Advanced Mathematics

for Rwandan Schools

Teacher's Guide

Senior Six

FOREWORD

Dear Teachers,

Rwanda Education Board is honoured to present the teacher's guide for senior six Mathematics of advanced level where Mathematics is a major subject. This book serves as a guide to competence-based teaching and learning to ensure consistency and coherence in the learning of the mathematics content. The Rwandan educational philosophy is to ensure that learners achieve full potential at every level of education which will prepare them to be well integrated in society and exploit employment opportunities.

In line with efforts to improve the quality of education, the government of Rwanda emphasizes the importance of aligning teaching and learning materials with the syllabus to facilitate their learning process. Many factors influence what they learn, how well they learn and the competences they acquire. Those factors include the relevance of the specific content, the quality of teachers' pedagogical approaches, the assessment strategies and the instructional materials.

The ambition to develop a knowledge-based society and the growth of regional and global competition in the jobs market has necessitated the shift to a competence-based curriculum. This book provides active teaching and learning techniques that engage student teachers to develop competences.

In view of this, your role is to:

- Plan your lessons and prepare appropriate teaching materials.
- Organize group discussions for students considering the importance of social constructivism suggesting that learning occurs more effectively when the students works collaboratively with more knowledgeable and experienced people.

- Engage students through active learning methods such as inquiry methods, group discussions, research, investigative activities and group and individual work activities.
- Provide supervised opportunities for students to develop different competences by giving tasks which enhance critical thinking, problem solving, research, creativity and innovation, communication and cooperation.
- Support and facilitate the learning process by valuing students' contributions in the class activities.
- Guide students towards the harmonization of their findings.
- Encourage individual, peer and group evaluation of the work done in the classroom and use appropriate competence-based assessment approaches and methods.

To facilitate you in your teaching activities, the content of this book is self explanatory so that you can easily use it. It is divided in 3 parts:

The part I explain the structure of this book and give you the methodological guidance;

The part II gives a sample lesson plan;

The part III details the teaching guidance for concepts given in the student book.

Even though this teacher's guide contains the guidance on solutions for all activities given in the learner's book, you are requested to work through each question before judging student's findings.

I wish to sincerely express my appreciation to the people who contributed towards the development of this book, particularly, REB staff, Teachers from general education and experts from Local and international Organizations for their technical support.

Dr. NDAYAMBAJE Irénée

Director General, REB

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Finally, my word of gratitude goes to the Rwanda Education Board staffs who were involved in the whole process of in-house textbook production.

Joan MURUNGI

Head of Curriculum, Teaching and Learning Resources Department

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PART I. GENERAL INTRODUCTION

1.1 The structure of the guide

The tutor's guide of Mathematics is composed of three parts:

The Part I concerns general introduction that discusses methodological guidance on how best to teach and learn Mathematics, developing competences in teaching and learning, addressing cross-cutting issues in teaching and learning and Guidance on assessment.

Part II presents a sample lesson plan. This lesson plan serves to guide the teacher on how to prepare a lesson in Mathematics.

The Part III is about the structure of a unit and the structure of a lesson. This includes information related to the different components of the unit. This part provides information and guidelines on how to facilitate student while working on learning activities. More other, many application activities from the textbook have answers in this part.

1.2 Methodological guidance

1.2.1 Developing competences

Since 2015 Rwanda shifted from a knowledge based to a competency-based curriculum for pre-primary, primary, secondary education. This called for changing the way of learning by shifting from teacher centred to a learner centred approach. Teachers are not only responsible for knowledge transfer but also for fostering learners' learning achievement and creating safe and supportive learning environment. It implies also that learners have to demonstrate what they are able to transfer the acquired knowledge, skills, values and attitude to new situations.

The competence-based curriculum employs an approach of teaching and learning based on discrete skills rather than dwelling on only knowledge or the cognitive domain of learning. It focuses on what learner can do rather than what learner knows. Learners develop competences through subject unit with specific learning objectives broken down into knowledge, skills and attitudes through learning activities.

In addition to the competences related to Mathematics, student teachers also develop generic competences which should promote the development of the higher order thinking skills and professional skills in Mathematics teaching. Generic competences are developed throughout all units of Mathematics as follows:

Generic competences	**Ways of developing generic competences**
Critical thinking	All activities that require learners to calculate, convert, interpret, analyse, compare and contrast, etc have a common factor of developing critical thinking into learners
Creativity and innovation	All activities that require learners to plot a graph of a given algebraic data, to organize and interpret statistical data collected and to apply skills in solving problems of economics have a common character of developing creativity into learners
Research and problem solving	All activities that require learners to make a research and apply their knowledge to solve problems from the real-life situation have a character of developing research and problem solving into learners.
Communication	During Mathematics class, all activities that require learners to discuss either in groups or in the whole class, present findings, debate …have a common character of developing communication skills into learners.

Co-operation, interpersonal relations and life skills	All activities that require learners to work in pairs or in groups have character of developing cooperation and life skills among learners.
Lifelong learning	All activities that are connected with research have a common character of developing into learners a curiosity of applying the knowledge learnt in a range of situations. The purpose of such kind of activities is for enabling learners to become life-long learners who can adapt to the fast-changing world and the uncertain future by taking initiative to update knowledge and skills with minimum external support.

The generic competences help learners deepen their understanding of Mathematics and apply their knowledge in a range of situations. As students develop generic competences, they also acquire the set of skills that employers look for in their employees, and so the generic competences prepare students for the world of work.

1.2.2 Addressing cross cutting issues

Among the changes brought by the competence-based curriculum is the integration of cross cutting issues as an integral part of the teaching learning process-as they relate to and must be considered within all subjects to be appropriately addressed. The eight cross cutting issues identified in the national curriculum framework are: *Comprehensive Sexuality Education, Environment and Sustainability, Financial Education, Genocide studies, Gender, Inclusive Education, Peace and Values Education, and Standardization Culture*.

Some cross-cutting issues may seem specific to particular learning areas/subjects but the teacher need to address all of them whenever an opportunity arises. In addition, learners should always be given an opportunity during the learning process to address these cross-cutting issues both within and out of the classroom.

Below are examples of how crosscutting issues can be addressed:

Cross-Cutting Issue	Ways of addressing cross-cutting issues
Comprehensive Sexuality Education: The primary goal of introducing Comprehensive Sexuality Education program in schools is to equip children, adolescents, and young people with knowledge, skills and values in an age appropriate and culturally gender sensitive manner so as to enable them to make responsible choices about their sexual and social relationships, explain and clarify feelings, values and attitudes, and promote and sustain risk reducing behaviour.	Using different charts and their interpretation, Mathematics teachers should lead students to discuss the following situation: "Alcohol abuse and unwanted pregnancies" and advise students on how they can fight those abuses. Some examples can be given when learning statistics, powers, logarithms and their properties.
Environment and Sustainability: Integration of Environment, Climate Change and Sustainability in the curriculum focuses on and advocates for the need to balance economic growth, society well-being and ecological systems. Learners need basic knowledge from the natural sciences, social sciences, and humanities to understand to interpret principles of sustainability.	Using Real life models or students' experience, Mathematics teacher should lead students to illustrate the situation of "population growth" and discuss its effects on the environment and sustainability.

Financial Education: The integration of Financial Education into the curriculum is aimed at a comprehensive Financial Education program as a precondition for achieving financial inclusion targets and improving the financial capability of Rwandans so that they can make appropriate financial decisions that best fit the circumstances of one's life.	Through different examples and calculations on interest rate problems, total revenue and total cost, Mathematics teacher can lead student to discuss how to make appropriate financial decisions.
Gender: At school, gender will be understood as family complementarities, gender roles and responsibilities, the need for gender equality and equity, gender stereotypes, gender sensitivity, etc.	Mathematics teacher should address gender as cross-cutting issue through assigning leading roles in the management of groups to both girls and boys and providing equal opportunity in the lesson participation and avoid any gender stereotype in the whole teaching and learning process.
Inclusive Education: Inclusion is based on the right of all learners to a quality and equitable education that meets their basic learning needs and understands the diversity of backgrounds and abilities as a learning opportunity.	Firstly, Mathematics teachers need to identify/ recognize students with special needs. Then by using adapted teaching and learning resources while conducting a lesson and setting appropriate tasks to the level of students, they can cater for students with special education needs. They must create opportunity where student teachers can discuss how to support colleagues with special educational needs.

Peace and Values Education: Peace and Values Education (PVE) is defined as education that promotes social cohesion, positive values, including pluralism and personal responsibility, empathy, critical thinking and action in order to build a more peaceful society.	Through a given lesson, a teacher should: ▪ Set a learning objective which is addressing positive attitudes and values, ▪ Encourage students to develop the culture of tolerance during discussion and to be able to instil it in colleagues and cohabitants; ▪ Encourage students to respect ideas for others.
Standardization Culture: Standardization Culture in Rwanda will be promoted through formal education and plays a vital role in terms of health improvement, economic growth, industrialization, trade and general welfare of the people through the effective implementation of Standardization, Quality Assurance, Metrology and Testing.	With different word problems related to the effective implementation of Standardization, Quality Assurance, Metrology and Testing, students can be motivated to be aware of health improvement, economic growth, industrialization, trade and general welfare of the people.

1.2.3 Guidance on how to help students with special education needs in classroom

In the classroom, students learn in different way depending to their learning pace, needs or any other special problem they might have. However, the teacher has the responsibility to know how to adopt his/her methodologies and approaches in order to meet the learning need of each student in the classroom. Also teachers need to understand that student with special needs, have to be taught differently or need some accommodations to enhance the learning environment. This will be done depending on the subject and the nature of the lesson.

In order to create a well-rounded learning atmosphere, teachers need to:

- Remember that learners learn in different ways so they have to offer a variety of activities (e.g. role-play, music and singing, word games and quizzes, and outdoor activities);
- Maintain an organized classroom and limits distraction. This will help learners with special needs to stay on track during lesson and follow instruction easily;
- Vary the pace of teaching to meet the needs of each child. Some learners process information and learn more slowly than others;
- Break down instructions into smaller, manageable tasks. Learners with special needs often have difficulty understanding long-winded or several instructions at once. It is better to use simple, concrete sentences in order to facilitate them understand what you are asking.
- Use clear consistent language to explain the meaning (and demonstrate or show pictures) if you introduce new words or concepts;
- Make full use of facial expressions, gestures and body language;
- Pair a learner who has a disability with a friend. Let them do things together and learn from each other. Make sure the friend is not over protective and does not do everything for the one with disability. Both learners will benefit from this strategy;
- Use multi-sensory strategies. As all learners learn in different ways, it is important to make every lesson as multi-sensory as possible. Learners with learning disabilities might have difficulty in one area, while they might excel in another. For example, use both visual and auditory cues.

Below are general strategies related to each main category of disabilities and how to deal with every situation that may arise in the classroom. However, the list is not exhaustive because each child is unique with different needs and that should be handled differently.

Strategy to help learners with developmental impairment:

- Use simple words and sentences when giving instructions;
- Use real objects that learners can feel and handle. Rather than just working abstractly with pen and paper;
- Break a task down into small steps or learning objectives. The learner should start with an activity that she/he can do already before moving on to something that is more difficult;
- Gradually give the learner less help;
- Let the learner with disability work in the same group with those without disability.

Strategy to help learners with visual impairment:

- Help learners to use other senses (hearing, touch, smell and taste) and carry out activities that will promote their learning and development;
- Use simple, clear and consistent language;
- Use tactile objects to help explain a concept;
- If the learner has some sight, ask him/her what he/she can see;
- Make sure the learner has a group of friends who are helpful and who allow him/her to be as independent as possible;
- Plan activities so that learners work in pairs or groups whenever possible;

Strategy to help learners with hearing disabilities or communication difficulties

- Always get the learner's attention before you begin to speak;
- Encourage the learner to look at your face;

- Use gestures, body language and facial expressions;
- Use pictures and objects as much as possible.
- Keep background noise to a minimum.

Strategies to help learners with physical disabilities or mobility difficulties:

- Adapt activities so that learners, who use wheelchairs or other mobility aids, can participate.
- Ask parents/caregivers to assist with adapting furniture e.g. the height of a table may need to be changed to make it easier for a learner to reach it or fit their legs or wheelchair under;
- Encourage peer support when needed;
- Get advice from parents or a health professional about assistive devices if the learner has one.

Adaptation of assessment strategies:

At the end of each unit, the tutor is advised to provide additional activities to help students achieve the key unit competence. These assessment activities are for remedial, consolidation and extension designed to cater for the needs of all categories of students; slow, average and gifted students respectively. Therefore, the tutor is expected to do assessment that fits individual student.

Remedial activities	After evaluation, slow students are provided with lower order thinking activities related to the concepts learnt to facilitate them in their learning. These activities can also be given to assist deepening knowledge acquired through the learning activities for slow students.
Consolidation activities	After introduction of any concept, a range number of activities can be provided to all students to enhance/ reinforce learning.

Extended activities	After evaluation, gifted and talented students can be provided with high order thinking activities related to the concepts learnt to make them think deeply and critically. These activities can be assigned to gifted and talented students to keep them working while other students are getting up to required level of knowledge through the learning activity.

1.2.4. Guidance on assessment

Assessment is an integral part of teaching and learning process. The main purpose of assessment is for improvement of learning outcomes. Assessment for learning/ Continuous/ formative assessment intend to improve students' learning and tutor's teaching whereas assessment of learning/summative assessment intends to improve the entire school's performance and education system in general.

Continuous/ formative assessment

It is an on-going process that arises during the teaching and learning process. It includes lesson evaluation and end of sub unit assessment. This formative assessment should play a big role in teaching and learning process. The teacher should encourage individual, peer and group evaluation of the work done in the classroom and uses appropriate competence-based assessment approaches and methods.

Formative assessment is used to:

- Determine the extent to which learning objectives are being achieved and competences are being acquired and to identify which students need remedial interventions, reinforcement as well as extended activities. The application activities are developed in the learner book and they are designed to be given as remedial, reinforcement, end lesson assessment, homework or assignment

- Motivate students to learn and succeed by encouraging students to read, or learn more, revise, etc.
- Check effectiveness of teaching methods in terms of variety, appropriateness, relevance, or need for new approaches and strategies. Mathematics tutors need to consider various aspects of the instructional process including appropriate language levels, meaningful examples, suitable methods and teaching aids/ materials, etc.
- Help students to take control of their own learning.

In teaching Mathematics, formative or continuous assessment should compare performance against instructional objectives. Formative assessment should measure the student's ability with respect to a criterion or standard. For this reason, it is used to determine what students can do, rather than how much they know.

Summative assessment

The assessment can serve as summative and informative depending to its purpose. The end unit assessment will be considered summative when it is done at end of unit and want to start a new one.

It will be formative assessment, when it is done in order to give information on the progress of learners and from there decide what adjustments need to be done.

The assessment done at the end of the term, end of year, is considered as summative assessment so that the teacher, school and parents are informed of the achievement of educational objective and think of improvement strategies. There is also end of level/ cycle assessment in form of national examinations.

When carrying out assessment?

Assessment should be clearly visible in lesson, unit, term and yearly plans.

- Before learning (diagnostic): At the beginning of a new unit or a section of work; assessment can be organized to find out what students already know / can do, and to check whether the students are at the same level.

- During learning (formative/continuous): When students appear to be having difficulty with some of the work, by using on-going assessment (continuous). The assessment aims at giving students support and feedback.
- After learning (summative): At the end of a section of work or a learning unit, the Mathematics Tutor has to assess after the learning. This is also known as Assessment of Learning to establish and record overall progress of students towards full achievement. Summative assessment in Rwandan schools mainly takes the form of written tests at the end of a learning unit or end of the month, and examinations at the end of a term, school year or cycle.

Instruments used in assessment.

- **Observation:** This is where the Mathematics teacher gathers information by watching students interacting, conversing, working, playing, etc. A teacher can use observations to collect data on behaviours that are difficult to assess by other methods such as attitudes, values, and generic competences and intellectual skills. It is very important because it is used before the lesson begins and throughout the lesson since the teacher has to continue observing each and every activity.
- **Questioning**

(a) Oral questioning: a process which requires a student to respond verbally to questions

(b) Class activities/ exercises: tasks that are given during the learning/ teaching process

(c) Short and informal questions usually asked during a lesson

(d) Homework and assignments: tasks assigned to students by their teacher to be completed outside of class.

Homework assignments, portfolio, project work, interview, debate, science fair, Mathematics projects and Mathematics competitions are also the different forms/instruments of assessment.

1.2.5. Teaching methods and techniques that promote active learning

The different learning styles for students can be catered for, if the teacher uses active learning whereby learners are really engaged in the learning process.

The main teaching methods used in mathematics are the following:

- **Inductive-deductive method:** Inductive method is to move from specific examples to generalization and deductive method is to move from generalization to specific examples.
- **Analytic-synthetic method:** Analytic method proceeds from unknown to known, 'Analysis' means 'breaking up' of the problem in hand so that it ultimately gets connected with something obvious or already known. Synthetic method is the opposite of the analytic method. Here one proceeds from known to unknown.
- **Skills Laboratory method:** Laboratory method is based on the maxim "learning by doing." It is a procedure for stimulating the activities of the students and to encourage them to make discoveries through practical activities.

Problem solving method, Project method and Seminar Method.

The following are some active techniques to be used in Mathematics:

- Group work
- Research
- Probing questions
- Practical activities (drawing, plotting, interpreting graphs)

- Modelling
- Brainstorming
- Quiz Technique
- Discussion Technique
- Scenario building Technique

What is Active learning?

Active learning is a pedagogical approach that engages learners in doing things and thinking about the things they are doing. Learners play the key role in the active learning process. They are not empty vessels to fill but people with ideas, capacity and skills to build on for effective learning. Thus, in active learning, learners are encouraged to bring their own experience and knowledge into the learning process.

The role of the teacher in active learning	The role of learners in active learning
- The teacher engages learners through active learning methods such as inquiry methods, group discussions, research, investigative activities, group and individual work activities. - He/she encourages individual, peer and group evaluation of the work done in the classroom and uses appropriate competence-based assessment approaches and methods. - He provides supervised opportunities for learners to develop different competences by giving tasks which enhance critical thinking, problem solving, research, creativity and innovation, communication and cooperation.	A learner engaged in active learning: - Communicates and shares relevant information with peers through presentations, discussions, group work and other learner-centred activities (role play, case studies, project work, research and investigation); - Actively participates and takes responsibility for his/her own learning; - Develops knowledge and skills in active ways; - Carries out research/ investigation by consulting print/online documents and resourceful people, and presents their findings;

- Teacher supports and facilitates the learning process by valuing learners' contributions in the class activities.	- Ensures the effective contribution of each group member in assigned tasks through clear explanation and arguments, critical thinking, responsibility and confidence in public speaking - Draws conclusions based on the findings from the learning activities.

Main steps for a lesson in active learning approach

All the principles and characteristics of the active learning process highlighted above are reflected in steps of a lesson as displayed below. Generally, the lesson is divided into three main parts whereby each one is divided into smaller steps to make sure that learners are involved in the learning process. Below are those main part and their small steps:

1) Introduction

Introduction is a part where the teacher makes connection between the current and previous lesson through appropriate technique. The teacher opens short discussions to encourage learners to think about the previous learning experience and connect it with the current instructional objective. The teacher reviews the prior knowledge, skills and attitudes which have a link with the new concepts to create good foundation and logical sequencing.

2) Development of the new lesson

The development of a lesson that introduces a new concept will go through the following small steps: discovery activities, presentation of learners' findings, exploitation, synthesis/summary and exercises/application activities.

❖ Discovery activity

Step 1

- The teacher discusses convincingly with learners to take responsibility of their learning
- He/she distributes the task/activity and gives instructions related to the tasks (working in groups, pairs, or individual to instigate collaborative learning, to discover knowledge to be learned).

Step 2

- The teacher let learners work collaboratively on the task;
- During this period the teacher refrains to intervene directly on the knowledge;
- He/she then monitors how the learners are progressing towards the knowledge to be learned and boosts those who are still behind (but without communicating to them the knowledge).

❖ **Presentation of learners' findings/productions**

- In this episode, the teacher invites representatives of groups to present their productions/findings.
- After three/four or an acceptable number of presentations, the teacher decides to engage the class into exploitation of learners' productions.

❖ **Exploitation of learner's findings/ productions**

- The teacher asks learners to evaluate the productions: which ones are correct, incomplete or false
- Then the teacher judges the logic of the learners' products, corrects those which are false, completes those which are incomplete, and confirms those which are correct.

- ❖ Institutionalization or harmonization (summary/conclusion/ and examples)

  - The teacher summarizes the learned knowledge and gives examples which illustrate the learned content.

- ❖ Application activities

  - Exercises of applying processes and products/objects related to learned unit/sub-unit
  - Exercises in real life contexts
  - Teacher guides learners to make the connection of what they learnt to real life situations. At this level, the role of teacher is to monitor the fixation of process and product/object being learned.

3) Assessment

In this step the teacher asks some questions to assess achievement of instructional objective. During assessment activity, learners work individually on the task/activity. The teacher avoids intervening directly. In fact, results from this assessment inform the teacher on next steps for the whole class and individuals. In some cases, the teacher can end with a homework/ assignment. Doing this will allow learners to relay their understanding on the concepts covered that day. Teacher leads them not to wait until the last minute for doing the homework as this often results in an incomplete homework set and/or an incomplete understanding of the concept.

PART II: EXAMPLE OF LESSON

When teaching any lesson, you can follow the following steps.

Introduction

Start by reviewing previous lesson through asking some questions to learners. If there is no previous lesson, ask them prerequisites related questions for the lesson of the day.

Lesson development

In this step, activities can be more than one (exploration activity, explanation activity and elaboration activity). For each one, give an activity to learners that will be done in groups or individually. After a while, invite one or more groups for presentation of their work to other groups. If the activity is individual, ask one or more learners to present his/her work to others. After activities, capture the main points from the presentation of the learners and guide the whole class to summarize them. After this, provide application activity in their respective groups. Request learners to correct them on chalkboard where you guide every student by addressing eventual misconception.

Evaluation

Give students an activity to be done individually as an assessment. Correct every one and provide related feedback.

Conclusion

Conclude the lesson and remember to assign a home work to students. This homework may include remedial activities, consolidation activities or extended activities depending on the feedback from the assessment. Sometimes when there is no problem in the assessment, a teacher can provide a homework which will arouse the curiosity of students for the next lesson.

Part II: Sample lesson plan

School: ……………………… Academic year: ………..

Teacher's name: ………………

Term	Date	Subject	Class	Unit No	Lesson No	Duration	Class size
1	….	Mathe-matics	S6 MEG	1	1 of 27	40 minutes	35

Type of Special Educational Needs and number of learners	• **3 low vision learners:** Give them places where they are able to see what is written on the blackboard. Avoid making their own group otherwise it can be considered as segregation. • **4 learners with extra abilities:** To encourage them to explain, to each other and help their classmates.
Unit title	Complex numbers
Key Unit Competence:	Perform operations on complex numbers in different forms and use complex numbers to solve related problems in Physics (voltage and current in alternating current), computer Science (fractals), Trigonometry (Euler's formula to transform trigonometric expressions).
Title of the lesson	Concepts of complex numbers
Instructional objective	Through examples, learners should be able to define a complex number, show real part and imaginary part of a complex number and show that two complex numbers are equal or not equal accurately.
Plan for this Class	Location: Classroom Learners are organised into groups
Learning Materials	Exercise book, pen, calculator, ruler
References	Advanced Mathematics for Rwanda Schools, Learner's Book Senior 6

Timing for each step	DESCRIPTION OF TEACHING AND LEARNING ACTIVITY		Competences and cross cutting issues to be addressed
	In groups, learners will do activity 1.1 in learner's book page 2, make presentation of group findings. In conclusion, learners will do questions 1.a), 1.b), 2.a) and 2.b) of exercise 1.1 in the Learner's Book in their respective groups and solve them on the chalkboard. Learners will do questions 1.c) and 2.c) of exercise 1.1 as individual quiz and questions 1.d) and 2.d) will be an assignment. At the end of the lesson, learners are also given another assignment to be discussed as an activity of the next lesson "Definition and properties of the number i".		
	Teacher's activities	**Learners' activities**	
Introduction **5 minutes**	Ask a question, on how to solve quadratic equations in set of real numbers using discriminant method (Including case where the discriminant is negative).	**Questions** By using discriminant method, solve in $\mathbb{R}$ 1. $x^2+7x+10=0$ 2. $x^2+4x+4=0$ 3. $x^2+x+4=0$ **Solution** 1. $x^2+7x+10=0$ $\Delta=49-40=9$ $x_1=\frac{-7+3}{2}=-2,\ x_2=\frac{-7-3}{2}=-5$ $S=\{-5,-2\}$ 2. $x^2+4x+4=0$ $\Delta=16-16=0$ $x_1=x_2=\frac{-4}{2}=-2$ $S=\{-2\}$ 3. $x^2+x+4=0$ $\Delta=1-16=-15<0$ No real solution	Students are developing communication skills when they are explaining and sharing ideas.

Development of the lesson **5 minutes** **10 minutes**	**Step 1:** Form groups • Ask learners to do activity 1.1 in Learner's Book page 2 in their groups. • Go round to check the progress of the discussion, and intervene where necessary. • Guide learners with special educational needs on how to do activity. **Step 2:** Ask groups to present their work on the chalkboard.	In their groups, learners will do activity 1.1. In their exercise book using the fact that $\sqrt{-1}=i$, they will find two numbers, a and b, whose sum is 6 and whose product is 18. Secretary presents the work. Learners interact through questions and comments. **Answers** Recall that a quadratic equation is written as $x^2-sx+p=0$, where s and p are the sum and product of two roots respectively. Then, we need to solve the equation $x^2-6x+18=0$ $\Delta=(-6)^2-4(18)$ $=36-72$ $=-36$ $x_1=a=\dfrac{6+\sqrt{-36}}{2}$	Cooperation and interpersonal management developed through working in groups. Communication: Learners communicate and convey information and ideas through speaking when they are presenting their work. Self confidence: Learners will gain self confidence competence when they are presenting their work. In group activities, the fact of being convinced without fighting, peace and education values are developed too.

		and $x_2 = b = \frac{6-\sqrt{-36}}{2}$ Now, if $\sqrt{-1} = i$ or $i^2 = -1$, we have $a = \frac{6+\sqrt{36\times i^2}}{2}$ $b = \frac{6-\sqrt{36\times i^2}}{2}$ or $a = 3+3i,\ b = 3-3i$	
Conclusion **5 minutes**	Ask learners to give the main points of the learned lesson in summary.	Summarise the learned lesson: A complex number is a number that can be put in the form $a+bi$, where a and b are real numbers and $i = \sqrt{-1}$ (i being the first letter of the word "imaginary"). The set of all complex numbers is denoted by $\mathbb{C}$ and is defined as $\mathbb{C} = \left\{\begin{matrix} z = a+bi : a,b \in \mathbb{R} \\ and \ \ i^2 = -1 \end{matrix}\right\}$ The number a of the complex number $z = a+bi$ is called the real part of z and denoted by $Re(z)$ or $\Re(z)$;	Learners develop critical thinking through generating a summary. Through group activities, cooperation is developed.

		the number b is called the **imaginary part** and denoted by $Im(z)$ or $\Im(z)$. A complex number whose **real part** is zero, is said to be purely imaginary, whereas a complex number whose **imaginary part** is zero, is said to be a real number or simply **real**. Thus, $\forall x \in \mathbb{R}, x \in \mathbb{C}$, which gives that $\mathbb{R} \subset \mathbb{C}$.	
5 minutes **5 minutes**	Request learners to do questions 1.a) and 2.a) of exercise 1.1 in their respective groups. Move around the class checking the progress of the discussion, and intervene where necessary. Request some learners to answer to questions 1.b), and 2.b) of exercise 1.1 on chalkboard. Ensures that learners understand the learned lesson and decide whether to repeat the lesson or to start a new a lesson next time.	Learners will do questions 1.a) and 2.a) of exercise 1.1, in their respective groups. Learners will present answers of questions 1.b) and 2.b) of exercise 1.1, on chalkboard.	Through presentation on chalkboard, communication skills are developed.

5 minutes	Give learners an individual evaluation (quiz) and homework in regard to the learned lesson. **Lead into next lesson:** Request learners to do activity 1.2 at home.	Learners will do questions 1.c) and 2.c) of exercise 1.1 as individual quiz; questions 1.d) and 2.d) as assignment.	
Teacher's self evaluation	Even if the objective has been achieved, some learners don't remember how to solve a quadratic equation using discriminant method. The time management has been disturbed by revising how to use discriminant method. For this reason, next time before any activity, learners will be given a task of revising the topics related to the given activity as homework.		

Icons

The icons used in this book are as follows:

Practical Activity icon

The hand indicates a practical activity such as curve sketching, draw figures and then presents the results or comments. The activity is done in groups.

Group Work icon

Group work means that learners are expected to discuss something in groups and report back on what their group discussed. In this way, they learn from each other and also learn how to work together as a group to address or solve a problem.

Pairing Activity icon

This means that they are required to do the activity in pairs, exchange ideas and write down the results.

Research Activity icon

Some activities require you to do research either by reading textbooks or using the internet.

Unit 1 Complex numbers

1.1. Key unit competence

Perform operations on complex numbers in different forms and use complex numbers to solve related problems in Physics (voltage and current in alternating current), Computer Science (fractals), Trigonometry (Euler's formulae to transform trigonometric expressions), …

1.2. Vocabulary or key words concepts

Argand diagram: Plane representing complex plane where $x-axis$ is called real axis and $y-axis$ is called imaginary axis.

Affix: Coordinates of a complex number in Argand diagram.

Modulus: The distance from origin to the affix of the complex number.

Argument: Argument of complex number z is the angle the segment oz makes with the positive real x-axis.

Polar form: A way of expressing a complex number using its modulus, its argument and basic trigonometric ratios (sine and cosine).

Exponential form: A way of expressing a given complex number using its modulus, its argument and the number e.

1.3. Guidance on the introductory activity

The problem statement is "*Solve in set of real number the following equations* $x^2+6x+8=0$ and $x^2+4=0$".

For first equation, we have solution in $\mathbb{R}$ but second equation does not have solution in $\mathbb{R}$. Therefore, the set $\mathbb{R}$ is not sufficient to contain solutions of some equations.

Since the square root of -4 does not exist in set of real number, for second equation we introduce new kind of number i such that $i^2 = -1$ and we write $x^2 = -4 = 4 \times i^2$ so that $x = \pm\sqrt{-4} = \pm\sqrt{4 \times i^2} = \pm 2i$

1.4. List of lessons

No	Lesson title	Number of periods
1	Concepts of complex numbers	1
2	Definition and properties of number i	1
3	Geometric representation of a complex number	1
4	Modulus of a complex number	1
5	Loci related to distances	1
6	Equality of complex numbers	1
7	Addition and subtraction of complex numbers	1
8	Conjugate and opposite of a complex number	1
9	Multiplication of complex numbers	1
10	Inverse and division of complex numbers	1
11	Square root of a complex number	1
12	Linear equations in set of complex numbers	1
13	Quadratic equations in set of complex numbers	1
14	Polynomials in set of complex numbers	2
15	Argument of a complex number	2
16	Loci related to angles	1
17	Polar form of a complex number	1
18	Multiplication and division of complex numbers in polar form	1

19	Powers of complex number in polar form	1
20	N^{th} roots of a complex number	2
21	Graphical representation of nth roots of a complex number	2
22	Construction of regular polygon	2
23	Exponential form of a complex number	1
24	Trigonometric number of a multiple of an angle	2
25	Linearisation of trigonometric expressions	2
26	Solving equation of the form $a\cos x + b\sin x = c$	2
27	Alternating current problem	2
Total periods		**36**

1.5. Lesson development

Lesson 1.1. Concepts of complex numbers

Learning objectives

Through examples, learners should be able to define a complex number, show real part and imaginary part of a complex number and show that two complex numbers are equal or not equal accurately.

Prerequisites

- Find the square root of a positive real number.
- The square root of a negative real number does not exist.

Teaching Aids

Exercise book, pen and calculator

Activity 1.1

In this lesson, the following generic competence and cross-cutting issues are to be addressed:

- Critical thinking
- Communication
- Self confidence
- Cooperation, interpersonal management and life skills
- Peace and values education
- Inclusive education

Answers

1. Here, we need to solve the equation $x^2 - 6x + 18 = 0$

   Calculating the discriminant for $x^2 - 6x + 18 = 0$, we get

   $\Delta = (-6)^2 - 4(18) = 36 - 72 = -36$

   $$x_1 = a = \frac{6+\sqrt{-36}}{2} \text{ and } x_2 = b = \frac{6-\sqrt{-36}}{2}$$

   As $\Delta < 0$, $x^2 - 6x + 18 = 0$ has no real solutions.

2. Now, if $\sqrt{-1} = i$ or $i^2 = -1$, we get

   $$a = \frac{6+\sqrt{-36}}{2} = \frac{6+\sqrt{36i^2}}{2} = \frac{6+6i}{2} = 3+3i$$

   and $b = \dfrac{6-\sqrt{-36}}{2} = \dfrac{6-\sqrt{36i^2}}{2} = \dfrac{6-6i}{2} = 3-3i$.

3. *a* and *b*, are not elements of $\mathbb{R}$ or $a, b \notin \mathbb{R}$

Synthesis

As conclusion, a complex number is a number that can be put in the form $a + bi$, where *a* and *b* are real numbers and $i = \sqrt{-1}$ (*i* being the first letter of the word "imaginary").

Application Activity 1.1

1. a) $\mathrm{Re}(z) = 0,\ \mathrm{Im}(z) = 45$ b) $\mathrm{Re}(z) = -3,\ \mathrm{Im}(z) = 0$

   c) $\mathrm{Re}(z) = -1,\ \mathrm{Im}(z) = 3$ d) $\mathrm{Re}(z) = -10,\ \mathrm{Im}(z) = 7$

2. a) Real   b) Purely imaginary

   c) Purely imaginary

   d) neither real nor purely imaginary

Lesson 1.2. Definition and properties of the number *i*

Learning objectives

Given powers of the number i with natural exponents, learners should be able to simplify them accurately.

Prerequisites

- Properties of powers in set of real numbers.

Teaching Aids

Exercise book, pen and calculator

Activity 1.2

In this lesson, the following generic competence and cross-cutting issues are to be addressed:

- Critical thinking
- Self confidence
- Communication
- Cooperation, interpersonal management and life skills
- Peace and values education
- Inclusive education

Answers

1. $i^3 = i^2 i^1 = -1i = -i$

   $i^4 = i^2 i^2 = (-1)\times(-1) = 1$

   $i^5 = i^2 i^2 i^1 = (-1)\times(-1)\times i = i$

   $i^6 = i^2 i^2 i^2 = (-1)\times(-1)\times(-1) = -1$

   $i^7 = i^2 i^2 i^2 i = (-1)\times(-1)\times(-1) i = -i$

   $i^8 = i^2 i^2 i^2 i^2 = (-1)\times(-1)\times(-1)\times(-1) = 1$

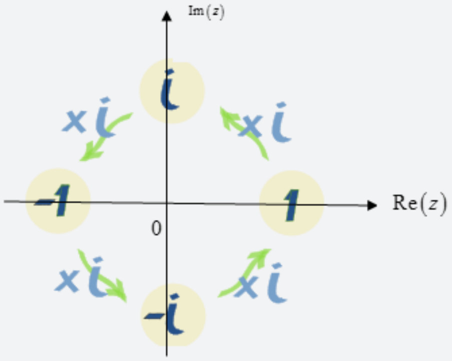
   $i^9 = i^2 i^2 i^2 i^2 i = (-1)\times(-1)\times(-1)\times(-1) i = i$

2. In general;

   $i^{4k} = 1,\ i^{4k+1} = i,\ i^{4k+2} = -1,\ i^{4k+3} = -i\ \ k = 0,1,2,3,4,5,...$

Synthesis

The imaginary unit, i, "cycles" through 4 different values each time we multiply as it is illustrated in the following figure .



The powers of imaginary unit can be generalised as follows:

$i^{4k}=1,\ i^{4k+1}=i,\ i^{4k+2}=-1,\ i^{4k+3}=-i$

Application Activity 1.2

1. -1	2. i	3. 1
4. $-i$	5. i	6. $-i$

Lesson 1.3. Geometric representation of a complex number

Learning objectives

Given complex numbers and using a ruler, learners should be able to represent those complex numbers in Argand diagram accurately.

Prerequisites

- Remember how to represent a point (x, y) in Cartesian plane.

Teaching Aids

Exercise book, pen, instruments of geometry

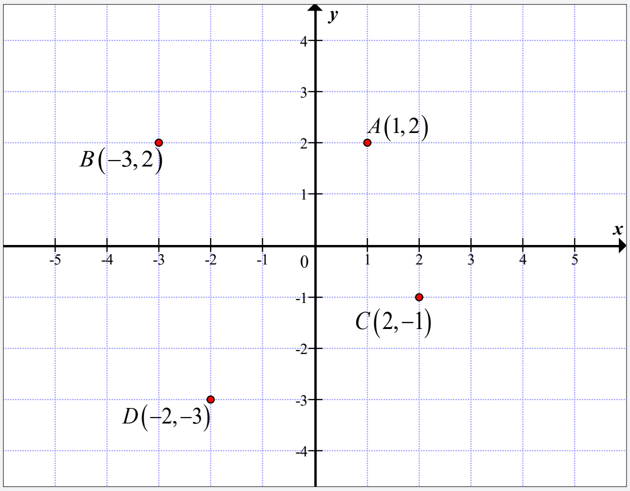
Activity 1.3

In this lesson, the following generic competence and cross-cutting issues are to be addressed:

- Critical thinking
- Communication
- Self confidence

- Cooperation, interpersonal management and life skills
- Peace and values education
- Inclusive education

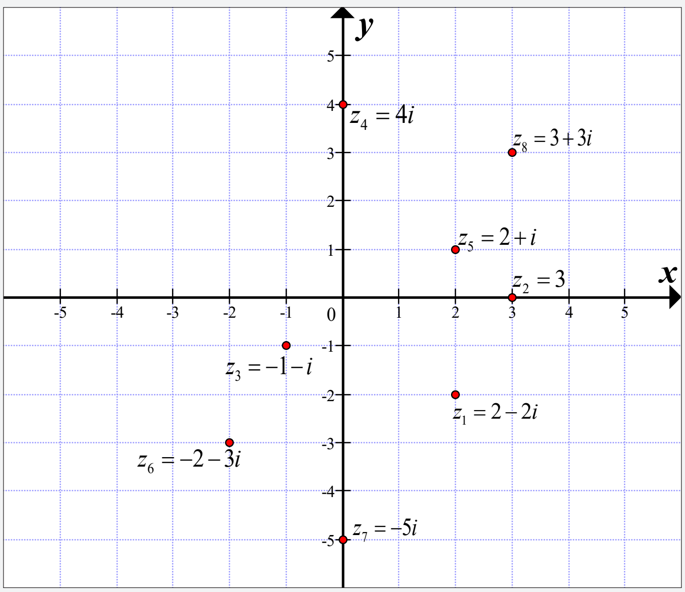
Answers



Synthesis

A complex number $z = a + bi$ can be visually represented as a pair of numbers (a,b) forming a vector from the origin or point on a diagram called **Argand diagram**.

Application Activity 1.3



Lesson 1.4. Modulus of a complex number

Learning objectives

Given a complex number, learners should be able to find its modulus correctly.

Prerequisites

- Finding distance between two points in Cartesian plane.

Teaching Aids

Exercise book, pen and calculator

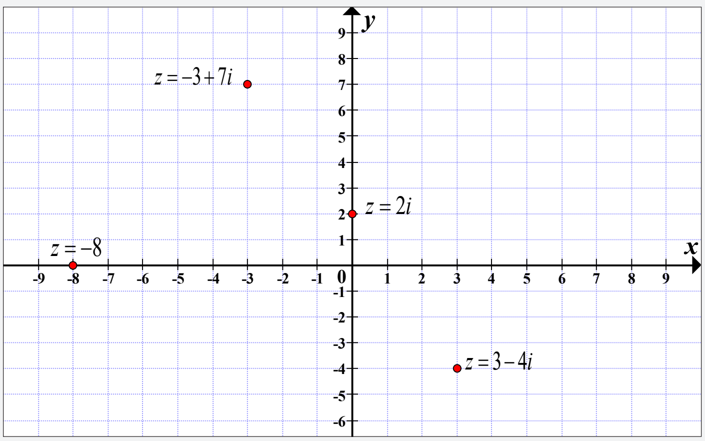
Activity 1.4

In this lesson, the following generic competence and cross-cutting issues are to be addressed:

- Critical thinking
- Communication
- Self confidence
- Cooperation, interpersonal management and life skills
- Peace and values education
- Inclusive education

Answers

Curve:



1. $z=-8=(-8,0)$

   Distance from origin is $\sqrt{(-8-0)^2+0}=8$

2. $z = 2i = (0,2)$

Distance from origin is $\sqrt{0+(2-0)^2} = 2$

3. $z = -3+7i = (-3,7)$

Distance from origin is $\sqrt{(-3-0)^2+(7-0)^2} = \sqrt{58}$

4. $z = 3-4i = (3,-4)$

Distance from origin is $\sqrt{(3-0)^2+(-4-0)^2} = 5$

Synthesis

As conclusion, the distance from the origin to point (x, y) corresponding to the complex number $z = x+yi$ is called the modulus of z and is denoted by $|z|$ or $|x+iy|$: $r = |z| = \sqrt{x^2+y^2}$

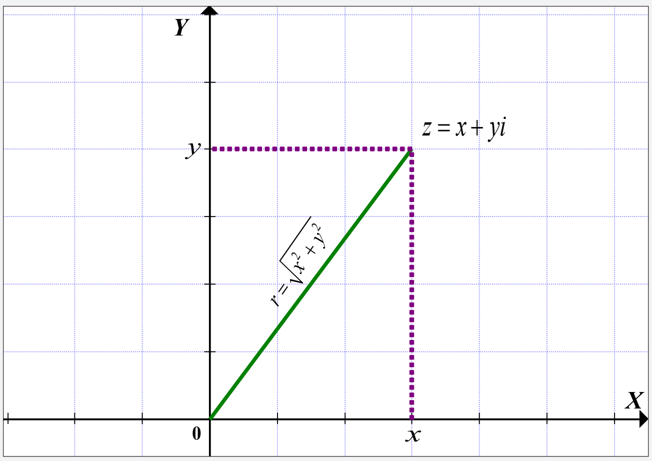


Figure 1.1: Modulus of a complex number

Application Activity 1.4

1) $\sqrt{5}$ 2) 5 3) 1

4. $\frac{\sqrt{2}}{2}$ 5) 1 6) $5\sqrt{5}$

Lesson 1.5. Loci related to distances on Argand diagram

Learning objectives

Given a condition, learners should be able to determine the locus on Argand plane precisely.

Prerequisites

- Finding modulus of a complex number.
- The general form of equation of a circle, a straight line,… in Cartesian plane.

Teaching Aids

Exercise book, pen, calculator and instrument of geometry

Activity 1.5

In this lesson, the following generic competence and cross-cutting issues are to be addressed:

- Critical thinking
- Communication
- Self confidence
- Cooperation, interpersonal management and life skills
- Peace and values education
- Inclusive education

Answers

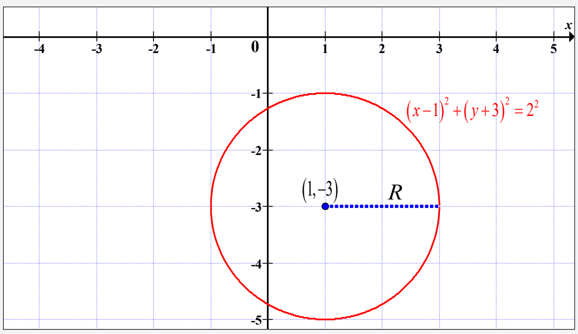
Let $z = x + yi$, we have

$$|x + yi - 1 + 3i| = 2 \Leftrightarrow |x - 1 + i(y+3)| = 2$$

$$\Leftrightarrow \sqrt{(x-1)^2 + (y+3)^2} = 2 \Leftrightarrow (x-1)^2 + (y+3)^2 = 2^2$$

which is the circle of centre $(1,-3)$ or $1-3i$ and radius $R = 2$.

Curve



Synthesis

As conclusion, $|z| = R$ represents a circle with centre P and radius R, $|z - z_1| = R$ represents a circle with centre z_1 and radius R and $|z - z_1| = |z - z_2|$ represents a straight line, the perpendicular bisector (mediator) of the segment joining the points z_1 and z_2.

Application Activity 1.5

1. Circle: $3x^2 + 3y^2 + 4x + 1 = 0$, radius is $\frac{1}{3}$ and centre is $\left(-\frac{2}{3}, 0\right)$
2. a) Circle: $x^2 + y^2 = 4$, radius 2, centre at origin; (0,0)
   b) Interior of the circle: $x^2 + y^2 = 4$, radius 2, centre at origin
   c) Exterior of the circle: $x^2 + y^2 = 4$, radius 2, centre at origin
   d) Circle: $(x+1)^2 + y^2 = 1$, radius 1, centre $(-1, 0)$
   e) Vertical line: $z_1 = -1$, mediator of the line segment joining points $z_1 = -1$ and $z_2 = 1$
   f) Circle: $(x-1)^2 + (y+3)^2 = 4$, radius 2, centre $(1, -3)$

Lesson 1.6. Equality of two complex numbers

Learning objectives

Given two complex numbers, learners should be able to show that they are equal or not and to use this concept to solve some equation accurately.

Prerequisites

- Solving a linear equation with one unknown.
- Solving a system of two linear equations with two unknown.

Teaching Aids

Exercise book, pen and calculator

Activity 1.6

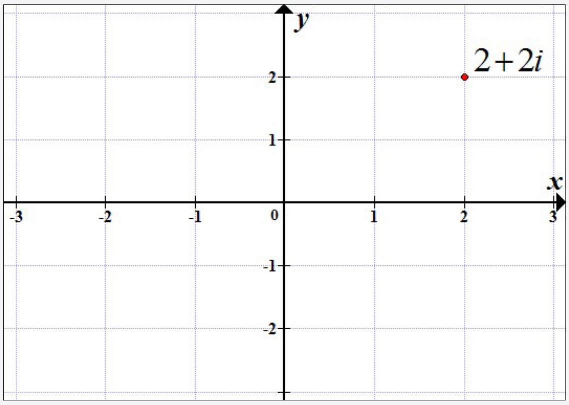
In this lesson, the following generic competence and cross-cutting issues are to be addressed:

- Critical thinking
- Communication
- Self confidence
- Cooperation, interpersonal management and life skills
- Peace and values education
- Inclusive education

Answers

1. $3+2i-1=3-1+2i=2+2i$ and $2+4i-2i=2+2i$.

Argand diagram



The two complex numbers are represented by the same point in Argand diagram.
From their real and imaginary parts, the two quantities have equal real parts and equal imaginary parts, so they are equal.

2. Combine like terms on the right: $x+2i=x+(2x-3)i$. Since the imaginary parts must be equal,

   $2=2x-3 \Rightarrow x=\frac{5}{2}$.

3. This is interesting: we have only one equation, but two variables; it doesn't seem like there is enough information to solve.

But since we can break this into a real part and an imaginary part, we can create two equations: $x=3y,\ y=-(2x-4)$. Doing substitution gives us $y=-6y+4 \Rightarrow y=\frac{4}{7}$, which gives $x=\frac{12}{7}$

Synthesis

As conclusion, if two complex numbers, say $a+bi$ and $c+di$ are equal, then their real parts are equal and their imaginary parts are equal. That is, $a+bi=c+di \iff a=c \text{ and } b=d$.

Application Activity 1.6

1. $x=4,\ y=-3$
2. $x=5,\ y=6$
3. $x=3,\ y=3$
4. $x=-6,\ y=9$
5. $x=2,\ y=5$
6. $x=3,\ y=1$
7. $x=6,\ y=-6$
8. $x=8,\ y=14$

Lesson 1.7. Addition and subtraction of complex numbers

Learning objectives

Given two complex numbers, learners should be able to add and subtract them correctly.

Prerequisites

- Simplification by combining like terms.

Teaching Aids

Exercise book, pen and calculator

Activity 1.7

In this lesson, the following generic competence and cross-cutting issues are to be addressed:

- Critical thinking
- Communication
- Self confidence
- Cooperation, interpersonal management and life skills

- Peace and values education
- Inclusive education

Answers

1. $z_1 + z_2 = (2+3i)+(5-4i) = 2+5+3i-4i = 7-i$
   $z_1 - z_2 = (2+3i)-(5-4i) = 2-5+3i+4i = -3+7i$
2. $\text{Re}(z_1+z_2) = 7,\ \text{Im}(z_1+z_2) = -1$
   $\text{Re}(z_1-z_2) = -3,\ \text{Im}(z_1-z_2) = 7$

Synthesis

As conclusion, two complex numbers are added (or subtracted) by adding (or subtracting) separately the two real and the two imaginary parts.

Application Activity 1.7

1. $z_1 + z_2 = -12,\ z_1 - z_2 = 12+6i$
2. $z_1 + z_2 = 16i,\ z_1 - z_2 = -10+8i$
3. $z_1 + z_2 = 5+3i,\ z_1 - z_2 = 1+5i$
4. $z_1 + z_2 = -2-24i,\ z_1 - z_2 = -44-4i$
5. $z_1 + z_2 = -2+8i,\ z_1 - z_2 = 8+12i$
6. $z_1 + z_2 = -2-24i,\ z_1 - z_2 = -44-4i$
7. $z_1 + z_2 = 35-13i,\ z_1 - z_2 = -9-15i$
8. $z_1 + z_2 = 4+9i,\ z_1 - z_2 = 2-11i$

Lesson 1.8. Conjugate and opposite of a complex number

Learning objectives

Given a complex number, learners should be able to find the conjugate and opposite moderately.

Prerequisites

- Plot a complex number in Argand plane

- Adding two complex numbers

Teaching Aids

Exercise book, pen and instruments of geometry

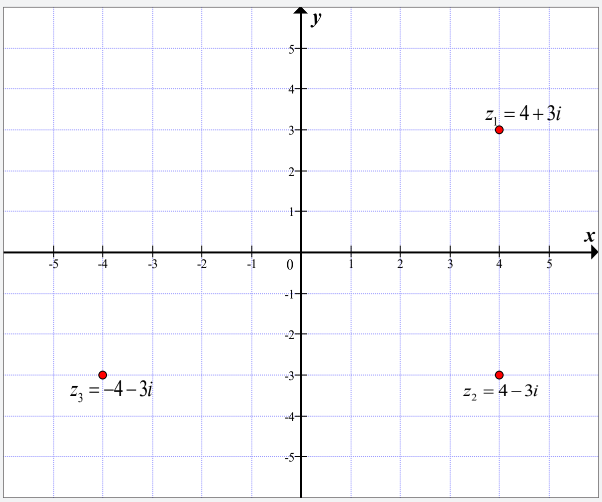
Activity 1.8

In this lesson, the following generic competence and cross-cutting issues are to be addressed:

- Critical thinking
- Communication
- Self confidence
- Cooperation, interpersonal management and life skills
- Peace and values education
- Inclusive education

Answers

1. Argand diagram of complex numbers



2. a) $\frac{1}{2}(z_1 + z_2) = \frac{1}{2}(4+3i+4-3i) = 4$

   b) $\frac{1}{2i}(z_1 - z_2) = \frac{1}{2i}(4+3i-4+3i) = 3$

3. $\frac{1}{2}(z_1 + z_2) = \text{Re}(z_1)$ and $\frac{1}{2i}(z_1 - z_2) = \text{Im}(z_1)$

Synthesis

As conclusion, the conjugate of the complex number $z = x + yi$, denoted by $\bar{z}$ or z^*, is obtained by changing the sign of the imaginary part. Hence, the complex conjugate of $z = x + yi$ is $\bar{z} = x - yi$.

Application Activity 1.8

1. -76	2. $9i$	3. $12+4i$
4. $3-i$	5. $-8-10i$	6. $3+i$
7. $3+5i$	8. $-5-5i$	

Lesson 1.9. Multiplication of complex numbers

Learning objectives

Given two complex numbers, learners should be able to multiply them perfectly.

Prerequisites

- Distributive property.
- Multiplication is distributive over addition.
- Relation $i^2 = -1$.

Teaching Aids

Exercise book, pen and calculator

Activity 1.9

In this lesson, the following generic competence and cross-cutting issues are to be addressed:

- Critical thinking
- Communication
- Self confidence
- Cooperation, interpersonal management and life skills
- Peace and values education
- Inclusive education

Answers

1. $z_1 \times z_2 = (2-3i)(3+2i)$

$$= 6+4i-9i-6i^2$$
$$= 6+6-5i$$
$$= 12-5i$$

2. $\mathrm{Re}(z_1 \times z_2) = 12, \quad \mathrm{Im}(z_1 \times z_2) = -5$

Synthesis

As conclusion, the multiplication of two complex numbers $z_1 = a+bi$ and $z_2 = c+di$ is defined by the following formula:

$$z_1 \times z_2 = (a+bi)(c+di)$$
$$= (ac-bd)+(bc+ad)i$$

Application Activity 1.9

1. $9-36i$	2. $-73+40i$	3. $10+5i$
4. $-3-10i$	5. $10-41i$	6. $-4-7i$

Lesson 1.10. Inverse and division of complex numbers

Learning objectives

Given complex numbers, learners should be able to find the inverse of a complex number and divide two complex numbers accurately.

Prerequisites

- Multiplication of complex numbers.

Teaching Aids

Exercise book, pen and calculator

Activity 1.10

In this lesson, the following generic competence and cross-cutting issues are to be addressed:

- Critical thinking
- Communication
- Self confidence
- Cooperation, interpersonal management and life skills
- Peace and values education
- Inclusive education

Answers

1. $z_1 \cdot \overline{z_1} = (2+i)(2-i) = 2^2 - 2i + 2i + 1^2 = 2^2 + 1^2 = 5$

   Hence, if $z = a + bi$ then, $\overline{z} = a - bi$ and $z \cdot \overline{z} = a^2 + b^2$.

2. $z_1 \cdot \overline{z_1} = 5 \Rightarrow \overline{z_1} = \frac{5}{z_1} \Rightarrow \frac{\overline{z_1}}{5} = \frac{1}{z_1}$

   Then,

   $\frac{1}{z_1} = \frac{\overline{z_1}}{5}$ or $\frac{1}{z_1} = \frac{\overline{z_1}}{a^2 + b^2}$

3. $\frac{1}{z_1} = \frac{\overline{z_1}}{a^2 + b^2}$

   Multiplying both sides by z_2, we get $\frac{z_2}{z_1} = \frac{z_2\,\overline{z_1}}{a^2 + b^2}$

Synthesis

The **inverse** of $z = a + bi$ is given by $z^{-1} = \frac{\overline{z}}{z \cdot \overline{z}} = \frac{\overline{z}}{a^2 + b^2}$ and the **division** of two complex

numbers is $\frac{z_1}{z_2} = \frac{z_1 \cdot \overline{z_2}}{z_2 \cdot \overline{z_2}} = \frac{a + bi}{c + di} = \left(\frac{ac + bd}{c^2 + d^2}\right) + i\left(\frac{bc - ad}{c^2 + d^2}\right)$

Application Activity 1.10

1. $\frac{1}{z_1} = -\frac{i}{3}, \frac{1}{z_2} = -\frac{4}{51} + \frac{i}{51}, \frac{z_1}{z_2} = -\frac{1}{17} - \frac{4}{17}i$
2. $\frac{1}{z_1} = -\frac{5}{169} - \frac{12}{169}i, \frac{1}{z_2} = \frac{5}{41} - \frac{4}{41}i, \frac{z_1}{z_2} = \frac{23}{41} + \frac{80}{41}i$
3. $\frac{1}{z_1} = \frac{3}{25} - \frac{4}{25}i, \frac{1}{z_2} = \frac{2}{5} + \frac{i}{5}, \frac{z_1}{z_2} = \frac{2}{5} + \frac{11}{5}i$

4. $\frac{1}{z_1}=-\frac{23}{725}+\frac{14}{725}i,\ \frac{1}{z_2}=\frac{21}{541}+\frac{10}{541}i,\ \frac{z_1}{z_2}=-\frac{343}{541}-\frac{524}{541}i$

5. $\frac{1}{z_1}=\frac{1}{10}+\frac{3}{10}i,\ \frac{1}{z_2}=-\frac{1}{5}-\frac{2}{5}i,\ \frac{z_1}{z_2}=-\frac{7}{5}+\frac{1}{5}i$

6. $\frac{1}{z_1}=-\frac{2}{5}-\frac{1}{5}i,\ \frac{1}{z_2}=-\frac{5}{29}-\frac{2}{29}i,\ \frac{z_1}{z_2}=\frac{12}{29}-\frac{1}{29}i$

7. $\frac{1}{z_1}=-\frac{1}{2}-\frac{1}{2}i,\ \frac{1}{z_2}=\frac{1}{2}+\frac{1}{2}i,\ \frac{z_1}{z_2}=-1$

8. $\frac{1}{z_1}=-\frac{1}{74}-\frac{3}{37}i,\ \frac{1}{z_2}=\frac{1}{101}-\frac{10}{101}i,\ \frac{z_1}{z_2}=\frac{118}{101}+\frac{32}{101}i$

Lesson 1.11. Square root of a complex number

Learning objectives

Given a complex number, learners should be able to find its square root accurately.

Prerequisites

- Solve a system of two linear equations with two unknowns.
- Use of the identity $(a+b)^2=a^2+2ab+b^2$.

Teaching Aids

Exercise book, pen, calculator

Activity 1.11

In this lesson, the following generic competence and cross-cutting issues are to be addressed:

- Critical thinking
- Communication
- Self confidence
- Cooperation, interpersonal management and life skills
- Peace and values education
- Inclusive education

Answers

$(x+yi)^2 = 8-6i$

$\Leftrightarrow x^2 - y^2 + 2xyi = 8-6i$

$\Leftrightarrow \begin{cases} x^2 - y^2 = 8 \\ 2xy = -6 \end{cases}$

Squaring both sides of each equation and adding two equations, gives

$$\Leftrightarrow \begin{cases} x^4 - 2x^2y^2 + y^4 = 64 \\ 4x^2y^2 = 36 \end{cases}$$
$$x^4 + 2x^2y^2 + y^4 = 100$$

Using algebraic identity, gives

$\Rightarrow (x^2 + y^2)^2 = 100$

$\Rightarrow x^2 + y^2 = 10$

Now,

$$\begin{cases} x^2 - y^2 = 8 \\ x^2 + y^2 = 10 \end{cases}$$
$$2x^2 = 18 \Rightarrow x^2 = 9 \text{ or } x = \pm 3$$

But $x^2 + y^2 = 10$, then, $y^2 = 10 - x^2 = 10 - 9 = 1$ or $y = \pm 1$

Thus, the square root of $z = 8-6i$ is $3-i$ or $-3+i$

We take different sign (for x and y) since the product xy is negative.

Synthesis

To get a square root of the complex number $a+bi$, we let $a+bi$ be a square root of the complex number $a+bi$, and solve the simultaneous equation

$$\begin{cases} (x+iy)^2 = a+bi \\ \left|(x+iy)^2\right| = |a+bi| \end{cases} \Rightarrow \begin{cases} x^2 - y^2 = a \\ 2xy = b \\ x^2 + y^2 = \sqrt{a^2+b^2} \end{cases}$$

$$\Leftrightarrow \begin{cases} x^2+y^2=\sqrt{a^2+b^2} \\ x^2-y^2=a \end{cases} \text{ and } 2xy=b$$

Notice:

In writing square root of the complex number $a+bi$, that is, $x+iy$, x and y must satisfy the condition $2xy=b$.

Application Activity 1.11

1) $\pm\left(\sqrt{7}+i\sqrt{7}\right)$ 2) $\pm(-4-6i)$ 3) $\pm(-4-6i)$

4) $\pm(-3+10i)$ 5) $\pm(3+2i)$ 6) $\pm(-6+2i)$

7) $\pm(-6-2i)$ 8) $\pm(-5-12i)$

Lesson 1.12. Linear equations

Learning objectives

Given linear equations with complex coefficients, learners will be able to solve them in the set of complex numbers accurately.

Prerequisites

- Solving linear equations in set or real numbers.

Teaching Aids

Exercise book, pen and calculator

Activity 1.12

In this lesson, the following generic competence and cross-cutting issues are to be addressed:

- Critical thinking
- Communication
- Self confidence
- Cooperation, interpersonal management and life skills
- Peace and values education
- Inclusive education

Answers

1. $z+3i-4=0 \Rightarrow z=4-3i$
2. $4-i+iz=4z-3i \Rightarrow iz-4z=-3i-4+i$

$$\Rightarrow z(i-4) = -4-2i$$

$$\Rightarrow z = \frac{-4-2i}{i-4}$$

$$\Rightarrow z = \frac{14}{17} + \frac{12}{17}i$$

3. $(1+i)(i+z) = 4i \Rightarrow i + z - 1 + iz = 4i$

$$\Rightarrow z(i+1) = 4i + 1 - i$$

$$\Rightarrow z(i+1) = 1 + 3i$$

$$\Rightarrow z = \frac{1+3i}{i+1} = 2 + i$$

4. $(1-i)z = 2+i \Rightarrow z = \dfrac{2+i}{1-i} \quad \Rightarrow z = \dfrac{1+3i}{2}$

Synthesis

As conclusion, in complex numbers also, we may need to find the complex number z that satisfies the given linear equation.

Application Activity 1.12

1. $2+2i$
2. $-4-2i$
3. $-7+6i$
4. $\frac{9}{5}+\frac{3}{5}i$

Lesson 1.13. Quadratic equations

Learning objectives

Given quadratic equations, learners should be able to solve them in the set of complex numbers correctly.

Prerequisites

- Discriminant method used to solve a quadratic equation.
- Relation $i = \sqrt{-1}$.

Teaching Aids

Exercise book, pen and calculator

Activity 1.13

In this lesson, the following generic competence and cross-cutting issues are to be addressed:

- Critical thinking
- Communication
- Self confidence
- Cooperation, interpersonal management and life skills
- Peace and values education
- Inclusive education

Answers

1. $x^2 + 2x + 3 = 0$

   $\Delta = 4 - 12 = -8$

   $$x_1 = \frac{-2+\sqrt{-8}}{2} = \frac{-2+2i\sqrt{2}}{2} = -1 + i\sqrt{2}$$

   $$x_2 = \frac{-2-\sqrt{-8}}{2} = \frac{-2-2i\sqrt{2}}{2} = -1 - i\sqrt{2}$$

   $S = \left\{-1 + i\sqrt{2}, -1 - i\sqrt{2}\right\}$

2. $x^2 + 2x + 1 + i = 0$

   $\Delta = 4 - 4(1+i) = 4 - 4 - 4i = -4i$

   $\sqrt{\Delta} = \sqrt{-4i}$

   $\sqrt{\Delta} = \sqrt{2} - i\sqrt{2}\ or\ -\sqrt{2} + i\sqrt{2}$

   $$x_1 = \frac{-2+\sqrt{2}-i\sqrt{2}}{2} = \frac{-2+\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}$$

   $$x_2 = \frac{-2-\sqrt{2}+i\sqrt{2}}{2} = \frac{-2-\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}$$

   $$S = \left\{\frac{-2+\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}, \frac{-2-\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right\}$$

Synthesis

In solving equation $az^2 + bz + c = 0$ where a, b and c are real numbers ($a \neq 0$), we get either:

- Two real roots (if $\Delta > 0$); $z_1 = \frac{-b+\sqrt{\Delta}}{2a}$ and $z_2 = \frac{-b-\sqrt{\Delta}}{2a}$
- One double real root (if $\Delta = 0$); $z_1 = z_2 = \frac{-b}{2a}$ or
- Two conjugate complex roots (if $\Delta < 0$):

  $z_1 = \frac{-b+i\sqrt{-\Delta}}{2a}$ and $z_2 = \frac{-b-i\sqrt{-\Delta}}{2a}$.

Application Activity 1.13

1. $S = \left\{\frac{2+i\sqrt{26}}{3}, \frac{2-i\sqrt{26}}{3}\right\}$
2. $S = \{5+3i, 5-3i\}$
3. $S = \left\{\frac{3}{2}+\frac{\sqrt{11}}{2}i, \frac{3}{2}-\frac{\sqrt{11}}{2}i\right\}$

Lesson 1.14. Polynomials in set of complex numbers

Learning objectives

Given a polynomial with complex coefficients, learners should be able to factorise completely it in set of complex numbers accurately.

Prerequisites

- Finding zero of a polynomial.
- Use of synthetic division.

Teaching Aids

Exercise book, pen and calculator

Activity 1.14

In this lesson, the following generic competence and cross-cutting issues are to be addressed:

- Critical thinking
- Communication
- Self confidence

- Cooperation, interpersonal management and life skills
- Peace and values education
- Inclusive education

Answers

1. a) $(z-2-3i)(z+3+i)=z^2+3z+iz-2z-6-2i-3zi-9i+3$

$=z^2+(1-2i)z-3-11i$

b) $(z-i)(z+3i)(z-4i)=(z^2+3zi-iz+3)(z-4i)$

$=(z^2+2iz+3)(z-4i)$

$=z^3-4iz^2+2iz^2+8z+3z-12i$

$=z^3-2iz^2+11z-12i$

2. $P(z)=z^3+(-2-i)z^2+(2+2i)z-4$ is divisible by $z+i$ if and only if $P(-i)=0$.

$P(-i)=(-i)^3+(-2-i)(-i)^2+(2+2i)(-i)-4$

$=i+2+i-2i+2-4$

$=0$

Thus, $P(z)=z^3+(-2-i)z^2+(2+2i)z-4$ is divisible by $z+i$.

Now, using synthetic division

	1	$-2-i$	$2+2i$	-4
$-i$		$-i$	$2i-2$	4
	1	$-2-2i$	$4i$	0

$P(z)=z^3+(-2-i)z^2+(2+2i)z-4$

$=(z+i)\left[z^2+(-2-2i)z+4i\right]$

Again, we factorise $z^2+(-2-2i)z+4i$ since

2 is a root, then

	1	$-2-2i$	$4i$
2		2	$-4i$
	1	$-2i$	0

$$z^2+(-2-2i)z+4i=(z-2)(z-2i)$$

Thus, $P(z)=(z+i)(z-2)(z-2i)$

3. If $P(z)=z^3-2z^2+(7+2i)z-6(2-i)$

$2-i$ is a factor of $-6(2-i)$,

$$P(2-i)=(2-i)^3-2(2-i)^2+(7+2i)(2-i)-6(2-i)$$
$$=2-11i-6+8i+16-3i-12+6i$$
$$=0$$

Other values are: $z=3i,\ z=-2i$

All roots can be found as follows:

$$P(z)=z^3-2z^2+(7+2i)z-12+6i$$

$$P(z)=0 \Leftrightarrow z^3-2z^2+(7+2i)z-12+6i=0$$

$z=3i$ is a root since

$$P(3i)=(3i)^3-2(3i)^2+(7+2i)(3i)-12+6i$$
$$=-27i+18+21i-6-12+6i$$
$$=0$$

Using Synthetic division, we have

	1	-2	$7+2i$	$-12+6i$
$3i$		$3i$	$-6i-9$	$-6i+12$
	1	$-2+3i$	$-2-4i$	0

$$P(z)=z^3-2z^2+(7+2i)z-12+6i$$
$$=(z-3i)\left[z^2+(-2+3i)z-2-4i\right]$$

$-2i$ is also a root

	1	$-2+3i$	$-2-4i$
$-2i$		$-2i$	$4i+2$
	1	$-2+i$	0

$$P(z)=z^3-2z^2+(7+2i)z-12+6i$$
$$=(z-3i)(z+2i)(z-2+i)$$

Then, $z=3i\ or\ z=-2i\ or\ z=2-i$

Synthesis

As conclusion, the process of finding the roots of a polynomial in set of complex numbers is similar to the case of real numbers remembering that the square root of a negative real number exist in set of complex numbers considering $\sqrt{-1}=i$. The methods used are synthetic division and factorisation.

Application Activity 1.14

1. a) $P(z)=(z+2)(z-1+3i)(z-1-3i);\ \{-2,1-3i,1+3i\}$

   b) $Q(z)=(z-2)(z-2+i)(z-2-i);\ \{2,2-i,2+i\}$

   c) $R(z)=(z-2+3i)(z-2-3i)\left(z+2-\sqrt{2}\right)\left(z+2+\sqrt{2}\right);$

   $$\left\{2-3i,2+3i,-2+\sqrt{2},-2-\sqrt{2}\right\}$$

   d) $M(z)=(z-3i)(z+2i)(z-2+i);\ \{3i,-2i,2-i\}$

2. $a=79,\ b=29$

3. $p(z)=-2z^3+8z^2-18z+20$

4. $S=\left\{3-i,-\dfrac{7}{2}+\dfrac{7}{2}i\right\}$

Lesson 1.15. Argument of a complex number

Learning objectives

Given a complex numbers, learners should be able to find its argument moderately.

Prerequisites

- Concepts of trigonometry.

Teaching Aids

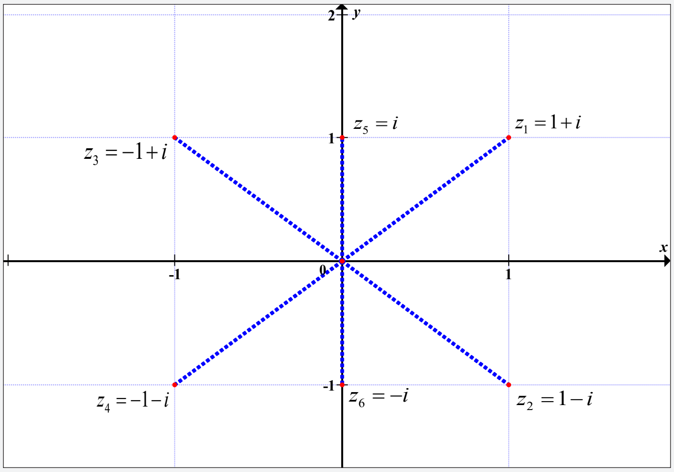
Exercise book, pen, scientific calculator and instruments of geometry

Activity 1.15

In this lesson, the following generic competence and cross-cutting issues are to be addressed:

- Critical thinking
- Communication
- Self confidence
- Cooperation, interpersonal management and life skills
- Peace and values education
- Inclusive education

Answers



For $z_1 = 1 + i$, $\tan\theta = \frac{1}{1} = 1 \Rightarrow \theta = \arctan(1) = \frac{\pi}{4}$. From the figure, this is the needed angle.

For $z_2 = 1 - i$, $\tan\theta = \frac{1}{-1} = -1 \Rightarrow \theta = \arctan(-1) = -\frac{\pi}{4}$. From the figure, this is the needed angle.

For $z_3 = -1 + i$, $\tan\theta = \frac{-1}{1} = -1 \Rightarrow \theta = \arctan(-1) = -\frac{\pi}{4}$. From the figure, this is not the needed angle. The needed angle is $\pi + \theta = \pi - \frac{\pi}{4} = \frac{3\pi}{4}$

For $z_4 = -1 - i$, $\tan\theta = \frac{-1}{-1} = 1 \Rightarrow \theta = \arctan(1) = \frac{\pi}{4}$. From the figure, this is not the needed angle. The needed angle is $\theta - \pi = \frac{\pi}{4} - \pi = -\frac{3\pi}{4}$.

For $z_5 = i$. From the figure, the needed angle is $\theta = \frac{\pi}{2}$

For $z_6 = -i$. From the figure, the needed angle is $\theta = -\frac{\pi}{2}$

Synthesis

Depending on the quadrant in which the argument of complex number $z = x + yi$ lies, we define $\arg(z)$ as follows:

$$\arg(z) = \begin{cases} \arctan\dfrac{y}{x}, \ \textit{if } z \textit{ lies in } 1^{st} \textit{ or } 4^{th} \textit{ quadrant or on positive } x-\textit{axis} \\ \pi + \arctan\dfrac{y}{x}, \ \textit{if } z \textit{ lies in } 2^{nd} \textit{ quadrant or on negative } x-\textit{axis} \\ -\pi + \arctan\dfrac{y}{x}, \ \textit{if } \ z \textit{ lies in } 3^{rd} \textit{ quadrant} \\ \dfrac{\pi}{2}, \ \textit{if } z \textit{ lies on positive } y-\textit{axis} \\ -\dfrac{\pi}{2}, \ \textit{if } z \textit{ lies on negative } y-\textit{axis} \\ \textit{undefined}, \ \textit{if } x = 0 \textit{ and } y = 0 \end{cases}$$

This is equivalent to

$$\arg(z) = \begin{cases} \arctan\dfrac{y}{x}, \quad \textit{if } x > 0 \\ \pi + \arctan\dfrac{y}{x}, \ \textit{if } x < 0 \textit{ and } y \geq 0 \\ -\pi + \arctan\dfrac{y}{x}, \textit{if } x < 0 \textit{ and } y < 0 \\ \dfrac{\pi}{2}, x = 0, \ \textit{if } y > 0 \\ -\dfrac{\pi}{2}, x = 0, \textit{if } y < 0 \\ \textit{undefined}, \textit{if } x = 0 \textit{ and } y = 0 \end{cases}$$

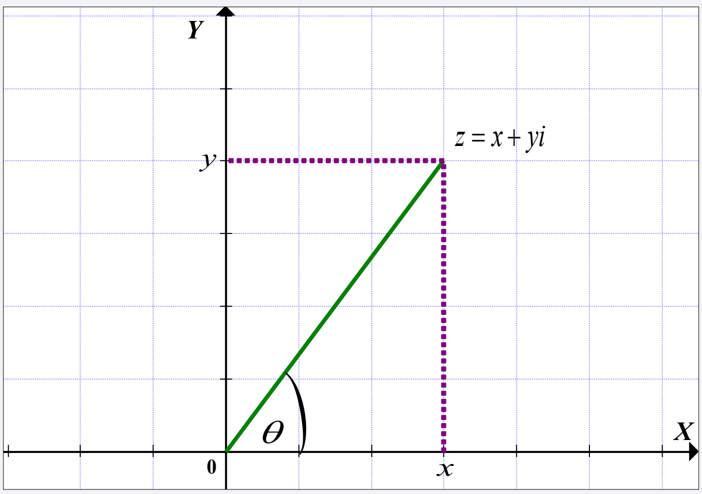


Figure 1.2: Argument of a complex number

Application Activity 1.15

1. $-\frac{\pi}{4}$
2. $-\frac{\pi}{6}$
3. $\frac{\pi}{4}$
4. $-\frac{\pi}{3}$
5. $-\frac{\pi}{6}$

Lesson 1.16. Loci related to the angles

Learning objectives

Given an argument condition, learners should be able to sketch on Argand diagram, the region satisfying that condition accurately.

Prerequisites

- Drawing angle with a given size.

Teaching Aids

Exercise book, pencil and instruments of geometry

Activity 1.16

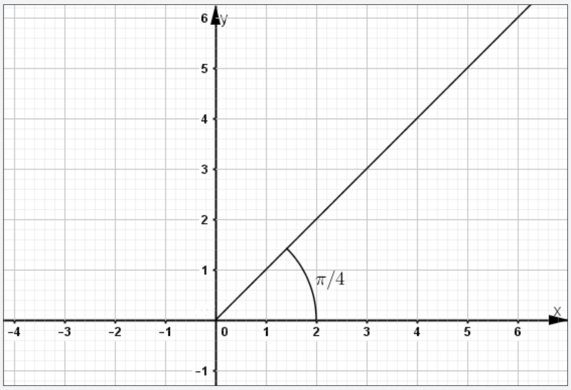
In this lesson, the following generic competence and cross-cutting issues are to be addressed:

- Critical thinking
- Communication
- Self confidence
- Cooperation, interpersonal management and life skills
- Peace and values education
- Inclusive education

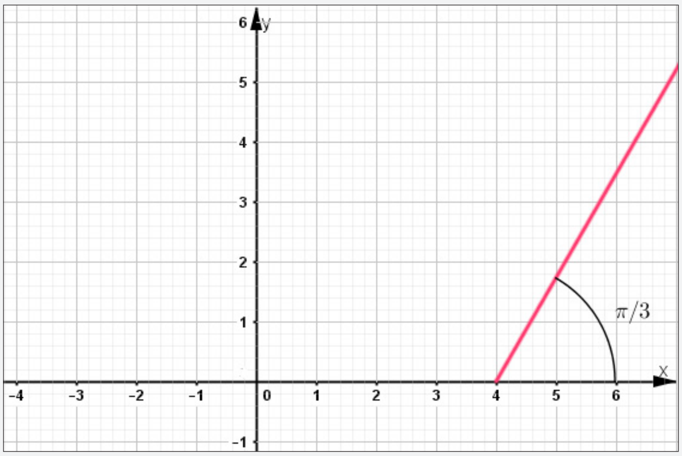
Answers

1. $\arg(z)=\frac{\pi}{4}\Rightarrow\arg(x+yi)=\frac{\pi}{4}$

   Here, we need all complex numbers lying on the half line passing through $(0,0)$ and makes an angle of $\frac{\pi}{4}$ with positive $x-axis$.



2. $\arg(z-4)=\frac{\pi}{3} \Rightarrow \arg(x+yi-4)=\frac{\pi}{3} \Rightarrow \arg(x-4+yi)=\frac{\pi}{3}$
Here, we need all complex numbers lying on the half line passing through $(4,0)$ and makes an angle of $\frac{\pi}{3}$ with positive $x-axis$.



Synthesis

As conclusion, $\arg(z)=\theta$ represents the half line through O inclined at an angle θ to the positive direction of $x-axis$.

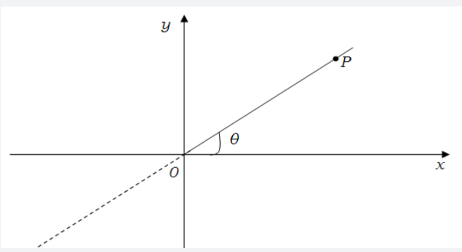


Figure 1.3: Locus as a half line through 0

$\arg(z - z_1) = \theta$ represents the half line through the point z_1 inclined at an angle θ to the positive direction of $x-axis$.

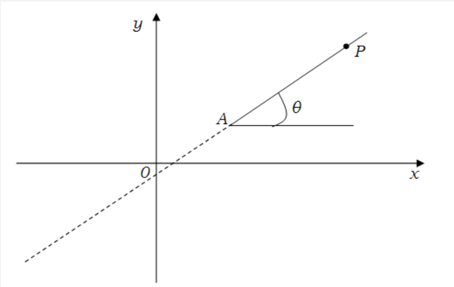


Figure 1.4: Locus as a half line through any point

$\theta \le \arg(z - z_1) \le \beta$ indicates that the angle between AP and the positive x-axis lies between θ and β, so that P can lie on or within the two half lines as shown in
Figure 5.1.

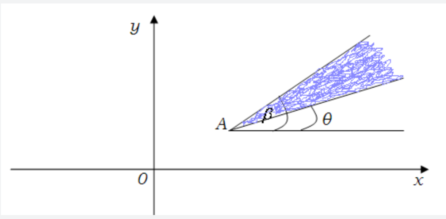
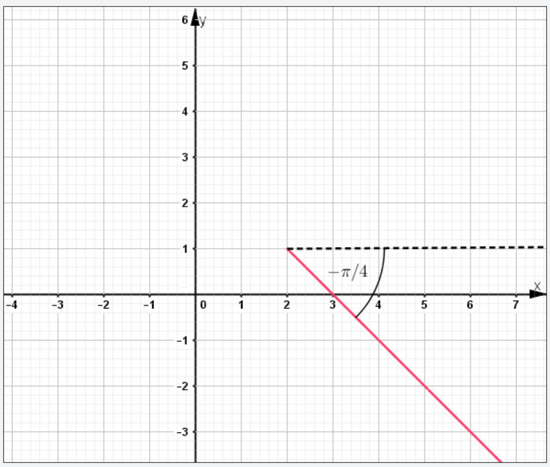


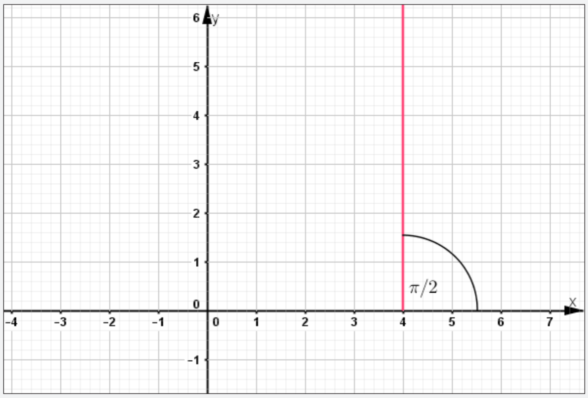
Figure 1.5: Locus between two half lines

Application Activity 1.16

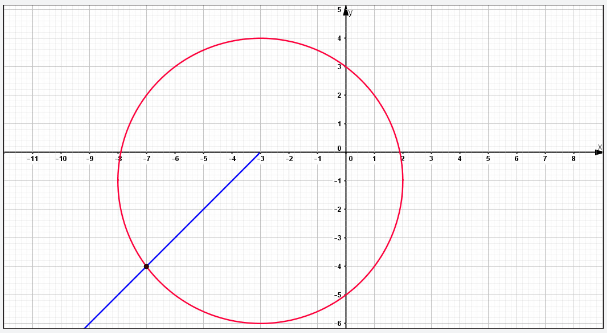
1. a)



b)



2.



a) From the graph, we see that there is only one point of intersection. Thus, there is only one complex number satisfying both conditions.

b) Putting $z=-7-4i$, we have

$|-7-4i+3+i|=|-4-3i|=\sqrt{16+9}=5$ also

$\arg(-7-4i+3)=\arg(-4-4i)=-\pi+\arctan(1)=-\pi+\frac{\pi}{4}=-\frac{3\pi}{4}$

Thus, $z=-7-4i$ verifies both conditions.

Lesson 1.17. Polar form of a complex number

Learning objectives

Given a complex number, learners should be able to express it in polar form accurately.

Prerequisites

- Finding the modulus of s complex number.
- Finding argument of a complex number.

