frac{\sum_{i=1}^{n}\left(x_i - \overline{x}\right)^2}{n}$$

Developing this formula we have

$$\begin{aligned}\sigma^2 &= \frac{\sum_{i=1}^{n}\left(x_i^2 - 2x_i\overline{x} + \left(\overline{x}\right)^2\right)}{n}\\ &= \frac{1}{n}\sum_{i=1}^{n}x_i^2 - \frac{1}{n}2\overline{x}\sum_{i=1}^{n}x_i + \frac{1}{n}\left(\overline{x}\right)^2\sum_{i=1}^{n}1\\ &= \frac{1}{n}\sum_{i=1}^{n}x_i^2 - 2\overline{x}\overline{x} + \left(\overline{x}\right)^2\\ &= \frac{1}{n}\sum_{i=1}^{n}x_i^2 - \left(\overline{x}\right)^2\end{aligned}$$

Thus, the variance is also defined by

$$\sigma^2 = \frac{1}{n}\sum_{i=1}^{n}x_i^2 - \left(\overline{x}\right)^2$$

Recall that the mean of the set of n values $x_1, x_2, x_3, \ldots, x_n$ is denoted and defined by

$$\overline{x} = \frac{x_1 + x_2 + x_3 + \ldots + x_n}{n} \qquad = \sum_{i=1}^{n}\frac{x_i}{n} \qquad = \frac{1}{n}\sum_{i=1}^{n}x_i$$

Examples

1) Calculate the variance of the following distribution: 9, 3, 8, 8, 9, 8, 9, 18

Solution

$$\overline{x} = \frac{9+3+8+8+9+8+9+18}{8} = 9$$

$$\sigma^2 = \frac{(9-9)^2 + (3-9)^2 + (8-9)^2 + (8-9)^2 + (9-9)^2 + (8-9)^2 + (9-9)^2 + (18-9)^2}{8} = 15$$

3. Standard deviation

The standard deviation has the same dimension as the data, and hence is comparable to deviations from the mean. We define the **standard deviation** to be the square root of the variance.

Thus, the standard deviation is denoted and defined by

$$\sigma = \sqrt{\frac{\sum_{i=1}^{n}\left(x_i - \overline{x}\right)^2}{n}} \quad \text{or} \quad \sigma = \sqrt{\frac{1}{n}\sum_{i=1}^{n} x_i^2 - \left(\overline{x}\right)^2}$$

The following results follow directly from the definitions of mean and standard deviation:

- When all the data values are multiplied by a constant a, the new mean and new standard deviation are equal to a times the original mean and standard deviation. That is, the mean of $ax_1, ax_2, ax_3, ..., ax_n$ is $a\left(\overline{x}\right)$ and the standard deviation is $a\sigma$.
- When a constant value, b, is added to all data values, then the new mean is increased by b. However standard deviation does not change. That is, the mean of $ax_1, ax_2, ax_3, ..., ax_n$ is $\overline{x} + b$ and the standard deviation is σ.

Examples

1) The six runners in a 200 meter race clocked times (in seconds) of 24.2, 23.7, 25.0, 23.7, 24.0,24.6.

   a) Find the mean and standard deviation of these times.

Solution

a) $\overline{x} = \dfrac{24.2 + 23.7 + 25.0 + 23.7 + 24.0 + 24.6}{6} = 24.2 \text{ seconds}$

$$\sigma = \sqrt{\frac{(24.2-24.2)^2 + (23.7-24.2)^2 + (25.0-24.2)^2 + (23.7-24.2)^2 + (24.0-24.2)^2 + (24.6-24.2)^2}{6}}$$

$= 0.473 \text{ seconds}$

The method which uses the formula for the standard deviation is not necessarily the most efficient.

Consider the following: $\dfrac{\sum\left(x-\bar{x}\right)^2}{n}=\dfrac{1}{n}\sum x^2-\left(\bar{x}\right)^2$

2) The heights (in meters) of six children are 1.42, 1.35, 1.37, 1.50, 1.38 and 1.30. Calculate the mean height and the standard deviation of the heights.

Solution:

$$\text{Mean}=\frac{1}{6}(1.42+1.35+1.37+1.50+1.38+1.30)=1.39\ \ m$$

$$\text{Variance}=\frac{1}{6}\left(1.42^2+1.35^2+1.37^2+1.50^2+1.38^2+1.30^2\right)-1.39^2=0.00386\ \ m^2$$

$$\text{Standard deviation}=\sqrt{0.00386\ \ m^2}=0.0621\ \ m$$

Variance and Standard Deviation for Grouped Data

The variance for grouped data is given by $\delta^2=\dfrac{\sum\left\{f\left(x-\bar{x}\right)^2\right\}}{\sum f}$ and the standard deviation for grouped data is given by $\delta=\sqrt{\dfrac{\sum\left\{f\left(x-\bar{x}\right)^2\right\}}{\sum f}}$

Example:

The frequency distribution result of Mathematics test on 30 marks of 48 students-teachers is as shown in the table below. Calculate the variance and the standard deviation from the mean marks of results

Class	Frequency
20.5-20.9	3
21.0-21.4	10
21.5-21.9	11
22.0-22.4	13
22.5-22.9	9
23.0-23.4	2
Total	48

Solution:

From the data the distribution mean value $\bar{x} = 21.92$,

The "$x - values$" are the class mid-points values,

i.e 20.7; 21.2; 21.7;22.2; 22.7 and 23.2

Thus, the $\left(x-\bar{x}\right)^2$ values are:

$(20.7-21.92)^2;(21.2-21.92)^2;(21.7-21.92)^2;(22.2-21.92)^2;(22.7-21.92)^2;(23.2-21.92)^2$

and the $f\left(x-\bar{x}\right)^2$ values are:

$3(20.7-21.92)^2;10(21.2-21.92)^2;11(21.7-21.92)^2;13(22.2-21.92)^2;9(22.7-21.92)^2;2(23.2-21.92)^2$

The $\sum f\left(x-\bar{x}\right)^2$ are:

$4.4652+5.1840+0.5324+1.0192+5.4756+3.2768=19.9532$

$\delta^2 = \dfrac{\sum\left\{f\left(x-\bar{x}\right)^2\right\}}{\sum f} = \dfrac{19.9532}{48} = 0.41569$ and the standard deviation is

$\delta = \sqrt{\dfrac{\sum\left\{f\left(x-\bar{x}\right)^2\right\}}{\sum f}} = \sqrt{0.41569} = 0.645$

Note that the standard deviation for the sample data is given by:

$\delta = \sqrt{\dfrac{\sum X^2 - [(\sum X)^2/n}{n-1}}$ and the standard deviation for grouped data is

given by: $\delta = \sqrt{\dfrac{\sum f.X_m{}^2 - [(\sum f.X_m)^2/n]}{n-1}}$

4. Coefficient of variation

The coefficient of variation measures variability in relation to the mean (or average) and is used to compare the relative dispersion in one type of data with the relative dispersion in another type of data. It allows us to compare the dispersions of two different distributions if their means are positive. The greater dispersion corresponds to the value of the coefficient of greater variation.

The coefficient of variation is a calculation built on other calculations: the standard deviation and the mean as follows:

$$Cv = \frac{\sigma}{\bar{x}} \times 100$$

Where:

- σ is the standard deviation
- $\bar{x}$ is the mean.

Example:

One data series has a mean of 140 and standard deviation 28.28. The second data series has a mean of 150 and standard deviation 24. Which of the two has a greater dispersion?

Solution:

$$Cv_1 = \frac{28.28}{140} \times 100 = 20.2\%$$

$$Cv_2 = \frac{24}{150} \times 100 = 16\%$$

The first data series has a higher dispersion.

5. Range

The range of a set of observations is the difference in values between the largest and the smallest observations in the set. In the case of grouped data the range is defined as the difference between the upper limit of the highest class and the lower limit of the smallest class.

Example

Calculate the range of the following set of the data set: 1, 3, 4, 5, 5, 6, 9, 14 and 21

Solution:

From the given series the lowest data is 1 and the highest data is 21

The Range = highest value − lowest value

$\text{Range} = 21 - 1 = 20$

Application activity 6.6

1) Out of 4 observations done by tutor of English, arranged in descending order, the 5^{th}, 7^{th}, 8^{th} and 10^{th} observations are respectively 89, 64, 60 and 49. Calculate the median of all the 4 observations.
2) In the following statistical series, calculate the standard deviation of the following set of data

   56,54,55,59,58,57,55
3) In the classroom of language education the first five student-teachers scored the following marks out of 10 in a quiz of French

   5, 6, 5, 2, 4, 7, 8, 9, 7, 5.

   a) Calculate the mean, median and the modal mark
   b) Calculate the quartiles and inter-quartile range
   c) Calculate the variance and the standard deviation
   d) Evaluate the coefficient of variation

6.7 Practical activity in statistics

Activity 6.7

Student teachers were interested in getting information on the number of hours patients spent in the hospital in a certain week. The table below summarizes the data they recorded where the frequency indicates the number of patients.

Class boundaries(of hours)	Frequency (number of patients)
7.5-12.5	3
12.5-17.5	5
17.5-22.5	15
22.5-27.5	5
27.5-32.5	2

Determine and interpret the average, the modal class, the median, the variance and the standard deviation related to the number of hours a sick person spends in that hospital.

What is the advice you can provide to the manager of that hospital if he/she has a few number of beds?

With observation, data are collected through direct observation. Information could also be collected using an existing table that shows types of data to collect in a certain period of time, this method of collecting data is reliable and accurate.

Once data has been collected, they may be presented or displayed in various ways. Such displays make it easier to interpret and compare the data.