Teaching Aids

Exercise book, pen and scientific calculator

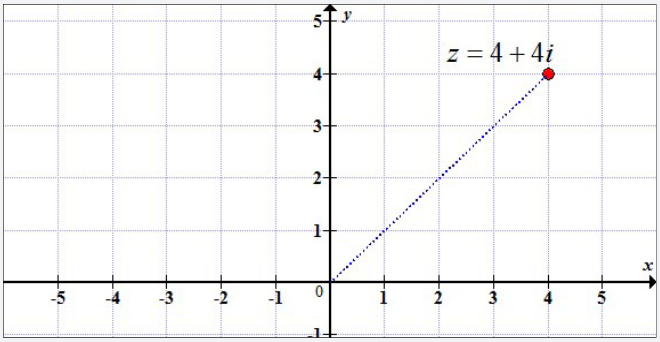
Activity 1.17

In this lesson, the following generic competence and cross-cutting issues are to be addressed:

- Critical thinking
- Communication
- Self confidence
- Cooperation, interpersonal management and life skills
- Peace and values education
- Inclusive education

Answers

1. Graph



2. $r = \sqrt{16+16} = 4\sqrt{2}$

3. $\theta = \arctan\left(\frac{4}{4}\right) = \frac{\pi}{4}$

4. $\cos\theta = \frac{x}{r} \Rightarrow x = r\cos\theta \Rightarrow 4 = 4\sqrt{2}\cos\frac{\pi}{4}$

   $\sin\theta = \frac{y}{r} \Rightarrow y = r\sin\theta \Rightarrow 4 = 4\sqrt{2}\sin\frac{\pi}{4}$

   From $z = 4 + 4i$, we have

   $$z = 4\sqrt{2}\cos\frac{\pi}{4} + i4\sqrt{2}\sin\frac{\pi}{4} = 4\sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$$

Synthesis

As conclusion, if r and θ are the modulus and principal argument of complex number z respectively, then the polar form of z is $z = r(\cos\theta + i\sin\theta)$.

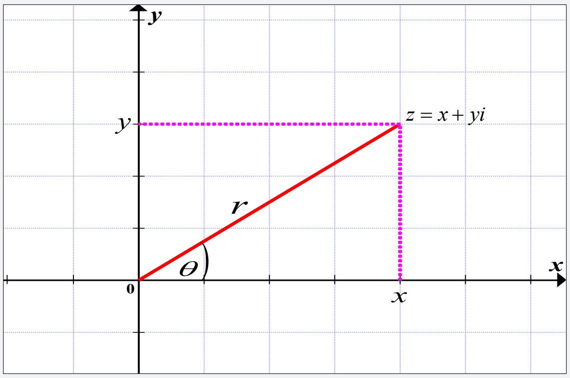


Figure 1.6: Modulus and argument of a complex number

Application Activity 1.17

1. a) $4\operatorname{cis}0$ b) $2\operatorname{cis}\frac{\pi}{2}$ c) $2\operatorname{cis}(\pi)$

   d) $5\operatorname{cis}\left(-\frac{\pi}{2}\right)$ e) $2\operatorname{cis}\frac{\pi}{6}$ f) $2\sqrt{3}\operatorname{cis}\left(-\frac{\pi}{3}\right)$

   g) $2\sqrt{3}\operatorname{cis}\left(-\frac{\pi}{6}\right)$

Here, remember that the notation $r\operatorname{cis}\theta$ is the same as $r(\cos\theta + i\sin\theta)$.

2. a) $1+i\sqrt{3}$ b) $\sqrt{2}(-2+2i)$ c) $1-i$

   d) $3i$ e) -4 f) $\frac{1}{2}\left(-\sqrt{3}-i\right)$

   g) $\sqrt{3}-i$

Lesson 1.18. Multiplication and division of complex numbers in polar form

Learning objectives

Given two complex numbers, learners should be able to multiply and divide them in polar form exactly.

Prerequisites

- Putting a complex number in polar form
- Addition formulae in trigonometry.

Teaching Aids

Exercise book, pen and scientific calculator

Activity 1.18

In this lesson, the following generic competence and cross-cutting issues are to be addressed:

- Critical thinking
- Communication
- Self confidence
- Cooperation, interpersonal management and life skills
- Peace and values education
- Inclusive education

Answers

1. $z_1z_2 = (1-i)\left(\sqrt{3}-i\right)$

$$= \sqrt{3}-i-i\sqrt{3}-1 = \sqrt{3}-1-\left(\sqrt{3}+1\right)i$$

$$\left|z_1z_2\right| = \sqrt{\left(\sqrt{3}-1\right)^2+\left(\sqrt{3}+1\right)^2} = \sqrt{3-2\sqrt{3}+1+3+2\sqrt{2}+1} = \sqrt{8} = 2\sqrt{2}$$

$$\arg\left(z_1z_2\right) = \arctan\left(\frac{-\sqrt{3}-1}{\sqrt{3}-1}\right) = -\frac{5\pi}{12}$$

Then, $z_1z_2 = 2\sqrt{2}\operatorname{cis}\left(\frac{-5\pi}{12}\right)$

2. $z_1 = 1-i$

$$\left|z_1\right| = \sqrt{2},\ \arg\left(z_1\right) = \arctan(-1) = -\frac{\pi}{4} \Rightarrow z_1 = \sqrt{2}\operatorname{cis}\left(-\frac{\pi}{4}\right)$$

$$z_2 = \sqrt{3}-i$$

$$\left|z_2\right| = 2,\ \arg\left(z_2\right) = \arctan\left(-\frac{1}{\sqrt{3}}\right) = -\frac{\pi}{6} \Rightarrow z_2 = 2\operatorname{cis}\left(-\frac{\pi}{6}\right)$$

$$z_1z_2 = \sqrt{2}\operatorname{cis}\left(-\frac{\pi}{4}\right)2\operatorname{cis}\left(-\frac{\pi}{6}\right) = 2\sqrt{2}\left[\cos\left(-\frac{\pi}{4}\right)+i\sin\left(-\frac{\pi}{4}\right)\right]\left[\cos\left(-\frac{\pi}{6}\right)+i\sin\left(-\frac{\pi}{6}\right)\right]$$

$$= 2\sqrt{2}\left[\cos\left(-\frac{\pi}{4}\right)\cos\left(-\frac{\pi}{6}\right)+i\cos\left(-\frac{\pi}{4}\right)\sin\left(-\frac{\pi}{6}\right)+i\sin\left(-\frac{\pi}{4}\right)\cos\left(-\frac{\pi}{6}\right)-\sin\left(-\frac{\pi}{4}\right)\sin\left(-\frac{\pi}{6}\right)\right]$$

$$= 2\sqrt{2}\left[\cos\left(-\frac{\pi}{4}\right)\cos\left(-\frac{\pi}{6}\right)-\sin\left(-\frac{\pi}{4}\right)\sin\left(-\frac{\pi}{6}\right)+i\left(\cos\left(-\frac{\pi}{4}\right)\sin\left(-\frac{\pi}{6}\right)+\sin\left(-\frac{\pi}{4}\right)\cos\left(-\frac{\pi}{6}\right)\right)\right]$$

$$= 2\sqrt{2}\left[\cos\left(-\frac{\pi}{4}+\left(-\frac{\pi}{6}\right)\right)+i\left(\sin\left(-\frac{\pi}{4}+\left(-\frac{\pi}{6}\right)\right)\right)\right],\quad \textit{from addition formulae in trigonometry}$$

$$= 2\sqrt{2}\left[\cos\left(-\frac{5\pi}{12}\right)+i\cos\left(-\frac{5\pi}{12}\right)\right] = 2\sqrt{2}\operatorname{cis}\left(\frac{-5\pi}{12}\right)$$

3. The two results are the same

4. $\dfrac{z_1}{z_2} = \dfrac{1-i}{\sqrt{3}-i}$

$$= \frac{(1-i)\left(\sqrt{3}+i\right)}{4} = \frac{\sqrt{3}+i-i\sqrt{3}+1}{4} = \frac{\sqrt{3}+1}{4} + \frac{1-\sqrt{3}}{4}i$$

$$\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|} = \frac{|1-i|}{\left|\sqrt{3}-i\right|} = \frac{\sqrt{2}}{2}$$

$$\arg\left(\frac{z_1}{z_2}\right) = \arctan\left(\frac{\frac{1-\sqrt{3}}{4}}{\frac{1+\sqrt{3}}{4}}\right) = -\frac{\pi}{12}$$

Then,

$$\frac{z_1}{z_2} = \frac{\sqrt{2}}{2}\operatorname{cis}\left(-\frac{\pi}{12}\right)$$

5. $z_1 = \sqrt{2}\operatorname{cis}\left(-\dfrac{\pi}{4}\right),\ z_2 = 2\operatorname{cis}\left(-\dfrac{\pi}{6}\right)$

$$\frac{z_1}{z_2} = \frac{\sqrt{2}\operatorname{cis}\left(-\frac{\pi}{4}\right)}{2\operatorname{cis}\left(-\frac{\pi}{6}\right)} = \frac{\sqrt{2}\left[\cos\left(-\frac{\pi}{4}\right)+i\sin\left(-\frac{\pi}{4}\right)\right]}{2\left[\cos\left(-\frac{\pi}{6}\right)+i\sin\left(-\frac{\pi}{6}\right)\right]}$$

$$= \frac{\sqrt{2}\left[\cos\left(-\frac{\pi}{4}\right)+i\sin\left(-\frac{\pi}{4}\right)\right]\left[\cos\left(-\frac{\pi}{6}\right)-i\sin\left(-\frac{\pi}{6}\right)\right]}{2\left[\cos\left(-\frac{\pi}{6}\right)+i\sin\left(-\frac{\pi}{6}\right)\right]\left[\cos\left(-\frac{\pi}{6}\right)-i\sin\left(-\frac{\pi}{6}\right)\right]}$$

$$= \frac{\sqrt{2}\left[\cos\left(-\frac{\pi}{4}\right)\cos\left(-\frac{\pi}{6}\right)-i\sin\left(-\frac{\pi}{6}\right)\cos\left(-\frac{\pi}{4}\right)+i\cos\left(-\frac{\pi}{6}\right)\sin\left(-\frac{\pi}{4}\right)+\sin\left(-\frac{\pi}{4}\right)\sin\left(-\frac{\pi}{6}\right)\right]}{2\left[\cos^2\left(-\frac{\pi}{6}\right)+\sin^2\left(-\frac{\pi}{6}\right)\right]}$$

$$= \frac{\sqrt{2}\left[\cos\left(-\frac{\pi}{4}\right)\cos\left(-\frac{\pi}{6}\right)+\sin\left(-\frac{\pi}{4}\right)\sin\left(-\frac{\pi}{6}\right)+i\left(\sin\left(-\frac{\pi}{4}\right)\cos\left(-\frac{\pi}{6}\right)-\cos\left(-\frac{\pi}{4}\right)\sin\left(-\frac{\pi}{6}\right)\right)\right]}{2\times 1}$$

$$= \frac{\sqrt{2}\left[\cos\left(-\frac{\pi}{4}-\left(-\frac{\pi}{6}\right)\right)+i\sin\left(-\frac{\pi}{4}-\left(-\frac{\pi}{6}\right)\right)\right]}{2}$$

$$= \frac{\sqrt{2}}{2}\left[\cos\left(-\frac{\pi}{12}\right)+i\sin\left(-\frac{\pi}{12}\right)\right] = \frac{\sqrt{2}}{2}\operatorname{cis}\left(-\frac{\pi}{12}\right)$$

6. The two results are the same.

Synthesis

Given two complex numbers $z_1 = r_1\left(cos\ \theta_1 + isin\ \theta_1\right)$ and $z_2 = r_2\left(cos\ \theta_2 + isin\ \theta_2\right)$ then,

$z_1z_2 = r_1r_2\left(cos\ \left(\theta_1+\theta_2\right) + isin\ \left(\theta_1+\theta_2\right)\right)$ and

$\frac{z_1}{z_2} = \frac{r_1}{r_2}\left(\cos\left(\theta_1-\theta_2\right) + i\sin\left(\theta_1-\theta_2\right)\right)$ with the provision that 2π may have to be added to, or substracted from $\theta_1+\theta_2$ (or $\theta_1-\theta_2$) if $\theta_1+\theta_2$ (or $\theta_1-\theta_2$) is outside the permitted range of the principal argument $\left]-\pi,\pi\right]$.

We note that;

$Arg\left(z_1z_2\right) = Arg\left(z_1\right) + Arg\left(z_2\right)$ and

$Arg\left(\frac{z_1}{z_2}\right) = Arg\left(z_1\right) - Arg\left(z_2\right)$

Application Activity 1.18

1. a) $zw = 2\sqrt{2}\,\text{cis}\left(-\frac{11\pi}{12}\right)$, $\frac{z}{w} = \frac{\sqrt{2}}{2}\text{cis}\left(-\frac{7\pi}{12}\right)$

   b) $zw = 4\,\text{cis}\left(\frac{\pi}{3}\right)$, $\frac{z}{w} = 4\,\text{cis}\left(-\pi\right)$

   c) $zw = 2\sqrt{2}\,\text{cis}\left(\frac{5\pi}{12}\right)$, $\frac{z}{w} = \frac{\sqrt{2}}{2}\text{cis}\left(\frac{\pi}{12}\right)$

   d) $zw = 4\sqrt{6}\,\text{cis}\left(\frac{5\pi}{12}\right)$, $\frac{z}{w} = \frac{\sqrt{6}}{3}\text{cis}\left(-\frac{11\pi}{12}\right)$

   e) $zw = \sqrt{2}\,\text{cis}\left(\frac{\pi}{12}\right)$, $\frac{z}{w} = \frac{\sqrt{2}}{2}\text{cis}\left(\frac{7\pi}{12}\right)$

   f) $zw = 2\,\text{cis}\left(\pi\right)$, $\frac{z}{w} = \text{cis}\left(\frac{\pi}{2}\right)$

2. $\sqrt{2}\,\text{cis}\left(-\frac{5\pi}{12}\right)$

Lesson 1.19. Powers of complex number in polar form

Learning objectives

Given a complex number, learners should be able to find its powers and use De Moivre's theorem accurately.

Prerequisites

- Putting a complex number in polar form.

Teaching Aids

Exercise book, pen and scientific calculator

Activity 1.19

In this lesson, the following generic competence and cross-cutting issues are to be addressed:

- Critical thinking
- Communication
- Self confidence
- Cooperation, interpersonal management and life skills
- Peace and values education
- Inclusive education

Answers

1. $z = \sqrt{3} + i$

   $|z| = 2,\ \arg(z) = \arctan\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6}$

   $z = 2\,\text{cis}\left(\frac{\pi}{6}\right)$

2. $z^2 = \left(\sqrt{3}+i\right)\left(\sqrt{3}+i\right) = \left(\sqrt{3}\right)^2 + 2i\sqrt{3} + (i)^2 = 3 + 2i\sqrt{3} - 1 = 2 + 2i\sqrt{3}$

   $|z^2| = \sqrt{4+12} = 4,\ \arg(z^2) = \arctan\left(\sqrt{3}\right) = \frac{\pi}{3}$

   $z^2 = 4\,\text{cis}\left(\frac{\pi}{3}\right)$

3. $z^3 = \left(2+2i\sqrt{3}\right)\left(\sqrt{3}+i\right) = 2\sqrt{3} + 2i + 6i - 2\sqrt{3} = 8i$

   $|z^3| = \sqrt{8^2} = 8,\ \arg(z^3) = \frac{\pi}{2}$

   $z^3 = 8\,\text{cis}\left(\frac{\pi}{2}\right)$

4. From 1 to 3, we see that

$$z^2 = 4\,\text{cis}\left(\frac{\pi}{3}\right) = 2^2\,\text{cis}\left(\frac{2\pi}{6}\right)$$

$$z^3 = 8\,\text{cis}\left(\frac{\pi}{2}\right) = 2^3\,\text{cis}\left(\frac{3\pi}{6}\right)$$

Hence, $z^n = r^n\,\text{cis}\left(n\theta\right)$ where r is modulus and θ is argument of z.

Synthesis

The power of a complex number in polar form is given by;

$z^n = \left(r\left(\cos\theta + i\sin\theta\right)\right)^n = r^n\left(\cos n\theta + i\sin n\theta\right);\ n\in\mathbb{Z}_0$ where r and θ are modulus and argument of z respectively.

Application Activity 1.19

1. a) $\frac{1}{2}\left(-1-i\sqrt{3}\right)$ b) -64 c) $-32i$

   d) $\frac{1}{2}\left(-1-i\sqrt{3}\right)$ e) -1 f) -512

   g) $-128-128i$

2. $m = 6k,\ k = 1,2,3,4,\ldots$

Lesson 1.20. n^{th} root of a complex number

Learning objectives

Given a complex number, learners should be able to find the n^{th} roots of that complex number accurately.

Prerequisites

- Putting a complex number in polar form.
- Evaluating powers in polar form.

Teaching Aids

Exercise book, pen and scientific calculator

Activity 1.20

In this lesson, the following generic competence and cross-cutting issues are to be addressed:

- Critical thinking
- Communication
- Self confidence
- Cooperation, interpersonal management and life skills
- Peace and values education
- Inclusive education

Answers

1. $|z| = 4,\ \arg(z) = \arctan(0) = 0 \Rightarrow z = 4\,\text{cis}\,0$

2. $(z_k)^4 = z$

   But $(z_k)^4 = (r')^4 \text{ cis}\,4\theta'$

   Then $(r')^4 \text{ cis}\,4\theta' = 4\,\text{cis}\,0$

   $$\begin{cases}(r')^4 = 4\\ 4\theta' = 2k\pi\end{cases} \Rightarrow \begin{cases}r' = \sqrt[4]{4} \Rightarrow r' = \sqrt{2}\\ \theta' = \dfrac{k\pi}{2}\end{cases}$$

   Now, $z_k = \sqrt{2}\,\text{cis}\left(\dfrac{k\pi}{2}\right)$

   If $k = 0,\ z_0 = \sqrt{2}\,\text{cis}\,0$

   If $k = 1,\ z_1 = \sqrt{2}\,\text{cis}\left(\dfrac{\pi}{2}\right)$

   If $k = 2,\ z_2 = \sqrt{2}\,\text{cis}\,\pi$

   If $k = 3,\ z_3 = \sqrt{2}\,\text{cis}\left(\dfrac{3\pi}{2}\right) = \sqrt{2}\,\text{cis}\left(-\dfrac{\pi}{2}\right)$

Synthesis

To find n^th roots of a complex number z, you start by expressing z in polar form $z = rcis\theta$, where r is modulus of z and θ argument of z.

Then, n^th roots of a complex number z is given by

$$z_k = \sqrt[n]{r}\ cis\left(\frac{\theta + 2k\pi}{n}\right) \qquad k = 0,1,2,3,\ldots\ldots,n-1$$

Application Activity 1.20

1. $z_0 = \text{cis}\left(\frac{\pi}{8}\right),\ z_1 = \text{cis}\left(\frac{5\pi}{8}\right),\ z_2 = \text{cis}\left(-\frac{7\pi}{8}\right),\ z_3 = \text{cis}\left(-\frac{3\pi}{8}\right)$
2. $z_0 = 1,\ z_1 = \text{cis}\left(\frac{2\pi}{5}\right),\ z_2 = \text{cis}\left(\frac{4\pi}{5}\right),\ z_3 = \text{cis}\left(-\frac{4\pi}{5}\right),\ z_4 = \text{cis}\left(-\frac{2\pi}{5}\right)$
3. $z_0 = 2,\ z_1 = 2\,\text{cis}\left(\frac{2\pi}{5}\right),\ z_2 = 2\,\text{cis}\left(\frac{4\pi}{5}\right),\ z_3 = 2\,\text{cis}\left(-\frac{4\pi}{5}\right),\ z_4 = 2\,\text{cis}\left(-\frac{2\pi}{5}\right)$
4. $z_0 = 2\,\text{cis}\left(\frac{\pi}{12}\right),\ z_1 = 2\,\text{cis}\left(\frac{7\pi}{12}\right),\ z_2 = 2\,\text{cis}\left(-\frac{11\pi}{12}\right),\ z_3 = 2\,\text{cis}\left(-\frac{5\pi}{12}\right)$
5. $\sin\frac{2\pi}{5} = \frac{\sqrt{10+2\sqrt{5}}}{4}$

Hint:

First, find $\cos\frac{2\pi}{5}$ and then use the relation

$$\sin\frac{2\pi}{5} = \sqrt{1-\cos^2\frac{2\pi}{5}}$$

Lesson 1.21. Graphical representation of nth roots of a complex number

Learning objectives

Given a complex number and using a ruler, learners should be able to represent n^{th} roots of that complex number in Argand plane correctly.

Prerequisites

- Finding n^{th} roots of a complex number.

Teaching Aids

Exercise book, pencil, instruments of geometry and calculator

Activity 1.21

In this lesson, the following generic competence and cross-cutting issues are to be addressed:

- Critical thinking
- Communication
- Self confidence

- Cooperation, interpersonal management and life skills
- Peace and values education
- Inclusive education

Answers

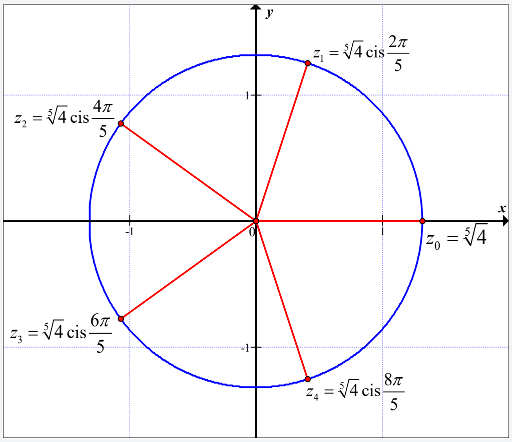
1. $z = 4,\ \arg(z) = 0$

$$z_k = \sqrt[5]{4}\,\text{cis}\,\frac{2k\pi}{5} \qquad z_2 = \sqrt[5]{4}\,\text{cis}\,\frac{4\pi}{5}$$

$$z_0 = \sqrt[5]{4}\,\text{cis}\,0 = \sqrt[5]{4} \qquad z_3 = \sqrt[5]{4}\,\text{cis}\,\frac{6\pi}{5}$$

$$z_1 = \sqrt[5]{4}\,\text{cis}\,\frac{2\pi}{5} \qquad z_4 = \sqrt[5]{4}\,\text{cis}\,\frac{8\pi}{5}$$

2. Representation of the obtained roots on Argand diagram and joining the obtained points.



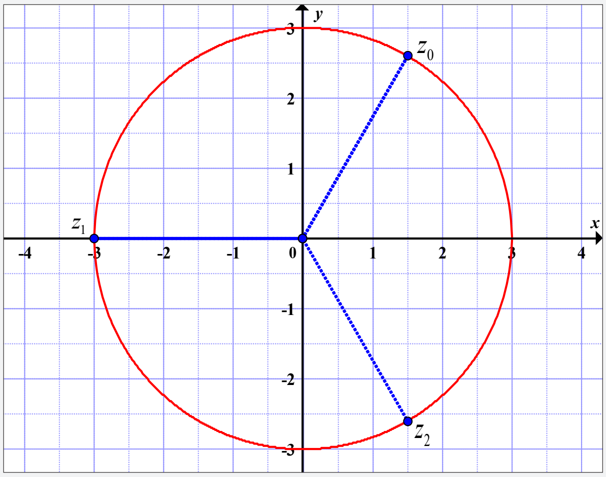
3. See diagram above.

Synthesis

As conclusion, if the complex number for which we are computing the n^{th} roots is $z = rcis\theta$, the radius of the circle will be $R = \sqrt[n]{r}$ and the first root z_0 corresponding to $k = 0$ will be at an amplitude of $\varphi = \frac{\theta}{n}$. This root will be followed by the $n-1$ remaining roots at equal distances apart.

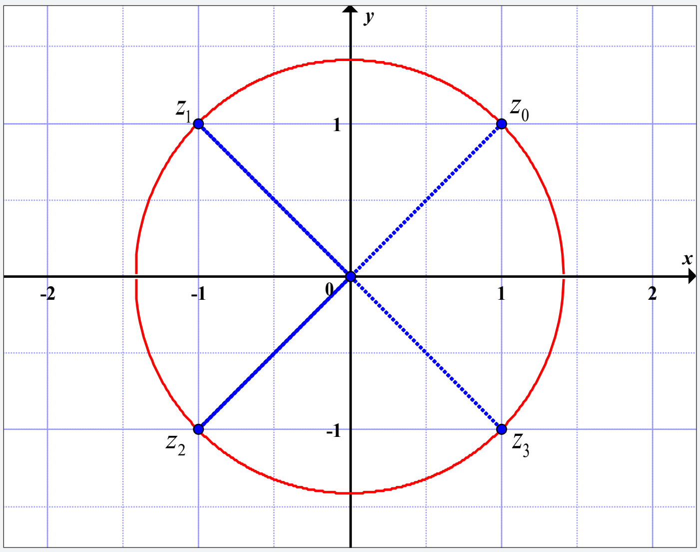
Application Activity 1.21

1. $z_0 = 3\,\mathrm{cis}\left(\frac{\pi}{3}\right)$, $z_1 = 3\,\mathrm{cis}(\pi)$, $z_2 = 3\,\mathrm{cis}\left(-\frac{\pi}{3}\right)$

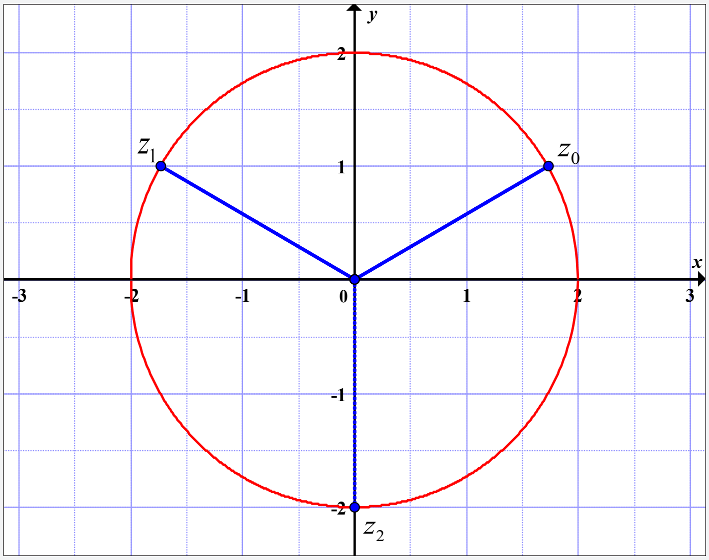


2. $z_0 = \sqrt{2}\,\mathrm{cis}\left(\frac{\pi}{4}\right)$, $z_1 = \sqrt{2}\,\mathrm{cis}\left(\frac{3\pi}{4}\right)$, $z_2 = \sqrt{2}\,\mathrm{cis}\left(-\frac{3\pi}{4}\right)$,

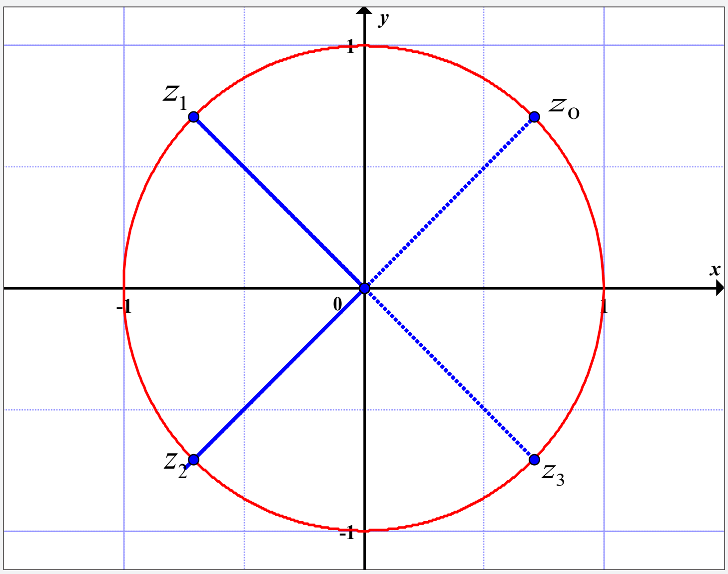
$z_3 = \sqrt{2}\,\mathrm{cis}\left(-\frac{\pi}{4}\right)$



3. $z_0 = 2\operatorname{cis}\left(\frac{\pi}{6}\right)$, $z_1 = 2\operatorname{cis}\left(\frac{5\pi}{6}\right)$, $z_2 = 2\operatorname{cis}\left(-\frac{\pi}{2}\right)$



4. $z_0 = \operatorname{cis}\left(\frac{\pi}{4}\right)$, $z_1 = \operatorname{cis}\left(\frac{3\pi}{4}\right)$, $z_2 = \operatorname{cis}\left(-\frac{3\pi}{4}\right)$, $z_3 = \operatorname{cis}\left(-\frac{\pi}{4}\right)$



Lesson 1.22. Construction of regular polygon

Learning objectives

Using n^{th} roots of unity and geometric instruments, learners should be able to construct a regular polygon in Argand plane accurately.

Prerequisites

- Finding n^{th} roots of unity
- Representation of n^{th} roots of unity in Argand diagram.

Teaching Aids

Exercise book, pencil, instrument of geometry and calculator

Activity 1.22

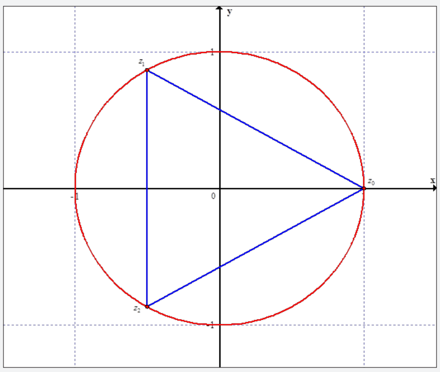
In this lesson, the following generic competence and cross-cutting issues are to be addressed:

- Critical thinking
- Communication
- Self confidence
- Cooperation, interpersonal management and life skills
- Peace and values education
- Inclusive education

Answers

1. $z_k = \text{cis}\dfrac{2k\pi}{3},\ k = 0,1,2$

   $z_0 = \text{cis}\,0 = 1,\ z_1 = \text{cis}\dfrac{2\pi}{3},\ z_2 = \text{cis}\dfrac{4\pi}{3}$
2. Graph



3. See the graph above.

4. The obtained figure is an equilateral triangle.

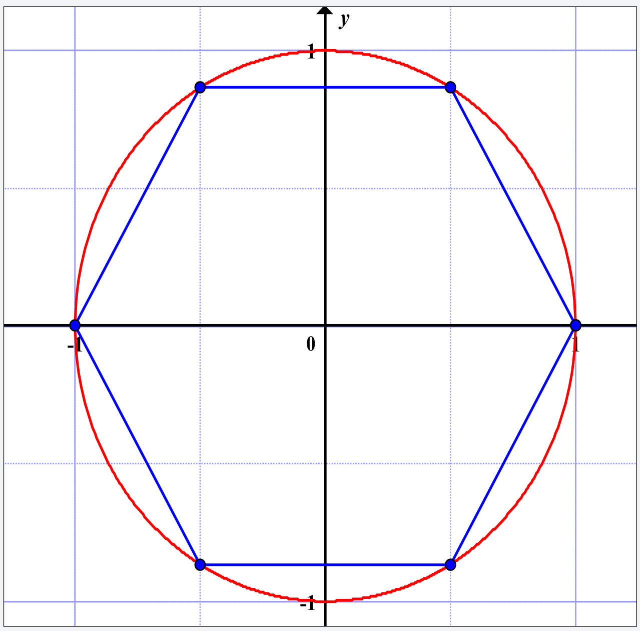
Synthesis

To draw a regular polygon with n sides follow the following steps:

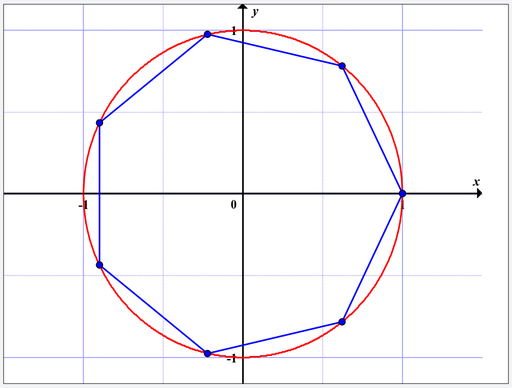
- Start by drawing a unit circle in Argand diagram. The radius and the centre of this circle will be the radius and centre of the regular polygon.
- Around the circle, place the points with affixes $z_k = \text{cis}\frac{2k\pi}{n},\ k = 0,1,2,.....,n-1$. Those points are the vertices of the polygon.
- Using a ruler, join the obtained points around the circle.
- The obtained figure is the needed regular polygon.

Application Activity 1.22

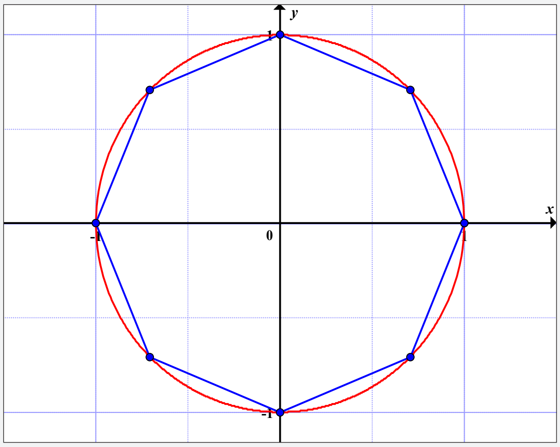
1. A regular hexagon (6 sides)



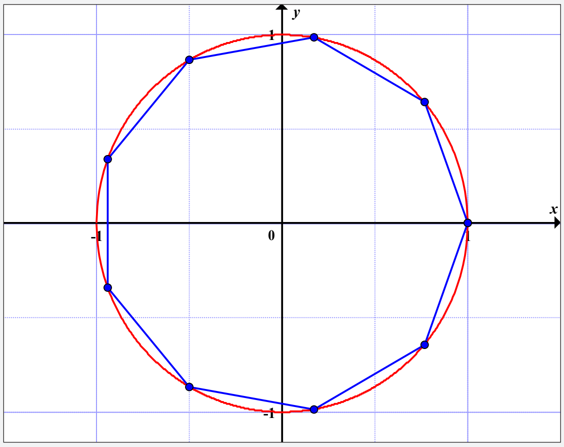
2. A regular heptagon (7 sides)



3. A regular octagon (8 sides)



4. A regular nonagon (9 sides)



Lesson 1.23. Exponential form of a complex number

Learning objectives

Given a complex number, learners should be able to express that complex number in exponential form accurately.

Prerequisites

- Finding modulus of a complex number.
- Finding argument of a complex number.

Teaching Aids

Exercise book, pen and calculator

Activity 1.23

In this lesson, the following generic competence and cross-cutting issues are to be addressed:

- Critical thinking
- Communication
- Self confidence
- Cooperation, interpersonal management and life skills
- Peace and values education
- Inclusive education

Answers

1. $$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \frac{x^7}{7!} + \ldots$$

   Replacing x with $i\theta$ gives

   $$e^{i\theta} = 1 + i\theta + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \frac{(i\theta)^5}{5!} + \frac{(i\theta)^6}{6!} + \frac{(i\theta)^7}{7!} + \ldots$$

   $$= 1 + i\theta - \frac{\theta^2}{2!} - \frac{i\theta^3}{3!} + \frac{\theta^4}{4!} + \frac{i\theta^5}{2!} - \frac{\theta^6}{6!} - \frac{i\theta^7}{7!} + \ldots$$

2. $$e^{i\theta} = 1 + i\theta - \frac{\theta^2}{2!} - \frac{i\theta^3}{3!} + \frac{\theta^4}{4!} + \frac{i\theta^5}{2!} - \frac{\theta^6}{6!} - \frac{i\theta^7}{7!} + \ldots$$

   $$= 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \ldots + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{2!} - \frac{\theta^7}{7!} + \ldots\right)$$

   Since $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} \ldots$ and

$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}...$, we can write

$e^{i\theta} = \cos\theta + i\sin\theta$

3. The right hand side of the expression obtained in 2) is the polar form of complex number having modulus 1 and argument θ.

Synthesis

Exponential form of a complex number z, can be simply found from its polar form $z = rcis\theta$.

For a complex number having modulus 1 and argument θ, we have the following equality; $e^{i\theta} = \cos\theta + i\sin\theta$, which leads to $r(\cos\theta + i\sin\theta) = re^{i\theta}$.

Therefore, $z = re^{i\theta}$ is exponential form of complex number $z = r(\cos\theta + i\sin\theta)$.

Application Activity 1.23

1. $e^{\frac{i\pi}{2}}$
2. $2e^{\frac{2i\pi}{3}}$
3. $2\sqrt{2}\,e^{-\frac{i\pi}{4}}$
4. $2\sqrt{3}\,e^{-\frac{i\pi}{6}}$
5. $5e^{i\pi}$
6. $3\sqrt{2}\,e^{\frac{i\pi}{4}}$
7. $5e^{0.9i}$
8. $13e^{-1.9i}$

Lesson 1.24. Trigonometric number of a multiple of an angle

Learning objectives

Given a multiple of an angle, learners should be able to find its trigonometric number accurately.

Prerequisites

- Binomial expansion.

Teaching Aids

Exercise book, pen and calculator

Activity 1.24

In this lesson, the following generic competence and cross-cutting issues are to be addressed:

- Critical thinking
- Communication
- Cooperation, interpersonal management and life skills
- Self confidence
- Peace and values education
- Inclusive education

Answers

1. Newton binomial expansion gives

$$(\cos x + i\sin x)^n = {}^nC_0\cos^n x + {}^nC_1\cos^{n-1}x(i\sin x) + {}^nC_2\cos^{n-2}x(i\sin x)^2 + \dots\dots + {}^nC_n(i\sin x)^n \quad (1)$$

$$= {}^nC_0\cos^n x + {}^nC_1 i\cos^{n-1}x\sin x - {}^nC_2\cos^{n-2}x\sin^2 x + \dots\dots + {}^nC_n i^n\sin^n x \quad (2)$$

2. Relations (1) and (2) are equivalent. Then,

$$\cos nx + i\sin nx = {}^nC_0\cos^n x - {}^nC_2\cos^{n-2}x\sin^2 x + \dots\dots\dots$$
$$+ i\left({}^nC_1\cos^{n-1}x\sin x - {}^nC_3\cos^{n-3}x\sin^3 x + \dots\right) \quad (3)$$

3. Recall that two complex numbers are equal if they have the same real parts and same imaginary parts. Thus, from (3), we have

$$\cos nx = {}^nC_0\cos^n x - {}^nC_2\cos^{n-2}x\sin^2 x + {}^nC_4\cos^{n-4}x\sin^4 x + \dots\dots$$
$$\sin nx = {}^nC_1\cos^{n-1}\sin x - {}^nC_3\cos^{n-3}x\sin^3 x + {}^nC_5\cos^{n-5}x\sin^5 x + \dots$$

Synthesis

Generally,

$$\cos nx = \sum_{0\le k\le n} {}^nC_k i^k\cos^{n-k}x\sin^k x,\ with\ k-even$$

$$i\sin nx = \sum_{1\le k\le n} {}^nC_k i^k\cos^{n-k}x\sin^k x,\ with\ k-odd \qquad {}^nC_k = \frac{n!}{k!(n-k)!}$$

Application Activity 1.24

1. $2\cos^2 x - 1$
2. $2\cos x \sin x$
3. $\dfrac{3\cot x - \cot^3 x}{1 - 3\cot^2 x}$
4. $\dfrac{3\cot x - \cot^3 x}{1 - 3\cot^2 x}$
5. $\dfrac{\tan^5 x - 10\tan^3 x + 5\tan x}{5\tan^4 x - 10\tan^2 x + 1}$
6. $32\cos^6 x - 48\cos^4 x + 18\cos^2 x - 1$
7. $-32\cos^3 x \sin x + 32\sin x \cos^5 x + 6\sin x \cos x$

Lesson 1.25. Linearisation of trigonometric expressions

Learning objectives

Given a trigonometric expression, learners should be able to linearise that trigonometric expression exactly.

Prerequisites

- Euler's formulae.

Teaching Aids

Exercise book, pen

Activity 1.25

In this lesson, the following generic competence and cross-cutting issues are to be addressed:

- Critical thinking
- Communication
- Self confidence
- Cooperation, interpersonal management and life skills
- Peace and values education
- Inclusive education

Answers

Euler's formulae are

$$\cos x = \frac{1}{2}\left(e^{ix} + e^{-ix}\right)$$

$$\sin x = \frac{1}{2i}\left(e^{ix} - e^{-ix}\right)$$

From these formulae, we have

$$\sin^2 x\cos x=\left(\frac{e^{ix}-e^{-ix}}{2i}\right)^2\left(\frac{e^{ix}+e^{-ix}}{2}\right)$$

$$=\left(\frac{e^{2ix}+e^{-2ix}-2}{-4}\right)\left(\frac{e^{ix}+e^{-ix}}{2}\right)$$

$$=\frac{e^{3ix}+e^{ix}+e^{-ix}+e^{-3ix}-2e^{ix}-2e^{-ix}}{-8}$$

$$=\frac{e^{3ix}+e^{-3ix}-e^{ix}-e^{-ix}}{-8}$$

$$=-\frac{1}{4}\left(\frac{e^{3ix}+e^{-3ix}}{2}-\frac{e^{ix}+e^{-ix}}{2}\right)$$

$$=-\frac{1}{4}\left(\cos 3x+\cos x\right)$$

Synthesis

To linearise trigonometric expression (product in sum), we use Euler's formulae

$$\cos\theta=\frac{1}{2}\left(e^{i\theta}+e^{-i\theta}\right) \qquad \sin\theta=\frac{1}{2i}\left(e^{i\theta}-e^{-i\theta}\right)$$

Application Activity 1.25

1. $\frac{1}{2}\cos\left(x-y\right)+\frac{1}{2}\cos\left(x+y\right)$
2. $\frac{1}{2}\cos\left(x-y\right)-\frac{1}{2}\cos\left(x+y\right)$
3. $\frac{1}{2}\sin 2x$
4. $\frac{1}{2}-\frac{1}{2}\cos 2x$
5. $\frac{1}{2}+\frac{1}{2}\cos 2x$
6. $\frac{3}{4}\sin x-\frac{1}{4}\sin 3x$
7. $\frac{3}{4}\cos x+\frac{1}{4}\cos 3x$
8. $\frac{1}{8}-\frac{1}{8}\cos 4x$

Lesson 1.26. Solving equation of the form $a\cos x + b\sin x = c,\ a,b,c \in \mathbb{R}$ and $a \cdot b \neq 0$

Learning objectives

Given equation of the form $a\cos x + b\sin x = c,\ a,b,c \in \mathbb{R}$ and $a \cdot b \neq 0$, learners should be able to solve it using complex numbers precisely.

Prerequisites

- Putting a complex number in polar form.
- Solving simple trigonometric equation.

Teaching Aids

Exercise book, pen and scientific calculator.

Activity 1.26

In this lesson, the following generic competence and cross-cutting issues are to be addressed:

- Critical thinking
- Communication
- Self confidence
- Cooperation, interpersonal management and life skills
- Peace and values education
- Inclusive education

Answers

1. $z = 1 - i\sqrt{3}$
2. $|z| = \sqrt{1+3} = 2$
3. $\arg(z) = \arctan\left(-\sqrt{3}\right) = -\frac{\pi}{3}$
4. $2\cos\left(x + \frac{\pi}{3}\right) = -1$

$$\Leftrightarrow \cos\left(x + \frac{\pi}{3}\right) = -\frac{1}{2} \Leftrightarrow x + \frac{\pi}{3} = \begin{cases} \frac{2\pi}{3} + 2k\pi \\ -\frac{2\pi}{3} + 2k\pi \end{cases}$$

$$\Rightarrow x = \begin{cases} \frac{\pi}{3} + 2k\pi \\ -\pi + 2k\pi \end{cases}$$

Synthesis

To solve the equation of the form

$a\cos x + b\sin x = c,\ \ a,b,c \in \mathbb{R}$ and $a\cdot b \neq 0$, follow these steps:

1. Reduction of $a\cos x + b\sin x \quad a,b \in \mathbb{R}$

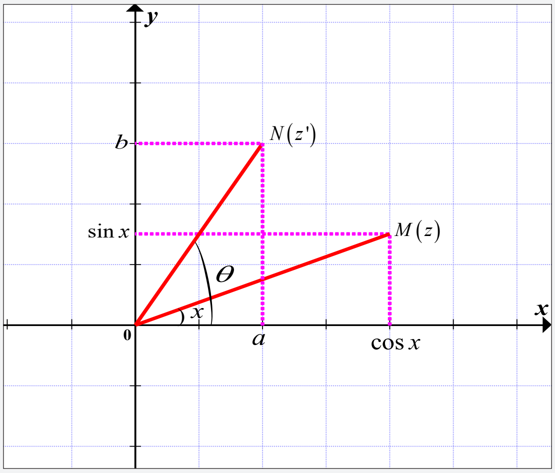


Figure 1.7: Reduction of a trigonometric expression

To get the expression equivalent to $a\cos x + b\sin x$, we use dot product expressed in terms of angle $\theta - x$ that is, between two vectors $\overrightarrow{OM} = (\cos x, \sin x)$ and $\overrightarrow{ON} = (a,b)$.

Or $\cos(\theta - x) = \dfrac{\overrightarrow{OM}\cdot\overrightarrow{ON}}{\left|\overrightarrow{OM}\right|\cdot\left|\overrightarrow{ON}\right|}$

$\Leftrightarrow \cos(\theta - x) = \dfrac{a\cos x + b\sin x}{\sqrt{a^2+b^2}\cdot\sqrt{\cos^2 x + \sin^2 x}}$

$\Leftrightarrow \cos(\theta - x) = \dfrac{a\cos x + b\sin x}{\sqrt{a^2+b^2}}$

$\Rightarrow a\cos x + b\sin x = \sqrt{a^2+b^2}\cos(\theta - x)$

Therefore, $a\cos x + b\sin x = c$

$\Leftrightarrow \sqrt{a^2+b^2}\cos(\theta - x) = c$

2. Solve reduction formula of $a\cos x + b\sin x = c$

$$a\cos x + b\sin x = c \Leftrightarrow \sqrt{a^2+b^2}\cos(\theta - x) = c$$

$$\Leftrightarrow \cos(\theta - x) = \frac{c}{\sqrt{a^2+b^2}},\ as\ \sqrt{a^2+b^2} \neq 0$$

Since $\forall \alpha \in \mathbb{R}, -1 \le \cos\alpha \le 1 \Leftrightarrow |\cos\alpha| \le 1$, thus,

$a\cos x + b\sin x = c$ has many solutions if and only if

$\left|\frac{c}{\sqrt{a^2+b^2}}\right| \le 1$ or $|c| \le \sqrt{a^2+b^2}$, otherwise, there is no solution.

Application Activity 1.26

1. $\left\{x = \frac{\pi}{6} + k\pi, x = \frac{\pi}{2}, k \in \mathbb{Z}\right\}$
2. $\left\{x = \frac{\pi}{4} + k\pi, k \in \mathbb{Z}\right\}$
3. $\pm\frac{3\pi}{4} - \frac{\pi}{4} + 2k\pi,\ k \in \mathbb{Z}$
4. $\frac{\pi}{6} \pm \frac{\pi}{4} + 2k\pi,\ k \in \mathbb{Z}$
5. $\frac{\pi}{6} \pm \frac{\pi}{3} + 2k\pi,\ k \in \mathbb{Z}$
6. $-\frac{\pi}{4} + 2k\pi,\ k \in \mathbb{Z}$

Lesson 1.27. Alternating current problem

Learning objectives

By leading textbooks or accessing internet, learners should be able to solve alternating current problems that involve complex numbers accurately.

Prerequisites

- Converting a complex number to different forms.
- Make a research by reading textbooks or accessing internet.

Teaching Aids

Exercise book, pen, textbooks or internet if available.

Activity 1.27

In this lesson, the following generic competence and cross-cutting issues are to be addressed:

- Critical thinking
- Communication
- Self confidence

- Cooperation, interpersonal management and life skills
- Research
- Peace and values education
- Inclusive education

Answers

$$Z = R + j\omega L + \frac{1}{j\omega C}$$

when $R = 10, L = 5, C = 0.04$ and $\omega = 4$ we have

$$Z = 10 + j(4)(5) + \frac{1}{j(4)(0.04)}$$

$$= 10 + j20 + \frac{1}{0.16j} = 10 + j20 + \frac{-j0.16}{(j0.16)(-j0.16)}$$

$$= 10 + j20 - \frac{j0.16}{0.0256} = 10 + j20 - j6.25 = 10 + j13.75$$

Thus, $Z = 10 + j13.75$ or $Z = 10 + 13.75j$

Synthesis

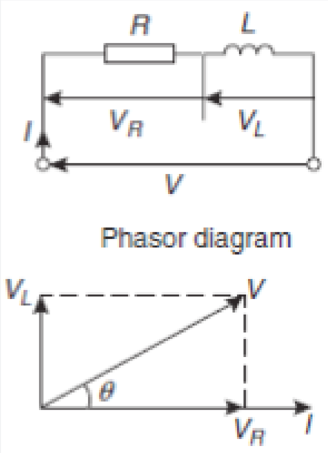
The voltage in an AC circuit can be represented as

$$V = V_0 e^{jwt}$$
$$= V_0(\cos wt + j\sin wt)$$

which denotes **Impedance**, V_o is peak value of impedance and $\omega = 2\pi f$ where *f* is the frequency of supply. To obtain the measurable quantity, the real part is taken:

$\text{Re}(V) = V_0 \cos wt$ and is called **Resistance** while imaginary part denotes **Reactance** (inductive or capacitive).

Briefly, the current, I (cosine function) leads the applied potential difference (p.d.), V (sine function) by one quarter of a cycle i.e. $\frac{\pi}{2}$ radians or 90^0.



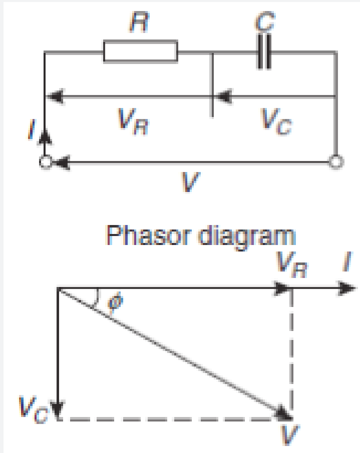


Figure 1.8: $R-L$ *series circuit*   *Figure1.9:* $R-C$ *series circuit*

In the **Resistance and Inductance** ($R-L$) *series* circuit, as shown in figure 1.8,

$V_R + jV_L = V$ as $V_R = IR, V_L = IX_L$ (where X_L is the inductive reactance $2\pi f L$ ohms) and $V=IZ$ (where Z is the impedance), then, $R + jX_L = Z$.

In the **Resistance and Capacitance** ($R-C$) circuit, as shown in figure 1.9,

$V_R - jV_C = V$, from which $R - jX_C = Z$

(where X_C is the capacitive reactance, $X_C = \frac{1}{2\pi fC}$ Ω).

Application Activity 1.27

1. a) $R = 3\Omega, L = 25.5mH$ b) $R = 2\Omega, L = 1061\mu F$

   c) $R = 0, L = 44.56mH$ d) $R = 4\Omega, L = 459.5\mu F$

2. $\left[15.76A, 23.20^0\ lagging\right]$

3. $\left[27.25A, 3.37^0\ lagging\right]$

4. a) $0.3A$ b) V leads I by 52^0

5. $Z_0 = 390.2cis\left(-10.43^0\right), \gamma = 0.1029cis61.92^0$

1.7. Summary of the unit

1. Concepts of complex numbers

A **complex number** is a number that can be put in the form $a+bi$, where *a* and *b* are real numbers and $i=\sqrt{-1}$.

The set of all complex numbers is denoted by $\mathbb{C}$ and is defined as $\mathbb{C}=\left\{z=a+bi:(a,b)\in\mathbb{R}^2 \text{ and } i^2=-1\right\}$.

The real number a of the complex number $z=a+bi$ is called the **real part** of z, and the real number b is often called the **imaginary part**. A complex number whose real part is zero is said to be **purely imaginary**, whereas a complex number whose imaginary part zero is said to be a **real number** or simply **real**.

2. Algebraic form of a complex number

Powers of i: $i^{4k}=1,\ i^{4k+1}=i,\ i^{4k+2}=-1,\ i^{4k+3}=-i$

$z=(a,b)$ is a **geometric** form of the complex number z. $z=a+bi$ is the **algebraic** (or **standard** or **Cartesian** or **rectangular**) form of the complex number z.

If two complex numbers, say $a+bi$ and $c+di$ are equal then, both their real and imaginary parts are equal. That is, $a+bi=c+di \Leftrightarrow a=c \text{ and } b=d$.

The **addition and subtraction** of two complex numbers $a+bi$ and $c+di$ is defined by the formula:

$$(a+bi)\pm(c+di)=(a\pm c)+(b\pm d)i$$

The **complex conjugate** of the complex number $z=x+yi$, denoted by $\bar{z}$ or z^*, is defined to be $\bar{z}=x-yi$.

The complex number $-z=-x-yi$ is the **opposite** of $z=x+yi$, symmetric of z with respect to 0.

The **multiplication** of two complex numbers $c+di$ and $c+di$ is defined by the formula: $(a+bi)(c+di)=(ac-bd)+(bc+ad)i$

The **inverse** of $z=a+bi$ is given by $\frac{1}{z}=z^{-1}=\frac{\bar{z}}{a^2+b^2}$

If $z_1 = a+bi$ *and* $z_2 = c+di$ then,

$$\frac{z_1}{z_2} = \frac{a+bi}{c+di} = \left(\frac{ac+bd}{c^2+d^2}\right) + \left(\frac{bc-ad}{c^2+d^2}\right)i$$

If a complex number $x+yi$ is a square root of the complex number $a+bi$, then, $\begin{cases} x = \pm\sqrt{\frac{1}{2}\left(a+\sqrt{a^2+b^2}\right)} \\ y = \pm\sqrt{\frac{1}{2}\left(\sqrt{a^2+b^2}-a\right)} \end{cases}$

Let a, b and c be real numbers ($a \neq 0$), then the equation $az^2+bz+c=0$ has either two real roots, one double real root or two conjugate complex roots.

a) If $\Delta > 0$, there are two distinct real roots:

$$z_1 = \frac{-b+\sqrt{\Delta}}{2a} \text{ and } z_2 = \frac{-b-\sqrt{\Delta}}{2a}.$$

b) If $\Delta = 0$, there is a double real root:

$$z_1 = z_2 = -\frac{b}{2a}$$

c) If $\Delta < 0$, there is no real roots. In this case, there are two conjugate complex roots:

$$z_1 = \frac{-b+i\sqrt{-\Delta}}{2a} \text{ and } z_2 = \frac{-b-i\sqrt{-\Delta}}{2a}.$$

Where, $\Delta = b^2-4ac$

$$z_1+z_2 = -\frac{b}{a},\ z_1 \cdot z_2 = \frac{c}{a}$$

Every polynomial of positive degree with coefficients in the system of complex numbers has a zero in the system of complex numbers. Moreover, every such polynomial can be factored linearly in the system of complex numbers.

3. Polar form of a complex number

The **absolute value (**or **modulus** or **magnitude)** of a complex number $z = x+yi$ is $r = |z| = \sqrt{x^2+y^2}$

Principal argument of a complex number $z = x + yi$

$$\arg(z) = \begin{cases} \arctan\frac{y}{x}, & if\ x > 0 \\ \pi + \arctan\frac{y}{x}, & if\ x < 0, y \geq 0 \\ -\pi + \arctan\frac{y}{x}, & if\ x < 0,\ y < 0 \\ \frac{\pi}{2}, & if\ x = 0,\ y > 0 \\ -\frac{\pi}{2}, & if\ x = 0,\ y < 0 \\ Undefined & if\ x = 0\, and\ y = 0 \end{cases}$$

Polar (or **modulus-argument**) **form** is $z = r(\cos\theta + i\sin\theta)$ or $z = r\,\text{cis}\theta$.

Given two complex numbers $z_1 = r_1(cos\ \theta_1 + isin\ \theta_1)$ and $z_2 = r_2(cos\ \theta_2 + isin\ \theta_2)$, the formulae for multiplication and division are $z_1 z_2 = r_1 r_2(cos\ (\theta_1 + \theta_2) + isin\ (\theta_1 + \theta_2))$ and $\frac{z_1}{z_2} = \frac{r_1}{r_2}(\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2))$ respectively.

Power of a complex number z is given by

$$z^n = (r(\cos\theta + i\sin\theta))^n = r^n(\cos n\theta + i\sin n\theta);\ n \in \mathbb{Z}_0$$

De Moivre's theorem: $(\cos\theta + i\sin\theta)^n = (\cos n\theta + i\sin n\theta)$

If $(z_k)^n = z$ for $z = rcis\theta$, then

$$z_k = \sqrt[n]{r}\ cis\left(\frac{\theta + 2k\pi}{n}\right) \quad k = 0,1,2,3,......,n-1$$

To draw a regular polygon with n sides, the steps followed are:

a) Start by drawing a unit circle in Argand diagram. The radius and the centre of this circle will be the radius and centre of the regular polygon.

b) Around the circle, place the points with affixes $z_k = \text{cis}\frac{2k\pi}{n},\ k = 0,1,2,.....,n-1$. Those points are the vertices of the polygon.

c) Using a ruler, join the obtained points around the circle.

d) The obtained figure is the needed regular polygon.

4. Exponential form of a complex number

The exponential form of a complex number z whose modulus is r and argument is θ, is $z = re^{i\theta}$

Euler's formulae (these formulae are used to linearise trigonometric expressions):

$$\cos\theta = \frac{1}{2}\left(e^{i\theta} + e^{-i\theta}\right)$$

$$\sin\theta = \frac{1}{2i}\left(e^{i\theta} - e^{-i\theta}\right)$$

5. Applications

Formulae for trigonometric number of a multiple of an angle

$$\cos nx = \sum_{0\le k\le n} C_k i^k \cos^{n-k} x \sin^k, \quad with\ k\ even$$

$$i\sin nx = \sum_{1\le k\le n} C_k i^k \cos^{n-k} x \sin^k, \quad with\ k\ odd \qquad {}^nC_k = \frac{n!}{k!(n-k)!}$$

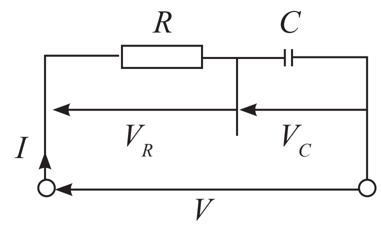
To solve the equation $a\cos x + b\sin x = c$, solve the equation

$$\cos(x-\theta) = \frac{c}{\sqrt{a^2+b^2}}, \quad \theta = \arg(a+bi)$$

Alternating current

Resistance and Capacitance (R-C)

Let a p.d. V be applied across a resistance R and a capacitance C in series. The same current I flows through each component and so the reference vector will be that representing I. The p.d. R across R is in phase with I, and V_C, that across C, lags on current I by 90^0.



Phasor diagram

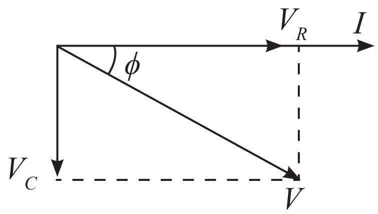


Figure showing Resistance and Capacitance in series

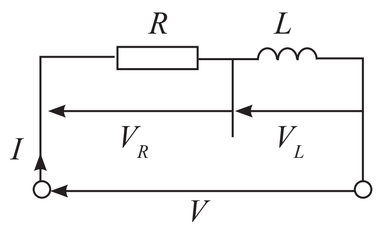
Vector sum of V_R and V_C is called Impedance and equals the applied p.d. V;

$Z = V_R + jV_C$ where V_R and V_C are known as resistance and reactance respectively.

But $V_R = IR$ and $V_C = IX_C$ where X_C is the capacitive reactance of C and equals $\dfrac{1}{\omega C}$.

Resistance and inductance (R-L)

The analysis is similar but here, the p.d. V_L across L leads on current I and the p.d. V_R across R is again in phase with I.



Phasor diagram

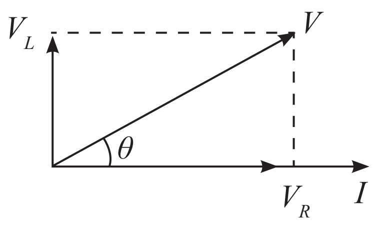


Figure showing Resistance and Inductance in series

$Z = V_R + jV_L$ where V_R and V_L are known as resistance and reactance respectively.

But $V_R = IR$ and $V_L = IX_L$ where X_L is the inductive reactance of L and equals ωL

or $\omega = 2\pi f$.

For the n-branch parallel circuit, Impedance Z is given by:

$$\frac{1}{Z} = \sum_{k=1}^{n} \frac{1}{Z_k}$$

1.8. End of Unit Assessment

1. a) $3-8i$ b) $5+5i$

   c) $100+200i$ d) $\frac{1-7i}{5}$

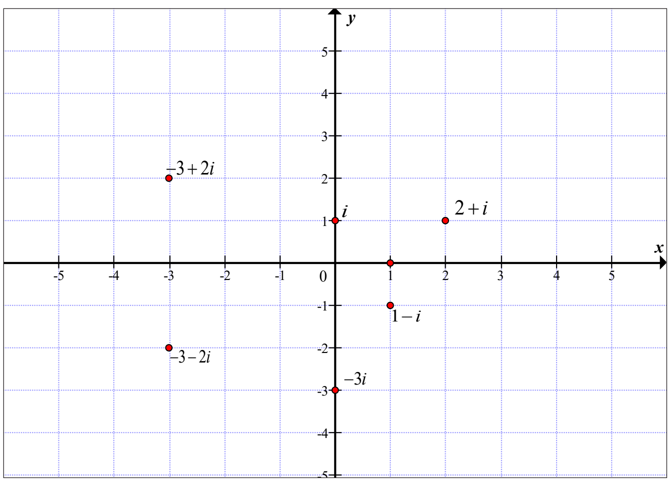
2. a) $x = \pm\sqrt{-4} = \pm\, 2\sqrt{-1} = \pm 2i$

   b) $x = \frac{-1 \pm i\sqrt{3}}{2} = -\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$

   c) $x = \frac{-6 \pm \sqrt{36-44}}{2} = \frac{-6 \pm i2\sqrt{2}}{2} = -3 \pm i\sqrt{2}$

   d) $x = 1, \quad -\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$

3. Plot



a) $1 = cis\, 0$ b) $i = cis\,\frac{\pi}{2}$

c) $-3i = 3cis\left(-\frac{\pi}{2}\right)$ d) $1-i = \sqrt{2}cis\left(-\frac{\pi}{4}\right)$

e) $2+i = \sqrt{5}cis\left(\arctan\left(\frac{1}{2}\right)\right)$

f) $-3-2i = \sqrt{13}cis\left(\pi - \arctan\left(\frac{2}{3}\right)\right)$

g) $-3+2i = \sqrt{13}cis\left(\pi + \arctan\left(-\frac{2}{3}\right)\right)$

4. a) $2\,cis\,0 = 2$ b) $3\,cis\,\pi = -3$

c) $cis\,\frac{\pi}{2} = i$ d) $3\,cis\,\frac{3\pi}{4} = -\frac{3\sqrt{2}}{2} + i\frac{3\sqrt{2}}{2}$

5. a) $i^7 = -i$ b) $(1+i)^5 = -4(1+i)$

c) $\left(\sqrt{3}-i\right)^{-4} = 2^{-4}\left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)$

6. a) $z_0 = cis\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2} + i\frac{1}{2}$, $z_1 = cis\left(\frac{5\pi}{6}\right) = -\frac{\sqrt{3}}{2} + i\frac{1}{2}$,

$z_2 = cis\left(\frac{3\pi}{2}\right) = -i$

b) $z_0 = \sqrt[8]{2}cis\left(\frac{\pi}{16}\right)$, $z_1 = \sqrt[8]{2}cis\left(\frac{9\pi}{16}\right)$, $z_2 = \sqrt[8]{2}cis\left(\frac{17\pi}{16}\right)$,

$z_3 = \sqrt[8]{2}cis\left(\frac{25\pi}{16}\right)$

7. Polar form: $\frac{z_1}{z_2} = cis\frac{7\pi}{12}$, Cartesian form:

$$\frac{z_1}{z_2} = \frac{\sqrt{2}}{4} - \frac{\sqrt{6}}{4} + i\left(\frac{\sqrt{2}}{4} + \frac{\sqrt{6}}{4}\right)$$

$$\begin{cases} \cos\frac{7\pi}{12} = \frac{\sqrt{2}-\sqrt{6}}{4} \\ \sin\frac{7\pi}{12} = \frac{\sqrt{2}+\sqrt{6}}{4} \end{cases}$$

8. $\left|e^{i\theta}\right| = \left|\cos\theta + i\sin\theta\right| = \sqrt{\cos^2\theta + \sin^2\theta} = 1$

9. $\left(e^{i\theta}\right)^{-1} = \left(\cos\theta + i\sin\theta\right)^{-1}$

$$= \frac{1}{\cos\theta + i\sin\theta} = \frac{\cos\theta - i\sin\theta}{\cos^2\theta + \sin^2\theta} = \cos\theta - i\sin\theta = \cos(-\theta) + i\sin(-\theta) = e^{-i\theta}$$

10. The n^{th} roots of unit are given by:

$$z_k = cis\frac{2k\pi}{n} \qquad k = 0,1,2,3,....,n-1$$

$$z_0 = cis\,0 = 1 \qquad z_1 = cis\left(\frac{2\pi}{n}\right)$$

$$z_2 = cis\left(\frac{4\pi}{n}\right) = \left(cis\left(\frac{2\pi}{n}\right)\right)^2 = z_1^2$$

$$z_3 = cis\left(\frac{6\pi}{n}\right) = \left(cis\left(\frac{2\pi}{n}\right)\right)^3 = z_1^3$$

$$\vdots$$

$$z_{n-1} = cis\left(\frac{2(n-1)\pi}{n}\right) = \left(cis\left(\frac{2\pi}{n}\right)\right)^{n-1} = z_1^{n-1}$$

The sum is $s_n = 1 + z_1 + z_1^2 + z_1^3 + ... + z_1^{n-1} \quad (1)$

Multiplying both sides by z_1 gives

$$z_1 s_n = z_1\left(1 + z_1 + z_1^2 + z_1^3 + ... + z_1^{n-1}\right)$$

$$\Leftrightarrow z_1 s_n = z_1 + z_1^2 + z_1^3 + z_1^4 + ... + z_1^n \qquad (2)$$

$(1)-(2)$ gives

$$\begin{aligned}(s_n - z_1 s_n) &= 1 + z_1 + z_1^2 + z_1^3 + ... + z_1^{n-1} \\ &\quad\; - z_1 - z_1^2 - z_1^3 - z_1^4 - .... - z_1^n \\ \hline s_n(1 - z_1) &= 1 - z_1^n\end{aligned}$$

$$\Leftrightarrow s_n = \frac{1 - z_1^n}{1 - z_1}$$

But $z_1^n = cis\left(\frac{2n\pi}{n}\right) = cis(2\pi) = 1$, then

$$s_n = \frac{1-z_1^n}{1-z_1} = \frac{1-1}{1-z_1} = \frac{0}{1-z_1} = 0$$

Thus, the sum of nth roots of unit is zero.

11. $\cos\frac{\pi}{5} = \frac{1+\sqrt{5}}{4}$

12. $z = -\frac{3}{2} + yi, \quad y \in \mathbb{R}$

13. $S = \{-3, -2i, 2i\}$

14. $\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)^{2001} = -1$

15. a) M is a point on circle of diameter $[AB]$ if $\overrightarrow{BM} \perp \overrightarrow{AM}$.

We need to check if $\frac{z_M - z_B}{z_M - z_A}$ is pure imaginary.

$$\frac{z_M - z_B}{z_M - z_A} = \frac{ie^{i\theta} - i}{ie^{i\theta} + i} = \frac{e^{i\theta} - 1}{e^{i\theta} + 1}$$

$$= \frac{e^{i\frac{\theta}{2}}\left(e^{i\frac{\theta}{2}} - e^{-i\frac{\theta}{2}}\right)}{e^{i\frac{\theta}{2}}\left(e^{i\frac{\theta}{2}} + e^{-i\frac{\theta}{2}}\right)} = \frac{2i\sin\frac{\theta}{2}}{2\cos\frac{\theta}{2}} = i\tan\frac{\theta}{2}$$

which is a pure imaginary.

Thus, M is a point on circle of diameter $[AB]$.

b) Rotation of centre o and angle of $\frac{\pi}{2}$ is $z' = e^{i\frac{\pi}{2}}z = iz$

$$z_{M'} = i\left(1 + ie^{i\theta}\right) = i - e^{i\theta}$$

$$\frac{z_M - z_B}{z_{M'} - z_B} = \frac{ie^{i\theta} - i}{-e^{i\theta} - 1} = \frac{i\left(e^{i\theta} - 1\right)}{i^2\left(e^{i\theta} + 1\right)} = \frac{\left(e^{i\theta} - 1\right)}{i\left(e^{i\theta} + 1\right)} = \frac{2i\sin\frac{\theta}{2}}{2i\cos\frac{\theta}{2}} = \tan\frac{\theta}{2}$$

which is real.

Thus, points B, M and M' are collinear.

16. Values of x are 2 and -5

17. a) The locus is the mediator of the segment $[AB]$ such that $z_A = 2$ and $z_B = -1$.

b) The locus is the mediator of the segment $[AB]$ such that $z_A = 2i \text{ and } z_B = -2$.

c) The locus is the circle of centre $1-3i$ and radius 2.

d) The locus is the circle of centre $-1+0i$ and radius 1.

e) The locus is the rectangular hyperbola.

f) The locus is the union of 2 bisectors of equations $y=-x \text{ and } y=x$ respectively.

18. The two complex numbers are $1+2i\sqrt{2}$ and $1-2i\sqrt{2}$

19. $z = cis\left(\frac{k\pi}{4}\right), k \in \mathbb{Z}$

20. $\text{Re}(z)=0$, $\text{Im}(z) = -\frac{\sqrt{3}}{3}$

21. $|z| = \cot\frac{\theta}{2}, \arg(z) = \theta - \frac{\pi}{2}$

22. $z = \left(\frac{-1+\sqrt{3}}{2}\right) + \left(\frac{-1+\sqrt{3}}{2}\right)i$ or
$z = \left(\frac{-1-\sqrt{3}}{2}\right) + \left(\frac{-1-\sqrt{3}}{2}\right)i$

23. $z = 6 + yi, y \in \mathbb{R}$

24. Isosceles triangle has two sides equal in length different from the third. We must check if there are two equal sides among $\left\|\overrightarrow{AB}\right\|$, $\left\|\overrightarrow{AC}\right\|$ and $\left\|\overrightarrow{BC}\right\|$

$$\left\|\overrightarrow{AB}\right\| = \|4-2i-1-2i\| = \|3-4i\| = \sqrt{9+16} = 5$$

$$\left\|\overrightarrow{AC}\right\| = \|1-6i-1-2i\| = \|-8i\| = \sqrt{64} = 8$$

$$\left\|\overrightarrow{BC}\right\| = \|1-6i-4+2i\| = \|-3-4i\| = \sqrt{9+16} = 5$$

Then, $\left\|\overrightarrow{AB}\right\| = \left\|\overrightarrow{BC}\right\| \neq \left\|\overrightarrow{AC}\right\|$ and hence the triangle is isosceles.

25. a) (i) $\left(\frac{\sqrt{3}-i}{1+i\sqrt{3}}\right)^9 = -i$ (ii) $\left(\frac{\sqrt{3}+i}{\sqrt{3}-i} + \frac{\sqrt{3}-i}{\sqrt{3}+i} - 2\right)^{30} = 1$

b) (i) $n = 6k, k \in \{1,2,3,4,5,.......\}$

(ii) $n = 3m, m \in \{1,3,5,7,9,.....\}$

c) Let $z_1 = 1+i$ $\quad z_2 = 1-i$

$|z_1| = \sqrt{2}$ $\quad |z_2| = \sqrt{2}$

$\arg(z_1) = \frac{\pi}{4}$ $\quad \arg(z_2) = -\frac{\pi}{4}$

$z_1 = \sqrt{2}cis\left(\frac{\pi}{4}\right)$ $\quad z_2 = \sqrt{2}cis\left(-\frac{\pi}{4}\right)$

Then,

$$(1+i)^n + (1-i)^n = z_1^n + z_2^n = \left[\sqrt{2}cis\left(\frac{\pi}{4}\right)\right]^n + \left[\sqrt{2}cis\left(-\frac{\pi}{4}\right)\right]^n$$

$$= \left(\sqrt{2}\right)^n \cos\left(\frac{n\pi}{4}\right) + i\left(\sqrt{2}\right)^n \sin\left(\frac{n\pi}{4}\right) + \left(\sqrt{2}\right)^n \cos\left(\frac{n\pi}{4}\right) + i\left(\sqrt{2}\right)^n \sin\left(-\frac{n\pi}{4}\right)$$

$$= \left(\sqrt{2}\right)^n \cos\left(\frac{n\pi}{4}\right) + i\left(\sqrt{2}\right)^n \sin\left(\frac{n\pi}{4}\right) + \left(\sqrt{2}\right)^n \cos\left(\frac{n\pi}{4}\right) - i\left(\sqrt{2}\right)^n \sin\left(\frac{n\pi}{4}\right)$$

$$= \left(\sqrt{2}\right)^n \cos\left(\frac{n\pi}{4}\right) + \left(\sqrt{2}\right)^n \cos\left(\frac{n\pi}{4}\right) = 2\left(\sqrt{2}\right)^n \cos\left(\frac{n\pi}{4}\right)$$

$$= 2\left(\sqrt{2}\right)^n \cos\left(\frac{n\pi}{4}\right) = 2(2)^{\frac{n}{2}} \cos\left(\frac{n\pi}{4}\right) = 2^{1+\frac{n}{2}} \cos\left(\frac{n\pi}{4}\right)$$

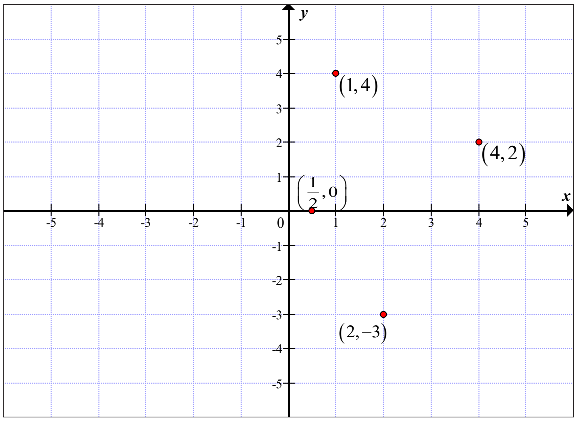
$= 2^{\frac{n+2}{2}} \cos\left(\frac{n\pi}{4}\right)$ as required.

26. a) $E(-1) = (-1)^3 + 2(-1)^2 + 2(-1) + 1 = 0$. Thus, -1 is a root of E.

b) $a = 1,\ b = 1,\ c = 1$

c) $S = \left\{-1, -\frac{1}{2} - i\frac{\sqrt{3}}{2}, -\frac{1}{2} + i\frac{\sqrt{3}}{2}\right\}$

27. a) Complex plane



b) $z_{\overline{AB}}=\frac{1}{2}-2+3i=-\frac{3}{2}+3i$, $z_{\overline{BC}}=1+4i-\frac{1}{2}=\frac{1}{2}+4i$

c) $E=\frac{5}{2}+i$

28. a) Polar form: $\frac{z_1}{z_2}=\frac{\sqrt{2}}{2}\operatorname{cis}\left(\frac{5\pi}{12}\right)$

Algebraic form: $\frac{z_1}{z_2}=\frac{\sqrt{3}-1}{4}+i\frac{\sqrt{3}+1}{4}$

b) $\begin{cases}\cos\frac{5\pi}{12}=\frac{\sqrt{6}-\sqrt{2}}{4}\\ \sin\frac{5\pi}{12}=\frac{\sqrt{6}+\sqrt{2}}{4}\end{cases}$

c) The lowest value of n is 12.

29. $11.86N,\ 146.77^0$ from force A

30. $8.394N,\ 208.68^0$ from force A

31. $(10+j20)\Omega,\ 22.36cis63.43^0\Omega$

32. $\pm\left(\frac{mh}{2\pi}\right)$

33. $\left[14.42A, 43.85^0\ lagging\right]$

34. $14.58\,A,\ 2.51^0\ leading$

35. Current $I=\frac{V}{Z}$

Impedance Z for three branch parallel circuit is given by

$$\frac{1}{Z}=\frac{1}{Z_1}+\frac{1}{Z_2}+\frac{1}{Z_3}$$

For our case, $Z_1=4+3j, Z_2=10, Z_3=12-5j$.

And then,

$$\frac{1}{Z}=\frac{1}{4+3j}+\frac{1}{10}+\frac{1}{12-5j}=\frac{2797}{8450}-\frac{764}{8450}=0.331-0904j$$

$$=0.343\left[\cos\left(-15^0 17'\right)+j\sin\left(-15^0 17'\right)\right]$$

$$I=\frac{V}{Z}=240\times 0.343\left[\cos\left(-15.28^0\right)+j\sin\left(-15.28^0\right)\right]$$

$$=82.32\left[\cos\left(-15.28^0\right)+j\sin\left(-15.28^0\right)\right]$$

Thus, $I=82.32A, with\, \theta=15.28^0\ lagging$.

Unit 2 Logarithmic and Exponential Functions

2.1. Key unit competence

Extend the concepts of functions to investigate fully logarithmic and exponential functions, finding the domain of definition, the limits, asymptotes, variations, graphs, and model problems about interest rates, population growth or decay, magnitude of earthquake, etc

2.2. Vocabulary or key words concepts

Depreciation: A negative growth (diminishing in value over a period of time).

Earthquake: A sudden violent shaking of the ground as a result of movements within the earth's crust.

Richter scale: A logarithmic scale for expressing the magnitude of an earthquake on the basis of seismograph oscillations.

2.3. Guidance on the introductory activity

The problem statement is "The population P of a city increases according to the formula $P = 500e^{at}$ where t is in years and $t = 0$ corresponds to 1980. In 1990, the population was 10,000. Find the value of the constant a; correct your answer to 3 decimal places."

From this problem, if $t = 0$ corresponds to1980, then 1990 corresponds to $t = 10$ and this gives the following equation: $500e^{at} = 1000$ or $e^{at} = 2$.

To find the value a, we take $\ln$ on both sides and we get

$\ln e^{10a} = \ln 2$ or $10a \ln e = \ln 2 \Rightarrow a = \frac{\ln 2}{10} = 0.069$. Such kind of problems are solved using logarithms.

2.4. List of lessons

No	Lesson title	Number of periods
1	Domain and range of natural logarithmic function	1
2	Limit and asymptotes for natural logarithmic function	1
3	Derivative of natural logarithmic function	1
4	Variation and curve of natural logarithmic function	2
5	Domain and range of logarithmic function with any base	1
6	Limit and asymptotes for logarithmic function with any base	1
7	Logarithmic differentiation	1
8	Further differentiation	1
9	Variation and curve of logarithmic function with any base	2
10	Domain and range of exponential function with base e	1
11	Limit and asymptotes for exponential function with base e	1
12	Derivative of exponential function with base e	1
13	Variation and curve of exponential function with base e	2
14	Domain and range of exponential function with any base	1
15	Limit and asymptotes for exponential function with any base	1
16	Derivative of exponential function with any base	1
17	Variation and curve of exponential function with any base	2

18	Compound interest problems	2
19	Mortgage amount problems	1
20	Population growth problems	1
21	Depreciation value problems	1
22	Earthquake problems	1
23	Carbon-14 dating	1
Total periods		28

2.5. Lesson development

Lesson 2.1. Domain and range of natural logarithmic functions

Learning objectives

Given any logarithmic function, learners should be able to find its domain and range accurately.

Prerequisites

- Finding domain of polynomial, rational/irrational and trigonometric functions.
- Finding range of polynomial, rational/irrational and trigonometric functions.

Teaching Aids

Exercise book, pen and calculator

Activity 2.1

In this lesson, the following generic competence and cross-cutting issues are to be addressed:

- Critical thinking
- Self confidence
- Communication
- Cooperation, interpersonal management and life skills
- Peace and values education
- Inclusive education
- Financial education

Answers

x	$\ln x$
-0.8	impossible
-0.6	impossible
-0.4	impossible
-0.2	impossible
0	impossible

x	$\ln x$
0.2	-1.61
0.4	-0.91
0.6	-0.51
0.8	-0.22
1	0

x	$\ln x$
1.5	0.40
2	0.69
2.5	0.91
3	1.09
3.5	1.25

1. (i) For negative x values and zero, $\ln$ is impossible.

   (ii) For x values between 0 and 1, $\ln$ is less than zero.

   (iii) For x values greater than 1, $\ln$ is greater than zero.

2. Curve

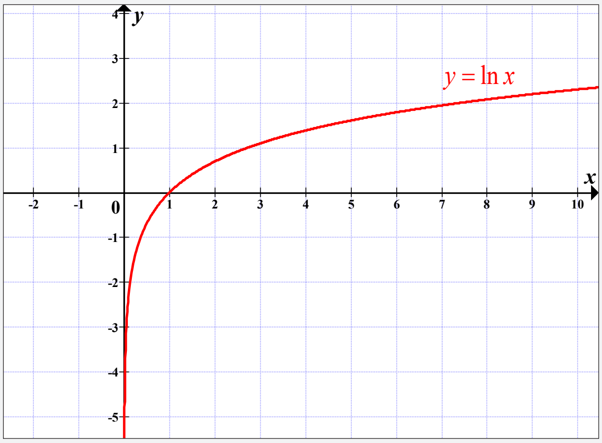


Figure 2.1: Curve of $y = \ln x$

Synthesis

From figure 2.1, $\ln x$ is defined on positive real numbers, $]0,+\infty[$ and its range is all real numbers that is $domf =]0,+\infty[$ and $\operatorname{Im} f =]-\infty,+\infty[$.

Application Activity 2.28

1. a) $]0,+\infty[$ b) $]0,4[$

   c) $]-\infty,+\infty[$ d) $]-\infty,3[$

2. 316.2

Lesson 2.2. Limit and asymptotes for natural logarithmic functions

Learning objectives

Given a natural logarithmic function, learners should be able to evaluate limits and deduce relative asymptotes accurately.

Prerequisites

- Evaluating limits.
- Finding relative asymptotes.

Teaching Aids

Exercise book, pen and calculator

Activity 2.2

In this lesson, the following generic competence and cross-cutting issues are to be addressed:

- Critical thinking
- Self confidence
- Communication
- Cooperation, interpersonal management and life skills
- Peace and values education
- Inclusive education

Answers

1. $\lim\limits_{x\to 0^-} \ln x$ does not exist since at the left of zero $\ln$ is impossible.
2. If x takes on values closer to 0 from the right, we have

x	$\ln x$
0.5	-0.69315
0.45	-0.79851
0.4	-0.91629
0.35	-1.04982
0.3	-1.20397
0.25	-1.38629
0.2	-1.60944
0.15	-1.89712
0.1	-2.30259
0.05	-2.99573

We see that if x takes on values closer to 0 from the right, $\ln x$ becomes smaller and smaller negative. Then $\lim_{x\to 0^+} \ln x = -\infty$. There is a vertical asymptote $x = 0$.

3. If we give to x the values of the form $10^n \left(n \in \mathbb{N}\right)$, $\ln 10^n = n\ln 10 \approx 2.30n$ and let n take values 1, 2, 3, 4, 5, 6, 7, 8, 9, 10,…, we have;

n	$x = 10^n$	$\ln x$	$\frac{\ln x}{x}$
1	10	2.302585	0.230258509
2	100	4.60517	0.046051702
3	1000	6.907755	0.006907755
4	10000	9.21034	0.000921034
5	100000	11.51293	0.000115129
6	1000000	13.81551	0.000013816
7	10000000	16.1181	0.000001612
8	100000000	18.42068	0.000000184
9	1000000000	20.72327	0.000000021
10	10000000000	23.02585	0.000000002

We see that if x takes on values of the form $10^n \left(n \in \mathbb{N}\right)$, $\ln x$ becomes larger and larger without bound and consequently approaches no fixed value. Then $\lim_{x\to +\infty} \ln\left(x\right) = +\infty$. There is no horizontal asymptote.

Also, that if x takes the values of the form $10^n \left(n \in \mathbb{N}\right)$, $\frac{\ln x}{x}$ becomes closer to zero.

Then, $\lim_{x\to +\infty} \frac{\ln x}{x} = 0$.

There is no oblique asymptote.

Synthesis

As conclusion, $\lim_{x\to +\infty} \ln x = +\infty$ and $\lim_{x\to 0^+} \ln x = -\infty$

There exists a vertical asymptote with equation $VA \equiv x = 0$

No horizontal asymptote.

Application Activity 2.29

1) $-\infty$ 2) 0 3) $+\infty$ 4) $+\infty$

Lesson 2.3. Derivative of natural logarithmic functions

Learning objectives

Given a natural logarithmic function, learners should be able to differentiate it accurately.

Prerequisites

- Definition of derivative.
- Differentiating a polynomial, rational/ irrational and trigonometric functions.

Teaching Aids

Exercise book and pen

Activity 2.3

In this lesson, the following generic competence and cross-cutting issues are to be addressed:

- Critical thinking
- Communication
- Self confidence
- Cooperation, interpersonal management and life skills
- Peace and values education
- Inclusive education

Answers

1. $$(\ln x)' = \lim_{h\to 0}\frac{\ln(x+h)-\ln x}{h} = \lim_{h\to 0}\frac{\ln\left(\frac{x+h}{x}\right)}{h}$$

$$= \lim_{h\to 0}\frac{1}{h}\ln\left(\frac{x+h}{x}\right) = \lim_{h\to 0}\ln\left(1+\frac{h}{x}\right)^{\frac{1}{h}}$$

Let $u = \frac{h}{x} \Rightarrow h = ux$

If $h \to 0, u \to 0$

$$\lim_{h\to 0}\ln\left(1+\frac{h}{x}\right)^{\frac{1}{h}} = \lim_{u\to 0}\ln(1+u)^{\frac{1}{ux}}$$

$$= \lim_{u\to 0}\frac{1}{x}\ln(1+u)^{\frac{1}{u}}$$

$$= \frac{1}{x}\lim_{u\to 0}\ln(1+u)^{\frac{1}{u}}$$

$$= \frac{1}{x}\ln\lim_{u\to 0}(1+u)^{\frac{1}{u}}$$

$$= \frac{1}{x}\ln e \qquad \text{since } \lim_{u\to 0}(1+u)^{\frac{1}{u}} = e$$

$$= \frac{1}{x}$$

Thus, $(\ln x)' = \frac{1}{x}$

2. $(\ln u)' = \frac{1}{u}u' = \frac{u'}{u}$. Thus, $(\ln u)' = \frac{u'}{u}$ where u is another differentiable function.

Synthesis

$(\ln x)' = \frac{1}{x}$; if u is another differentiable function of x then, $(\ln u)' = \frac{u'}{u}$.

Application Activity 2.30

1. $\frac{2\ln x}{x}$
2. $\frac{\tan^2 x+1}{\tan x}$
3. $\frac{x}{x^2-1}$
4. $\frac{2}{x^2-1}$
5. $\frac{x\tan x-\ln(\sin x)}{x^2}$
6. $-\tan x+\frac{1}{x}$
7. $\tan^2 x-\frac{1-\ln x}{3x^2}+1$
8. $\frac{-2\ln\left(\sqrt{x+1}\right)+1}{2(x+1)^2}$

Lesson 2.4. Variation and curve sketching of natural logarithmic functions

Learning objectives

Given a natural logarithmic function, learners should be able to study the variation and sketch its curve perfectly.

Prerequisites

- Finding domain and limits at the boundaries of the domain.
- Deducing relative asymptotes.
- Finding first and second derivative.
- Variation and concavity of a function.
- Sketch a curve in Cartesian plane given some points.

Teaching Aids

Exercise book, pencil, instrument of geometry and calculator

Activity 2.4

In this lesson, the following generic competence and cross-cutting issues are to be addressed:

- Critical thinking
- Communication
- Self confidence
- Cooperation, interpersonal management and life skills
- Peace and values education
- Inclusive education

Answers

1. $f(x)=\ln x$. The domain is $]0,+\infty[$.

   $\lim\limits_{x\to 0^+} f(x)=-\infty$. There is a vertical asymptote $x=0$

   $\lim\limits_{x\to +\infty} f(x)=+\infty$. There is no horizontal asymptote

   $\lim\limits_{x\to +\infty} \frac{f(x)}{x}=\lim\limits_{x\to +\infty} \frac{\ln x}{x}=0$. There is no oblique asymptote.

2. $f(x)=\ln x \Rightarrow f'(x)=\frac{1}{x}$. Since $x\in]0,+\infty[$, $f'(x)$ is always positive and hence $f(x)=\ln x$ increases on its domain.

Since $f'(x)=\frac{1}{x}\neq 0$, there is no extrema (no maximum, no minimum).

3. $f'(x)=\frac{1}{x}\Rightarrow f''(x)=-\frac{1}{x^2}$. $f''(x)$ is always negative and hence the concavity of $f(x)=\ln x$ is turning down on its domain. Since $f''(x)=-\frac{1}{x^2}\neq 0$, there is no inflection points.

4. Completed table of variation

x	0	1		e	$+\infty$
Sign of $f'(x)$	$\|$ +	+	+	+	+
Sign of $f''(x)$	$\|$ -	-	-	-	-
Variation of $f(x)$	$-\infty \nearrow 0 \nearrow 1 \nearrow +\infty$				
Concavity of $f(x)$					

5. Intersection of $f(x)$ with axes of co-ordinates:

There is no intersection of $f(x)$ with $y-axis$ since this axis is an asymptote.

Intersection with $x-axis$:

$f(x)=0\Leftrightarrow \ln x=0\Rightarrow \ln x=\ln 1\Rightarrow x=1$. Then,

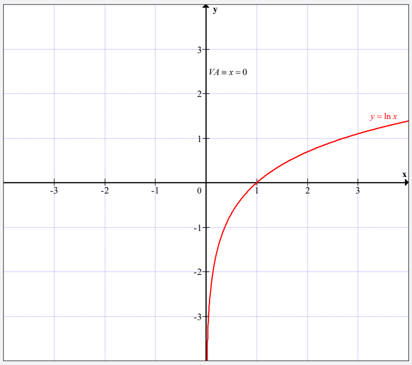
$f(x)\cap ox=\{(1,0)\}$.

6. Additional points

x	0.1	0.3	0.5	0.7	0.9	1.1	1.3	1.5	1.7	1.9	2.1
y	-2.3	-1.2	-0.7	-0.4	-0.1	0.1	0.3	0.4	0.5	0.6	0.7

x	2.3	2.5	2.7	2.9	3.1	3.3	3.5	3.7	3.9
y	0.8	0.9	1.0	1.1	1.1	1.2	1.3	1.3	1.4

Graph



Synthesis

To sketch a function, follow the following steps:

- Find domain of definition.
- Evaluate limits at the boundary of domain and deduce relative asymptotes.
- Find first derivative. Deduce maxima and draw variation table.
- Find second derivative. Deduce inflection points and draw concavity table.
- Find x and y intercepts.
- Find additional points.
- Sketch the curve.

For other function, you may need to study party and periodicity. Also, you may need to find tangent lines at remarkable points (maxima, inflection points, x and y intercepts)

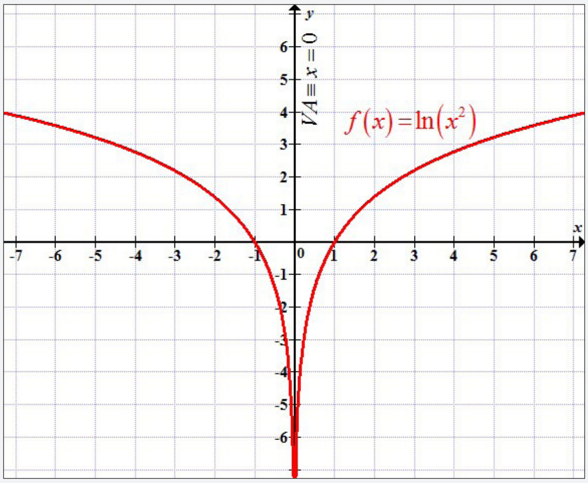
Application Activity 2.31

1. Domain of definition: $]-\infty,0[\cup]0,+\infty[$

   Vertical asymptote $x=0$

   $f(x)$ decreases on interval $]-\infty,0[$ and increases on interval $]0,+\infty[$.

Curve

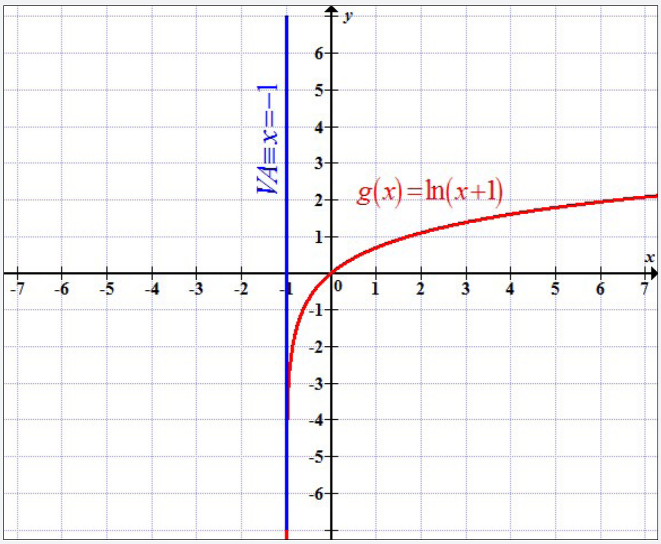


2. Domain of definition: $]-1, +\infty[$

   Vertical asymptote: $x = -1$

   $g(x)$ increases on its domain

   Curve

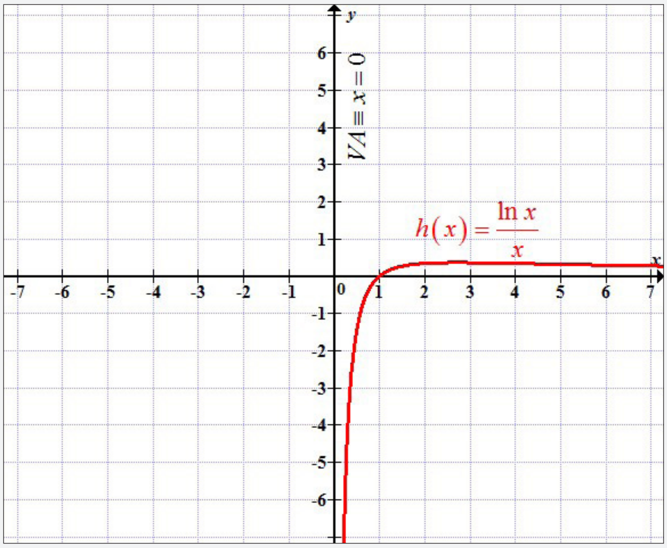


3. Domain of definition: $]0, +\infty[$

   Vertical asymptote: $x = 0$

   $h(x)$ increases on its domain

Curve

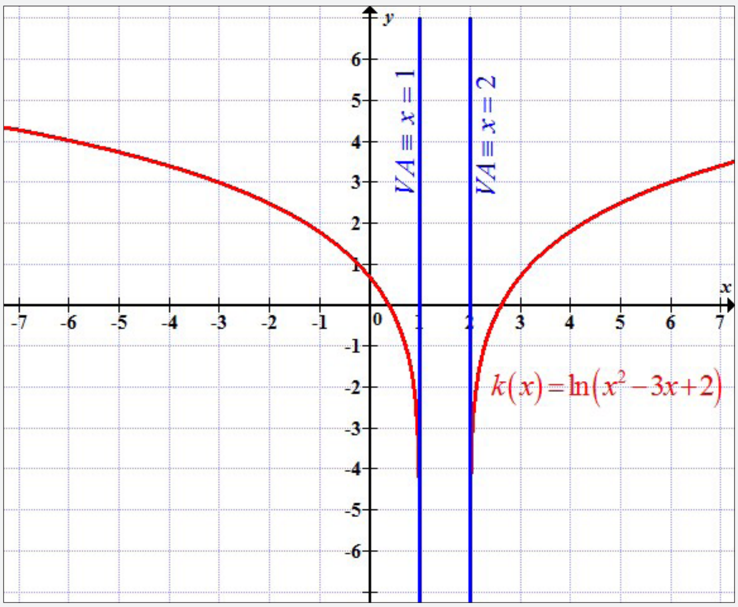


4. Domain of definition: $]-\infty,1[\cup]2,+\infty[$

   Vertical asymptote: $x=1$ and $x=2$

   $k(x)$ decreases on interval $]-\infty,1[$ and increases on interval $]2,+\infty[$

   Curve



Lesson 2.5. Domain and range of logarithmic function with any base

Learning objectives

Given a logarithmic function with any base, learners should be able to find the domain and range accurately.

Prerequisites

- Domain of a natural logarithmic function.
- Range of a natural logarithmic function.

Teaching Aids

Exercise book, pen and calculator

Activity 2.5

In this lesson, the following generic competence and cross-cutting issues are to be addressed:

- Critical thinking
- Communication
- Self confidence
- Cooperation, interpersonal management and life skills
- Peace and values education
- Inclusive education
- Financial education

Answers

1. Existence condition: $x > 0$

   Hence, $Domf =]0, +\infty[$

   Limit on boundaries

   $$\lim_{x \to 0^+} \frac{\ln x}{\ln 2} = -\infty \qquad \lim_{x \to \infty} \frac{\ln x}{\ln 2} = +\infty$$

   From limits on boundaries, we get that range of $f(x)$ is $]-\infty, +\infty[$.

2. Existence condition: $x > 0$

   Hence, $Domg =]0, +\infty[$

   Limit on boundaries

   $$\lim_{x \to 0^+} \frac{\ln x}{\ln \frac{1}{2}} = +\infty \qquad \lim_{x \to \infty} \frac{\ln x}{\ln \frac{1}{2}} = \lim_{x \to \infty} \frac{\ln x}{-\ln 2} = -\infty$$

From limits on boundaries, we get that range of $g(x)$ is $]-\infty,+\infty[$.

Synthesis

Logarithm of a real number x with base a is the number denoted $\log_a x$ defined by $\log_a x=\dfrac{\ln x}{\ln a}$, $x\in\mathbb{R}_0^+$, $a\in\mathbb{R}_0^+\setminus\{1\}$

By letting $a=2$, the curve of $f(x)=\log_2 x$ is the following

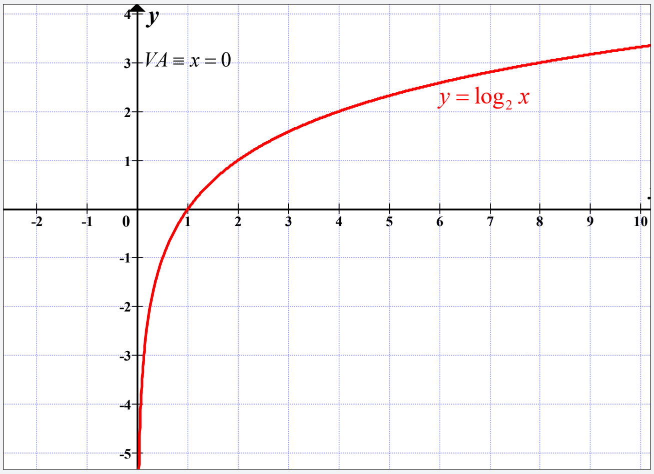


Figure 2.2: Curve of $y=\log_2 x$

By letting $a=\dfrac{1}{2}$, you get the curve of $f(x)=\log_{\frac{1}{2}} x$ as illustrated in figure 2.2.

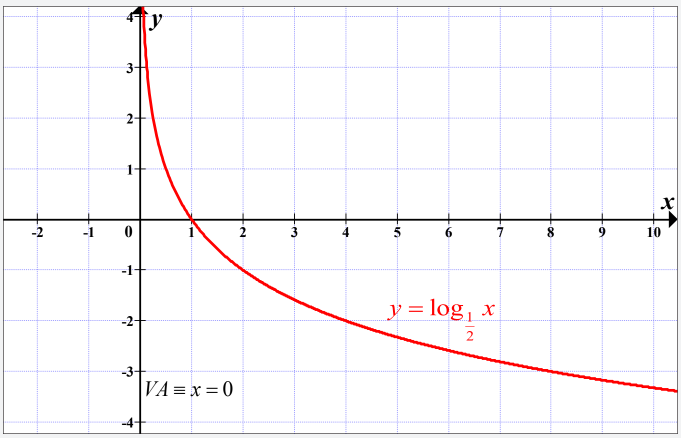


Figure 2.3: Curve of $y=\log_{\frac{1}{2}} x$

Application Activity 2.32

1. $\left]0,+\infty\right[$
2. $\left]-\infty,-1\right[\cup\left]1,+\infty\right[$
3. $\left]-\infty,-1\right[\cup\left]4,+\infty\right[$
4. $\left]-5,-2\right[\cup\left]-2,0\right[$

Lesson 2.6. Limit of logarithmic function with any base

Learning objectives

Given a logarithmic function with any base, learners should be able to find limits and deduce its relative asymptotes accurately.

Prerequisites

- Limit of natural logarithmic function.

Teaching Aids

Exercise book, pen

Activity 2.6

In this lesson, the following generic competence and cross-cutting issues are to be addressed:

- Critical thinking
- Communication
- Self confidence
- Cooperation, interpersonal management and life skills
- Peace and values education
- Inclusive education

Answers

1. $Domf=\left]0,+\infty\right[$ and $\ln 3>0$,

$$\lim_{x\to 0^+} f(x)=\lim_{x\to 0^+}\frac{\ln x}{\ln 3}=\frac{\lim_{x\to 0^+}\ln x}{\ln 3}=\frac{-\infty}{\ln 3}=-\infty$$

There is a vertical asymptote $VA\equiv x=0$

$$\lim_{x\to +\infty} f(x)=\lim_{x\to +\infty}\frac{\ln x}{\ln 3}=\frac{\lim_{x\to +\infty}\ln x}{\ln 3}=\frac{+\infty}{\ln 3}=+\infty$$

There is no horizontal asymptote

$$\lim_{x\to+\infty}\frac{f(x)}{x}=\lim_{x\to+\infty}\frac{\ln x}{x\ln 3}=\frac{\lim\limits_{x\to+\infty}\frac{\ln x}{x}}{\ln 3}=\frac{0}{\ln 3}=0$$

There is no oblique asymptote

2. For $0<\frac{1}{3}<1,\ \ln\frac{1}{3}<0$,

$$\lim_{x\to0^+}f(x)=\lim_{x\to0^+}\frac{\ln x}{\ln\frac{1}{3}}=\frac{\lim\limits_{x\to0^+}\ln x}{\ln\frac{1}{3}}=\frac{-\infty}{\ln\frac{1}{3}}=+\infty$$

There is a vertical asymptote $VA\equiv x=0$

$$\lim_{x\to+\infty}f(x)=\lim_{x\to+\infty}\frac{\ln x}{\ln\frac{1}{3}}=\frac{\lim\limits_{x\to+\infty}\ln x}{\ln\frac{1}{3}}=\frac{+\infty}{\ln\frac{1}{3}}=-\infty$$

There is no horizontal asymptote

$$\lim_{x\to+\infty}\frac{f(x)}{x}=\lim_{x\to+\infty}\frac{\ln x}{x\ln\frac{1}{3}}=\frac{\lim\limits_{x\to+\infty}\frac{\ln x}{x}}{\ln\frac{1}{3}}=\frac{0}{\ln\frac{1}{3}}=0$$

There is no oblique asymptote.

Synthesis

Figure 2.4 and figure 2.5, are helpful to note that

$\lim\limits_{x\to0}\log_3 x=-\infty$ and $\lim\limits_{x\to+\infty}\log_3 x=+\infty$

$\lim\limits_{x\to0}\log_{\frac{1}{3}} x=+\infty$ and $\lim\limits_{x\to+\infty}\log_{\frac{1}{3}} x=-\infty$.

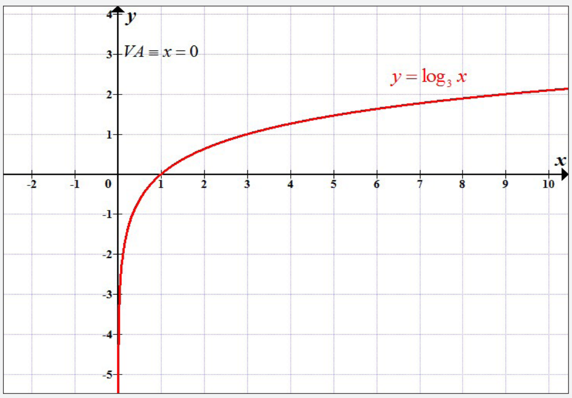


Figure 2.4: Curve of $y=\log_3 x$

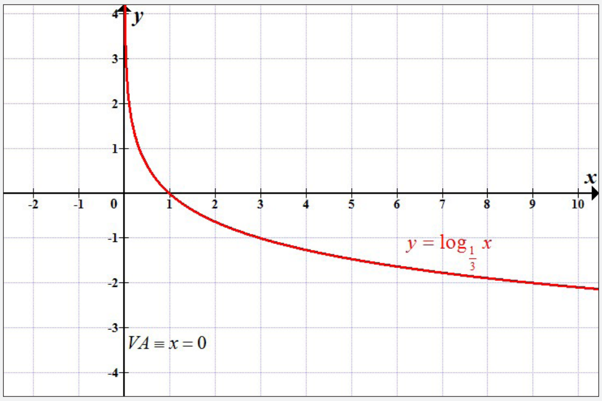


Figure 2.5: Curve of $y = \log_{\frac{1}{3}} x$

Generally, calculating limit of logarithmic function with any base, for example $\log_a x$, from definition $\log_a x = \frac{\ln x}{\ln a}$, you get the following results:

- $\lim\limits_{x \to 0^+} f(x) = \begin{cases} -\infty & if \;\; a > 1 \\ +\infty & if \;\; 0 < a < 1 \end{cases}$

  Thus, there is a vertical asymptote $VA \equiv x = 0$

- $\lim\limits_{x \to +\infty} f(x) = \begin{cases} +\infty & if \;\; a > 1 \\ -\infty & if \;\; 0 < a < 1 \end{cases}$

  Then, there is no horizontal asymptote. In addition, no oblique asymptote.

Application Activity 2.33

1. $+\infty$ 2. $+\infty$ 3. $-\infty$ 4. $+\infty$

Lesson 2.7. Logarithmic differentiation

Learning objectives

Given a logarithmic function with any base, learners should be able to find its derivative accurately.

Prerequisites

- Differentiation of natural logarithmic functions.

Teaching Aids

Exercise book and pen

Activity 2.7

In this lesson, the following generic competence and cross-cutting issues are to be addressed:

- Critical thinking
- Communication
- Self confidence
- Cooperation, interpersonal management and life skills
- Peace and values education
- Inclusive education

Answers

1. $f(x)=\frac{\ln x}{\ln 2}\Rightarrow f'(x)=\frac{\frac{1}{x}}{\ln 2}=\frac{1}{x\ln 2}$

2. $\left(\frac{\ln x^2}{\ln 2}\right)'=\frac{\frac{1}{x^2}\left(x^2\right)'}{\ln 2}=\frac{2x}{x^2\ln 2}=\frac{2}{x\ln 2}$

Synthesis

As conclusion, $(\log_a x)'=\frac{1}{x\ln a}$. Also, if u is another differentiable function of x, then

$(\log_a u)'=\frac{u'}{u\ln a}$

Application Activity 2.34

1. $\frac{2x+2}{\left(x^2+2x+1\right)\ln 10}$

2. $-\frac{6}{\left(x^2-4x-5\right)\ln 2}$

3. $\frac{-3x^2-2}{\left(2x^3+4x-16\right)\ln 2}$

4. $-\frac{\sqrt{x}\sin\sqrt{x}}{2x\ln 3\cos\sqrt{x}}$

Lesson 2.8. Further logarithmic differentiation

Learning objectives

Given a function containing more complicated products and quotients, learners should be able to differentiate it moderately.

Prerequisites

- The laws of logarithms,
- The derivative of logarithmic functions, and
- The differentiation of implicit functions.

Teaching Aids

Exercise book and pen.

Activity 2.8

In this lesson, the following generic competence and cross-cutting issues are to be addressed:

- Critical thinking
- Communication
- Self confidence
- Cooperation, interpersonal management and life skills
- Peace and values education
- Inclusive education

Answers

1. $y = \dfrac{x+1}{x-3}$

   Taking $\ln$ on both sides gives $\ln y = \ln \dfrac{x+1}{x-3}$
   Applying laws of logarithms, we get

   $\ln y = \ln(x+1) - \ln(x-3)$

2. Differentiating with respect to x yields

   $\dfrac{1}{y}\dfrac{dy}{dx} = \dfrac{1}{x+1} - \dfrac{1}{x-3}$.

   Rearranging gives $\dfrac{dy}{dx} = y\left(\dfrac{1}{x+1} - \dfrac{1}{x-3}\right)$

   Substituting for y gives $\dfrac{dy}{dx} = \left(\dfrac{x+1}{x-3}\right)\left(\dfrac{1}{x+1} - \dfrac{1}{x-3}\right)$

Synthesis

For functions containing more complicated products and quotients, differentiation is often made easier if the logarithm of the function is taken before differentiating. And then apply the property of differentiation of implicit functions.

Application Activity 2.35

1. $\frac{dy}{dx}=\frac{(x-2)(x+1)}{(x-1)(x+3)}\left(\frac{1}{x-2}+\frac{1}{x+1}-\frac{1}{x-1}-\frac{1}{x+3}\right)$

2. $\frac{dy}{dx}=\frac{(2x-1)\sqrt{x+2}}{(x-3)\sqrt{(x+1)^3}}\left(\frac{2}{2x-1}+\frac{1}{2(x+1)}-\frac{1}{x-3}-\frac{3}{2(x+1)}\right)$

3. $\frac{dy}{d\theta}=3\theta\sin\theta\cos\theta\left(\frac{1}{\theta}+\tan\theta-\cot\theta\right)$

4. $\frac{dy}{dx}=\frac{x^3\ln 2x}{e^x\sin x}\left(\frac{3}{x}+\frac{1}{x\ln 2x}-1-\cot x\right)$

5. $\frac{dy}{dx}=\frac{2x^4\tan x}{e^{2x}\ln 2x}\left(\frac{4}{x}+\frac{1}{\sin x\cos x}-2-\frac{1}{x\ln 2x}\right)$

Lesson 2.9. Variation and curves of logarithmic functions with any base

Learning objectives

Given a logarithmic function with any base, learners should be able to study the variation and sketch its curve accurately.

Prerequisites

- Finding domain and limits at the boundaries of the domain.
- Deducing relative asymptotes.
- Finding first and second derivative.
- Variation and concavity of a function.
- Sketch a curve in Cartesian plane given some points.

Teaching Aids

Exercise book, pencil, instrument of geometry and calculator

Activity 2.9

In this lesson, the following generic competence and cross-cutting issues are to be addressed:

- Critical thinking
- Communication
- Self confidence
- Cooperation, interpersonal management and life skills
- Peace and values education
- Inclusive education

Answers

1. $f(x) = \log_2 x$

e) From Activity 2.6

$\lim_{x \to 0^+} f(x) = -\infty$

There is a vertical asymptote $VA \equiv x = 0$

$\lim_{x \to +\infty} f(x) = +\infty$

There is no horizontal asymptote. In addition, no oblique asymptote.

f) $f'(x) = \left(\frac{\ln x}{\ln 2}\right)' = \frac{\frac{1}{x}}{\ln 2} = \frac{1}{x \ln 2}$

For $\ln 2 > 0$, $f'(x) = \frac{1}{x \ln 2} > 0$ since $x > 0$

The function $f(x) = \log_2 x$ increases on its domain

$f'(x) = \frac{1}{x \ln 2} \neq 0$, for $\forall x > 0$, no extrema.

g) $f''(x) = \left(\frac{1}{x \ln 2}\right)' = -\frac{1}{x^2 \ln 2}$

For $\ln 2 > 0$, $f''(x) = -\frac{1}{x^2 \ln 2} < 0$ since $x^2 > 0$

The concavity of function $f(x) = \log_2 x$ turns downward on domain of $f(x)$.

$f''(x) = -\frac{1}{x^2 \ln 2} \neq 0$, for $\forall x > 0$, no inflection points.

h) Intersection of $f(x)$ with axes of co-ordinates:

No intersection with $y-axis$ since this axis is a vertical asymptote.

Intersection with $x-axis$:

$$\log_2 x = 0 \Leftrightarrow \frac{\ln x}{\ln 2} = 0 \Leftrightarrow \ln x = 0 \Rightarrow x = 1.$$

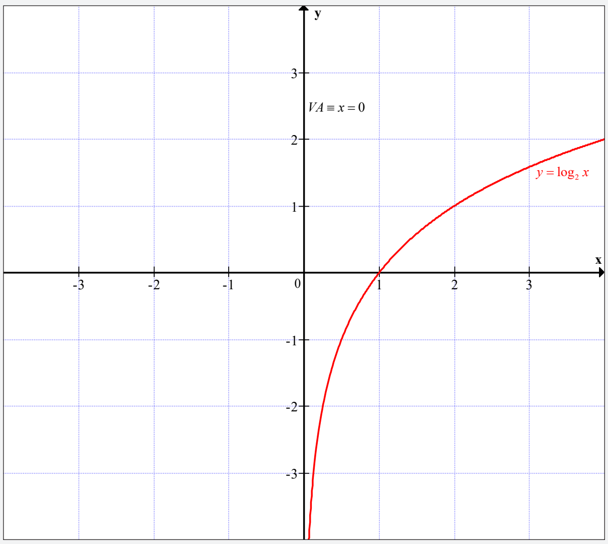
Hence, $f(x) \cap ox = \{(1,0)\}$

i) Additional points for $f(x) = \log_2 x$

x	0.1	0.3	0.5	0.7	0.9	1.1	1.3	1.5	1.7	1.9
y	-3.3	-1.7	-1.0	-0.5	-0.2	0.1	0.4	0.6	0.8	0.9

x	2.1	2.3	2.5	2.7	2.9	3.1	3.3	3.5	3.7	3.9
y	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9	2.0

Curve



2. $g(x) = \log_{\frac{1}{2}} x$

a) From Activity 2.6

$\lim_{x \to 0^+} g(x) = +\infty$

There is a vertical asymptote $VA \equiv x = 0$

$\lim_{x \to +\infty} g(x) = -\infty$

There is no horizontal asymptote. In addition, no oblique asymptote.

b) $g'(x)=\left(\dfrac{\ln x}{\ln\frac{1}{2}}\right)^{/}=\dfrac{\frac{1}{x}}{\ln\frac{1}{2}}=\dfrac{1}{x\ln\frac{1}{2}}=\dfrac{2}{-x\ln 2}$,

since $\ln\frac{1}{2}=-\ln 2$

For $\ln 2>0$, $g'(x)=\dfrac{1}{-x\ln 2}<0$ since $x>0$

The function $g(x)=\log_{\frac{1}{2}} x$ decreases on its domain

$g'(x)=\dfrac{1}{-x\ln 2}\neq 0$, no extrema.

c) $g''(x)=\left(\dfrac{1}{-x\ln 2}\right)^{/}=\dfrac{1}{x^2\ln 2}$

For $\ln 2>0$, $g''(x)=\dfrac{1}{x^2\ln 2}>0$ since $x^2>0$

The concavity of function $g(x)=\log_{\frac{1}{2}} x$ turns upward on domain of $g(x)$.

$g''(x)=\dfrac{1}{x^2\ln 2}\neq 0$, no inflection points.

d) Intersection of $f(x)$ with axes of co-ordinates

No intersection with $y-axis$ since this axis is a vertical asymptote.

Intersection with $x-axis$:

$\log_{\frac{1}{2}} x=0\Leftrightarrow\dfrac{\ln x}{\ln 2}=0\Leftrightarrow\ln x=0\Rightarrow x=1$.

Hence, $f(x)\cap ox=\{(1,0)\}$.

e) Additional points for $g(x)=\log_{\frac{1}{2}} x$

x	0.1	0.3	0.5	0.7	0.9	1.1	1.3	1.5	1.7	1.9
y	3.3	1.7	1.0	0.5	0.2	-0.1	-0.4	-0.6	-0.8	-0.9

x	2.1	2.3	2.5	2.7	2.9	3.1	3.3	3.5	3.7	3.9
y	-1.1	-1.2	-1.3	-1.4	-1.5	-1.6	-1.7	-1.8	-1.9	-2.0

Curve

Synthesis

To sketch a function, follow the following steps:

- Find domain of definition.
- Evaluate limits at the boundary of domain and deduce relative asymptotes.
- Find first derivative. Deduce maxima and draw variation table.
- Find second derivative. Deduce inflection points and draw concavity table.
- Find x and y intercepts.
- Find additional points.
- Sketch the curve.

For other function, you may need to study parity and periodicity. Also, you may need to find tangent lines at remarkable points (maxima, inflection points, x and y intercepts).

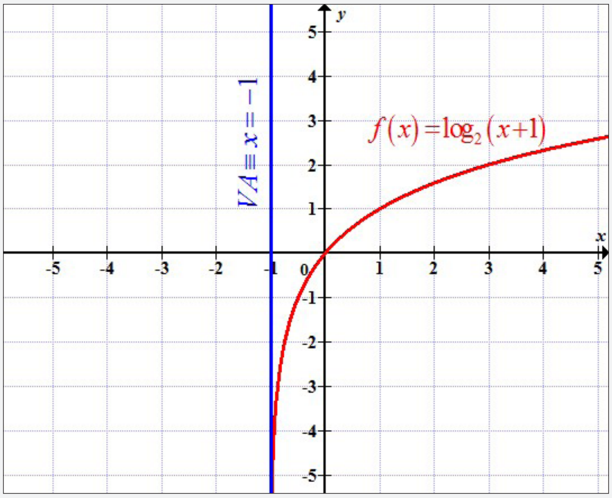
Application Activity 2.36

1. Domain: $]-1,+\infty[$

   Vertical asymptote: $x=-1$

   $f(x)$ increases on its domain

   Curve

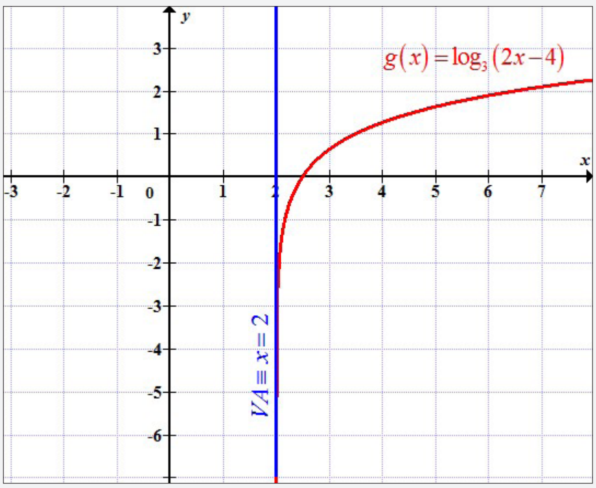


2. Domain: $]2,+\infty[$

   Vertical asymptote: $x=2$

   $g(x)$ increases on its domain

   Curve

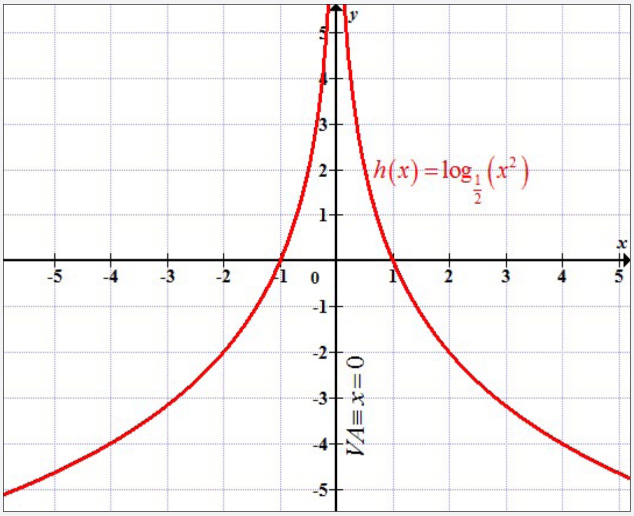


3. Domain: $]-\infty, 0[\cup]0, +\infty[$

   Vertical asymptote: $x = 0$

   $h(x)$ increases on interval $]-\infty, 0[$ and decreases on interval $]0, +\infty[$

   Curve

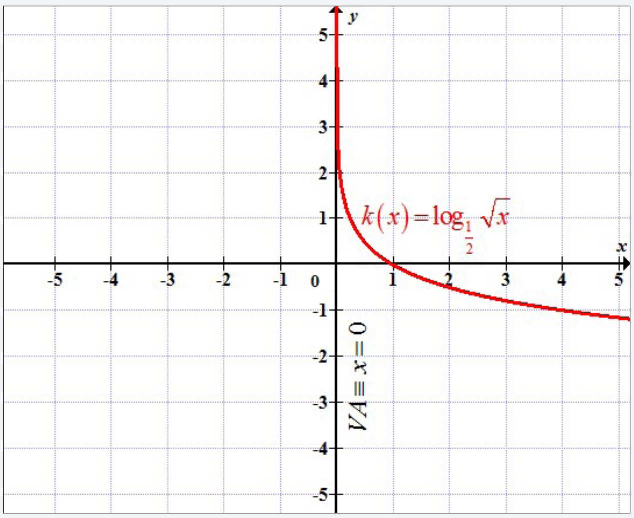


4. Domain: $]0, +\infty[$

   Vertical asymptote: $x = 0$

   $k(x)$ decreases on its domain

   Curve



Lesson 2.10. Domain and range of exponential functions with base "e"

Learning objectives

Given an exponential function with base "e", learners should be able to find domain and range accurately.

Prerequisites

- Domain of natural logarithmic function.
- Range of natural logarithmic function.

Teaching Aids

Exercise book and pen

Activity 2.10

In this lesson, the following generic competence and cross-cutting issues are to be addressed:

- Critical thinking
- Communication
- Self confidence
- Cooperation, interpersonal management and life skills
- Peace and values education
- Inclusive education
- Financial education

Answers

We saw that the domain of $f(x)=\ln x$ is $]0,+\infty[$ and its range is $\mathbb{R}$. Since $g(x)$ is the inverse of $f(x)$, the domain of $g(x)$ is $\mathbb{R}$ and its range is $]0,+\infty[$.

Synthesis

The domain of definition of $y=e^x$ is $]-\infty,+\infty[$ and its range is $]0,+\infty[$ as illustrated in figure 2.6.

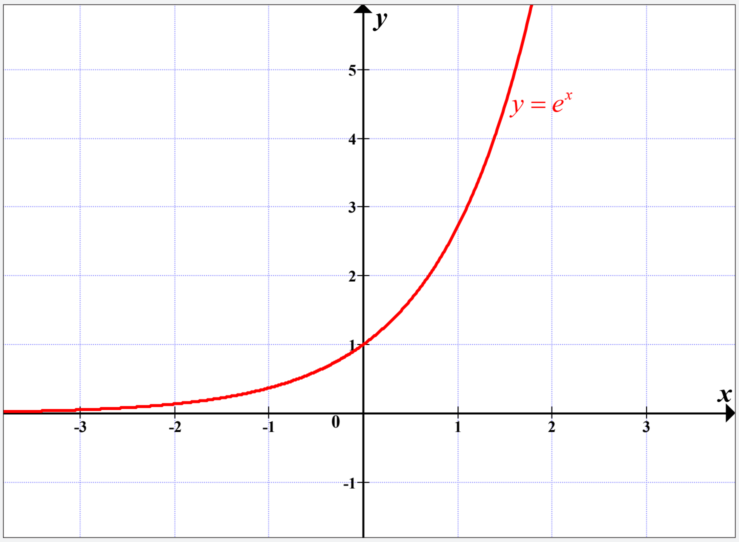


Figure 2.6: Curve of $y=e^x$

Application Activity 2.37

1) $\mathbb{R}\setminus\{2,5\}$ 2) $\mathbb{R}$ 3) $]0,+\infty[$ 4) $[4,+\infty[$

Lesson 2.11. Limit of exponential functions with base "e"

Learning objectives

Given an exponential function with base "e", learners should be able to find limit and deduce relative asymptote accurately.

Prerequisites

- Finding limits using table of values.
- Deduction of relative asymptotes.

Teaching Aids

Exercise book, pen, calculator and instruments of geometry

Activity 2.11

In this lesson, the following generic competence and cross-cutting issues are to be addressed:

- Critical thinking
- Communication

- Self confidence
- Cooperation, interpersonal management and life skills
- Peace and values education
- Inclusive education

Answers

1. Completed table

x	e^x
-1	0.36787944117144
-2	0.13533528323661
-5	0.00673794699909
-15	0.00000030590232
-30	0.00000000000009

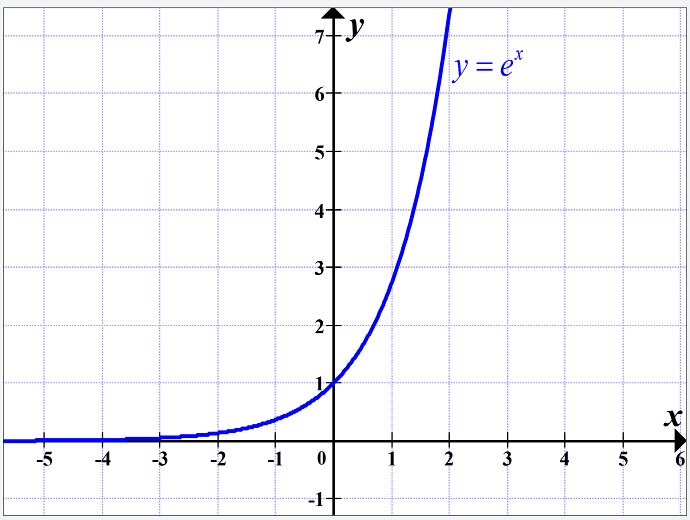
x	e^x
1	2.7182818
2	7.3890561
5	148.4131591
15	3269017.3724721
30	10686474581524.5

2. From table in 1), when x takes values approaching to $-\infty$, e^x takes value closed to zero. Hence, $\lim_{x \to -\infty} e^x = 0$.

   There exists a horizontal asymptote $y = 0$, no oblique asymptote.

   Also, when x takes value approaching to $+\infty$, e^x increases without bound. Hence, $\lim_{x \to +\infty} e^x = +\infty$. There is no horizontal asymptote.

3. Graph



Synthesis

From the above figure, it is clear that

$\lim_{x \to -\infty} e^x = 0$ and $\lim_{x \to +\infty} e^x = +\infty$.

There exists horizontal asymptote: $H.A \equiv y = 0$.

$\lim_{x \to -\infty} \frac{e^x}{x} = 0, \lim_{x \to +\infty} \frac{e^x}{x} = +\infty$

There is no oblique asymptote.

Application Activity 2.38

1. $\sqrt[3]{e}$
2. 0
3. $+\infty$
4. 0
5. 0

Lesson 2.1. Derivative of exponential functions with base "*e*"

Learning objectives

Given an exponential functions with base "e", learners should be able to differentiate it correctly.

Prerequisites

- Use the derivative of natural logarithmic function.
- Rule of differentiating inverse functions.

Teaching Aids

Exercise book and pen

Activity 2.12

In this lesson, the following generic competence and cross-cutting issues are to be addressed:

- Critical thinking
- Communication
- Self confidence
- Cooperation, interpersonal management and life skills
- Peace and values education
- Inclusive education

Answers

1. $f'(x)=\dfrac{1}{\left(f^{-1}\right)^{/}(y)}$

$f(x)=e^x \Rightarrow f^{-1}(x)=\ln x$ but $(\ln x)'=\dfrac{1}{x}$

$f'(x)=\dfrac{1}{\dfrac{1}{e^x}}=e^x$

Thus, $\left(e^x\right)^{/}=e^x$

2. $(f\circ g)'=f'(g)g'$. Then, $\left(e^u\right)^{/}=u'e^u$

Synthesis

$\left(e^x\right)'=e^x$ and if u is another differentiable function of x, $\left(e^u\right)'=u'e^u$.

Or from the definition of differentiation,

$\dfrac{df(x)}{dx}=\lim\limits_{h\to0}\dfrac{f(x+h)-f(x)}{h}$, thus,

$$\frac{d\left(e^x\right)}{dx}=\lim_{h\to0}\frac{e^{x+h}-e^x}{h}$$

$$=\lim_{h\to0}\frac{e^x\left(e^h-1\right)}{h}=e^x\lim_{h\to0}\left(\frac{e^h-1}{h}\right)$$

$$=e^x\ln e\quad\left[as\ \lim_{h\to0}\left(\frac{e^h-1}{h}\right)=\ln e\right]$$

$$=e^x\ln e=e^x$$

Therefore, $\left(e^x\right)'=\dfrac{d\left(e^x\right)}{dx}=e^x$

Application Activity 2.39

1. $2e^{2x-1}$
2. $2\left(e^{2x}+e^{-2x}\right)$
3. $\left(1+\tan^2 x\right)e^{\tan x}$
4. $\dfrac{(x-2)e^x}{(x-1)|x-1|}$

Lesson 2.2. Variation and curve of exponential functions with base "e"

Learning objectives

For an exponential function with base "e", learners should be able to study the variation and sketch its curve accurately.

Prerequisites

- Reflecting a curve about the first bisector.
- Properties of inverse functions.

Teaching Aids

Exercise book, pencil, calculator and instruments of geometry.

Activity 2.13

In this lesson, the following generic competence and cross-cutting issues are to be addressed:

- Critical thinking
- Communication
- Self confidence
- Cooperation, interpersonal management and life skills
- Peace and values education
- Inclusive education

Answers

When reflecting the curve of $f(x) = \ln x$ about the first bisector, we obtain

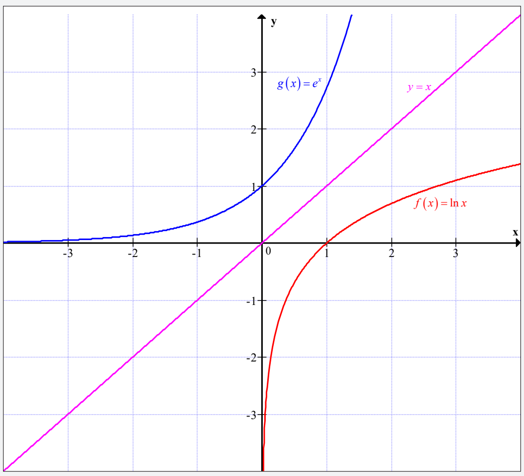


Figure 2.6: Reflection of $y = \ln x$ about first bisector

Synthesis

Since e^x is the inverse of $\ln x$, the curve of $g(x)=e^x$ is the image of the curve of $f(x)=\ln x$ with respect to the first bisector, $y=x$. Then, the coordinates of the points for $f(x)=\ln x$ are reversed to obtain the coordinates of the points for $g(x)=e^x$.

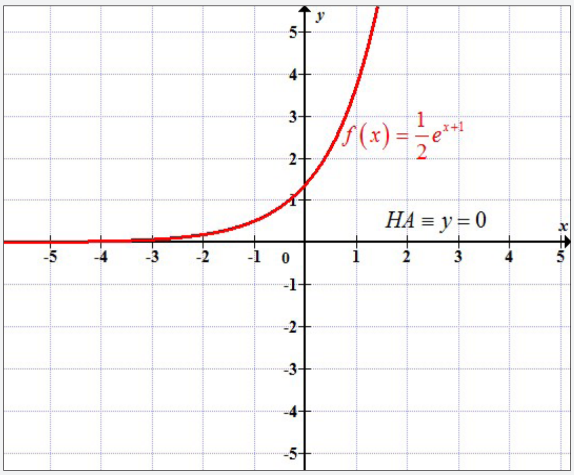
Application Activity 2.40

1. Domain: $]-\infty,+\infty[$

   Horizontal asymptote: $y=0$

   $f(x)$ increases on its domain

   Curve

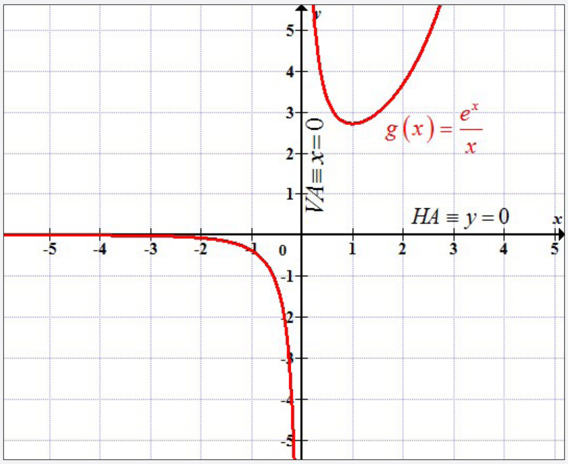
2. Domain: $]-\infty,0[\cup]0,+\infty[$

   Vertical asymptote: $x=0$ and horizontal asymptote: $y=0$

   $g(x)$ increases on interval $]1,+\infty[$

   $g(x)$ decreases on intervals: $]-\infty,0[$ and $]0,1[$

Curve

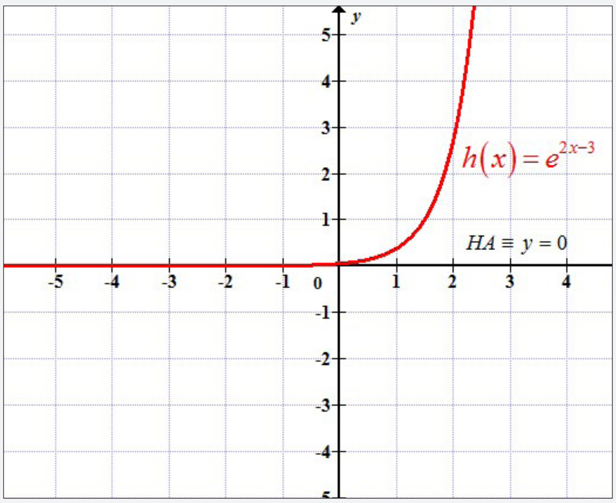


3. Domain: $]-\infty,+\infty[$

   Horizontal asymptote: $y=0$

   $h(x)$ increases on its domain

Curve

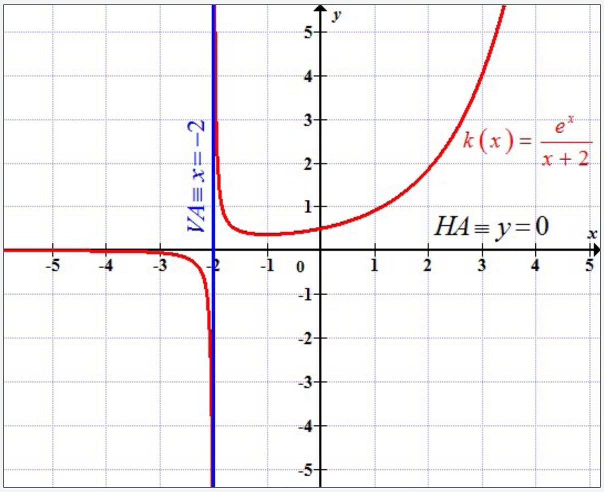


4. Domain: $]-\infty,-2[\cup]-2,+\infty[$

   Vertical asymptote: $x=-2$ and horizontal asymptote: $y=0$

   $k(x)$ decreases on interval $]-\infty,-2[\cup]-2,-1[$ and increases on interval $]-1,+\infty[$

Curve



Lesson 2.3. Domain and range of exponential functions with any base

Learning objectives

Given an exponential functions with any base, learners should be able to find the domain and range accurately.

Prerequisites

- Domain of logarithmic function with any base.
- Range of logarithmic function with any base.

Teaching Aids

Exercise book and pen

Activity 2.14

In this lesson, the following generic competence and cross-cutting issues are to be addressed:

- Critical thinking
- Communication
- Self confidence
- Cooperation, interpersonal management and life skills
- Peace and values education
- Inclusive education
- Financial education

Answers

We know that the domain of $f(x)=\log_a x$ is $]0,+\infty[$ and its range is $\mathbb{R}$. Since $g(x)$ is the inverse of $f(x)$, the domain of $g(x)$ is $\mathbb{R}$ and its range is $]0,+\infty[$.

Synthesis

The domain of $f(x)=a^x$ with $a>0$ and $a\neq 1$, is the set of real numbers and its image is the positive real numbers.

Application Activity 2.41

1. $\mathbb{R}\setminus\{-5,-2\}$
2. $\mathbb{R}$
3. $]-\infty,-1]\cup]3,+\infty[$
4. $]-\infty,-3[\cup]-2,+\infty[$

Lesson 2.4. Limit of exponential functions with any base

Learning objectives

Given an exponential functions with any base, learners should be able to evaluate limit and deduce relative asymptotes accurately.

Prerequisites

- Finding limits using table of values.
- Deduction of relative asymptotes.

Teaching Aids

Exercise book, pen and calculator

Activity 2.15

In this lesson, the following generic competence and cross-cutting issues are to be addressed:

- Critical thinking
- Communication
- Self confidence
- Cooperation, interpersonal management and life skills
- Peace and values education
- Inclusive education

Answers

1. a) Table of values

x	2^x
-1	0.5
-2	0.25
-5	0.03125
-15	0.0000305176
-30	0.0000000009

x	2^x
1	2
2	4
5	32
15	32768
30	1073741824

b) From table in a), when x takes values approaching to $-\infty$, 2^x takes values closed to zero. Hence, $\lim_{x\to-\infty} 2^x = 0$.

There exists a horizontal asymptote $y=0$, no oblique asymptote.

Also, when x takes values approaching to $+\infty$, 2^x increases without bound. Hence, $\lim_{x\to+\infty} 2^x = +\infty$. There is no horizontal asymptote.

2. a) Table of values

x	$\left(\frac{1}{2}\right)^x$
-1	2
-2	4
-5	32
-15	32768
-30	1073741824

x	$\left(\frac{1}{2}\right)^x$
1	0.5
2	0.25
5	0.03125
15	0.0000305176
30	0.0000000009

b) From table in a), when x takes values approaching to $-\infty$, $\left(\frac{1}{2}\right)^x$ increases without bound. Hence,

$\lim_{x\to-\infty}\left(\frac{1}{2}\right)^x = +\infty$. There is no horizontal asymptote.

Also, when x takes values approaching to $+\infty$, $\left(\frac{1}{2}\right)^x$ takes values closed to zero. Hence, $\lim_{x\to+\infty}\left(\frac{1}{2}\right)^x = 0$.

There exists a horizontal asymptote $y=0$, no oblique asymptote.

Synthesis

If $a>1$, $\lim\limits_{x\to-\infty} a^x = 0$ and $\lim\limits_{x\to+\infty} a^x = +\infty$

If $0<a<1$, $\lim\limits_{x\to-\infty} a^x = +\infty$ and $\lim\limits_{x\to+\infty} a^x = 0$

There is horizontal asymptote $y=0$.

No vertical asymptote since the domain is the set of real numbers. In addition there is no oblique asymptote.

Application Activity 2.42

1. 1	2. e^4	3. e^2
4. e	5. e^k	6. e^k

Lesson 2.5. Derivative of exponential functions with any base

Learning objectives

Given an exponential functions with any base, learners should be able to differentiate it accurately.

Prerequisites

- Derivative of logarithmic function with any base.
- Rule of differentiating inverses functions.

Teaching Aids

Exercise book and pen

Activity 2.16

In this lesson, the following generic competence and cross-cutting issues are to be addressed:

- Critical thinking
- Communication
- Self confidence
- Cooperation, interpersonal management and life skills
- Peace and values education

- Inclusive education

Answers

1. $f'(x) = \dfrac{1}{\left(f^{-1}\right)'(y)}$

   $f(x) = 3^x \Rightarrow f^{-1}(x) = \log_3 x$ but $(\log_3 x)' = \dfrac{1}{x\ln 3}$

   $f'(x) = \dfrac{1}{\dfrac{1}{3^x \ln 3}} = 3^x \ln 3$

   Thus, $\left(3^x\right)' = 3^x \ln 3$

2. $(f \circ g)' = f'(g)g'$. Then, $\left(3^{\cos x}\right)' = (\cos x)'\left(3^{\cos x}\right)(\ln 3)$

   Or $\left(3^{\cos x}\right)' = -\sin x\left(3^{\cos x}\right)(\ln 3)$

Synthesis

As conclusion, $\left(a^x\right)' = a^x \ln a$. Also, if u is another differentiable function of x, we have $\left(a^u\right)' = u' a^u \ln a$

Application Activity 2.43

1. a) $-2(0.3)^x \ln(0.3)$ b) $10^x\left(\dfrac{1}{x} + \ln x \ln 10\right)$

   c) $\sin x(\sin x + 2x\cos x)$ d) $x(4)^{\ln x}(2 + \ln 4)$

2. a) $\ln 2$ b) $e\left(e^e + 1\right)$ c) $\dfrac{1}{2}$ d) $\dfrac{1}{2}$

Lesson 2.6. Variation and curve of exponential functions with any base

Learning objectives

Given an exponential function with base any base, learners should be able to study the variation and sketch its curve accurately.

Prerequisites

- Reflecting a curve about the first bisector.
- Properties of inverse functions.

Teaching Aids

Exercise book, pencil, calculator and instruments of geometry

Activity 2.17

In this lesson, the following generic competence and cross-cutting issues are to be addressed:

- Critical thinking
- Communication
- Self confidence
- Cooperation, interpersonal management and life skills
- Peace and values education
- Inclusive education

Answers

1. When reflecting the curve of $f(x)=\log_2 x$ about the first bisector, we obtain

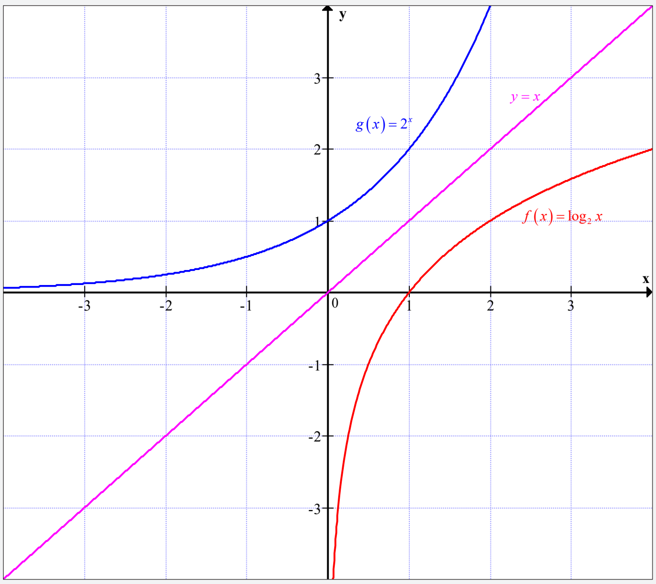


Figure 2.8: Reflection of $y=\log_2 x$ about first bisector

2. When reflecting the curve of $f(x)=\log_{\frac{1}{2}} x$ about the first bisector, we obtain

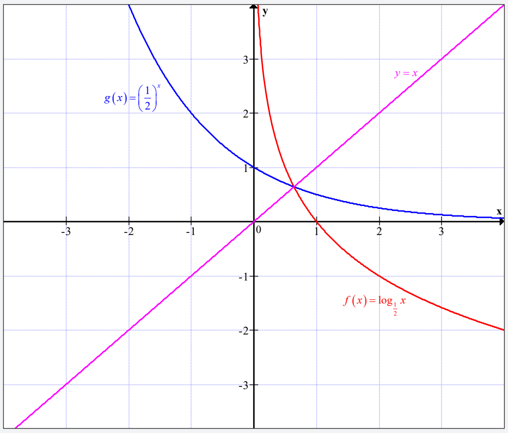


Figure 2.9: Reflection of $\log_{\frac{1}{2}} x$ *about first bisector*

Synthesis

As a^x is the inverse of $\log_a x$, we can obtain a curve of a^x by symmetry with respect to the first bisector $y=x$.

If $a=2$, we have $f(x)=2^x$.

x	-4	-3.6	-3.2	-2.8	-2.4	-2	-1.6	-1.2	-0.8	-0.4
y	0.06	0.08	0.11	0.14	0.19	0.25	0.33	0.44	0.57	0.76

x	0	0.4	0.8	1.2	1.6	2	2.4	2.8	3.2	3.6	4
y	1.00	1.32	1.74	2.30	3.03	4.00	5.28	6.96	9.19	12.13	16.00

Curve:

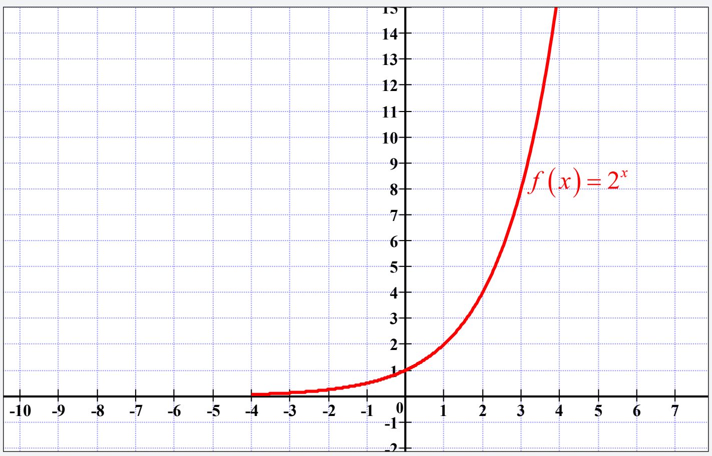


Figure 2.10: Curve of 2^x

If $a=\frac{1}{2}$, we have $f(x)=\left(\frac{1}{2}\right)^x$

x	-4	-3.6	-3.2	-2.8	-2.4	-2	-1.6	-1.2	-0.8	-0.4	0
y	16	12.13	9.19	6.96	5.28	4.00	3.03	2.30	1.74	1.32	1.00

x	0.4	0.8	1.2	1.6	2	2.4	2.8	3.2	3.6	4
y	0.76	0.57	0.44	0.33	0.25	0.19	0.14	0.11	0.08	0.06

Curve:

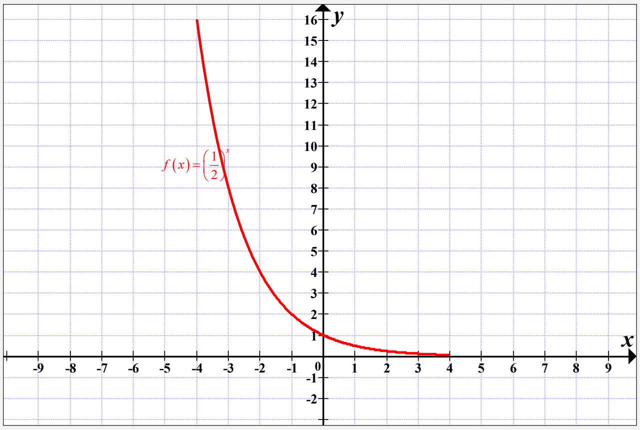


Figure 2.11: Curve of $\left(\frac{1}{2}\right)^x$

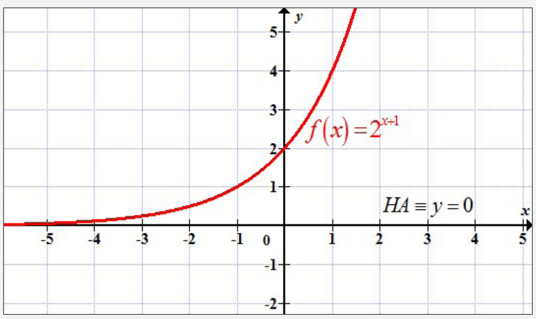
Application Activity 2.44

1. Domain: $]-\infty,+\infty[$

   Horizontal asymptote: $y=0$

   $f(x)$ increases on its domain

   Curve

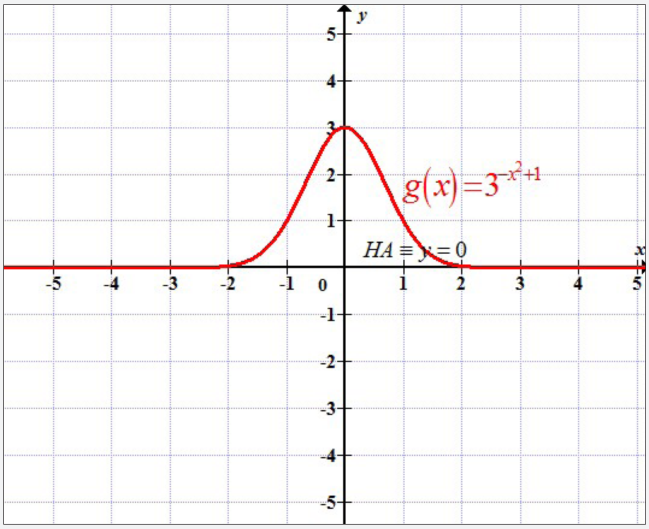


2. Domain: $]-\infty,+\infty[$

   Horizontal asymptote: $y=0$

   $g(x)$ increases on interval $]0,+\infty[$ and decreases on interval: $]0,+\infty[$

   Curve

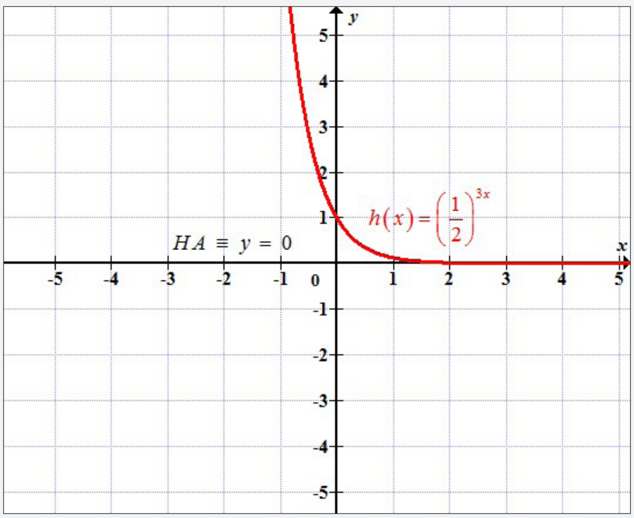


3. Domain: $]-\infty,+\infty[$

   Horizontal asymptote: $y=0$

   $h(x)$ decreases on its domain

   Curve

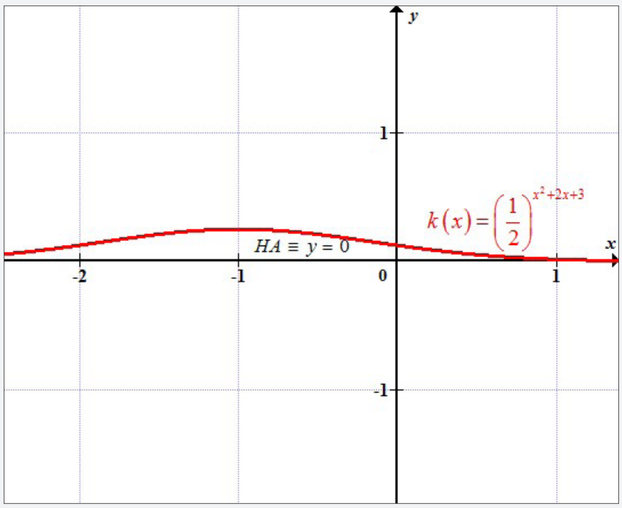


4. Domain: $]-\infty,+\infty[$

   Horizontal asymptote: $y=0$

   $k(x)$ increases on interval $]-\infty,-1[$ and increases on interval $]-1,+\infty[$

   Curve



Lesson 2.7. Compound interest problems

Learning objectives

By reading text books or accessing internet, learners should be able to use logarithmic and exponential functions to solve compound interest problems accurately.

Prerequisites

- Use of logarithmic and exponential functions.

Teaching Aids

Exercise book, pen and textbooks or internet if available.

Activity 2.18

In this lesson, the following generic competence and cross-cutting issues are to be addressed:

- Critical thinking
- Communication

- Self confidence
- Cooperation, interpersonal management and life skills
- Research
- Peace and values education
- Inclusive education
- Financial education

Answers

If P is the principal, n is the number of years, r is the interest rate per period, k is the number of periods per year, and A the total amount at the end of periods, then $A = P\left(1+\frac{r}{k}\right)^{kn}$

Here, $P = 4000,\ r = 0.06,\ k = 4,\ n = 5$

Then,

$$A = 4000\left(1+\frac{0.06}{4}\right)^{4\times 5} = 4000(1.015)^{20} = 5387.42$$

After 5 years there, will be $5{,}387.42\ FRW$ on the account.

Synthesis

If P is the principal, n is the number of years, k is the interest rate per period, k is the number of periods per year, and A the total amount at the end of periods, then $A = P\left(1+\frac{r}{k}\right)^{kn}$

Application Activity 2.45

1) 11,358.24 FRW
2) 7,007.08 FRW
3) Approximately 7.9 years
4) Approximately 6.3 years
5) Approximately 23.1 years

Lesson 2.8. Mortgage amount problems

Learning objectives

By reading text books or accessing internet, learners should be able to use logarithmic and exponential functions to solve mortgage amount problems accurately.

Prerequisites

- Use of logarithmic and exponential functions.

Teaching Aids

Exercise book, pen and textbooks or internet if available.

Activity 2.19

In this lesson, the following generic competence and cross-cutting issues are to be addressed:

- Critical thinking
- Communication
- Self confidence
- Cooperation, interpersonal management and life skills
- Research
- Peace and values education
- Inclusive education
- Financial education

Answers

The following formula illustrates the relationship:

$$P = \frac{\frac{rM}{n}}{1-\left(1+\frac{r}{n}\right)^{-nt}}$$

Where

$P = \text{the payment}$, r the annual rate, $M = \text{the mortgage amount}$, $t = \text{the number of years}$ and $n = \text{the number of payments per year}$

Here, $P = 800$, $M = 100000$, $n = 12$, $r = 0.09$ and we need t.

Now,

$$800=\frac{\frac{0.09\times 100000}{12}}{1-\left(1+\frac{0.09}{12}\right)^{-12t}}$$

$$\Leftrightarrow 800=\frac{750}{1-(1.0075)^{-12t}} \qquad \Leftrightarrow (1.0075)^{-12t}=-\frac{750}{800}+1$$

$$\Leftrightarrow 1-(1.0075)^{-12t}=\frac{750}{800} \qquad \Leftrightarrow (1.0075)^{-12t}=\frac{-750+800}{800}$$

$$\Leftrightarrow -(1.0075)^{-12t}=\frac{750}{800}-1 \qquad \Leftrightarrow (1.0075)^{-12t}=\frac{1}{16}$$

Take natural logarithm both sides

$$\Leftrightarrow \ln(1.0075)^{-12t}=\ln\frac{1}{16} \quad \Leftrightarrow -12t\ln(1.0075)=\ln\frac{1}{16}$$

$$\Leftrightarrow -12t=\frac{\ln(0.0625)}{\ln(1.0075)} \Leftrightarrow -12t=-371.06 \Rightarrow t=30.92$$

Then, you have to make payments to pay off the mortgage in approximately 30 years and 11 months. You would have 370 payments of 800 FRW and the last payment would be 850.40 FRW. The interest paid over the term of the mortgage would be 216,850.40 FRW.

Synthesis

There is a relationship between the mortgage amount M, the number of payments per year n, the amount of the payment P, how often the payment is made t, and the interest rate r. The following formula illustrates the relationship: $P=\dfrac{\frac{rM}{n}}{1-\left(1+\frac{r}{n}\right)^{-nt}}$

Application Activity 2.46

1. 2,400,000 FRW
2. 2,400,000 FRW
3. 12,719.89 FRW
4. 8.42%

Lesson 2.9. Population growth problems

Learning objectives

By reading text books or accessing internet, learners should be able to use logarithmic and exponential functions to solve population growth problems accurately.

Prerequisites

- Use of logarithmic and exponential functions.

Teaching Aids

Exercise book, pen and textbooks or internet if available.

Activity 2.20

In this lesson, the following generic competence and cross-cutting issues are to be addressed:

- Critical thinking
- Communication
- Self confidence
- Cooperation, interpersonal management and life skills
- Research
- Peace and values education
- Inclusive education

Answers

If P_0 is the population at the beginning of a certain period and $r\%$ is the constant rate of growth per period, the population n periods will be $P_n = P_0(1+r)^n$

Here, $P_0 = 1{,}000,\ r = 0.5,\ n = 5$

$$P_5 = P_0(1+r)^5 = 1{,}000(1+0.5)^5 = 1{,}000(1.5)^5 = 7{,}593.75$$

Thus, the population of bacteria in flask at the start of day 5 is 7,593.75.

Synthesis

If P_0 is the population at the beginning of a certain period and $r\%$ is the constant rate of growth per period, the population for n periods will be $P_n = P_0(1+r)^n$.

Application Activity 2.47

1. a) 4200   b) 4%   c) 5109
2. a) 1,726,458.24   b) 2020

Lesson 2.10. Depreciation value problems

Learning objectives

By reading text books or accessing internet, learners should be able to use logarithmic and exponential functions to solve depreciation value problems moderately.

Prerequisites

- Use of logarithmic and exponential functions

Teaching Aids

Exercise book, pen and textbooks or internet if available.

Activity 2.21

In this lesson, the following generic competence and cross-cutting issues are to be addressed:

- Critical thinking
- Communication
- Self confidence
- Cooperation, interpersonal management and life skills
- Research
- Peace and values education
- Inclusive education

Answers

If V_0 is the value at a certain time, and $r\%$ is the rate of depreciation per period, the value V_t at the end of t periods is $V_t = V_0(1-r)^t$.

Here, $V_0 = 2.3\times10^{30}$, $t = 5$, $r = 0.5$ since the number of bacteria halves every second.

Then,

$$V_5 = 2.3\times10^{30}\left(1-0.5\right)^5$$
$$= 2.3\times10^{30}\left(0.5\right)^5 = 2.3\times10^{30}\times0.03125$$
$$= 0.071875\times10^{30} = 7.2\times10^{28}$$

Thus, 7.2×10^{28} bacteria were left after 5 seconds.

Synthesis

Depreciation (or decay) is negative growth. If V_0 is the value at a certain time, and $r\%$ is the rate of depreciation per period, the value V_t at the end of t periods is $V_t = V_0\left(1-r\right)^t$.

Application Activity 2.48

1. $V = x\left(0.75\right)^t$
2. 19 years

Lesson 2.11. Earthquake problems

Learning objectives

By reading text books or accessing internet, learners should be able to use logarithmic and exponential functions to solve earthquake problems.

Prerequisites

- Use of logarithmic and exponential functions.

Teaching Aids

Exercise book, pen and textbooks or internet if available.

Activity 2.22

In this lesson, the following generic competence and cross-cutting issues are to be addressed:

- Critical thinking
- Communication
- Self confidence
- Cooperation, interpersonal management and life skills
- Research

- Peace and values education
- Inclusive education

Answers

The formula $M=\log\frac{I}{S}$ determines the magnitude of an earthquake, where I is the intensity of the earthquake and S is the intensity of a "standard earthquake."

Here,

$$8=\log\frac{I_1}{S},\ 6=\log\frac{I_2}{S}$$

$$10^8=\frac{I_1}{S},\ 10^6=\frac{I_2}{S}$$

$$\frac{10^8}{10^6}=\frac{\frac{I_1}{S}}{\frac{I_2}{S}}\Rightarrow 100=\frac{I_1}{I_2}$$

So, the earthquake will be a hundred times stronger.

Synthesis

The magnitude of an earthquake is given by $M=\log\frac{I}{S}$ where I is the intensity of the earthquake and S is the intensity of a ''standard earthquake''

Application Activity 2.49

1. 5
2. 2.6
3. a) 39.8 times more intense b) 7.2
4. 1.26 times more intense

Lesson 2.12. Carbon-14 dating problems

Learning objectives

By reading text books or accessing internet, learners should be able to use logarithmic and exponential functions to solve carbon-14 dating problems accurately.

Prerequisites

- Use of logarithmic and exponential functions

Teaching Aids

Exercise book, pen and textbooks or internet if available.

Activity 2.23

In this lesson, the following generic competence and cross-cutting issues are to be addressed:

- Critical thinking
- Communication
- Self confidence
- Cooperation, interpersonal management and life skills
- Research
- Peace and values education
- Inclusive education

Answers

A formula used to calculate how old a sample is by carbon-14 dating is: $t = \dfrac{\ln\left(\dfrac{N_f}{N_0}\right)}{-0.693} \times t_{\frac{1}{2}}$

where $\dfrac{N_f}{N_0}$ is the percent of carbon-14 in the sample compared to the amount in living tissue, and $t_{\frac{1}{2}}$ is the half-life of carbon-14 ($5,730 \pm 30$ years).

Then, $t = \dfrac{\ln(0.10)}{-0.693} \times 5700 = 18,940\ \textit{years old}$

Synthesis

Carbon dating is used to work out the age of organic material – in effect, any living thing. The technique hinges on carbon-14, a radioactive isotope of the element that, unlike other more stable forms of carbon, decays away at a steady rate. The half-life of a substance is the amount of time it takes for half of that substance to decay.

A formula to calculate how old a sample is by carbon-14 dating is:

$$t = \frac{\ln\left(\frac{N_f}{N_0}\right)}{-0.693} \times t_{\frac{1}{2}}$$

where $\frac{N_f}{N_0}$ is the percent of carbon-14 in the sample compared to the amount in living tissue, and $t_{\frac{1}{2}}$ is the half-life of carbon-14 ($5,730 \pm 30$ years).

Application Activity 2.50

1. 8,260 years
2. 9,953 years
3. 0.239 mg
4. 3.2 per minutes per gram
5. 3,870 years
6. a) A common rule of thumb is that a radioactive dating method is good out to about 10 half-lives. Given a Carbon-14 half-life of 5730 years, you can see that Carbon-14 dating is (theoretically) good out to around 60,000 years (more-or-less). In fact, due to fluctuations in the carbon amount in the atmosphere, modern Carbon-14 dating needs to be correlated to dates determined by analysis of tree-ring records (dendrochronology).

   b) A skull does not have very much (if any) carbon in it after 73 million years. It would not be dated using Carbon-14 dating. In fact, the value of 73 million years is not arrived at by directly testing the skull. Minerals containing radioactive elements are dated and the age of the skull would be assumed to be of the same age as the strata in which it was discovered.

2.6. Summary of the unit

1. Logarithmic functions

- Domain of definition and range:

  The Natural logarithm of x is denoted as $\ln x$ or $\log_e x$ and defined on positive real numbers, $]0,+\infty[$, its range is all real numbers.

  $\forall x \in]1,+\infty[$, $\ln x > 0$ and $\forall x \in]0,1[$, $\ln x < 0$

  The equation $\ln x = 1$ has, in interval $]0,+\infty[$, a unique solution, a rational number

  $2.718281828459045235360....$. This number is denoted by e.

  Hence $\ln x = 1 \Leftrightarrow x = e$.

  Generally $e = \lim_{x\to+\infty}\left(1+\frac{1}{x}\right)^x$

- Limits on boundaries:

  Logarithmic function $f(x) = \ln x$ being defined on $]0,+\infty[$,

  $\lim_{x\to+\infty} \ln x = +\infty$ and $\lim_{x\to0^+} \ln x = -\infty$.

  From $\lim_{x\to+\infty} \ln x = +\infty$, we deduce that there is no horizontal asymptote.

  From $\lim_{x\to0^+} \ln x = -\infty$, we deduce that there exists a vertical asymptote with equation $VA \equiv x = 0$

- Derivative of natural logarithmic functions or logarithmic derivative:

  $x \in \mathbb{R}_0^+, (\ln x)' = \frac{1}{x}$ and $(\ln x)' > 0$

  Also, if u is differentiable function at x then,

  $(\ln u)' = \frac{u'}{u}$

  With certain functions containing more complicated products and quotients, differentiation is often made easier if the logarithm of the function is taken before differentiating.

- Domain and limits on boundaries of a logarithmic function with any base:

  Logarithm function of a real number x with base a is a function f denoted $f(x)=\log_a x$ and defined by

  $$\log_a x=\frac{\ln x}{\ln a},\ x\in\mathbb{R}_0^+,\ a\in\mathbb{R}_0^+\setminus\{1\}$$

  $$\forall x\in\mathbb{R}_0^+,\ \log_a x=y\Leftrightarrow x=a^y$$

  $$\lim_{x\to 0^+} f(x)=\begin{cases}-\infty & if\ \ a>1\\ +\infty & if\ \ 0<a<1\end{cases}$$

  There is a vertical asymptote $VA\equiv x=0$

  $$\lim_{x\to +\infty} f(x)=\begin{cases}+\infty & if\ \ a>1\\ -\infty & if\ \ 0<a<1\end{cases}$$

  There is no horizontal asymptote nor oblique asymptote.

- Logarithmic Differentiation:

  If $f(x)=\log_a x$, then $f'(x)=\dfrac{1}{x\ln a}$

  Also, if u is another differentiable function in x, then

  $$(\log_a u)'=\frac{u'}{u\ln a}$$

2. Exponential functions

Exponential function with base "e"

- **Domain and range of exponential functions with base "e"**

  The domain of definition of $y=e^x$ is $]-\infty+\infty[$ and its range is $]0,+\infty[$.

  Then, $\forall x\in\,]0,+\infty[\,, y\in\,]-\infty,+\infty[\,:\ y=\ln x\Leftrightarrow x=e^y$.

- **Limit of exponential functions with base "e"**

  $\lim\limits_{x\to-\infty} e^x=0$ and $\lim\limits_{x\to+\infty} e^x=+\infty$

  There exists horizontal asymptote: $H.A\equiv y=0$

- **Derivative of exponential functions with base "e"**

  $\forall x\in\mathbb{R},\ (e^x)'=e^x$

If u is another differentiable function at x,

$(e^u)' = u'e^u$

Remarks

1. $\forall y > 0,\ y = e^{\ln y}$

   In particular, $a^x = e^{\ln a^x}$ means $a^x = e^{x\ln a}$.

   Hence, to study the function $y = u^v$ is the same as to study the function $y = e^{v\ln u}$ where u and v are two other functions.

2. Whenever an expression to be differentiated contains a term raised to a power which is itself a function of the variable, then logarithmic differentiation must be used. For example, the differentiation of expressions such as $|x|^x, (1-x)^{1-x^2}$, $\sqrt[x]{x+2}, (x)^{\sin x}$ and so on can only be achieved using logarithmic differentiation.