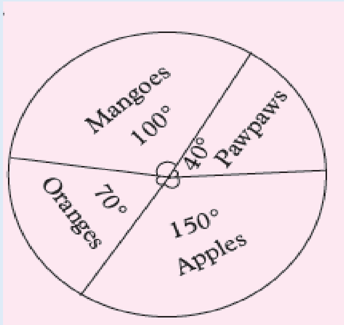
After the collection of data there is need of interpreting them and here there are some tips:

- Collect your data and make it as clean as possible.
- Choose the type of analysis to perform: qualitative or quantitative
- Analyze the data through various statistical methods such as mean, mode, standard deviation or Frequency distribution tables
- Reflect on your own thinking and reasoning and analyze your data and then interpret them referring to the reality.

During interpretation, avoid subjective bias, false information and inaccurate decisions.

Application activity 6.7

1) After selling fruits in a market, Martha had a total of 144 fruits remaining. The pie chart below shows each type of fruit that remained.



   a) Find the total cost of mangoes and pawpaws remaining if a mango sells at 30 FRW and a pawpaw at 160

   b) Which type of fruit remained the most?

   c) What was the median number of fruit that remained?

   d) Draw a frequency table to display the information on the pie chart.

2) Work in groups and use tailor's meters to correct data on the height for 20 student teachers. (You can do it to collect data on their weights by the use of a balance).

   a) Organize the corrected data with a frequency distribution

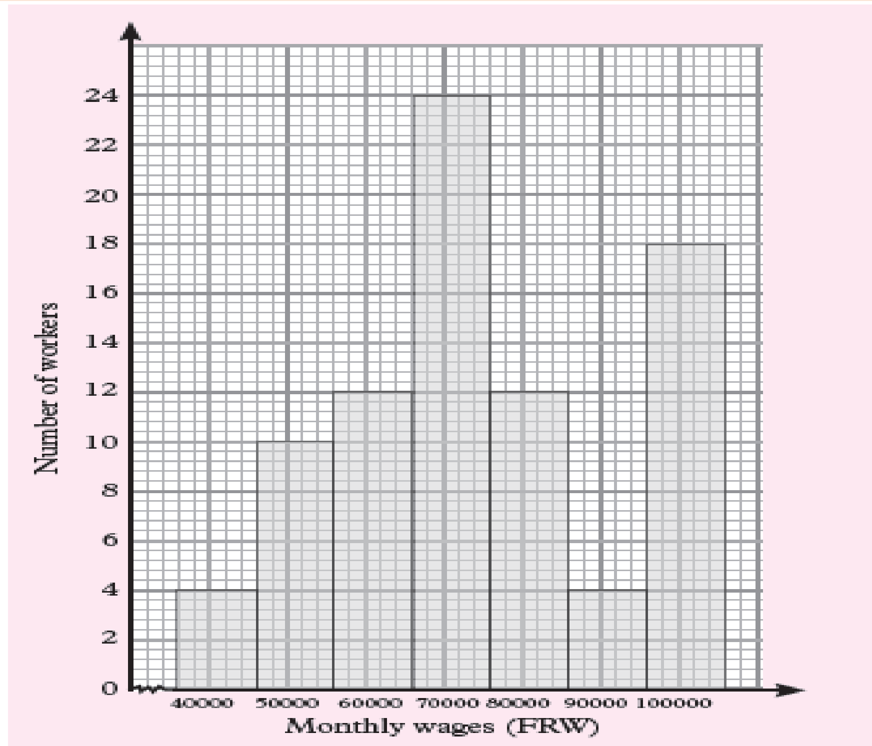
   b) Determine and interpret the mean, the mode and the median for the data.

   c) Determine and interpret the range, the quartiles and the standard deviation.

   d) Organize data into a grouped data distribution of 10 groups and determine: the modal class and the standard deviation. Compare this standard deviation with the one found in (c).

6.8. END UNIT ASSESSMENT

1. Use the graph below to answer the questions that follows



a) Use the graph to estimate the mode.
b) State the range of the distribution.
c) Draw a frequency distribution table from the graph
d) Draw the frequency polygon represented by the histogram

2. In test of mathematics, 10 student-teachers got the following marks:

6, 7, 8, 5, 7, 6, 6, 9, 4, 8

a) Calculate the mean, mode, quartiles and inter-quartile range
b) Calculate the variance and standard deviation
c) Calculate the coefficient of variation.

UNIT: 7

ELEMENTARY PROBABILITY

Key unit competence : Use combinations and permutations to determine probabilities of occurrence of an event.

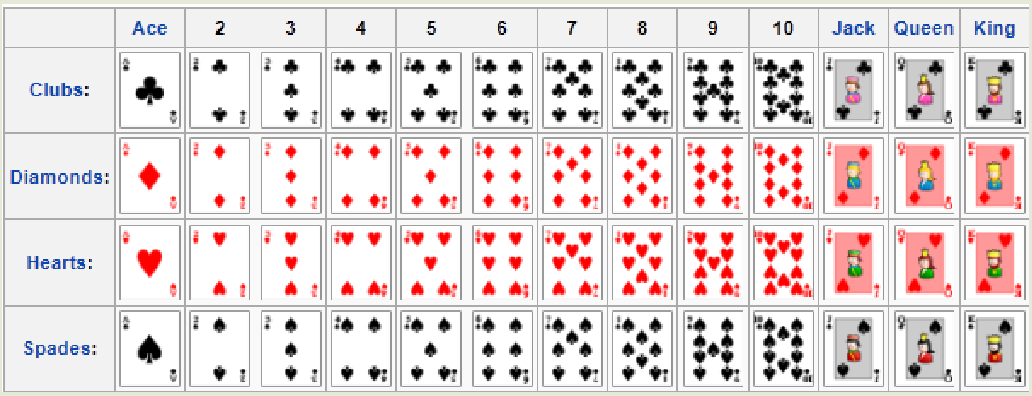
7.0. Introductory Activity

A woman applying the family planning program considers the assumption that one boy or one girl can be born at each delivery. If she wishes to have 3 children including two girls and one boy, she knows that this is a case among other cases which can happen for the 3 children she can have. Discuss all these cases and deduce the chance the woman has for having a girl at the first and the second delivery and a boy at the third delivery.

7.1 Concepts of probability: Sample space and Events

Activity 7.1

Consider the deck of 52 playing cards



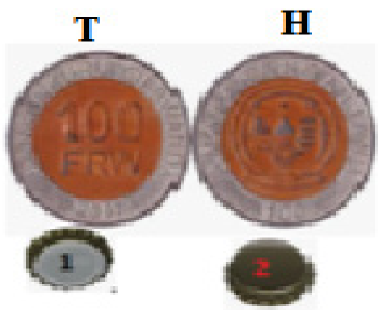
1) Suppose that you are choosing one card

a) How many possibilities to choose this card are there? b) How many possibilities to choose a king are there? c)How many possibilities to choose aces of hearts are there? 2) If "selecting a queen is an example of event, give other examples of events.

Probability is the chance that something will happen.

The concept of probability can be illustrated in the context of a game of 52 playing cards. In a park of deck of 52 playing cards, cards are divided into four suits of 13 cards each. If a player selects a card at random (by simple random sampling), then each card has the same chance or same probability of being selected.

When a coin is tossed, it may show Head (H- face with logos) or Tail (T-face with another symbol).



We cannot say beforehand whether it will show head up or tail up. That depends on chance. The same, a card drawn from a well shuffled pack of 52 cards can be red or black. That depends on chance. Such phenomena are called probabilistic. The theory of probability is concerned with this type of phenomena.

Probability is a concept which numerically measures the degree of uncertainty and therefore of certainty of occurrence of events.

In most sampling situations we are generally not concerned with sampling a specific individual but instead we concern ourselves with the probability of sampling certain types of individuals.

Random experiments and Events

A **random experiment** is an experiment whose outcome cannot be predicted or determined in advance.

Example of experiments:

- Tossing a coin,
- Throwing a dice
- Selecting a card from a pack of paying cards, etc.

In all these cases there are a number of possible results (outcomes) which can occur but there is an uncertainty as to which one of them will actually occur.

Each performance in a random experiment is called a **trial**. The result of a trial in a random experiment is called an outcome, an elementary event, or a sample point. The totality of all possible outcome (or sample points) of a random experiment constitutes the sample space which is denoted by Ω. Sample space may be discrete or continuous.

Discrete sample space:

- Firstly, the number of possible outcomes is **finite.**
- Secondly, the number of possible outcomes is **countably infinite,** which means that there is an infinite number of possible outcomes, but the outcomes can be put in a one-to-one correspondence with the positive integers.

Example:

If a die is rolled and the number that show up is noted, then $\Omega = \{1, 2, 3, ..., 6\}$.

If a die is rolled until a "6" is obtained, and the number of rolls made before getting first "6" is counted, then we have that $\Omega = \{0, 1, 2, 3, ...\}$.

Continuous sample space: If the sample space contains one or more intervals, the sample space is then **uncountable infinite.**

Example:

A die is rolled until a "6" is obtained and the time needed to get this first "6" is recorded. In this case, we have that $\Omega = \{t \in \mathbb{R} : t > 0\} = (0, \infty)$.

An **event** is a set of outcomes of a probability experiment, it is a subset of the sample space. The null set ϕ is thus an event known as the **impossible event**. The sample space Ω corresponds to the **sure event**.

In particular, every elementary outcome is an event, and so is the sample space itself.

Remarks

- An elementary outcome is sometimes called a **simple event**, whereas a **compound** event is made up of at least two elementary outcomes.
- To be precise we should distinguish between the elementary outcome *w*, which is an element of Ω and the elementary event $\{w\} \subset \Omega$.
- The events are denoted by capital letters such as *A*, *B*, *C* and so on.

Example

Consider the experiment that consists in rolling a die and recording the number that shows up. Let A be the event "the even number is shown" and B be the event "the odd number less than 5 is shown". Define the events A and B.

Solution

We have the sample space $\Omega = \{1,2,3,4,5,6\}$.

$A = \{2,4,6\}$ and $B = \{1,3\}$

Definitions

- Two or more events which have an equal probability (same chance) of occurrence are said to be **equally likely**, i.e. if on taking into account all the conditions, there should be no reason to except any one of the events in preference over the others. Equally likely events may be simple or compound events.
- Two events, *A* and *B* are said to be **incompatible** (or **mutually exclusive**) if their intersection is empty. We then write that $A \cap B = \varnothing$.
- Two events, *A* and *B* are said to be **exhaustive** if they satisfy the condition $A \cup B = \Omega$.
- An event is said to be impossible if it cannot occur.

Example

Consider the experiment that consists in rolling a die and recording the number that shows up.

We have that $\Omega = \{1,2,3,4,5,6\}$.

We define the events

$A=\{1,2,4\}$, $B=\{2,4,6\}$, $C=\{3,5\}$, $D=\{1,2,3,4\}$ and $E=\{3,4,5,6\}$

We have

$A\cup B=\{1,2,4,6\}$,

$A\cap B=\{2,4\}$,

$A\cap C=\varnothing$ and

$D\cup E=\Omega$.

Therefore, *A* and *C* are incompatible events.

D and *E* are exhaustive events.

Moreover, we may write that $A'=\{3,5,6\}$, where the symbol A' denotes the **complement** of the event *A*.

This suggests the following definition:

If *E* is an event, then E' is the event which occurs when *E* does not occur. Events *E* and E' are said to be **complementary events.**

Example of event and sample spaces

- Tossing a coin: there are two possible outcomes, you gain Head up or Tail up. Then, $\Omega=\{H,T\}$ - throwing a dice and noting the number of its uppermost face. There are 6 possible outcomes: one number from 1 to 6 can be up. Then, $\Omega=\{1,2,3,4,5,6\}$.
- Two coins are thrown simultaneously. $\Omega=\{HH,HT,TH,TT\}$
- Tree coins are thrown simultaneously. $\Omega=\{HHH,HHT,HTH,THH,HTT,THT,TTH,TTT\}$
- Two dice are thrown simultaneously. The sample space consists of 36 points: .....

$\Omega=\{(1,1),(1,2),....\}$. Please complete other points!

Note: The determination of sample space for some events such as the one for dice thrown simultaneously requires the use of complex reasoning but it can be facilitated by different counting techniques.

Application activity 7.1

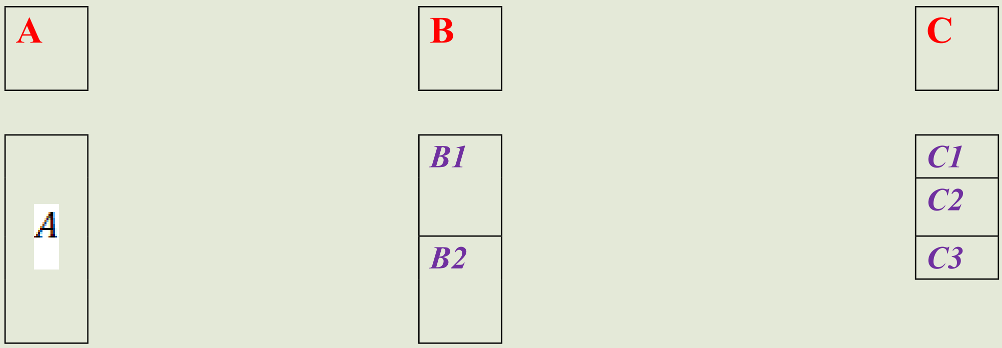
Two dice are thrown simultaneously and the sum of points is noted, determine the sample space.

7.2 Counting techniques

7.2.1 Simple counting techniques

Activity 7.2.1

1. Use the library or the internet to search on counting techniques used to determine outcomes for different random experiments.
2. There are 2 roads joining A and B and 3 roads joining B and C. Write down different roads from A to C via B. How many are they?



CONTENT SUMMARY

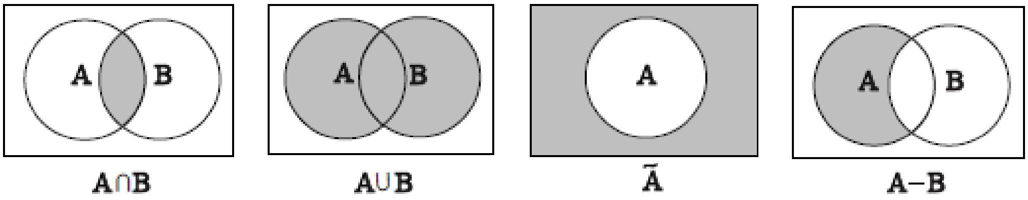
The main counting techniques are related to arrangement and combination rules. Before these rules, let us have a recall on Venn diagram, Tree diagrams and Contingency table.

1. Use of Venn diagram

As studied in senior two, Venn diagram, Tree diagrams and Contingency table can be used to determine all the possible outcomes of some events.

A Venn diagram refers to representing mathematical or logical sets pictorially as circles or closed curves within an enclosing rectangle (the universal set), common elements of the sets represented by intersections of the circles.

In many cases, events can be described in terms of other events through the use of the standard constructions of set theory. We will briefly review definitions of these constructions. The reader is referred to the following figure for Venn diagrams which illustrate these constructions.



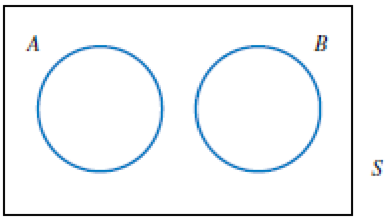
Let A and B be two sets. Then the union of A and B is the set $A \cup B = \{x / x \in A \, or \, x \in B\}$

The intersection of a and b is the set $A \cap B = \{x / x \in A \, and \, x \in B\}$

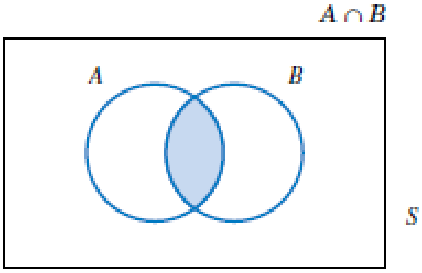
The difference of A and B is the set $A - B = \{x / x \in A \, and \, x \notin B\}$.

The set A is a subset of B, written $A \subset B$, if every element of A is also an element of B.

When $A \cap B = \phi$, the two events are said to be mutually exclusive. This means that they cannot occur at the same time, they do not have outcomes in common.

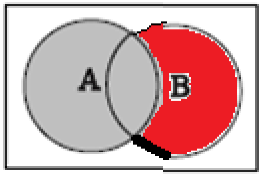


When $A \cap B \neq \phi$, the two events are not mutually exclusive. This means that they have some outcomes in common.



The complement of an event is A

It is the set of outcomes in the sample space Ω that are not included in the outcomes of event A. The complement of A is denoted A'



Finally, the complement of A is the set $\bar{A} = \{x / x \in \Omega \text{ and } x \notin A\}$

Example:

1. Determine which events are mutually exclusive and which are not when a single die is rolled.

a) Getting an odd number and getting an even number.

b) Getting a 3 and getting an odd number.

c) Getting an odd number and getting a number less than4

d) Getting a number greater than 4 and getting a number less than 4.

Solution:

a) Events are mutually exclusive.

b) Events are not mutually exclusive.

c)Events are not mutually exclusive.

d)Events are mutually exclusive

2. A survey involving 120 people about their preferred breakfast showed that;

55 eat eggs for breakfast.

40 drink juice for breakfast.

25 eat both eggs and drink juice for breakfast.

(a) Represent the information on a Venn diagram.

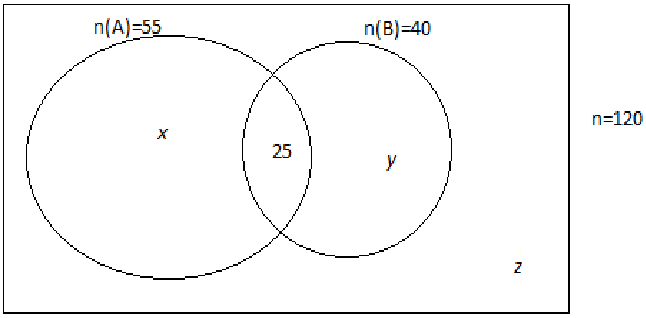
(b) Calculate the following probabilities.

(i) A person selected at random takes only one type for breakfast.

(ii) A person selected at random takes neither eggs nor juice for breakfast.

Solution:

a) Let A= Those who eat eggs, B = Those who take juice and z represent those who did not take any.



Here, we can now solve for the number of people who take eggs only for breakfast.

x = 55 – 25=30

So 30 people took Eggs only

Also, y = 40 – 25 = 15

So, 15 people took Juice only.

Hence 30 + 25 + 15 + z = 120

Z = 120 – (30+15+25)

Z = 120 – 70

Z = 50 . The number of people who did not take anything for breakfast is 50.

i)The number of people who took one type of breakfast is: 30+15=45, the corresponding probability is $P(E)=\frac{45}{120}$. and ii) $(A\cup B)'$ is the event 'takes neither eggs nor juice for breakfast, then $P\left[(A\cup B)'\right]=\frac{50}{120}$

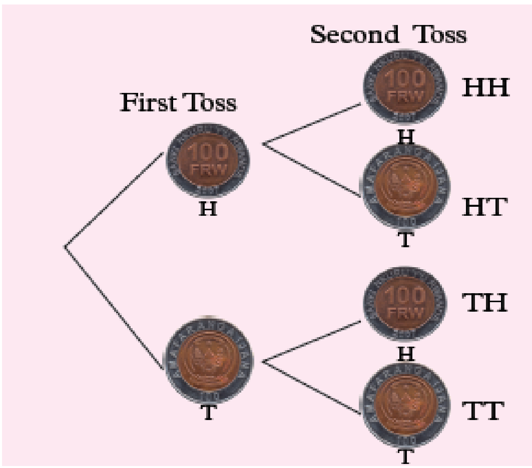
2. Use of tree diagrams

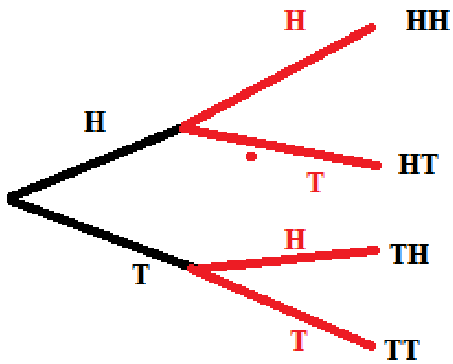
A **tree diagram** is simply a way of representing a sequence of events. Tree diagrams are particularly useful in probability since they record all possible outcomes in a clear and uncomplicated manner.