3. Applications

a) Compound interest problems

If P is the principal, n is the number of years, r is the interest rate per period, k in the number of periods per year, and A the total amount at the end of periods, then $A = P\left(1+\frac{r}{k}\right)^{kn}$.

b) Population growth problems

If P_0 is the population at the beginning of a certain period and $r\%$ is the constant rate of growth per period, the population after n periods will be $P_n = P_0(1+r)^n$.

c) Depreciation value problems

Depreciation (or decay) is negative growth. If V_0 is the value at a certain time, and $r\%$ is the rate of depreciation per period, the value V_t at the end of t periods is $V_t = V_0(1-r)^t$.

d) Earthquake problems

Charles Richter defined the magnitude of an earthquake to be $M = \log \frac{I}{S}$ where I is the intensity of the earthquake (measured by the amplitude of a seismograph reading taken $100\,km$ from the epicentre of the earthquake) and S is the intensity of a "standard earthquake" (whose amplitude is $1\,micron = 10^{-4}\,cm$).

e) Carbon-14 dating problems

Carbon dating is used to work out the age of organic material – in effect, any living thing. By measuring the ratio of the radio isotope to non-radioactive carbon, the amount of carbon-14 decay can be worked out, thereby giving an age for the specimen in question.

Through research, scientists have agreed that the half-life of C^{14} is approximately 5700 years.

A formula to calculate how old a sample is by carbon-14 dating is:

$$t = \frac{\ln\left(\frac{N_f}{N_0}\right)}{-0.693} \times t_{\frac{1}{2}}$$

where $\frac{N_f}{N_0}$ is the percent of carbon-14 in the sample compared to the amount in living tissue, and $t_{\frac{1}{2}}$ is the half-life of carbon-14 ($5,730 \pm 30$ years).

2.7. End of Unit Assessment

1. $]1,+\infty[$
2. $]0,1[\cup]1,+\infty[$
3. $]-4,1[\cup]2,+\infty[$
4. $\mathbb{R}\setminus\{2-\sqrt{3},\ 2+\sqrt{3}\}$
5. $]0,1[\cup]1,+\infty[$
6. $\mathbb{R}\setminus\{-2,-1,0\}$
7. $\mathbb{R}$
8. $\mathbb{R}\setminus\{1\}$
9. 0
10. $+\infty$

11. 0

12. $+\infty$

13. 1

14. 0

15. $+\infty$

16. 0

17. $+\infty$

18. $+\infty$

19. $\frac{2}{3}$

20. e

21. $\frac{1}{\sqrt[3]{e^2}}$

22. 0

23. x^2e^x

24. $\frac{1}{2}\left(\tan\frac{x}{2}+\cot\frac{x}{2}\right)$

25. $\frac{1}{x\ln x}$

26. $\frac{\sqrt{x^2+a^2}}{x^2+a^2}$

27. $\frac{2e^x}{e^{2x}+2e^x+1}$

28. $\frac{-2\sqrt{x^2+1}}{x^2+1}$

29. $\frac{x^x\ln x}{e^x}$

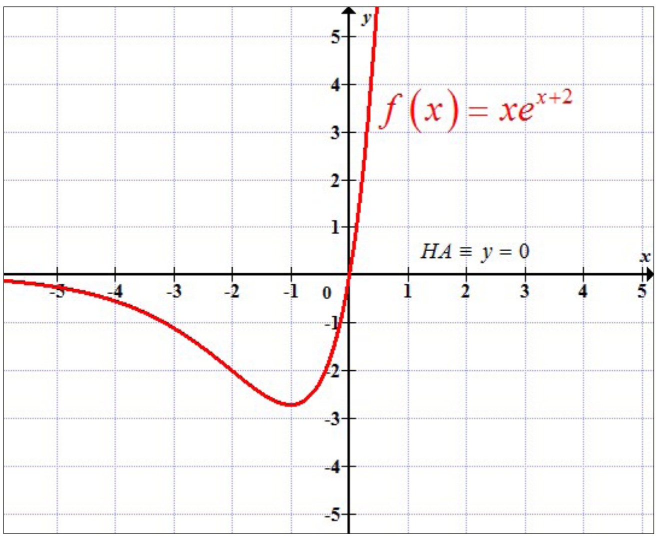
30. $(\cos x)^x\left[\ln(\cos x)-x\tan x\right]$

31.Domain: $]-\infty,+\infty[$

Horizontal asymptote: $y=0$

$f(x)$ increases on intervals $]-\infty,-2[$ and $]0,+\infty[$, it decreases on interval $]-2,0[$

Curve

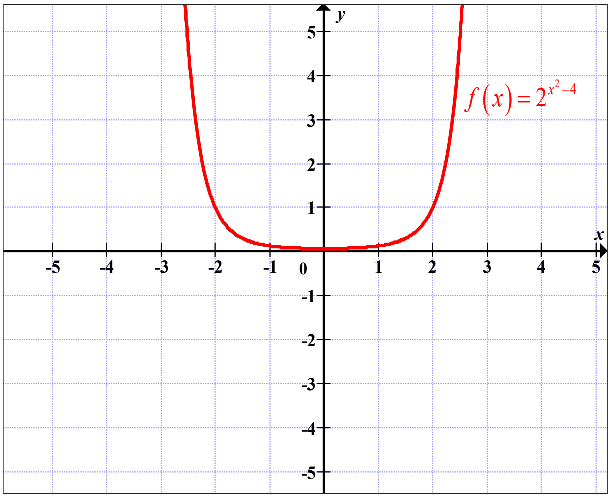


32. Domain: $]-\infty,+\infty[$

No asymptote

$f(x)$ decreases on interval $]-\infty,0[$, it increases on interval $]0,+\infty[$.

Curve

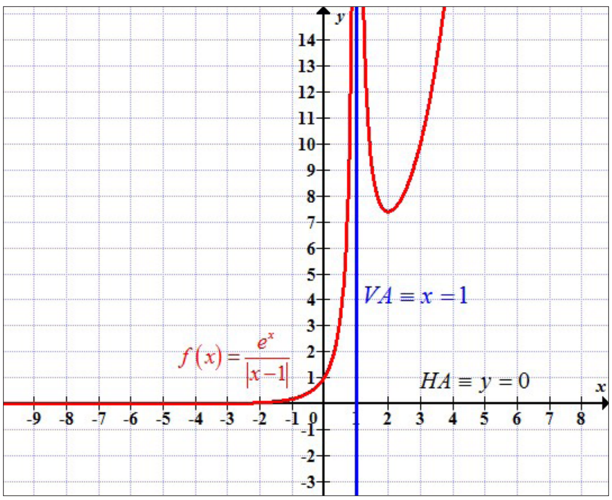


33. Domain: $]-\infty,+\infty[$

Vertical asymptote $x=1$ and horizontal asymptote $y=0$

$f(x)$ increases on its domain

Curve

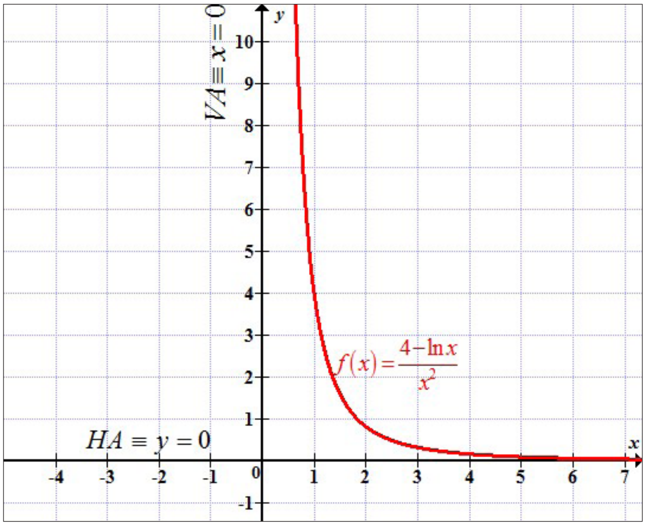


34. Domain: $]0,+\infty[$

Vertical asymptote $x=0$ and horizontal asymptote $y=0$

$f(x)$ increases on interval $\left]0,\sqrt{e}\right[$ and decreases on interval $\left]\sqrt{e},+\infty\right[$.

Curve



35. a) $f(t) = 1,000,000(0.9)^t$ b) $t = 109.27\,\text{min}$

36. a) $f(t) = 75,000 \times e^{0.98083t}$ b) $2,942,490$

37. $3,315.53

38. Monthly payment is $550.32. Interest is $123,115.20

39. $72,537.23

40. a) $f(t) = 3^t$ b) 4.239×10^{28}

41. a) $f(t) = 100,000\left(\frac{1}{2}\right)^t$ b) 97.65625

42. a) 7.3 b) 125,892,451 as greater as A_0

43. 8.43

44. a) $3.16 \times 10^{-3}\, mol/l$ b) 12,589 times more acidic

45. 1,000,000 times more intense

46. 70dB

Unit 3 Taylor and Maclaurin's Expansions

3.1. Key unit competence

Use Taylor and Maclaurin's expansion to solve problems about approximations, limits, …

Extend the Maclaurin's expansion to Taylor series.

3.2. Vocabulary or key words concepts

Power series: Infinite series of the form $\sum_{n=0}^{\infty} a_n (x-c)^n$.

Taylor series of function $f(x)$ at point x_0: The infinite series of the form $\sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n$.

Maclaurin series: The special case of the Taylor series when $x_0 = 0$.

Lagrange remainder: The remainder function in Taylor series.

3.3. Guidance on the introductory activity

The problem statement is

"Suppose that we need to complete the table below.

Angle, x	0^0	1^0	2^0	3^0	4^0	5^0
$\sin x$						

For $x = 0^0$ is very easy since this angle is a remarkable angle. But, what about other angles, $1^0, 2^0, 3^0, 4^0, 5^0$? How can we find their sine without using sine button on scientific calculator?

To solve this problem, we need the Maclaurin series of $\sin x$ and then x will be replaced by its value, remembering that all angles must be expressed in radian.

3.4. List of lessons

No	Lesson title	Number of periods
1	Finite series	1
2	Infinite series	1
3	Test for convergence of series	1
4	Power series	1
5	Taylor and Maclaurin series	2
6	Taylor series by using Maclaurin series	1
7	Calculation of limits	1
8	Estimation of the number e	1
9	Estimation of the number π	1
10	Estimation of trigonometric number of an angle	1
11	Estimation of an irrational number	1
12	Estimation of a natural logarithm number	1
13	Estimation of roots of equations	1
Total periods		14

3.5. Lesson development

Lesson 3.1. Finite series

Learning objectives

Given a finite series, learners should be able to sum that series accurately.

Prerequisites

- Terms of a series.
- General term of a series.
- Sigma notation.

Teaching Aids

Exercise book and pen

Activity 3.1

In this lesson, the following generic competence and cross-cutting issues are to be addressed:

- Critical thinking
- Communication
- Self confidence
- Cooperation, interpersonal management and life skills
- Peace and values education
- Inclusive education

Answers

1. $u_k = f(k) - f(k+1)$

   For $k=1$, $u_1 = f(1) - f(2)$

   For $k=2$, $u_2 = f(2) - f(3)$

   For $k=3$, $u_3 = f(3) - f(4)$

   For $k=4$, $u_4 = f(4) - f(5)$

   For $k=5$, $u_5 = f(5) - f(6)$

   $\vdots$

   For $k=n-1$, $u_{n-1} = f(n-1) - f(n)$

   For $k=n$, $u_n = f(n) - f(n+1)$

2. Adding obtained terms we have

$$\begin{aligned} &u_1 + u_2 + u_3 + u_4 + u_5 + \ldots + u_{n-1} + u_n = \\ &= f(1) - f(2) + f(2) - f(3) + f(3) - f(4) + f(4) \\ &\quad - f(5) + f(5) - f(6) + \ldots. + f(n-1) - f(n) + f(n) - f(n+1) \\ &= f(1) - f(n+1) \end{aligned}$$

Thus, adding these terms, on the right hand side, nearly all the terms cancel out leaving just $f(1) - f(n+1)$ and on the left hand side, is the required sum of the series. Thus, $\sum_{k=1}^{n} u_n = f(1) - f(n+1)$.

Synthesis

As conclusion, the sum of the series $u_1, u_2, u_3, ..., u_n$ is given by $\sum_{k=1}^{n} u_n = f(1) - f(n+1)$ where $f(k)$ is a function of k.

Application Activity 3.51

1. $1-\frac{1}{n+1}$
2. $\frac{1}{2}-\frac{1}{4n+2}$
3. $\frac{1}{4}n^4+\frac{3}{2}n^3+\frac{11}{4}n^2+\frac{3}{2}n$
4. $\frac{1}{2}\left(-\frac{1}{n+1}-\frac{1}{n+2}+\frac{3}{2}\right)$

Lesson 3.2. Infinite series

Learning objectives

Given an infinite series or a repeating decimal, learners should be able to find the sum of infinite series or find a rational number represented by the repeating decimal accurately.

Prerequisites

- Evaluating limits

Teaching Aids

Exercise book and pen

Activity 3.2

In this lesson, the following generic competence and cross-cutting issues are to be addressed:

- Critical thinking
- Communication
- Self confidence
- Cooperation, interpersonal management and life skills
- Peace and values education
- Inclusive education

Answers

1. $S_n = \frac{7}{10} + \frac{7}{10^2} + \frac{7}{10^3} + \frac{7}{10^4} + \ldots + \frac{7}{10^n} + \ldots$

$$\frac{1}{10} S_n = \frac{1}{10}\left(\frac{7}{10} + \frac{7}{10^2} + \frac{7}{10^3} + \frac{7}{10^4} + \ldots + \frac{7}{10^n} + \ldots\right)$$

$$\Rightarrow \frac{1}{10} S_n = \frac{7}{10^2} + \frac{7}{10^3} + \frac{7}{10^4} + \frac{7}{10^5} + \ldots + \frac{7}{10^{n+1}} + \ldots$$

2. Subtracting, we have

$$S_n = \frac{7}{10} + \frac{7}{10^2} + \frac{7}{10^3} + \frac{7}{10^4} + \ldots + \frac{7}{10^n} + \ldots$$

$$-\frac{1}{10} S_n = -\frac{7}{10^2} - \frac{7}{10^3} - \frac{7}{10^4} - \frac{7}{10^5} - \ldots - \frac{7}{10^{n+1}} - \ldots$$

$$S_n - \frac{1}{10} S_n = \frac{7}{10} - \frac{7}{10^{n+1}}$$

$$\Rightarrow \frac{9}{10} S_n = \frac{7}{10}\left(1 - \frac{7}{10^n}\right)$$

$$\Rightarrow S_n = \frac{7}{10} \times \frac{10}{9}\left(1 - \frac{7}{10^n}\right) \Rightarrow S_n = \frac{7}{9}\left(1 - \frac{7}{10^n}\right)$$

3. Taking limit as $n \to +\infty$.

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{7}{9}\left(1 - \frac{7}{10^n}\right) = \frac{7}{9}$$

Synthesis

It is impossible to add up infinitely many numbers, thus, we will deal with infinite sums by limiting process involving sequences.

An **infinite series** is an expression of the form $u_1 + u_2 + u_3 + \ldots + u_k + \ldots$ or in sigma notation $\sum_{k=1}^{+\infty} u_k$. The terms $u_1, u_2, u_3, \ldots$ are called **terms** of the series.

To carry out this summation process, we proceed as follows:

Let s_n denote the sum of the first *n* terms of the series. Thus,

$s_1 = u_1$

$s_2 = u_1 + u_2$

$s_3 = u_1 + u_2 + u_3$

$\vdots$

$$s_n = u_1 + u_2 + u_3 + \ldots + u_n = \sum_{k=1}^{n} u_k$$

The number s_n is called the n^{th} **partial sum** of the series and the sequence $\{s_n\}_{n=1}^{+\infty}$ is called the **sequence of partial sums**.

Application Activity 3.52

1. a) $\frac{1}{2}n^2 + \frac{1}{2}n$ b) $\frac{1}{3}n^3 + \frac{1}{2}n^2 + \frac{1}{6}n$

   c) $\frac{1}{4}n^4 + \frac{1}{2}n^3 + \frac{1}{4}n^2$ d) $\frac{1}{3}n^3 + n^2 + \frac{2}{3}n$

2. a) $\frac{3}{11}$ b) $\frac{5}{6}$ c) $\frac{49}{396}$

Lesson 3.3. Tests for convergence of series

Learning objectives

Given a series and by using comparison test, limit comparison test, the ratio test or the n^{th} root test, learners should be able to test for convergence accurately.

Prerequisites

- Evaluating limits.
- Compare two expressions.
- Compare real numbers.

Teaching Aids

Exercise book and pen

Activity 3.3

In this lesson, the following generic competence and cross-cutting issues are to be addressed:

- Critical thinking
- Communication
- Self confidence
- Cooperation, interpersonal management and life skills
- Peace and values education
- Inclusive education

Answers

1. a) $\lim_{n\to\infty}\frac{u_{n+1}}{u_n}=\lim_{n\to\infty}\frac{\frac{3^{n+1}+1}{5^{n+1}}}{\frac{3^n+1}{5^n}}=\lim_{n\to\infty}\frac{3^{n+1}+1}{5^n\times 5}\times\frac{5^n}{3^n+1}=\lim_{n\to\infty}\frac{3^n\times 3+1}{5\left(3^n+1\right)}$

$$=\lim_{n\to\infty}\frac{3^n\left(3+\frac{1}{3^n}\right)}{5\times 3^n\left(1+\frac{1}{3^n}\right)}=\lim_{n\to\infty}\frac{3+\frac{1}{3^n}}{5\times\left(1+\frac{1}{3^n}\right)}=\frac{3+0}{5(1+0)}=\frac{3}{5}$$

   b) $\lim_{n\to\infty}\sqrt[n]{u_n}=\lim_{n\to\infty}\frac{\sqrt[n]{n}}{3}=\lim_{n\to\infty}\frac{n^{\frac{1}{n}}}{3}=\frac{n^0}{3}=\frac{1}{3}$

2. Taking $\frac{1}{2n-1}$, if we add 1 to the denominator, we get $\frac{1}{2n}$ and then by comparison methods for rational numbers, $\frac{1}{2n-1}>\frac{1}{2n}$ since the numerators are the same and denominator $2n>2n-1$

Synthesis

Comparison test

Let $\sum_{n=1}^{\infty}a_n$ be a series with positive terms;

a) $\sum_{n=1}^{\infty}a_n$ converges if there exists a convergent series

$\sum_{n=1}^{\infty} b_n$ such that $a_n \le b_n$ for all $n > N$, where N is some positive integer.

b) $\sum_{n=1}^{\infty} a_n$ diverges if there exists a divergent series $\sum_{n=1}^{\infty} c_n$ such that $a_n \ge c_n$ for all $n > N$, where N is some positive integer.

Limit comparison test

If the series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are two series with positive terms, and $\lim_{n\to\infty} \frac{a_n}{b_n}$ is finite, both series converge or diverge.

The ratio test

Let $\sum_{n=1}^{\infty} u_n$ be a series with positive terms and let $\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = L$, then;

a) the series converges if $L < 1$.
b) the series diverges if $L > 1$.
c) the series may or may not converge if $L = 1$ (i.e. the test is inconclusive).

The n^{th} root test

Let $\sum_{n=1}^{\infty} u_n$ be a series with positive terms and let $\lim_{n\to\infty} \sqrt[n]{u_n} = L$, then;

a) the series converges if $L < 1$.
b) the series diverges if $L > 1$.
c) the test is inconclusive $L = 1$.

Application Activity 3.53

1. Converges	2. Converges	3. Converges
4. Diverges	5. Converges	6. Converges

Lesson 3.4. Power series

Learning objectives

Through examples, learners should be able to define a power series and to find radius of convergence accurately.

Prerequisites

- Test for convergence of series.

Teaching Aids

Exercise book and pen

Activity 3.4

In this lesson, the following generic competence and cross-cutting issues are to be addressed:

- Critical thinking
- Communication
- Self confidence
- Cooperation, interpersonal management and life skills
- Peace and values education
- Inclusive education

Answers

1. $u_n = (-1)^{n+1}\dfrac{x^n}{n}$

   n^{th} root test:

   $$\lim_{n\to\infty}\sqrt[n]{u_n} = \lim_{n\to\infty}\sqrt[n]{\left|(-1)^{n+1}\frac{x^n}{n}\right|} = \lim_{n\to\infty}\sqrt[n]{\left|\frac{x^n}{n}\right|} = \lim_{n\to\infty}\frac{\sqrt[n]{|x^n|}}{\sqrt[n]{|n|}} = \lim_{n\to\infty}\frac{|x|}{n^{\frac{1}{n}}} = |x|$$

   The series is convergence for $|x|<1$ (and divergence for $|x|>1$)

2. $u_n = \dfrac{x^n}{n!}$ and

   $$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{x^{n+1}}{(n+1)!}\times\frac{n!}{x^n}\right| = \lim_{n\to\infty}\frac{|x|}{n+1} = 0<1$$

   for all x. Therefore, the series is absolutely convergence.

Synthesis

Power series is like an infinite polynomial. It has the form

$$\sum_{n=0}^{\infty} a_n (x-c)^n = a_0 + a_1(x-c) + a_2(x-c)^2 + ... + a_n(x-c)^n + ...$$

- The power series converges at $x = c$. Here, the radius of convergence is zero.
- The power series converges for all x, i.e $]-\infty, +\infty[$. Here, the radius of convergence is infinity.
- There is a number R called the **radius of convergence** such that the series converges for all $c - R < x < c + R$ and the series diverges outside this interval.

Application Activity 3.54

1. $-3 < x < -1,\ R = 1$	2. $-\frac{2}{3} < x < 0,\ R = \frac{1}{3}$
3. $-2 < x < -1,\ R = \frac{1}{2}$	4. $-1 < x < 3,\ R = 2$
5. All $x, R \to \infty$	6. $-1 < x < 1,\ R = 1$
7. All $x, R \to \infty$	8. $-8 < x < -2,\ R = 3$
9. $x = 3,\ R = 0$	10. All $x, R \to \infty$

Lesson 3.5. Taylor and Maclaurin series

Learning objectives

Using power series, learners should be able to give general form of a Taylor and Maclaurin series without errors.

Prerequisites

- Power series

Teaching Aids

Exercise book and pen

Activity 3.5

In this lesson, the following generic competence and cross-cutting issues are to be addressed:

- Critical thinking
- Communication
- Self confidence
- Cooperation, interpersonal management and life skills
- Peace and values education
- Inclusive education

Answers

1. $f(x)=c_0+c_1(x-a)+c_2(x-a)^2+c_3(x-a)^3+c_4(x-a)^4+...+c_n(x-a)^n+...$

   $f(a)=c_0$

2. $f'(x)=c_1+2c_2(x-a)+3c_3(x-a)^2+4c_4(x-a)^3+...+nc_n(x-a)^{n-1}+...$

   $f'(a)=c_1$

3. $f''(x)=2\times c_2+3\times 2\times c_3(x-a)+4\times 3\times c_4(x-a)^2+...+n(n-1)c_n(x-a)^{n-2}+...$

   $f''(a)=2!c_2 \Rightarrow c_2=\dfrac{f''(a)}{2!}$

4. $f'''(x)=3\times 2\times c_3+4\times 3\times 2\times c_4(x-a)+...+n(n-1)(n-2)c_n(x-a)^{n-3}+...$

   $f'''(a)=3\times 2\times c_3=3!c_3 \Rightarrow c_3=\dfrac{f'''(a)}{3!}$

5. $f^{(iv)}(x)=4\times 3\times 2\times c_4+...+n(n-1)(n-2)(n-3)c_n(x-a)^{n-4}+...$

   $f^{(iv)}(a)=4\times 3\times 2\times c_4=4!c_4 \Rightarrow c_2=\dfrac{f^{(iv)}(a)}{4!}$

6. Now, we can see the pattern. If we continue to differentiate and substitute $x=a$, we obtain $f^{(n)}(a)=n(n-1)(n-2)(n-3)...1\times c_n$ or using factorial notation; $f^{(n)}(a)=n!c_n$

   Solving we get $c_n=\dfrac{f^{(n)}(a)}{n!}$

7. Now,

$$f(x)=f(a)+\frac{f'(a)}{1!}(x-a)+\frac{f''(a)}{2!}(x-a)^2+\frac{f'''(a)}{3!}(x-a)^3$$
$$+\frac{f^{(iv)}(a)}{4!}(x-a)^4+\dots+\frac{f^{(n)}(a)}{n!}(x-a)^n+\dots$$

Using sigma notation, we can write,

$$f(x)=\sum_{n=0}^{\infty}\frac{f^{(n)}(a)}{n!}(x-a)^n$$

Synthesis

As conclusion, the Taylor series for $f(x)$ is given by

$f(x)=\sum_{n=0}^{\infty}\frac{f^{(n)}(x_0)}{n!}(x-x_0)^n$ and the Maclaurin series is given by

$$f(x)=\sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}x^n$$
$$=f(0)+\frac{f'(0)}{1!}x+\frac{f''(0)}{2!}x^2+\frac{f'''(0)}{3!}x^3+\dots+\frac{f^{(n)}(0)}{n!}x^n+\dots$$

Application Activity 3.55

a) $6-11(x+2)+6(x+2)^2-(x+2)^3+\dots$

b) $\sum_{n=0}^{\infty}\frac{(-1)^n}{2^{k+1}}(x-2)^n$

c) $\sum_{n=0}^{\infty}\frac{(-1)^n}{k!e}(2x-1)^n$

d) $\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!\sqrt{2}}\left(x-\frac{\pi}{4}\right)^{2n}+\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!\sqrt{2}}\left(x-\frac{\pi}{4}\right)^{2n+1}$

Lesson 3.6. Taylor series by using Maclaurin series

Learning objectives

By using Maclaurin series $(x_0=0)$ without necessary using Taylor's formula, learners should be able to find the Taylor series for other functions accurately.

Prerequisites

- Maclaurin series

Teaching Aids

Exercise book and pen

Activity 3.6

In this lesson, the following generic competence and cross-cutting issues are to be addressed:

- Critical thinking
- Communication
- Self confidence
- Cooperation, interpersonal management and life skills
- Peace and values education
- Inclusive education

Answers

1. a) $\sin x$

$$\begin{array}{ll} f(x)=\sin x & f(0)=0 \\ f'(x)=\cos x & f'(0)=1 \\ f''(x)=-\sin x & f''(0)=0 \\ f'''(x)=-\cos x & f'''(0)=-1 \\ f^{(4)}(x)=\sin x & f^{(4)}(0)=0 \\ \vdots & \end{array}$$

Since the derivatives repeat in a cycle of four, we can write the Maclaurin series as follows:

$$\begin{aligned} \sin x &= 0+\frac{1}{1!}x+\frac{0}{2!}x^2+\frac{-1}{3!}x^3+\frac{0}{4!}x^4+\ldots \\ &= x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\ldots \\ &= \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!} \end{aligned}$$

b) $\cos x$

$$\begin{array}{ll} f(x)=\cos x & f(0)=1 \\ f'(x)=-\sin x & f'(0)=0 \\ f''(x)=-\cos x & f''(0)=-1 \\ f'''(x)=\sin x & f'''(0)=0 \\ f^{(4)}(x)=\cos x & f^{(4)}(0)=1 \\ \vdots & \end{array}$$

Since the derivatives repeat in a cycle of four, we can write the Maclaurin series as follows:

$$\cos x = 1 + \frac{0}{1!}x + \frac{-1}{2!}x^2 + \frac{0}{3!}x^3 + \frac{1}{4!}x^4 + ...$$
$$= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + ...$$
$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$$

Alternative method

Since $\cos x = \frac{d}{dx}(\sin x)$, we can differentiate the Maclaurin series for $\sin x$ obtained in a) to get one for $\cos x$. That, is,

$$\cos x = \frac{d}{dx}(\sin x) = \frac{d}{dx}\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + ... \right)$$
$$= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + ...$$
$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$$

c) $\ln(1+x)$

$f(x) = \ln(1+x)$ $\qquad f(0) = 0$

$f'(x) = \frac{1}{1+x}$ $\qquad f'(0) = 1$

$f''(x) = -\frac{1}{(1+x)^2}$ $\qquad f''(0) = -1$

$f'''(x) = \frac{2}{(1+x)^3}$ $\qquad f'''(0) = 2$

$f^{(4)}(x) = -\frac{6}{(1+x)^4}$ $\qquad f^{(4)}(0) = -6$

$\vdots$

Now,

$$\ln(1+x) = 0 + \frac{1}{1!}x + \frac{-1}{2!}x^2 + \frac{2}{3!}x^3 + \frac{-6}{4!}x^4 + ...$$
$$= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + ...$$
$$= \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}$$

2. From results in 1),

$$\sin 2x = 2x - \frac{(2x)^3}{3!} + \frac{(2x)^5}{5!} - \frac{(2x)^7}{7!} + \dots$$

$$= 2x - \frac{4}{3}x^3 + \frac{4}{15}x^5 - \frac{8}{315}x^7 + \dots$$

$$\cos 2x = 1 - \frac{(2x)^2}{2!} + \frac{(2x)^4}{4!} - \frac{(2x)^6}{6!} + \dots$$

$$= 1 - 2x^2 + \frac{2}{3}x^4 - \frac{4}{45}x^6 + \dots$$

$$\ln(1+2x) = 2x - \frac{(2x)^2}{2} + \frac{(2x)^3}{3} - \frac{(2x)^4}{4} + \dots$$

$$= 2x - 2x^2 + \frac{8}{3}x^3 - 4x^4 + \dots$$

Synthesis

As conclusion, in calculating the limit of some functions, find the Maclaurin series for the transcendental functions contained in the given function, simplify and then evaluate the limit.

Application Activity 3.56

1. a) $\frac{1}{3} - \frac{(x-3)}{3^2} + \frac{(x-3)^2}{3^3} + \dots + (-1)^n \frac{(x-3)^n}{3^{n+1}} + \dots$

   b) $1 + \frac{x}{2!} + \frac{x^2}{3!} + \dots + \frac{x^{n-1}}{n!} + \dots$

   c) $1 - \frac{1}{3!}x^2 + \frac{1}{5!}x^4 - \dots + (-1)^{n+1}\frac{1}{(2n-1)!}x^{2n-2} + \dots$

   d) $1 - \frac{\pi^2}{4^2 2!}(x-2)^2 + \frac{\pi^4}{4^4 4!}(x-2)^4 - \dots + (-1)^n \frac{\pi^{2n}}{4^{2n}(2n)!}(x-2)^{2n} + \dots$

2. a) $1 + \sum_{n=1}^{\infty} (-1)^n \frac{2^{2n-1}}{(2n)!} x^{2n}$ b) $\sum_{n=0}^{\infty} \frac{x^{n+2}}{k!}$

   c) $1 - \frac{1}{2}x^3 - \frac{1}{8}x^6 - \dots$ d) $1 + 2x + \frac{5}{2}x^2 + \dots$

Lesson 3.7. Calculation of limits

Learning objectives

Given a function involving transcendental functions and by using Maclaurin series, learners should be able to evaluate its limit at a given point correctly.

Prerequisites

Maclaurin series of some functions like

$e^x, \ \sin x, \ \cos x, \tan x, \ln x, \ldots$

Limits concepts.

Teaching Aids

Exercise book, pen and calculator

Activity 3.7

In this lesson, the following generic competence and cross-cutting issues are to be addressed:

- Critical thinking
- Communication
- Self confidence
- Cooperation, interpersonal management and life skills
- Peace and values education
- Inclusive education

Answers

1. $\cos 4x = 1 - 8x^2$ and $\sin 3x = 3x - \frac{9}{2}x^3$

2. $\frac{1-\cos 4x + x\sin 3x}{x^2} = \frac{1-\left(1-8x^2\right)+x\left(3x-\frac{9}{2}x^3\right)}{x^2}$

$$= \frac{1-1+8x^2+3x^2-\frac{9}{2}x^4}{x^2} = \frac{11x^2-\frac{9}{2}x^4}{x^2} = 11 - \frac{9}{2}x^2$$

Then $\lim\limits_{x\to 0}\frac{1-\cos 4x + x\sin 3x}{x^2} = \lim\limits_{x\to 0}\left(11-\frac{9}{2}x^2\right) = 11$

Synthesis

As conclusion, find the Maclaurin series for the transcendental functions contained in the given function, simplify and then evaluate the limit.

Application Activity 3.57

1) $-\frac{1}{2}$ 2) 2 3) $\frac{1}{2}$ 4) 0

Lesson 3.8. Estimation of the number e

Learning objectives

Given number e and by using Maclaurin series of e^x, learners should be able to estimate this number to some decimal places perfectly.

Prerequisites

Maclaurin series of e^x

Teaching Aids

Exercise book, pen and calculator

Activity 3.8

In this lesson, the following generic competence and cross-cutting issues are to be addressed:

- Critical thinking
- Communication
- Self confidence
- Cooperation, interpersonal management and life skills
- Peace and values education
- Inclusive education

Answers

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \frac{x^7}{7!} + \frac{x^8}{8!} + \frac{x^9}{9!} + \frac{x^{10}}{10!} + \frac{x^{11}}{11!} + \frac{x^{12}}{12!} + \ldots + \frac{x^n}{n!} + \ldots$$

Putting $x = 1$, we have

$$e = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \frac{1}{6!} + \frac{1}{7!} + \frac{1}{8!} + \frac{1}{9!} + \frac{1}{10!} + \frac{1}{11!} + \frac{1}{12!} + \ldots$$

Since we need this number to 8 decimal places, we will stop when we reach the decimal term less than 10^{-8}. Here, $\frac{1}{12!}=2\times10^{-9}<10^{-8}$, so we will stop at $\frac{1}{12!}$

Then,

$$\begin{aligned} e &\approx 2+0.5+0.1666667+0.04166667+0.00833333+0.00138889 \\ &\quad +0.00019841+0.00002480+0.00000275+0.00000027 \\ &\quad +0.00000003+0.00000000 \approx 2.71828182 \end{aligned}$$

Synthesis

By putting $x=1$ in the development of e, we can easily estimate the value of the number e to desired decimal places.

Application Activity 3.58

1. $e\approx 2.71$
2. $e\approx 2.7182$
3. $e\approx 2.718281$
4. $e\approx 2.7182818284$

Lesson 3.9. Estimation of the number π

Learning objectives

Given number π and by using Maclaurin series of $\arctan x$, learners should be able to estimate this number to some decimal places perfectly.

Prerequisites

- Maclaurin series of $\arctan x$.
- Change degrees to radians.
- Find trigonometric number of an angle.

Teaching Aids

Exercise book, pen and calculator

Activity 3.9

In this lesson, the following generic competence and cross-cutting issues are to be addressed:

- Critical thinking
- Communication

- Self confidence
- Cooperation, interpersonal management and life skills
- Peace and values education
- Inclusive education

Answers

1. Let $f(x)=\arctan x$

$$f'(x)=\frac{1}{x^2+1} \Rightarrow f'(0)=1$$

$$f''(x)=\frac{-2x}{(x^2+1)^2} \Rightarrow f''(0)=0$$

$$f'''(x)=\frac{6x^2-2}{(x^2+1)^3} \Rightarrow f'''(0)=-2$$

$$f^{(4)}(x)=\frac{-24x^3+24x}{(x^2+1)^4} \Rightarrow f^{(4)}(0)=0$$

$$f^{(5)}(x)=\frac{120x^4-240x^2+24}{(x^2+1)^5} \Rightarrow f^{(5)}(0)=24$$

$$f^{(6)}(x)=\frac{-720x^5+2400x^3-720x}{(x^2+1)^6} \Rightarrow f^{(6)}(0)=0$$

$$f^{(7)}(x)=\frac{5040x^6-25200x^4+15120x^2-720}{(x^2+1)^7} \Rightarrow f^{(7)}(0)=-720$$

Then, $\arctan x = x-\frac{2}{3!}x^3+\frac{24}{5!}x^5-\frac{720}{7!}x^7+\ldots$

Or

$$\arctan x = x-\frac{1}{3}x^3+\frac{1}{5}x^5-\frac{1}{7}x^7+\ldots+(-1)^n\frac{x^{2n+1}}{2n+1}+\ldots$$

The general term is $(-1)^n\frac{x^{2n+1}}{2n+1}$

2. $\tan x = \frac{\sqrt{3}}{3} \Rightarrow x = \arctan\frac{\sqrt{3}}{3} \Rightarrow x = \frac{\pi}{6}$

Synthesis

By using the series

$$\arctan x = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \ldots + (-1)^n \frac{x^{2n+1}}{2n+1} + \ldots,$$

and $x = \frac{\pi}{6} \Rightarrow \pi = 6x$ we get that

$$\pi = 6\left[\frac{\sqrt{3}}{3} - \frac{\left(\frac{\sqrt{3}}{3}\right)^3}{3} + \frac{\left(\frac{\sqrt{3}}{3}\right)^5}{5} - \frac{\left(\frac{\sqrt{3}}{3}\right)^7}{7} + \cdots + (-1)^n \frac{\left(\frac{\sqrt{3}}{3}\right)^{2n+1}}{2n+1} + \cdots\right]$$

or $\pi = 6\frac{\sqrt{3}}{3} - 6\frac{\left(\frac{\sqrt{3}}{3}\right)^3}{3} + 6\frac{\left(\frac{\sqrt{3}}{3}\right)^5}{5} - 6\frac{\left(\frac{\sqrt{3}}{3}\right)^7}{7} + \ldots + (-1)^n 6\frac{\left(\frac{\sqrt{3}}{3}\right)^{2n+1}}{2n+1} + \ldots$

we can easily estimate the number π.

Application Activity 3.59

1. $\pi \approx 3.141$
2. $\pi \approx 3.14159$
3. $\pi \approx 3.1415926$
4. $\pi \approx 3.141592653$

Lesson 3.10. Estimation of trigonometric number of an angle

Learning objectives

Given an angle and by using Maclaurin series of trigonometric functions, learners should be able to estimate the trigonometric number of that angle accurately.

Prerequisites

- Maclaurin series of trigonometric functions

Teaching Aids

Exercise book, pen and calculator

Activity 3.10

In this lesson, the following generic competence and cross-cutting issues are to be addressed:

- Critical thinking
- Communication
- Self confidence
- Cooperation, interpersonal management and life skills
- Peace and values education
- Inclusive education

Answers

1. $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + \ldots$

2. If $x = \frac{\pi}{4}$, we get

$$\sin\frac{\pi}{4} = \frac{\pi}{4} - \frac{\left(\frac{\pi}{4}\right)^3}{3!} + \frac{\left(\frac{\pi}{4}\right)^5}{5!} - \frac{\left(\frac{\pi}{4}\right)^7}{7!} + \ldots + (-1)^n \frac{\left(\frac{\pi}{4}\right)^{2n+1}}{(2n+1)!} + \ldots$$

Since we need $\sin\frac{\pi}{4}$ to 4 decimal places, we will stop when we reach the decimal term less than 10^{-4}.

Here, $\frac{\left(\frac{\pi}{4}\right)^7}{7!} = 3\times10^{-5} < 10^{-4}$, so we will stop at $\frac{\left(\frac{\pi}{4}\right)^7}{7!}$

Then,

$$\begin{aligned}\sin\frac{\pi}{4} &= \frac{\pi}{4} - \frac{\left(\frac{\pi}{4}\right)^3}{3!} + \frac{\left(\frac{\pi}{4}\right)^5}{5!} - \frac{\left(\frac{\pi}{4}\right)^7}{7!}\\ &= 0.7854 - 0.0807 + 0.0024 - 0.0000\\ &= 0.7071\end{aligned}$$

Remember that on the right hand side π is replaced by $3.1415\ldots$ not 180^0

Synthesis

x being expressed in radian, we can approximate the value of any trigonometric number using the series of trigonometric functions.

Application Activity 3.60

1. 0.866
2. 0.017452
3. 0.4226
4. -0.70711

Lesson 3.11. Estimation of an irrational number

Learning objectives

Given an irrational number and by using Maclaurin series of $(1+x)^m$, learners should be able to estimate correctly that irrational number to some decimal places accurately.

Prerequisites

- Maclaurin series of $(1+x)^m$.

Teaching Aids

Exercise book, pen and calculator

Activity 3.11

In this lesson, the following generic competence and cross-cutting issues are to be addressed:

- Critical thinking
- Communication
- Self confidence
- Cooperation, interpersonal management and life skills
- Peace and values education
- Inclusive education

Answers

1. $1, 4, 9, 16, 25, 36, 49, 64, 81, 100, 121, ...$
2. $2, 8, 18, 32, 50, 72, 98, 128, 162, 200, 242, ...$

3. Take 49 and 50. Their ratio is 0.98, closed to 1. We can also take 121 and 128. If we extend the sequence, we can get other two numbers, 289 and 288. Their ratio is 1.003.

4. Now take 289 and 288, try to transform $\sqrt{2}$. Knowing that $288 = 2\times 144$, we have

$$\sqrt{2} = \sqrt{\frac{289\times 2\times 144}{289\times 144}} = \frac{17}{12}\sqrt{\frac{2\times 144}{289}} = \frac{17}{12}\sqrt{\frac{288}{289}} = \frac{17}{12}\sqrt{1-\frac{1}{289}}$$

Synthesis

Using the Maclaurin series of $(1+x)^m$ for $|x|<1$, we can estimate any irrational number like $\sqrt{2}, \sqrt{3}, \sqrt[3]{5}, \ldots$

Procedure:

Suppose that we need to estimate the value of $\sqrt[n]{a}$ to 6 decimal places.

1. Write down a sequence of natural numbers to the power n (as we need nth root).
2. Multiply each term in obtained sequence from 1) by the radicand (here radicand is a).
3. Take two numbers from sequence in 1) and another from sequence in 2) such that their ratio is closed to 1.

Using the obtained numbers from 3), transform the radicand so that it differs little from 1, then use expansion of $(1+x)^n$ to get $\sqrt[n]{a}$.

Application Activity 3.61

1. $\sqrt{3} \approx 1.732$
2. $\sqrt{5} \approx 2.2361$
3. $\sqrt[3]{2} = 1.259921$
4. $\sqrt[3]{4} = 1.587401$

Lesson 3.12. Estimation of roots of equations

Learning objectives

Given an equation and by using Maclaurin series, learners should be able to estimate the roots of that equations accurately.

Prerequisites

- Maclaurin series of transcendental functions

Teaching Aids

Exercise book, pen and calculator

Activity 3.12

In this lesson, the following generic competence and cross-cutting issues are to be addressed:

- Critical thinking
- Communication
- Self confidence
- Cooperation, interpersonal management and life skills
- Peace and values education
- Inclusive education

Answers

1. $\ln(1+x)=x-\frac{x^2}{2}$
2. The equation $\ln(1+x)+x=0$ becomes $x-\frac{x^2}{2}+x=0$ or $2x-\frac{x^2}{2}=0$
3. $2x-\frac{x^2}{2}=0 \Rightarrow x\left(2-\frac{x}{2}\right)=0$

   $x=0 \ \ or \ \ 2-\frac{x}{2}=0 \Rightarrow x=0 \ \ or \ \ x=4$
4. If $x=0$, $\ln(1+0)+0=0 \Leftrightarrow 0=0$ TRUE

   If $x=4$, $\ln(1+4)+0=0 \Leftrightarrow \ln 5=0$ FALSE

   Hence, $S=\{0\}$

Synthesis

The n^{th} order Maclaurin polynomial is helpful to estimate the roots of a given equation involving transcendental functions.

Application Activity 3.62

1. $S=\left\{-\frac{\sqrt{10}}{5},\frac{\sqrt{10}}{5}\right\}$

2. $S=\left\{\frac{1}{2}\right\}$

3. $S=\left\{0,\frac{4}{7}\right\}$

3.6. Summary of the unit

1. Generalities on series

Definitions

A finite series is an expression of the form $u_1+u_2+u_3+...+u_n$ or in sigma notation $\sum_{k=1}^{n}u_k$,
where the index of summation, k, takes consecutive integer values from the lower limit, 1, to the upper limit, n. The terms $u_1,u_2,u_3,...,u_n$ are called terms of the series and the term u_n is the general term.

To obtain $\sum_{k=1}^{n}u_k$, the method of difference is usually used i.e. $\sum_{k=1}^{n}u_n=f(1)-f(n+1)$ where $u_k=f(k)-f(k+1)$, with $f(k)$ a function of k.

Convergence and divergence of a series

Let $\{s_n\}$ be the sequence of partial sums of the series $\sum_{k=1}^{+\infty}u_k$. If the sequence $\{s_n\}$ converges to a limit S, then the series is said to converge and S is called the sum of the series.

We denote this by writing $S = \sum_{k=1}^{+\infty} u_k$.

If the sequence of partial sums of a series diverges, then the series is said to diverge. A divergent series has no sum.

Comparison test

Let $\sum_{n=1}^{\infty} a_n$ be a series with positive terms.

a) $\sum_{n=1}^{\infty} a_n$ converges if there exists a convergent series $\sum_{n=1}^{\infty} b_n$ such that $a_n \le b_n$ for all $n > N$, where N is some positive integer.

b) $\sum_{n=1}^{\infty} a_n$ diverges if there exists a divergent series $\sum_{n=1}^{\infty} c_n$ such that $a_n \ge c_n$ for all $n > N$, where N is some positive integer.

Limit comparison test

If the series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are two series with positive terms, and $\lim_{n\to\infty} \frac{a_n}{b_n}$ is finite, both series converge or diverge.

The ratio test

Let $\sum_{n=1}^{\infty} u_n$ be a series with positive terms and let

$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = L$, then,

a) the series converges if $L < 1$,

b) the series diverges if $L > 1$,

c) the series may or may not converge if $L = 1$ (i.e., the test is inconclusive).

The n^{th} root test

Let $\sum_{n=1}^{\infty} u_n$ be a series with positive terms and let $\lim_{n\to\infty} \sqrt[n]{u_n} = L$, then,

a) the series converges if $L<1$,

b) the series diverges if $L>1$,

c) the test is inconclusive $L=1$.

2. Power series

Power series is like an infinite polynomial. It has the form

$$\sum_{n=0}^{\infty} a_n (x-c)^n = a_0 + a_1(x-c) + a_2(x-c)^2 + ... + a_n(x-c)^n + ...$$

Here, c is any real number and a series of this form is called a **power series** centred at c.

Let $f(x) = \sum_{n=0}^{\infty} a_n (x-c)^n$ be the function defined by this power series. $f(x)$ is only defined if the power series converges, so we will consider the domain of the function f to be the set of x values for which the series converges. There are **three** possible cases:

The power series converges at $x=c$. Here, the radius of convergence is zero.

The power series converges for all x, i.e $]-\infty,+\infty[$. Here, the radius of convergence is infinity.

There is a number R called the radius of convergence such that the series converges for all $c-R<x<c+R$ and the series diverges outside this interval.

3. Taylor and Maclaurin series

If $f(x)$ is a function defined on the open interval (a,b), and which can be differentiated $(n+1)$ times on (a,b), then the equality

$$f(x)=\sum_{n=0}^{\infty}\frac{f^{(n)}(x_0)}{n!}(x-x_0)^n+R_{n+1}(x)$$

for any values of x and x_0 in (a,b) is called **Taylor's formula**

$R_{n+1}(x)$ is called the **remainder function**.

The resulting function (without $R_{n+1}(x)$) is called the Taylor expansion of $f(x)$ with respect to about the point $x=x_0$ of order n.

One of the most common forms of the remainder function is the Lagrange form:

$$R_{n+1}(x)=\frac{(x-x_0)^{n+1}}{(n+1)!}f^{(n+1)}\left(x_0+\theta(x-x_0)\right) \text{ where } 0<\theta<10\,.$$

If $\lim_{n\to\infty}R_{n+1}(x)=0$ for some terms in

$f(x)=\sum_{n=0}^{\infty}\frac{f^{(n)}(x_0)}{n!}(x-x_0)^n+R_{n+1}(x)$, then the infinite

series

$$f(x)=f(x_0)+\sum_{n=1}^{\infty}\frac{f^{(n)}(x_0)}{n!}(x-x_0)^n$$

is called the Taylor series for $f(x)$.

A **Maclaurin series** is a Taylor series with $x_0=0$

Note that if $f(x)$ is a polynomial of degree, then it will have utmost only n non-zero derivatives; all other higher-order derivatives will be identically equal to zero.

The following series are very important. All of them are Maclaurin series $(x_0=0)$ and, it is possible to find the Taylor series for other functions by using these formulae without necessarily using Taylor's formula.

a) $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots$

b) $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots + (-1)^{n-1} \frac{x^{2n-1}}{(2n-1)!} + \cdots$

c) $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots + (-1)^n \frac{x^{2n}}{(2n)!}$

d) If $-1 < x < 1$, then

$$(1+x)^m = 1 + mx + \frac{m(m-1)}{2!}x^2 + \frac{m(m-1)(m-2)x^3}{3!} + \cdots$$
$$+ \frac{m(m-1)(m-2)\ldots(m-n+1)x^n}{n!} + \cdots$$

Particularly, if $|x| < 1$, then

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \ldots + (-1)^n x^n + \ldots$$

If $-1 < x \le 1$, then $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} + \ldots + \frac{(-1)^{n-1} x^n}{n} + \ldots$

3.7. End of Unit Assessment

1. a) $\frac{1}{3}n^3 + \frac{5}{2}n^2 + \frac{13}{6}n$ b) $\frac{1}{4}n^4 + \frac{5}{2}n^3 + \frac{37}{4}n^2 + 15n$

   c) $-\frac{1}{3}\left(\frac{1}{n+4} + \frac{1}{n+5} + \frac{1}{n+6} - \frac{37}{60}\right)$

   d) $\frac{1}{3}n^3 + \frac{3}{2}n^2 + \frac{7}{6}n$ e) $-\frac{1}{2}\left(\frac{1}{n+1} - \frac{1}{n+2} - \frac{1}{2}\right)$

2. a) $\frac{233}{990}$ b) $\frac{50}{99}$ c) $\frac{11}{999}$

3. a) $-\frac{7}{2} < x < \frac{1}{2}, r = 2$ b) $-1 < x < 1, r = 1$

   c) $-1 \le x \le 1, r = 1$ d) $-2 < x < 2, r = 2$

   e) $-2 \le x < 2, r = 2$ f) $-\frac{3}{2} < x < \frac{3}{2}, r = \frac{3}{2}$

4. a) $(x-1)-\frac{(x-1)^2}{2}+\frac{(x-1)^3}{3}-\frac{(x-1)^4}{2}+\ldots$

b) $1-(x-1)+(x-1)^2-(x-1)^3+\ldots$

c) $\frac{\sqrt{2}}{2}\left[1+\left(x-\frac{\pi}{4}\right)-\frac{\left(x-\frac{\pi}{4}\right)^2}{2}-\frac{\left(x-\frac{\pi}{4}\right)^3}{6}-\ldots\right]$

5. $1-\pi^2\frac{\left(x-\frac{1}{2}\right)^2}{2!}+\pi^4\frac{\left(x-\frac{1}{2}\right)^4}{4!}+\ldots$

6. $5(x-1)+6(x-1)^2+4(x-1)^3+(x-1)^4$

7. $(x-1)e+(x-1)^2 e+\frac{(x-1)^3}{2}e+\frac{(x-1)^4}{6}e$

8. a) $x^2-\frac{x^4}{3}+\frac{2x^6}{45}$ b) $x+\frac{x^3}{2}+\frac{3x^5}{8}$

c) $x-x^2+\frac{x^3}{2}$ d) $x-x^3+x^6$

9. $x-\frac{x^2}{2}+\frac{x^3}{3}-\ldots$; $2\left(x+\frac{x^3}{3}+\frac{x^5}{5}+\ldots\right)=2\sum_{n=1}^{\infty}\frac{x^{2n-1}}{2n-1}$

10. $\ln(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}$. The absolute value of the remainder term in Lagrange form is $\frac{6x^4}{(1+c)^4\,4!}=\frac{x^4}{4(1+c)^4}$ where $0<c<x$. The maximum value of the remainder term is obtained where $c=0$ and so, equals $\frac{x^4}{4}$. We must then have $\frac{x^4}{4}<5\times10^{-4}$ and so $x<0.211$.

11. a) $\sin\pi=0$ b) $\cos e$

12. $x^2-\frac{x^4}{3!}+\frac{x^6}{5!}-\ldots$

13. $\gamma=\dfrac{1}{\sqrt{1-\dfrac{v^2}{c^2}}}=\left(1-\dfrac{v^2}{c^2}\right)^{-\frac{1}{2}}=1+\dfrac{1}{2}\dfrac{v^2}{c^2}+\dfrac{3}{8}\left(\dfrac{v^2}{c^2}\right)^2+\dots$

And so

$$K=\left(1+\frac{1}{2}\frac{v^2}{c^2}+\frac{3}{8}\left(\frac{v^2}{c^2}\right)^2+\dots-1\right)mc^2=\frac{1}{2}mv^2+\frac{3}{8}mv^2\left(\frac{v^2}{c^2}\right)^2+\dots$$

This is approximately $K=\dfrac{1}{2}mv^2$ if $v<<c$ since the neglected terms are small.

14. $1-\dfrac{x^2}{2}+\dfrac{5x^4}{24}+\dots$; $\dfrac{1}{2}$

15. $x-\dfrac{x^3}{3}+\dfrac{x^5}{10}+\dots$; $\dfrac{1}{3}$

16. $-1+\sqrt{2}$

17. $x-\dfrac{x^2}{2}+\dfrac{x^3}{3}-\dfrac{x^4}{4}+\dots$ and $1-\dfrac{x}{2}+\dfrac{x^2}{3}-\dfrac{x^3}{4}+\dots$

18. $1-\dfrac{x}{2}+\dfrac{3x^2}{8}+\dots$

19. $1+x^2+x^4+\dots$

20. $\dfrac{1}{6}+\dfrac{11x}{36}+\dfrac{49x^2}{216}+\dfrac{179x^3}{1296}+\dots$

21. $x-\dfrac{7x^3}{6}+\dfrac{27x^5}{40}+\dots$, limit is $-\dfrac{7}{6}$

22. $1+x+\dfrac{x^2}{2}-\dfrac{x^4}{8}+\dots$

23. $e^{i\theta}=1+i\theta-\dfrac{\theta^2}{2!}-i\dfrac{\theta^3}{3!}+\dfrac{\theta^4}{4!}+i\dfrac{\theta^5}{5!}+\dots$,

$$e^{i\theta}=1-\frac{\theta^2}{2!}+\frac{\theta^4}{4!}+\dots+i\left(\theta-\frac{\theta^3}{3!}+\frac{\theta^5}{5!}+\dots\right)$$

Substituting $\theta=\pi$ gives

$e^{i\pi}=\cos\pi+i\sin\pi=-1\Rightarrow e^{i\pi}+1=0$

24. a) $1+\dfrac{x}{3}-\dfrac{2x^2}{9}+\dots$

b) $\sqrt[3]{n^3+1} = n\sqrt[3]{1+\frac{1}{n^3}}$. In a) put $x = \frac{1}{n^3}$,

$$\sqrt[3]{n^3+1} = n\sqrt[3]{1+\frac{1}{n^3}} = n\left(1+\frac{1}{3n^3}-\frac{2}{9n^6}+\ldots\right)$$

$$= n+\frac{1}{3n^2}-\frac{2}{9n^5}+\ldots$$

and,

$$\sqrt[3]{n^3+1}-n = n+\frac{1}{3n^2}-\frac{2}{9n^5}+\ldots-n$$

$$= \frac{1}{3n^2}-\frac{2}{9n^5}+\ldots$$

$$\approx \frac{1}{3n^2} \qquad \textit{when } n \textit{ is large}$$

c) Use the limit comparison test with the series $\frac{1}{3n^2}$

Unit 4 Integration

4.1. Key unit competence

Use integration as the inverse of differentiation and as the limit of a sum then apply it to find area of plane surfaces, volumes of solid of revolution, lengths of curved lines.

4.2. Vocabulary or key words concepts

Primitive function of function $f(x)$: Is a function $F(x)$ such that $F'(x)=f(x)$.

Integration: Process of finding primitive functions (or anti derivative functions).

Indefinite integrals: Primitive functions.

Definite integrals: Primitive functions evaluated at a given closed interval.

Improper integrals: Definite integrals involving infinity limits or a discontinuous point in the interval of integration.

Volume of revolution: Volume obtained when a curve of a function or a surface between two curves is revolved around an axis.

4.3. Guidance on the introductory activity

Organize groups of students, then assign them to do the introductory activity from the student's book. As they are working, move around to each group and ask them probing questions leading them to the right way. After a while, invite some group to present their findings in a whole class discussion. Guide students to harmonize their answers and arouse their curiosity to the content of this unit.

4.4. List of lessons

No	Lesson title	Number of periods
1	Differentials	2
2	Definition of indefinite integrals	1
3	Properties of integrals	1
4	Integration by substitution	1
5	Integration of rational function where numerator is expressed in terms of derivative of denominator	2
6	Integration of rational function where degree of numerator is greater or equal to the degree of denominator	2
7	Integration of rational function where denominator is factorised into linear factors	2
8	Integration of rational function where denominator is a quadratic factor	2
9	Integral of the form $\int \sin mx \cos nx\, dx$, $\int \cos mx \cos nx\, dx$, $\int \sin mx \sin nx\, dx$	2
10	Integral of the form $\int \sin^m x \cos^n x\, dx$	2
11	Integral of the form $\int \tan^m x \sec^n x\, dx$	2
12	Integral containing $\sin x,\ \cos x,\ \tan x$ on denominator	2
13	Integral containing $\sin^2 x, \cos^2 x$ on denominator	2
14	Integral containing $\sqrt[n]{ax+b}$	2
15	Integral containing $\sqrt{ax^2+bx+c}$	2
16	Integration by parts	2
17	Integration by reduction formulae	2
18	Integration by Maclaurin series	1
19	Definition of definite integrals	1

20	Properties of definite integrals	1
21	Improper integrals: Infinite limits of integration	1
22	Discontinuous integrand	1
23	Calculation of area of plane surface	2
24	Calculation of volume of solid of revolution	2
25	Calculation of arc length of curved lines	2
Total periods		42

4.5. Lesson development

Lesson 4.1. Differentials

Learning objectives

Given a function, learners should be able to find differential of that function and the percentage error perfectly.

Prerequisites

- Differentiation of a function

Teaching Aids

Exercise book and pen

Activity 4.1

In this lesson, the following generic competence and cross-cutting issues are to be addressed:

- Critical thinking
- Self confidence
- Communication
- Cooperation, interpersonal management and life skills
- Peace and values education
- Inclusive education

Answers

a) If $y=\sin x,\ x=\frac{\pi}{3},\ \Delta x=0.006$ then,

$$dy=\cos x dx=\cos\left(\frac{\pi}{3}\right)dx=\frac{1}{2}(0.006)=0.003$$

Thus, the change in the value of $\sin x$ is approximately 0.003.

b) $\sin\left(\frac{\pi}{3}+0.006\right)\approx\sin\frac{\pi}{3}+0.003=\frac{\sqrt{3}}{2}+0.003=0.869$

Synthesis

As conclusion,

Differential dy is given by $dy=f'(x)dx$ for $y=f(x)$

Whenever one makes an approximation, it is wise to try and estimate how big the error might be. Relative change in x is $\frac{\Delta x}{x}$ and percentage change in x is $100\times\frac{\Delta x}{x}$.

Application Activity 4.63

1. a) $df=(2x-3)dx$ b) $df=-\frac{4}{x^2+4x+4}dx$

   c) $df=-\frac{3}{8\sqrt{2-x}}dx$

2. $+2$ 3. 1.75% 4. ± 10

Lesson 4.2. Definition of indefinite integrals

Learning objectives

Through examples, learners should be able to define indefinite integrals rightfully.

Prerequisites

- Derivative of a function

Teaching Aids

Exercise book, calculator and pen

Activity 4.2

In this lesson, the following generic competence and cross-cutting issues are to be addressed:

- Critical thinking
- Self confidence
- Communication
- Cooperation, interpersonal management and life skills
- Peace and values education
- Inclusive education

Answers

a) $x^3 + c, \quad c \in \mathbb{R}$

b) $\frac{x^2}{2} + c, \quad c \in \mathbb{R}$

c) $2\sqrt{x}, \quad c \in \mathbb{R}$

d) $\frac{1}{x} + c, \quad c \in \mathbb{R}$

Synthesis

As conclusion, the function $F(x)$ is an indefinite integral of $f(x)$ if $F'(x) = f(x)$.

Application Activity 4.64

1. $2x^2 - 5x + c$
2. $2x^3 + 2x^2 + 3x + c$
3. $\frac{1}{4}x^4 + \frac{1}{3}x^3 + \frac{1}{2}x^2 + c$
4. $3x^3 - 12x^2 + 16x + c$
5. $5x + c$
6. $-\frac{8}{3}x^3 + 9x^2 - 9x + c$
7. $\frac{2}{5}x^5 - x^3 - 5x + c$
8. $\frac{4}{5}x^5 - \frac{4}{3}x^3 + x + c$
9. $\frac{1}{4}x^4 - 2x^3 + 6x^2 - 8x + c$

Lesson 4.3. Properties of integrals

Learning objectives

Through examples, learners should be able to use properties of indefinite integrals accurately.

Prerequisites
Integrals of simple functions.

Teaching Aids
Exercise book, calculator and pen

Activity 4.3

In this lesson, the following generic competence and cross-cutting issues are to be addressed:

- Critical thinking
- Self confidence
- Communication
- Cooperation, interpersonal management and life skills
- Peace and values education
- Inclusive education

Answers

1. $\int f(x)dx = \int \cos x dx = \sin(x) + c$

   $$\frac{d\int f(x)dx}{dx} = \frac{d\left[\sin(x)+c\right]}{dx} = \cos x$$

   Observation: $\frac{d}{dx}\int f(x)dx = f(x)$

2. Differential of $f(x)$ is $df = \cos x dx$

   $\int df = \int \cos x dx = \sin x + c$

   Observation: $\int df(x) = f(x) + c$

3. $\int 3x\, dx = \frac{3x^2}{2} + c$ and

   $$3\int x\, dx = 3\left(\frac{x^2}{2} + k\right) = \frac{3x^2}{2} + 3k = \frac{3x^2}{2} + c$$

Observation: $\int kf(x)dx = k\int f(x)dx$, $k \in \mathbb{R}$

4. $\int f(x)dx + \int g(x)dx = \int(x^3+3x-1)dx + \int(x^2+2x+2)$

$$= \frac{x^4}{4} + \frac{3x^2}{2} - x + \frac{x^3}{3} + x^2 + 2x + c$$

$$= \frac{x^4}{4} + \frac{x^3}{3} + \frac{5x^2}{2} + x + c$$

$$\int[f(x)+g(x)]dx = \int(x^3+3x-1+x^2+2x+2)dx$$

$$= \int(x^3+x^2+5x+1)dx$$

$$= \frac{x^4}{4} + \frac{x^3}{3} + \frac{5x^2}{2} + x + c$$

Observation:

$$\int[f(x)\pm g(x)]dx = \int f(x)dx \pm \int g(x)dx$$

5. $\frac{d}{dx}\cos(2x+3) = -(2x+3)'\sin(2x+3) = -2\sin(2x+3)$

$$\int[-\sin(2x+3)]dx = \frac{1}{2}\cos(2x+3) + c$$

6. Hence;

$$\int f(ax+b)dx = \frac{1}{a}F(ax+b) + c \qquad a,b,c \in \mathbb{R}, a \neq 0$$

Synthesis

1. The derivative of the indefinite integral is equal to the function to be integrated.

$$\frac{d}{dx}\int f(x)dx = f(x)$$

2. The integral of differential of a function is equal to the sum of that function and an arbitrary constant.

$$\int df(x) = f(x) + c$$

3. Each constant function may be pulled out of integral sign.

$$\int k\,f(x)dx = k\int f(x)dx,\ k \in \mathbb{R}$$

4. The indefinite integral of the algebraic sum of two functions is equal to the algebraic sum of the indefinite integrals of those functions.

$$\int\left[f(x)\pm g(x)\right]dx=\int f(x)\,dx\pm\int g(x)\,dx$$

5. If $F(x)$ is a primitive function of $f(x)$, then, the integral

$$\int f(ax+b)\,dx=\frac{1}{a}F(ax+b)+c \qquad a,b,c\in\mathbb{R},a\neq 0$$

Application Activity 4.65

1. $4\int f(x)\,dx=4\left(x^2+2x+c\right)=4x^2+8x+k$ Where $4c=k$ Which is constant.

2. $$\begin{aligned}\frac{2}{5}\int\left[g(x)-6\right]dx&=\frac{2}{5}\left[\int g(x)\,dx-\int 6dx\right]\\&=\frac{2}{5}\left[\int g(x)\,dx-6\int dx\right]\\&=\frac{2}{5}\left(x^3-3x^2-4x+k-6x\right)\\&=\frac{2}{5}\left(x^3-3x^2-10x+k\right)\\&=\frac{2}{5}x^3-\frac{6}{5}x^2-4x+\frac{2}{5}k\\&=\frac{2}{5}x^3-\frac{6}{5}x^2-4x+c\end{aligned}$$

3. $$\begin{aligned}\int\left[f(x)+3g(x)\right]dx&=\int f(x)\,dx+3\int g(x)\,dx\\&=x^2+2x+c+3\left(x^3-3x^2-4x+k\right)\\&=x^2+2x+c+3x^3-9x^2-12x+3k\\&=3x^3-8x^2-10x+d\end{aligned}$$

where d is a constant.

4. $$\begin{aligned}\frac{d}{dx}\int\left[2f(x)-3g(x)\right]dx&=2f(x)-3g(x)\\&=2(2x+2)-3\left(3x^2-6x-4\right)\\&=4x+4-9x^2+18x+12\\&=-9x^2+22x+16\end{aligned}$$

Lesson 4.4. Integration by substitution

Learning objectives

Through examples, learners should be able to find integrals by substitution method correctly.

Prerequisites

- Differentiation of a function.

Teaching Aids

Exercise book and pen

Activity 4.4

In this lesson, the following generic competence and cross-cutting issues are to be addressed:

- Critical thinking
- Communication
- Self confidence
- Cooperation, interpersonal management and life skills
- Peace and values education
- Inclusive education

Answers

$$u = 5x + 2 \Rightarrow du = 5dx$$

$$\int e^{5x+2}\,dx = \int e^{u}\frac{du}{5} = \frac{1}{5}\int e^{u}du = \frac{1}{5}e^{u} + c = \frac{1}{5}e^{5x+2} + c$$

Synthesis

Integration by substitution is based on rule for differentiating composite functions. Substitution means to let $f(x)$ be a function of another function.

In $\int f(x)dx$, let x be $x(t)$; thus, $dx = x'(t)dt$ and then we get $\int f(x)dx = \int f(x(t))x'(t)dt$ that is a formula of integration by substitution.

Application Activity 4.66

1. a) $e^x - \frac{x^{e+1}}{e+1} + c$ b) $\frac{1}{3}e^{x^3} + c$

   c) $\frac{1}{2}e^{2x} + 2e^x + x + c$ d) $-\frac{1}{2}e^{\frac{1}{x^2}} + c$

   e) $\sin\left(e^x\right) + c$ f) $f)\int e^{3\cos 2x}\sin 2xdx$

   by letting $t = 3\cos 2x$ to get $\int -\frac{e^t}{6}dt = -\frac{1}{6}e^t + c =$

   $-\frac{1}{6}e^{3\cos 2x} + c$

   g) $\sin\left(\ln x\right) + c$ h) $\frac{1}{36}\left(4x^3 - 12\right)^3 + c$

2. $100\,m$

Lesson 4.5. Integration of rational functions where numerator is expressed in terms of derivative of denominator

Learning objectives

Given a rational function where numerator is expressed in terms of derivative of denominator, learners should be able to find primitive function moderately.

Prerequisites

- Derivative of $\ln\left[g\left(x\right)\right]$.
- Derivative of $\arctan\left[g\left(x\right)\right]$.

Teaching Aids

Exercise book, calculator and pen

Activity 4.5

In this lesson, the following generic competence and cross-cutting issues are to be addressed:

- Critical thinking
- Communication
- Self confidence
- Cooperation, interpersonal management and life skills
- Peace and values education
- Inclusive education

Answers

1. We see that $\left(1-x^2\right)'=-2x \Rightarrow x=-\frac{1}{2}\left(1-x^2\right)'$. So, we can write

$$\int \frac{x}{\left(1-x^2\right)^2}dx=\int \frac{-\frac{1}{2}\left(1-x^2\right)'}{\left(1-x^2\right)^2}dx$$

$$=-\frac{1}{2}\int \frac{\left(1-x^2\right)'}{\left(1-x^2\right)^2}dx$$

$$=\frac{1}{2\left(1-x^2\right)}+c \qquad \text{since } \left(\frac{1}{g}\right)'=-\frac{g'}{g^2}$$

2. We see that

$$\left(3x^2-3x+1\right)'=6x-3=3\left(2x-1\right)\Rightarrow 2x-1=\frac{1}{3}\left(3x^2-3x+1\right)'$$

So, we can write

$$\int \frac{2x-1}{3x^2-3x+1}dx=\int \frac{\frac{1}{3}\left(3x^2-3x+1\right)'}{3x^2-3x+1}dx$$

$$=\frac{1}{3}\int \frac{\left(3x^2-3x+1\right)'}{3x^2-3x+1}dx$$

$$=\frac{1}{3}\ln\left|3x^2-3x+1\right|+c \qquad \text{since } \left(\ln u\right)'=\frac{u'}{u}$$

Synthesis

The following basic integration formulae are most helpful:

$\int \frac{u'}{u}dx=\ln|u|+c$, $\int \frac{u'}{u^2}dx=-\frac{1}{u}+c$ and $\int \frac{u'}{u^2+1}dx=\arctan u+c$

Application Activity 4.67

1. $-\frac{1}{2\left(x^2+2x+3\right)}+c$
2. $\frac{1}{2\left(1-x^2\right)}+c$
3. $-\frac{1}{6\left(2x^3+5\right)}+c$
4. $-\frac{1}{4\left(x^2+2x+5\right)^2}+c$

Lesson 4.6. Integration of rational functions where degree of numerator is greater or equal to the degree of denominator

Learning objectives

Given an irrational function where degree of numerator is greater or equal to the degree of denominator, learners should be able to find primitive function accurately.

Prerequisites

- Long division of polynomials.

Teaching Aids

Exercise book, calculator and pen

Activity 4.6

In this lesson, the following generic competence and cross-cutting issues are to be addressed:

- Critical thinking
- Communication
- Self confidence
- Cooperation, interpersonal management and life skills
- Peace and values education
- Inclusive education

Answers

1. $\frac{2x+4}{5x-3}=\frac{2}{5}+\frac{26}{25x-15}$; $\frac{2}{5}x+\frac{26}{25}\ln|5x-3|+c$
2. $\frac{x^2-3x+2}{x^2+1}=1+\frac{1-3x}{x^2+1}$; $x-\frac{3}{2}\ln\left(x^2+1\right)-\arctan x+1+c$
3. $\frac{x^2+1}{x-1}=x+1+\frac{2}{x-1}$; $\frac{x^2}{2}+x+2\ln|x-1|+c$
4. $\frac{x^3+2x-4}{x^2+2}=x-\frac{4}{x^2+2}$; $\frac{x^2}{2}-2\sqrt{2}\arctan\frac{\sqrt{2}x}{2}+c$

Synthesis

If we want to find $\int\frac{f(x)}{g(x)}dx$ when the degree of $f(x)$ is greater than the degree of $g(x)$, we proceed by long division to find $\int \frac{f(x)}{g(x)}dx = \int q(x)dx + \int \frac{r(x)}{g(x)}dx$ where $q(x)$ is the quotient, $r(x)$ the remainder and then integrate the new expression on the right hand side.

Application Activity 4.68

1. $\frac{1}{2}x^2 - \frac{1}{2}\ln\left(x^2+1\right) - 2\arctan x + c$
2. $-\frac{1}{3}\ln|x-1| - \frac{2}{3}\ln|x+2| + x + c$
3. $\frac{1}{6}\ln|x| - \frac{13}{54}\ln|3x-2| - \frac{1}{9}x + c$
4. $\frac{1}{3}\left(x^3 + a^3\ln\left|x^3 - a^3\right|\right) + c$
5. $\frac{1}{2}x^2 - 26\ln|x+3| + 63\ln|x+4| - 7x + c$

Lesson 4.7. Integration of rational functions where denominator is factorised into linear factors

Learning objectives

Given an irrational function where denominator is factorised into linear factors, learners should be able to find primitive function accurately.

Prerequisites

- Factorise completely a polynomial.

Teaching Aids

Exercise book, calculator and pen

Activity 4.7

In this lesson, the following generic competence and cross-cutting issues are to be addressed:

- Critical thinking
- Communication
- Self confidence

- Cooperation, interpersonal management and life skills
- Peace and values education
- Inclusive education

Answers

1. $x^2+2x=x(x+2)$

$$\frac{x-2}{x^2+2x}=\frac{A}{x}+\frac{B}{x+2}=\frac{A(x+2)+Bx}{x^2+2x}$$

$$x-2=A(x+2)+Bx$$

Solving, we get

$$\begin{cases} A=-1 \\ B=2 \end{cases}$$

Then,

$$\frac{x-2}{x^2+2x}=-\frac{1}{x}+\frac{2}{x+2}$$

And $\int \frac{x-2}{x^2+2x}dx=-\int \frac{dx}{x}+\int \frac{2}{x+2}dx$

$$=-\ln|x|+2\ln|x+2|+c$$

$$=\ln|x|+\ln(x+2)^2+c=\ln\frac{(x+2)^2}{|x|}+c$$

2. $x^2+3x+2=(x+1)(x+2)$

$$\frac{x}{x^2+3x+2}=\frac{A}{x+1}+\frac{B}{x+2}=\frac{A(x+2)+B(x+1)}{x^2+3x+2}$$

$$x=A(x+2)+B(x+1)$$

Solving we get,

$$\begin{cases} A=-1 \\ B=2 \end{cases}$$

Then,

$$\frac{x}{x^2+3x+2}=-\frac{1}{x+1}+\frac{2}{x+2}$$

Therefore;

$$\int \frac{x}{x^2+3x+2}dx = -\int \frac{dx}{x+1} + \int \frac{2}{x+2}dx$$

$$= -\ln|x+1| + 2\ln|x+2| + c$$

$$= \ln|x+1| + \ln(x+2)^2 + c = \ln\frac{(x+2)^2}{|x+1|} + c$$

3. $\dfrac{2}{x^2-1} = \dfrac{A}{x-1} + \dfrac{B}{x+1}$

$$\frac{2}{x^2-1} = \frac{A(x+1)+B(x-1)}{x^2-1}$$

$$2 = A(x+1) + B(x-1)$$

Solving, we get

$$\begin{cases} A = 1 \\ B = -1 \end{cases} \qquad \frac{2}{x^2-1} = \frac{1}{x-1} - \frac{1}{x+1}$$

Then, $\displaystyle\int \frac{2}{x^2-1}dx = \int \frac{dx}{x-1} - \int \frac{dx}{x+1}$

$$= \ln|x-1| - \ln|x+1| + c = \ln\frac{x-1}{x+1} + c$$

4. $\dfrac{2x-3}{x^2-x-2}$

$$x^2-x-2 = (x-2)(x+1)$$

$$\frac{2x-3}{x^2-x-2} = \frac{A}{x-2} + \frac{B}{x+1} = \frac{A(x+1)+B(x-2)}{x^2-x-2}$$

$$2x-3 = A(x+1) + B(x-2)$$

Solving, we get

$$\begin{cases} A = \dfrac{1}{3} \\ B = \dfrac{5}{3} \end{cases} \qquad \frac{2x-3}{x^2-x-2} = \frac{1}{3(x-2)} + \frac{5}{3(x+1)}$$

Finally,

$$\int \frac{2x-3}{x^2-x-2}dx = \int \frac{dx}{3(x-2)} + \int \frac{5}{3(x+1)}dx$$

$$=\frac{1}{3}\ln|x-2|+\frac{5}{3}\ln|x+1|+c=\frac{1}{3}\ln(x-2)(x+1)^5+c$$

Synthesis

For integration of rational function where denominator is factorised into linear factors, before integrating, note that to each factor $ax+b$ occurring once in the denominator of a proper rational fraction, there corresponds a single partial fraction of the form $\frac{A}{ax+b}$ where A is a constant

to be found, but to each factor $ax+b$ occurring n times in the denominator of a proper rational fraction, there corresponds a sum of n partial fractions $\frac{A_1}{ax+b}+\frac{A_2}{(ax+b)^2}+...+\frac{A_n}{(ax+b)^n}$ where A_n are constants

to be found, and then integrate the new expression.

Application Activity 4.69

1. $\ln\left|\frac{x-1}{x+1}\right|+c$

2. $\ln\left|\frac{(x+2)^2}{x+1}\right|+c$

3. $\ln\sqrt{\frac{x-2}{x^3}}+c$

4. $\ln(|x+1|)+\frac{1}{x+1}+c, c\in\mathbb{R}$

5. $\int\frac{3x}{x^2-4x+4}dx=\int\frac{3x}{(x-2)^2}dx$; $t=(x-1)$ let ; then

$$=3\ln(|x-2|)-\frac{6}{x-2}+c$$

6. $$6.\int\frac{8x^2-19x}{x^3-3x^2+4}dx=\int\left(\frac{3}{x+1}+\frac{5}{x-2}-\frac{2}{(x-2)^2}\right)dx$$

$$=3\ln(|x+1|)+5\ln(|x-2|)+\frac{2}{x-2}+c, c\in\mathbb{R}$$

Lesson 4.8. Integration of rational functions where denominator is a quadratic factor

Learning objectives

Given an irrational function where denominator is a quadratic factor, learners should be able to find primitive function correctly.

Prerequisites

Use of the relation $ax^2+bx+c=a\left[\left(x+\frac{b}{2a}\right)^2-\frac{b^2-4ac}{4a^2}\right]$.

Teaching Aids

Exercise book, calculator and pen

Activity 4.8

In this lesson, the following generic competence and cross-cutting issues are to be addressed:

- Critical thinking
- Communication
- Self confidence
- Cooperation, interpersonal management and life skills
- Peace and values education
- Inclusive education

Answers

1. Relation to be used; $ax^2+bx+c=a\left[\left(x+\frac{b}{2a}\right)^2-\frac{b^2-4ac}{4a^2}\right]$

$$\int\frac{dx}{x^2+3x+2},\ a=1,\ b=3,\ c=2$$

$$\int\frac{dx}{x^2+3x+2}=\int\frac{dx}{\left(x+\frac{3}{2}\right)^2-\frac{1}{4}}$$

$$\int\frac{dx}{x^2+3x+2}=\int\frac{dx}{\left(x+\frac{3}{2}\right)^2-\frac{1}{4}}\quad \text{Let } u=x+\frac{3}{2}\Rightarrow du=dx$$

$$\int\frac{dx}{x^2+3x+2}=\int\frac{du}{u^2-\frac{1}{4}}=\int\frac{du}{u^2-\left(\frac{1}{2}\right)^2}$$

Using the formula $\int\frac{dx}{x^2-k^2}=\frac{1}{2k}\ln\left|\frac{x-k}{x+k}\right|+d$, we have

$$\int \frac{dx}{x^2+3x+2} = \int \frac{du}{u^2 - \left(\frac{1}{2}\right)^2} = \frac{1}{2\times\frac{1}{2}} \ln\left|\frac{u-\frac{1}{2}}{u+\frac{1}{2}}\right| + d$$

$$= \ln\left|\frac{x+\frac{3}{2}-\frac{1}{2}}{x+\frac{3}{2}+\frac{1}{2}}\right| + d = \ln\left|\frac{x+1}{x+2}\right| + d$$

2. $\int \frac{dx}{x^2-4x+4}$, $a=1,\ b=-4,\ c=4$

$$x^2-4x+4 = \left(x-\frac{4}{2}\right)^2 - \frac{(-4)^2-4\times 4}{4} = (x-2)^2$$

$$\int \frac{dx}{x^2-4x+4} = \int \frac{dx}{(x-2)^2}$$

Let $u = x-2 \Rightarrow du = dx$

$$\int \frac{dx}{x^2-4x+4} = \int \frac{du}{u^2}$$

Using the formula $\int \frac{u'}{u^2} du = -\frac{1}{u} + d$, we have

$$\int \frac{dx}{x^2-4x+4} = \int \frac{du}{u^2} = -\frac{1}{u} + d = -\frac{1}{x-2} + d$$

3. $\int \frac{dx}{x^2-6x+18}$, $a=1,\ b=-6,\ c=18$

$$x^2-6x+18 = \left(x-\frac{6}{2}\right)^2 - \frac{(-6)^2-4\times 18}{4} = (x-3)^2+9$$

$$\int \frac{dx}{x^2-6x+18} = \int \frac{dx}{(x-3)^2+9}$$

Let $u = x-3 \Rightarrow du = dx$

$$\int \frac{dx}{x^2+3x+2} = \int \frac{du}{u^2+9} = \int \frac{du}{u^2+3^2}$$

Using the formula $\int \frac{dx}{x^2+k^2} = \frac{1}{k}\arctan\frac{x}{k} + d$, we have

$$\int \frac{dx}{x^2+3x+2} = \int \frac{du}{u^2+3^2} = \frac{1}{3}\arctan\frac{u}{3} + d = \frac{1}{3}\arctan\frac{x-3}{3} + d$$

Synthesis

For the integral of the form $\int \frac{dx}{ax^2+bx+c}$,

- If $b^2-4ac=0$, then,

  $$\int \frac{dx}{ax^2+bx+c}=\frac{1}{a}\int \frac{dx}{\left(x+\frac{b}{2a}\right)^2}$$ and we let $u=x+\frac{b}{2a}$

- If $b^2-4ac>0$, then,

  $$\int \frac{dx}{ax^2+bx+c}=\frac{1}{a}\int \frac{dx}{\left(x+\frac{b}{2a}\right)^2+\frac{b^2-4ac}{4a^2}}$$. We let

  $$u=x+\frac{b}{2a},\ k^2=\frac{b^2-4ac}{4a^2}$$

  and use the standard integral

  $$\int \frac{dx}{x^2-k^2}=\frac{1}{2k}\ln\left|\frac{x-k}{x+k}\right|+d$$

- If $b^2-4ac<0$, then,

  $$\int \frac{dx}{ax^2+bx+c}=\frac{1}{a}\int \frac{dx}{\left(x+\frac{b}{2a}\right)^2-\frac{b^2-4ac}{4a^2}}$$. We let

  $u=x+\frac{b}{2a},\ -k^2=\frac{b^2-4ac}{4a^2}$ and use the standard integral $\int \frac{dx}{x^2+k^2}=\frac{1}{k}\arctan\frac{x}{k}+d$

Application Activity 4.70

1. $$\int \frac{1}{x^2+x+2}dx=\int \frac{1}{x^2+x+\frac{1}{4}+\frac{7}{4}}dx=$$

$$\int \frac{1}{\left(x+\frac{1}{2}\right)^2+\frac{7}{4}}dx=\int \frac{1}{t^2+\frac{7}{4}}dt=\frac{1}{\sqrt{\frac{7}{4}}}\arctan\left(\frac{t}{\sqrt{\frac{7}{4}}}\right)+c$$

$$=\frac{1}{\sqrt{\frac{7}{4}}}\arctan\left(\frac{x+\frac{1}{2}}{\sqrt{\frac{7}{4}}}\right)+c=\frac{2}{\sqrt{7}}\arctan\left(\frac{2x+1}{\sqrt{7}}\right)+c$$

2. $-\frac{1}{9}\arctan(3x+1)+\frac{1}{18}\ln\left(9x^2+6x+2\right)+c$

3. $-\frac{\sqrt{2}}{2}\arctan\left(\sqrt{2}\,x\right)+3\ln|x-2|+c$

4. $\frac{1}{2}\ln\left(x^2+2\right)-\ln|2x+1|+c$

Lesson 4.9. Integrals of the form $\int \sin mx\cos nx\,dx,\ \int \cos mx\cos nx\,dx,\ \int \sin mx\sin nx\,dx$

Learning objectives

Given integrals of the form $\int \sin mx\cos nx\,dx,\ \int \cos mx\cos nx\,dx,\ \int \sin mx\sin nx\,dx$, learners should be able to find primitive function accurately.