It has branches and sub-branches which help us to see the sequence of events and all the possible outcomes at each stage.

Example: Using a tree diagram, determine all the possible outcomes that can be obtained when a coin is tossed twice.

Solution: In the first toss, we get either a head (H) or a tail (T). On getting a H in the first toss, we can get a H or T in the second toss. Likewise, after getting a T in the first toss, we can get a H or T in the second toss.





$$S = \{HH, HT, TH, TT\}$$

3. Use of a table

A **table** is simply a way of representing a sequence of events. It is a rectangular array in which the fist column has elements of the first set while the first low has elements of the second set to be associated with the first.

Example:

1) Find the space for rolling two dice.

Solution: As each die cam land in 6 different ways, and two dice are rolled, the sample space can be presented as a rectangular array.

Die 2 D1	1	2	3	4	5	6
1	(1,1)	(1,2)	(1,3)	(1,4)	(1,5)	(1,6)
2	(2,1)	(2,2)	(2,3)	(2,4)	(2,5)	(2,6)
3	(3,1)	(3,2)	(3,3)	(3,4)	(3,5)	(3,6)
4	(4,1)	(4,2)	(4,3)	(4,4)	(4,5)	(4,6)
5	(5,1)	(5,2)	(5,3)	(5,4)	(5,5)	(5,6)
6	(6,1)	(6,2)	(6,3)	(6,4)	(6,5)	(6,6)

Thus the total number of all possible outcomes while rolling two dice is 36.

There is a technique of counting without necessarily listing the total number of all possible outcomes.

This is known as **Basic product principle of counting:**

If a sequence of n events in which the first one has n_1 possibilities, the second with n_2 possibilities the third with n_3 possibilities, and so forth until n_k, the total number of possibilities of the sequence will be:

Total number $= n_1 \times .n_2 \times .n_3 ... \times n_k$.

Application activity 7.2.1

1. Use a tree diagram to find the gender for 3 children in a family.
2. A coin is tossed and a die is rolled. Find the number of outcomes for the sequence of events.

7.2.2 Arrangement of n unlike objects in a row

Activity 7.2.2

Consider three letters R, E and B written in a row, one after another.

Form all possible different words from three letters R, E and B (not necessarily sensible).

In fact, each arrangement is a possible permutation of the letters R, B and E; for example REB, RBE, …

How many arrangements are possible for three letters R, E and B?

From different arrangement of three letters ***R, E and B***, the first letter to be written down can be chosen in 3 ways. The second letter can then be chosen in 2 ways because there are 2 remaining letters to be written down and the third letter can be chosen in 1 way because it is only one letter remain to be written down. Thus, the three operations can be performed in $3\times2\times1=6$ ways.

This arrangement of letters is the same as sitting different people on the same bench. A permutation is an arrangement of n objects in a specific order.

Example: Give all different ways three students: Aloys, Emmanuel and Alexis can be sit on the same bench. Two wyas were given in the table, complete others.

Aloys	Emmanuel	Alexis
Aloys	Alexis	Emmanuel
..	..	..

This suggests the following fact:

The number of different permutations of n different objects (unlike objects) in a row is

$$n\times(n-1)\times(n-2)\times(n-3)\times\ldots\times2\times1$$

This corresponds also to the number of ways of arranging n unlike objects in a row.

A useful short hand of writing this operation is $n!$ (read ***n*** **factorial)**. Then,

$$n!=n\times(n-1)\times(n-2)\times(n-3)\times\ldots\times2\times1$$

Thus, $1!=1$, $2!=2\times1=2$, $3!=3\times2\times1=6$, $4!=4\times3\times2\times1=24$, $5!=5\times4\times3\times2\times1=120$ and so on.

Note that $0!=1$

Other calculatons are for example: 1) $\frac{6!}{2!4!}=\frac{6\times5\times4\times3\times2\times1}{2\times1\times4\times3\times2\times1}=15$

2) $\frac{7!}{4!2!}=\frac{7\times6\times5\times4\times3\times2\times1}{4\times3\times2\times1\times2\times1}=105$

3) Five children have to be seated on a bench. In how many ways they can be seated? How many arrangements are they, if the youngest child has to sit at the left end of the bench?

Solution

Since there are five children, the first child can be chosen in 5 ways, the next child in 4 ways, the next in 3 ways, the next in 2 ways and the last in 1 way. Then, the number of ways is $5!=5\times4\times3\times2\times1=120$.

Now, if the youngest child has to sit at the left end of the bench, this place can be filled in only 1 way. The next child can then be chosen in 4 ways, the next in 3 ways and so on. Thus, the number of total arrangement is $1\times4!=1\times4\times3\times2\times1=24$.

3. Three different mathematics books and five other books are to be arranged on a bookshelf. Find :

a) the number of possible arrangements of the book.

b) the number of possible arrangements if the three mathematics books must be kept together?

Solution:

We have 8 books altogether.

a) Since we have 8 books altogether, the first book can be chosen in 8 ways, the next in 7 ways, the next in 6 ways and so on. Thus, the total arrangement is $8! = 40320$

b) Since the 3 mathematics books have to be together, consider these bound together as one book. There are now 6 books to be arranged and these can be performed in $6! = 720$.

Note that we have taken the three mathematics book as one book; these three books can be arranged in $3! = 6$ ways. Thus, the total number of arrangements is $720 \times 6 = 4320$.

Application activity 7.2.2

1. Simplify : a. $\frac{5!}{2!}$ b. $\frac{10!}{6!7!}$

2. Four different English books, five different Biology books and ten other books are to be arranged on a bookshelf. Find

a) The number of possible arrangements of the books.

b) The number of possible arrangements if the three Biology books must be kept together?

7.2.3 Arrangement of of indistinguishable objects (Permutation with repetition)

Activity 7.2.3

1) Make a list of all arrangements formed by 4 numbers: 1,2,3,4. How many arrangements are they possible?

2) Consider the arrangements of four letters in the word "BOOM".

- Write down all possible different arrangements.
- How many arrangements are they possible of four letters in the word "BOOM"?

Consider the arrangements of six letters in the word "**AVATAR**" (a title used for the movie).

We see that there are three **A**'s (or 3 alike letters).

- Let the three **A**'s in the word be distinguished as $\boldsymbol{A_1}$, $\boldsymbol{A_2}$ and $\boldsymbol{A_3}$ respectively. Then all the six letters are different, so the number of permutations of them (called labeled permutations) is $n! = 6!$.
- However, consider each of the real permutations without distinguishing the three **A**'s, for example **W=RATAVA.**
- The following are all of the 6 (=3!) labeled permutations among the 6! ones, which come from permuting the three labeled **A**'s in **W=RATAVA**:

$$\mathbf{RA_1TA_2VA_3, RA_1TA_3VA_2, RA_2TA_1VA_3, RA_2TA_3VA_1, RA_3TA_1VA_2, RA_3TA_2VA_1}$$

- All these six labeled permutations should be considered as an identical real permutation, which is **W=RATAVA**.
- Since each real permutation has six of such labeled permutations coming from the three **A**'s, we conclude that the desired number of real permutations is just

$$\frac{6!}{3!} = \frac{6\times5\times4\times3!}{3!} = 6\times5\times4 = 120$$

This suggests the following fact:

*The number of different permutations of n **indistinguishable** objects with* n_1 *alike,* n_2 *alike, …, is* $\dfrac{n!}{n_1!n_2!\ldots}$.

This corresponds also to the number of ways for arranging in a line n objects of which n_1 *of one type are alike,* n_2 *of the second type are alike and so on.*

Note that **alike** means that the objects in a group are indistinguishable from one another.

Example:

1) How many distinguishable six digit numbers can be formed from the digits 5, 4, 8, 5, 5, 4?

Solution: There are 6 letters in total with three 5's and two 4's. Then the required numbers are $\dfrac{6!}{3!2!} = \dfrac{720}{12} = 60$

2) How many arrangements can be made from the letters of the word **TERRITORY**?

Solution

There are 9 letters in total with three **R**'s and two **T**'s.

Thus, we have $\frac{9!}{3!\times 2!}=\frac{9\times 8\times 7\times 6\times 5\times 4}{2\times 1}=\frac{60,480}{2}=30,240\, arrangements$.

3) In how many different ways can 4 identical red balls, 3 identical green balls and a yellow ball be arranged in a row?

Solution

There are 8 balls in total with 4 red, 3 green and one yellow.

Thus, we have $\frac{8!}{4!\times 3!}=\frac{8\times 7\times 6\times 5}{3\times 2\times 1}=280\, ways$.

Application activity 7.2.3

1. How many different arrangements can be made from the letters of the word

   a) ENGLISH b) MATHEMATICS
2. How many arrangements can be made from the letters of English alphabet?
3. In how many ways can 4 red, 3 yellow and 2 green discs be arranged in a row, if discs of the same colour are indistinguishable?

7.2.4 Circular arrangements

Activity 7.2.4

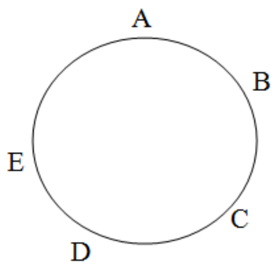
Take 5 different note books:

- Put them on a circular table,
- Fix one note book, for example A;
- Try to interchange other 4 note books as possible .
- How many different ways obtained?

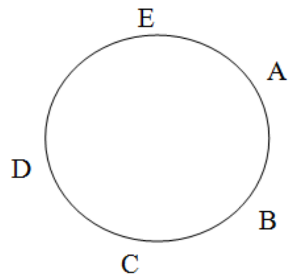
Remember that there is one note book that will not change its place.

We have seen that if we wish to arrange n different things in a row, we have $n!$ possible arrangements. Suppose that we wish to arrange n things around a circular table. The number of possible arrangements will no longer be $n!$ because there is now no distinction between certain arrangements that were distinct when written in a row.

For example ABCDE is different arrangement from EABCD, but



is not a different arrangement from



With circular arrangement of this type, it is the relative positions of the items being arranged which is important. One item can therefore be fixed and the remaining items arranged around it.

The number of arrangements of n unlike things in a circle will therefore be $(n-1)!$. *In those cases where clockwise and anticlockwise arrangements are not considered to be different, this reduces to* $\frac{1}{2}(n-1)!$.

Example:

1. Four men Peter, Rogers, Smith and Thomas are to be seated at a circular table. In how many ways can this be done?

Solution

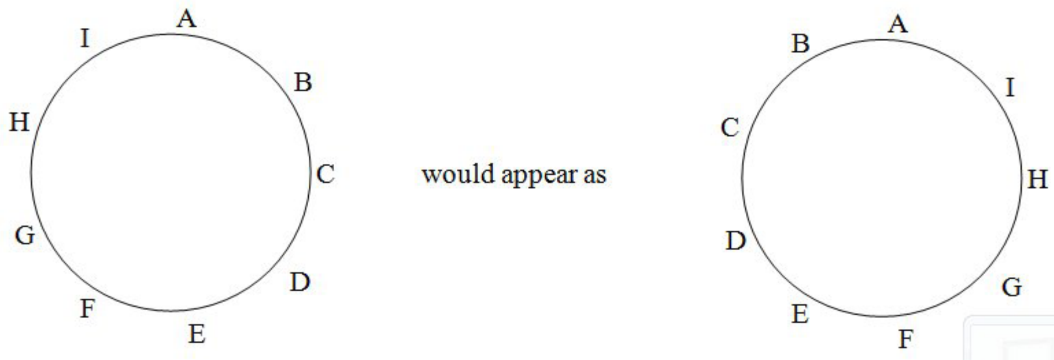
Suppose Peter is seated at some particular place. The seats on his left can be filled in 3 ways, the next seat can then be filled in 2 ways and the remaining seat in 1 way.

Thus, total number of arrangements is $3!=6$.

2. Nine beads, all of different colors are to be arranged on a circular wire. Two arrangements are not considered to be different if they appear the same when the ring is turned over. How many different possible arrangements are there?

Solution

When the ring is turned over, the arrangements



When viewed from one side, these arrangements are only different in that one is a clockwise arrangement and the other is anticlockwise. If one bead is fixed, there are $(9-1)!=8!$ ways of arranging the remaining beads relative to the fixed one.

But, half of these arrangements will appear the same as the other half when the ring is turned over, because for every clockwise arrangement there is a similar anticlockwise arrangement. Hence, number of arrangements is $\frac{1}{2}8!=20160$.

Application activity 7.2.4

1. Five men Eric, Peter, John, Smith and Thomas are to be seated at a circular table. In how many ways can this be done?
2. Eleven different books are placed on a circular table. In how many ways can this be done?

7.2.5 Basic sum principle of counting for mutually exclusive situations

Activity 7.2.5

1. Suppose that you go to a restaurant and you are allowed a soup or juice. Will you pick one, the other or both?
2. How many different four digits numbers, end in a 3 or a 4, can be formed from the figures 3,4,5,6 if each figure is used only once in each number.

Two experiments 1 and 2 are mutually exclusive, if when experiment 1 occurs, experiment 2 cannot occur. Likewise, if experiment 2 occurs, experiment 1 cannot occur.

Basic sum principle of counting

In such cases, the number of permutations of either experiment 1 or experiment 2occuring can be obtained by adding the number of permutations of experiment 1 to the number of permutations of experiment 2.

This suggests the following result:

"*If* the first *experiment has* **m** *possible outcomes and if* the second *experiment has* **n** *possible outcomes, then an experiment which might be* "the first *experiment or* the second *experiment*", *called* **experiment 1 or 2**, *has* $(m+n)$ *possible outcomes*."

Example:

1) In tossing an object which might be a coin (with two sides H and T) or a die (with six sides 1 through 6), how many possible outcomes will appear?

Solution

- The experiment may be tossing a coin (experiment 1) or tossing a die (experiment 2), or just experiment 1 or 2.
- So the number of outcomes is $2+6=8$ according to the above basic sum principle of counting.

The number of permutations in which a certain experiment 1 occurs will clearly be mutually exclusive with those permutations in which that experiment does not occur. Thus,

$$\text{Number of permutations in which experiment 1 does not occur}$$
$$= \text{total number of permutation} - \text{number of permutations in which experiment 1 occurs}$$

2) In how many ways can five people Smith, James, Clark, Brown and John, be arranged around a circular table in each of following cases:

a) Smith must sit next Brown?

b) Smith must not sit next Brown?

Solution

There are five people.

a) Since Smith and Brown must sit next to each other, consider these two bound together as one person. There are now, 4 people to seat. Fix on of these, and then the remaining 3 people can be seated in $3\times2\times1=6$ ways relative to the one who was fixed.

In each of these arrangements Smith and Brown are seated together in a particular way. Smith and Brown could now change the seats giving another 6 ways of arranging the five people. Total number of arrangements is $6\times2=12$.

b) If Smith must not sit next Brown, then this situation is a mutually exclusive with the situation in a). Total number of arrangements of 5 people at a circular table is $(5-1)!=4!=24$.

Thus, if Smith must not sit next Brown, the number of arrangements is $24-12=12$.

Generalized sum principle of counting

"*If Experiments* 1 *through* k *have respectively* n_1 *through* n_k *outcomes*,
then the experiment 1 *or* 2 *or* $\ldots$ *or* k *has* $n_1+n_2+\ldots+n_k$ *outcomes.*

Example:

How many even numbers containing one or more digits can be formed from the digits 2, 3, 4, 5, 6 if no digit may be repeated?

Solution

Since the required numbers are even, last digit must 2 or 4 or 6. Note that there are 5 digits.

So we can form one digit, two digits, three digits, four digits or five digits as follow

One digit: 2 or 4 or 6. That is 3 numbers

Two digits: 3 ways to choose the last and 4 ways to choose the first. That is $3\times4=12$ numbers.

Three digits: 3 ways to choose the last, 4 ways to choose the first and 3 ways to choose the second. That is $3\times4\times3=36$

Four digits: 3 ways to choose the last, 4 ways to choose the first, 3 ways to choose the second and 2 ways to choose the fourth. That is $3\times4\times3\times2=72$

Five digits: 3 ways to choose the last, 4 ways to choose the first, 3 ways to choose the second, 2 ways to choose the fourth and 1 way to choose the fifth. That is $3\times4\times3\times2\times1=72$

Adding we have $3+12+36+72+72=195$ even numbers in total.

Application activity 7.2.5

1. How many even numbers containing 2 digits can be formed from the digits 2, 3, 4 if no digit may be repeated?

7.2.6 Permutation of distinguishable objects

Activity 7.2.6

Make a selection of any three letters from the word "*PRODUCT*" and fill them in 3 empty spaces

Use a box like this for empty spaces		

Write down all different possible permutations of 3 letters selected from the letters of the word "*PRODUCT*". How many are they?

We are going to determine the number of permutations of r unlike objects selected from n different objects.

Consider the number of ways of placing 3 of the letters A, B, C, D, E, F, G in 3 empty spaces.

The first space can be filled in 7 ways, the second in 6 ways and the third in 5 ways. Therefore there are $7\times6\times5$ ways of arranging 3 letters taken from 7 letters. This is the number of permutations of 3 objects taken from 7 and it is written ${}^7P_3$.

So ${}^7P_3 = 7\times6\times5 = 210$.

Note that the order in which the letters are arranged is important: ABC is a different permutation from ACB.

Now, $7\times6\times5$ could be written $\frac{7\times6\times5\times4\times3\times2\times1}{4\times3\times2\times1}$

i.e ${}^7P_3 = \frac{7\times6\times5\times4\times3\times2\times1}{4\times3\times2\times1} = \frac{7!}{4!} = \frac{7!}{(7-3)!}$

This suggests the following fact:

The number of different permutations (ways) of r unlike objects selected from n different objects is $^{n}P_{r}=\frac{n!}{(n-r)!}$ *or we can use the denotation* $P_{r}^{n}=\frac{n!}{(n-r)!}$ *or*

$$P(n,r)=\frac{n!}{(n-r)!}$$

Note that if $r=n$, we have $^{n}P_{n}=\frac{n!}{(n-n)!}=\frac{n!}{0!}=n!$ which is the ways of arranging *n* unlike objects.

Example:

1. *How many permutations are there of 3 letters chosen from eight letters of the word* ***RELATION*** *?*

Solution

We see that all those eight letters are distinguishable (unlike). So the required arrangements are given by

$$^{8}P_{3}=\frac{8!}{(8-3)!}=\frac{8!}{5!}=336$$

2. *How many permutations of 2 letters chosen from letters A, B, C, D, E are there ?*

Solution

There are 5 letters which are distinguishable (unlike). So the required arrangements are given by $^{5}P_{2}=\frac{5!}{(5-2)!}=\frac{5!}{3!}=20$.