Prerequisites

- Identities:

$$\sin A\cos B=\frac{1}{2}\left[\sin(A-B)+\sin(A+B)\right]$$

$$\sin A\sin B=\frac{1}{2}\left[\cos(A-B)-\cos(A+B)\right]$$

$$\cos A\cos B=\frac{1}{2}\left[\cos(A-B)+\cos(A+B)\right]$$

Teaching Aids

Exercise book and pen

Activity 4.9

In this lesson, the following generic competence and cross-cutting issues are to be addressed:

- Critical thinking
- Communication
- Self confidence

- Cooperation, interpersonal management and life skills
- Peace and values education
- Inclusive education

Answers

1. $\sin 2x\cos x = \frac{1}{2}(\sin x + \sin 3x) = \frac{1}{2}\sin x + \frac{1}{2}\sin 3x$

$$\Rightarrow \int \sin 2x \cos x dx = \int\left(\frac{1}{2}\sin x + \frac{1}{2}\sin 3x\right)dx = -\frac{1}{2}\cos x - \frac{1}{6}\cos 3x + c$$

2. $\sin x\sin 5x = \frac{1}{2}(\cos 4x - \cos 6x) = \frac{1}{2}\cos 4x - \frac{1}{2}\cos 6x$

$$\Rightarrow \int \sin x \sin 5x dx = \int\left(\frac{1}{2}\cos 4x - \frac{1}{2}\cos 6x\right)dx = \frac{1}{8}\sin 4x - \frac{1}{12}\sin 6x + c$$

3. $\cos 2x\cos 3x = \frac{1}{2}(\cos x + \cos 5x) = \frac{1}{2}\cos x + \frac{1}{2}\cos 5x$

$$\Rightarrow \int \cos 2x \cos 3x dx = \int\left(\frac{1}{2}\cos x + \frac{1}{2}\cos 5x\right)dx = \frac{1}{2}\sin x + \frac{1}{10}\sin 5x + c$$

4. $\sin x\sin 3x = \frac{1}{2}\left[\cos(-2x) - \cos 4x\right] = \frac{1}{2}\cos 2x - \frac{1}{2}\cos 4x$

$$\sin x\sin 3x\sin 4x = \frac{1}{2}\cos 2x\sin 4x - \frac{1}{2}\cos 4x\sin 4x$$

$$= \frac{1}{2}\left[\frac{1}{2}(\sin 2x + \sin 6x)\right] - \frac{1}{2}\left[\frac{1}{2}(\sin 0 + \sin 8x)\right]$$

$$= \frac{1}{4}\sin 2x + \frac{1}{4}\sin 6x - \frac{1}{4}\sin 8x$$

$$\Rightarrow \int \sin x\sin 3x\sin 4x\, dx = \int\left(\frac{1}{4}\sin 2x + \frac{1}{4}\sin 6x - \frac{1}{4}\sin 8x\right)dx$$

$$= -\frac{1}{8}\cos 2x + \frac{1}{32}\cos 8x - \frac{1}{24}\cos 6x + c$$

Synthesis

To evaluate the integral of the form $\int \sin mx \cos nx dx$ or $\int \cos mx \cos nx dx$ or $\int \sin mx \sin nx dx$, we express the product into sum by using the corresponding identities:

$$\sin A\cos B=\frac{1}{2}\left[\sin\left(A-B\right)+\sin\left(A+B\right)\right]$$

$$\sin A\sin B=\frac{1}{2}\left[\cos\left(A-B\right)-\cos\left(A+B\right)\right]$$

$$\cos A\cos B=\frac{1}{2}\left[\cos\left(A-B\right)+\cos\left(A+B\right)\right]$$

And then integrate the new expression.

Application Activity 4.71

1. $-\frac{1}{2}\cos x-\frac{1}{10}\cos 5x+c$
2. $\frac{1}{2}\cos x-\frac{1}{10}\cos 5x+c$
3. $-\frac{1}{12}\sin 6x+\frac{1}{2}x+c$
4. $\frac{1}{2}\sin^2 x+c$
5. $\frac{1}{12}\sin 6x+\frac{1}{2}x+c$
6. $\frac{1}{12}\sin 6x+\frac{1}{16}\sin 8x+c$

Lesson 4.10. Integrals of the form $\int\sin^m x\cos^n x\,dx$

Learning objectives

Given an integral of the form $\int\sin^m x\cos^n x\,dx$, learners should be able to find primitive function accurately.

Prerequisites

- Derivative of $\cos x$ and $\sin x$.
- Identity $\cos^2 x+\sin^2 x=1$.
- Identity $\cos 2x=\cos^2 x-\sin^2 x$.
- Identity $\sin 2x=2\sin x\cos x$.

Teaching Aids

Exercise book and pen

Activity 4.10

In this lesson, the following generic competence and cross-cutting issues are to be addressed:

- Critical thinking
- Communication
- Self confidence
- Cooperation, interpersonal management and life skills
- Peace and values education

- Inclusive education

Answers

1. $u=\cos x \Rightarrow du=-\sin x dx$

$$\int \sin x\cos^2 x dx=\int \cos^2 x\sin x dx=-\int u^2 du=-\frac{u^3}{3}+c=-\frac{\cos^3 x}{3}+c$$

2. $\int \sin^2 x\cos^2 x dx=\int \frac{1}{2}(1-\cos 2x)\frac{1}{2}(1+\cos 2x)dx$

$$=\frac{1}{4}\int\left(1+\cos 2x-\cos 2x-\cos^2 2x\right)dx$$

$$=\frac{1}{4}\int\left(1-\cos^2 2x\right)dx$$

$$=\frac{1}{4}\int\left(1-\frac{1}{2}(1+\cos 4x)\right)dx,$$

since $\cos^2 x=\frac{1}{2}(1+\cos 2x)\Rightarrow \cos^2 2x=\frac{1}{2}(1+\cos 4x)$

$$=\frac{1}{4}\int\left(1-\frac{1}{2}-\frac{1}{2}\cos 4x\right)dx$$

$$=\frac{1}{4}\int\left(\frac{1}{2}-\frac{1}{2}\cos 4x\right)dx=\frac{1}{8}x-\frac{1}{32}\sin 4x+c$$

Synthesis

To integrate an integral of the form $\int \sin^m x\cos^n x\, dx$, we have two cases:

a) If m or $\boldsymbol{n}$ is odd, save one cosine factor (or one sine factor) and use the relation $\cos^2 x=1-\sin^2 x$ (or $\sin^2 x=1-\cos^2 x$). Let $u=\sin x\Rightarrow du=\cos x dx$ (or let $u=\cos x\Rightarrow du=-\sin x dx$).

b) If m and n are even, we use the identities:

$\sin^2 x=\frac{1}{2}(1-\cos 2x)$ and $\cos^2 x=\frac{1}{2}(1+\cos 2x)$.

Application Activity 4.72

1. $-\frac{1}{4}\cos^4 x+c$
2. $\frac{1}{10}\sin^5 2x+c$
3. $\frac{1}{3}\cos^3 x-\cos x+c$
4. $-\frac{1}{12}\sin^3 4x+\frac{1}{4}\sin 4x+c$
5. $\frac{1}{6}\cos^6 x-\frac{1}{4}\cos^4 x+c$
6. $-\frac{1}{16}\cos^8 2x+\frac{1}{6}\cos^6 2x-\frac{1}{8}\cos^4 2x+c$

Lesson 4.11. Integrals of the form $\int \tan^m x \sec^n x\, dx$

Learning objectives

Given an integral of the form $\int \tan^m x \sec^n x\, dx$, learners should be able to find primitive function accurately.

Prerequisites

- Derivative of $\sec x$.
- Identity $\sec^2 x = 1+\tan^2 x$.

Teaching Aids

Exercise book and pen

Activity 4.11

In this lesson, the following generic competence and cross-cutting issues are to be addressed:

- Critical thinking
- Communication
- Self confidence
- Cooperation, interpersonal management and life skills
- Peace and values education
- Inclusive education

Answers

1. $u=\tan x \Rightarrow du=\left(1+\tan^2 x\right)dx=\left(1+u^2\right)dx$

   $\Rightarrow dx=\frac{du}{1+u^2}$

$$\int \tan^2 x dx = \int \frac{u^2}{1+u^2} du$$
$$= \int \left(1 - \frac{1}{1+u^2}\right) du$$
$$= u - \arctan u + c$$
$$= \tan x - \arctan(\tan x) + c$$
$$= \tan x - x + c$$

2. $u = \sec x \Rightarrow du = \sec x \tan x dx$

$$dx = \frac{du}{\sec x \tan x} = \frac{du}{u \tan x}$$
$$\int \tan^5 x dx = \int \tan^4 x \tan x dx = \int (\tan^2 x)^2 \tan x dx$$
$$= \int (\sec^2 x - 1)^2 \tan x dx$$
$$= \int (\sec^4 x - 2\sec^2 x + 1) \tan x dx$$
$$= \int (u^4 - 2u^2 + 1) \tan x \frac{du}{u \tan x}$$
$$= \int \frac{u^4 - 2u^2 + 1}{u} du$$
$$= \int \left(u^3 - 2u + \frac{1}{u}\right) du$$
$$= \frac{u^4}{4} - u^2 + \ln|u| + c = \frac{\sec^4 x}{4} - \sec^2 x + \ln|\sec x| + c$$

Synthesis

Integration of the form $\int \tan^m x \sec^n x\, dx$, is in two types:

a) If the power of secant is even, save a factor of $\sec^2 x$ and use $\sec^2 x = 1 + \tan^2 x$ to express the remaining factors in term of $\tan x$. Then substitute $u = \tan x$.

b) If the power of tangent is odd, save a factor of $\sec x \tan x$ and use $\tan^2 x = \sec^2 x - 1$ to express the remaining factors in terms of $\sec x$. Then substitute $u = \sec x$.

Application Activity 4.73

1. $\frac{1}{2}\sec^2 x + c$
2. $\frac{1}{4}\ln\left(\frac{1-\sin x}{1+\sin x}\right) + \frac{1}{2}\sec x \tan x + c$
3. $\frac{1}{3}\sec^3 x + c$
4. $-\frac{1}{3}\sec^3 x + \frac{1}{5}\sec^5 x + c$
5. $\frac{1}{3}\tan^3 x + c$
6. $\frac{1}{5}\tan^5 x + \frac{1}{3}\tan^3 x + c$

Lesson 4.12. Integrals containing $\sin x, \cos x, \tan x$ on denominator

Learning objectives

Given a function containing $\sin x, \cos x, \tan x$ on denominator, learners should be able to find primitive function moderately.

Prerequisites

- Identities

$$\sin x = \frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}},\ \cos x = \frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}},\ \tan x = \frac{2\tan\frac{x}{2}}{1-\tan^2\frac{x}{2}}$$

from double angle formulae.

Teaching Aids

Exercise book and pen

Activity 4.12

In this lesson, the following generic competence and cross-cutting issues are to be addressed:

- Critical thinking
- Communication
- Self confidence
- Cooperation, interpersonal management and life skills
- Peace and values education
- Inclusive education

Answers

$$\sin x + \cos x + 1 = \frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}} + \frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}} + 1$$

$$= \frac{2\tan\frac{x}{2} + 1 - \tan^2\frac{x}{2} + 1 + \tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}} = \frac{2\tan\frac{x}{2}+2}{1+\tan^2\frac{x}{2}}$$

$$\frac{1}{\sin x + \cos x + 1} = \frac{1+\tan^2\frac{x}{2}}{2\tan\frac{x}{2}+2}$$

Let $u = \tan\frac{x}{2} \Rightarrow \arctan u = \frac{x}{2} \Rightarrow \frac{du}{1+u^2} = \frac{dx}{2} \Rightarrow dx = \frac{2du}{1+u^2}$

$$\int \frac{1}{\sin x + \cos x + 1}\,dx = \int \frac{1+u^2}{2u+2} \times \frac{2du}{1+u^2}$$

$$= \int \frac{du}{u+1}$$

$$= \ln|u+1| + c$$

$$= \ln\left|\tan\frac{x}{2} + 1\right| + c$$

Synthesis

To find an integral containing $\sin x,\ \cos x,\ \tan x$ on denominator, use the formulae

$$\sin x = \frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}},\ \cos x = \frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}},\ \tan x = \frac{2\tan\frac{x}{2}}{1-\tan^2\frac{x}{2}}$$

and let $u = \tan\frac{x}{2} \Rightarrow \arctan u = \frac{x}{2}$

Application Activity 4.74

1. $\frac{2}{\sqrt{3}}\arctan\left(\frac{1}{\sqrt{3}}\tan\frac{x}{2}\right)+c$ 2. $\frac{2}{\sqrt{5}}\arctan\left(\frac{3\tan\frac{x}{2}-2}{\sqrt{5}}\right)+c$

3. $\frac{1}{\sqrt{3}}\arctan\left(\frac{2\tan x+1}{\sqrt{3}}\right)+c$ 4. $\frac{1}{2}\arctan\left(2\tan\frac{x}{2}\right)+c$

5. $\frac{2}{\sqrt{11}}\arctan\left(\frac{3\tan\frac{x}{2}+4}{\sqrt{11}}\right)+c$ 6. $-\frac{1}{\tan\frac{x}{2}-2}+c$

Lesson 4.13. Integrals containing $\sin^2 x, \cos^2 x$ on denominator

Learning objectives

Given an integral containing $\sin^2 x, \cos^2 x$ on denominator, learners should be able to find primitive function accurately.

Prerequisites

- Identities $\cos x = \frac{1}{\sqrt{1+\tan^2 x}}$ and $\sin x = \frac{\tan x}{\sqrt{1+\tan^2 x}}$.

Teaching Aids

Exercise book and pen

Activity 4.13

In this lesson, the following generic competence and cross-cutting issues are to be addressed:

- Critical thinking
- Communication
- Self confidence
- Cooperation, interpersonal management and life skills
- Peace and values education
- Inclusive education

Answers

We know that $\sec^2 x = 1 + \tan^2 x$ or $\sec x = \pm\sqrt{1+\tan^2 x}$. Then,

$$\cos x = \pm\frac{1}{\sqrt{1+\tan^2 x}} \text{ and } \sin x = \pm\frac{\tan x}{\sqrt{1+\tan^2 x}}$$

Now, $\dfrac{1}{\cos^2 x} = \dfrac{1}{\left(\pm\dfrac{1}{\sqrt{1+\tan^2 x}}\right)^2} = \dfrac{1}{\dfrac{1}{1+\tan^2 x}} = 1+\tan^2 x$

$$\int \frac{1}{\cos^2 x}dx = \int \left(1+\tan^2 x\right)dx$$

Let $u = \tan x \Rightarrow x = \arctan u \Rightarrow dx = \dfrac{du}{1+u^2}$

$$\int \frac{1}{\cos^2 x}dx = \int \left(1+u^2\right)\frac{du}{1+u^2} = \int du = u + c = \tan x + c$$

Synthesis

To integrate an integral containing $\sin^2 x, \cos^2 x$ on denominator, use identities $\cos x = \dfrac{1}{\sqrt{1+\tan^2 x}}$ and $\sin x = \dfrac{\tan x}{\sqrt{1+\tan^2 x}}$ and let $u = \tan x \Rightarrow x = \arctan u$

Application Activity 4.75

1) $\frac{1}{3}\tan^3 x + \tan x + c$

2) $\frac{1}{5}\tan^5 x + \frac{2}{3}\tan^3 x + \tan x + c$

3) $-\cot x - \frac{2}{3}\cot^3 x - \frac{1}{5}\cot^5 x + c$

4) $\frac{1}{7}\tan^7 x + \frac{3}{5}\tan^5 x + \tan^3 x + \tan x + c$

Lesson 4.14. Integrals containing $\sqrt[n]{ax+b}$

Learning objectives

Given an integral containing $\sqrt[n]{ax+b}$, learners should be able to find primitive function accurately.

Prerequisites

- Properties of radicals.

Teaching Aids

Exercise book and pen

Activity 4.14

In this lesson, the following generic competence and cross-cutting issues are to be addressed:

- Critical thinking
- Communication
- Self confidence
- Cooperation, interpersonal management and life skills
- Peace and values education
- Inclusive education

Answers

$u^2 = 3x - 1 \Rightarrow u = \sqrt{3x-1}$

$\Rightarrow 2udu = 3dx \Rightarrow dx = \dfrac{2udu}{3}$

$$\int \sqrt{3x-1}\,dx = \int u\frac{2udu}{3} = \frac{2}{3}\int u^2 du$$

$$= \frac{2u^3}{9} + c = \frac{2}{9}\left(\sqrt{3x-1}\right)^3 + c$$

$$= \frac{2}{9}\left(\sqrt{3x-1}\right)^2 \sqrt{3x-1} + c = \frac{2}{9}(3x-1)\sqrt{3x-1} + c$$

Synthesis

For integral containing $\sqrt[n]{ax+b},\ a \neq 0$, let $u^n = ax+b$

Application Activity 4.76

1. $\frac{1}{3}(2x+1)\sqrt{6x+3}+c$
2. $\frac{2}{25}(5x-2)^2\sqrt{5x-2}+c$
3. $\frac{3}{16}\sqrt[3]{(8x+1)^2}+c$
4. $\dfrac{2}{3\sqrt{2-3x}}+c$
5. $\frac{1}{3}\sqrt{(2x+5)^3}+c$
6. $\frac{1}{4}(3x-8)\sqrt[3]{3x-8}+c$
7. $4\sqrt{2x+3}+c$
8. $\dfrac{4}{3\sqrt{1-3x}}+c$

Lesson 4.15. Integrals containing $\sqrt{ax^2+bx+c}$

Learning objectives

Given an integral containing $\sqrt{ax^2+bx+c}$, learners should be able to find primitive function accurately.

Prerequisites

- Use of the relation $ax^2+bx+c=a\left[\left(x+\frac{b}{2a}\right)^2-\frac{b^2-4ac}{4a^2}\right]$.

Teaching Aids

Exercise book and pen

Activity 4.15

In this lesson, the following generic competence and cross-cutting issues are to be addressed:

- Critical thinking
- Communication
- Self confidence
- Cooperation, interpersonal management and life skills
- Peace and values education
- Inclusive education

Answers

The relation to be used is

$$ax^2+bx+c=a\left[\left(x+\frac{b}{2a}\right)^2-\frac{b^2-4ac}{4a^2}\right]$$

1. $\int\frac{dx}{\sqrt{x^2-2x+1}}$

$$x^2-2x+1=(x-1)^2-\frac{(-2)^2-4\times 1}{4}=(x-1)^2$$

Let $u=x-1\Rightarrow du=dx$

$$\int\frac{dx}{\sqrt{x^2-2x+1}}=\int\frac{du}{\sqrt{u^2}}=\int\frac{du}{u}$$

Using formula $\int\frac{u'}{u}du=\ln|u|+d$, we have

$$\int\frac{dx}{\sqrt{x^2-2x+1}}=\int\frac{du}{u}=\ln|u|+d=\ln|x-1|+d$$

2. $\int \frac{dx}{\sqrt{x^2 - 5x + 6}}$

$$x^2 - 5x + 6 = \left(x - \frac{5}{2}\right)^2 - \frac{(-5)^2 - 4 \times 6}{4} = \left(x - \frac{5}{2}\right)^2 - \frac{1}{4}$$

Let $u = x - \frac{5}{2} \Rightarrow du = dx$

$$\int \frac{dx}{\sqrt{x^2 - 5x + 6}} = \int \frac{du}{\sqrt{u^2 - \frac{1}{4}}}$$

Using formula $\int \frac{dx}{\sqrt{x^2 \pm k^2}} = \ln\left|x + \sqrt{x^2 \pm k^2}\right| + d$,

we have

$$\int \frac{dx}{\sqrt{x^2 - 5x + 6}} = \int \frac{du}{\sqrt{u^2 - \frac{1}{4}}}$$

$$= \ln\left|u + \sqrt{u^2 - \frac{1}{4}}\right| + d = \ln\left|x - \frac{5}{2} + \sqrt{\left(x - \frac{5}{2}\right)^2 - \frac{1}{4}}\right| + d$$

$$= \ln\left|\frac{2x - 5 + 2\sqrt{x^2 - 5x + 6}}{2}\right| + d$$

$$= \ln\left|2x - 5 + 2\sqrt{x^2 - 5x + 6}\right| + e$$

3. $\int \frac{dx}{\sqrt{x^2 - 6x + 18}}$

$$x^2 - 6x + 18 = (x - 3)^2 - \frac{(-6)^2 - 4 \times 16}{4}$$

$$= (x - 3)^2 + 9$$

Let $u = x - 3 \Rightarrow du = dx$

$$\int \frac{dx}{\sqrt{x^2 - 5x + 6}} = \int \frac{du}{\sqrt{u^2 + 9}}$$

Using formula $\int \frac{dx}{\sqrt{x^2 \pm k^2}} = \ln\left|x + \sqrt{x^2 \pm k^2}\right| + d$,

we have

$$\int \frac{dx}{\sqrt{x^2 - 6x + 18}} = \int \frac{du}{\sqrt{u^2 + 9}}$$
$$= \ln\left|u + \sqrt{u^2 + 9}\right| + d$$
$$= \ln\left|x - 3 + \sqrt{(x-3)^2 + 9}\right| + d$$
$$= \ln\left|x - 3 + \sqrt{x^2 - 6x + 18}\right| + d$$

Synthesis

For the integrals of the form $\int \frac{dx}{\sqrt{ax^2 + bx + c}}$,

first, transform $ax^2 + bx + c$ in the form $a\left[\left(x + \frac{b}{2a}\right)^2 - \frac{b^2 - 4ac}{4a^2}\right]$ and then;

- If $b^2 - 4ac = 0$, then

$$\int \frac{dx}{\sqrt{ax^2 + bx + c}} = \frac{1}{\sqrt{a}} \int \frac{dx}{x + \frac{b}{2a}}$$ and we let $u = x + \frac{b}{2a}$

- If $b^2 - 4ac > 0$, then

$$\int \frac{dx}{\sqrt{ax^2 + bx + c}} = \frac{1}{\sqrt{a}} \int \frac{dx}{\sqrt{\left(x + \frac{b}{2a}\right)^2 + \frac{b^2 - 4ac}{4a^2}}}$$

We let $u = x + \frac{b}{2a}$, $k^2 = \frac{b^2 - 4ac}{4a^2}$ finally, use the integral

$$\int \frac{dx}{\sqrt{x^2 \pm k^2}} = \ln\left|x + \sqrt{x^2 \pm k^2}\right| + d$$

- If $b^2 - 4ac < 0$, then

$$\int \frac{dx}{\sqrt{ax^2 + bx + c}} = \frac{1}{\sqrt{a}} \int \frac{dx}{\sqrt{\left(x + \frac{b}{2a}\right)^2 - \frac{b^2 - 4ac}{4a^2}}}$$

We let $u = x + \frac{b}{2a}$, $k^2 = -\frac{b^2 - 4ac}{4a^2}$ and use the integral

$$\int \frac{dx}{\sqrt{x^2 \pm k^2}} = \ln\left|x + \sqrt{x^2 \pm k^2}\right| + d$$

Application Activity 4.77

1. $\ln\left|x + 1 + \sqrt{x^2 + 2x + 5}\right| + c$
2. $\arcsin\frac{x+1}{\sqrt{5}} + c$
3. $\ln\left|x + 2 + \sqrt{x^2 + 4x + 2}\right| + c$
4. $\arcsin\frac{x-3}{2} + c$
5. $\frac{2x-1}{4}\sqrt{x - x^2} + \frac{1}{8}\arcsin(2x-1) + c$

Lesson 4.16. Integration by parts

Learning objectives

Through examples, learners should be able to integrate by parts accurately.

Prerequisites

Product rule differentiation.

Teaching Aids

Exercise book and pen

Activity 4.16

In this lesson, the following generic competence and cross-cutting issues are to be addressed:

- Critical thinking
- Communication
- Self confidence
- Cooperation, interpersonal management and life skills
- Peace and values education
- Inclusive education

Answers

1. $\frac{d}{dx}f(x)=e^x\frac{d}{dx}(x-1)+(x-1)\frac{d}{dx}e^x$

$$=e^x+(x-1)e^x=xe^x$$

2. From 1), $\int xe^x dx=(x-1)e^x+c$

3. Let $u=x, v=e^x$,

Thus, $\int uvdx=\int xe^x dx$ while $\int udx\int vdx=\int xdx\int e^x dx$

From 2) we have

$$\int uvdx=\int xe^x dx \ =(x-1)e^x+c$$

Let us find $\int udx\int vdx$:

$$\int udx\int vdx=\int xdx\int e^x dx \ =\frac{x^2}{2}e^x+c$$

Therefore, $\int uvdx\neq\int udx\int vdx$

Synthesis

Integration by parts use the formula $\int udv=uv-\int vdu$

The following table can be used:

u	v'
Logarithmic function	Polynomial function
Polynomial function	Exponential function
Polynomial function	Trigonometric function
Exponential function	Trigonometric function
Trigonometric function	Exponential function
Inverse trigonometric function	Polynomial function

Application Activity 4.78

1. $\frac{1}{2}x\sin 2x+\frac{1}{4}\cos 2x+c$
2. $\frac{1}{9}(3x-1)e^{3x}+c$
3. $\frac{1}{16}\sin 4x-\frac{1}{4}x\cos 4x+c$
4. $\frac{1}{9}x^3(3\ln x-1)+c$
5. $(x+1)e^{2x}+c$
6. $-\frac{1}{4}(2x+1)e^{-2x}+c$

Lesson 4.17. Integration by reduction formulae

Learning objectives

Given integral I_m and by using integration by parts, learners should be able to find a reduction formula for I_m rightfully.

Prerequisites

- Integration by parts.

Teaching Aids

Exercise book and pen

Activity 4.17

In this lesson, the following generic competence and cross-cutting issues are to be addressed:

- Critical thinking
- Communication
- Self confidence
- Cooperation, interpersonal management and life skills
- Peace and values education
- Inclusive education

Answers

$I_m = \int x^m \cos bx\,dx,\; J_m = \int x^m \sin bx\,dx$

For $J_m = \int x^m \sin bx\,dx$, let $u = x^m \Rightarrow du = mx^{m-1}dx$

$$dv = \sin bx\,dx \Rightarrow v = -\frac{1}{b}\cos bx$$

$$J_m = -\frac{x^m \cos bx}{b} - \int\left(-\frac{1}{b}\cos bx\right)mx^{m-1}dx$$

$$\Rightarrow J_m = -\frac{x^m \cos bx}{b} + \frac{m}{b}\underbrace{\int \cos bx x^{m-1}dx}_{I_{m-1}}$$

$$\Rightarrow J_m = -\frac{x^m \cos bx}{b} + \frac{m}{b}I_{m-1}$$

$$\Rightarrow bJ_m = -x^m \cos bx + mI_{m-1} \Rightarrow bJ_m - mI_{m-1} = -x^m \cos bx$$

Synthesis

Knowing integral I_m, we can establish a general relation, integration by parts, which will help us to reduce the power and find $I_{m-1}, I_{m-2}, I_{m-3}, ..., I_0$.

Application Activity 4.79

1. $\frac{1}{a}x^n e^{ax} - \frac{n}{a}I_{n-1}$
2. $I_n = \frac{\tan^{n-1}x}{n-1} - I_{n-2}$ and

   $\int \tan^5 x\,dx = I_5 = \frac{\tan^4 x}{4} - \frac{\tan^2 x}{2} + \ln|\sec x| + c$
3. $I_n = -\frac{1}{n}\sin^{n-1}x\cos x + \frac{n-1}{n}I_{n-2}$
4. $I_n = \frac{1}{n}\cos^{n-1}x\sin x + \frac{n-1}{n}I_{n-2}$
5. $I_n = x(\ln x)^n - nI_{n-1}$

Lesson 4.18. Integration by Maclaurin series

Learning objectives

Using Maclaurin series, learners should be able to find primitive functions of some functions accurately.

Prerequisites

- Maclaurin series of a function.

Teaching Aids

Exercise book and pen

Activity 4.18

In this lesson, the following generic competence and cross-cutting issues are to be addressed:

- Critical thinking
- Communication
- Self confidence
- Cooperation, interpersonal management and life skills

- Peace and values education
- Inclusive education

Answers

1. $\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \ldots$
2. $\int \ln(1+x)\,dx = \int\left(x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \ldots\right)dx$

$$= \frac{1}{2}x^2 - \frac{1}{6}x^3 + \frac{1}{12}x^4 - \frac{1}{20}x^5 + \ldots + c$$

Synthesis

For some integrals, we proceed also by Maclaurin series of the function to be integrated.

Application Activity 4.80

1. $\int e^{-3x}dx = \int\left(1 - 3x + \frac{9}{2}x^2 - \frac{9}{2}x^3 + \ldots\right)dx = x - \frac{3}{2}x^2 + \frac{3}{2}x^3 - \frac{9}{8}x^4 + \ldots + c$
2. $\int \sin x\,dx = \int\left(x - \frac{1}{6}x^3 + \frac{1}{120}x^5 + \ldots\right)dx = \frac{1}{2}x^2 - \frac{1}{24}x^4 + \frac{1}{720}x^6 + \ldots + c$
3. $\int \cos x\,dx = \int\left(1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \frac{1}{720}x^6 + \ldots\right)dx = x - \frac{1}{6}x^3 + \frac{1}{120}x^5 - \frac{1}{5040}x^7 + \ldots + c$
4. $\int \tan x\,dx = \int\left(x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \ldots\right)dx = \frac{1}{2}x^2 + \frac{1}{12}x^4 + \frac{1}{45}x^6 + \ldots + c$
5. $\int \sqrt{1+x}\,dx = \int\left(1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 + \ldots\right)dx = x + \frac{1}{4}x^2 - \frac{1}{24}x^3 + \frac{1}{64}x^4 + \ldots + c$

Lesson 4.19. Definite integrals

Learning objectives

By the end of this lesson, learners should be able to define definite integrals.

Prerequisites

- Indefinite integrals.

Teaching Aids

Exercise book, calculator and pen

Activity 4.19

In this lesson, the following generic competence and cross-cutting issues are to be addressed:

- Critical thinking
- Communication
- Self confidence
- Cooperation, interpersonal management and life skills
- Peace and values education
- Inclusive education

Answers

1. $F(x)=\int(x^2-2x+3)\,dx=\frac{1}{3}x^3-x^2+3x+c$

2. $$\begin{aligned}F(1)-F(-1)&=\left[\frac{1}{3}(1)^3-(1)^2+3(1)+c\right]\\&-\left[\frac{1}{3}(-1)^3-(-1)^2+3(-1)+c\right]\\&=\left(\frac{1}{3}-1+3+c\right)-\left(-\frac{1}{3}-1-3+c\right)\\&=\frac{1}{3}+2+c+\frac{1}{3}+4-c\\&=\frac{20}{3}\end{aligned}$$

Synthesis

The area S_i of the strip between x_{i-1} and x_i is approximately equal to the area of a rectangle with width $l=\Delta x$ and length $L=f(x_i)$ i.e. as illustrated in figure 4.1.

The total area A is $\sum_{i=1}^{n}S_i=\sum_{i-1}^{n}f(x_i)\cdot\Delta x$ or $A=\lim_{n\to\infty}\sum_{i=1}^{n}f(x_i)\cdot\Delta x$; this is known as **Sum of Riemann**.

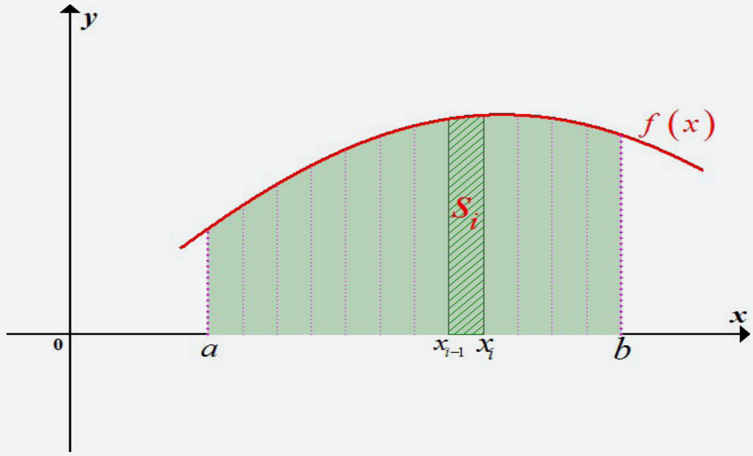


Figure 4.1: Definite integral of the function

We define the definite integrals of the function $f(x)$ with respect to x from a to b to be

$\int_a^b f(x)dx = [F(x)]_a^b = F(b) - F(a)$.Where $F(x)$ is the anti-derivative of $f(x)$.

Application Activity 4.81

1. $\frac{9}{2}$
2. $\frac{5}{6}$
3. -2
4. $\frac{3}{4}$
5. $\frac{46}{3}$
6. 4
7. 25
8. 145

Lesson 4.20. Properties of definite integrals

Learning objectives

Through examples, learners should be able to use properties of definite integrals accurately.

Prerequisites

- Definition of definite integrals.

Teaching Aids

Exercise book, calculator and pen

Activity 4.20

In this lesson, the following generic competence and cross-cutting issues are to be addressed:

- Critical thinking
- Communication
- Self confidence
- Cooperation, interpersonal management and life skills
- Peace and values education
- Inclusive education

Answers

1. $\int_{-3}^{0} f(x)\,dx = \int_{-3}^{0} x^2 dx = \left[\frac{x^3}{3}\right]_{-3}^{0} = 9$

   $\int_{0}^{-3} f(x)\,dx = \int_{0}^{-3} x^2 dx = \left[\frac{x^3}{3}\right]_{0}^{-3} = -9$

   Observation: $\int_{-3}^{0} f(x) = -\int_{0}^{-3} f(x)\,dx$

2. $\int_{-2}^{2} x^2 dx = \left[\frac{1}{3}x^3\right]_{-2}^{2} = \frac{8}{3} + \frac{8}{3} = \frac{16}{3}$

   $\int_{-2}^{0} x^2 dx + \int_{0}^{2} x^2 dx = \left[\frac{1}{3}x^3\right]_{-2}^{0} + \left[\frac{1}{3}x^3\right]_{0}^{2} = 0 + \frac{8}{3} + \frac{8}{3} - 0 = \frac{16}{8}$

   Observation:

   $\int_{-2}^{2} x^2 dx = \int_{-2}^{0} x^2 dx + \int_{0}^{2} x^2 dx$

Synthesis

- **Permutation of bounds**: If $f(x)$ is defined on (a,b) except may be at a finite number of points, then

  $\int_a^b f(x)dx = -\int_b^a f(x)dx$

- **Chasles relation:** For any arbitrary numbers a and b and any $c \in [a,b]$

  $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$

- **Positivity**: Let f be a continuous function on interval $I=[a,b]$ the elements of I

  If $f \ge 0$ on I and if $a \le b$ then $\int_a^b f(x)\,dx \ge 0$

  Also, if $f(x) \le g(x)$ on $[a,b]$, then, $\int_a^b f(x)dx \le \int_a^b g(x)dx$

Application Activity 4.82

1. 4 2. $\frac{7}{3}$ 3. $\frac{1}{32}\pi^2$ 4. $\frac{1}{2}\pi - 1$

Lesson 4.21. Improper integrals, Infinite limits of integration

Learning objectives

Given an improper integral with infinite limits, learners should be able to determine whether it converges or diverges correctly.

Prerequisites

- Limits concepts.

Teaching Aids

Exercise book and pen

Activity 4.21

In this lesson, the following generic competence and cross-cutting issues are to be addressed:

- Critical thinking
- Communication
- Self confidence
- Cooperation, interpersonal management and life skills
- Peace and values education
- Inclusive education

Answers

1. $\lim\limits_{n\to+\infty}\int_0^n \frac{dx}{x^2+4} = \lim\limits_{n\to+\infty}\left[\frac{1}{2}\arctan\left(\frac{1}{2}x\right)\right]_0^n$

$$= \lim_{n\to+\infty}\left[\frac{1}{2}\arctan\left(\frac{1}{2}n\right)\right]$$

$$=\frac{1}{4}\pi$$

2. $$\lim_{n\to-\infty}\int_n^{-4}\frac{xdx}{\sqrt{1+3x^2}} = \lim_{n\to-\infty}\left[\frac{1}{3}\sqrt{3x^2+1}\right]_n^{-4}$$

$$= \lim_{n\to-\infty}\frac{-\sqrt{3n^2+1}+7}{3}$$

$$=-\infty$$

Synthesis

We define the improper integral as $\int_a^{+\infty} f(x)dx = \lim_{n\to+\infty}\int_a^n f(x)dx$ or $\int_{-\infty}^b f(x)dx = \lim_{n\to-\infty}\int_n^b f(x)dx$

If the limit exists, we say that the integral converges, otherwise it diverges.

Application Activity 4.83

1) Convergent to π

2) Convergent to $\frac{1}{2}$

3) Convergent to $\frac{1}{4}\pi$

4) Convergent to $\frac{1}{2}$

5) divergent

6) Convergent to 0

Lesson 4.22. Discontinuous integrand

Learning objectives

Given an improper integral with discontinuous integrand, learners should be able to determine whether it converges or diverges correctly.

Prerequisites

- Limits concepts.

Teaching Aids

Exercise book, calculator and pen

Activity 4.22

In this lesson, the following generic competence and cross-cutting issues are to be addressed:

- Critical thinking
- Communication
- Self confidence
- Cooperation, interpersonal management and life skills
- Peace and values education
- Inclusive education

Answers

1. Since this function is a rational function, the denominator cannot be zero. Then, $x-1 \neq 0$ or $x \neq 1$. Thus, considering the given interval, the given function is discontinuous at $x=1$.
2. Since this function is a rational function, the denominator cannot be zero. Then, $x^2-3x-10 \neq 0$ or $x \neq -2$ and $x \neq 5$. Thus, considering the given interval, the given function is discontinuous at $x=-2$.
3. Since there is natural logarithm, then, $x>0$ also $\ln x \neq 0$ or $x \neq 1$. Thus, considering the given interval, the given function is discontinuous at $x=1$.

Synthesis

For a function $f(x)$ which is continuous on the interval $[a,b[$, we define the improper integral as $\int_a^b f(x)\,dx = \lim_{t\to b}\int_a^t f(x)\,dx$.

Also, if $f(x)$ is continuous on the interval $]a,b]$, we have the improper integral $\int_a^b f(x)\,dx = \lim_{t\to a}\int_t^b f(x)\,dx$.

If $f(x)$ is a continuous function for all real numbers x in the interval $]a,b[$, except for some point $c \in]a,b[$, then,

$$\begin{aligned}\int_a^b f(x)dx &= \int_a^c f(x)dx + \int_c^b f(x)dx \\ &= \lim_{t\to c}\int_a^t f(x)dx + \lim_{t\to c}\int_t^b f(x)dx\end{aligned}$$

Application Activity 4.84

1. Diverges
2. a) converges to 3 b) diverges

   c) diverges d) converges to $5+5\sqrt[5]{2}$

Lesson 4.23. Calculation of area of plane surfaces

Learning objectives

Given two functions and by using integration, learners should be able to find the area between two curves in a given interval precisely.

Prerequisites

- Curve sketching.
- Definite integrals.

Teaching Aids

Exercise book, calculator and pen

Activity 4.23

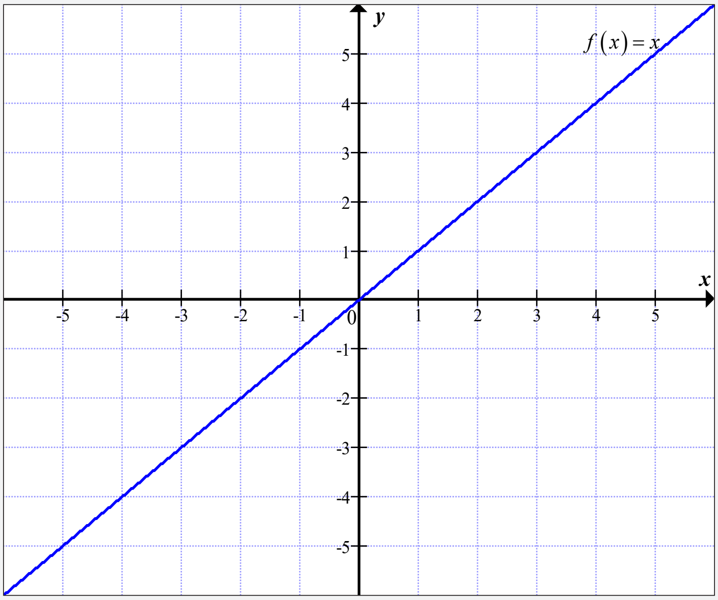
In this lesson, the following generic competence and cross-cutting issues are to be addressed:

- Critical thinking
- Communication
- Self confidence
- Cooperation, interpersonal management and life skills
- Peace and values education
- Inclusive education

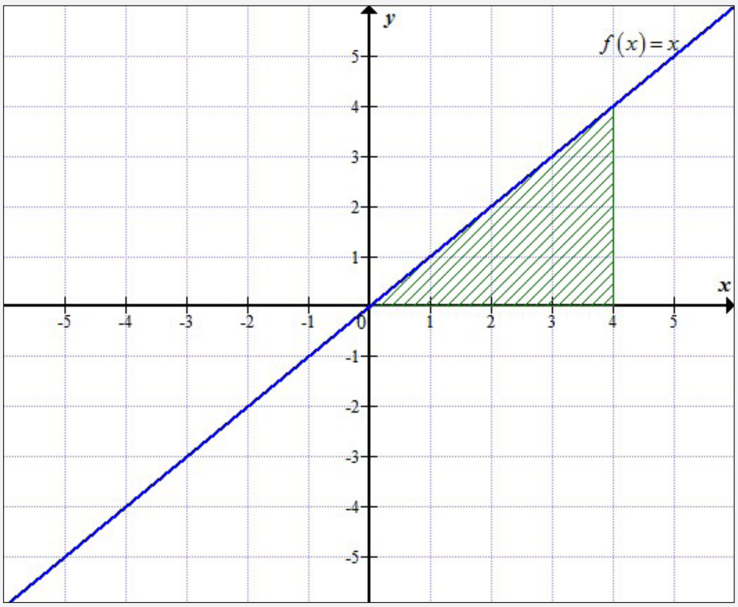
Answers

1. Curve

x	0	4
$f(x)$	0	4



2. Curve with shaded region



3. We see that the shaded region is a triangle whose base is 4 units and height is 4 units. Then, we know the area of a triangle with base B and height H is $A=\frac{B\times H}{2}$. Then, the area of the shaded region is $A=\frac{4\times 4}{2}=8\ sq.\ units$.

4. $\int_0^4 f(x)\,dx=\int_0^4 x\,dx$
$$=\left[\frac{x^2}{2}\right]_0^4$$
$$=\frac{4^2}{2}$$
$$=8$$

5. Results in 3) and 4) are the same.

Synthesis

Given function $f(x)$ which lies above the x-axis, the area enclosed by the curve of $f(x)$ and x-axis in interval $[a,b]$ is given by;

$$A=\int_a^b f(x)\,dx$$

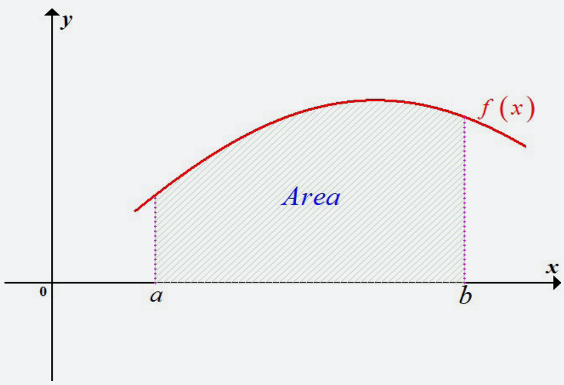


Figure 4.2: Area enclosed by a curve of a function and x-axis

The area between two functions $f(x)$ and $g(x)$ where $f(x)\leq g(x)$ in $[a,b]$ is given by

$$\int_a^b [g(x)-f(x)]dx=\int_a^b g(x)dx-\int_a^b f(x)dx$$

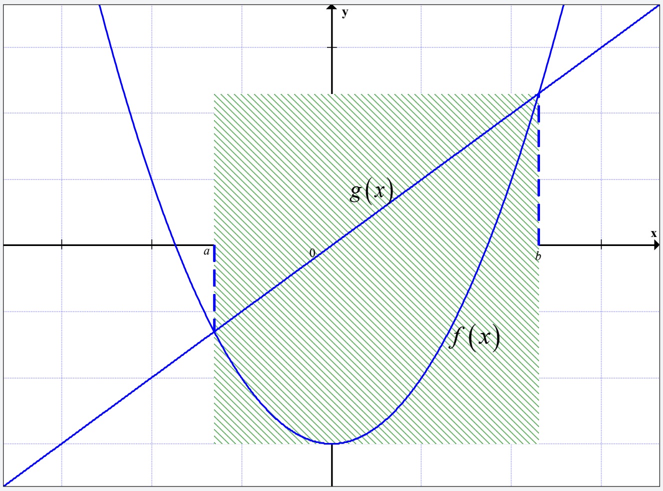


Figure 4.3: Area between two curves

Application Activity 4.85

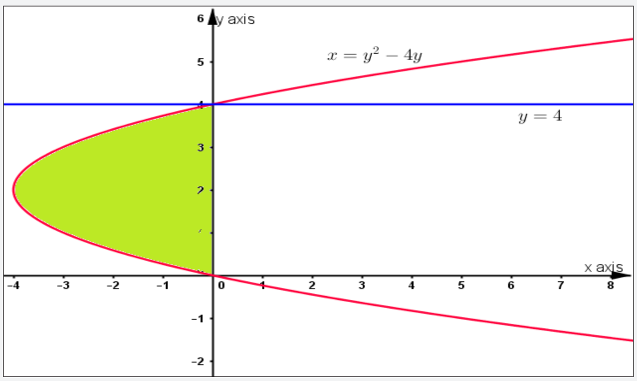
1. a) $\frac{2}{3}$ sq. units b) $\frac{1}{2}$ sq. units c) $33\frac{1}{3}$ sq. units

   d) $\frac{4}{3}$ sq. units e) 9 sq. units

2) $\frac{16}{15}a^5$ sq. units 3) $143\frac{5}{6}$ sq. units

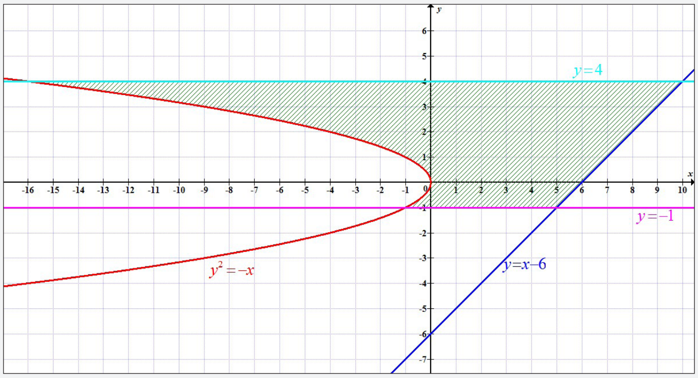
4) 3.75 sq. units 5) $4\sqrt{2}$ sq. units

6) a) Graph



Area is $\frac{32}{3}sq.units$

b) Graph



Area is $\frac{355}{6}\, sq.units$

Lesson 4.24. Calculation of volume of a solid of revolution

Learning objectives

Given a function and by using integration, learners should be able to find the volume of a solid obtained when a curve of a function is revolved around an axis precisely.

Prerequisites

- Curve sketching.
- Definite integrals.

Teaching Aids

Exercise book, calculator and pen

Activity 4.24

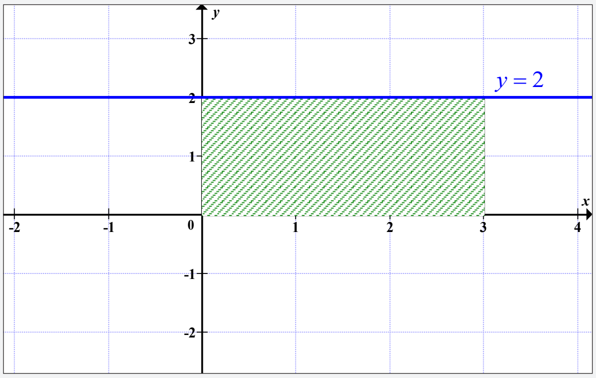
In this lesson, the following generic competence and cross-cutting issues are to be addressed:

- Critical thinking
- Communication
- Self confidence
- Cooperation, interpersonal management and life skills
- Peace and values education
- Inclusive education

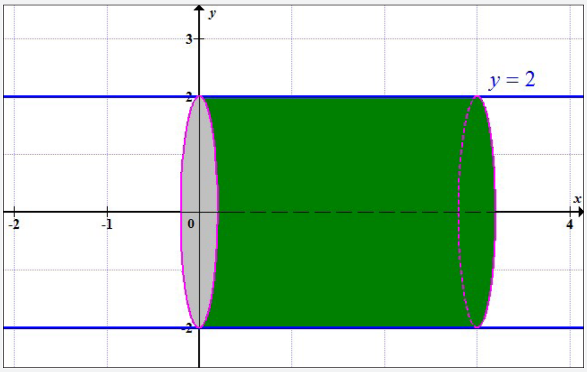
Answers

1. $y=2$ for $0\le x\le 3$

   a) The region enclosed by the curve $y=2$ for $0\le x\le 3$ and $x-axis$



   b) The region for which the area in (a) is rotated 360^0 about the x-axis



   c) Solid of revolution obtained in (b) is a cylinder of radius 2 and height 3.

      Volume of cylinder is

      $\pi r^2 h=\pi(2)^2(3)=12\pi\, cubic\, units$

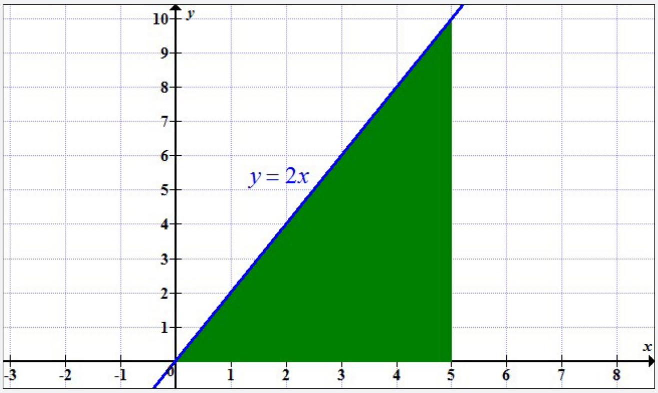
   d) Volume $V=\lim\limits_{\delta x\to 0}\sum\limits_{x=0}^{x=3}\pi y^2\delta x=\int_0^3 \pi y^2\,dx$

      $=\int_0^3 \pi(2)^2\,dx=\int_0^3 4\pi\,dx\ =4\pi[x]_0^3=12\pi\, cubic\, units$

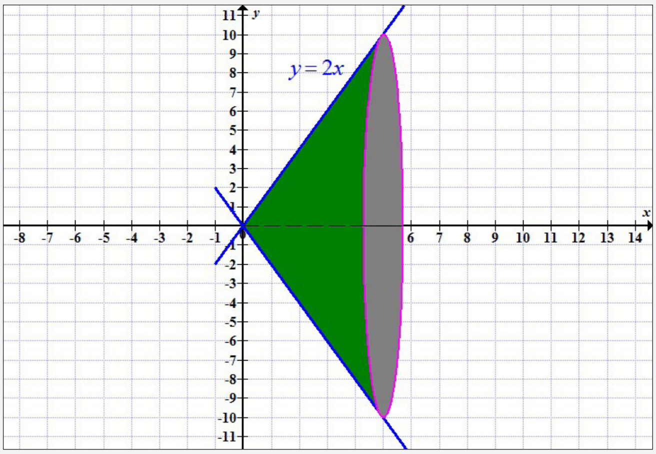
   e) The results obtained in (c) and (d) are equal.

2. $y = 2x$ for $0 \le x \le 5$

a) The region enclosed by the curve $y = 2x$ for $0 \le x \le 5$ and $x - axis$



b) The region for which the area in (a) is rotated 360^0 about the x-axis



c) Solid of revolution obtained in (b) is a cone of radius 10 and height 5.

Volume of cone is

$$\frac{1}{3}\pi r^2 h = \frac{1}{3}\pi(10)^2(5) = \frac{500}{3}\pi \, cubic\, units$$

d) Volume $V = \lim_{\delta x \to 0} \sum_{x=0}^{x=5} \pi y^2 \delta x = \int_0^5 \pi y^2 \, dx$

$$= \int_0^5 \pi (2x)^2 \, dx = \int_0^5 4\pi x^2 \, dx$$

$$= 4\pi\left[\frac{x^3}{3}\right]_0^5 = \frac{500}{3}\pi\, cubic\, units$$

e) The results obtained in (c) and (d) are equal.

Synthesis

The volume of the solid of revolution bound by the curve $f(x)$ about the $x-axis$ calculated from $x=a$ to $x=b$, is given $V = \pi\int_a^b f^2(x)dx$.

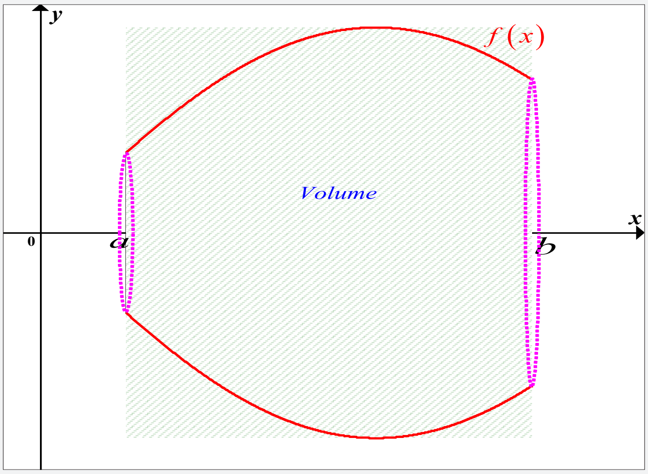


Figure 4.4: Volume of revolution

Application Activity 4.86

1. a) $\frac{32\pi}{5}$ cubic units b) $\frac{373\pi}{14}$ cubic units

   c) $\frac{1296\pi}{5}$ cubic units

2. a) $\frac{3\pi}{5}$ cubic units b) 8π cubic units

   c) 2π cubic units

Application Activity 4.87

1) $\frac{1}{243}\left(85\sqrt{85}-8\right)$ cubic units

2) $\frac{1}{27}\left(80\sqrt{10}-13\sqrt{13}\right)$ cubic units

3) $\frac{17}{6}$ cubic units

4) $\frac{1}{27}\left(13\sqrt{13}+80\sqrt{10}-16\right)$ cubic units

Lesson 4.25. Calculation of arc length of a curved surface

Learning objectives

Given a function and by using integration, learners should be able to find arc length of a curve in a given interval precisely.

Prerequisites

- Curve sketching.
- Definite integrals.

Teaching Aids

Exercise book, calculator and pen

Activity 4.25

In this lesson, the following generic competence and cross-cutting issues are to be addressed:

- Critical thinking
- Communication
- Self confidence
- Cooperation, interpersonal management and life skills
- Peace and values education
- Inclusive education

Answers

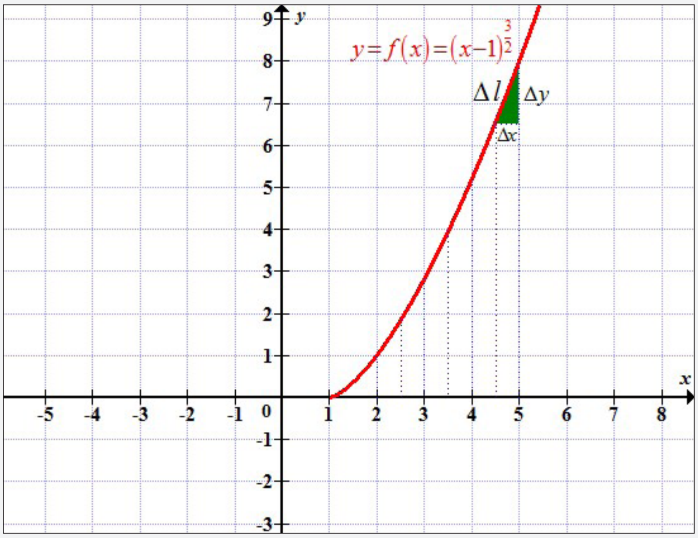


Figure 4.5: Arc length of a curved line

1. $(\Delta l)^2 = (\Delta x)^2 + (\Delta y)^2$

$$\Delta l = \sqrt{(\Delta x)^2 + (\Delta y)^2}$$

2. $\Delta l = \sqrt{(\Delta x)^2 + (\Delta y)^2} \Rightarrow dl = \sqrt{(dx)^2 + (dy)^2}$

$$dl = \sqrt{(dx)^2 + (dy)^2}$$

$$= \sqrt{(dx)^2 \left(1 + \frac{(dy)^2}{(dx)^2}\right)}$$

$$= \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

We recognise the ratio inside the square root as the derivative, $\dfrac{dy}{dx} = f'(x)$, then we can rewrite this as

$$dl = \sqrt{1 + [f'(x)]^2}\, dx$$

But $f(x) = (x-1)^{\frac{3}{2}} \Rightarrow f'(x) = \frac{3}{2}(x-1)^{\frac{1}{2}}$, then,

$$dl = \sqrt{1+\left[\frac{3}{2}(x-1)^{\frac{1}{2}}\right]^2}\,dx$$

$$= \sqrt{1+\frac{9}{4}(x-1)}\,dx$$

$$= \sqrt{\frac{4+9x-9}{4}}\,dx$$

$$= \sqrt{\frac{9x-5}{4}}\,dx$$

3. $\int dl = \int_2^5 \sqrt{\frac{9x-5}{4}}\,dx \Rightarrow l = \int_2^5 \sqrt{\frac{9x-5}{4}}\,dx$

$$= \frac{1}{2}\int_2^5 \sqrt{9x-5}\,dx$$

But $\frac{1}{2}\int_2^5 \sqrt{9x-5}\,dx = \left[\frac{1}{27}(9x-5)^{\frac{3}{2}}\right]_2^5$

Then,

$$l = \frac{1}{27}\left[(9x-5)^{\frac{3}{2}}\right]_2^5$$

$$= \frac{1}{27}\left((45-5)^{\frac{3}{2}}-(18-5)^{\frac{3}{2}}\right)$$

$$= \frac{1}{27}\left((40)^{\frac{3}{2}}-(13)^{\frac{3}{2}}\right)$$

$$= \frac{1}{27}\left(\sqrt{(40)^3}-\sqrt{(13)^3}\right)$$

$$= \frac{1}{27}\left(\sqrt{40\times(40)^2}-\sqrt{13\times(13)^3}\right)$$

$$= \frac{1}{27}\left(\sqrt{4\times 10\times(40)^2}-\sqrt{13\times(13)^2}\right)$$

$$= \frac{1}{27}\left(80\sqrt{10}-13\sqrt{13}\right) \text{ units of length}$$

Synthesis

Arc length of a curve of function $f(x)$ in interval $]a,b[$ is given by $L = \int_a^b \sqrt{1+[f'(x)]^2}\,dx$.

Application Activity 4.88

1) $\frac{85\sqrt{85}-8}{243}$ units of length

2) $\frac{80\sqrt{10}-13\sqrt{13}}{27}$ units of length

3) $\ln\left(\sqrt{2}+1\right)$ units of length

4) $\frac{14}{3}$ units of length

4.6. Summary of the unit

1. Differentials

The exact change, Δy, in y is given by $\Delta y = f\left(x+\Delta x\right)-f\left(x\right)$.

But if the change Δx is small, then we can get a good approximation to Δy by using the fact that $\frac{\Delta y}{\Delta x}$ is approximately the derivative $\frac{dy}{dx}$. Thus,

$$\Delta y = \frac{\Delta y}{\Delta x}\Delta x \approx \frac{dy}{dx}\Delta x = f'\left(x\right)\Delta x$$

If we denote the change of x by dx instead of Δx, then the change Δy in y is approximated by the differential dy, that is, $\Delta y \approx dy = f'\left(x\right)dx$

Whenever one makes an approximation, it is wise to try and estimate how big the error might be.

Relative change in x is $\frac{\Delta x}{x}$

Percentage change in x is $100\times\frac{\Delta x}{x}$

2. Indefinite integrals

Integration can be defined as the inverse process of differentiation.

If $y = f\left(x\right)$ then

$$\frac{dy}{dx} = f'(x) \Leftrightarrow \int \frac{dy}{dx} dx = f(x) + c$$

Or equivalently

$$\int \frac{dy}{dx} dx = y + c$$

This is called indefinite integration and c is the constant of integration.

3. Basic integration formula

Exponential functions

1. $\int x^n dx = \frac{x^{n+1}}{n+1} + c,\ n \neq -1$
2. $\int e^x dx = e^x + c$
3. $\int a^x dx = \frac{a^x}{\ln a} + c$

Rational functions

1. $\int \frac{1}{x} dx = \ln|x| + c$
2. $\int \frac{dx}{a^2 + x^2} = \frac{1}{a} \arctan \frac{x}{a} + c$
3. $-\int \frac{dx}{a^2 + x^2} = \frac{1}{a} \operatorname{arc\,cot} \frac{x}{a} + c$
4. $\int \frac{dx}{x^2 - a^2} = \frac{1}{2a} \ln \left| \frac{x-a}{x+a} \right| + c$
5. $\int \frac{dx}{a^2 - x^2} = \frac{1}{2a} \ln \left| \frac{a+x}{a-x} \right| + c$

Irrational functions

1. $\int \frac{dx}{\sqrt{a^2 - x^2}} = \arcsin \frac{x}{a} + c$
2. $-\int \frac{dx}{\sqrt{a^2 - x^2}} = \arccos \frac{x}{a} + c$
3. $\int \frac{dx}{\sqrt{x^2 + a^2}} = \ln \left| \frac{x + \sqrt{x^2 + a^2}}{a} \right| + c$
4. $\int \frac{dx}{\sqrt{x^2 - a^2}} = \ln \left| \frac{x + \sqrt{x^2 - a^2}}{a} \right| + c$

5. $\int \frac{dx}{x\sqrt{x^2-a^2}} = \frac{1}{a}\operatorname{arcsec}\frac{x}{a} + c$

6. $-\int \frac{dx}{x\sqrt{x^2-a^2}} = \frac{1}{a}\operatorname{arccsc}\frac{x}{a} + c$

Trigonometric functions

1. $\int \sin x\,dx = -\cos x + c$
2. $\int \cos x\,dx = \sin x + c$
3. $\int \sec^2 x\,dx = \tan x + c$
4. $\int \csc^2 x\,dx = -\cot x + c$
5. $\int \tan x\,dx = -\ln|\cos x| + c$
6. $\int \cot x\,dx = \ln|\sin x| + c$
7. $\int \sec x dx = \ln|\sec x + \tan x| + c = \ln\left|\tan\left(\frac{x}{2}+\frac{\pi}{4}\right)\right|$
8. $\int \csc x dx = \ln|\csc x - \cot x| + c = \ln\left|\tan\frac{x}{2}\right|$
9. $\int \sec x \tan x\,dx = \sec x + c$
10. $\int \csc x \cot x\,dx = -\csc x + c$

4. Non-basic integration

I) Integration by substitution

In evaluating $\int f(x)dx$ when $f(x)$ is not a basic function:

if $f(x) = g'(x)g(x)$ or $f(x) = \frac{g'(x)}{g(x)}$ or

$f(x) = h(g(x))g'(x)$, you let $u = g(x)$.

II) Integration by parts

To integrate a product of functions, try the formula for integration by parts $\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx$.

An effective strategy is to choose for $\frac{dv}{dx}$ the most complicated factor that can readily be integrated. Then we differentiate the other part, u, to find $\frac{du}{dx}$.

The following table can be used:

u	v'
Logarithmic function	Polynomial function
Polynomial function	Exponential function
Polynomial function	Trigonometric function
Exponential function	Trigonometric function
Trigonometric function	Exponential function
Inverse trigonometric function	Polynomial function

Applying the method of integration by parts, the power of integrand is reduced and the process is continued till we get a power whose integral is known or which can be easily integrated. This process is called **Reduction formula**.

III) Integration by partial fractions

Remember that:

A **rational function** is a function of the form $f(x)=\frac{P(x)}{Q(x)}$, where $P(x)$ and $Q(x)$ are polynomials.

A **proper rational function** is a rational function in which the degree of $P(x)$ is strictly less than the degree of $Q(x)$.

The problem of integrating rational functions is really the problem of integrating proper rational functions since **improper rational functions** (i.e. those in which the degree of $P(x)$ is greater than or equal to the degree of $Q(x)$) and can always be rewritten as the sum of a polynomial and a proper rational function.

The integrals of proper rational functions are found by **partial fraction expansion** of the integrand into simple fractions.

There are 4 types of simple fractions:

a) Fractions of the type $\frac{A}{x-a}$.

The integrals of such fractions are easily found:

$$\int \frac{A}{x-a}dx = A\ln|x-a|+c$$

b) Fractions of the type $\frac{A}{(x-a)^n}$, where n is a natural number greater than 1.

The integrals of such fractions are easily found:

$$\int \frac{A}{(x-a)^n}\,dx = A\int (x-a)^{-n}\,dx = \frac{A}{1-n}(x-a)^{1-n} + c$$

c) Fractions of the type $\frac{Ax+B}{x^2+px+q}$, where $p^2-4q<0$

The integrals of such fractions are found by completing the square in the denominator and subsequent substitution which lead to rational integrals of the form

$$\int \frac{du}{u^2+k^2} \ or \int \frac{du}{u^2-k^2} \ or \ \int \frac{du}{k^2+u^2}.$$

d) Fractions of the type $\frac{Ax+B}{(x^2+px+q)^n}$,

where $p^2-4q<0$ and n is a natural number greater than 1. Integration of this type of fraction will not be considered in this course.

Expansion of proper rational functions in partial fractions is achieved by first factoring the denominator and then writing the type of partial fraction (with unknown coefficients in the numerator) that corresponds to each term in the denominator:

(i) if the denominator contains $(x-a)$, then the partial fraction expansion will contain $\frac{A}{x-a}$;

(ii) if the denominator contains $(x-a)^n$, then the partial fraction expansion will contain

$$\frac{A}{(x-a)^n} + \frac{B}{(x-a)^{n-1}} + \frac{C}{(x-a)^{n-2}} + \ldots + \frac{Z}{(x-a)}$$

(iii) if the denominator contains (x^2+px+q) where $p^2-4q<0$, then the partial fraction expansion will contain $\frac{Ax+B}{x^2+px+q}$.

The unknown coefficients (A, B, etc.) are then found by one of two ways: by inserting concrete values of , or by using the method of undetermined coefficients.

4. Integration of irrational functions

a) Integrals of the form $\int \frac{dx}{\sqrt{ax^2+bx+c}}$

The integrals of such fractions are found by completing the square in the denominator and subsequent substitution which leads to irrational integrals of the form

$$\int \frac{du}{\sqrt{u^2+k^2}} \ or \int \frac{du}{\sqrt{u^2-k^2}} \ or \int \frac{du}{\sqrt{k^2+u^2}}$$

a) Integrals of the form $\int \frac{px+q}{\sqrt{ax^2+bx+c}}dx$

The numerator is written as the sum of two parts. One part is the derivative of radicand and the other part is a constant only, i.e.

$$\int \frac{px+q}{\sqrt{ax^2+bx+c}}dx = k_1 \int \frac{\frac{d}{dx}\left(ax^2+bx+c\right)}{\sqrt{ax^2+bx+c}}dx + k_2 \int \frac{dx}{\sqrt{ax^2+bx+c}}$$

a) Integrals of the form $\int \frac{px^2+qx+r}{\sqrt{ax^2+bx+c}}dx$

The numerator is written as the sum of three parts. One part is the same as radicand, the second part is derivative of radicand and the last part is a constant only, i.e.

$$\int \frac{px^2+qx+r}{\sqrt{ax^2+bx+c}}dx = k_1 \int \frac{ax^2+bx+c}{\sqrt{ax^2+bx+c}}dx + k_2 \int \frac{\frac{d}{dx}\left(ax^2+bx+c\right)}{\sqrt{ax^2+bx+c}}dx + k_3 \int \frac{dx}{\sqrt{ax^2+bx+c}}$$

5. Integration of trigonometric functions

a) Integrals of the form $\int \frac{dx}{a\sin x + b\cos x + c}$

You can use t-formulae by letting $t = \tan\frac{x}{2}$.

b) Integrals of the form $\int \frac{dx}{a+b\cos^2 x}$ or $\int \frac{dx}{a+b\sin^2 x}$

Here also you can use t-formulae

In integrating the trigonometric functions containing product or power, transforming product or power into sum (or difference) leads to basic integration.

6. Definite integration

Remember that integrals containing an arbitrary constant *c* in their results are called **indefinite integrals** since their precise value cannot be determined without further information.

a) **Definite integrals** are those in which limits are applied.

If an expression is written as $\left[F(x)\right]_a^b$, 'b' is called the upper limit and 'a' the lower limit.

The operation of applying the limits is defined as:

$\left[F(x)\right]_a^b = F(b)-F(a)$

For example the increase in the value of the integral $f(x)$ as x increases from 1 to 3 is written as $\int_1^3 f(x)dx$.

The **definite integral**, from $x = a$ to $x = b$, is defined as the area under the curve between those two values.

This is written as $\int_a^b f(x)dx$

b) The mean value of a function $y=f(x)$ over the range $]a,b[$ is the value the functions would have if it were constant over the range but with the same area under the graph. The mean value of $y=f(x)$ over the range $]a,b[$ is $\overline{f(x)}=\frac{1}{b-a}\int_a^b f(x)dx$.

c) The **root mean square** value (R.M.S. value) is the square root of the mean value of the square of y.

The r.m.s. value from $x = a$ to $x = b$ is given by;

$$R.M.S. = \sqrt{\frac{\int_a^b f^2(x)\,dx}{b-a}}$$

d) Improper integral

The definite integral $\int_a^b f(x)dx$ is called an improper integral if one of two situations occurs:

- The limit a or b (or both bounds) are infinites.
- The function $f(x)$ has one or more points of discontinuity in the interval $[a,b]$.

Let $f(x)$ be a continuous function on the interval $[a,+\infty[$ or $]-\infty,b]$

We define the improper integral as $\int_a^{+\infty} f(x)dx = \lim_{n\to+\infty}\int_a^n f(x)dx$

Or $\int_{-\infty}^b f(x)dx = \lim_{n\to-\infty}\int_n^b f(x)dx$ respectively.

If these limits exist and are finite, then we say that the improper integrals are convergent, otherwise, the integrals are divergent.

Let $f(x)$ be a continuous function for all real numbers. By Chasles theorem $\int_{-\infty}^{+\infty} f(x)dx = \int_{-\infty}^{c} f(x)dx + \int_{c}^{+\infty} f(x)dx$

If for real number c, both integrals on the right side are convergent, then we say that the integral $\int_{-\infty}^{+\infty} f(x)dx$ is also **convergent**; otherwise it is **divergent**.

7. Applications

Integration has many applications, some of which are listed below:

a) The area between two functions $f(x)$ and $g(x)$ where $f(x) \le g(x)$ in $[a,b]$ is given by

$$\int_a^b [g(x) - f(x)]dx = \int_a^b g(x)dx - \int_a^b f(x)dx$$

b) Volume

The volume of a solid of revolution can be found using one of the following methods:

- disc method,
- washer method, and

- shell method.

In any of the methods, when finding volume, it is necessary to integrate along the axis of revolution; if the region is revolved about a horizontal line, integrate by x, and if the region is revolved about a vertical line, integrate with respect to y.

(i) **Disc method**

The volume of the solid of revolution bound by the curve $f(x)$ about the $x-axis$ calculated from $x=a$ to $x=b$, is given by $\pi\int_a^b y^2dx$.

Volume of the solid generated by revolution of the area bound by the curve $y=f(x)$ about the $y-axis$ is given by $\pi\int_a^b x^2dy$.

If the axis of revolution is the line parallel to $x-axis$ (say $y=k$), the volume will be

$$\pi\int_a^b (y-k)^2\,dx$$

(ii) **Washer method**

If the region bound by outer radius $y_U=g(x)$ (on top) and inner radius $y_L=f(x)$ and then lines $x=a$, $x=b$ is revolved about $x-axis$, then the volume of revolution is given by:

$$V=\pi\int_a^b\left([g(x)]^2-[f(x)]^2\right)dx$$

(iii) **Shell method**

The volume of the solid generated by revolving the region between the curve $x-axis, y=f(x)\geq 0, L\leq a\leq x\leq b$, about a vertical line $x=L$ is

$$V=2\pi\int_a^b\begin{pmatrix}shell\\radius\end{pmatrix}\begin{pmatrix}shell\\height\end{pmatrix}dx$$

HINT for shell method:

Regardless of the position of the axis of revolution (horizontal or vertical), the steps for implementing the shell method are the following:

- Draw the region and sketch a line segment across it, parallel to the axis of revolution. Label the segment's height or length (shell height) and distance from the axis of revolution (shell radius).
- Find the limits of integration for the thickness variable.
- Integrate the product $2\pi\begin{pmatrix} shell \\ radius \end{pmatrix}\begin{pmatrix} shell \\ height \end{pmatrix}$ with respect to the thickness variable $(x \, or \, y)$ to find the volume.
- Length of arc of the curve $y = f(x)$ between the points whose abscissas are a and b is $$s = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$$
- The **work** done by a variable force $F(x)$ in the direction of motion along the $x-axis$ over the interval $[a,b]$ is $$W = \int_a^b F(x) dx.$$

Hook's law says that the force required to hold a stretched or compressed spring x units beyond its equilibrium position pulls back with a force $F(x) = kx$ where k is constant called **spring constant** (or **force constant**).

4.7. End of Unit Assessment

1. a) $\frac{x^2}{2} + 9x + 125\ln|x-5| - 64\ln|x-4| + c$

   b) $x^3 + x^2 - 5x + 18\ln|x+3| + c$

   c) $-\frac{16}{x-2} - \frac{2}{3}\ln|x-2| + \frac{5}{3}\ln|x+1| + c$

   d) $\ln|x-1| - \frac{1}{2}\ln(x^2+1) + \tan^{-1} x + c$

e) $4\ln|x+2|-\frac{3}{2}\ln\left(x^2+2x+2\right)+\tan^{-1}(x+1)+c$

f) $\frac{1}{2}\ln\left(x^2+1\right)-\ln|x+1|-\frac{3}{x+1}+c$

g) $x\sin x+\cos x+c$

h) $\frac{5}{4}e^{4x}\left(x-\frac{1}{4}\right)+c$

i) $\frac{x^2}{4}(2\ln x-1)+c$

j) $\frac{\cos 3x}{27}\left(2-9x^2\right)+\frac{2}{9}x\sin 3x+c$

k) $\frac{e^{ax}}{a^2+b^2}(\mathrm{b}\sin bx+a\cos bx)+c$

l) $\frac{e^{ax}}{a^2+b^2}(a\sin bx-a\cos bx)+c$

2. a) $\frac{4}{3}$ b) $\frac{10}{9}$ c) $\ln 2$

d) $\frac{\pi}{4}$ e) $\frac{\sqrt{2}}{2}$ f) 1

3. To be proved

4. $20\frac{5}{6}$

5. 4

6. $\frac{16}{3}$

7. $\frac{56\pi}{27}$

8. $\frac{32\pi}{3}$

9. $\frac{208\pi}{15}$

10. a) $\frac{\pi}{30}$ b) $\frac{\pi}{6}$ c) $\frac{\pi}{2}$

d) $\frac{5\pi}{6}$ e) $\frac{11\pi}{30}$ f) $\frac{19\pi}{30}$

11. a) $\frac{48\pi}{5}$ b) $\frac{24\pi}{5}$

12. a) $kt = \frac{1}{2a}\ln|a-x| - \ln|a+3x|$

b) $x = \frac{a\left(e^{2akt}-1\right)}{3e^{2akt}-1}$ c) $x \to \frac{a}{3}, t \to \infty$

13. $0.632N_o$

14. 7.26

15. $1.17J$

16. $\frac{9}{5}$

17. a) $30\,N/m$ b) $60\,J$ c) $1.5\,m$

18. a) $926,640$ b) 0.468

Unit 5

Differential Equations

5.1. Key unit competence

Use ordinary differential equations of first and second order to model and solve related problems in Physics, Economics, Chemistry, Biology.

5.2. Vocabulary or key words concepts

Differential equation (D.E): Equation that involves a function and its derivatives.

First order differential equation: Differential equation containing only first derivatives apart from dependent variable.

Second order differential equation: Differential equation containing second derivatives (and possibly first derivative also).

Particular solution: A solution found at particular values.

5.3. Guidance on the introductory activity

Organize groups of students, and then assign them to do the introductory activity from the student's book. As they are working, move around to each group and ask them probing questions leading them to the right way. After a while, invite some group to present their findings in a whole class discussion. Guide students to harmonize their answers and arouse their curiosity to the content of this unit.

5.4. List of lessons

No	Lesson title	Number of periods
1	Definition and classification	1
2	Differential equations with separable variables	1

3	Simple homogeneous equations	2
4	Linear equations	2
5	Particular solution	1
6	Second order differential equations: Definition	1
7	Second order differential equations with constant coefficient: two distinct real roots	1
8	Characteristic equation has a double root	1
9	Characteristic equation has complex roots	1
10	Non-homogeneous linear differential equations of the second order with constant coefficients	2
11	Non-homogeneous linear differential equations of the second order with the right hand side $r(x) = Pe^{\alpha x}$	2
12	Non-homogeneous linear differential equations of the second order with the right hand side $r(x) = Pe^{\alpha x}\cos\beta x + Qe^{\alpha x}\sin\beta x$	2
13	Application: Newton's law of cooling	2
14	Application: Electrical circuits	2
Total periods		**21**

5.5. Lesson development

Lesson 5.1. Definition and classification

Learning objectives

Through examples, learners should be able to define and classify given differential equations correctly.

Prerequisites

- Differentiation

Teaching Aids

Exercise book and pen

Activity 5.1

In this lesson, the following generic competence and cross-cutting issues are to be addressed:

- Critical thinking
- Communication
- Self confidence
- Cooperation, interpersonal management and life skills
- Peace and values education
- Inclusive education

Answers

1. On differentiation; $\frac{dy}{dx} = A$

   The given equation becomes $y = x\frac{dy}{dx} + \left(\frac{dy}{dx}\right)^2$

   Order of the highest derivative is 1.

2. On differentiation; $\frac{dy}{dx} = -A\sin x + B\cos x$

   Again differentiating:

   $$\frac{d^2y}{dx^2} = -\left(A\cos x + B\sin x\right) = -y$$

   Or $\frac{d^2y}{dx^2} + y = 0$

   Order of the highest derivative is 2.

3. On differentiation $2y\frac{dy}{dx} = 2Ax + B$

   Again differentiating $2y\frac{d^2y}{dx^2} + 2\left(\frac{dy}{dx}\right)^2 = 2A$

   On differentiating again:

   $$y\frac{d^3y}{dx^3} + \frac{dy}{dx}\frac{d^2y}{dx^2} + 2\frac{dy}{dx}\frac{d^2y}{dx^2} = 0$$

Or $y\dfrac{d^3y}{dx^3}+3\dfrac{dy}{dx}\dfrac{d^2y}{dx^2}=0$

Order of the highest derivative is 3.

Synthesis

An equation involving a differential coefficient i.e.

$\dfrac{dy}{dx},\dfrac{d^2y}{dx^2},\dfrac{dr}{dt}$ and so on is called a **"differential equation".**

Order of the differential equation is the highest derivative of function that appears in a differential equation and is said to be the **order of differential equation**.

Given a function with arbitrary constants, you form differential equation by eliminating its arbitrary constants using differentiation.

Application Activity 5.89

1. a) $y^2\left(\dfrac{dy}{dx}\right)^2+2xy\dfrac{dy}{dx}-y^2=0$ b) $xy\dfrac{d^2y}{dx^2}+x\left(\dfrac{dy}{dx}\right)^2-y\dfrac{dy}{dx}=0$

   c) $\dfrac{d^2y}{dx^2}-4\dfrac{dy}{dx}+3y=0$ d) $\dfrac{d^2y}{dx^2}-2\dfrac{dy}{dx}+2y=0$

   e) $\dfrac{dy}{dx}=-\tan\left(x+3\right)$

2. a) This DE has **order 2** (the highest derivative appearing is the **second** derivative) and **degree 1** (the **power** of the highest derivative is 1).
   b) This DE has **order 1** (the highest derivative appearing is the **first** derivative) and **degree 4** (the **power** of the highest derivative is 4).
   c) This DE has **order 2** (the highest derivative appearing is the **second** derivative) and **degree 3** (the **power** of the highest derivative is 3).
   d) order 2; degree 1
   e) order 2; degree 1
   f) order 3; degree 1
   g) order 2; degree 1
   h) order 2; degree 3

Lesson 5.2. Differential equations with separable variables

Learning objectives

Through examples, learners should be able to identify and solve differential equations with separable variables accurately.

Prerequisites

- Integration

Teaching Aids

Exercise book and pen

Activity 5.2

In this lesson, the following generic competence and cross-cutting issues are to be addressed:

- Critical thinking
- Communication
- Self confidence
- Cooperation, interpersonal management and life skills
- Peace and values education
- Inclusive education

Answers

1. $\frac{dy}{dx}=\frac{x}{y}\Rightarrow ydy=xdx \Rightarrow \int ydy=\int xdx \Rightarrow \frac{y^2}{2}=\frac{x^2}{2}+k$

   Or $y^2=x^2+2k \Rightarrow y^2=x^2+c, \quad c\in\mathbb{R}$

2. $\frac{dy}{dx}=x^2y^3 \Rightarrow \frac{dy}{y^3}=x^2dx \Rightarrow \int\frac{dy}{y^3}=\int x^2dx$

   $\Rightarrow \int y^{-3}dy=\int x^2dx \Rightarrow \frac{y^{-2}}{-2}=\frac{x^3}{3}+k \Rightarrow -\frac{1}{2y^2}=\frac{x^3}{3}+k, \quad k\in\mathbb{R}$

Synthesis

To solve the integral $\frac{dy}{dx}=g(x)h(y)$, we write it in the separated form $\frac{dy}{h(y)}=g(x)dx$ and integrate.