Note: When determining the number of all possible permutations of r objects selected from a mixture of n alike and unlike objects, start by determining all possible mutually exclusive events from the given experiment that may occur and then apply basic sum principle (See the following example 3).

3. *How many different arrangements are there of 3 letters chosen from the word COMBINATION?*

Solution

There are 11 letters including two O's, two I's and two N's. to find the total number of different arrangements we consider the possible arrangements as four mutually exclusive situations.

a) Arrangements in which all 3 letters are different: there are ${}^{8}P_{3} = 336$

b) Arrangements containing two O's and one other letter: the other letter can be one of seven letters (C, M, B, I, N, A or T) and can appear in any of the three positions (before the two O'S, between the two O's, or after the two O's). i.e $3 \times 7 = 21$ arrangements containing two O's and one other letter.

c) Arrangements containing two I's and one other letter: by the same reasoning in b) there will be $3 \times 7 = 21$ arrangements containing two I's and one other letter.

d) Arrangements containing two N's and one other letter: by the same reasoning in b) there will be $3 \times 7 = 21$ arrangements containing two N's and one other letter.

Thus the total number of arrangements of 3 letters chosen from the word COMBINATION will be $336 + 21 + 21 + 21 = 399$

4. *How many different ways can a chairperson and an assistant chairperson be selected for a research project if there are seven scientists available?*

Solution It is ${}_{7}P_{2} = \dfrac{7!}{(7-2)!} = 42$

Application activity 7.2.6

1. How many permutations are there of 4 letters chosen from letters of the word ENGLISH?
2. How many permutations are there of 2 letters chosen from letters of the word PACIFIC?
3. How many permutations are there of 5 letters chosen from letters A, B, C, D, E, F, and G.
4. How many permutations are there of 10 letters chosen from English alphabet.

NOTE: Ordered Samples

Many problems are concerned with choosing an element from a set S , say ,with n elements. When we choose one element after another, say, r times, we call the choice an ordered sample of size r . We consider two cases.

(1) Sampling with replacement: Here the element is replaced in the set before the next element is chosen. Thus, each time there are n ways to choose an element (repetitions are allowed). The Product rule tells us that the number of such samples is: $n \times n \times n \times ... \times n \ (r \ factors) = n^r$

(2) Sampling without replacement

Here the element is not replaced in the set S before the next element is chosen. Thus, there is no repetition in the ordered sample. Such a sample is simply an $r-permutation$. Thus the number of such samples is:

$$P(n,r) \text{ or } {}^nP_r = n(n-1)(n-2)......(n-r+1) = \frac{n!}{(n-r)!}$$

Example: Three cards are chosen one after the other from a 52-card deck. Find the number m of ways this can be done:

(a) with replacement; (b) without replacement.

Solution: (a) Each card can be chosen in 52 ways. Thus
$m = (52)(52)(52) = 140608$

(b) Here there is no replacement, then $m = (52,3) = 52(51)(50) = 132600$

7.2.7. Combinations

Activity 7.2.7

Take 8 different Mathematics books and form different groups each containing 2 mathematics books. How many groups obtained?

From permutation of r unlike objects selected from n different objects, we have seen that the order in which those objects are placed is important. But when considering the number of combinations of r unlike objects selected from n different objects, the order in which they are placed is not important.

For example, the one combination **ABC** gives rise to 3! permutations: **ABC, ACB, BCA, BAC, CAB, CBA**.

Consider the number of permutations of 3 letters selected from the 7 letters **A, B, C, D, E, F, G**.

That is $\quad {}^{7}P_{3}=\frac{7!}{(7-3)!}=\frac{7!}{4!}$.

If we need the combinations of 3 letters selected from those 7 letters, we will take this number of permutations divided by $3!$ because each permutation gives rise to 3! permutations.

That is, the number of combinations of 3 letters selected from those 7 letters is

$$\frac{{}^{7}P_{3}}{3!}=\frac{\frac{7!}{4!}}{3!}=\frac{7!}{4!3!}=\frac{7!}{(7-3)!3!}.$$

This number is denoted by ${}^{7}C_{3}$.

Thus, the number of combinations of 3 letters selected from those 7 unlike letters is

$${}^{7}C_{3}=\frac{7!}{(7-3)!3!}=\frac{7!}{4!3!}=35.$$

This suggests the following fact:

The number of different groups of r items that could be formed from a set of n distinct objects with the order of selections being ignored is

$${}^{n}C_{r}=\frac{n!}{(n-r)!r!}.$$

We can write ${}^{n}C_{r}=\frac{{}^{n}P_{r}}{r!}$

${}^{n}C_{r}$ is sometimes denoted by C_{r}^{n} or ${}_{n}C_{r}$ or $\binom{n}{r}$ or $C(n,r)$.

Note that the objects selected to be in a group are regarded as indistinguishable (unlike).

Example:

1. *From a group of 5 men and 7 women, how many different committees consisting of 2 men and 3 women can be formed?*

Solution

- Experiment 1: select 2 men from 5.
- Experiment 2: select 3 women from 7.
- Experiment of forming a committee: experiment 1 & 2.
- Number of possible outcomes of experiment 1 is ${}^{5}C_{2}=\frac{5!}{2!3!}=\frac{5\times4\times3!}{2\times3!}=10$
- Number of possible outcomes of experiment 2 is ${}^{7}C_{3}=\frac{7!}{3!4!}=\frac{7\times6\times5\times4!}{6\times4!}=35$
- Number of possible outcomes of experiment 1 and 2is ${}^{5}C_{2}\times{}^{7}C_{3}=10\times35=350$ by the basic product principle of counting
- That is, the desired number of possible outcomes of the experiment of forming a committee is 350.

2. *A committee of three men and one woman is obtained from five men and three women. In how many ways can the members be chosen?*

Solution: Three men can be selected from five men, i.e ${}^{5}C_{3}=\frac{5!}{(5-3)!3!}=\frac{5!}{2!3!}$ ways

One woman can be selected from three women, i.e ${}^{3}C_{1}=\frac{3!}{(3-1)!1!}=\frac{3!}{2!1!}$ ways

By the basic product principle of counting, there are ${}^{5}C_{3}\times{}^{3}C_{1}=\frac{5!}{2!3!}\times\frac{3!}{2!1!}=\frac{5!}{2!2!}=30$ ways of selecting the committee.

The following two identities are true:

a) ${}^{n}C_{r}={}^{n}C_{n-r}$

b) Pascal's identity: ${}^{n+1}C_{r}={}^{n}C_{r}+{}^{n}C_{r-1}$

Application activity 7.2.7

1. A committee of four men and two women is obtained from 10 men and 12 women. In how many ways can the members be chosen?
2. A group containing 4 Mathematics books and 5 Physics books is formed from 9 Mathematics books and 10 Physics books. How many groups can be formed?

7.3 Binomial expansion and Pascal's triangles

Activity 7.3

Expand the expressions

$(a+b)^2$

Since $(a+b)^3=(a+b)^2(a+b)$ *and* $(a+b)^4=(a+b)^3(a+b)$

Expand $(a+b)^3$ and $(a+b)^4$

Once more find the expansion of $(a+b)^5$.

Complete the following table

Power	Coefficient of powers of a and b					Binomial expression
0						$(a+b)^0$
1						$(a+b)^1$
2						$(a+b)^2$
3						$(a+b)^3$
4						$(a+b)^4$

Try to generalize the form of coefficients for each term of a binomial expansion using combinations learnt in the previous lessons.

Pascal's Triangle

Pascal's Triangle is a triangular array of the binomial coefficients. The rows of Pascal's Triangle are conventionally enumerated starting with row $n = 0$ at the top. The entries in each row are numbered from the left beginning with $r = 0$ and are usually staggered relative to the numbers in the adjacent rows.

The elements of Pascal's Triangle are the number of combinations of r objects chosen from n unlike objects. That is ${}^{n}C_{r}$. This triangle is constructed by the Pascal's identity:

$$ {}^{n+1}C_{r} = {}^{n}C_{r} + {}^{n}C_{r-1} \quad \text{or} \quad {}^{n}C_{r} = {}^{n-1}C_{r-1} + {}^{n-1}C_{r} \quad \text{or} \quad {}^{n+1}C_{r+1} = {}^{n}C_{r} + {}^{n}C_{r+1} $$

$$\begin{array}{ccc} \boxed{x} & + & \boxed{y} \\ & & \| \\ & & \boxed{z} \end{array}$$

Here, $z = {}^{n+1}C_{r}$,

$y = {}^{n}C_{r}$ and $x = {}^{n}C_{r-1}$

r / n	0	1	2	3	4	5	…
0	${}^{0}C_{0} = 1$						
1	${}^{1}C_{0} = 1$	${}^{1}C_{1} = 1$					
2	${}^{2}C_{0} = 1$	${}^{2}C_{1} = 2$	${}^{2}C_{2} = 1$				
3	${}^{3}C_{0} = 1$	${}^{3}C_{1} = 3$	${}^{3}C_{2} = 3$	${}^{3}C_{3} = 1$			
4	${}^{4}C_{0} = 1$	${}^{4}C_{1} = 4$	${}^{4}C_{2} = 6$	${}^{4}C_{3} = 4$	${}^{4}C_{4} = 1$		
5	${}^{5}C_{0} = 1$	${}^{5}C_{1} = 5$	${}^{5}C_{2} = 10$	${}^{5}C_{3} = 10$	${}^{5}C_{4} = 5$	${}^{5}C_{5} = 1$	
⋮							

A simple construction of this triangle proceeds in the following manner:

- On row 0, write only the number 1.
- Then, to construct the elements of following rows, add the number above and to the left with the number above to the right to find the new value.