Application Activity 5.90

1. $y = c\sqrt{x}$, $c \in \mathbb{R}$
2. $x^3 - y^3 = c$, $c \in \mathbb{R}$
3. $\arctan y = x + c$, $c \in \mathbb{R}$
4. $\tan^{-1} y = x - \ln|1 + x| + c$, $c \in \mathbb{R}$

Lesson 5.3. Simple homogeneous equations

Learning objectives

Through examples, learners should be able to identify and solve simple homogeneous equations accurately.

Prerequisites

- Integration

Teaching Aids

Exercise book and pen

Activity 5.3

In this lesson, the following generic competence and cross-cutting issues are to be addressed:

- Critical thinking
- Communication
- Self confidence
- Cooperation, interpersonal management and life skills
- Peace and values education
- Inclusive education

Answers

1. $f(tx,ty) = (tx)^2 + (tx)(ty)$

$$f(tx,ty) = t^2x^2 + t^2xy$$

$$f(tx,ty) = t^2\left(x^2 + xy\right)$$

$$f(tx,ty) = t^2 f\left(x, y\right)$$

The value of n is 2.

2. $z=\dfrac{y}{x}\Rightarrow y=zx$

$\dfrac{dy}{dx}=z+x\dfrac{dz}{dx}$

$\dfrac{dy}{dx}=f(x,y)$ becomes $z+x\dfrac{dz}{dx}=f\left(x,y\right)$

But $f\left(x,y\right)=f\left(1,z\right)$

3. $f\left(tx,ty\right)=f\left(x,y\right)$

But $t=\dfrac{1}{x}$, then $f\left(1,\dfrac{y}{x}\right)=f\left(x,y\right)\Rightarrow f\left(1,z\right)=f\left(x,y\right)$

Then, $z+x\dfrac{dz}{dx}=f\left(1,\mathrm{z}\right)$

Separating variables, we have

$$x\frac{dz}{dx}=f\left(1,\mathrm{z}\right)-z\ \Rightarrow x\frac{dz}{f\left(1,\mathrm{z}\right)-z}=dx\ \Rightarrow \frac{dz}{f\left(1,\mathrm{z}\right)-z}=\frac{dx}{x}$$

Synthesis

A function $f(x,y)$ is called homogeneous of degree n if $f(tx,ty)=t^n f(x,y)$ for all suitably restricted x, y and t.

The differential equation $M\left(x,y\right)dx+N\left(x,y\right)dy=0$ is said to be homogeneous if M and N are homogeneous functions of the same degree.

This equation can be written in the form $\dfrac{dy}{dx}=f(x,y)$.

Where $f(x,y)=\dfrac{-M(x,y)}{N(x,y)}$ is clearly homogeneous of degree 0. We solve this equation by letting $z=\dfrac{y}{x}$.

Application Activity 5.91

1. $x^2-y^2=c$, $c\in\mathbb{R}$ 2. $x^2+y^2=cx$, $c\in\mathbb{R}$
3. $y^2=x^2\left(cx^2-4\right)$, $c\in\mathbb{R}$
4. $\left(y-x\right)^2+2\left(y-x\right)=2x+c$, $c\in\mathbb{R}$

Lesson 5.4. Linear differential equations

Learning objectives

Through examples, learners should be able to identify and solve linear differential equations accurately.

Prerequisites

- Differentiation
- Integration

Teaching Aids

Exercise book and pen

Activity 5.4

In this lesson, the following generic competence and cross-cutting issues are to be addressed:

- Critical thinking
- Communication
- Self confidence
- Cooperation, interpersonal management and life skills
- Peace and values education
- Inclusive education

Answers

1. $y = uv$

   $$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

2. Now, $\frac{dy}{dx} + py = q$ becomes $u\frac{dv}{dx} + v\frac{du}{dx} + p(uv) = q$

3. $u\frac{dv}{dx} + v\frac{du}{dx} + p(uv) = q \Rightarrow u\frac{dv}{dx} + p(uv) + v\frac{du}{dx} = q$

   $$\Rightarrow u\left(\frac{dv}{dx} + pv\right) + v\frac{du}{dx} = q$$

   If $\frac{dv}{dx} + pv = 0$, $dv + pvdx = 0 \Rightarrow dv = -pvdx$

   Separating variables, we have $\frac{dv}{v} = -pdx$

Integrating both sides, we have $\int \frac{dv}{v} = \int -pdx$

$\ln|v| = -\int pdx + c \Rightarrow \ln|v| = \ln e^{-\int pdx} + \ln k$

$\Rightarrow \ln|v| = \ln ke^{-\int pdx} \Rightarrow |v| = ke^{-\int pdx}$

Take $v = e^{-\int pdx}$

4. Now, the equation $u\left(\frac{dv}{dx} + pv\right) + v\frac{du}{dx} = q$ becomes

$e^{-\int pdx}\frac{du}{dx} = q$ since $\frac{dv}{dx} + pv$ is assumed to zero.

$e^{-\int pdx}\frac{du}{dx} = q \Rightarrow \frac{du}{dx} = \frac{q}{e^{-\int pdx}}$

$\Rightarrow \frac{du}{dx} = q\,e^{\int pdx} \Rightarrow du = q\,e^{\int pdx}\,dx$

Integrating both sides gives

$\Rightarrow \int du = \int q\,e^{\int pdx}\,dx \Rightarrow u = \int q\,e^{\int pdx}\,dx$

Synthesis

The general solution to the equation $\frac{dy}{dx} + py = q$ where p and q are functions in x or constants, is $y = uv$ where $u = \int q\,e^{\int pdx}\,dx$ and $v = e^{-\int pdx}$.

Short cut method:

The solution of $\frac{dy}{dx} + py = q$ is simply given by formula $y = e^{-\int pdx}\int qe^{\int pdx}dx$.

Application Activity 5.92

1. $y = \frac{x}{2} + \frac{c}{x}$
2. $y = x^2 - 2 + ce^{-\frac{x^2}{2}}$
3. $y = (x+1)e^x + c(x+1)$
4. $y = \cos x + c\cos^2 x$

Lesson 5.5. Particular solution

Learning objectives

Given a differential equation and initial condition, learners should be able to find a particular solution for that differential equation accurately.

Prerequisites

- Integration

Teaching Aids

Exercise book and pen

Activity 5.5

In this lesson, the following generic competence and cross-cutting issues are to be addressed:

- Critical thinking
- Communication
- Self confidence
- Cooperation, interpersonal management and life skills
- Peace and values education
- Inclusive education

Answers

1. $\frac{dy}{dx} = x+4$

   $\Rightarrow dy = (x+4)\,dx \Rightarrow \int dy = \int (x+4)\,dx$

   $\Rightarrow y = \frac{x^2}{2} + 4x + c$

2. If $x = 2$ then $y = 4$

   $\Rightarrow 4 = \frac{2^2}{2} + 4\times 2 + c \Rightarrow 4 = 10 + c \Rightarrow c = -6$

   New solution is $y = \frac{x^2}{2} + 4x - 6$

Synthesis

If we want to determine a function, $y(x)$, such that the given equation is satisfied for $y(x_0) = y_0$ or $y|_{x=x_0} = y_0$, this equation is referred to as an **initial value problem** for the obvious reason that out of the totality of all solution of the differential equation, we are looking for the one solution which initially (at, x_0) has the value y_0.

Application Activity 5.93

1. $y = \cos x - 2\cos^2 x$
2. $y = \tan\left(\tan^{-1} x + \frac{\pi}{4}\right)$
3. $y = \frac{1}{\sqrt{1-x^2}}$
4. $y^2 = x^2 - 2\ln x + 3$
5. $e^y = \frac{1}{2}e^{2x} + \frac{1}{2}$

Lesson 5.6. Second order differential equations: Definition

Learning objectives

Through examples, learners should be able to define a second order differential equation accurately.

Prerequisites

- Differentiation

Teaching Aids

Exercise book and pen

Activity 5.6

In this lesson, the following generic competence and cross-cutting issues are to be addressed:

- Critical thinking
- Communication
- Self confidence
- Cooperation, interpersonal management and life skills
- Peace and values education
- Inclusive education

Answers

Answer may vary; here are some.

1. Second order differential equation with degree greater than 1 is of the form

$$\left(\frac{d^2y}{dx^2}\right)^n + p(x)\left(\frac{dy}{dx}\right)^k + q(x)y = r(x),$$

where $p(x)$, $q(x)$ and $r(x)$ are functions of x alone (or perhaps constants) and $n, k \in \mathbb{Z}$ with $n=1$.

2. Second order differential equation with degree 1 is of the form

$$\left(\frac{d^2y}{dx^2}\right) + p(x)\left(\frac{dy}{dx}\right)^k + q(x)y = r(x),$$

where $p(x)$, $r(x)$ and $r(x)$ are functions of x alone (or perhaps constants) and $k \in \mathbb{Z}$.

Synthesis

The general second order linear differential equation is of the form $\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y = r(x)$ or more simply, $y'' + p(x)y' + q(x)y = r(x)$;

where $p(x)$, $q(x)$ and $r(x)$ are functions of x alone (or perhaps constants).

Lesson 5.7. Second order differential equations with constant coefficient: two distinct real roots

Learning objectives

Given a second order differential equations with constant coefficient where characteristic equation has two distinct real roots, learners should be able to find its general solution perfectly.

Prerequisites

- Solving quadratic equation.

Teaching Aids

Exercise book and pen

Activity 5.7

In this lesson, the following generic competence and cross-cutting issues are to be addressed:

- Critical thinking
- Communication
- Self confidence
- Cooperation, interpersonal management and life skills
- Peace and values education
- Inclusive education

Answers

1. $y'+ky=0 \Rightarrow \frac{dy}{dx}=-ky \Rightarrow \frac{dy}{y}=-kdx$

   $\Rightarrow \int \frac{dy}{y}=\int -kdx \Rightarrow \ln|y|=-kx$

   $\Rightarrow \ln|y|=\ln e^{-kx} \Rightarrow y=e^{-kx}$

2. $y''+py'+qy=0$

   But $y=e^{-kx} \Rightarrow y'=-ke^{-kx}$ and $y''=(-k)^2 e^{-kx}$

   $y''+py'+qy=0$ becomes

   $(-k)^2 e^{-kx}+p\left(-ke^{-kx}\right)+qe^{-kx}=0$

   $\Rightarrow \left[(-k)^2-kp+q\right]e^{-kx}=0$

   This relation is true if $(-k)^2-kp+q=0$ since e^{-kx} cannot be zero.

   Then $(-k)^2-kp+q=0$. Putting $m=-k$, we have $m^2+mp+q=0$

   Thus, the solution of $y'+ky=0$ is also a solution of $y''+py'+qy=0$ if m satisfy the auxiliary equation $m^2+mp+q=0$ for $m=-k$.

   Therefore, the solution of the form e^{mx} is the solution of $y''+py'+qy=0$.

Synthesis

In solving homogeneous linear equation of second order $y''+py'+qy=0$, we first determine its characteristic equation which is $m^2+mp+q=0$.

If m_1 and m_2 are solutions of the characteristic equation, then the general solution of $y''+py'+qy=0$ is $y=c_1e^{m_1x}+c_2e^{m_2x}$ where $m_1,m_2=\dfrac{-p\pm\sqrt{p^2-4q}}{2}$.

Application Activity 5.94

1. $y=c_1e^{3x}+c_2e^{5x}$
2. $y=c_1e^{-2x}+c_2e^{x}$
3. $y=c_1e^{5x}+c_2e^{-6x}$
4. $y=c_1e^{-3x}+c_2e^{-7x}$

Lesson 5.8. Characteristic equation with a double root

Learning objectives

Given a second order differential equations with constant coefficient where characteristic equation has one double root, learners should be able to find its general solution correctly.

Prerequisites

- Solving quadratic equation.

Teaching Aids

Exercise book and pen

Activity 5.8

In this lesson, the following generic competence and cross-cutting issues are to be addressed:

- Critical thinking
- Communication
- Self confidence
- Cooperation, interpersonal management and life skills
- Peace and values education
- Inclusive education

Answers

1. Characteristic equation:

$m^2 + 2m + 1 = 0$

$\Delta = 4 - 4 = 0$

Thus, $m_1 = m_2 = -\frac{2}{2} = -1$

One of solutions of $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = 0$ is $y_1 = e^{-x}$

2. $y_2 = xy_1 \Rightarrow y_2 = xe^{-x}$

Since $y_2 = xe^{-x}$, then

$$\frac{dy_2}{dx} = \frac{d\left(xe^{-x}\right)}{dx} = e^{-x} - xe^{-x}$$

$$\frac{d^2y_2}{dx^2} = \frac{d\left(e^{-x} - xe^{-x}\right)}{dx} = -e^{-x} - e^{-x} + xe^{-x} = -2e^{-x} + xe^{-x}$$

Substituting y by $y_2 = xe^{-x}$ in $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = 0$ yield

$$\left(-2e^{-x} + xe^{-x}\right) + 2\left(e^{-x} - xe^{-x}\right) + xe^{-x}$$

$$= \cancel{-2e^{-x}} + \cancel{xe^{-x}} + \cancel{2e^{-x}} - \cancel{2xe^{-x}} + \cancel{xe^{-x}} = 0$$

We note that $y_2 = xe^{-x}$ is also a solution of

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = 0$$

The ratio $\frac{y_1}{y_2} = \frac{e^{-x}}{xe^{-x}} = \frac{1}{x}$ is not constant, thus, $y_1 = e^{-x}$ and $y_2 = xe^{-x}$ are linearly independent and $y = c_1e^{-x} + c_2xe^{-x}$ is the general solution of $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = 0$

(c_1 and c_2 being arbitrary constants).

Synthesis

In solving homogeneous linear equation of second order $y'' + py' + qy = 0$, if the characteristic equation $m^2 + mp + q = 0$ has a double root equal to m, the general solution of equation $y'' + py' + qy = 0$ will be $y = c_1e^{mx} + c_2xe^{mx}$.

Application Activity 5.95

1. $y = c_1 e^{3x} + c_2 x e^{3x}$
2. $y = c_1 e^{-3x} + c_2 x e^{-3x}$
3. $y = e^{-4x}(c_1 + c_2 x)$
4. $y = (c_1 + x c_2) e^{\frac{1}{6}x}$

Lesson 5.9. Characteristic equation with complex roots

Learning objectives

Given a second order differential equations with constant coefficient where the characteristic equation has complex roots, learners should be able to find its general solution correctly.

Prerequisites

- Solving quadratic equation in complex numbers.

Teaching Aids

Exercise book and pen

Activity 5.9

In this lesson, the following generic competence and cross-cutting issues are to be addressed:

- Critical thinking
- Communication
- Self confidence
- Cooperation, interpersonal management and life skills
- Peace and values education
- Inclusive education

Answers

1. Characteristic equation is $m^2 - 4m + 25 = 0$

   $\Delta = 16 - 100 = -64$ $\quad\quad$ $\sqrt{\Delta} = \pm 8i$

   $m_1 = \dfrac{4+8i}{2} = 2 + 4i$ $\quad\quad$ $m_2 = \dfrac{4-8i}{2} = 2 - 4i$

   a) The basis of $\dfrac{d^2y}{dx^2} - 4\dfrac{dy}{dx} + 25y = 0$ are

   $y_1 = c_1 e^{(2+4i)x}$ and $y_2 = c_2 e^{(2-4i)x}$

b) Its general solution is $y=c_1e^{(2+4i)x}+c_2e^{(2-4i)x}$

2. $y=c_1e^{(2+4i)x}+c_2e^{(2-4i)x}$

$\Leftrightarrow y=c_1e^{2x+4ix}+c_2e^{2x-4ix} \Leftrightarrow y=e^{2x}\left(c_1e^{4ix}+c_2e^{-4ix}\right)$

$\Leftrightarrow y=e^{2x}\left[c_1\left(\cos 4x+i\sin 4x\right)+c_2\left(\cos 4x-i\sin 4x\right)\right]$

$\Leftrightarrow y=e^{2x}\left[\left(c_1+c_2\right)\cos 4x+\left(c_1-c_2\right)i\sin 4x\right]$

$\Leftrightarrow y=e^{2x}\left(c_1+c_2\right)\cos 4x+e^{2x}\left(c_1-c_2\right)i\sin 4x$

$\Leftrightarrow y=e^{2x}\left[\left(c_1+c_2\right)\cos 4x+\left(c_1-c_2\right)i\sin 4x\right]$

3. Real basis are $y_1=Ae^{2x}\cos 4x$ and $y_2=Be^{2x}\sin 4x$

4. General solution is $y=e^{2x}\left(\mathrm{A}\cos 4x+B\sin 4x\right)$

Synthesis

If the characteristic equation has complex roots, $\alpha \pm i\beta$ then, the general solution is $y=e^{\alpha x}\left(\mathrm{A}\cos \beta x+B\sin \beta x\right)$, where α and β are respectively ,real and imaginary part of root of characteristic equation.

Application Activity 5.96

1. $y=e^{-2x}\left(A\cos 3t+B\sin 3t\right)$
2. $y=e^{-2x}\left(A\cos x+B\sin x\right)$
3. 3. $y=e^{x}\left(A\cos x+B\sin x\right)$
4. $y=-e^{\pi-2x}\sin 3x$
5. $y=\dfrac{2}{9}e^{2x}-\dfrac{2}{9}e^{-\frac{5x}{2}}$

Lesson 5.10. Non- homogeneous linear differential equations of the second order with constant coefficients

Learning objectives

Through examples, learners should be able to identify a non-homogeneous linear differential equation of the second order with constant coefficients and solve it where possible correctly.

Prerequisites

- Solving homogeneous differential equation of second order.

Teaching Aids

Exercise book, calculator and pen

Activity 5.10

In this lesson, the following generic competence and cross-cutting issues are to be addressed:

- Critical thinking
- Communication
- Self confidence
- Cooperation, interpersonal management and life skills
- Peace and values education
- Inclusive education

Answers

1. $\frac{dy}{dx}-\frac{y}{x+1}=e^x(x+1)$ is a linear differential equation of 1st order.

   Let $y=u\cdot v$, then $\frac{dy}{dx}=u\frac{dv}{dx}+v\frac{du}{dx}$.

   Substituting in the given equation, we get

   $u\frac{dv}{dx}+v\frac{du}{dx}-\frac{u\cdot v}{x+1}=e^x(x+1)$

   Or $u\frac{dv}{dx}-\frac{u\cdot v}{x+1}+v\frac{du}{dx}=e^x(x+1)$

   $\Leftrightarrow u\left(\frac{dv}{dx}-\frac{v}{x+1}\right)+v\frac{du}{dx}=e^x(x+1)$

   Taking $\frac{dv}{dx}-\frac{v}{x+1}=0,$ you get $\ln v=\ln(x+1)$ or $v=x+1$.

   As $\frac{dv}{dx}-\frac{v}{x+1}=0$, $u\left(\frac{dv}{dx}-\frac{v}{x+1}\right)+v\frac{du}{dx}=e^x(x+1)$

   $(x+1)\frac{du}{dx}=e^x(x+1)\Leftrightarrow\frac{du}{dx}=e^x$ or $u=e^x+c$.

   The solution of the given equation is then,

   $y=(x+1)(e^x+c)$ or $y=(x+1)e^x+c(x+1)$

2. $\frac{d^2y}{dx^2}-4\frac{dy}{dx}=5y \Leftrightarrow \frac{d^2y}{dx^2}-4\frac{dy}{dx}-5y=0$

is homogeneous linear equation of second order.

Characteristic equation

$m^2-4m-5=0 \qquad \Delta=16+20=36$

$m_1=\frac{4-6}{2}=-1, m_2=\frac{4+6}{2}=5$

General solution is $y=c_1e^{-x}+c_2e^{5x}$

3. $\frac{d^2y}{dx^2}-4\frac{dy}{dx}-5y=x$ is non-homogeneous

linear equation of second order.

At this level, it is impossible for most learners to solve this type of equation.

[General solution is given by $y=\overline{y}+y^*$.

From 1) Complementary solution is $y=c_1e^{-x}+c_2e^{5x}$.

Let $y^*=Ax+B$ be particular solution of the given equation.

Then $y^{*'}=A$ and $y^{*''}=0$.

Putting $y^*=Ax+B$ and its derivatives in

$\frac{d^2y}{dx^2}-4\frac{dy}{dx}-5y=x$, gives

$0-4A-5(Ax+B)=x \Leftrightarrow -5Ax-4A-5B=x$

Identifying the coefficients, we get

$-5A=1$ and $-4A-5B=0$

Or $A=-\frac{1}{5}$ and $B=\frac{4}{25}$.

Thus, particular solution is $y^*=-\frac{1}{5}x+\frac{4}{25}$

The general solution is $y=c_1e^{-x}+c_2e^{5x}-\frac{1}{5}x+\frac{4}{25}$

Synthesis

The general solution of the second order non-homogeneous linear equation $y''+py'+qy=r(x)$ can be expressed in the form $y = \overline{y} + y^*$ where y^* is any specific function that satisfies the non-homogeneous equation, and $\overline{y} = c_1 y_1 + c_1 y_1$ is a general solution of the corresponding homogeneous equation $y'' + py' + qy = 0$.

Lesson 5.11. Differential equations of the second order with the right hand side $r(x) = Pe^{\alpha x}$

Learning objectives

Given a differential equation of second order where the right hand side is of the form $r(x) = Pe^{\alpha x}$, learners should be able to find its general solution correctly.

Prerequisites

- Solving homogeneous differential equation of second order..

Teaching Aids

Exercise book, calculator and pen

Activity 5.11

In this lesson, the following generic competence and cross-cutting issues are to be addressed:

- Critical thinking
- Communication
- Self confidence
- Cooperation, interpersonal management and life skills
- Peace and values education
- Inclusive education

Answers

1. Characteristic equation: $m^2 - 2m + 1 = 0$

   $\Delta = 4 - 4 = 0$

   $m_1 = m_2 = \frac{2-0}{2} = 1 \qquad \overline{y} = c_1 e^x + c_2 e^x$

2. The right hand side can be written as $e^x = 1e^{1x}$.

   $P = 1$ and $\alpha = 1$

   $\alpha = 1$, is double root of characteristic equation, so $k = 2$

   $y^* = Ax^2e^x$, $Q(x) = A$ as $P = 1$ on right hand side, $Q(x)$ has degree zero.

3. $y^{*/} = 2Axe^x + Ax^2e^x$

   $y^{*//} = 2Ae^x + 2Axe^x + 2Axe^x + Ax^2e^x$

   $\Rightarrow\ 2Ae^x + 2Axe^x + 2Axe^x + Ax^2e^x - 4Axe^x - 2Ax^2e^x + Ax^2e^x = e^x$

   $\Rightarrow\ 2A + 2Ax + 2Ax + Ax^2 - 4Ax - 2Ax^2 - Ax^2 = 1$

   $\Rightarrow -2Ax^2 + 2Ax^2 + x(2A + 2A - 4A) - 2A = 1$

   $2A = 1 \Rightarrow A = \frac{1}{2}$

   Thus, $y^* = \frac{1}{2}x^2e^x$

Synthesis

If the right hand side of the equation $y'' + py' + qy = r(x)$ is $r(x) = Pe^{\alpha x}$ where P is a polynomial, we take the particular solution to be

$y^* = x^kQ_n(x)e^{\alpha x}$, $Q_n = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + ... + a_n$

Here, k- is the number of roots of the associated homogeneous equation equal to α.

α; coefficient of x in $e^{\alpha x}$ on the right hand side,

n; degree of $Q(x)$, the same as degree of $P(x)$ on right hand side.

3 cases arise

- If α is not a root of characteristic equation $k = 0$
- If α is a simple root of characteristic equation $k = 1$
- If α is a double root of characteristic equation $k = 2$

Note that the simple root or double root in the last 2 cases must be real numbers.

Application Activity 5.97

1. $y = c_1 e^{-3x} + c_2 x e^{-3x} + \frac{5e^{3x}}{36}$
2. $c_1 e^{x} + c_2 e^{2x} + \frac{e^{3x}}{2}$
3. $y = c_1 e^{-x} + c_2 e^{-2x} + \frac{1}{4} e^{2x}$
4. $y = c_1 e^{x} + c_2 e^{2x} + \frac{1}{2} e^{3x}$
5. $y = (c_1 + c_2 x) e^{3x} + \frac{x^2}{2} e^{3x}$

Lesson 5.12. Differential equations of the second order with the right hand side $r(x) = Pe^{\alpha x}\cos\beta x + Qe^{\alpha x}\sin\beta x$

Learning objectives

Given differential equations of second order where the right hand side is of the form $r(x) = Pe^{\alpha x}\cos\beta x + Qe^{\alpha x}\sin\beta x$, learners should be able to find its solution correctly.

Prerequisites

- Solving homogeneous differential equations of the second order.

Teaching Aids

Exercise book, calculator and pen

Activity 5.12

In this lesson, the following generic competence and cross-cutting issues are to be addressed:

- Critical thinking
- Communication
- Self confidence
- Cooperation, interpersonal management and life skills
- Peace and values education
- Inclusive education

Answers

1. Characteristic equation: $m^2 + 4 = 0$

   $m_1 = 2i, m_2 = -2i$

   $\bar{y} = c_1 \cos 2x + c_2 \sin 2x$

2. The right hand side of the given equation is written as $\cos 2x = 1e^{0x}\cos 2x + 0e^{0x}\sin 2x$

$P=1,\ Q=0,\ \alpha=0,\ \beta=2$,

$\alpha+\beta i = 0+2i = 2i$ is a root of characteristic equation, so $r=1$

Highest degree of P and Q is zero since $P=1,\ Q=0$

Then, $u=A,\ v=B$ and $y^{*} = x\left(A\cos 2x + B\sin 2x\right)$

3. $y^{*/} = A\cos 2x + B\sin 2x + x\left(-2A\sin 2x + 2B\cos 2x\right)$

$= A\cos 2x + B\sin 2x + 2x\left(-A\sin 2x + B\cos 2x\right)$

$$y^{*//} = -2A\sin 2x + 2B\cos 2x - 2A\sin 2x + 2B\cos 2x + 2x\left(-2A\cos 2x - 2B\sin 2x\right)$$

$$= 4\left(-A\sin 2x + B\cos 2x\right) + 4x\left(-A\cos 2x - B\sin 2x\right)$$

$$\Rightarrow\ 4\left(-A\sin 2x + B\cos 2x\right) + 4x\left(-A\cos 2x - B\sin 2x\right) + 4x\left(A\cos 2x + B\sin 2x\right) = \cos 2x$$

$$\Rightarrow\ -4A\sin 2x + 4B\cos 2x - 4x\left(A\cos 2x + B\sin 2x\right) + 4x\left(A\cos 2x + B\sin 2x\right) = \cos 2x$$

$$\Rightarrow\ -4A\sin 2x + 4B\cos 2x = \cos 2x$$

$$\begin{cases} -4A = 0 \\ 4B = 1 \end{cases} \Rightarrow \begin{cases} A = 0 \\ B = \dfrac{1}{4} \end{cases}$$

$$y^{*} = x\left(0\cos 2x + \frac{1}{4}\sin 2x\right) = \frac{1}{4}x\sin 2x$$

Synthesis

If the right hand side of the equation $y'' + py' + qy = r(x)$ is $r(x) = Pe^{\alpha x}\cos\beta x + Qe^{\alpha x}\sin\beta x$ where P and Q are polynomials, two cases arise:

- If $\alpha + i\beta$ is not a root of characteristic equation, the particular solution is

  $y^{*} = Ue^{\alpha x}\cos\beta x + Ve^{\alpha x}\sin\beta x$

- If $\alpha + i\beta$ is a root of characteristic equation, the particular solution becomes,

  $y^* = x\left[Ue^{\alpha x}\cos\beta x + Ve^{\alpha x}\sin\beta x\right]$

In all cases, U and V are polynomial for which their degree is equal to the highest degree of P and Q.

Application Activity 5.98

1. $y = c_1 e^x + c_2 x e^x + \frac{1}{2}\left(x\cos x + \cos x - \sin x\right)$
2. $y = c_1 e^x + c_2 x e^x - e^x\left(x\sin x + 2\cos x\right)$
3. $y = c_1 e^x + c_2 e^{-x} - \frac{1}{10}\left(\frac{3}{5}\cos 3x + x\sin 3x + 5\cos x\right)$
4. $y = c_1\cos x + c_2\sin x - x\cos x + \sin x\ln\left|\sin x\right|$
5. $y = c_1\cos x + c_2\sin x - \cos x\ln\left|\sec x + \tan x\right|$

Lesson 5.13. Applications: Newton's law of cooling

Learning objectives

By reading textbooks or accessing internet, learners should be able to use differential equations to solve problems involving Newton's law of cooling accurately.

Prerequisites

- Solving differential equations

Teaching Aids

Exercise book, calculator, library or internet if available and pen

Activity 5.13

In this lesson, the following generic competence and cross-cutting issues are to be addressed:

- Critical thinking
- Communication
- Self confidence
- Cooperation, interpersonal management and life skills

- Peace and values education
- Inclusive education
- Research

Answers

To formulate a model, we need to know something about how a liquid cools.

Experimental evidence shows that the rate at which temperature changes is proportional to the difference in temperature between the liquid and the surrounding (ambient) air. If T is the temperature of the liquid at time t then in this case;

$\frac{dT}{dt} = -k(T-20)$ where k is the constant of proportionality and the negative sign shows that the temperature is reducing.

When coffee is made, its temperature is $90^0 C$. So $T = 90^0 C$ when $t = 0$.

In formulating this model, we assume that;

- The temperature throughout the coffee is uniform.
- The temperature of surrounding air is constant.
- The rate of cooling of a body is proportional to the temperature of the body above that of the surrounding air.

Synthesis

Newton's law of cooling states that the rate at which an object cools is proportional to the difference between the temperature at the surface of the body, and the ambient air temperature.

Thus, if T is the surface temperature at time t and T_a is the ambient temperature, then $\frac{dT}{dt} = -\lambda(T - T_a)$ where $\lambda > 0$ is some experimentally determined constant of proportionality, and T_0 is the initial temperature.

Lesson 5.14. Applications: Electrical circuits

Learning objectives

By reading textbooks or accessing internet, learners should be able to use differential equations in solving electrical circuit problems accurately.

Prerequisites

- Solving differential equations.
- Alternating current.

Teaching Aids

Exercise book, calculator, library or internet if available and pen

Activity 5.14

In this lesson, the following generic competence and cross-cutting issues are to be addressed:

- Critical thinking
- Communication
- Self confidence
- Research
- Cooperation, interpersonal management and life skills
- Peace and values education
- Inclusive education

Answers

a) Rearranging $E-L\left(\frac{di}{dt}\right)=Ri$, gives $\frac{di}{dt}=\frac{E-Ri}{L}$ and separating the variables, we get $\frac{di}{E-Ri}=\frac{dt}{L}$

Integrating both sides gives:

$$\int\frac{di}{E-Ri}=\int\frac{dt}{L}\Rightarrow-\frac{1}{R}\ln\left(E-Ri\right)=\frac{t}{L}+c$$

When $t=0$, $i=0$, thus $-\frac{1}{R}\ln E=c$

Thus, the particular solution is:

$$-\frac{1}{R}\ln\left(E-Ri\right)=\frac{t}{L}-\frac{1}{R}\ln E$$

Rearranging gives:

$$\Leftrightarrow -\frac{1}{R}\ln\left(E-Ri\right)+\frac{1}{R}\ln E=\frac{t}{L}$$

$$\Leftrightarrow -\frac{1}{R}\ln\frac{E-Ri}{E}=\frac{t}{L}$$

$$\Leftrightarrow \frac{1}{R}\ln\frac{E}{E-Ri}=\frac{t}{L} \Leftrightarrow \ln\frac{E}{E-Ri}=\frac{Rt}{L}$$

from which

$$\frac{E}{E-Ri}=e^{\frac{Rt}{L}} \Leftrightarrow \frac{E-Ri}{E}=e^{-\frac{Rt}{L}}$$

$$\Leftrightarrow E-Ri=Ee^{-\frac{Rt}{L}} \Rightarrow Ri=E-Ee^{-\frac{Rt}{L}}$$

Therefore, $i=\frac{E}{R}\left(1-e^{-\frac{Rt}{L}}\right)$

b) $i=\frac{E}{R}\left(1-e^{-3}\right)\approx 0.95\frac{E}{R}$

c) $i=\frac{E}{R}\left(1-e^{-2}\right)\approx 0.86\frac{E}{R}$ i.e. 86%

Synthesis

In the R–L series circuit shown in figure 5.1, the supply p.d., E, is given by

$E=V_R+V_L$, $V_R=iR$ and $V_L=L\frac{di}{dt}$

Hence $E=iR+L\frac{di}{dt}$. From which $E-L\frac{di}{dt}=iR$

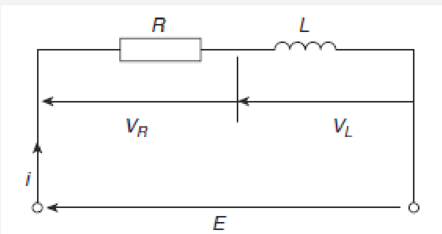


Figure 5.1: R-L series

The corresponding solution is $i=\frac{E}{R}\left(1-e^{-\frac{Rt}{L}}\right)$ which represents the law of growth of current in an inductive circuit as shown in figure 5.2

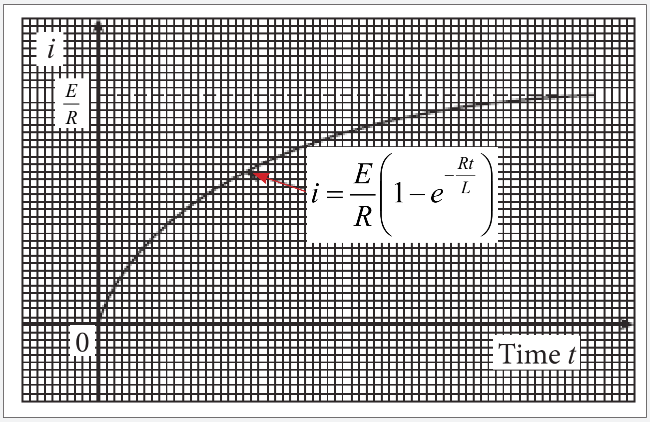


Figure 5.2: Law of growth of current

The growth of the current in the RL circuit is the current's steady-state value. The number $t=\frac{L}{R}$ is the time constant of the circuit. The current gets to within 5% of its steady-state value in 3 times constant.

Application Activity 5.99

1. P indicates number of rabbits, t time in months.

   Differential equation:

   $\frac{dP}{dt}=0.7P$, $P=10$ when $t=0$

2. To be proved

3. a) $\frac{dT}{dt}=k(1-32.2)$

   $T(0)=34.8, T(1)=34.1$

   T is temperature in ^{o}C, t is time hours after 2:30 a.m. and k is constant.

   Assume that the rate of temperature change is proportional to the difference between body temperature and room temperature. Assume room temperature is constant.

   b) $T=32.2+2.6e^{-0.31t}$

   c) 0:33 a.m. $(t=-117\,\text{minutes})$

4. (a) $Q = Q_o e^{-\frac{t}{CR}}$ (b) $9.30C, 5.81C$

5. $273.3N, 2.31rads$

5.6. Summary of the unit

1. Definition and classification

An equation involving one or more differential coefficients i.e. $\frac{dy}{dx}, \frac{d^2y}{dx^2}, \frac{dr}{dt}$ is called a **differential equation**.

Order of the highest derivative of function that appears in a differential equation is said to be the **order of differential equation**.

The general ordinary differential equation of the n^{th} order is

$F\left(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, ..., \frac{d^ny}{dx^n}\right) = 0$, OR

$F\left(x, y, y', y'', ..., y^{(n)}\right) = 0$

2. First order differential equations

The general differential equation of the 1st order is

$F\left(x, y, \frac{dy}{dx}\right) = 0$ or $\frac{dy}{dx} = f(x, y)$

The simplest is that in which the variables are separable:

$\frac{dy}{dx} = g(x)h(y)$.

A homogeneous equation of degree 0 can be expressed as a function of $z = \frac{y}{x}$.

The general solution to the equation $\frac{dy}{dx} + py = q$ where p and q are functions in x or constants, is $y = uv$ where $u = \int q e^{\int pdx} dx$ and $v = e^{-\int pdx}$.

3. Second order differential equations

The general second order linear differential equation is of the form

$$\frac{d^2y}{dx^2}+p(x)\frac{dy}{dx}+q(x)y=r(x)$$

Let $y''+py'+qy=0$ be a homogeneous linear equation of second order (right hand side is equal to zero) where p and q are constants.

The equation $m^2+pm+q=0$ is called the **characteristic auxiliary equation.**

- If characteristic equation has two distinct real roots then, $y=c_1e^{m_1x}+c_2e^{m_2x}$ is the general solution of $y''+py'+qy=0$.
- If characteristic equation has a real double root then, $y=c_1e^{mx}+c_2xe^{mx}$ is the general solution of $y''+py'+qy=0$.
- If characteristic equation has complex roots then, $y=e^{ax}\left(c_1\cos bx+c_2\sin bx\right)$ is the general solution of $y''+py'+qy=0$.

Let $y''+py'+qy=r(x)$ be a non-homogeneous linear equation of second order (right hand side is different from zero) where p and q are real numbers.

- If the right hand side of the equation $y''+py'+qy=r(x)$ has the form $r(x)=Pe^{\alpha x}$ where P is a polynomial, then the particular solution will be $y^*=x^kQ_n(x)e^{\alpha x},\ Q_n=a_0x^n+a_1x^{n-1}+a_2x^{n-2}+.....+a_n$,

where k - is the number of real roots of the associated homogeneous equation that equals to α;

α is the coefficient of x in $e^{\alpha x}$ in the right hand side and n is the degree of $Q(x)$ that is the same as the degree of $P(x)$ for $r(x)$.

- If the right hand side of the equation $y''+py'+qy=r(x)$ is of the form $r(x)=Pe^{\alpha x}\cos\beta x+Qe^{\alpha x}\sin\beta x$ where P and Q are polynomials, two cases arise:

  $\alpha+i\beta$ *is not a root of characteristic equation.*
  Here, the particular solution will be
  $y^{*}=Ue^{\alpha x}\cos\beta x+Ve^{\alpha x}\sin\beta x$
  $\alpha+i\beta$ *is a root of characteristic equation;*
  Then, the particular solution is
  $y^{*}=x\left[Ue^{\alpha x}\cos\beta x+Ve^{\alpha x}\sin\beta x\right]$.

In all cases, U and V are polynomials of degree that is equal to the highest degree of P and Q.

Alternative method: Variation of parameters

We know that the general solution of the characteristic equation associated with the equation $y''+py'+qy=r(x)$ is found to be $\overline{y}=c_1y_1(x)+c_2y_2(x)$.

From $\overline{y}=c_1y_1(x)+c_2y_2(x)$, we can get particular solution y^{*} as follows:

- We determine $W(y_1,y_2)$ known as **Wronskian** of two functions y_1 and y_2 defined by

  $W(y_1,y_2)=\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}\neq 0$, since y_1 and y_2 are linearly independent.

- We find out $v_1=\int\frac{-y_2r(x)}{W(y_1,y_2)}$, and $v_2=\int\frac{y_1r(x)}{W(y_1,y_2)}$ where $r(x)$ is the right hand side of the given equation.

  Then, particular solution y^{*} is given by
  $y^{*}=v_1(x)y_1(x)+v_2(x)y_2(x)\cdot$
  Therefore, the general solution is $y=\overline{y}+y^{*}$
  Or $y=c_1y_1(x)+c_2y_2(x)+v_1(x)y_1(x)+v_2(x)y_2(x)$

4. Applications

There are a number of well-known applications of first order equations which provide classic prototypes for mathematical modeling. These mainly rely on the interpretation of $\frac{dy}{dt}$ as a rate of change of a function y with respect to time t. In everyday life, there are many examples of the importance of rates of change – speed of moving particles, growth and decay of populations and materials, heat flow, fluid flow, and so on. In each case, we can construct models of varying degrees of sophistication to describe given situations.

5.7. End of unit assessment

1. a) $\frac{x^2}{2}+1$ b) $\frac{1}{2}\left(x^2+1\right)+\ln x$ c) $-\cos-\frac{1}{\pi}x+1$

   d) $ce^{-x}+\frac{1}{3}e^{2x}$ e) e^x+ce^{-2x} f) $\frac{x^3+c}{x-1}$

   g) $cx^2-\frac{x^2}{2}e^{-2x}$

2. a) $e^{3y}=\frac{3}{2}e^{2x}+c$ b) $y=cx^xe^{-x}$

   c) $\left(y^2-1\right)^2=cx$ d) $y=x(cx-3)$

   e) $x^2+y^2=kx$ f) $(2y-x)^4=c(x+y)$

   g) $y=\frac{1}{5}e^{2x}+ce^{-3x}$ h) $y=\sqrt{1-x^2}+c\left(1-x^2\right)$

3. a) $\ln\left(x^2y\right)=2x-y-1$ b) $y=x(2-\ln x)$

   c) $y=x\sqrt{8x+1}$ d) $y=e^x(x-1)$

   e) $y=\tan x+2\sec x$ f) $y=c_1e^{-2x}+c_2e^x$

4. a) $y=c_1e^{-2x}+c_2e^x$ b) $y=e^{-2x}\left(c_1\cos x+c_2\sin x\right)$

   c) $y=c_1e^{2x}+c_2xe^{2x}$ d) $c_1\cos 2x+c_2\sin 2x$

   e) $y=c_1e^{-3x}+c_2e^{3x}$

5. a) $y=c_1e^{-x}+c_2e^{-3x}+\frac{1}{8}(11-4x)$

b) $y = c_1 \cos 2x + c_2 \sin 2x - \frac{e^{2x}}{4}(x+2)$

c) $y = e^{-x}(c_1 \cos x + c_2 \sin x) - \frac{14}{85}\cos 3x + \frac{12}{85}\sin 3x$

d) $y = c_1 \cos 2x + c_2 \sin 2x + \frac{1}{10}e^{-x}(2\cos x - \sin x)$

e) $y = c_1 \cos 2x + c_2 \sin 2x + \frac{1}{5}e^{x} - \frac{x}{4}\cos 2x$

f) $y = c_1 e^{x} + c_2 e^{3x} + \frac{1}{2}e^{3x}(x^2 - x) + \frac{3}{8}e^{3x}(\sin 2x - \cos 2x)$

6. a) $\frac{2}{9}\left(e^{2x} - e^{-\frac{5x}{2}}\right)$ b) $-e^{\pi - 2x}\sin 3x$

7. a) $\frac{1}{4}(e^{x} - e^{-3x})$ b) $-\frac{1}{4}\cos 2x - \frac{\pi}{16}\sin 2x + \frac{1}{4}x + \frac{1}{4}$

c) $\frac{2}{3}\sin x - \frac{1}{3}\sin 2x$ d) $\frac{1}{6}e^{3x} - \frac{3}{2}e^{x} + x + \frac{4}{3}$

e) $y = \frac{4}{3}e^{5x} - \frac{10}{3}e^{2x} - \frac{1}{3}xe^{2x} + 2$

f) $y = 2e^{-\frac{3}{2}x} - 2e^{2x} + \frac{3}{29}e^{x}(3\sin x - 7\cos x)$

g) $y = e^{x}(3\cos x + \sin x) - e^{x}\cos 2x$

8. m is mass and t is time.

$\frac{dm}{dt} = -km, k$ is a constant.

9. $\frac{dP}{dt} = kP(1{,}500 - P), k$ is a constant.

10. h is height in cm, t is time in days, $\frac{dh}{dt} = 0.25h,\ h = 2$ when $t = 0$.

11. a) $q(t) = EC + (q_o - EC)e^{-\frac{t}{RC}}$ b) EC

c) $-RC\ln\left(\frac{0.01EC}{q_o - EC}\right)$

12. $47.22^{o}C$

13. $77.9^{0}C$

Unit 6

Intersection and Sum of Subspaces

6.1. Key unit competence

Relate the sum and the intersection of subspaces of a vector space by the dimension formula.

6.2. Vocabulary or key words concepts

Dimension: Number of vectors of the basis of a vector space (or a subspace).

Grassmann's formula: Relation connecting dimensions of subspaces.

6.3. Guidance on the introductory activity

Organize groups of students, and then assign them to do the introductory activity from the student's book. As they are working, move around to each group and ask them probing questions leading them to the right way. After a while, invite some group to present their findings in a whole class discussion. Guide students to harmonize their answers and arouse their curiosity to the content of this unit.

6.4. List of lessons

No	Lesson title	Number of periods
1	Definition of subspaces	4
2	Intersection of subspaces	2
3	Dimension of intersection of subspaces	2
4	Sum of subspaces	2
5	Dimension of sum of subspaces	2
6	Grassmann's formula of dimension for subspaces	2
Total periods		14

6.5. Lesson development

Lesson 6.1. Definition of subspaces

Learning objectives

Through examples, learners should be able to verify that a subset V of $\mathbb{R}^n$ is a subspace of $\mathbb{R}^n$ accurately.

Prerequisites

- Vector space

Teaching Aids

Exercise book and pen

Activity 6.1

In this lesson, the following generic competence and cross-cutting issues are to be addressed:

- Critical thinking
- Communication
- Self confidence
- Cooperation, interpersonal management and life skills
- Peace and values education
- Inclusive education

Answers

1. $(2x,0,5x)=(0,0,0)$

   $$\begin{cases}2x=0\\5x=0\end{cases}\Rightarrow x=0$$

   Thus, the value of x is 0.

2. $\alpha\vec{u}+\beta\vec{v}=\alpha(2a,0,5a)+\beta(2b,0,5b)$

   $$=(2\alpha a,0,5\alpha a)+(2\beta b,0,5\beta b)=(2\alpha a+2\beta b,0,5\alpha a+5\beta b)$$
   $$=(2(\alpha a+\beta b),0,5(\alpha a+\beta b))=(2x,0,5x)\quad for\quad x=\alpha a+\beta b$$

   Hence, $\alpha\vec{u}+\beta\vec{v}\in V$

3. From results in 1) and 2) and since V is a subset of $\mathbb{R}^3$, we conclude that V is a sub-vector space.

Synthesis

A subset V of $\mathbb{R}^n$ is called a **sub-vector space**, or just a **subspace** of $\mathbb{R}^n$ if it has the following properties:

- The null vector belongs to V.
- V is closed under vector addition, i.e if $\vec{u},\vec{v}\in V$ then $\vec{u}+\vec{v}\in V$.
- V is closed under scalar multiplication, i.e if $\alpha\in\mathbb{R},\ \vec{u}\in V$, $\alpha\vec{u}\in V$.

Application Activity 6.100

1. a) No, S is not closed under multiplication:
   $$\begin{bmatrix}1\\1\end{bmatrix}\in S\ \ but-\begin{bmatrix}1\\1\end{bmatrix}=\begin{bmatrix}-1\\-1\end{bmatrix}\notin S$$
   b) Yes, all properties are verified.
2. a) This is a subspace. It contains $(1,0,0)$ and $(2,\ 1,\ 0)$.
   b) This is a subspace. It contains $(2,\ 1,\ 0)$ and $(3,\ 0,-3)$.
   c) This is not a subspace. It doesn't contain $(0,0,0)$.
3. From results in 1) and 2) and since V is a subset of $\mathbb{R}^3$, we conclude that V is a sub-vector space.

Lesson 6.2. Intersection of subspaces

Learning objectives

Given two subspaces, learners should be able to find their intersection and verify that this intersection is also a subspace correctly.

Prerequisites

- Subspace properties

Teaching Aids

Exercise book and pen

Activity 6.2

In this lesson, the following generic competence and cross-cutting issues are to be addressed:

- Critical thinking
- Communication
- Self confidence
- Cooperation, interpersonal management and life skills
- Peace and values education
- Inclusive education

Answers

$$\begin{cases} 2x - y + 3z = 0 \\ x + y + z = 0 \end{cases}$$

$$3x + 4z = 0 \Rightarrow x = -\frac{4}{3}z$$

$$y = -x - z = -\left(-\frac{4}{3}z\right) - z = \frac{4}{3} - z = \frac{1}{3}z$$

Then,

$$H \cap K = \left\{\left(-\frac{4}{3}z, \frac{1}{3}z, z\right) : z \in \mathbb{R}\right\}$$ or

$$H \cap K = \{(-4x, x, 3x) : x \in \mathbb{R}\}$$

Synthesis

Let U and W be subspaces of a vector space V. The intersection of U and W, written $U \cap W$, consists of all vectors $\vec{u}$ where $\vec{u} \in U$ and $\vec{u} \in W$.

Any intersection of subspaces of a vector space V is a subspace of V.

Properties:

- For any two subspaces U and W, $U \cap W = W \cap U$
- If U and W are subspaces of a vector space V, then $U \cap W$ is also a subspace of V.

Application Activity 6.101

1. $U \cap W = \left\{ \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} : a \in \mathbb{R} \right\}$
2. $H \cap K = \left\{ functions\ f\ \ on\,\mathbb{R} : f(2) = f(1) = 0 \right\}$
3. $U \cap V = \{(0,0)\}$
4. $U_1 \cap U_2 = \{(0, y, 0) : y \in \mathbb{R}\}$
5. $U_1 \cap U_2 = \{(0,0,0)\}$

Lesson 6.3. Dimensions of intersection of subspaces

Learning objectives

Given two subspaces, learners should be able to find their intersection and the dimension of the intersection accurately.

Prerequisites

- Intersection of two subspaces.

Teaching Aids

Exercise book and pen

Activity 6.3

In this lesson, the following generic competence and cross-cutting issues are to be addressed:

- Critical thinking
- Communication
- Self confidence
- Cooperation, interpersonal management and life skills
- Peace and values education
- Inclusive education

Answers

a) $U \cap W$

$$\begin{cases} x = x \\ y = y \\ 0 = z \end{cases}$$

b) $U \cap W = \{(x, y, 0) : x, y \in \mathbb{R}\}$

$U \cap W = \{(x, y, 0) : x, y \in \mathbb{R}\}$

$(x, y, 0) = (x, 0, 0) + (0, y, 0)$

$= x(1,0,0) + y(0,1,0)$

The vectors $(1,0,0)$ and $(0,1,0)$ are linearly independent. Then basis of $U \cap W$ is $\{(1,0,0),(0,1,0)\}$ and hence $\dim(U \cap W) = 2$.

Synthesis

A finite set S of vectors in a vector space V is called a basis for V provided that;

- The vectors in S are linearly independent.
- The vector in S span V (or S is a generating set of V).

The unique number of vectors in each basis for V is called the dimension of V and is denoted by $\dim(V)$.

The dimension of $U \cap W$ is the number of vectors of the basis for $U \cap W$.

Application Activity 6.102

1. $\dim(U \cap W) = 1$
2. $\dim(U \cap W) = 1$
3. $\dim(H \cap K) = 2$

Lesson 6.4. Sum of subspaces

Learning objectives

Given two subspaces, learners should be able to find their sum and verify if the sum is a subspace accurately.

Prerequisites

- Properties of subspaces

Teaching Aids

Exercise book and pen

Activity 6.4

In this lesson, the following generic competence and cross-cutting issues are to be addressed:

- Critical thinking
- Communication
- Self confidence
- Cooperation, interpersonal management and life skills
- Peace and values education
- Inclusive education

Answers

For $U=\{(a,0,c):a,c\in\mathbb{R}\}$ and $W=\{(0,b,b):b\in\mathbb{R}\}$

$$U+W=\{(a,0,c)+(0,b,b):a,b,c\in\mathbb{R}\}$$

$$=\{(a,b,c+b):a,b,c\in\mathbb{R}\}$$

Clearly, $(0,0,0)\in U+W$. Let $\vec{u}=(x,y,z),\vec{w}=(x',y',z')\in U+W$ and $\alpha,\beta\in U+W$

$$\alpha\vec{u}+\beta\vec{v}=\alpha(x,y,z)+\beta(x',y',z')$$

$$=(\alpha x,\alpha y,\alpha z)+(\beta x',\beta y',\beta z')$$

$$=(\alpha x+\beta x',\alpha y+\beta y',\alpha z+\beta z')$$

$$=(a,b,c)\in U+W$$

$$\text{for } a=\alpha x+\beta x',\ b=\alpha y+\beta y',\ c=\alpha z+\beta z'$$

Thus, $U+W$ is a sub space of $\mathbb{R}^3$

Synthesis

Let U and W be subspaces of a vector space V. The sum of U and W, written $U+W$, consists of all sums $x+y$ where $x\in U$ and $y\in W$.

- The sum $U+W$ of the subspaces U and V is also a subspace of V.
- W_1 and W_2 are subspace of V, then W_1+W_2 is the smallest subspace that contains both W_1 and W_2.

Application Activity 6.103

1. Let $\vec{v} \in U+W$. Then $\vec{v} = \vec{u} + \vec{w},\ \vec{u} \in U$ and $\vec{w} \in W$. Since $\{u_i\}$ generates U, $\vec{u}$ is a linear combination of $\overrightarrow{u_i}\text{'}s$; and since $\{\overrightarrow{w_j}\}$ generates W, $\vec{w}$ is a linear combination of $\overrightarrow{w_j}\text{'}s$.

   Thus
   $\vec{v} = \vec{u} + \vec{w} = a_1\overrightarrow{u_{i1}} + a_2\overrightarrow{u_{i2}} + ... + a_n\overrightarrow{u_{in}} + b_1\overrightarrow{w_{j1}} + b_2\overrightarrow{w_{j2}} + ... + b_m\overrightarrow{w_{jm}}$
   and so $\{\overrightarrow{u_i}, \overrightarrow{w_j}\}$ generates $U+V$.
2. $U+W = \left\{\begin{pmatrix} e & b \\ d & 0 \end{pmatrix} : b, d, e \in \mathbb{R}\right\}$
3. $\{(2a-b, 3a+4b, 5a+3b) : a, b \in \mathbb{R}\}$

Lesson 6.5. Dimension of sum of subspaces

Learning objectives

Given two subspaces, learners should be able to find their sum and the dimension of the sum accurately.

Prerequisites

- Sum of subspaces.

Teaching Aids

Exercise book and pen

Activity 6.5

In this lesson, the following generic competence and cross-cutting issues are to be addressed:

- Critical thinking
- Communication
- Self confidence
- Cooperation, interpersonal management and life skills
- Peace and values education
- Inclusive education

Answers

1. $U+W=\{(a,0,0)+(0,b,0):a,b\in\mathbb{R}\}=\{(a,b,0):a,b\in\mathbb{R}\}$
2. $U+W=\{(a,b,0):a,b\in\mathbb{R}\}$

   $(a,b,0)=(a,0,0)+(0,b,0)$

   $\quad\quad\quad\;=a(1,0,0)+b(0,1,0)$

   The vectors $(1,0,0)$ and $(0,1,0)$ are linearly independent. Then, basis of $U+W$ is $\{(1,0,0),(0,1,0)\}$ and hence $\dim(U+W)=2$.

Synthesis

A finite set S of linearly independent vectors in the sum $U+V$ is called a basis for $U+V$ and the number of vectors in set S is the dimension of $U+V$.

Application Activity 6.104

1. $\dim(H+K)=2$
2. $\dim(U+V)=2$
3. $\dim(U_1+U_2)=2$

Lesson 6.6. Grassmann's formula of dimension for subspaces

Learning objectives

Given two subspaces, learners should be able to use Grassmann's formula to find the dimension of the sum or intersection correctly.

Prerequisites

- Sum of subspaces.
- Intersection of subspaces.

Teaching Aids

Exercise book and pen

Activity 6.6

In this lesson, the following generic competence and cross-cutting issues are to be addressed:

- Critical thinking
- Communication
- Self confidence
- Cooperation, interpersonal management and life skills
- Peace and values education
- Inclusive education

Answers

1. a) $\dim(F)=2$ and $\dim(G)=2$

b) $\dim(F)+\dim(G)=2+2=4$

c) $F\cap G=\{(0,0,z): z\in\mathbb{R}\}$ and $\dim(F\cap G)=1$

d) $\dim(F)+\dim(G)-\dim(F\cap G)=4-1=3$

e) $F+G=\{(x,y,z): x,y,z\in\mathbb{R}\}$ and $\dim(F+G)=3$

2. From results in d) and e),

$$\dim(F+G)=\dim(F)+\dim(G)-\dim(F\cap G)$$

Synthesis

- If $(\mathbb{R},F,+)$ and $(\mathbb{R},G,+)$ are two sub-vector spaces of $(\mathbb{R},E,+)$, we have,

  $\dim(F+G)=\dim(F)+\dim(G)-\dim(F\cap G)$.

- If $\dim(F\cap G)=0$, then $\dim(F+G)=\dim(F)+\dim(G)$. In this case, F and G are said to be **complementary** and the sum $F+G$ is said to be a direct sum; and it is denoted by $F\oplus G$. Otherwise, F and G are said to be **supplementary**.

Application Activity 6.105

1. $\dim(V)=3,\ \dim(W)=2,\ \dim(V\cap W)=1$

   $\dim(V+W)=\dim(V)+\dim(W)-\dim(V\cap W)=3+2-1=4$

2. $\mathbb{R}^3$ cannot be uniquely represented as a direct sum of W_1 and W_2.

3. $F=W_1+W_2$ is a direct sum. i.e, $F=W_1\oplus W_2$.

4. No, since $\dim(F\cap G)=3\neq 0$.

5. Since U is not equal to W, the basis for U must have at least one vector linearly independent from U, so $\dim(U+W)$ is at least 4. But they are subspaces of $\mathbb{R}^4$, so $\dim(U+W)=4$. Using the fact that $\dim(U+W)=\dim(U)+\dim(W)-\dim(U\cap W)$.

   Then, $4=3+3-\dim(U\cap W)\Rightarrow \dim(U\cap W)=6-4=2$.

6.6. Summary of the unit

1. Definition

If $(\mathbb{R},F,+)$ is a subspace of $(\mathbb{R},E,+)$, then

- $F\subset E$
- $\vec{0}\in F$
- $\vec{u},\vec{v}\in F,\alpha,\beta\in\mathbb{R};\quad \alpha\vec{u}+\beta\vec{v}\in F$

2. Intersection and sum of two vector spaces

Let U and W be subspaces of a vector space V. The intersection of U and W, written $U\cap W$, consists of all vectors $\vec{u}$ where $\vec{u}\in U$ and $\vec{u}\in W$.

Any intersection of subspaces of a vector space V is a subspace of V. W_1 and W_2 are subspaces of V, then $W_1\cup W_2$ is a subspace $\Leftrightarrow W_1\subseteq W_2\ or\ W_2\subseteq W_1$.

If F and G are two sub-vector spaces of E, then, the sum of F and G is also a sub-vector space of E. It is denoted as $F+G=\{x+y, x\in F, y\in G\}$.

Grassmann's formula of dimensions.

If $(IR, F, +)$ and $(\mathbb{R}, G, +)$ are two sub-vector spaces of $(\mathbb{R}, E, +)$, we have,

$\dim(F+G)=\dim(F)+\dim(G)-\dim(F\cap G)$.

Remark

If $\dim(F\cap G)=0$, then $\dim(F+G)=\dim(F)+\dim(G)$. In this case, F and G are said to be **complementary** and the sum $F+G$ is said to be a **direct sum**; and it is denoted by $F\oplus G$.

Otherwise, F and G are said to be **supplementary**.

6.7. End of Unit Assessment

1.a)Yes, this is a plane through origin because $x=y$ if and only if $x=0$ *and* $y=0$, then we have null vector $(0,0)$

b) No, this does not contain the origin; because the line $(1+t, 2t, 3t)$ does not respect null vector. when $t=0$, we have $(1,0,0)$

c) Yes, this is just the zero point,; because the locus $x^2+y^2+z^2=0$ it respect null vector if $x=0, y=0, z=0$ and only if and also respect vector addition.

d) No, because this locus does $x^2+y^2-z^2=0$ not respect vector space addition.

6. $\dim(E)=1$, $\dim(F)=2$

7. $dim(W)=3$

8. Since U and W are distinct, $U+W$ properly contains U and W; hence $\dim(U+W)>4$. Since $\dim(V)=6$, $\dim(U+W)$ cannot be greater than 6.

Hence, there are two possibilities:

a) $\dim(U+W)=5 \Leftrightarrow 5=4+4-\dim(U\cap W) \Rightarrow \dim(U\cap W)=3$

b) $\dim(U+W)=6 \Leftrightarrow 6=4+4-\dim(U\cap W) \Rightarrow \dim(U\cap W)=2$

9. a) $\dim(U+W)=3$ b) $\dim(U\cap W)=1$

10. The set of the symmetric matrices W_1 and the set of the skew symmetric matrices W_2 are both subspaces of $M_{n\times n}$.

$A\in W_1\cap W_2,\ A=A^t=-A^t \Rightarrow A=0$, $\therefore W_1\cap W_2=\{0\}$

Let $\begin{cases} B=\dfrac{1}{2}\left(A+A^t\right) \\ C=\dfrac{1}{2}\left(A-A^t\right) \end{cases}$

Then,

$B\in W_1,\ C\in W_2$, $\therefore M_{n\times n}=W_1\oplus W_2$

11. Yes, since $\dim(F\cap G)=0$.

Unit 7 Transformation of Matrices

7.1. Key unit competence

Transform matrices to an echelon form or to diagonal matrix and use the results to solve simultaneous linear equations or to calculate the n^{th} power of a matrix.

7.2. Vocabulary or key words concepts

Elementary row/column operations: Operations performed on row/column of a matrix (addition, scalar multiplication, and interchanging rows/columns) to obtain a new matrix.

Characteristic equation: Polynomial $|A-\lambda I|=0,\ \lambda\in\mathbb{R}$ where A is a given matrix and I is identity matrix of the same order as A.

Eigenvalue: The real number λ that is a root in the characteristic polynomial $|A-\lambda I|=0,\ \lambda\in\mathbb{R}$

Eigenvector: The vector $\vec{u}$ such that $(A-\lambda I)\vec{u}=\vec{0}$.

Row echelon form: Matrix is in row echelon form when the first non-zero element in each row (called the leading entry) is 1 and this leading entry is in a column to the right of the leading entry in the previous row. Rows with all zero elements, if any, are below rows having a non-zero element.

Reduced row echelon form: A matrix is in reduced row echelon form when it is in row echelon form and the leading entry in each row is the only non-zero entry in its column.

7.3. Guidance on the introductory activity

Organize groups of students, and then assign them to do the introductory activity from the student's book. As they are working, move around to each group and ask them probing questions leading them to the right way. After a while, invite some group to present their findings in a whole class discussion. Guide students to harmonize their answers and arouse their curiosity to the content of this unit.

7.4. List of lessons

No	Lesson title	Number of periods
1	Kernel and range	4
2	Elementary row/column operations	3
3	Eigenvalues and eigenvectors	4
4	Diagonalisation of a matrix	3
5	Echelon matrix	3
6	Inverse matrix	3
7	Rank of a matrix	3
8	Solving system of linear equations	3
9	Power of a matrix	3
Total periods		29

7.5. Lesson development

Lesson 7.1. Kernel and range

Learning objectives

By the end of this lesson, learners should be able to find kernel and range of a linear transformation.

Prerequisites

- Operation on vectors.
- Operations on matrices.

Teaching Aids

Exercise book and pen

Activity 7.1

In this lesson, the following generic competence and cross-cutting issues are to be addressed:

- Critical thinking

- Communication
- Self confidence
- Cooperation, interpersonal management and life skills
- Peace and values education
- Inclusive education

Answers

$(3x+y+2, 3x-y+1)=(0,0)$

$$\Rightarrow \begin{cases} 3x+y+2=0 \\ 3x-y+1=0 \end{cases}$$

$$\begin{cases} 3x+y+2=0 \\ \underline{3x-y+1=0} \end{cases}$$

$$6x \quad +3=0 \Rightarrow x=-\frac{1}{2}$$

$$3\left(-\frac{1}{2}\right)+y+2=0$$

$$\Rightarrow y=-2+\frac{3}{2}=-\frac{1}{2}$$

Thus, $(x,y)=\left(-\frac{1}{2},-\frac{1}{2}\right)$

Synthesis

- The **kernel** of a linear mapping $f:E\to F$ denoted $Ker(f)$ is a subset of E whose image by f is 0-vector of F. i.e, $Ker(f)=\{v\in E: f(v)=0\}$.
- The **nullity** of f denoted $n(f)$ is the dimension of $Ker(f)$. i.e, $n(f)=\dim Ker(f)$.
- The **image** or **range** of a linear mapping $f:E\to F$ is the set of vectors in F to which points in E are mapped on. i.e, $\operatorname{Im} f=\{u\in F: f(v)=u\},\ v\in E$.
- The **rank** of f denoted $rank(f)$ or $r(f)$ is the dimension of image of f. i.e, $rank(f)=\dim(\operatorname{Im} f)$.
- If $f:E\to F$, $\dim[Ker(f)]+\dim[range(f)]=\dim(E)$.

Application Activity 7.106

1. a) $\text{ImF} = \{(x,y,z): z=0\} = xy-plane$

   b) $KerF = \{(x,y,z): \text{x}=0, \text{y}=0\} = z-axis$.

2. a) Basis is $\{(1,0,1),(0,1,-1)\}$ and dimension is 2.

   b) Basis is $\{(3,-1,1)\}$ and dimension is 1.

3. a) Basis is $\{(1,1,1),(0,1,2)\}$ and dimension is 2.

   b) Basis is $\{(2,1,-1,0),(1,2,0,1)\}$ and dimension is 2.

Lesson 7.2. Elementary row/column operations

Learning objectives

Given a matrix, learners should be able to use row/column operations to transform correctly.

Prerequisites

- Adding row/column of a matrix.
- Multiplying a row/column of a matrix by a real number.

Teaching Aids

Exercise book, pen and calculator

Activity 7.2

In this lesson, the following generic competence and cross-cutting issues are to be addressed:

- Critical thinking
- Communication
- Self confidence
- Cooperation, interpersonal management and life skills
- Peace and values education
- Inclusive education

Answers

1. $\begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 3 & 2 & 1 \end{pmatrix}$

2. $\begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 0 & 2 & -5 \end{pmatrix}$

3. $\begin{pmatrix} 1 & 2 & 0 \\ 0 & 3 & 1 \\ 0 & -5 & 2 \end{pmatrix}$

4. $\begin{pmatrix} 1 & 2 & -\frac{2}{3} \\ 0 & 3 & 0 \\ 0 & -5 & \frac{11}{3} \end{pmatrix}$

Synthesis

Common row/column and their notations are;

Operation description		Notation
Row operations		
1. Interchange row i and j	$\to$	$r_i \leftrightarrow r_j$
2. Multiply row i by $s \neq 0$	$\to$	new $r_i \to sr_i$
3. Add s times row i to row j	$\to$	new $r_j \to r_j + sr_i$
Column operations		
1. Interchange column i and j	$\to$	$c_i \leftrightarrow c_j$
2. Multiply column i by $s \neq 0$	$\to$	new $c_i \to sc_i$
3. Add s times column i to column j	$\to$	new $c_j \to c_j + sc_i$

Application Activity 7.107

1. a) $\begin{pmatrix} 1 & 2 & -3 & 0 \\ 0 & 0 & 4 & 2 \\ 3 & 6 & -4 & 3 \end{pmatrix}$ b) $\begin{pmatrix} 1 & 2 & -3 & 0 \\ 0 & 0 & 4 & 2 \\ 0 & 0 & 5 & 3 \end{pmatrix}$

c) $\begin{pmatrix} 1 & 2 & -3 & 0 \\ 0 & 0 & 4 & 2 \\ 0 & 0 & 0 & 2 \end{pmatrix}$

2. a) $\begin{pmatrix} 6 & 9 & 0 & 7 & -2 \\ 0 & 0 & 3 & 2 & 5 \\ 0 & 0 & 0 & 0 & 2 \end{pmatrix}$ b) $\begin{pmatrix} 6 & 9 & 0 & 7 & 0 \\ 0 & 0 & 3 & 2 & 5 \\ 0 & 0 & 0 & 0 & 2 \end{pmatrix}$

c) $\begin{pmatrix} 6 & 9 & 0 & 7 & 0 \\ 0 & 0 & 6 & 4 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{pmatrix}$

Lesson 7.3. Eigenvalues and eigenvectors

Learning objectives

Given a matrix, learners should be able to find eigenvalues and eigenvectors accurately.

Prerequisites

- Operation on matrices.
- Matrix determinant.
- Solving equation of second/third degree.

Teaching Aids

Exercise book, pen and calculator

Activity 7.3

In this lesson, the following generic competence and cross-cutting issues are to be addressed:

- Critical thinking
- Communication
- Self confidence
- Cooperation, interpersonal management and life skills
- Peace and values education
- Inclusive education

Answers

1. $\det(A-\lambda I)=\left|\begin{pmatrix}4 & 2\\ 3 & -1\end{pmatrix}-\lambda\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\right|$

$$=\begin{vmatrix}4-\lambda & 2\\ 3 & -1-\lambda\end{vmatrix}$$

$$=(4-\lambda)(-1-\lambda)-6$$

$$=\lambda^2-3\lambda-10$$

2. $\lambda^2-3\lambda-10=0 \Leftrightarrow (\lambda+2)(\lambda-5)=0 \Rightarrow \lambda=-2 \; or \; \lambda=5$

3. $(A-\lambda I)\vec{u}=\vec{0}$

For $\lambda=-2$

$$\left(\begin{pmatrix}4 & 2\\ 3 & -1\end{pmatrix}+2\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\right)\begin{pmatrix}u_1\\ u_2\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix}$$

$$\begin{pmatrix}6 & 2\\ 3 & 1\end{pmatrix}\begin{pmatrix}u_1\\ u_2\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix}$$

$$\begin{cases}6u_1+2u_2=0\\ 3u_1+u_2=0\end{cases} \Rightarrow u_2=-3u_1$$

Thus, $\vec{u}=k\begin{pmatrix}1\\ -3\end{pmatrix},\; k\in\mathbb{R}_0$

For $\lambda=5$

$$\left(\begin{pmatrix}4 & 2\\ 3 & -1\end{pmatrix}-5\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\right)\begin{pmatrix}u_1\\ u_2\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix}$$

$$\begin{pmatrix}-1 & 2\\ 3 & -6\end{pmatrix}\begin{pmatrix}u_1\\ u_2\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix} \qquad \begin{cases}-u_1+2u_2=0\\ 3u_1-6u_2=0\end{cases} \Rightarrow u_1=2u_2$$

Thus, $\vec{u}=k\begin{pmatrix}2\\ 1\end{pmatrix},\; k\in\mathbb{R}_0$

Synthesis

The eigenvalues of square matrix A, are the roots of the polynomial $\det(A-\lambda I)$. The homogeneous system $(f-\lambda I)\vec{u}=\vec{0}$ gives the eigenvector $\vec{u}$ associated with eigenvalue λ.

Cayley and Hamilton theorem

The Cayley–Hamilton theorem (named after the mathematicians Arthur Cayley and William Rowan Hamilton) states that every square matrix over a commutative ring (such as the real or complex field) satisfies its own characteristic equation.

Note that an eigenvector cannot be 0, but an eigenvalue can be 0. If 0 is an eigenvalue of A, there must be some non-trivial vector $\vec{u}$ for which $\left(A-\vec{0}\right)\vec{u}=\vec{0}$.

Application Activity 7.108

1. a) Eigenvalues: 7 and -4, eigenvectors: $\vec{u}=\begin{pmatrix}3\\1\end{pmatrix}$ and $\vec{v}=\begin{pmatrix}2\\-3\end{pmatrix}$

   b) Eigenvalues: 7 and -4, eigenvectors: $\vec{u}=\begin{pmatrix}3\\1\end{pmatrix}$ and $\vec{v}=\begin{pmatrix}2\\-3\end{pmatrix}$

   c) No eigenvalues, no eigenvectors

2. a) $\lambda_1=2, u=(1,-1,0), v=(1,0,-1); \lambda_2=6, w=(1,2,1)$

   b) $\lambda_1=3, u=(1,1,0), v=(1,0,1); \lambda_2=, w=(2,-1,1)(\ )$

   c) $\lambda=1, u=(1,0,0), v=(0,0,1)$

3. Characteristic equation of matrix A is

   $|A-\lambda I|=0.$ (1)

   Characteristic equation of matrix A^t is

   $|A^t-\lambda I|=0.$ (2)

   Clearly, both (1) and (2) are the same, as we know that $|A|=|A^t|$.

   Therefore, A and A^t have the same eigenvalues.

4. Let us consider the triangular matrix,

$$A=\begin{pmatrix} a_{11} & 0 & \cdots & 0 \\ a_{21} & a_{22} & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$$

Characteristic equation is $|A-\lambda I|=0.$

$$\Leftrightarrow \begin{vmatrix} a_{11}-\lambda & 0 & \cdots & 0 \\ a_{21} & a_{22}-\lambda & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn}-\lambda \end{vmatrix}=0$$

$$\Leftrightarrow (a_{11}-\lambda)(a_{22}-\lambda)\cdots(a_{nn}-\lambda)=0$$

Therefore, $\lambda=a_{11},a_{22},\ldots a_{nn}$ are the elements of diagonal entries.

5. $\frac{1}{9}\begin{pmatrix} 7 & 2 & -10 \\ -2 & 2 & -1 \\ -1 & 1 & 4 \end{pmatrix}$

6. $A^5-4A^4-7A^3+11A^2-A-10I \;\; A+5I$

$$\begin{aligned}
&=\left(A^5-4A^4-5A^3\right)-2A^3+11A^2-A-10I \\
&=A^3\left(A^2-4A-5\right)-2A^3+11A^2-A-10I \\
&=0-2A^3+11A^2-A-10I \\
&=-\left(2A^3-8A^2-10A\right)+3A^2-11A-10I \\
&=0+3A^2-11A-10I \\
&=\left(3A^2-12A-15I\right)+A+5I \\
&=3\left(A^2-4A-5I\right)+A+5I=A+5I
\end{aligned}$$

Therefore, $A^5-4A^4-7A^3+11A^2-A-10I=A+5I$.

Lesson 7.4. Diagonalisation of a matrix

Learning objectives

Given a matrix, learners should be able to diagonalise that matrix accurately.

Prerequisites

- Operation on matrices

Teaching Aids

Exercise book, pen and calculator

Activity 7.4 Learner's Book page <?>

In this lesson, the following generic competence and cross-cutting issues are to be addressed:

- Critical thinking
- Communication
- Self confidence
- Cooperation, interpersonal management and life skills
- Peace and values education
- Inclusive education

Answers

1. $A=\begin{pmatrix}-4 & -6\\ 3 & 5\end{pmatrix}$

   Eigenvalues:

   $$\left|\begin{pmatrix}-4 & -6\\ 3 & 5\end{pmatrix}-\lambda\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\right|=0 \Rightarrow \begin{vmatrix}-4-\lambda & -6\\ 3 & 5-\lambda\end{vmatrix}=0$$

   $$\Rightarrow(-4-\lambda)(5-\lambda)+18=0$$

   $$\lambda^2-\lambda-2=0\Rightarrow\lambda=-1 \text{ or } \lambda=2$$

   Eigenvalues are -1 and 2

   Eigenvectors:

   For $\lambda=-1$

   $$\left[\begin{pmatrix}-4 & -6\\ 3 & 5\end{pmatrix}+\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\right]\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} -3 & -6 \\ 3 & 6 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad \begin{cases} -3x - 6y = 0 \\ 3x + 6y = 0 \end{cases}$$

$$\Rightarrow \begin{pmatrix} -3x - 6y \\ 3x + 6y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad \Rightarrow 3x = -6y \Rightarrow x = -2y$$

Eigenvector associated to $\lambda = -1$ has the form

$\begin{pmatrix} -2y \\ y \end{pmatrix}$, $y \in \mathbb{R}_0$. Take $\vec{u} = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$

For $\lambda = 2$

$$\left[\begin{pmatrix} -4 & -6 \\ 3 & 5 \end{pmatrix} - \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}\right]\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} -6 & -6 \\ 3 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad \begin{cases} -6x - 6y = 0 \\ 3x + 3y = 0 \end{cases}$$

$$\Rightarrow \begin{pmatrix} -6x - 6y \\ 3x + 3y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad \Rightarrow 3x = -3y \Rightarrow x = -y$$

Eigenvector associated to $\lambda = 2$ has the form

$\begin{pmatrix} -y \\ y \end{pmatrix}$, $y \in \mathbb{R}_0$. Take $\vec{u} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

2. From 1), $P = \begin{pmatrix} -2 & -1 \\ 1 & 1 \end{pmatrix}$

3. $P^{-1} = \dfrac{1}{-1}\begin{pmatrix} 1 & 1 \\ -1 & -2 \end{pmatrix} = \begin{pmatrix} -1 & -1 \\ 1 & 2 \end{pmatrix}$

4. $D = P^{-1}AP = \begin{pmatrix} -1 & -1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} -4 & -6 \\ 3 & 5 \end{pmatrix}\begin{pmatrix} -2 & -1 \\ 1 & 1 \end{pmatrix}$

$$= \begin{pmatrix} 4-3 & 6-5 \\ -4+6 & -6+10 \end{pmatrix}\begin{pmatrix} -2 & -1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 2 & 4 \end{pmatrix}\begin{pmatrix} -2 & -1 \\ 1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} -2+1 & -1+1 \\ -4+4 & -2+4 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 2 \end{pmatrix}$$

Matrix D is a diagonal matrix. Also, elements of the leading diagonal are the eigenvalues obtained in 1).

Synthesis

To diagonalise matrix A, we perform the following steps:

1. Find the eigenvalues.
2. If there is a non-real eigenvalue, the matrix cannot be diagonalised.
3. If all eigenvalues are real, find their associated eigenvectors (they must be linearly independent).
4. If the number of eigenvectors is not equal to the order of matrix A, then this matrix cannot be diagonalised.
5. If the number of eigenvectors is equal to the order of matrix A, form matrix P whose columns are elements of eigenvectors.
6. Find the inverse of P.
7. Find D, diagonal matrix of A by relation; $D = P^{-1}AP$.

Theorem

A $n \times n$ matrix is diagonalisable if and only if it has n linearly independent eigenvectors.

Application Activity 7.109

1. a) $\begin{pmatrix} 8 & 0 \\ 0 & -2 \end{pmatrix}$ b) $\begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix}$ c) $\begin{pmatrix} 7 & 0 \\ 0 & -3 \end{pmatrix}$

   d) $\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$ e) $\begin{pmatrix} -5 & 0 & 0 \\ 0 & -5 & 0 \\ 0 & 0 & 16 \end{pmatrix}$

2. a) 2 and -4 b) $S = \{(1,1,0),(0,1,1)\}$

   c) A is not diagonalisable.

Lesson 7.5. Echelon matrix

Learning objectives

Given a matrix and using elementary row/column operations, learners should be able to transform that matrix into its echelon form accurately.

Prerequisites

- Elementary row/column operations.

Teaching Aids

Exercise book, pen and calculator

Activity 7.5

In this lesson, the following generic competence and cross-cutting issues are to be addressed:

- Critical thinking
- Communication
- Self confidence
- Cooperation, interpersonal management and life skills
- Peace and values education
- Inclusive education

Answers

$$A = \begin{pmatrix} 8 & 3 \\ 1 & 2 \end{pmatrix}$$

Change first element of first row to 1.

$$\xrightarrow{r_1 = r_1 - 7r_2} \begin{pmatrix} 1 & -11 \\ 1 & 2 \end{pmatrix}$$

The first non-zero element in second row is 1 but it is not in a column to the right of the other in first row. So, this has to be changed to 0.

$$\xrightarrow{r_2 = r_2 - r_1} \begin{pmatrix} 1 & -11 \\ 0 & -13 \end{pmatrix}$$

Now, second element in second row has to be changed to 1.

$$\xrightarrow{r_2 = r_2 - \frac{14}{11} r_1} \begin{pmatrix} 1 & -11 \\ 0 & 1 \end{pmatrix}$$

Now, the first two conditions are satisfied.

For the third condition:

The first non-zero element in second row is not the only non-zero entry in its column. So -11, in first row, has to be changed to 0.

$$\xrightarrow{r_1 = r_1 + 11r_2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Now, all conditions are satisfied.

Synthesis

A matrix is in **row echelon form** (ref) when it satisfies the following conditions:

- The first non-zero element in each row, called the **leading entry (or pivot)**, is 1.
- Each leading entry is in a column to the right of the leading entry in the previous row.
- Rows with all zero elements, if any, are below rows having a non-zero element.

A matrix is in **reduced row echelon form** (rref) when it satisfies the following conditions:

- The matrix is in row echelon form (i.e., it satisfies the three conditions listed above).
- The leading entry in each row is the only non-zero entry in its column.

Application Activity 7.110

1. $\begin{pmatrix} 1 & -1 & 2 & 0 \\ 0 & 1 & 4 & -5 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 & 6 & 0 \\ 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

2. $\begin{pmatrix} 1 & -2 & 3 & 9 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 2 & 4 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{pmatrix}$

3. $\begin{pmatrix} 2 & -2 & 4 & -2 \\ 0 & 3 & 6 & 9 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 & 4 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

4. $\begin{pmatrix} 1 & -3 & 4 & 10 \\ 0 & 1 & -\frac{8}{7} & -\frac{23}{7} \\ 0 & 0 & 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 & \frac{4}{7} & 0 \\ 0 & 1 & -\frac{8}{7} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

5. $\begin{pmatrix} 1 & 2 & 3 & -2 & 3 \\ 0 & 1 & 1 & -1 & \frac{4}{3} \\ 0 & 0 & 0 & 1 & \frac{4}{9} \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 & 1 & 0 & \frac{1}{3} \\ 0 & 1 & 1 & 0 & \frac{16}{9} \\ 0 & 0 & 0 & 1 & \frac{4}{9} \end{pmatrix}$

6. $\begin{pmatrix} 1 & \tan\theta \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

Lesson 7.6. Matrix inverse

Learning objectives

Given a square matrix and using elementary row/column operations, learners should be able to find the inverse of that matrix correctly.

Prerequisites

- Use of elementary row operations.
- Properties of inverse matrix.

Teaching Aids

Exercise book, pen and calculator

Activity 7.6

In this lesson, the following generic competence and cross-cutting issues are to be addressed:

- Critical thinking
- Communication
- Self confidence
- Cooperation, interpersonal management and life skills
- Peace and values education
- Inclusive education

Answers

1. $M = \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 1 & 2 & 4 & 0 & 0 & 1 \end{array}\right)$

   We need to transform matrix A such that all elements of leading diagonal become 1 and other elements become zero

$$\xrightarrow{r_3 \to r_3 - r_1} \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & 1 & 3 & -1 & 0 & 1 \end{array}\right) \xrightarrow{r_3 \to r_3 - r_2} \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & -1 & 1 \end{array}\right)$$

$$\xrightarrow{r_1 \to r_1 - r_2} \left(\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & -1 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & -1 & 1 \end{array}\right)$$

$$\xrightarrow{r_1 \to r_1 + r_3} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & -2 & 1 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & -1 & 1 \end{array}\right)$$

$$\xrightarrow{r_2 \to r_2 - 2r_3} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & -2 & 1 \\ 0 & 1 & 0 & 2 & 3 & -2 \\ 0 & 0 & 1 & -1 & -1 & 1 \end{array}\right)$$

2. Matrix obtained in 1) is $\begin{pmatrix} 0 & -2 & 1 \\ 2 & 3 & -2 \\ -1 & -1 & 1 \end{pmatrix}$. Multiplying it by

the given matrix, gives

$$\begin{pmatrix} 0 & -2 & 1 \\ 2 & 3 & -2 \\ -1 & -1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 1 & 2 & 4 \end{pmatrix} = \begin{pmatrix} 0+0+1 & 0-2+2 & 0-4+4 \\ 2+0-2 & 2+3-4 & 2+6-8 \\ -1+0+1 & -1-1+2 & -1-2+4 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Observation: Multiplying matrix obtained in 1) by matrix A gives identity matrix.

Therefore, the new matrix is the inverse of the matrix A.

Synthesis

To calculate the inverse of A, denoted as A^{-1}, follow these steps:

Construct a matrix of type $M = (A \mid I)$, that is to say, A is in the left half of M and the identity matrix I is on the right.

Using elementary row operations, transform the left half, A, to the identity matrix located to the right, and the matrix that results in the right side will be the inverse of matrix.

Application Activity 7.111

1. $\begin{pmatrix} -1 & 2 & -4 \\ 1 & -1 & 3 \\ 0 & 0 & 1 \end{pmatrix}$

2. No inverse

3. $\begin{pmatrix} 1 & -3 & 2 \\ -3 & 3 & -1 \\ 2 & -1 & 0 \end{pmatrix}$

4. No inverse

Lesson 7.7. Rank of matrix

Learning objectives

Given a matrix and by using elementary row/column operations, learners should be able to find the rank of that matrix accurately.

Prerequisites

- Transformation of matrix using elementary row/column operations.

Teaching Aids

Exercise book, pen and calculator

Activity 7.7

In this lesson, the following generic competence and cross-cutting issues are to be addressed:

- Critical thinking
- Communication
- Self confidence
- Cooperation, interpersonal management and life skills
- Peace and values education
- Inclusive education

Answers

1. $\begin{pmatrix} 4 & -6 & 0 \\ -6 & 0 & 1 \\ 0 & 9 & -1 \\ 0 & 1 & 4 \end{pmatrix} \xrightarrow{r_2 \to r_2 + \frac{3}{2} r_1} \begin{pmatrix} 4 & -6 & 0 \\ 0 & -9 & 1 \\ 0 & 9 & -1 \\ 0 & 1 & 4 \end{pmatrix}$

$$\begin{array}{l} \xrightarrow{r_3 \to r_3 + r_2} \\ r_4 \to r_4 + \frac{1}{9} r_2 \end{array} \Bigg\rangle \begin{pmatrix} 4 & -6 & 0 \\ 0 & -9 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{37}{9} \end{pmatrix} \xrightarrow{r_3 \leftrightarrow r_4} \begin{pmatrix} 4 & -6 & 0 \\ 0 & -9 & 1 \\ 0 & 0 & \frac{37}{9} \\ 0 & 0 & 0 \end{pmatrix}$$

2. There are three non-zero rows.

Synthesis

To find rank of matrix,

- Transform matrix in its row echelon form using elementary row operations.
- The number of non-zero rows is the rank of matrix.

Application Activity 7.112

1. Rank 4
2. Rank 3
3. Rank 3
4. Rank 2

Lesson 7.8. Solving system of linear equations

Learning objectives

Given a system of linear equations and by using Gaussian elimination method, learners should be able to find the solution of that system correctly.

Prerequisites

- Elementary row operations.

Teaching Aids

Exercise book, pen and calculator

Activity 7.8

In this lesson, the following generic competence and cross-cutting issues are to be addressed:

- Critical thinking
- Communication
- Self confidence
- Cooperation, interpersonal management and life skills
- Peace and values education
- Inclusive education

Answers

1. The# system $\begin{cases} x+y+z=6 \\ 2x+y-z=1 \\ 3x+2y+z=10 \end{cases} \Leftrightarrow \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ 3 & 2 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 6 \\ 1 \\ 10 \end{pmatrix}$

   Thus, $A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ 3 & 2 & 1 \end{pmatrix}$

2. $\begin{pmatrix} 1 & 1 & 1 & : & 6 \\ 2 & 1 & -1 & : & 1 \\ 3 & 2 & 1 & : & 10 \end{pmatrix}$

3. From $\begin{pmatrix} 1 & 1 & 1 & : 6 \\ 2 & 1 & -1 & : & 1 \\ 3 & 2 & 1 & : 10 \end{pmatrix}$, using elementary row operations; $r_2 \to r_2 - 2r_1, r_3 \to r_3 - 3r_1$, we get

$$\begin{pmatrix} 1 & 1 & 1 & : & 6 \\ 0 & -1 & -3 & : & -11 \\ 0 & -1 & -2 & : & -8 \end{pmatrix}$$

Now, $r_3 \to r_3 - r_1$, yields $\begin{pmatrix} 1 & 1 & 1 & : & 6 \\ 0 & -1 & -3 & : & -11 \\ 0 & 0 & 1 & : & 3 \end{pmatrix}$

4. We have the system

$$\begin{cases} x + y \quad + z = 6 \\ \quad - y - 3z = -11 \\ \qquad\qquad z = 3 \end{cases}$$

Then,

$-y - 9 = -11 \Rightarrow y = 2$

$x + 2 + 3 = 6 \Rightarrow x = 1$ $\qquad S = \{(1,2,3)\}$

Synthesis

For the system

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = c_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = c_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n = c_m \end{cases}$$

The Gauss elimination method is used to transform a system of equations into an equivalent system that is in row echelon form.

To facilitate the calculation, transform the system into a matrix and place the coefficients of the variables and the independent terms into the matrix as follows:

$$(A:C)=\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} & :c_1 \\ a_{21} & a_{22} & \dots & a_{2n} & :c_2 \\ \vdots & \vdots & \dots & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} & :c_m \end{pmatrix}$$

And then, transform the system in the form where the elements above and below the leading diagonal of matrix A become zeros. The system is now reduced to the simplest system.

Application Activity 7.113

1. $S=\{(-4,-5,2)\}$
2. No solution
3. Infinity number of solution.
4. Infinity number of solution.

Lesson 7.9. Power of matrix

Learning objectives

Given a square matrix and by using diagonalisation method, learners should be able to find the power of that matrix accurately.

Prerequisites

- Finding eigenvalues and eigenvectors of a matrix.

Teaching Aids

Exercise book and pen

Activity 7.9

In this lesson, the following generic competence and cross-cutting issues are to be addressed:

- Critical thinking
- Communication
- Self confidence
- Cooperation, interpersonal management and life skills
- Peace and values education
- Inclusive education

Answers

1. $A^2 = AA = PDP^{-1}PDP^{-1} = PD^2P^{-1}$

2. $A^3 = A^2A = PD^2P^{-1}PDP^{-1} = PD^3P^{-1}$

3. $A^4 = A^3A = PD^3P^{-1}PDP^{-1} = PD^4P^{-1}$

4. $A^5 = A^4A = PD^4P^{-1}PDP^{-1} = PD^4DP^{-1} = PD^5P^{-1}$

   $A^n = A^{n-1}A = PD^{n-1}P^{-1}PDP^{-1} = PD^nP^{-1} \Rightarrow A^n = PD^nP^{-1}$

Synthesis

The power of matrix A is given by $A^n = PD^nP^{-1}$ for an invertible matrix P whose columns are elements of eigenvectors of matrix A, and D is diagonal matrix of A.

Where,

$$D^n = \begin{pmatrix} \lambda_1^n & 0 & \cdots & 0 \\ 0 & \lambda_2^n & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_k^n \end{pmatrix}$$

λ_k are eigenvalues

Application Activity 7.114

1. $P = \begin{pmatrix} -3 & 4 \\ 4 & 3 \end{pmatrix}$, $D = \begin{pmatrix} 20 & 0 \\ 0 & -5 \end{pmatrix}$, $A^3 = \begin{pmatrix} 2800 & -3900 \\ -3900 & 5075 \end{pmatrix}$

2. $P = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}$, $D = \begin{pmatrix} 3 & 0 \\ 0 & -1 \end{pmatrix}$, $A^5 = \begin{pmatrix} -245 & 488 \\ -244 & 487 \end{pmatrix}$

3. $P = \begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix}$, $D = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$, $A^{20} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

4. $P = \begin{pmatrix} 1 & 3 & 3 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$, $D = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}$, $A^5 = \begin{pmatrix} 32 & 0 & 633 \\ 0 & 243 & 0 \\ 0 & 0 & 243 \end{pmatrix}$

7.6. Summary of the unit

1. Kernel and range

- The kernel of a linear mapping $f: E \to F$ denoted $Ker(f)$ is a subset of E whose image f is 0-vector of F. i.e,
  $Ker(f) = \{v \in E : f(v) = 0\}$.

  A linear transformation f is called singular if there exists a non-zero vector whose image is zero vector. Thus, it is non-singular if the only zero vector has zero vector as image, or equivalently, if its kernel consists only of the zero vector: $Ker(f) = \{0\}$.

  A linear transformation $f: E \to F$ is one-to-one (1-1) if and only if $Ker(f) = \{0\}$.
- The nullity of f denoted $n(f)$ is the dimension of $Ker(f)$ .i.e, $n(f) = \dim Ker(f)$.
- The image or range of a linear mapping $f: E \to F$ is the set of points in F to which points in E are mapped on. i.e,
  $\operatorname{Im} f = \{u \in F : f(v) = u\},\ \ v \in E$.

  A linear transformation $f: E \to F$ is onto if the range is equal to F.
- The rank of f denoted $rank(f)$ or $r(f)$ is the dimension of image of f.
  i.e, $rank(f) = \dim(\operatorname{Im} f)$.

2. Elementary row/column operations

When these operations are performed on rows, they are called elementary row operations; and when they are performed on columns, they are called elementary column operations.

Operation description		Notation
Row operations		
1. Interchange row i and j	$\rightarrow$	$r_i \leftrightarrow r_j$
2. Multiply row i by $s \neq 0$	$\rightarrow$	$new\, r_i \rightarrow sr_i$
3. Add s times row i to row j	$\rightarrow$	$new\, r_j \rightarrow r_j + sr_i$
Column operations		
1. Interchange column i and j	$\rightarrow$	$c_i \leftrightarrow c_j$
2. Multiply column i by $s \neq 0$	$\rightarrow$	$new\, c_i \rightarrow sc_i$
3. Add s times column i to column j	$\rightarrow$	$new\, c_j \rightarrow c_j + sc_i$

Two matrices are said to be row equivalent (or column equivalent) if one can be changed to the other by a sequence of elementary row (or column) operations.

Two matrices A and B are said to be similar if $B = P^{-1}AP$ for some invertible matrix P.

3. Diagonalisation of matrices

a) Eigenvalues and eigenvectors

The eigenvalues of f are the roots (in K) of the polynomial; $\det(f - \lambda I)$. This polynomial is a polynomial associated with f and is called characteristic polynomial. For any square matrix A, the polynomial $\det(A - \lambda I)$ is its characteristic polynomial. The homogeneous system $(f - \lambda I)\vec{u} = \vec{0}$ gives the eigenvector $\vec{u}$ associated with eigenvalue λ.

b) Diagonalisation

To diagonalise matrix A, we perform the following steps:

1. Find the eigenvalues.
2. If there is a non-real eigenvalue, the matrix cannot be diagonalised.
3. If all eigenvalues are real, find their associated eigenvectors (they must be linearly independent).

4. If the number of eigenvectors is not equal to the order of matrix A, then this matrix cannot be diagonalised.
5. If the number of eigenvectors is equal to the order of matrix A, form matrix P whose columns are elements of eigenvectors.
6. Find the inverse of P.
7. Find D, diagonal matrix of A by relation $D = P^{-1}AP$.

4. Applications

a) Echelon matrix

A matrix is in **row echelon form** (ref) when it satisfies the following conditions:

- The first non-zero element in each row, called the leading entry, is 1.
- Each leading entry is in a column to the right of the leading entry in the previous row.
- Rows with all zero elements, if any, are below rows having a non-zero element.

  A matrix is in **reduced row echelon form** (rref) when it satisfies the following conditions:

- The matrix is in row echelon form (i.e., it satisfies the three conditions listed above).
- The leading entry in each row is the only non-zero entry in its column.

b) Matrix inverse

A is a square matrix of order n. To calculate the inverse of A, denoted as A^{-1}, follow these steps:

- Construct a matrix of type $M = (A \mid I)$, that is to say, A is in the left half of M and the identity matrix I is on the right.
- Using the Gaussian elimination method, transform the left half, A, to the identity matrix located to the right, and the

matrix that results in the right side will be the inverse of matrix A.

c) Rank of matrix

The rank of matrix is the number of linearly independent rows or columns. Using this definition, the Gaussian elimination method is used to find the rank.

A line can be discarded if:

- All the coefficients are zeros.
- There are two equal lines.
- A line is proportional to another.
- A line is a linear combination of others.

  In general, eliminate the maximum possible number of lines, and the rank is the number of non-zero rows.

d) Solving system of linear equations

Consider the following system;

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n = c_1 \\ a_{21}x_1 + a_{22}x_2 + ... + a_{2n}x_n = c_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + ... + a_{mn}x_n = c_m \end{cases}$$

The Gauss elimination method is to transform a system of equations into an equivalent system that is in triangular form.

To facilitate the calculation, transform the system into a matrix and place the coefficients of the variables and the independent terms into the matrix as follows:

$$(A:C) = \begin{pmatrix} a_{11} & a_{12} \dots a_{1n} & :c_1 \\ a_{21} & a_{22} \dots a_{2n} & :c_m \\ \vdots & \vdots \dots \vdots & \vdots \\ a_{m1} & a_{m2} \cdots a_{mn} & :c_m \end{pmatrix}$$

Where

The matrix $(A:C)$ is called **augmented matrix**.

Remarks

- If $rank(A) \neq rank(A:C)$, the system is said to be inconsistent and there is no solution.
- If $rank(A) = rank(A:C) = r$, the system is said to be consistent and there is solution.
  - » If $r = n$, as there are n unknowns, then the system has a unique solution.
  - » If $r < n$, the system has infinite solutions. (It is undetermined system).

5. Power of matrix

The power of matrix A is given by $A^n = PD^nP^{-1}$ for an invertible matrix P whose columns are elements of eigenvectors of matrix A and D is diagonal matrix of A.

Where,

$$D^n = \begin{pmatrix} \lambda_1^n & 0 & \cdots & 0 \\ 0 & \lambda_2^n & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_k^n \end{pmatrix}$$

λ_k are eigenvalues

7.7. End of Unit Assessment

1. a) $\lambda^2 - 5\lambda + 1$ b) $\lambda^2 - 3\lambda - 18$

   c) $\lambda^2 + 9$ d) $\lambda^3 + \lambda^2 - 8\lambda + 62$

   e) $\lambda^3 - 6\lambda^2 - 35\lambda - 38$

2. a) Eigenvalues: 2 and -5, eigenvectors: $\vec{u} = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$ and $\vec{v} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

   b) $P = \begin{pmatrix} 4 & 1 \\ 1 & 2 \end{pmatrix}$, $D = \begin{pmatrix} 2 & 0 \\ 0 & -5 \end{pmatrix}$

3. a) Eigenvalues: 1 and 4, eigenvectors: $\vec{u} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$ and $\vec{v} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

   b) $P = \begin{pmatrix} 2 & 1 \\ -1 & 1 \end{pmatrix}$, $D = \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix}$

   c) $A^6 = \begin{pmatrix} 1366 & 2230 \\ 1365 & 2731 \end{pmatrix}$, $f(A) = \begin{pmatrix} 1 & 2 \\ -1 & 0 \end{pmatrix}$

   d) $\frac{1}{3}\begin{pmatrix} 2+\sqrt[3]{4} & -2+2\sqrt[3]{4} \\ -1+\sqrt[3]{4} & 1+2\sqrt[3]{4} \end{pmatrix}$

4. a) Eigenvalues: 3 and 5

   b) $S = \{(1,-1,0),(1,0,1),(1,2,1)\}$

   c) $\begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{pmatrix}$

5. $A = \begin{pmatrix} 2 & 1 & -2 \\ 2 & 3 & -4 \\ 1 & 1 & -1 \end{pmatrix}$, $D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$

6. To be proved

7. a) $\begin{pmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ b) $\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

   c) $\begin{pmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$ d) $\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \end{pmatrix}$

8. a) Rank 2 b) Rank 2

   c) Rank 3 d) Rank 2

9. a) $S = \{(-2,5,10)\}$ b) No solution

10. a) $\frac{1}{5}\begin{pmatrix} -2 & -3 & 5 \\ 1 & 4 & -5 \\ -6 & -29 & 40 \end{pmatrix}$ b) $\frac{1}{11}\begin{pmatrix} 2 & 5 & -14 \\ 1 & -3 & 4 \\ -1 & 3 & 7 \end{pmatrix}$

c) $\frac{1}{2}\begin{pmatrix} 2 & 1 & 0 \\ 2 & 0 & 0 \\ -4 & -1 & 2 \end{pmatrix}$

11. Characteristic equation: $\lambda^3 - 5\lambda^2 + 7\lambda - 3 = 0$

From Cayley-Hamilton theorem, we have

$A^3 - 5A^2 + 7A - 3I = 0$.

Now, $A^8 - 5A^7 + 7A^6 - 3A^5 + A^4 - 5A^3 + 8A^2 - 2A + I$

$= A^5\left(A^3 - 5A^2 + 7A - 3I\right) + A\left(A^3 - 5A^2 + 7A - 3A\right) + A^2 + A + I$

$= 0 + 0 + A^2 + A + I$

$$= \begin{pmatrix} 5 & 4 & 4 \\ 0 & 1 & 0 \\ 4 & 4 & 5 \end{pmatrix} + \begin{pmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 8 & 5 & 5 \\ 0 & 3 & 0 \\ 5 & 5 & 8 \end{pmatrix}$$

Unit 8 Conics

8.1. Key unit competence

- Determine the characteristics and the graph of a conic given by its Cartesian, parametric or polar equation.
- Find the Cartesian, parametric and polar equations of a conic from its characteristics.

8.2. Vocabulary or key words concepts

Conic section: Curve obtained by intersecting a double cone with a plane.

Parabola: Conic section obtained when the plane is parallel to generator but not along the generator.

Ellipse: Conic section obtained when the plane cuts the cone obliquely.

Hyperbola: Conic section obtained when the plane is parallel to the axis but not along the axis.

8.3. Guidance on the introductory activity

Organize groups of students, and then assign them to read the text of the introductory activity from the student's book and to do the related questions. As they are working, move around to each group and ask them probing questions leading them to the right way. After a while, invite some group to present their findings in a whole class discussion. Guide students to harmonize their answers and arouse their curiosity to the content of this unit.

8.3. List of lessons

No	Lesson title	Number of periods
1	Generalities on conic sections	3
2	Definition and equation of a parabola	3
3	Tangent line and normal line on a parabola	3
4	Definition and equation of an ellipse	3
5	Tangent line and normal line on an ellipse	3
6	Definition and equation of a hyperbola	3
7	Tangent line and normal line on a hyperbola	3
8	Definition of polar coordinates	3
9	Polar equation of a conic	3
10	Polar equation of a straight line	3
11	Polar equation of a circle	3
12	Applications of conics	2
Total periods		35

8.4. Lesson development

Lesson 8.1. Generalities on conic sections

Learning objectives

Given a double cone and a plane, learners should be able to define a conic and draw the shape of conic sections accurately.

Prerequisites

- Double cone
- Plane

Teaching Aids

Exercise book, pen, calculator and instruments of geometry

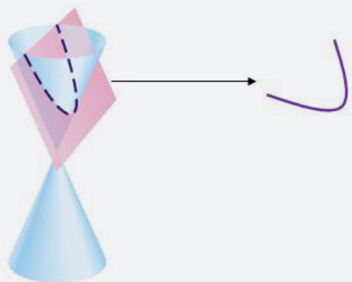
Activity 8.1

In this lesson, the following generic competence and cross-cutting issues are to be addressed:

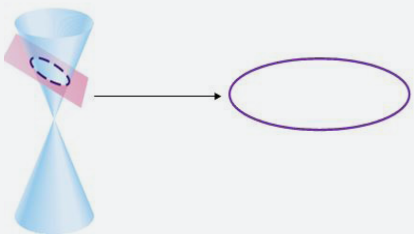
- Critical thinking
- Communication
- Self confidence
- Cooperation, interpersonal management and life skills
- Peace and values education
- Inclusive education

Answers

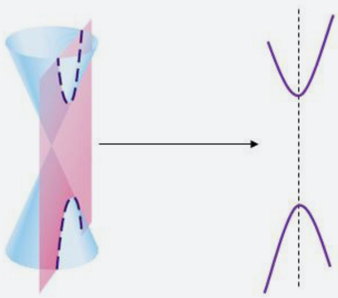
1. The plane is parallel to a generator of the cone but not along the generator



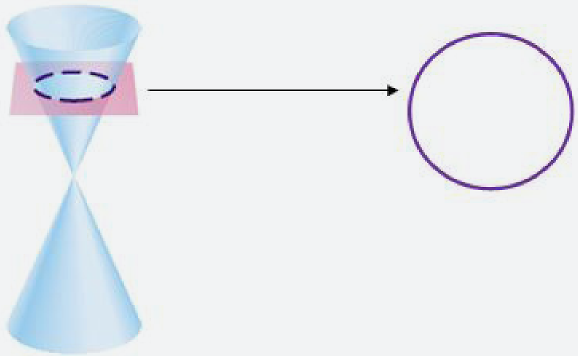
2. The plane cuts the cone obliquely



3. The plane is parallel to the axis but not along the axis



4. The plane is parallel to the base but does not pass through the vertex



Synthesis

Conic is the name given to the shapes that we obtain by taking different plane slices through a double cone. The sections of a right circular cone by different planes give curves of different shapes: parabola, ellipse, hyperbola, circle, single point, single line, pair of lines.

A conic section is the set of all points which move in a plane such that its distance from a fixed point and a fixed line not containing the fixed point are in a constant ratio.

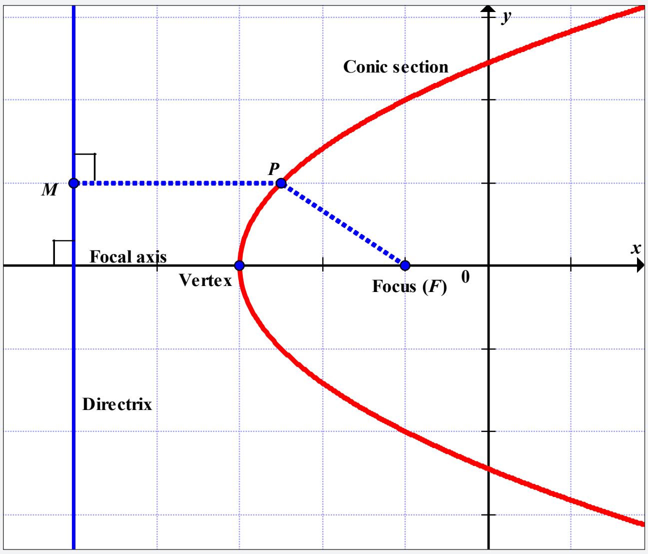


Figure 8.8: Conic section

From figure 8.8, we have

$\left|\overline{PF}\right| = e\left|\overline{PM}\right|$ where M is a foot of perpendicularity of line joining P to directrix, P point lying on conic and F focal point.

Focal axis is a line passing through the focus and perpendicular to the directrix.

Vertex is a point where the conic intersects its axis.

Application Activity 8.115

1. Single point: This is formed when the plane passes through the vertex horizontally, i.e. parallel to the base.
2. Single line: This is formed when the plane passes through the vertex and along the generator.
3. Pair of lines: This is the section formed when the plane passes through the vertex. In this case, the section is a pair of straight lines passing through the vertex.

Lesson 8.2. Definition and equation of a parabola

Learning objectives

Through examples, learners should be able to define a parabola and determine its equation accurately.

Prerequisites

- Distance between two points.
- Distance from appoint to a straight line.
- Curve sketching.

Teaching Aids

Exercise book, pen, calculator and instruments of geometry

Activity 8.2

In this lesson, the following generic competence and cross-cutting issues are to be addressed:

- Critical thinking
- Communication
- Self confidence
- Cooperation, interpersonal management and life skills
- Peace and values education
- Inclusive education

Answers

1. Choose some point having coordinates (x, y). The distance between this point and $(5,3)$ is given by $\sqrt{(x-5)^2+(y-3)^2}$.

   The distance between point (x, y) and $(2,1)$ is given by $\sqrt{(x-2)^2+(y-1)^2}$.

   Equating these distances, as the point (x, y) is to be equidistant from the two given points, we have

   $$\sqrt{(x-5)^2+(y-3)^2}=\sqrt{(x-2)^2+(y-1)^2}$$

   Squaring both sides, we get

   $$(x-5)^2+(y-3)^2=(x-2)^2+(y-1)^2$$

   Expanding, we have

   $$x^2-10x+25+y^2-6y+9=x^2-4x+4+y^2-2y+1$$

   Cancelling and combining like terms, we get

   $$4y+5=-6x+34$$

   Or

   $$4y=-6x+29$$

   This is the equation of a straight line with slope $-\frac{3}{2}$ and y intercept $\frac{29}{4}$.

2. Choose some point on the curve having coordinates (x, y).

The distance from the point (x, y) on the curve to the line $x=-3$ is $\sqrt{(x+3)^2+(y-y)^2}=\sqrt{(x+3)^2}$.

The distance from the point (x, y) on the curve to the point $(3,0)$ is $\sqrt{(x-3)^2+(y-0)^2}=\sqrt{(x-3)^2+y^2}$.

Equating the two distances yields

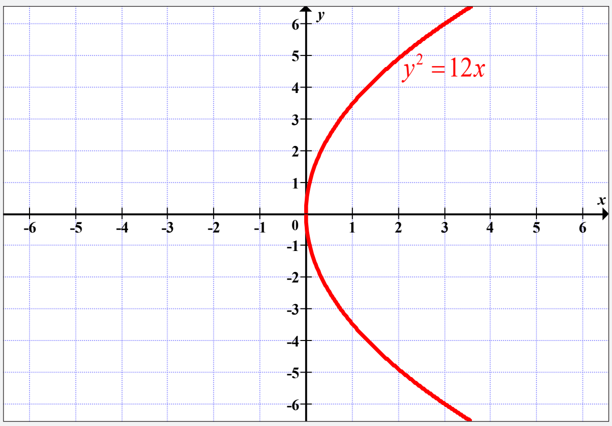
$\sqrt{(x+3)^2}=\sqrt{(x-3)^2+y^2}$

Squaring and expanding both sides, we get

$x^2+6x+9=x^2-6x+9+y^2$

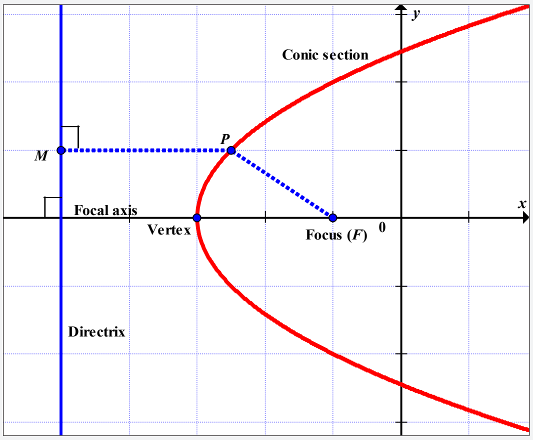
Cancelling and collecting like terms yields $y^2=12x$ which is an equation of a curve.

3. Curve



Synthesis

A **parabola** is set of points $P(x, y)$ in the plane equidistant from a fixed point F, called focus and a fixed line d, called **directrix**. In the figure below $\overline{PF}=\overline{PM}$, where $M \in d$.



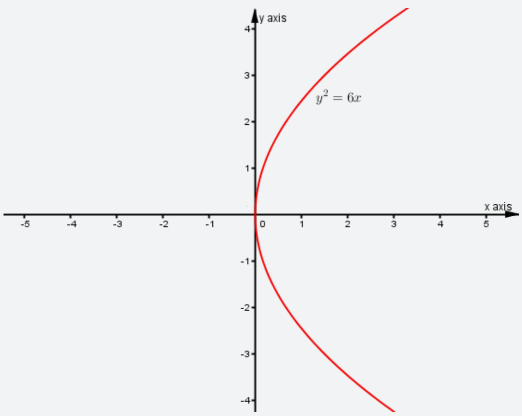
The equation of a parabola, whose focus at point $(a,0)$ and directrix with equation $x=-a$, is given by $y^2=4ax$.

The standard forms of the equation of parabola with vertices at the point $V(h,k)$ are as follows:

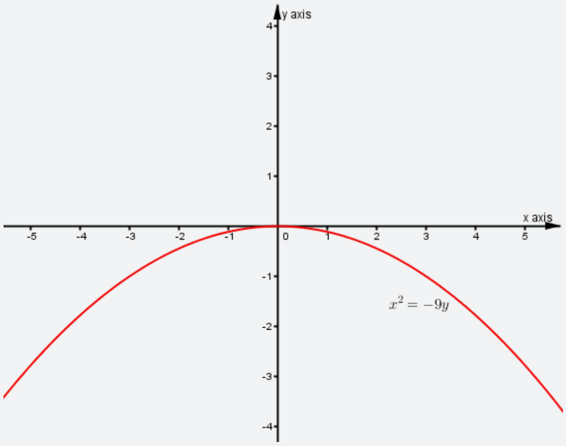
1. $(y-k)^2=4a(x-h)$, parabola opens to the right.
2. $(y-k)^2=-4a(x-h)$, parabola opens to the left.
3. $(x-h)^2=4a(y-k)$, parabola opens upward.
4. $(x-h)^2=-4a(y-k)$, parabola opens downward.

Application Activity 8.116

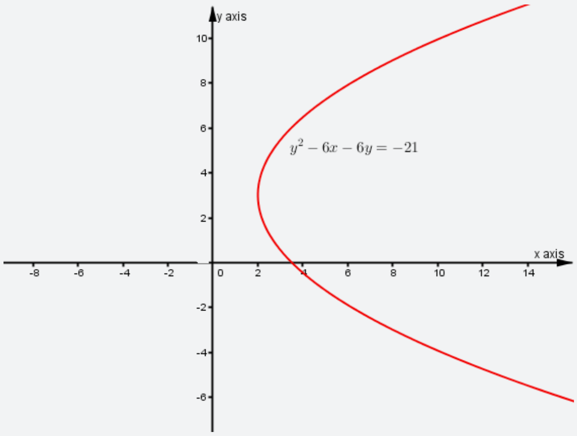
1. Focus is $(-2,0)$, directrix is $x=2$
2. a) Sketch:



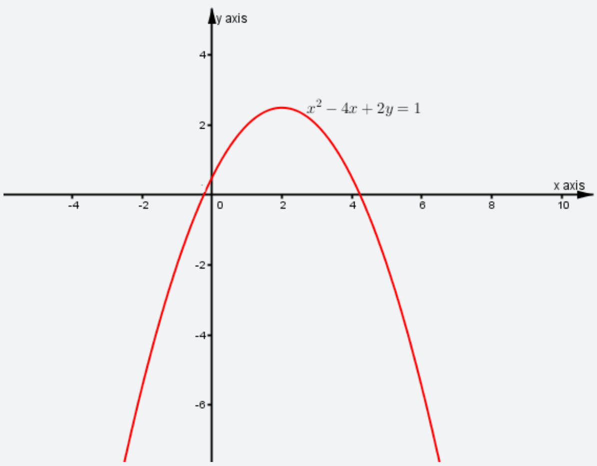
b) Sketch:



c) Sketch:



d) Sketch:



3. a) Focus $\left(\frac{25}{4},0\right)$; directrix $x=-\frac{25}{4}$; length of latus rectum 25; equation of latus rectum $x=\frac{25}{4}$; ends of latus rectum $\left(\frac{25}{4},-\frac{25}{2}\right)$ and $\left(\frac{25}{4},-\frac{25}{2}\right)$.

b) Focus $(0,2)$; directrix $y=-2$; length of latus rectum 8; equation of latus rectum $y=2$; ends of latus rectum $(4,2)$ and $(-4,2)$.

c) Focus $\left(0,-\frac{5}{4}\right)$; directrix $y=\frac{5}{4}$; length of latus rectum 5; equation of latus rectum $y=-\frac{5}{4}$; ends of latus rectum $\left(\frac{5}{2},-\frac{5}{4}\right)$ and $\left(-\frac{5}{2},-\frac{5}{4}\right)$

4. $(y-2)^2=12(x-1)$

5. a) $x^2=-8y$ b) $y^2=4(x+4)$

c) $4x^2+4xy+y^2+4x+32y+16=0$

6. a) $\frac{7}{4}$ b) $\left(\frac{41}{16},1\right)$; $(3,1)$

7. $(1,3)$

Lesson 8.3. Tangent line and normal line on a parabola

Learning objectives

Given equation of parabola, learners should be able to find equation of tangent line and normal line at a given point and draw them accurately.

Prerequisites

- Equation of tangent at a point on a curve.
- Equation of normal line at a point on a curve.
- Differentiation.

Teaching Aids

Exercise book, pen and calculator

Activity 8.3

In this lesson, the following generic competence and cross-cutting issues are to be addressed:

- Critical thinking
- Communication
- Self confidence
- Cooperation, interpersonal management and life skills
- Peace and values education
- Inclusive education

Answers

1. $T \equiv y - y_o = m(x - x_o)$

   Differentiating with respect to x yields

   $2y\dfrac{dy}{dx} = 4a \Rightarrow \dfrac{dy}{dx} = \dfrac{2a}{y}$ and $m = \dfrac{2a}{y_o}$

   Then,

   $$T \equiv y - y_o = \frac{2a}{y_o}(x - x_o)$$

   $$\Leftrightarrow y_o y - y_o y_o = 2a(x - x_o) \Leftrightarrow y_o y - 4ax_o = 2ax - 2ax_o$$

   $$\Leftrightarrow y_o y - 4ax_o = 2ax - 2ax_o \text{ since } y_0 y_0 = y_0^2 = 4ax_0$$

   $$\Leftrightarrow y_o y = 2ax - 2ax_o + 4ax \Leftrightarrow y_o y = 2ax + 2ax_o \Leftrightarrow y_o y = 2a(x + x_o)$$

   Therefore, $T \equiv y_o y = 2a(x + x_o)$

2. Equation of normal line:

   $$N \equiv y - y_o = -\frac{1}{m}(x - x_o), \ with\ m = \left.\frac{dy}{dx}\right|_{x=x_o}$$

   Then,

   $$N \equiv y - y_o = -\frac{y_o}{2a}(x - x_o) \Leftrightarrow 2ay - 2ay_o = -y_o x + y_o x_o$$

   $$\Rightarrow 2ay_o y - 2ay_o y_o = -y_o y_o x + y_o y_o x_o$$

   $$\Rightarrow 2ay_o y - 2ay_0^2 = -y_0^2 x + y_0^2 x_o$$

$\Rightarrow 2ay_o y - 8a^2 x_o = -4ax_o x + 4ax_0^2$

$\Leftrightarrow y_o y - 4ax_o = -2x_o x + 2x^2{}_o \Leftrightarrow y_o y - y_o y_o = -2x_o(x - x_o)$

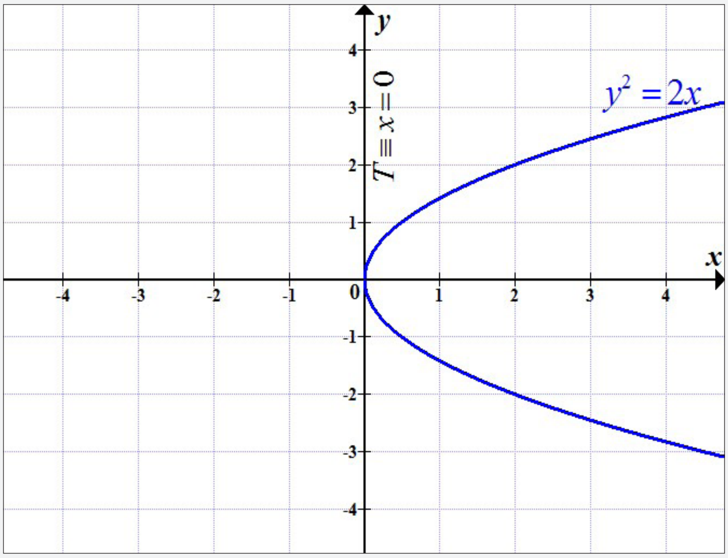
Therefore, $N \equiv y_o y - y_o y_o = -2x_o(x - x_o)$.

3. The tangent line of $y^2 = 2x$ at $(0,0)$:

Since the tangent line at point (x_0, y_0), on parabola $y^2 = 4ax$, is given by

$T \equiv y_0 y = 2a(x + x_0)$, here, $y^2 = 4\left(\frac{1}{2}\right)x$. So $a = \frac{1}{2}$.

Then the tangent line is $T \equiv 0y = 2\left(\frac{1}{2}\right)(x+0)$ or tangent line has equation $x = 0$.



The line $x = 0$ touches the parabola $y^2 = 2x$ once at $(0,0)$, as it is its tangent at $(0,0)$.

Synthesis

The tangent line at point (x_0, y_0), on parabola $y^2 = 4ax$, is given by $T \equiv y_0 y = 2a(x + x_0)$, and the normal line at the same point is $N \equiv y_o y - y_o y_o = -2x_o(x - x_o)$.

Application Activity 8.117

1. -6
2. Focus is $F(-2,4)$, vertex is $V(-2,-3)$, equation of axis is $x+2=0$, equation of directrix is $y+2=0$, equation of tangent at vertex is $y+3=0$.
3. $2y=x+8,\ y+2x+1=0$
4. $y+tx=2at+at^3;0,\pm 2$

Lesson 8.4. Definition and equation of an ellipse

Learning objectives

Through examples, learners should be able to define an ellipse and determine its equation accurately.

Prerequisites

- Distance between two points.
- Distance from a point to a straight line.
- Curve sketching.

Teaching Aids

Exercise book, pen, calculator and instruments of geometry

Activity 8.4

In this lesson, the following generic competence and cross-cutting issues are to be addressed:

- Critical thinking
- Communication
- Self confidence
- Cooperation, interpersonal management and life skills
- Peace and values education
- Inclusive education

Answers

1. Choose some point on the curve having coordinates (x, y).

   The distance from the point (x, y) on the curve to the point $(3,0)$ is $d_1 = \sqrt{(3-x)^2 + (0-y)^2} = \sqrt{(3-x)^2 + y^2}$.

   The distance from the point (x, y) to the line $x = \frac{25}{3}$ is

   $$d_2 = \sqrt{\left(\frac{25}{3} - x\right)^2 + (y-y)^2} = \frac{25}{3} - x.$$

   Since $\frac{d_1}{d_2} = \frac{3}{5} \Rightarrow d_1 = \frac{3}{5} d_2$, then,

   $$\sqrt{(3-x)^2 + y^2} = \frac{3}{5}\left(\frac{25}{3} - x\right)$$

   Squaring both sides and expanding, we get

   $$9 - 6x + x^2 + y^2 = 25 - 6x + \frac{9}{25}x^2$$

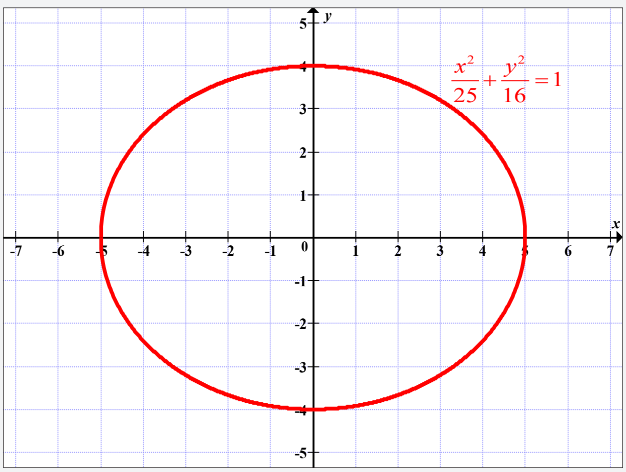
   Collecting like terms and transposing give

   $$\frac{16}{25}x^2 + y^2 = 16$$

   Dividing each term by 16, we see that $\frac{x^2}{25} + \frac{y^2}{16} = 1$.

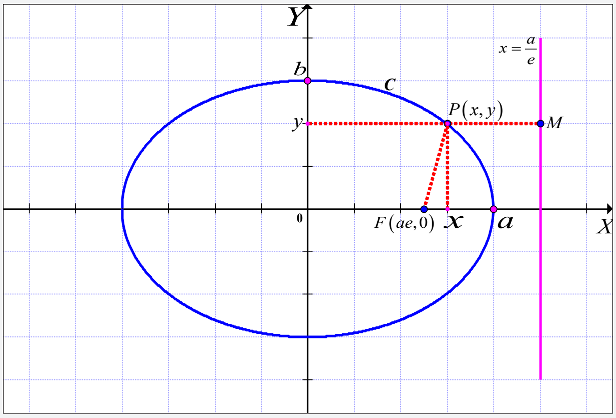
   This equation is of an **ellipse**.

2. Sketch of the curve:



Synthesis

We define an ellipse with eccentricity e (where $0<e<1$) to be the set of points P in the plane whose distance from a fixed point F is e times their distance from a fixed line.



For any point $P(x,y)$ on the ellipse, we have

$\overline{PF}=e\overline{PM}$, $M\in D$, where D is the directrix with equation $x=\dfrac{a}{e}$.

$$\overline{PF}=d_1=\sqrt{(ae-x)^2+(0-y)^2}=\sqrt{(ae-x)^2+y^2}.$$

$$\overline{PM}=d_2=\sqrt{(ae-x)^2+(y-y)^2}=\frac{a}{e}-x.$$

Since $\overline{PF}=e\overline{PM}$, then,

$$\sqrt{(ae-x)^2+y^2}=a-ex$$

Squaring both sides and expanding, we get

$$a^2e^2-\cancel{2aex}+x^2+y^2=a^2-\cancel{2aex}+e^2x^2$$

Collecting terms

$$\left(1-e^2\right)x^2+y^2=a^2\left(1-e^2\right)$$

Dividing each term by $a^2\left(1-e^2\right)$, we get

$$\frac{x^2}{a^2}+\frac{y^2}{a^2\left(1-e^2\right)}=1$$

Writing $b^2 = a^2\left(1-e^2\right)$, this gives

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

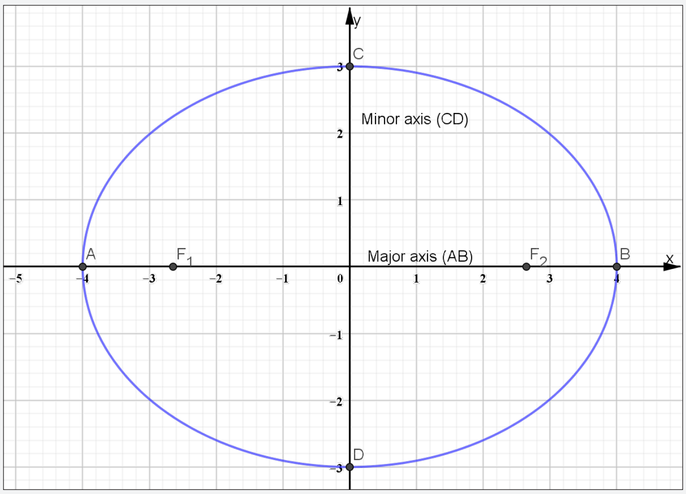
This is equation ellipse centred at $(0,0)$ in standard form.

For the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,\ b^2 = a^2\left(1-e^2\right)$ with $e<1$, we have two foci at $(\pm ae, 0)$ and two directices $x=\pm\frac{a}{e}$.

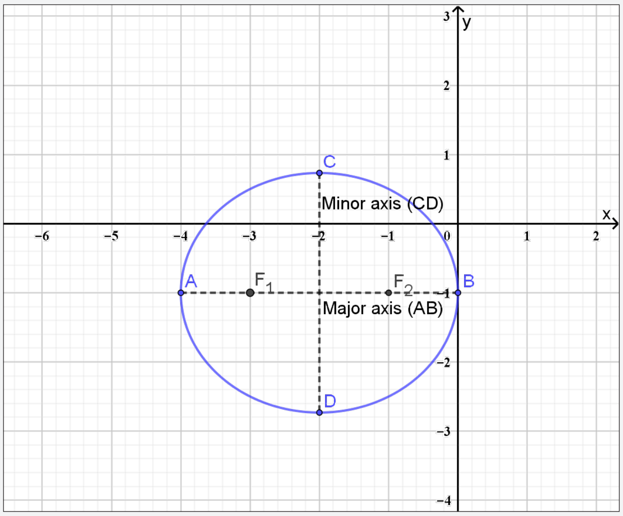
When the centre of ellipse is located at some point other than $(0,0)$, say the point (x_o, y_o), the equation of ellipse in standard form is $\frac{(x-x_o)^2}{a^2}+\frac{(y-y_o)^2}{b^2}=1$.

Application Activity 8.118

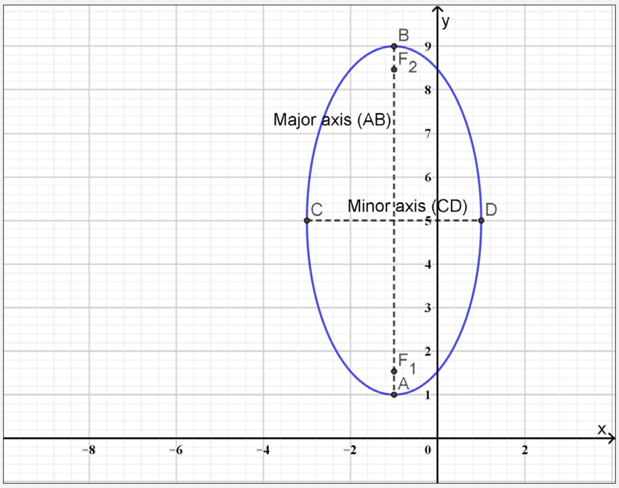
1. $\left(0,\sqrt{2}\right)$ and $\left(0,-\sqrt{2}\right)$
2. $3x^2+7y^2=115$
3. $\frac{x^2}{12}+\frac{y^2}{16}=1$
4. a) Sketch



b) Curve



c) Curve



5. $16x^2 + 9y^2 - 64x - 54y + 1 = 0$

$$\Rightarrow \frac{(x-2)^2}{9} + \frac{(y-3)^2}{16} = 1$$

Foci are $\left(2, 3+\sqrt{7}\right)$ and $\left(2, 3-\sqrt{7}\right)$

6. a) $e = \frac{\sqrt{2}}{2}$ b) $e = \frac{\sqrt{3}}{3}$ c) $e = \frac{\sqrt{3}}{2}$

7. $7 \, or \, 13$

8. $13:5$

Lesson 8.5. Tangent line and normal line on ellipse

Learning objectives

Given equation of ellipse, learners should be able to find tangent line and normal line at a given point and draw them accurately.

Prerequisites

- Equation of tangent line.
- Equation of normal line.
- Differentiation.

Teaching Aids

Exercise book, pen, calculator and instruments of geometry

Activity 8.5

In this lesson, the following generic competence and cross-cutting issues are to be addressed:

- Critical thinking
- Communication
- Self confidence
- Cooperation, interpersonal management and life skills
- Peace and values education
- Inclusive education

Answers

1. Equation of tangent line $T \equiv y - y_o = m(x - x_o)$ where

$$m = \left.\frac{dy}{dx}\right|_{x=x_o}$$

Differentiating with respect to x gives

$$\frac{2}{a^2}x + \frac{2}{b^2}y\frac{dy}{dx} = 0 \text{ or } \frac{dy}{dx} = -\frac{b^2}{a^2}\frac{x}{y}$$

Then, $m = -\frac{b^2}{a^2}\frac{x_o}{y_o}$

And $T \equiv y - y_o = -\frac{b^2}{a^2}\frac{x_o}{y_o}(x - x_o)$

$$\Leftrightarrow a^2 y_o y - a^2 y_o y_o = -b^2 x_o (x - x_o)$$

$$\Leftrightarrow a^2 y_o y + b^2 x_o x = b^2 x_o x_o + a^2 y_o y_o,$$

since $b^2x_ox_o + a^2y_oy_o = a^2b^2$

Thus $a^2y_oy + b^2x_ox = a^2b^2$

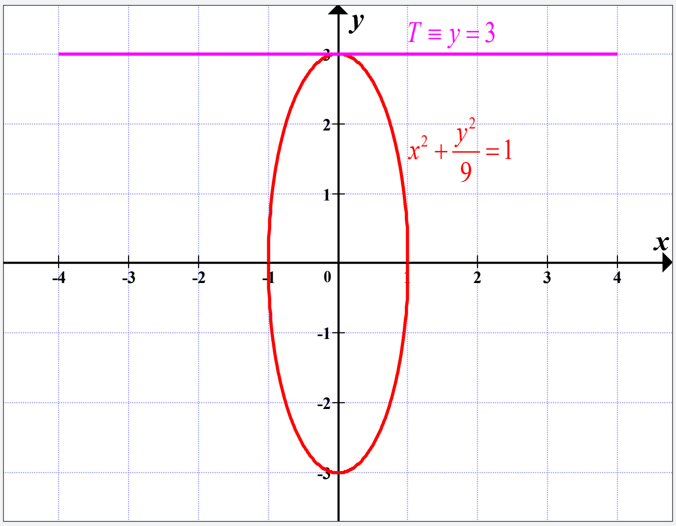
Dividing each term by a^2b^2, we get

$\frac{y_oy}{b^2} + \frac{x_ox}{a^2} = 1$ or $\frac{x_ox}{a^2} + \frac{y_oy}{b^2} = 1$

Therefore, $T \equiv \frac{x_ox}{a^2} + \frac{y_oy}{b^2} = 1$

2. The tangent line to the curve $x^2 + \frac{y^2}{9} = 1$ at $(0,3)$ is given by $T \equiv \frac{(0)x}{1} + \frac{3y}{9} = 1$ or $T \equiv y = 3$.

Curve:



Synthesis

The tangent line at point (x_0, y_0), on ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, is given by: $T \equiv \frac{x_0x}{a^2} + \frac{y_0y}{b^2} = 1$

Application Activity 8.119

1. $\pm\frac{3}{2}$
2. $y = -1.155x + 4$
3. $k = -4\,at\left(-2,-1\right);\; k = 4\,at\left(2,1\right)$
4. $At\left(0,0\right): T \equiv y = -2x$, $at\left(0,2\right): T \equiv y = 2x + 2,$
5. a) $\frac{6-\sqrt{6}}{4} < m < \frac{6+\sqrt{6}}{4}$ b) $m = \frac{6\pm\sqrt{6}}{4}$

   c) $m < \frac{6-\sqrt{6}}{4}$ or $m > \frac{6+\sqrt{6}}{4}$

Lesson 8.6. Definition and equation of a hyperbola

Learning objectives

Through examples, learners should be able to define hyperbola and find its equation accurately.

Prerequisites

- Distance between two points.
- Distance from a point to a straight line.
- Differentiation.

Teaching Aids

Exercise book, pen, calculator and instruments of geometry

Activity 8.6

In this lesson, the following generic competence and cross-cutting issues are to be addressed:

- Critical thinking
- Communication
- Self confidence
- Cooperation, interpersonal management and life skills
- Peace and values education
- Inclusive education

Answers

If the difference of the distances from any point $P(x,y)$ on conic to the two foci is $2a$, thus

$$PF_1 - PF_2 = 2a \Leftrightarrow \sqrt{(c-x)^2+(0-y)^2} - \sqrt{(-c-x)^2+(0-y)^2} = 2a$$

$$\Leftrightarrow \sqrt{(c-x)^2+y^2} - \sqrt{(c+x)^2+y^2} = 2a$$

Transposing one term from the left side to the right side and squaring, we get

$$\Leftrightarrow (c-x)^2+y^2 = 4a^2+4a\sqrt{(c+x)^2+y^2}+(c+x)^2+y^2$$

$$\Leftrightarrow c^2-2cx+x^2 = 4a^2+4a\sqrt{(c+x)^2+y^2}+c^2+2cx+x^2$$

$$\Leftrightarrow -2cx = 4a^2+4a\sqrt{(c+x)^2+y^2}+2cx$$

$$\Leftrightarrow -4\left(cx+a^2\right) = 4a\sqrt{(c+x)^2+y^2} \Leftrightarrow -\left(cx+a^2\right) = a\sqrt{(c+x)^2+y^2}$$

Squaring again both sides and expanding, we have

$$\Leftrightarrow c^2x^2+2cxa^2+a^4 = a^2\left(c^2+2cx+x^2+y^2\right)$$

$$\Leftrightarrow c^2x^2+2cxa^2+a^4 = a^2c^2+2cxa^2+a^2x^2+a^2y^2$$

$$\Leftrightarrow c^2x^2-a^2x^2-a^2y^2 = a^2c^2-a^4 \Leftrightarrow x^2\left(c^2-a^2\right)-a^2y^2 = a^2\left(c^2-a^2\right)$$

Since $c=\sqrt{a^2+b^2}$ thus $c^2-a^2=b^2$ and

$$x^2\left(c^2-a^2\right)-a^2y^2 = a^2\left(c^2-a^2\right) \Leftrightarrow x^2b^2-a^2y^2 = a^2b^2$$

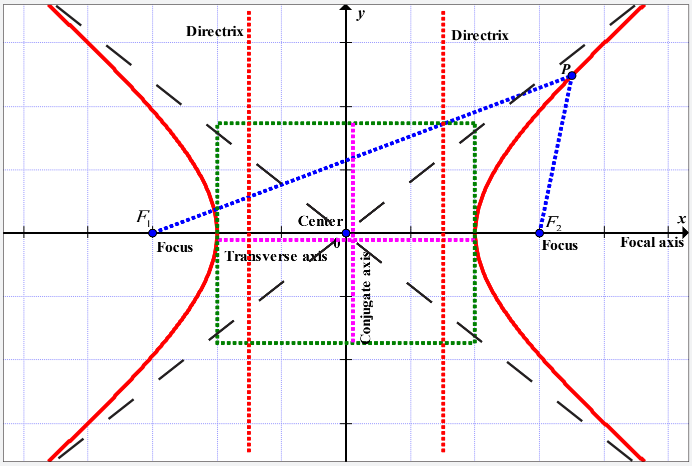
Dividing both sides by a^2b^2, we get

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$$

This is equation of hyperbola.

Synthesis

We define a hyperbola to be the set of all points P in the plane, the difference of whose distances from two fixed points, called **foci**, is a constant equal to $2a$.



In standard form a hyperbola is given by the equation

$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$; this hyperbola has two foci $(\pm c,0), c^2=a^2+b^2$

and two directrices $x=\pm\frac{a^2}{a}$.

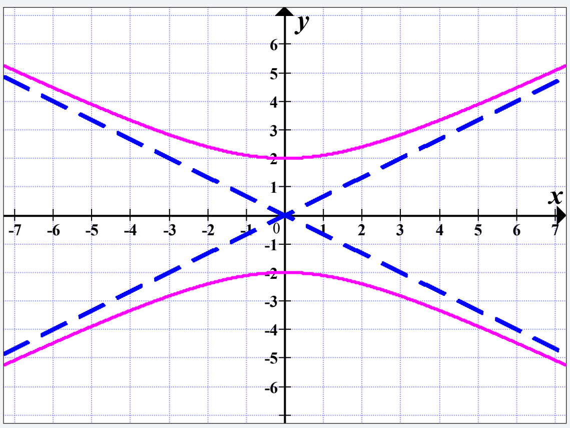
Eccentricity of the hyperbola is $e=\frac{c}{a}=\frac{\sqrt{a^2+b^2}}{a}, e>1$.

If the hyperbola has centre at (h,k), then the equation is

$\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1$.

Application Activity 8.120

1. a) Curve



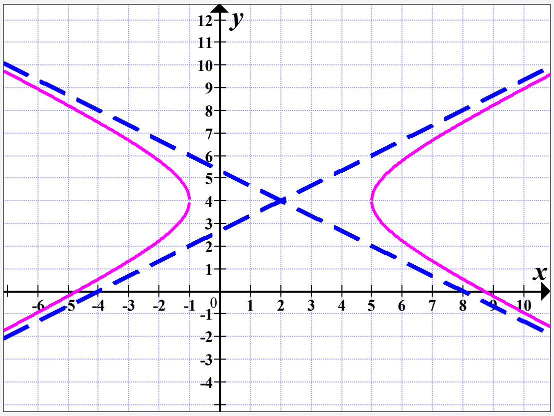
Vertices: $(0, \pm 2)$

Eccentricity is $e = \dfrac{\sqrt{13}}{2}$

Foci: $\left(0, \sqrt{13}\right)$ and $\left(0, -\sqrt{13}\right)$

Asymptotes: $y = \dfrac{2}{3}x$ and $y = -\dfrac{2}{3}x$.

b) Curve



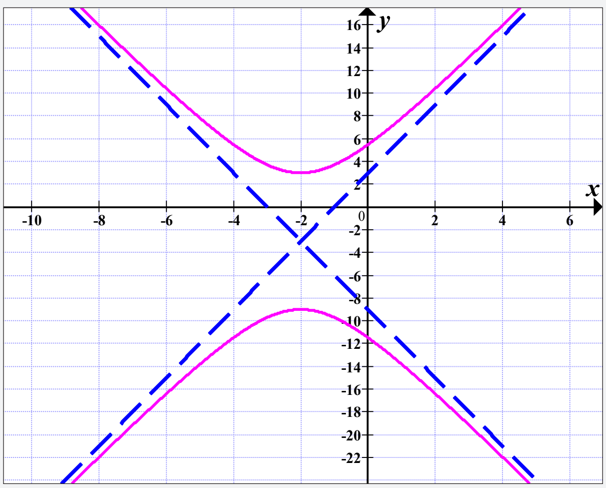
Vertices: $(5, 4)$ and $(-1, 4)$

Eccentricity is $e = \dfrac{\sqrt{13}}{3}$

Foci: $\left(2 + \sqrt{13}, 4\right)$ and $\left(2 - \sqrt{13}, 4\right)$

Asymptotes: $y = \dfrac{2}{3}x + \dfrac{8}{3}$ and $y = -\dfrac{2}{3}x + \dfrac{16}{3}$

c) Curve



Vertices: $(-2,3)$ and $(-2,-9)$

Eccentricity is $e=\frac{\sqrt{10}}{3}$

Foci: $\left(-2,-3+2\sqrt{10}\right)$ and $\left(-2,-3-2\sqrt{10}\right)$

Asymptotes: $y=3x+3$ and $y=-3x-9$

2. Foci are $(0,-5)$ and $(0,5)$, vertices are $(0,-4)$ and $(0,4)$, asymptotes: $y=\pm\frac{4}{3}x$

3. a) Length of transverse axis is $2a=6$; conjugate axis is $2b=8$.

   Eccentricity $\frac{\sqrt{a^2+b^2}}{a}=\frac{5}{3}$, coordinates of foci $(\pm ae,0)=(\pm 5,0)$, coordinates of vertices $V(\pm a,0)=(\pm 3,0)$.

   b) Length of transverse axis is $2a=2\sqrt{3}$; conjugate axis is $2b=2\sqrt{2}$.

   Eccentricity $\frac{\sqrt{a^2+b^2}}{a}=\frac{\sqrt{15}}{3}$, coordinates of foci $(\pm ae,0)=\left(\pm\sqrt{5},0\right)$, coordinates of vertices $V(\pm a,0)=\left(\pm\sqrt{3},0\right)$.

   c) Length of transverse axis is $2b=8$; conjugate axis is $2a=2$.

   Eccentricity $\frac{\sqrt{a^2+b^2}}{a}=\frac{\sqrt{17}}{4}$, coordinates of foci $(0,\pm be)=\left(0,\pm\sqrt{17}\right)$, coordinates of vertices $V(0,\pm b)=(0,\pm 4)$.

4. $\frac{y^2}{64}-\frac{x^2}{36}=1$

5. $x^2-y^2-4x+8y-21=0$

   $\Rightarrow \frac{(x-2)^2}{9}-\frac{(y-4)^2}{9}=1$

Vertices: $(-1,4),(5,4)$

Foci: $\left(2-3\sqrt{2},4\right), \left(2+3\sqrt{2},4\right)$

Asymptotes: $y-4=\pm(x-2)$

6. a) $7x^2+24xy-56x-6y+68=0$

   b) $9x^2-16y^2-36x+96y-252=0$

7. $21x^2-4y^2-84=0$

Lesson 8.7. Tangent line and normal line on hyperbola

Learning objectives

Given equation of hyperbola, learners should be able to find equation of tangent line and normal line at a given point and draw them accurately.

Prerequisites

- Equation of tangent line.
- Equation of normal line.
- Differentiation.

Teaching Aids

Exercise book, pen, calculator and instruments of geometry

Activity 8.7

In this lesson, the following generic competence and cross-cutting issues are to be addressed:

- Critical thinking
- Communication
- Self confidence
- Cooperation, interpersonal management and life skills
- Peace and values education
- Inclusive education

Answers

Equation of tangent line $T \equiv y-y_o=m(x-x_o)$ where

$$m=\left.\frac{dy}{dx}\right|_{x=x_o}$$

Differentiating with respect to x gives

$$\frac{2}{a^2}x-\frac{2}{b^2}y\frac{dy}{dx}=0 \text{ or } \frac{dy}{dx}=\frac{b^2}{a^2}\frac{x}{y}$$

Then, $m=\frac{b^2}{a^2}\frac{x_o}{y_o}$

And $T\equiv y-y_o=\frac{b^2}{a^2}\frac{x_o}{y_o}(x-x_o)$

$\Leftrightarrow a^2y_oy-a^2y_oy_o=b^2x_o(x-x_o)$

$\Leftrightarrow a^2y_oy-b^2x_ox=-b^2x_ox_o+a^2y_oy_o$, as $b^2x_ox_o-a^2y_oy_o=a^2b^2$

$a^2y_oy-b^2x_ox=-a^2b^2$

Dividing each term by $-a^2b^2$, we get

$$-\frac{y_oy}{b^2}+\frac{x_ox}{a^2}=1 \text{ or } \frac{x_ox}{a^2}-\frac{y_oy}{b^2}=1$$

Therefore, $T\equiv\frac{x_ox}{a^2}-\frac{y_oy}{b^2}=1$

Synthesis

The tangent line at point (x_0,y_0), on hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, is given by $T\equiv\frac{x_0x}{a^2}-\frac{y_0y}{b^2}=1$.

Application Activity 8.121

1. 4
2. $y=-\frac{1}{4}x+5$
3. $15x+4y+9=0$
4. $\left(\frac{3}{2}\sqrt{13},-9\right),\left(-\frac{3}{2}\sqrt{13},-9\right)$

Lesson 8.8. Definition of polar coordinates

Learning objectives

Through examples, learners should be able to define polar coordinates, convert polar coordinates to Cartesian coordinates, and sketch a curve in polar form accurately.

Prerequisites

- Polar form of a complex number.
- Converting a complex number from polar form to algebraic form.
- Curve sketching.

Teaching Aids

Exercise book, pen, calculator and instruments of geometry

Activity 8.8

In this lesson, the following generic competence and cross-cutting issues are to be addressed:

- Critical thinking
- Communication
- Self confidence
- Cooperation, interpersonal management and life skills
- Peace and values education
- Inclusive education

Answers

1. a) $|z|=\sqrt{1+1}=\sqrt{2}$ b) $|z|=\sqrt{1+1}=\sqrt{2}$

2. $\cos\theta=\frac{1}{\sqrt{2}}\Rightarrow\theta=\pm\frac{\pi}{4}+2k\pi$

$$\sin\theta=\frac{1}{\sqrt{2}}\Rightarrow\theta=\begin{cases}\frac{\pi}{4}+2k\pi\\ \frac{3\pi}{4}+2k\pi\end{cases}$$

As $-\pi<\theta\leq\pi$, we take $\theta=\frac{\pi}{4}$.

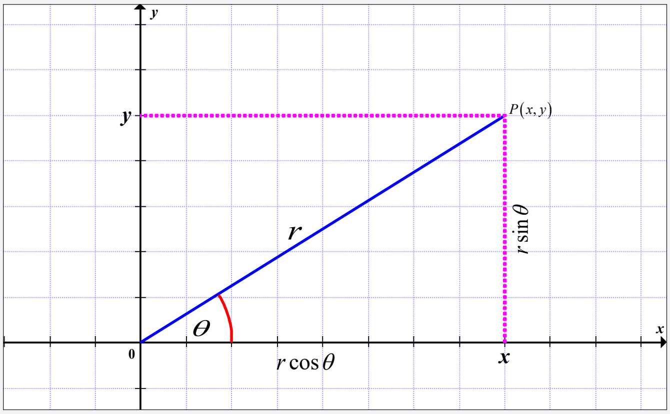
Synthesis

To form a polar coordinate system in the plane, we fix a point 0 called the **pole** (or origin) and construct from 0 an initial ray called the **polar axis.** Then each point P in the plane can be assigned **polar coordinates** (r,θ) as follows:

- r is the directed distance from O to P.

- θ is the directed angle, counterclockwise from polar axis to the segment $\overline{OP}$.

To convert rectangular coordinates (a,b) to polar coordinates is the same as to find the modulus and argument of complex number $z = a + bi$.



From the above figure $r = \overline{OP}$, $x = r\cos\theta$, $y = r\sin\theta$; ox is a polar axis.

Application Activity 8.122

1. a) and e); b and g; c) and h); d) and f)
2. a) $(2, 2k\pi)$ and $(-2, (2k+1)\pi), k \in \mathbb{Z}$

   b) $(2, (2k+1)\pi)$ and $(-2, 2k\pi), k \in \mathbb{Z}$

   c) $\left(2, \frac{\pi}{2} + 2k\pi\right)$ and $\left(-2, \frac{\pi}{2} + (2k+1)\pi\right), k \in \mathbb{Z}$

   d) $\left(2, \frac{3\pi}{2} + 2k\pi\right)$ and $\left(-2, \frac{3\pi}{2} + (2k+1)\pi\right), k \in \mathbb{Z}$
3. a) $(3,0)$ b) $(-3,0)$ c) $(-1,\sqrt{3})$

   d) $(1,\sqrt{3})$ e) $(3,0)$ f) $(1,\sqrt{3})$

   g) $(-3,0)$ h) $(-1,\sqrt{3})$
4. a) $x + y = 1$ b) $x = 3$ c) $x = y$

d) $x-3y=3$ e) $x^2+y^2=9$ f) $\frac{x^2}{16}+\frac{y^2}{25}=1$

g) $x^2+(y-2)^2=4$ h) $(x-3)^2+(y+1)^2=4$

i) $y^2=3x$

Lesson 8.9. Polar equation of a conic

Learning objectives

Through examples, learners should be able to find polar equation of a conic or change from polar equation to Cartesian equation accurately.

Prerequisites

- Cartesian equation of a conic.
- Conversion formulae from polar form to Cartesian form and vice versa.

Teaching Aids

Exercise book, pen and calculator

Activity 8.9

In this lesson, the following generic competence and cross-cutting issues are to be addressed:

- Critical thinking
- Communication
- Self confidence
- Cooperation, interpersonal management and life skills
- Peace and values education
- Inclusive education

Answers

1. Polar coordinates

$$x=r\cos\theta,\quad y=r\sin\theta,\quad r^2=x^2+y^2,\quad \tan\theta=\frac{y}{x},\ x\neq 0$$

$$y^2=1+2x \Leftrightarrow y^2=2x+1 \Leftrightarrow y^2-2x=1$$

$$\Leftrightarrow r^2\sin^2\theta-2r\cos\theta=1$$

2. a) $r=\dfrac{6}{2+\cos\theta}\Leftrightarrow 2r+r\cos\theta=6$

$\Leftrightarrow 2\sqrt{x^2+y^2}+x=6$

$\Leftrightarrow 2\sqrt{x^2+y^2}=6-x$

Squaring both sides gives

$4\left(x^2+y^2\right)=\left(6-x\right)^2 \Leftrightarrow 4x^2+4y^2=36-12x+x^2$

$\Leftrightarrow 3\left(x^2+4x\right)+4y^2=36 \Leftrightarrow 3\left(x^2+4x\right)+4y^2=36$

$\Leftrightarrow 3\left(x+2\right)^2+4y^2=48 \Leftrightarrow 3\left(x+2\right)^2+4y^2=48$

Dividing each term by 48, we get

$\Leftrightarrow \dfrac{\left(x+2\right)^2}{16}+\dfrac{y^2}{12}=1$

b) This is equation of a horizontal ellipse of centre $\left(-2,0\right)$, major axis 8, minor axis $4\sqrt{3}$, eccentricity

$e=\dfrac{\sqrt{a^2-b^2}}{a}=\dfrac{\sqrt{16-12}}{4}=\dfrac{1}{2}$, vertices $\left(2,0\right),\left(-6,0\right)$, foci $\left(0,0\right),\left(-4,0\right)$.

Synthesis

Using polar coordinates, there is an alternative way to define a conic. In polar equation of a conic, the pole is the focus of the conic. We use the following relations:

$$x=r\cos\theta,\quad y=r\sin\theta,\quad r^2=x^2+y^2,\quad \tan\theta=\frac{y}{x},\ x\neq 0$$

The polar equation of conic with eccentricity e, focus at the origin, whose directrix $x=-p$ has equation $r=\dfrac{ep}{1+e\cos\theta}$ where $\left(r,\theta\right)$ are polar coordinates of any point P lying on the conic.

The given conic is an **ellipse** if $e<1$, a circle if $e=0$, a **parabola** if $e=1$, a hyperbola if $e>1$.

Application Activity 8.123

1. a) $r=\dfrac{1}{1+\cos\theta}$ b) $r=\dfrac{2}{1-2\sin\theta}$

2. a) $3x^2+4y^2-4x=2$ b) $x^2=2-4y$

3. $r=\dfrac{k}{1+e\cos\theta}\Leftrightarrow r+er\cos\theta=k$

   $\Leftrightarrow\sqrt{x^2+y^2}+ex=k\ \Leftrightarrow\sqrt{x^2+y^2}=k-ex$

   Squaring both sides, we get

   $x^2+y^2=k^2-2kex+e^2x^2\ \Leftrightarrow x^2-e^2x^2+y^2+2kex-k^2=0$

   $\Leftrightarrow\left(1-e^2\right)x^2+y^2+2kex-k^2=0$ as required.

Lesson 8.10. Polar equation of a straight line

Learning objectives

Given Cartesian equation of a straight line, learners should be able to find polar equation of that straight line or change from polar equation to Cartesian equation accurately.

Prerequisites

- Cartesian equation of a straight line.
- Conversion formulae from polar form to Cartesian form and vice versa.

Teaching Aids

Exercise book, pen and calculator

Activity 8.10

In this lesson, the following generic competence and cross-cutting issues are to be addressed:

- Critical thinking
- Communication
- Self confidence
- Cooperation, interpersonal management and life skills
- Peace and values education
- Inclusive education

Answers

Polar coordinates are $x = r\cos, y = r\sin\theta$.

Replacing x and y by their values from polar coordinates in $3x - 2y + 6 = 0$, we get

$3(r\cos\theta) - 2(r\sin\theta) + 6 = 0 \Leftrightarrow 3r\cos\theta - 2r\sin\theta = -6$

$\Leftrightarrow r(3\cos\theta - 2\sin\theta) = -6 \Leftrightarrow -r(2\sin\theta - 3\cos\theta) = -6$

$\Leftrightarrow 2\sin\theta - 3\cos\theta = \frac{6}{r} \Leftrightarrow \frac{1}{r} = \frac{1}{3}\sin\theta - \frac{1}{2}\cos\theta$

Synthesis

The polar equation of a straight line is

$\frac{1}{r} = A\cos\theta + B\sin\theta,\ A, B \in \mathbb{R}$ and A and B are not all zero.

From general equation of a line in Cartesian plane, we get the polar equation of the given line.

In fact, $Ax + By + C = 0 \Leftrightarrow Ax + By = -C \Leftrightarrow Ax + By = -C$

$$\Leftrightarrow Ar\cos\theta + Br\sin\theta = -C$$

$$\Leftrightarrow r(A\cos\theta + B\sin\theta) = -C$$

$$\Leftrightarrow A\cos\theta + B\sin\theta = -\frac{C}{r}$$

Therefore, the polar equation of a straight line $Ax + By + C = 0$ is $A\cos\theta + B\sin\theta = -\frac{C}{r}$.

Application Activity 8.124

1. $r = \dfrac{4}{\cos\theta + \sqrt{3}\sin\theta}$

2. $r = \dfrac{2}{\cos\theta - \sin\theta}$

3. $r = \dfrac{\sqrt{3}}{\sqrt{3}\cos\theta - 2\sin\theta}$

4. $r = \dfrac{\sqrt{5}}{\cos\theta - 2\sin\theta}$

Lesson 8.11. Polar form of a circle

Learning objectives

Given a Cartesian equation of a circle, learners should be able to find polar equation of that circle or change from polar equation to Cartesian equation correctly.

Prerequisites

- Cartesian equation of a circle.
- Conversion formulae from polar form to Cartesian form and vice versa.

Teaching Aids

Exercise book, pen and calculator

Activity 8.11

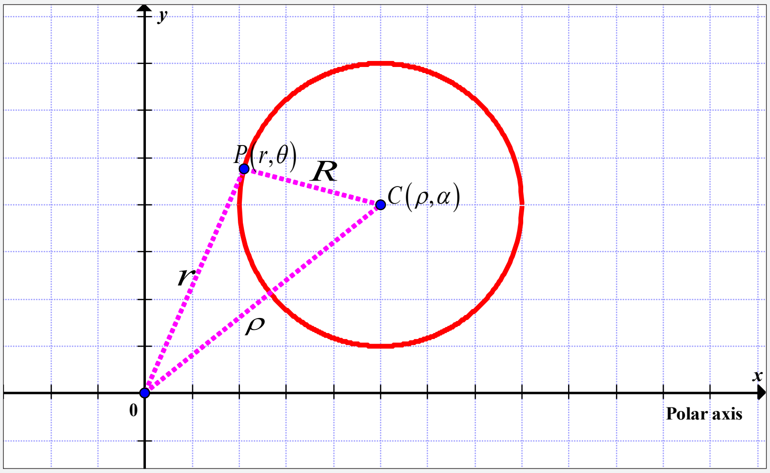
In this lesson, the following generic competence and cross-cutting issues are to be addressed:

- Critical thinking
- Communication
- Self confidence
- Cooperation, interpersonal management and life skills
- Peace and values education
- Inclusive education

Answers

From the figure below,

$\overline{OC} = \rho, \overline{OP} = r, \overline{CP} = R$



Cosine law here is: $\overline{CP}^2 = \overline{CO}^2 + \overline{OP}^2 - 2\overline{OC}\,\overline{OP}\cos(\theta - \alpha)$

$\Rightarrow R^2 = \rho^2 + r^2 - 2\rho r\cos(\theta - \alpha)$

$\Rightarrow r^2 = R^2 - \rho^2 + 2\rho r\cos(\theta - \alpha)$

Synthesis

The polar equation of a circle with centre (ρ, α) and radius R is

$r^2 = R^2 - \rho^2 + 2r\rho\cos(\theta - \alpha)$

Application Activity 8.125

1. $r = 6\cos\theta$
2. $r = 4\sin\theta$
3. $r = -\cos\theta$
4. $r = -2\sin\theta$

Lesson 8.12. Applications of conics

Learning objectives

By reading textbooks or accessing internet, learners should be able to apply conics in real life problems perfectly.

Prerequisites

- Equations of conics.

Teaching Aids

Exercise book and pen

Activity 8.12

In this lesson, the following generic competence and cross-cutting issues are to be addressed:

- Critical thinking
- Communication
- Self confidence
- Cooperation, interpersonal management and life skills
- Research and problem solving
- Peace and values education
- Inclusive education

Answers

The Pythagorean concept of a spherical Earth offers a simple surface that is mathematically easy to deal with. Many astronomical and navigational computations use it as a surface representing the Earth.

The idea of a planar or flat surface for Earth, however, is still sufficient for surveys of small areas, as the local topography is far more significant than the curvature. Plane-table surveys are made for relatively small areas,

and no account is taken of the curvature of the Earth. A survey of a city would likely be computed as though the Earth were a plane surface; the size of the city. For such small areas, exact positions can be determined relative to each other without considering the size and shape of the entire Earth.

The simplest model for the shape of the entire Earth is a sphere. The Earth's radius is the distance from Earth's centre to its surface, about 6,371 kilometres (3,959 mi). While "radius" normally is a characteristic of perfect spheres, the Earth deviates from a perfect sphere by only a third of a percent, sufficiently close to treat it as a sphere in many contexts and justifying the term "the radius of the Earth".

Synthesis

The orbits of planets are ellipses with the sun at one focus. For most planets, these ellipses have very small eccentricity, so they are nearly circular. However, the Mercury and Pluto, the innermost and outermost known planets, have visibly elliptical orbits.

Application Activity 8.126

1. a) 6 cm b) 10 cm
2. a) (i) $7\sqrt{5}$ ii) $7\sqrt{5}-13=133\,mm$
   b) (i) 1 ii) AC has equation $y=-x+10$
3. $(x-5)^2+(y-13)^2=9$
4. $\sqrt{5}\,m$
5. The minimum altitude is 272 miles above the Earth
   The maximum altitude is 648 miles above the Earth.

6. $\frac{x^2}{900}-\frac{y^2}{14400.3636}=1$

7. Then the equation for the elliptical ceiling is:

   $\frac{x^2}{625}+\frac{(y-5)^2}{400}=1$ and the height of the ceiling above each whispering point is $y=21$.

Summary of the unit

1. Generalities on conic sections

Parabolas, circles, ellipses and hyperbolas are called conics because they are curves in which planes intersect right circular cones.

2. Parabola

A parabola is the set of all points in plane that are equidistant from a fixed line (called directrix) and a fixed point (called focus) not on the line.

Important result relating to different parabolas

Equation	$y^2=4ax$	$x^2=4ay$
Focus	$(a,0)$	$(0,a)$
Directrix	$x=-a$	$y=-a$
Principal axis(the line through the focus perpendicular to the directrix)	$y=0$	$x=0$
Vertex (point where the parabola crosses its principal axis)	$(0,0)$	$(0,0)$

Length of latus rectum (length of chord through a focus and perpendicular to the principal axis)	$4a$	$4a$
Equation of latus rectum	$x = a$	$y = a$
Ends of latus rectum	$(a, \pm 2a)$	$(\pm 2a, a)$

Replacing x with *(x−h)* has the effect of shifting the graph of an equation by $|h|$ units to the right if *h* is positive, to the left if *h* is negative.

Similarly, replacing y with $(y-k)$ has the effect of shifting the graph by $|k|$ units up if k is positive and down if k is negative.

Equation	$(y-k)^2 = 4p(x-h)$	$(x-h)^2 = 4p(y-k)$
Focus	$(\mathrm{h}+p, k)$	$(h, k+p)$
Directrix	$x = h - p$	$y = k - p$
Principal axis(the line through the focus perpendicular to the directrix)	$y = k$	$x = h$
Vertex (point where the parabola crosses its principal axis)	(h, k)	(h, k)

Parametric equations of parabola are

$\begin{cases} x = at^2 \\ y = 2at \end{cases}$ where t is a parameter.

The tangent line at point (x_0, y_0), on parabola $y^2 = 4ax$, is given by

$T \equiv y_0 y = 2a(x + x_0)$

3. Ellipse

Ellipse is a set of all points in the plane, the sum of whose distances from two fixed points (called foci) is a given positive constant.

Important facts to different ellipses

Equation of Standard form	$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,\ a>b>0$	$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,\ 0<a<b$
Coordinates of centre	$(0,0)$	$(0,0)$
Coordinates of vertices	$(a,0)$ and $(-a,0)$	$(0,b)$ and $(0,-b)$
Length of major axis	$2a$	$2b$
Equation of major axis	$y=0$	$x=0$
Length of minor axis	$2b$	$2a$
Equation of minor axis	$x=0$	$y=0$
Eccentricity (ratio of semi-focal separation and the semi-major axis)	$b^2=a^2\left(1-e^2\right)\Rightarrow e=\frac{\sqrt{a^2-b^2}}{a}$	$a^2=b^2\left(1-e^2\right)\Rightarrow e=\frac{\sqrt{b^2-a^2}}{b}$
Coordinates of foci	$(ae,0)$ and $(-ae,0)$ $\Leftrightarrow\left(\pm\sqrt{a^2-b^2},0\right)$	$(0,be)$ and $(0,-be)$ $\Leftrightarrow\left(0,\pm\sqrt{b^{2,}-a^2}\right)$
Equation of directrices	$x=\pm\frac{a}{e}$	$y=\pm\frac{b}{e}$
Length of latus rectum	$\frac{2b^2}{a}$	$\frac{2a^2}{b}$
Equations of latus rectum	$x=\pm ae$	$y=\pm be$

Parametric equations of ellipse with centre (x_o,y_o) are

$\begin{cases} x=x_o+a\cos t \\ y=y_o+b\sin t \end{cases}$ where t is a parameter and $t\in(-\pi,\pi]$.

The tangent line at point (x_0, y_0), on ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, is given by

$$T \equiv \frac{x_0 x}{a^2}+\frac{y_0 y}{b^2}=1$$

4. Hyperbola

Hyperbola is a set of all points in the plane, the difference of whose distances from two fixed points (foci) is a given positive constant

Important facts to different hyperbolas

Equation of Standard form	$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$	$\frac{x^2}{a^2}-\frac{y^2}{b^2}=-1$
Coordinates of centre	$(0,0)$	$(0,0)$
Coordinates of vertices	$(a,0)$ and $(-a,0)$	$(0,b)$ and $(0,-b)$
Length of transverse axis	$2a$	$2b$
Equation of transverse axis	$y=0$	$x=0$
Equation of conjugate axis	$x=0$	$y=0$
Coordinates of foci	$(ae,0)$ and $(-ae,0)$ $\Leftrightarrow \left(\pm\sqrt{a^2+b^2},0\right)$	$(0,be)$ and $(0,-be)$ $\Leftrightarrow \left(0,\pm\sqrt{a^2+b^2}\right)$
Equation of directrices	$x=\pm\frac{a}{e}$	$y=\pm\frac{b}{e}$
Length of latus rectum	$\frac{2b^2}{a}$	$\frac{2a^2}{b}$
Equations of latus rectum	$x=\pm ae$	$y=\pm be$

Eccentricity	$b^2 = a^2\left(1-e^2\right)$	$a^2 = b^2\left(1-e^2\right)$

Parametric equations of hyperbola whose centre $\left(x_o, y_o\right)$ are

$\begin{cases} x = x_o + a\sec t \\ y = y_o + b\tan t \end{cases}$ where t is a parameter and

$$t \in \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[\cup \left]\frac{\pi}{2}, \frac{3\pi}{2}\right[$$

The tangent line at point $\left(x_0, y_0\right)$, on hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$,
is given by

$$T \equiv \frac{x_0 x}{a^2} - \frac{y_0 y}{b^2} = 1$$

5. Polar coordinates

To form a polar coordinate system in the plane, we fix a point 0 called the pole (or origin) and construct from 0 an initial ray called the polar axis. Then, each point P in the plane can be assigned polar coordinates (r, θ) as follows:

- r is the directed distance from 0 to *P*.
- θ is the directed angle, counterclockwise from polar axis to the segment $\overline{OP}$.