- If either the number to the right or left is not present, substitute a zero in its place.

For example, the first number in the first row is $0+1=1$, whereas the numbers 1 and 3 in the third row are added to produce the number 4 in the fourth row.

An element of Pascal's Triangle, ${}^nC_r$, is the coefficients of any term in the expansion of $(a+b)^n$ where r is the exponent of either a or b

Consider the product $(a+b)^n=(a+b)(a+b)(a+b)\ldots(a+b)$. If this product is multiplied out, each term of the answer will be of the form $c_1c_2c_3\ldots c_k\ldots c_n$ where, for all k, c_k is either a or b.

Thus, if $c_k=a$ for all k we obtain the term a^k. If $c_k=b$ for one of the terms and $c_k=a$ for the rest, we obtain terms such as $b\times a\times a\times\ldots\times a\times a$, $a\times b\times a\times\ldots\times a\times a$, …, $a\times a\times a\times\ldots\times b\times a$, $b\times a\times a\times\ldots\times a\times b$, and their sum is $na^{n-1}b$.

If $c_k=b$ for r of the terms and $c_k=a$ for the rest we obtain a number of terms of the form $a^{n-r}b^r$.

The number of such terms is the number of ways in which r of the form $c_1c_2c_3\ldots c_n$ can be selected as equal to b. This number is $\binom{n}{r}$, which is $\binom{n}{r}={}^nC_r=\dfrac{n!}{(n-r)!r!}$.

Thus, ${}^nC_r=\dfrac{n!}{(n-r)!r!}$ is the coefficient of $a^{n-r}b^r$ in the expansion of $(a+b)^n$.

This suggests the following theorem.

Binomial theorem

For every integer $n\geq 1$, $(a+b)^n=\sum_{r=0}^{n}{}^nC_r a^{n-r}b^r$

The following properties of the expansion of $(a+b)^n$ should be observed:

a) There are $n+1$ terms.

b) The sum of the exponents of a and b in each term is n.

c) The exponents of a decrease term by term from n and 0; the exponent of b increase term by term from 0 to n.

d) The coefficient of any term is ${}^{n}C_{r}$ where r is the exponent of either a or b.

e) The coefficients of terms equidistant from the end are equal.

Examples

1) $(a+b)^2 = \sum_{r=0}^{2} {}^{2}C_{r}a^{2-r}b^{r} = {}^{2}C_{0}a^{2}b^{0} + {}^{2}C_{1}a^{2-1}b^{1} + {}^{2}C_{2}a^{2-2}b^{2} = a^2 + 2ab + b^2$

2)

$$(a-b)^3 = (a+(-b))^3 = \sum_{r=0}^{3} {}^{3}C_{r}a^{3-r}(-b)^{r} = {}^{3}C_{0}a^{3}(-b)^{0} + {}^{3}C_{1}a^{3-1}(-b)^{1} + {}^{3}C_{2}a^{3-2}(-b)^{2} + {}^{3}C_{3}a^{3-3}(-b)^{3}$$

$$= a^3 - 3a^2b + 3ab^2 - b^3$$

3) *Find the coefficient of* x^3 *in the expansion of* $(2x-1)^5$

Solution

The term in x^r is ${}^{5}C_{r}(2x)^{5-r}(-1)^{r} = {}^{5}C_{r}2^{5-r}x^{5-r}(-1)^{r}$ and so the term in x^3 has $r=2$.

The coefficient of this term is ${}^{5}C_{3}2^{3}(-1)^{2} = 80$.

4) *Find the coefficient of* x^3 *in the expansion of* $\left(x^2 - \frac{1}{x}\right)^6$

Solution

The term in x^r is will be given by ${}^{6}C_{r}\left(x^2\right)^{6-r}\left(-\frac{1}{x}\right)^{r}$ which can be written as ${}^{6}C_{r}x^{12-2r}\frac{(-1)^{r}}{x^{r}} = {}^{6}C_{r}x^{12-3r}(-1)^{r}$ and so the term in x^3 has $r=3$.

The coefficient of this term is ${}^{6}C_{3}(-1)^{3} = -20$.

Application activity 7.3

Find the coefficient of

1) x^2 in the expansion of $(4x+1)^6$

2) x^3 in the expansion of $\left(x+\frac{1}{x}\right)^4$

3) x^6 in the expansion of $(9x-3)^{10}$ 4) Expand $(x+4)^7$ 5) Expand $(2x-3)^3$

7.4 Determination of Probability of an event, properties and formulas

Activity 7.4

When a card is selected from an ordinary deck of 52 cards, one assumes that the deck has been shuffled, and each card has the same chance of being selected. Let A be the event of selecting a black card,

a) If n is the number of black cards in the pack, what is the value of n?

b) Calculate the value P (A) = $\frac{n}{number\ of\ allcards}$

c) If P (A) is the probability of electing a black card, deduce the definition of probability for any event E.

The probability of an event $A \subset \Omega$, is a real number obtained by applying to A the function P defined by

$$P(A) = \frac{Number\ of\ outcomes\ \text{in}\ A}{Total\ \text{nu}\ mber\ of\ outcomes\ \text{in}\ the\ sample\ space} = \frac{n(A)}{n(\Omega)}$$

This is the formula for the classical probability, it uses the sample space Ω.

Probability can be expressed as a fraction, decimal or percentage.

Example:

1. For a card drawn from an ordinary deck, find the probability of getting a queen.

Solution: there are 52 cards in a deck and there are 4 queens, $P(\text{queen}) = \dfrac{n(\text{queen})}{n(\Omega)} = \dfrac{4}{52}$

2. A letter is chosen from the letters of the word "**MATHEMATICS**". What is the probability that the letter chosen is an "A"?

Solution

Since two of the eleven letters are A's, we have two favourable outcomes.

There are eleven letters, so we have 11 possible outcomes.

Thus, the probability of choosing a letter A is $\dfrac{2}{11}$.

Basic probability rules

(i) The probability cannot be negative or greater than 1

Suppose that an experiment has only a finite number of equally likely outcomes. If A is an event, then $0 \leq P(A) \leq 1$.

(ii) The probability of a certain event

If the event A is certain to occur, $A = \Omega$, and $P(A) = 1$ and $P(\Omega) = 1$.

(iii) Probability of impossible event

The event that cannot occur is an impossible event $A = \varnothing$, and if $A = \varnothing$ then $P(A) = 0$.

Example:

When a single die is rolled, find the probability of getting a 9.

Solution: $\Omega = \{1, 2, 3, 4, 5, 6\}$, it is impossible to get a 9. $A = \varnothing$

P(getting a 9) = 0.

(iv)The sum of the probabilities of all the outcomes in the sample space is 1.

(v) Probability of complementary event

When E and E' are complementary events, $P(E)=1-P(E')$.

Consider two different events, A and B, which may occur when an experiment is performed.

- The event $A\cup B$ is the event which occurs if A or B or both A and B occur, i.e, at least one of A and B occurs.
- The event $A\cap B$ is the event which occurs if A and B occur.
- The event $A-B$ is the event which occurs when A occurs and B does not occur.
- The event A' is the event which occurs when A does not occur.

Example:

1. When a die is rolled, the event E of getting odd number is that $E=\{1,3,5\}$ and $P(E)=\frac{3}{6}=\frac{1}{2}$

The event F of not getting an odd number is a complementary of E. $F=E'=\{2,4,6\}$

As $P(\Omega)=1, P(F)=1-\frac{1}{2}=\frac{1}{2}$.

2. If the probability that a person lives in an industrialized country of the world is $\frac{1}{4}$, find the probability that a person does not live in an industrialized country.

Solution: P(not living in an industrialized country) = 1- P (living in an industrialized

country) $P=1-\frac{1}{5}=\frac{4}{5}$.

Properties of probabilities

Referring to rules mentioned above, the probabilities assigned to events on a sample space Ω can be summarized in the following properties:

a) 1. $P(E)\geq 0$ *for every* $E\subset\Omega$

b) $P(\Omega)=1$

c) *If* $E\subset F\subset\Omega$, *then* $P(E)\leq P(F)$

d) If A and b are disjoint subsets of Ω ,

then $If\ E \subset F \subset \Omega, P(A \cup B) = P(A) + P(B)$

e) $P(A') = 1 - P(A)$ for every $A \subset \Omega$..

Formula for empirical (classical) probability

Given a frequency distribution, the probability of an event being in a given class is

$$P(A) = \frac{frequency\ for\ the\ class}{Total\ frequency\ in\ the\ distribution} = \frac{f}{n}$$

This probability is called empirical probability and is based on observation.

Examples:

1. A researcher asked 25 staff of an institution if they liked the way their breakfast is prepared. The responses were classified as "Yes", "No", and "Undecided". The results were categorized in a frequency distribution as follows:

Response	Frequency
Yes	15
No	8
Undecided	2
Total	25

What is the probability of selecting a person who disliked the way the breakfast is prepared.

Solution:

$$P(E) = \frac{f}{n} = \frac{8}{25}$$

2. Hospital records indicated that maternity patients stayed in the hospital for the number of days shown in the distribution:

Number of days stayed	**Frequency**
3	15

4	32
5	56
6	19
7	5
Total	**127**

Find these probabilities:

a) A patient stayed exactly 5 days

b) A patient stayed less than 6 days

c) A patient stayed at most 4 days

d) A patient stayed at least 5 days.

Solution:

a) $P(\mathrm{A}) = \frac{56}{127}$

b) Less than 6 days means 3,4 and 5 days;

$$P(\text{less than 6 days}) = \frac{15}{127} + \frac{32}{127} + \frac{56}{127} = \frac{103}{127}$$

c) At most 4 days means 3 or 4 days; $P(\mathrm{A}) = \frac{47}{127}$

d) At least 5 days means 5, 6, or 7 days; $P(\mathrm{A}) = \frac{80}{127}$.

Probability of mutually exclusive or incompatible or events

When $A \cap B = \phi$, A and B are mutually exclusive or disjoint and $P(A \cup B) = P(A) + P(B)$

Note that if $A = A_1 \cup A_2 \cup A_3 \cup ... \cup A_n$, where $A_1, A_2, A_3, ..., A_n$ are incompatible events, then we may write that $P(A) = \sum_{i=1}^{n} P(A_i)$ for $n = 2, 3, ...$

This is also called the addition rule for exclusive events.

The sum of the probability of outcomes

In the sample space, the sum of the probability of outcomes is 1.

The addition rule for non-exclusive events

When two events A and B are not mutually exclusive, $A \cap B \neq \phi$ *, the probability that A or B occurs is given by:*

$$P(A \cup B) = \mathrm{P(A)} + \mathrm{P(B)} - \mathrm{P}(A \cap B)$$

If two events A and B are such that $A \cup B = \Omega$ then $P(A \cup B) = 1$ and then these two events are said to be **exhaustive**.

Generally, Given a finite sample space, say $\Omega = \{a_1, a_2, a_3, ..., a_n\}$, we can find a finite probability by assigning to each point $a_i \in \Omega$ a real number P_i, called the probability of a_i, satisfying the following:

a) $P_i \geq 0$ for all integers i, $1 \leq i \leq n$;

b) $\sum_{i=1}^{n} P_i = 1$.

If E is an event, then the probability $P(E)$ is defined to be the sum of the probabilities of the sample points in E.

Example:

1. A coin is weighted so that heads is three times as likely to appear as tails. Find $P(H)$ and $P(T)$.

Solution

Let $P(T) = p_1$, then $P(H) = 3p_1$.

But $P(H) + P(T) = 1$

Therefore $3p_1 + p_1 = 1 \Leftrightarrow 4p_1 = 1 \Rightarrow p_1 = \dfrac{1}{4}$

Thus, $P(H) = \dfrac{3}{4}$ and $P(T) = \dfrac{1}{4}$.

2. A die is thrown once. Let A be the event: "the number obtained is less than 5" and B be the event: "the number obtained is greater than 3". Find probability of $A \cup B$.

Solution

Here $A = \{1,2,3,4\}$ and $B = \{4,5,6\}$, then $A \cup B = \{1,2,3,4,5,6\}$ and then

$P(A \cup B) = P(\Omega) = 1$ Or

$P(A) = \frac{4}{6}$, $P(A) = \frac{3}{6}$, $A \cap B = \{1\}$, then $P(A \cap B) = \frac{1}{6}$

Therefore,

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$
$$= \frac{4}{6} + \frac{3}{6} - \frac{1}{6}$$
$$= 1$$

3. Events A and B are such that they are both mutually exclusive and exhaustive. Find the relation between these two events.

Solution

If A and B are mutually exclusive then

$$P(A \cup B) = P(A) + P(B)$$

If A and B are mutually exclusive then

$$P(A \cup B) = 1$$

Therefore, $P(A) + P(B) = 1$ $\quad P(B) = 1 - P(A)$

But, $P(A') = 1 - P(A)$

Therefore, $P(B) = P(A')$ i.e. $B = A'$

Similarly, $A = B'$

Thus, if events A and B are such that they are both mutually exclusive and exhaustive, then they are complementary.

4. A pen is drawn from a basket containing 10 pens of which 5 are red and 3 are black. If A is the event: "a pen is red" and B is the event: "a pen is black", find $P(A), P(B), P(A \cup B)$.

Solution

There are 5 red pens, then $P(A) = \frac{5}{10} = \frac{1}{2}$

There are 3 black pens, then $P(B) = \frac{3}{10}$

Since the pen cannot be red and black at the same time, then $A \cap B = \varnothing$ and two events are mutually exclusive so

$$P(A \cup B) = P(A) + P(B)$$

$$= \frac{1}{2} + \frac{3}{10} = \frac{4}{5}$$

Application activity 7.4

Work out the following questions

1. If A and B are mutually exclusive events, given the probability of A and B as $\frac{1}{5}$ and $\frac{1}{3}$ respectively, find the probability of at least any one event occurring at a time.

2. If X and Y are two events, the probability of the happening of X or Y is $\frac{7}{10}$ and the probability of X is $\frac{1}{3}$. If X and Y are mutually exclusive, find the probability of Y.

3. In a class of a certain school, there are 12 girls and 20 boys. If a teacher want to choose one student to answer the asked question
   a) What is the probability that the chosen student is a girl?
   b) What is the probability that the chosen student is a boy?
   c) If teacher doesn't care on the gender, what is the probability of choosing any student?

7.5 Examples of Events in real life and determination of related probability

Activity 7.5

1. Two football teams in Rwanda " Rayon Sport" and "APR FC" had to play 3 matches. Two boys Matayo and Manasseh made a betting in the following ways in which the winner should be given 400,000Frw when his event succeeds.

Matayo said that APR will gain the first match only and Rayon Sport will gain the second and the third. Manasseh said that APR will gain at least two matches.

   a) Between Matayo and Manasseh, discuss and determine the boy who has more chances of winning that money.

   b) Is there any risk in betting? Referring to the results obtained in a) what are the points of advice you can give to the youth who spend their money in betting?

2. Carry out a research in the library or on internet to find other applications of probability in real life and present them in the classroom discussion.

Many people don't care about the risks involved in some activities since they do not understand the concept of probability. On the other hand, people may fear activities that involve little risk to health or life because these activities have been sensationalized by the press and media.

We have to think big before taking decision regarding our engagement in the games of chance. Such games are for example: betting on card games, slot machine (ikiryabarezi), lotteries, and weather forecasting. In such games, predictions are based on probability and hypotheses are tested by using probability.

The following are example of applications of rules of probability to solve some problems we can meet in our life experience.

Examples:

1. A box contains 3 blue marbles, 4 red marbles and 5 yellow marbles. If a person selects one marble at a random, find the probability that it is either a blue or yellow marble.

Solution:

The total of marbles is 12. Since there are 3 blue and 5 yellow marbles,

$$P(\text{blue or yellow}) = \text{P(blue)} + \text{P(yellow)} = \frac{3}{12} + \frac{5}{12} = \frac{8}{12}$$

2. In a political rally, there are 200 republicans, 130 Democrats and 60 independents. If a person is selected at random, find the probability that he or she is either democrat or independent.

Solution:

$$P(\text{Democrat or } independent) = \text{P(Democrat)} + \text{P(independent)} = \frac{130}{390} + \frac{60}{390} = \frac{19}{39}$$

3. A single card is drawn from a deck. Find the probability that it is a king or a club.

Solution:

As the king of clubs is counted twice, one of the two probabilities must be subtracted (it a part of intersection)

$$P(\text{king or } c\text{lub}) = \text{P(king)} + \text{P(club)} - \text{P(king of clubs)} = \frac{4}{52} + \frac{13}{52} - \frac{1}{52} = \frac{4}{13}$$

4. In a hospital unit there are 8 nurses and 5 physicians; 7 nurses and 3 physicians are females. If a staff person is selected, find the probability that the subject is a nurse or a male.

Solution:

The sample space is:

Staff	**Females**	**Males**	**Total**
Nurses	7	1	8
Physicians	3	2	5
Total	**10**	**3**	**13**

The probability is

$$P(\text{nurse or } male) = \text{P(nurse)} + \text{P(male)} - \text{P(male nurse)} = \frac{8}{13} + \frac{3}{13} - \frac{1}{13} = \frac{10}{13}.$$

5. A card is drawn from a deck of 52 playing cards. If A is an event of drawing an ace and B is an event of drawing a spade. Find $P(A), P(B), P(A \cap B), P(A \cup B)$

Solution

There are 4 aces, then $P(A) = \frac{4}{52} = \frac{1}{13}$

There are 13 spades, then $P(B)=\frac{13}{52}=\frac{1}{4}$

There is 1 ace of spades, then $\#(A\cap B)=1$ and $P(A\cap B)=\frac{1}{52}$

$$P(A\cup B)=P(A)+P(B)-P(A\cap B)=\frac{1}{13}+\frac{1}{4}-\frac{1}{52}=\frac{4}{13}$$

Alternatevely, there are 4 aces and 13 spades but also 1 ace of spades. Then

$\#(A\cup B)=16$ and $P(A\cup B)=\frac{16}{52}=\frac{4}{13}$

Application activity 7.5

1. In one state of America, the probability that a student owns a car is 0,65, and the probability that a student owns a computer is 0.82. If the probability that a student owns both is 0.55, what is the probability that a given student owns neither a car nor a computer?
2. At a particular school with 200 male students, 58 play football, 40 play basketball, and 8 play both. What is the probability that a randomly selected male student plays neither sport?

7.7. END UNIT ASSESSMENT 11

Lottery

An urn contains 20 lottery tickets numbered from 1 to 20.

To buy a ticket, each one is selected at random and replaced before the next selection. The organizer of the lottery decided to pay 1000Frw to the one who will select a number divisible by 4 and 3 at the same time. He will pay also 500Frw to the one who will select a number which is divisible by 5 and 2 at the same time.

1. Given that the 20 tickets numbered from 1 to 20 were bought at 200Frw per ticket, do the following:

   a) Play this lottery in your class and observe its outcomes. Is the game fair or not?

   b) The money received by the organizer of the lottery

   c) The probability for participants to win 1000Fr

   d) The probability for participants to win 500Frw

   e) The money to be made by the organizer of the lottery.

2. The parents of your friend Anne Marie gave her 200Frw for buying two pens, however, she wants to participate in the lottery to get more money before buying pens. What can you advise her?

Hint: *Use the following events: A: selecting a number divisible by 4; B: selecting the number divisible by 3; C: selecting the number divisible by 5, and D: Selecting the number divisible by 2.*

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