In polar coordinate system, the coordinates (r, θ),
$(r, \theta + 2k\pi)$, $k \in \mathbb{Z}$ and $\left(-r, \theta + (2k+1)\pi\right)$ represent the same point.

Coordinate conversion

The polar coordinates (r, θ) of a point are related to the rectangular coordinates (x, y) as follows:

$$x = r\cos\theta, \quad y = r\sin\theta, \quad r^2 = x^2 + y^2, \quad \tan\theta = \frac{y}{x},\ x \neq 0$$

Polar equation of a conic

A conic curve with eccentricity e, focus at the origin, whose directrix $x=-p$ has equation $r=\dfrac{ep}{1+e\cos\theta}$ where (r,θ) are polar coordinates of any point P lying on the conic.

It is an **ellipse** if $e<1$, a parabola if $e=1$, a **hyperbola** if $e>1$.

6. Applications

Eccentricities of orbits of the planets

The orbits of planets are ellipses with the sun at one focus. For most planets, these ellipses have very small eccentricity, so they are nearly circular. However, the Mercury and Pluto, the innermost and outermost known planets, have visibly elliptical orbits.

End of Unit Assessment

1. a) Parabola, $focus(0,-2)$, principal axis is $x=0$.

   b) Ellipse, $foci(\pm 1,0)$, semi-major axis $\sqrt{2}$, semi-minor axis 1.

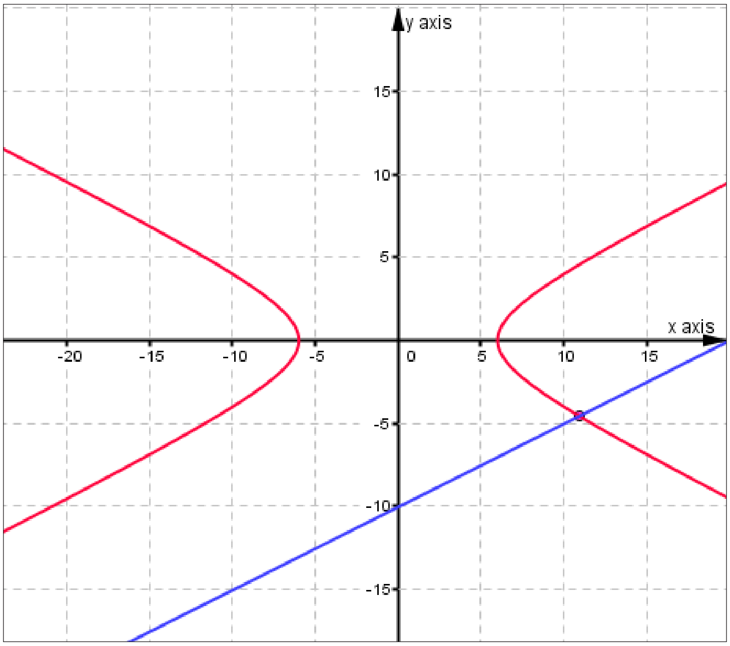
   c) Parabola, $focus\left(-\dfrac{1}{12},0\right)$, principal axis is $y=0$.

   d) Hyperbola, $foci(0,\pm 4)$, transverse axis $x=0$, conjugate axis $y=0$, asymptotes $y=\pm x$.

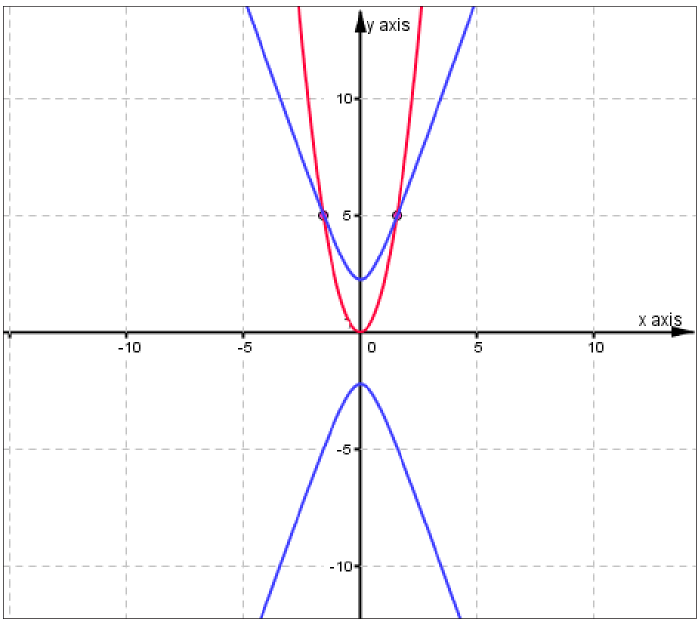
   e) Parabola, $focus\left(\dfrac{15}{4},1\right)$, principal axis is $y=1$.

   f) Hyperbola, $foci\left(\pm\sqrt{10},0\right)$, transverse axis is $y=0$, conjugate axis $x=0$, asymptotes $y=\pm 2x$.

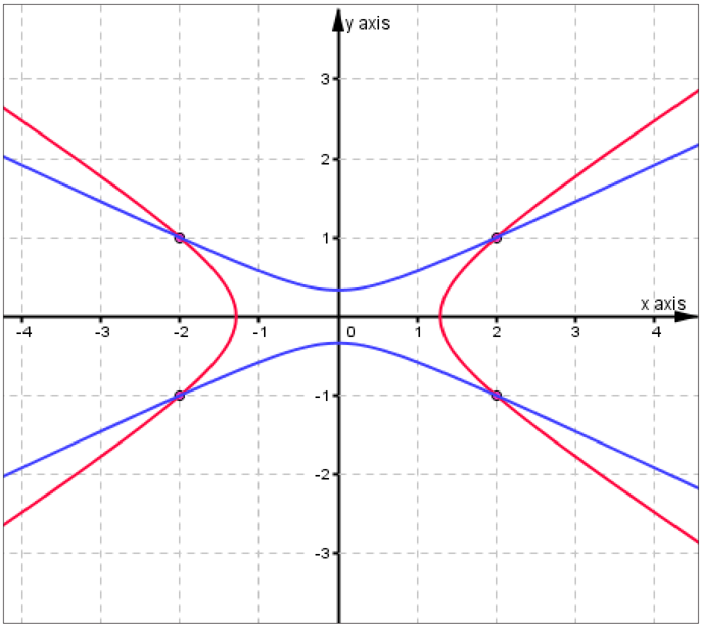
2. a) Intersection: $\left\{\left(\frac{109}{10},-\frac{91}{20}\right)\right\}$



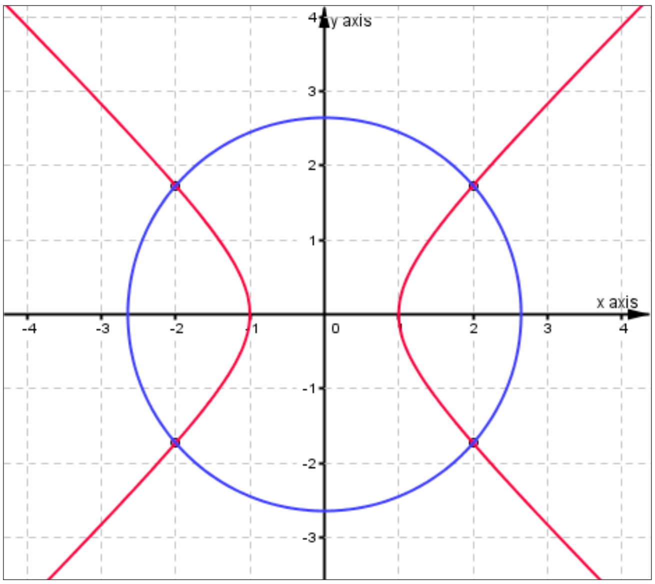
b) Intersection: $\left\{\left(\frac{\sqrt{10}}{2},5\right),\left(-\frac{\sqrt{10}}{2},5\right)\right\}$



c) Intersection: $\{(-2,1),(-2,-1),(2,1),(2,-1)\}$



d) Intersection: $\left\{\left(2,\sqrt{3}\right),\left(2,-\sqrt{3}\right),\left(-2,\sqrt{3}\right),\left(-2,-\sqrt{3}\right)\right\}$



3. $\dfrac{(x-4)^2}{32}+\dfrac{(y-3)^2}{36}=1$

4. $8xy-4x-4y+1=0$

5. a) The line $y = kx$ intersects the given conic once, when there is a unique solution.

As $AC - B^2 = 0$, the given conic is a parabola.

Solving the equations of line and conic taken together, we get equation $(2-k)^2 x^2 + 6x + 1 = 0$.

If $k = 2$, we get a linear equation: $6x + 1 = 0$.

$\Rightarrow x = -\frac{1}{6}$ and $y = -\frac{1}{3}$

That is, the line touches the conic at $\left(-\frac{1}{6}, -\frac{1}{3}\right)$.

If $k \neq 2$, we have quadratic equation which can be solved using discriminant;

$\Delta = b^2 - 4ac \Rightarrow \Delta = 9 - (2-k)^2 = (5-k)(1+k)$.

We have a unique solution if $\Delta = 0$, i.e. $k = 5$ *or* $k = -1$.

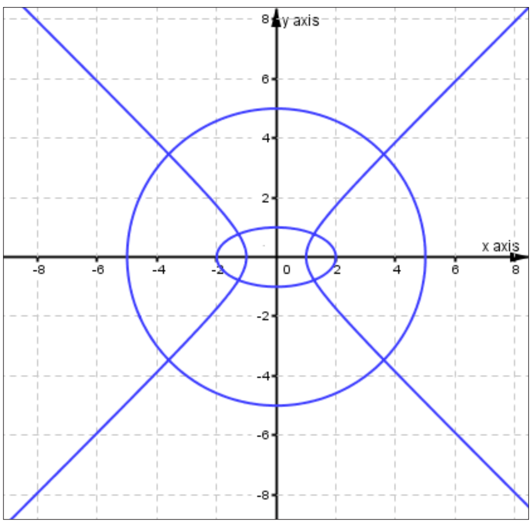
Therefore, the line $y = kx$ intersects the given conic once if $k = 2$ or $k = 5$ or $k = -1$.

b) The line $y = kx$ cuts the given conics in two points if $\Delta > 0$, $-1 < k < 5$ and $k \neq 2$.

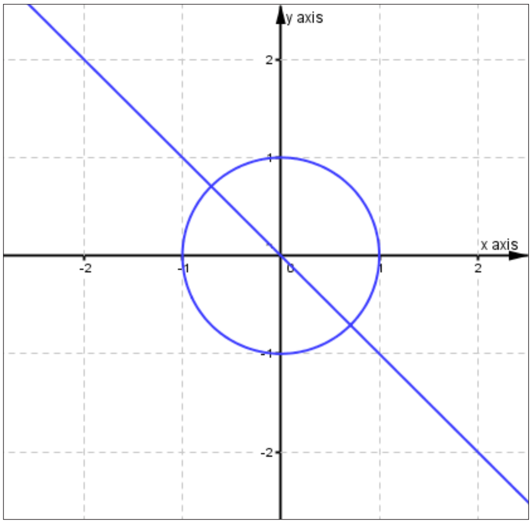
c) The line $y = kx$ does not intersect the given conics if $\Delta < 0$.

Thus, $k < -1$ *or* $k > 5$.

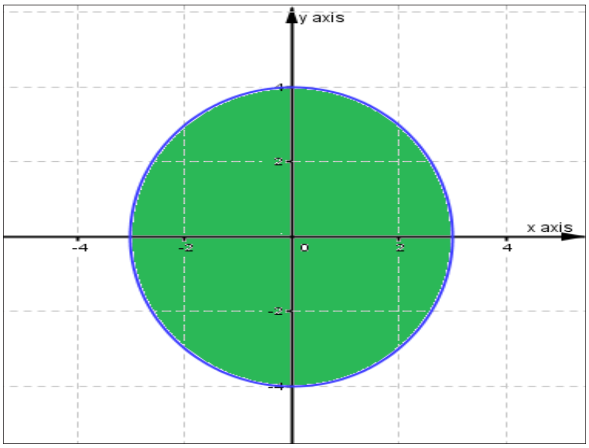
6. a)



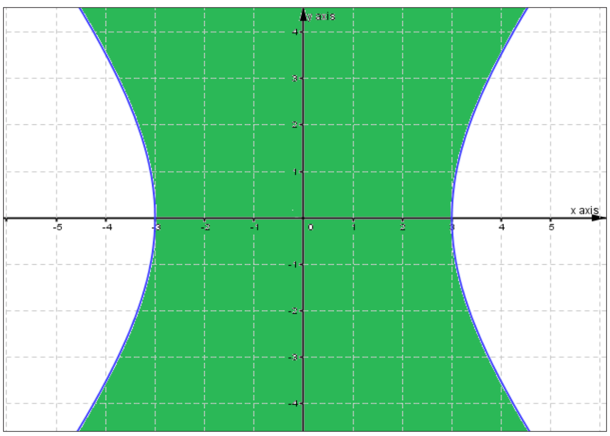
b)



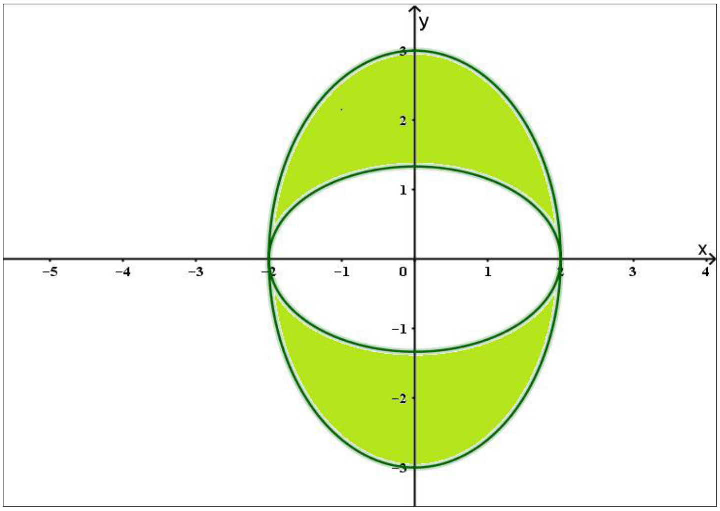
c)



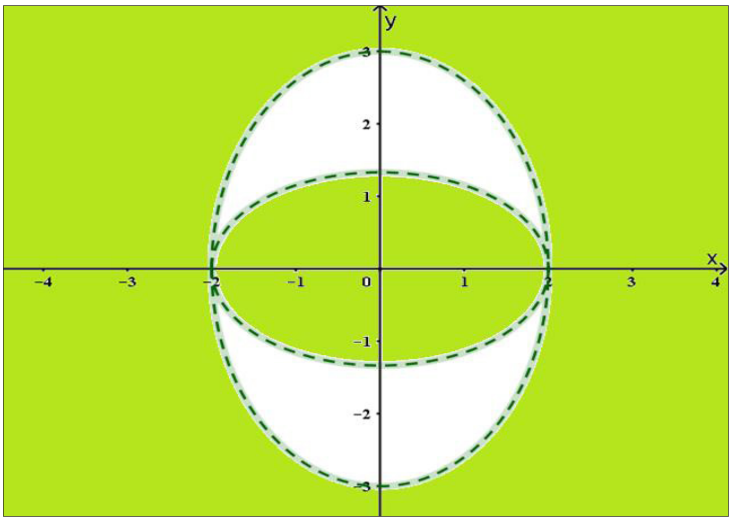
d)



e)



f)



7. a) $(y+2)^2 = 4(x+3), V(-3,-2), F(-1,-3), D \equiv x = -3$

b) $(x-1)^2 = 8(y+7), V(1,-7), F(1,-5), D \equiv y = -9$

c) $\dfrac{(x+2)^2}{6} + \dfrac{(y+1)^2}{9} = 1, F\left(-2, \pm\sqrt{3}-1\right), V(-2, \pm 3 - 1), C(-2,-1)$

d) $\dfrac{(x-2)^2}{3} + \dfrac{(y-3)^2}{2} = 1, F(3,3) \; and \; F(1,3), V\left(\pm\sqrt{3}+2, 3\right), C(2,3)$

e) $\dfrac{(x-2)^2}{4} - \dfrac{(y-2)^2}{5} = 1, F(5,2) \; and \; F(-1,2), V(4,2) \; and \; V(0,2), C(2,2),$

$A \equiv (y-2) = \pm\frac{\sqrt{5}}{2}(x-2)$

f) $(y+1)^2-(x+1)^2=1, F\left(-1,\sqrt{2}-1\right) and\ F\left(-1,-\sqrt{2}-1\right),$

$V(-1,0) and\ V(-1,-2)$

$C(-1,-1), A \equiv (y+1) = \pm(x+1)$

8. a) $\left(3\sqrt{3},3\right)$ b) $\left(-\frac{7}{2},\frac{7\sqrt{3}}{2}\right)$ c) $\left(4\sqrt{2},4\sqrt{2}\right)$

d) $(5,0)$ e) $\left(-\frac{7\sqrt{3}}{2},\frac{7}{2}\right)$ f) $(0,0)$

9. (i) a) $(5,\pi)$ b) $\left(4,\frac{11\pi}{6}\right)$ c) $\left(2,\frac{3\pi}{2}\right)$

d) $\left(8\sqrt{2},\frac{5\pi}{4}\right)$ e) $\left(6,\frac{2\pi}{3}\right)$ f) $\left(\sqrt{2},\frac{\pi}{4}\right)$

(ii) a) $(-5,0)$ b) $\left(-4,\frac{5\pi}{6}\right)$ c) $\left(-2,\frac{\pi}{2}\right)$

d) $\left(-8\sqrt{2},\frac{\pi}{4}\right)$ e) $\left(-6,\frac{5\pi}{3}\right)$ f) $\left(-\sqrt{2},\frac{5\pi}{4}\right)$

10. a) $x^2+y^2=5;$ circle b) $y=4;$ straight line

c) $y^2=1+2x;$ parabola d) $x^2-3y^2-8y=4;$ hyperbola

e) $3y-4x=5;$ straight line

f) $3x^2+4y^2-12x=36;$ ellipse

g) $x^2+y^2+4x=0;$ circle

11. a) Proof

b)

Planets	**Perihelion (astronomical units)**	**Aphelion (astronomical units)**
Mercury	0.3075	0.4667
Venus	0.7184	0.7282
Earth	0.9833	1.0167
Mars	1.3817	1.6663
Jupiter	4.9512	5.4548
Saturn	9.0210	10.0570
Uranus	18.2977	20.0623
Neptune	29.8135	30.3065

c)

Planets	Polar equation for the ellipse with eccentricity e and semi-major axis a: $r=\dfrac{a\left(1-e^2\right)}{1+e\cos\theta}$
Mercury	$r=\dfrac{0.3707}{1+0.2056\cos\theta}$
Venus	$r=\dfrac{0.7233}{1+0.0068\cos\theta}$
Earth	$r=\dfrac{0.9997}{1+0.0167\cos\theta}$
Mars	$r=\dfrac{1.5107}{1+0.0934\cos\theta}$
Jupiter	$r=\dfrac{5.1908}{1+0.0484\cos\theta}$

Saturn	$r = \dfrac{9.5109}{1 + 0.0543\cos\theta}$
Uranus	$r = \dfrac{19.1394}{1 + 0.0460\cos\theta}$
Neptune	$r = \dfrac{30.0580}{1 + 0.082\cos\theta}$

12. $7.25\,m$

13. $\dfrac{x^2}{25} + \dfrac{y^2}{9} = 1$

14. $\dfrac{x^2}{16} + \dfrac{y^2}{7} = 1$

15. $10\sqrt{15}$ inches

16. $\dfrac{x^2}{1.1025} - \dfrac{y^2}{7.8975}$

Unit 9 Random Variables

9.1. Key unit competence

Calculate and interpret the parameters of a random variable (discrete or continuous) including binomial and the Poisson distributions.

9.2. Vocabulary or key words concepts

Random variable: A variable which can assume numerical values each of which can correspond to one and only one of the events.

Discrete random variable: Random variable which takes only finite values between its limits.

Binomial distribution: Probability distribution for which probabilities are of successive terms of the binomial expansion $(q+p)^n$.

Poisson distribution: Discrete distribution used as a model for the number of events in a specific time period.

Continuous random variable: Random variable for which the possible values are all real values in some interval.

9.3. Guidance on the introductory activity

Organize groups of students, and then assign them to do the introductory activity from the student's book. As they are working, move around to each group and ask them probing questions leading them to the right way. After a while, invite some group to present their findings in a whole class discussion. Guide students to harmonize their answers and arouse their curiosity to the content of this unit.

This problem is solved using a special distribution called Poisson distribution that will be studied in this unit.

9.4. List of lessons

No	Lesson title	Number of periods
1	Probability density function of a discrete random variable	3
2	Expected value, variance and standard deviation of a discrete random variable	3
3	Cumulative distribution function of a discrete random variable	3
4	Binomial distribution	4
5	Expected value, variance and standard deviation of a binomial distribution	4
6	Poisson distribution	4
7	Probability density function of a continuous random variable	4
8	Cumulative distribution function of a continuous random variable	4
9	Expected value, variance and standard deviation of a continuous random variable	4
Total periods		33

9.5. Lesson development

Lesson 9.1. Probability density function of a discrete random variable

Learning objectives

Through examples, learners should be able to identify a discrete random variable and to find its probability distribution correctly.

Prerequisites

Finding probability of an event.

Teaching Aids

Exercise book, pen and calculator

Activity 9.1

In this lesson, the following generic competence and cross-cutting issues are to be addressed:

- Critical thinking
- Communication
- Self confidence
- Cooperation, interpersonal management and life skills
- Peace and values education
- Inclusive education

Answers

All balls are 6 with 4 red balls and 2 black balls

Probability of 0 red balls: $P(BBB)=\frac{2}{6}\times\frac{2}{6}\times\frac{2}{6}=\frac{1}{27}$

Probability of 1 red ball:

$$P(RBB)+P(BRB)+P(BBR)=\frac{4}{6}\times\frac{2}{6}\times\frac{2}{6}+\frac{2}{6}\times\frac{4}{6}\times\frac{2}{6}+\frac{2}{6}\times\frac{2}{6}\times\frac{4}{6}=\frac{2}{9}$$

Probability of 2 red balls:

$$P(RRB)+P(RBR)+P(BRR)=\frac{4}{6}\times\frac{4}{6}\times\frac{2}{6}+\frac{4}{6}\times\frac{2}{6}\times\frac{4}{6}+\frac{2}{6}\times\frac{4}{6}\times\frac{4}{6}=\frac{4}{9}$$

Probability of 3 red balls: $P(RRR)=\frac{4}{6}\times\frac{4}{6}\times\frac{4}{6}=\frac{8}{27}$

x	0	1	2	3
P	$\frac{1}{27}$	$\frac{2}{9}$	$\frac{4}{9}$	$\frac{8}{27}$

x takes on whole numbers only. The sum of obtained probabilities is 1. This is the probability of the sample space.

Synthesis

A variable X which can assume numerical values each of which can correspond to one and only one of the events is called a **random variable** (or **stochastic variable**). A random variable X is said to be a discrete random variable, if it takes only finite values between its limits.

It is convenient to introduce the probability function $P(X = x_i) = p_i$, also called the **probability distribution** satisfying

1. $P(X = x_i) \geq 0$
2. $\sum_i P(X = x_i) = 1$, where the sum is taken over all values of x_i.

The **probability density function** (p.d.f), $F(x)$, is a function that allocates probabilities to all distinct values that X can take on.

Application Activity 9.127

1. X is a random variable if $\sum_{i=1}^{3} P(X = x_i) = 1$.

   $\sum_{i=1}^{3} P(X = x_i) = P(X = 2) + P(X = 3) + P(X = 4) = \frac{1}{6} + \frac{2}{6} + \frac{3}{6} = 1$

   Therefore, X is a random variable.

2. a) $p = \frac{1}{10}$ b) $P(X \geq 2) = \frac{4}{5}$

3. $a = \frac{1}{2}$

Lesson 9.2. Cumulative distribution of discrete random variable

Learning objectives

Given a discrete random variable, learners should be able to find its cumulative distribution precisely.

Prerequisites

- Probability density function of a discrete random variable.

Teaching Aids

Exercise book, pen and calculator

Activity 9.2

In this lesson, the following generic competence and cross-cutting issues are to be addressed:

- Critical thinking
- Communication
- Self confidence
- Cooperation, interpersonal management and life skills
- Peace and values education
- Inclusive education

Answers

Cumulative probabilities are found by adding the probability up to each column of the table. In the table, we find the cumulative probability for one head by adding the probabilities for zero and one. The cumulative probability for two heads is found by adding the probabilities for zero, one, and two. We continue with this procedure until we reach the maximum number of heads, in this case four, which should have a cumulative probability of 1.00 because 100% of trials must have four or fewer heads.

Then,

Heads	0	1	2	3	4
Probability	$\frac{1}{16}$	$\frac{4}{16}$	$\frac{6}{16}$	$\frac{4}{16}$	$\frac{1}{16}$
Cumulative Probability	$\frac{1}{16}$	$\frac{5}{16}$	$\frac{11}{16}$	$\frac{15}{16}$	$\frac{16}{16}=1$

Synthesis

To find a **cumulative probability** we add the probabilities for all values qualifying as "less than or equal" to the specified value. Then,

The cumulative distribution function of a random variable X is the function $F(x)=P(X\le x)$.

Application Activity 9.128

1. Cumulative distribution function

$$F(x)=\begin{cases}0, & x<0\\ \frac{1}{4}, & 0\le x<1\\ \frac{3}{4}, & 1\le x<2\\ 1, & x\ge 2\end{cases}$$

2. Cumulative distribution

x	2	3	4	5	…	12
$F(x)$	$\frac{1}{36}$	$\frac{3}{36}$	$\frac{6}{36}$	$\frac{10}{36}$	…	1

3. Probability distribution

x	1	2	3	4	5
$P(X=x)$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$

Lesson 9.3. Expected value, variance and standard deviation of a discrete random variable

Learning objectives

Given a discrete random variable, learners should be able to calculate the expected value, variance and standard deviation correctly.

Prerequisites

- Probability distribution of a discrete random variable.

Teaching Aids

Exercise book, pen and calculator

Activity 9.3

In this lesson, the following generic competence and cross-cutting issues are to be addressed:

- Critical thinking
- Communication
- Self confidence
- Cooperation, interpersonal management and life skills
- Peace and values education
- Inclusive education

Answers

x	$P(X=x)$	$xP(X=x)$	$x^2P(X=x)$
1	0.2	0.2	0.2
2	0.5	1	2.0
3	0.3	0.9	2.7
Sum	1.0	2.1	4.9

Synthesis

The **expected value** of random variable X, which is the **mean** of the probability distribution of X, is denoted and defined by

$$\mu = E(X) = \sum_{i=1}^{n} x_i P(X = x_i).$$

The variance of random variable X is denoted and defined by

$$\sigma^2 = Var(X) = E(X^2) - [E(X)]^2$$ or

$$\text{var}(X) = E(X^2) - [E(X)]^2$$

The standard deviation of random variable X, is

$$\sigma = SD(X) = \sqrt{Var(X)}.$$

Application Activity 9.129

1. a) 1.9 b) 2.4 c) 0.4 d) 9.23
2. a) $\frac{2}{3}$ b) $\frac{26}{9}$ c) $\frac{152}{81}$ d) $\frac{2\sqrt{38}}{9}$
3. Expected value: 3.5, variance: 2.9, standard deviation: 1.7

Lesson 9.4. Binomial distribution

Learning objectives

Through examples, learners should be able to identify a binomial distribution and find its probability distribution accurately.

Prerequisites

- Powers.
- Combination of n objects taken from n objects.

Teaching Aids

Exercise book, pen and calculator

Activity 9.4

In this lesson, the following generic competence and cross-cutting issues are to be addressed:

- Critical thinking
- Communication
- Self confidence
- Cooperation, interpersonal management and life skills
- Peace and values education
- Inclusive education

Answers

1. Probability of the sequence $SSSSFFFFFF$ is

$$P(S)P(S)P(S)P(S)P(F)P(F)P(F)P(F)P(F)P(F) = ppppqqqqqq$$
$$= p^4q^6$$

2. From 1),

$$\underbrace{P(S)P(S)....}_{r\,times}\times\underbrace{P(F)P(F)...}_{n-r\,times}=\underbrace{pp....}_{r\,times}\times\underbrace{qq...}_{n-r\,times}$$

$$=p^r q^{n-r}$$

3. Different combinations that produce 4 heads are given by ${}^{10}C_4\, p^4 q^6$.
4. Different combinations that produce r heads in n trials are given by ${}^{n}C_r\, p^r q^{n-r}$.

Synthesis

The probability of obtaining r successes in n independent trials is $b(r:n,p)={}^{n}C_r p^r q^{n-r}$ for $0\le r\le n$ where p is the probability of a success in each trial. This probability distribution is called the binomial distribution since the values of the probabilities are successive terms of the binomial expansion of $(q+p)^n$; that is why $b(r:n,p)={}^{n}C_r p^r q^{n-r}$.

Each trial has two possible outcomes: success (p) and failure (q).

The outcome of the n trials are mutually independent and there will be r successes and $n-r$ failures.

Application Activity 9.130

1. $\frac{15}{64}$	2. 0.92	3. 0.65536
4. 0.19	5. 0.51	
6. a) 0.39	b) 0.35	c) 0.93
7. a) 0.26	b) $\frac{41}{1679616}$	c) 0.77
8. a) $\frac{8183}{8192}$	b) $\frac{16807}{32768}$	

Lesson 9.5. Expected value, variance and standard deviation of a binomial distribution

Learning objectives

Given a binomial random variable, learners should be able to find expected value, variance and standard deviation correctly.

Prerequisites

- Binomial distribution

Teaching Aids

Exercise book, pen and calculator

Activity 9.5

In this lesson, the following generic competence and cross-cutting issues are to be addressed:

- Critical thinking
- Communication
- Self confidence
- Cooperation, interpersonal management and life skills
- Peace and values education
- Inclusive education

Answers

1. $E(X)=\sum_{i=1}^{n} x_i P(X=x_i)=0\times(1-p)+1\times p=p$

   For n trials, $E(X)=np$

2. $E(X^2)=0^2\times(1-p)+1^2 p=p$ and for n trials

   $E(X^2)=np$

   $$\begin{aligned} Var(X)&=E(X^2)-[E(X)]^2\\ &=np-[np]^2\\ &=np-n^2p^2\\ &=np(1-p)\\ &=npq \end{aligned}$$

Synthesis

Basing on the results from activity 9.5, the expected value (or mean) of a binomial distribution of a discrete random variable X is denoted and defined by $\mu = E(X) = np$ where n is the number of trials and p is the probability of success.

The variance of a binomial distribution of a discrete random variable X is denoted and defined by $\sigma^2 = Var(X) = npq$ where n is the number of trials, p is the probability of success and q is the probability of failure.

The standard deviation of a binomial distribution of a discrete random variable X is denoted and defined by $\sigma = \sqrt{Var(X)} = \sqrt{npq}$.

Application Activity 9.131

1. $np > npq, as\, q < 1$
2. Mean is 30, standard deviation is 5
3. a) $E(X) = 2.5$ b) $\text{var}(X) = 1.875$
4. a) 0.117 b) 0.974
5. approximately 1 6. 11 7. ≈ 17

Lesson 9.6. Poisson distribution

Learning objectives

Through examples, learners should be able to identify a poison distribution and solve problems using poison distribution correctly.

Prerequisites

- Use of exponential and factorial notation.

Teaching Aids

Exercise book, pen and calculator

Activity 9.6

In this lesson, the following generic competence and cross-cutting issues are to be addressed:

- Critical thinking
- Communication
- Self confidence
- Cooperation, interpersonal management and life skills
- Peace and values education
- Inclusive education

Answers

1. Dividing both sides of (1) by e^{θ}

$$1=\frac{\theta^0}{0!e^{\theta}}+\frac{\theta^1}{1!e^{\theta}}+\frac{\theta^2}{2!e^{\theta}}+\frac{\theta^3}{3!e^{\theta}}+...+\frac{\theta^n}{n!e^{\theta}}+...$$

$$=\frac{\theta^0 e^{-\theta}}{0!}+\frac{\theta^1 e^{-\theta}}{1!}+\frac{\theta^2 e^{-\theta}}{2!}+\frac{\theta^3 e^{-\theta}}{3!}+...+\frac{\theta^n e^{-\theta}}{n!}+..$$

2. If we take $\lambda=\theta$, we have

$$1=\frac{\lambda^0}{0!e^{\lambda}}+\frac{\lambda^1}{1!e^{\lambda}}+\frac{\lambda^2}{2!e^{\lambda}}+\frac{\lambda^3}{3!e^{\lambda}}+...+\frac{\lambda^n}{n!e^{\lambda}}+..$$

3. Using the general term, $\frac{\lambda^n}{n!e^{\lambda}}$, and putting $n=x$, we have $P(X=x)=\frac{\lambda^x}{x!e^{\lambda}}$ or $P(X=x)=\frac{e^{-\lambda}\lambda^x}{x!}$

Synthesis

The probability density function of Poisson distribution is denoted $X \sim P(\lambda)$ and defined by

$$P(X=x)=\frac{e^{-\lambda}\lambda^x}{x!},\quad x=0,1,2,...$$

Where λ is a parameter which indicates the average number (the expected value) of events in the given time interval and $e \approx 2.718...$

For Poisson distribution with parameter λ, $E(x)=\lambda$ and $Var(x)=\lambda$.

Application Activity 9.132

1. Wrong statement, because $\sigma = \sqrt{\lambda}$
2. 0.827008
3. 0.052129
4. 0.160623
5. 0.128387
6. 0.00000546
7. $\sum_{x=0}^{14} \frac{e^{-20}(20)^x}{x!}$
8. Considering the given table:

x	0	1	2	3	4	5	6	Total
f	103	143	98	42	8	4	2	400
$x \cdot f$	0	143	126	126	32	20	12	529

$$Mean = \frac{529}{400} = 1.32$$

Number of cells	Probability $P(x) = \frac{0.2674 \times (1.32)^x}{x!}$	Theoretical frequency
0	$\frac{0.2674 \times (1.32)^0}{0!} = 0.2674$	$0.2674 \times 400 \simeq 107$
1	$\frac{0.2674 \times (1.32)^1}{1!} = 0.353$	$0.353 \times 400 \simeq 141$
2	$\frac{0.2674 \times (1.32)^2}{2!} = 0.233$	$0.233 \times 400 \simeq 93$
3	$\frac{0.2674 \times (1.32)^3}{3!} = 0.1025$	$0.1025 \times 400 \simeq 41$
4	$\frac{0.2674 \times (1.32)^4}{4!} = 0.0338$	$0.0338 \times 400 \simeq 14$

5	$\frac{0.2674\times(1.32)^5}{5!}=0.00893$	$0.00893\times 400\simeq 4$
6	$\frac{0.2674\times(1.32)^6}{6!}=0.00196$	$0.00196\times 400\simeq 1$

The expected (theoretical) frequencies are
$107, 141, 93, 41, 14, 4, 1$.

Lesson 9.7. Probability density function of a continuous random variable

Learning objectives

Through examples, learners should be able to identify a continuous random variable, and to find its probability distribution accurately.

Prerequisites

- Integration.
- Curve sketching.

Teaching Aids

Exercise book, pen, calculator, instruments of geometry.

Activity 9.7

In this lesson, the following generic competence and cross-cutting issues are to be addressed:

- Critical thinking
- Communication
- Self confidence
- Cooperation, interpersonal management and life skills
- Peace and values education
- Inclusive education

Answers

1. The area under the curve of $f(x)$ is given by;

$\int_{-1}^{0} k(x+1)^2\,dx + \int_{0}^{1} kx dx$. Then

$$\int_{-1}^{0} k(x+1)^2\,dx + \int_{0}^{1} kdx = 1 \Rightarrow k\int_{-1}^{0}\left(x^2+2x+1\right)dx + k\int_{0}^{1} dx = 1$$

$$\Rightarrow k\left[\frac{x^3}{3}+x^2+x\right]_{-1}^{0} + k\left[x\right]_0^1 = 1$$

$$\Rightarrow k\left(\frac{1}{3}-1+1\right)+k=1 \;\Rightarrow \frac{4}{3}k=1 \text{ or } k=\frac{3}{4}$$

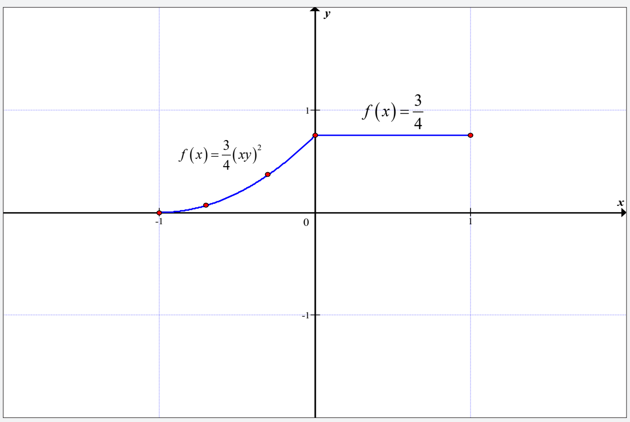
2. Graph of $f(x)=\begin{cases}\frac{3}{4}(x+1)^2 & -1\le x\le 0\\ \frac{3}{4} & 0<x\le 1\end{cases}$

$f(x)=\frac{3}{4}(x+1)^2 \qquad -1\le x\le 0$

x	-1	-0.3	-0.7	0
$f(x)=y$	0	0.37	0.07	0.43

$f(x)=\frac{3}{4} \qquad 0<x\le 1$

x	0	1
$f(x)=y$	0.43	0.43



Synthesis

A random variable X is said to be continuous if its possible values are all real values in some interval.

A function $f(x)$ defined on an interval $[a,b]$ is a probability density function for a continuous random variable X distributed on $[a,b]$ if, whenever x_1 and x_2 satisfy $a \le x_1 \le x_2 \le b$, we have

$$p(x_1 \le X \le x_2) = \int_{x_1}^{x_2} f(x)\,dx$$

If X is a continuous random variable, then the probability that the values of X will fall between the values a and b is given by the area of the region lying below the graph of $f(x)$ and above the $x-axis$ between a and b and this area is equal to 1.

Normally, $\int_{-\infty}^{+\infty} f(x)\,dx = 1$.

Application Activity 9.133

1. a) $c = \frac{3}{2}$ b) $P\left(X \ge \frac{1}{2}\right) = \frac{7}{16}$
2. 0.693
3. a) $k = \frac{1}{4}$

   b) i) $p(x<1) = \frac{1}{4}$ ii) $p(x=1) = 0$

   iii) $p(x > 2.5) = 0.3125$

   iv) $p\left[(0<x<2)/x \ge 1\right] = \dfrac{p(1 \le x \le 2)}{p(1 \le x \le 3)}$

$$= \frac{\int_1^2 \frac{1}{4}dx}{\int_1^2 \frac{1}{4}dx + \int_2^3 \frac{1}{4}(2x-3)\,dx} = \frac{1}{3}$$

Lesson 9.8. Cumulative distribution of continuous random variable

Learning objectives

Given a continuous random variable, learners should be able to find its cumulative distribution accurately.

Prerequisites

- Finite integrals

Teaching Aids

Exercise book, pen and calculator

Activity 9.8

In this lesson, the following generic competence and cross-cutting issues are to be addressed:

- Critical thinking
- Communication
- Self confidence
- Cooperation, interpersonal management and life skills
- Peace and values education
- Inclusive education

Answers

1. For $x<1$, $f(x)=0$ and then $F(x)=\int_1^x 0\,dx=0$
2. For $1\le x\le 3$, $\frac{1}{4}x$ and then
$$F(x)=\int_1^x \frac{1}{4}x\,dx=\left[\frac{1}{8}x^2\right]_1^x=\frac{1}{8}x^2-\frac{1}{8}=\frac{x^2-1}{8}$$
3. For $x>3$, $f(x)=0$ and then
$$F(x)=F(3)+\int_1^x 0\,dx=\frac{x^2-1}{8} \text{ and } F(3)=\frac{3^2-1}{8}=1$$
4. Hence,
$$F(x)=\begin{cases}0, & x<1\\ \dfrac{x^2-1}{8}, & 1\le x\le 3\\ 1, & x>3\end{cases}$$

Synthesis

The **cumulative distribution** function of a continuous random variable X is defined as: $F(x)=\int_{-\infty}^{x} f(t)\,dt$.

Where $F(x)=0$, for $x\to-\infty$ and $F(x)=1$, for $x\to+\infty$.

Application Activity 9.134

1. $F(x)=\begin{cases}0, & x\le -1\\ \frac{1}{2}(x+1)^2, & -1<x\le 0\\ 1-\frac{(1-x)^2}{2}, & 0<x<1\\ 1, & x\ge 1\end{cases}$

2. $F(x)=\begin{cases}0, & x<0\\ \frac{x^2}{16}, & 0\le x\le 4\\ 1, & x>4\end{cases}$

3. $F(x)=\begin{cases}0, & x<0\\ \frac{x^2}{6}, & 0\le x<2\\ \frac{-x^2}{3}+2x-2, & 2\le x<3\\ 1, & x\ge 3\end{cases}$

Lesson 9.9. Variance and standard deviation of a continuous random variable

Learning objectives

Given a continuous random variable, learners should be able to find expected value, variance and standard deviation correctly.

Prerequisites

- Probability density function of a continuous random variable.

Teaching Aids

Exercise book, pen and calculator

Activity 9.9

In this lesson, the following generic competence and cross-cutting issues are to be addressed:

- Critical thinking
- Communication
- Self confidence
- Cooperation, interpersonal management and life skills
- Peace and values education
- Inclusive education

Answers

1. $A=\int_0^1 xf(x)\,dx=6\int_0^1 x^2(1-x)\,dx$

$$=6\int_0^1\left(x^2-x^3\right)dx=6\left[\frac{1}{3}x^3-\frac{1}{4}x^4\right]_0^1$$

$$=6\left[\frac{1}{3}-\frac{1}{4}\right]=0.5$$

2. $B=\int_0^1 x^2 f(x)\,dx=6\int_0^1 x^3(1-x)\,dx$

$$=6\int_0^1\left(x^3-x^4\right)dx=6\left[\frac{1}{4}x^4-\frac{1}{5}x^5\right]_0^1$$

$$=6\left[\frac{1}{4}-\frac{1}{5}\right]=0.3$$

3. $B-A^2=0.3-0.25=0.05$

Synthesis

The **mean,** μ, (or **expected value,** $E(X)$), of X is denoted and defined by;

$$\mu=E(X)=\int_a^b xf(x)\,dx$$

Also, expectation of function g of X is

$$E(g(x))=\int_a^b g(x)f(x)\,dx$$

The variance $Var(x)$ or σ^2 is denoted and defined by

$\sigma^2 = Var(X) = E(X^2) - [E(x)]^2$.

The standard deviation is

$\sigma = SD = \sqrt{Var(X)}$.

Properties of $E(X)$ and $Var(X)$

$\forall a, b \in \mathbb{R}$

1. $E(a) = a$
2. $E(aX) = aE(X)$
3. $E(aX + b) = aE(X) + b$
4. $E(X + Y) = E(X) + E(Y)$
5. $E(aX^2 + b) \neq aE(X^2) + b$
6. $\text{var}(a) = 0$
7. $\text{var}(aX + b) = a^2 \text{ var}(X)$
8. $\text{var}(aX + b) = a^2 \text{ var}(X)$

Application Activity 9.135

1. a) $k = 1.44$

   b) mean is 0.443, variance is 0.0827
2. a) $c = \frac{3}{2}$ b) $E(X) = 0, \text{ var}(X) = \frac{3}{5}$
3. a) $a = 0.01$

   b) $E(x) = 10$, $Var(x) = 16.6667$

   $E(9x) = 9E(x) = 90$ $Var(9x) = 9^2 Var(x) = 1350$

9.6. Summary of the unit

1. Discrete and finite random variables

1. Probability density function

A random variable X is said to be a **discrete random variable**, if it takes only finite values between its limits; for example, the number of learners appearing in a festival consisting of 400 learners is a discrete random variable which can assume values other than 0, 1, 2, …,400.

The **probability density function** (p.d.f), $F(x)$, is a function that allocates probabilities to all distinct values that X can take on.

If the initial probability is known, you can find successive probabilities using the following recurrence relation

$$P(X=x+1)=\left(\frac{n-x}{x+1}\right)\left(\frac{p}{q}\right)P(X=x).$$

2. Expectation, variance and standard deviation

The expected value of random variable X, which is the mean of the probability distribution of X, is denoted and defined by

$$\mu=E(X)=\sum_{i=1}^{n}x_iP(X=x_i).$$

Also, the expectation of any function $g(X)$ of the random variable X is

$$\mu=E(g(X))=\sum_{i=1}^{n}g(x)P(X=x_i).$$

The variance of random variable X is denoted and defined by

$$\sigma^2=Var(X)=\sum_{i=1}^{n}[x_i-\mu]^2P(X=x_i).$$

This can be simplified to

$$\sigma^2=Var(X)=E(X^2)-\mu^2$$

The standard deviation of random variable X, denoted by $\mathrm{SD}(X)$, is the square root of the variance. That is $\sigma = SD(X) = \sqrt{Var(X)}$.

Properties for mean and variance

$\forall a, b \in \mathbb{R}$

1. $E(a) = a$
2. $E(aX) = aE(X)$
3. $E(aX+b) = aE(X)+b$
4. $E(X+Y) = E(X)+E(Y)$
5. $E(aX^2+b) \neq aE(X^2)+b$
6. $\mathrm{var}(a) = 0$
7. $\mathrm{var}(aX) = a^2 \mathrm{var}(X)$
8. $\mathrm{var}(aX+b) = a^2 \mathrm{var}(X)$

3. Binomial distribution (Law of Bernoulli)

For binomial probability distribution, we are interested in the probabilities of obtaining r successes in n trials, in other word r successes and n-r failures in n attempts.

Binomial distribution is denoted

$b(r:n,p) = {}^nC_r P^r q^{n-r}, r = 0,1,2,\ldots,n$

The constant n, p, q are called parameters of the binomial distribution.

The following assumptions are made:

- ***A*** There is a fixed number (n) of trials.
- ***A*** The probability of success is the same for each trial.
- ***A*** Each trial is independent of all other trials.

Note that $p+q=1$

For N set of n trial, the successes 0, 1, 2, ...r, ..., n are given by $N(p+q)^n$, which is called binomial distribution.

The expected value (or mean) of a binomial distribution of a discrete random variable X is denoted and defined by $\mu = E(X) = np$ where n is the number of trials and p is the probability of success.

The variance of a binomial distribution of a discrete random variable X is denoted and defined by $\sigma^2 = Var(X) = npq$ where n is the number of trials, p is the probability of success and q is the probability of failure.

The standard deviation of a binomial distribution of a discrete random variable X is denoted and defined by $\sigma = \sqrt{Var(X)} = \sqrt{npq}$.

4. Uncountable infinite discrete case: Poisson distribution

The **Poisson distribution** is a discrete distribution often used as a model for the number of events (such as the number of customers in waiting lines, number of defects in a given surface area, airplane arrivals, or the number of accidents at an intersection) in a specific time period.

Poisson distribution is a limiting form of the binomial distribution $(p+q)^n$ under the following **conditions**:

(i) $n \to \infty$, i.e., the number of trials is indefinitely large.

(ii) $p \to 0$, i.e., the constant probability of success for each trial is indefinitely small.

(iii) np is a finite quantity, say λ .

Typical events which could have a Poisson distribution:

(i) Number of customers arriving at a supermarket checkout per minute.
(ii) Number of suicides or deaths by heart attack in a minute.
(iii) Number of accidents that take place on a busy road in time t.
(iv) Number of printing mistakes at each unit of the book.
(v) Number of cars passing a certain street in time t.
(vi) Number of $\alpha-$particles emitted per second by a radioactive sources.
(vii) Number of faulty blades in a packet of 1000.
(viii) Number of person born blind per year in a certain village.

(ix) Number of telephone calls received at a particular switch board in a minute.

(x) Number of times a teacher is late for class in a given week.

The probability density function of Poisson distribution is defined by

$$P(X=x)=\frac{e^{-\lambda}\lambda^{x}}{x!},\ x=0,1,2,...$$

where λ is a parameter which indicates the average number (the expected value) of events in the given time interval. We write $X\sim \mathrm{Po}(\lambda)$

- If the initial probability is known, you can find successive probabilities using the following recurrence relation; $P(X=x+1)=\frac{\lambda}{x+1}P(X=x)$.
- For a Poisson distribution of a discrete random variable X, the mean μ (or expected value) and the variance σ^2 are the same and equal to λ. Thus, $\mu=\sigma^2=\lambda$

5. Continuous random variables

a) Probability density function

A function defined on an interval $[a,b]$ is a probability density function for a continuous random variable X distributed on $[a,b]$ if, whenever x_1 and x_2 satisfy $a\le x_1\le x_2\le b$, we have $p(x_1\le X\le x_2)=\int\limits_{x_1}^{x_2} f(x)dx$.

Properties of p.d.f $f(x)$

a) $f(x)>0$ for all x

b) $\int\limits_{all\,x} f(x)dx=1$

The **cumulative distribution** function of a continuous random variable X is defined as: $F(x)=\int_{-\infty}^{x} f(t)\,dt$

where $F(x)=0,$ for $x\to-\infty$ and $x\to+\infty$ for $x\to+\infty$.

b) Expected value, variance and standard deviation

The mean μ (or expected value $E(X)$) of X is denoted and defined by

$$\mu = E(X) = \int_a^b xf(x)dx$$

Also, expectation of function g of X is

$$E(g(x)) = \int_a^b g(x)f(x)dx$$

The variance $Var(x)$ or σ^2 is denoted and defined by

$$\sigma^2 = Var(X) = E(X^2) - [E(x)]^2 .$$

The standard deviation is

$$\sigma = SD = \sqrt{Var(X)} .$$

Properties of $E(X)$ and $Var(X)$

$\forall a, b \in \mathbb{R}$

1. $E(a) = a$
2. $E(aX) = aE(X)$
3. $E(aX + b) = aE(X) + b$
4. $E(X + Y) = E(X) + E(Y)$
5. $E(aX^2 + b) \neq aE(X^2) + b$
6. $\text{var}(a) = 0$
7. $\text{var}(aX) = a^2 \text{var}(X)$
8. $\text{var}(aX + b) = a^2 \text{var}(X)$

9.7. End of Unit Assessment

1. $\frac{29}{32}$
2. a) 0.0145 b) 0.1887
   c) 0.0000000000000000000001
3. 0.0863
4. Probability that it will work (0 defective components) is 0.896 .
   Probability that it will not work perfectly is 0.104
5. 0.00038

6. a) $\{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$

   b) $p(HHH) = 0.166375$,

   $p(HHT) = p(HTH) = p(THH) = 0.136125$,

   $p(HTT) = p(THT) = p(TTH) = 0.111375$,

   $p(TTT) = 0.091125$

   c) $f(0) = 0.911125,\ f(1) = 0.334125, f(2) = 0.408375,$

   $f(3) = 0.166375$

   d) 0.908875 e) 1.650000

7. $\frac{176}{1024} \times 100 \simeq 17$

8. 0.99863

9. 0.0376

10. $\frac{na}{a+b}$

11. a) 0.5905 b) $E(X) = 7, Var(x) = 6.3$

12. a) 0.9997 b) 0.005

13. $\frac{3}{4}$

14. a) $4.8,\ 0.98$ b) 0.655

15. a) 0.0498×1000 b) 0.3526×1000

16. 2.3026

17. $\frac{e^{-100}(100)^x}{x!}$

18. 0.51

19. a) 0.147 b) 0.0408 c) 0.762

20. a) 0.122 b) 0.138 c) 0.224 d) 0.0273

21.

x	0	1	2	3	4	Total
f	122	60	15	2	1	200
$x \cdot f$	0	60	30	6	4	100

$$Mean = \frac{100}{200} = 0.5$$

The number of x deaths is given by $200\times\frac{(e)^{-0.5}(0.5)^x}{x!}$ for 0, 1, 2, 3, 4.

Death/Frequencies	0	1	2	3	4
Probabilities $P(x)=\frac{e^{-0.5}\times(0.5)^x}{x!}$	0.27	0.35	0.23	0.10	0
Expected (Theoretical) frequency $200\times\frac{(e)^{-0.5}(0.5)^x}{x!}$	121	61	15	3	0

The expected frequencies are $121, 61, 15, 3$ and 0.

22. 0.9998

23. 0.5620

24.

x	0	1	2	3	4	5	6	Total
f	143	90	42	12	9	3	1	300
$x\cdot f$	0	90	84	36	36	15	6	267

$$Mean=\frac{267}{300}=0.89$$

The number of x mistakes per day is given by

$300\times\frac{(e)^{-0.89}(0.89)^x}{x!}$ for 0, 1, 2, 3, 4,5, 6.

Mistakes per day	1	1	2	3	4	5	6
Probabilities $P(x)=\frac{e^{-0.5}\times(0.5)^x}{x!}$	0.41	0.37	0.16	0.05	0.01	0	0

Expected (Theoretical) frequency $200\times\frac{(e)^{-0.5}(0.5)^x}{x!}$	123	111	48	15	3	0	0

The expected frequencies are $123, 111, 48, 15, 3$ and 0.

25.a) 0.25 b) $\frac{2}{3}$

26.a) $k=\frac{1}{2}$ b) 0.75

27.a) 0.125 b) 0.727

28.a) $k=\frac{3}{16}$ b) $E(x)=1\frac{3}{4}$

c) $P(1\le x\le 3)=\frac{11}{16}$

29.a) (i) $k=\frac{2}{5}$ (ii) $E(x)=1\frac{4}{15}$

(iii) $\sigma=0.75$ b) $p(x<\mu-\sigma)=0.207$

30.(i) a) $k=\frac{2}{9}$ b) $\mu=E(x)=2,\ \sigma^2=\frac{1}{2},\ \sigma=\frac{\sqrt{2}}{2}$

c) $\frac{4\sqrt{2}}{9}\approx 0.63$

(ii) a) $k=3$ b) $\mu=E(x)=\frac{3}{4},\ \sigma^2=\frac{3}{80},\ \sigma=\frac{3\sqrt{5}}{20}$

c) $\frac{207\sqrt{5}}{400}\approx 0.668$

iii) a) $k=6$ b) $\mu=E(x)=\frac{1}{2},\ \sigma^2=\frac{1}{20},\ \sigma=\frac{\sqrt{5}}{20}$

c) $\frac{7\sqrt{5}}{25}\approx 0.626$

31.a) $a=12$, $b=1$ b) 0.0523

Answers for Summative Evaluation One

1. $\left(\frac{1}{8}\right)^{x-2} = 4^{3-2x} \Leftrightarrow \left(2^{-3}\right)^{x-2} = \left(2^2\right)^{3-2x}$

   $\Leftrightarrow 2^{-3x+6} = 2^{6-4x} \Rightarrow -3x+6 = 6-4x$
   or $x=0$

   $S=\{0\}$

2. A quadratic function has a double root if and only if $\Delta = 0$.

   For our case, $\Delta = b^2 - 4ac = 9-4m$.

   $\Delta = 0 \Leftrightarrow 9-4m=0$ or $m=\frac{9}{4}$.

   Therefore, $x^2+3x+m=0$ admits a double root when $m=\frac{9}{4}$.

   For $m=\frac{9}{4}$, $x^2+3x+m=0 \Rightarrow x^2+3x+\frac{9}{4}=0$

   The root is $x=-\frac{3}{2}$.

3. If the angle between $\vec{u}=(k,3)$ and $\vec{v}=(4,0)$ is 45^o, thus

   $$\cos 45^o = \frac{4k}{\sqrt{k^2+9}\sqrt{16}}$$

   Or

   $$\cos 45^o = \frac{4k}{4\sqrt{k^2+9}} \Leftrightarrow \cos 45^o = \frac{k}{\sqrt{k^2+9}}$$

   Since $\cos 45^o = \frac{\sqrt{2}}{2}$, then,

   $$\frac{\sqrt{2}}{2} = \frac{k}{\sqrt{k^2+9}} \Leftrightarrow 2k = \sqrt{2}\sqrt{k^2+9}$$

   Squaring both sides yields

   $$4k^2 = k^2+18 \Leftrightarrow k^2 = 9 \Rightarrow k = \pm 3$$

   The value of k is 3 since $\cos 45^o > 0$.

4. $2\cos^2 x - \cos x - 1 = 0 \Leftrightarrow 2\cos^2 x - 2\cos x + \cos x - 1 = 0$

   $\Leftrightarrow 2\cos x(\cos x - 1) + \cos x - 1 = 0$

   $\Leftrightarrow (\cos x - 1)(2\cos x + 1) = 0 \quad \cos x - 1$

   $\Rightarrow \cos x - 1 = 0$ or $2\cos x + 1$

$\Leftrightarrow \cos x = 1$ or $\cos x = -\dfrac{1}{2}$

$\Rightarrow x = 2k\pi$ or $x = \pm\dfrac{2\pi}{3} + 2k\pi, k \in \mathbb{Z}$

$\Rightarrow x = 2k\pi$ or $x = \dfrac{2\pi}{3} + 2k\pi$ or $x = -\dfrac{2\pi}{3} + 2k\pi \equiv \dfrac{4\pi}{3} + 2k\pi$

Hence, $S = \left\{2k\pi, \dfrac{2\pi}{3} + 2k\pi, \dfrac{4\pi}{3} + 2k\pi, k \in \mathbb{Z}\right\}$.

5. $\lim\limits_{x\to 0} \dfrac{1-\cos x}{\sin x} = \dfrac{1-\cos 0}{\sin 0} = \dfrac{1-1}{0} = \dfrac{0}{0}\ I.F.$

Remove this indeterminate form by Hospital's rule

$$\lim_{x\to 0} \frac{1-\cos x}{\sin x} = \lim_{x\to 0} \frac{(1-\cos x)'}{(\sin x)'} = \lim_{x\to 0} \frac{\sin x}{\cos x} = 0$$

Then, $\lim\limits_{x\to 0} \dfrac{1-\cos x}{\sin x} = 0$

6. From sequence $\{u_n\}$ where $u_{n+1} = 3(u_n + 2)$ and $u_0 = 0$, we list;

$u_o = 0$

$u_1 = 3(u_0 + 2) = 6$

$u_2 = 3(u_1 + 2) = 3(6+2) = 24$

$u_3 = 3(u_2 + 2) = 3(24+2) = 78$

$u_4 = 3(u_3 + 2) = 3(78+2) = 240$.

Therefore, the first five terms of the given sequence are $0, 6, 24, 78$ and 240.

The sequence $\{u_n\}$ is arithmetic if $u_{n+1} - u_n = d, d \in \mathbb{R}$ and is geometric if $\dfrac{u_{n+1}}{u_n} = r,\ r \in \mathbb{R}$

Since $u_{n+1} = 3(u_n + 2)$, thus $u_{n+1} - u_n = 3(u_n + 2) - u_n = 2u_n + 6$ and this is not a constant.

So, $\{u_n\}$ is not an arithmetic sequence.

$\dfrac{u_{n+1}}{u_n} = \dfrac{3(u_n + 2)}{u_n} = 3 + \dfrac{6}{u_n}$ and this is not a constant.

Thus, $\{u_n\}$ is not a geometric sequence.

Therefore, $\{u_n\}$ is neither arithmetic nor geometric sequence.

7. Let $f(x)=\sqrt{x^2+2\sqrt{x^2-1}}-\sqrt{x^2-2\sqrt{x^2-1}}$

   a) Existence condition: $x^2-1\ge 0$ and $x^2-2\sqrt{x^2-1}\ge 0$

   $x^2-1\ge 0 \Leftrightarrow x\in(-\infty,-1]\cup[1,+\infty)$

   $x^2-2\sqrt{x^2-1}\ge 0 \Leftrightarrow x^2\ge 2\sqrt{x^2-1}$

   $\Leftrightarrow x^4\ge 4x^2-4 \;\Leftrightarrow x^4-4x^2+4\ge 0 \;\Leftrightarrow (x^2-2)^2\ge 0$

   $\Leftrightarrow \forall x\in\mathbb{R},(x^2-2)^2\ge 0$

   Hence,

   $Domf=(-\infty,-1]\cup[1,+\infty)$

   b) $f(x)=\sqrt{x^2+2\sqrt{x^2-1}}-\sqrt{x^2-2\sqrt{x^2-1}}$

   $f^2(x)=\left(x^2+2\sqrt{x^2-1}\right)-2\sqrt{x^2+2\sqrt{x^2-1}}\sqrt{x^2-2\sqrt{x^2-1}}+\left(x^2-2\sqrt{x^2-1}\right)$

   $f^2(x)=2x^2-2\sqrt{x^4-4x^2+4}$

   $f^2(x)=2x^2-2\sqrt{(x^2-2)^2}$

   $f^2(x)=2x^2-2|x^2-2|$

   $$f^2(x)=\begin{cases}2x^2-2(x^2-2),\ x\in\,]-\infty,-\sqrt{2}]\cup[\sqrt{2},+\infty[\\ 2x^2+2(x^2-2),\ x\in\,]-\sqrt{2},-1]\cup[1,\sqrt{2}[\end{cases}$$

   $$f^2(x)=\begin{cases}4,\ x\in\,]-\infty,-\sqrt{2}]\cup[\sqrt{2},+\infty[\\ 4x^2-4,\ x\in\,]-\sqrt{2},-1]\cup[1,\sqrt{2}[\end{cases}$$

   $$\Rightarrow f(x)=\begin{cases}2,\ x\in\,]-\infty,-\sqrt{2}]\cup[\sqrt{2},+\infty[\\ 2\sqrt{x^2-1},\ x\in\,]-\sqrt{2},-1]\cup[1,\sqrt{2}[\end{cases}$$

   From $Domf=(-\infty,-1]\cup[1,+\infty)$, we get that

   $$\Rightarrow f(x)=\begin{cases}2,\ x\in\,]-\infty,-\sqrt{2}]\cup[\sqrt{2},+\infty[\\ 2\sqrt{x^2-1},\ x\in\,]-\sqrt{2},-1]\cup[1,\sqrt{2}[\end{cases}$$

8. Let $P(T)=p$, then $P(H)=3p$.

But $P(T)+P(H)=1$.

Therefore, $4p=1$ or $p=\frac{1}{4}$.

Thus, $P(T)=\frac{1}{4}$ and $P(H)=\frac{3}{4}$.

9. Tangent line:

$T \equiv y-y_o = y_0'(x-x_o)$

Here, $x_o=2$ and $y_o=4$;

$f'(x)=3x^2-4x$

$y_0' = f'(2)=3(4)-4(2)=12-8=4$

Then, $T \equiv y-4=4(x-2) \Leftrightarrow y=4x-8+4 \Leftrightarrow y=4x-4$

Normal line:

$N \equiv y-y_o = -\frac{1}{y_0'}(x-x_o)$;

Thus, $N \equiv y-4=-\frac{1}{4}(x-2) \Leftrightarrow y=4x+\frac{1}{2}+4$

$\Leftrightarrow y=4x+\frac{9}{4}$

10. a) Equation of sphere S whose centre (x_o, y_o, z_o) and radius r has equation $(x-x_o)^2+(y-y_o)^2+(z-z_o)^2=r^2$

For our case,

$S \equiv (x-6)^2+(y-5)^2+(z+2)^2=70$

Or $S \equiv x^2+y^2+z^2-12x-10y+4z=70-36-25-4$

Or $S \equiv x^2+y^2+z^2-12x-10y+4z=5$

b) $x^2+y^2+z^2+4x-8y+6z+7=0 \Leftrightarrow x^2+4x+y^2-8y+z^2+6z=-7$

$\Leftrightarrow (x+2)^2+(y-4)^2+(z+3)^2=-6+4+16+9$

$\Leftrightarrow (x+2)^2+(y-4)^2+(z+3)^2=22$

Centre is $C(-2,4,-3)$, radius $r=\sqrt{22}$.

c) Let us find $\overrightarrow{AB}=\begin{pmatrix}1\\-4\\5\end{pmatrix}$, the line $AB \equiv \frac{x-1}{1}=\frac{y-1}{-4}=\frac{z+1}{5}$

$$\Rightarrow AB \equiv \begin{cases} -4x+4 = y+1 \\ 5x-5 = z+1 \end{cases} \Leftrightarrow AB \equiv \begin{cases} y = -4x+3 \\ z = 5x-6 \end{cases}$$

Substituting y, z with their values in S gives

$$x^2 + (-4x+3)^2 + (5x-6)^2 + 4x - 8(-4x+3) + 6(5x-6) + 7 = 0$$

$$\Leftrightarrow x^2 + 16x^2 - 24x + 9 + 25x^2 - 60x + 36 + 4x + 32x - 24 + 30x - 36 + 7 = 0$$

$$\Leftrightarrow 42x^2 - 18x - 8 = 0 \Leftrightarrow 21x - 9x - 4 = 0$$

$$\Delta = 81 + 336 = 417$$

$$x_{1,2} = \frac{9 \pm \sqrt{417}}{42}$$

$x_1 = 5.7$ and $x_2 = -14.7$

For $x = 5.7$, we have $y = -19.8$ and $z = 22.5$.

Intersection point is then, $(5.7, -19.8, 22.5)$

For $x = -14.7$, we have $y = 61.8$ and $z = 79.5$

Intersection point is then, $(-14.7, 61.8, 79.5)$

11. Let $q(t) = q_o e^{-tk}$

Here $q_o = 50$ and $q(5) = 20$.

$$q(5) = 20 \Rightarrow 20 = 50e^{-5k}$$

$$\Leftrightarrow \frac{2}{5} = e^{-5k} \Leftrightarrow -5k = \ln\frac{2}{5} \Leftrightarrow k = -\frac{1}{5}\ln\frac{2}{5} \Leftrightarrow k = 0.18326.$$

90% of the sugar being dissolved, it means that 10% of the sugar left i.e. 5kg.

Thus,

$$q(t) = 5 \Rightarrow 50e^{-0.18326t} = 5 \quad \Leftrightarrow e^{-0.18326t} = \frac{1}{10}$$

$$\Leftrightarrow -0.18326t = \ln\frac{1}{10} \Leftrightarrow 0.18326t = -\ln 10 \Leftrightarrow t = 12.5647$$

12. In fact, $\sin y \cos(x-y) + \cos y \sin(x-y)$

$$= \sin y(\cos x \cos y + \sin x \sin y) + \cos y(\sin x \cos y - \sin y \cos x)$$

$$= \cancel{\sin y \cos x \cos y} + \sin x \sin^2 y + \sin x \cos^2 y - \cancel{\cos y \sin y \cos x}$$

$$= \sin x \sin^2 y + \sin x \cos^2 y = \sin x(\sin^2 y + \cos^2 y)$$

$= \sin x$ as required.

13. $\lim\limits_{x\to 0}\frac{\sqrt[5]{(1+x)^3-1}}{x}=\frac{0}{0}$, $I.F.$

Remove this $I.F.$ by Hospital's rule.

$$\lim_{x\to 0}\frac{\sqrt[5]{(1+x)^3-1}}{x}=\lim_{x\to 0}\frac{\left(\sqrt[5]{(1+x)^3-1}\right)'}{x'}=\lim_{x\to 0}\frac{1}{5}\left[(1+x)^3-1\right]^{-\frac{4}{5}}3(1+x)^2$$

$$=\frac{1}{5}\left[(1+0)^3-1\right]^{-\frac{4}{5}}3(1+0)^2=+\infty$$

14. If the mean is 50, thus, $\frac{56+37+54+52+x+48}{6}=50$

$$\Leftrightarrow 247+x=300 \Leftrightarrow x=300-247 \Rightarrow x=53$$

15. Intersection points for $y^2=2px$ and $x^2=2py$:

$y^2=2px \Rightarrow y=\sqrt{2px}$ and $x^2=2py \Rightarrow y=\frac{x^2}{2p}$

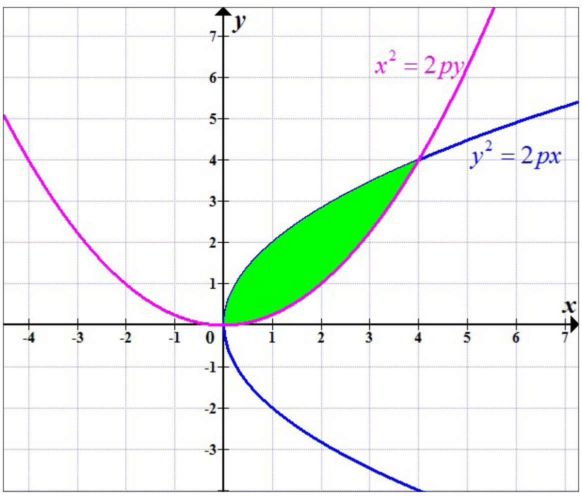
Then, $\sqrt{2px}=\frac{x^2}{2p} \Leftrightarrow 2px=\frac{x^4}{4p^2}$

$\Leftrightarrow 8p^3x=x^4 \Leftrightarrow 8p^3x-x^4=0 \Leftrightarrow x\left(8p^3-x^3\right)=0$

$\Rightarrow x=0$ or $8p^3-x^3=0$

$\Rightarrow x=0$ or $x=2p$

To be able to sketch the curve, let $p=2$. Then we have $y^2=4x$ and $x^2=4y$



$$A=\int_0^{2p}\left(\sqrt{2px}-\frac{x^2}{2p}\right)dx$$

$$A=\sqrt{2p}\int_0^{2p}\sqrt{x}dx-\frac{1}{2p}\int_0^{2p}x^2dx$$

$$A=\sqrt{2p}\left[\frac{2}{3}x^{\frac{2}{3}}\right]_0^{2p}-\frac{1}{2p}\left[\frac{x^3}{3}\right]_0^{2p}$$

$$A=\frac{2}{3}\sqrt{2p}\cdot 2p\cdot\sqrt{2p}-\frac{8p^3}{6p}$$

$$A=\frac{8p^2}{3}-\frac{4p^3}{3p}=\frac{4p^3}{3p}=\frac{4p^2}{3}$$

Therefore, the area enclosed by the curves

$y^2=2px$ and $x^2=2py$ is $\frac{4p^2}{3}$ $sq.\ unit$

16.a) $r=\frac{Cov(x,y)}{\sigma_x\sigma_y}$

$$Cov(x,y)=\frac{1}{n}\sum_{i=1}^{n}(x_i-\overline{x})^2(y_i-\overline{y})^2$$

$$\sigma_x=\sqrt{Var(x)}=\sqrt{\frac{1}{n}\sum_{i=1}^{n}(x_i-\overline{x})^2}$$

$$\sigma_y=\sqrt{Var(y)}=\sqrt{\frac{1}{n}\sum_{i=1}^{n}(y_i-\overline{y})^2}$$

Mean:

$$\overline{x}=\frac{7+8+9+11+15}{5}=10$$

$$\overline{y}=\frac{33+25+17+9+6}{5}=18$$

x_i	y_i	$x_i-\overline{x}$	$y_i-\overline{y}$	$(x_i-\overline{x})^2$	$(y_i-\overline{y})^2$	$(x_i-\overline{x})(y_i-\overline{y})$
7	33	–3	–12	9	225	–45
8	25	–2	–9	4	49	–14
9	17	–1	–1	1	1	1
11	9	1	7	1	81	–9
15	6	5	15	25	144	–60
SUM				40	500	–127

$$Cov(x,y)=\frac{1}{n}\sum_{i=1}^{n}(x_i-\overline{x})^2(y_i-\overline{y})^2$$

$$\sigma_x=\sqrt{\frac{1}{n}\sum_{i=1}^{n}(x_i-\overline{x})^2}=\sqrt{\frac{40}{5}}=\sqrt{8}=2\sqrt{2}$$

$$\sigma_y=\sqrt{\frac{1}{n}\sum_{i=1}^{n}(y_i-\overline{y})^2}=\sqrt{\frac{500}{5}}=\sqrt{100}=10$$

$$Cov(x,y)=\frac{1}{n}\sum_{i=1}^{n}(x_i-\overline{x})^2(y_i-\overline{y})^2=\frac{-127}{5}$$

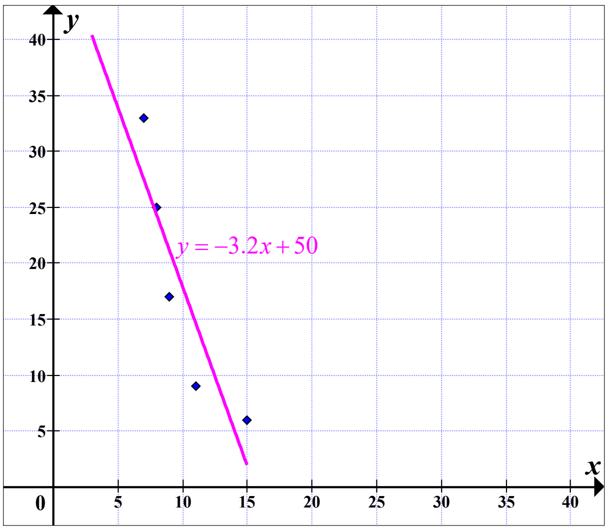
$$r=\frac{Cov(x,y)}{\sigma_x\sigma_y}=\frac{-127}{5\left(2\sqrt{2}\right)10}=-0.89$$

b) $L_{y/x}\equiv y-\overline{y}=\dfrac{Cov(x,y)}{\sigma_x^2}(x-\overline{x})$

$$L_{y/x}\equiv y-18=\frac{-127}{40}(x-10)$$

$$\Leftrightarrow y-18=-3.2(x-10)\ \Leftrightarrow y=-3.2x+50$$

c) Scatter diagram



17.a) $\begin{cases}3x+2y-5z=2\\ x+2y\qquad\ =3\\ 2x-\ y+\ z=-3\end{cases}\Leftrightarrow\begin{pmatrix}3&2&-5\\1&2&0\\2&-1&1\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}2\\3\\-3\end{pmatrix}$

If $\Delta\neq 0$, then $x=\dfrac{\Delta_y}{\Delta},y=\dfrac{\Delta_y}{\Delta},z=\dfrac{\Delta_z}{\Delta}$

$$\Delta = \begin{vmatrix} 3 & 2 & -5 \\ 1 & 2 & 0 \\ 2 & -1 & 1 \end{vmatrix} = -\begin{vmatrix} 2 & -5 \\ -1 & 1 \end{vmatrix} + 2\begin{vmatrix} 3 & -5 \\ 2 & 1 \end{vmatrix} = -2+5+6+20 = 29$$

$$\Delta_x = \begin{vmatrix} 2 & 2 & -5 \\ 3 & 2 & 0 \\ -3 & -1 & 1 \end{vmatrix} = -3\begin{vmatrix} 2 & -5 \\ -1 & 1 \end{vmatrix} + 2\begin{vmatrix} 2 & -5 \\ -3 & 1 \end{vmatrix} = -6+15+4-30 = -17$$

$$\Rightarrow x = -\frac{17}{29}$$

$$\Delta_y = \begin{vmatrix} 3 & 2 & -5 \\ 1 & 3 & 0 \\ 2 & -3 & 1 \end{vmatrix} = -\begin{vmatrix} 2 & -5 \\ -3 & 1 \end{vmatrix} + 3\begin{vmatrix} 3 & -5 \\ 2 & 1 \end{vmatrix} = -2+15+9+30 = 52$$

$$\Rightarrow y = \frac{52}{29}$$

$$\Delta_z = \begin{vmatrix} 3 & 2 & 2 \\ 1 & 2 & 3 \\ 2 & -1 & -3 \end{vmatrix} = -\begin{vmatrix} 2 & 2 \\ -1 & -3 \end{vmatrix} + 2\begin{vmatrix} 3 & 2 \\ 2 & -3 \end{vmatrix} - 3\begin{vmatrix} 3 & 2 \\ 2 & -1 \end{vmatrix} \quad 6 \quad 2 \quad 18 \quad 8 \quad 9 \quad 12 \qquad 1$$

$$= 6-2-18-8+9+12 = -1 \;\Rightarrow z = -\frac{1}{29}$$

$$S = \left\{ \left(-\frac{17}{29}, \frac{52}{29}, -\frac{1}{29} \right) \right\}$$

b) The area of a parallelogram whose adjacent sides are $\vec{a} = 6\vec{i} + 3\vec{j} - 2\vec{k}$ and $\vec{b} = 3\vec{i} - 2\vec{j} + 6\vec{k}$ is given by

$$A = \left\| \vec{a} \times \vec{b} \right\|$$

Or $\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{j} \\ 6 & 3 & -2 \\ 3 & -2 & 6 \end{vmatrix} = \vec{i}\begin{vmatrix} 3 & -2 \\ -2 & 6 \end{vmatrix} - \vec{j}\begin{vmatrix} 6 & -2 \\ 3 & 6 \end{vmatrix} + \vec{k}\begin{vmatrix} 6 & 3 \\ 3 & -2 \end{vmatrix}$

$$= \vec{i}(18-4) - \vec{j}(36+6) + \vec{k}(-12-9) = 14\vec{i} - 42\vec{j} - 21\vec{k}$$

$$A = \sqrt{(14)^2 + (-42)^2 + (-21)^2} = \sqrt{196+1768+441}$$

$$= \sqrt{2401} = 49$$

Therefore, the area of the given parallelogram is $49\,sq.unit$

18. We are given $F(x)=\begin{cases} cx(6-x)^2\,;\ 0\le x\le 6 \\ 0, \qquad\qquad elsewhere \end{cases}$

a) Since x is a random variable, thus, $\int\limits_{All} f(x)\,dx=1$

$$\Rightarrow 1=\int_0^6 cx(6-x)^2\,dx \iff 1=\int_0^6 cx\left(36-12x+x^2\right)dx$$

$$\iff 1=c\left[18x^2-4x^3+\frac{x^4}{4}\right]_0^6 \iff 1=c\left[18(36)-4(216)+324\right]$$

$$\iff 108c=1$$

$$\Rightarrow c=\frac{1}{108}$$

b) i) The mean is $E(x)=\int\limits_{All} xf(x)\,dx$

$$\Rightarrow E(x)=\frac{1}{108}\int_0^6 x^2(6-x)^2\,dx$$

$$\iff E(x)=\frac{1}{108}\int_0^6\left(36x^2-12x^3+x^4\right)dx$$

$$\iff E(x)=\frac{1}{108}\left[12x^3-3x^4+\frac{x^5}{5}\right]_0^6$$

$$\iff E(x)=\frac{1}{108}(2592-3888+1555.2)=2.4$$

$$\iff E(x)=\frac{1}{108}(259.2)=2.4$$

$$E(x)=2.4$$

ii) The variance $Var(x)=E\left(x^2\right)-\left[E(x)\right]^2$

or $E\left(x^2\right)=\int\limits_{All} x^2 f(x)\,dx \Rightarrow E\left(x^2\right)=\frac{1}{108}\int_0^6 x^3(6-x)^2\,dx$

$$\iff E\left(x^2\right)=\frac{1}{108}\int_0^6\left(36x^3-12x^4+x^5\right)dx$$

$$\iff E\left(x^2\right)=\frac{1}{108}\left[9x^4-12\frac{x^5}{5}+\frac{x^6}{6}\right]_0^6$$

$$\Leftrightarrow E\left(x^2\right)=\frac{1}{108}\left[9(6)^4-12\frac{(6)^5}{5}+\frac{(6)^6}{6}\right]_0^6=7.2$$

Then, $Var(x)=7.2-(2.4)^2=1.44$

iii) Standard deviation of x is $\sigma=\sqrt{Var(x)}=\sqrt{1.44}=1.2$

19. a) From $I_n=\int_0^{\frac{\pi}{2}}e^{-nx}\sin xdx$, let $u=\sin x$ and $dv=e^{-nx}dx$.

Hence, $du=-\sin xdx$ and $v=-\frac{1}{n}e^{-nx}$.

Therefore, $I_n=\left[-\frac{1}{n}e^{-nx}\sin x\right]_0^{\frac{\pi}{2}}+\frac{1}{n}\int_0^{\frac{\pi}{2}}e^{-nx}\cos xdx$

$\Leftrightarrow I_n=-\frac{1}{n}e^{-\frac{n\pi}{2}}+\frac{1}{n}J_n$

$\Leftrightarrow nI_n-J_n=-e^{-\frac{n\pi}{2}}$ (1)

From $J_n=\int_0^{\frac{\pi}{2}}e^{-nx}\cos xdx$, let $t=\cos x$ and $dz=e^{-nx}dx$.

Then, $dt=\cos xdx$ and $z=-\frac{1}{n}e^{-nx}$.

Therefore, $J_n=\left[-\frac{1}{n}e^{-nx}\cos x\right]_0^{\frac{\pi}{2}}-\frac{1}{n}\int_0^{\frac{\pi}{2}}e^{-nx}\sin xdx$

$\Leftrightarrow J_n=\frac{1}{n}-\frac{1}{n}I_n$

$\Leftrightarrow nJ_n+I_n=1$ (2)

Equation (1) and (2) give the simultaneous equations

$$\begin{cases}nI_n-J_n=-e^{-\frac{n\pi}{2}}\\ nJ_n+I_n=1\end{cases}\qquad(3)$$

And (3) indicates two relations between I_n and J_n.

b) Multiply first equation of (3) by n to eliminate J_n

$$\begin{cases}n^2I_n-nJ_n=-ne^{-\frac{n\pi}{2}}\\ nJ_n+I_n=1\end{cases}\Rightarrow n^2I_n+I_n=1-ne^{-\frac{n\pi}{2}}$$

Which gives $I_n = \dfrac{1 - ne^{-\frac{n\pi}{2}}}{n^2 + 1}$ (4)

From (1), $J_n = nI_n + e^{-\frac{n\pi}{2}}$

Then $J_n = \dfrac{n - n^2 e^{-\frac{n\pi}{2}} + n^2 e^{-\frac{n\pi}{2}} + e^{-\frac{n\pi}{2}}}{n^2 + 1}$

Or $J_n = \dfrac{n + e^{-\frac{n\pi}{2}}}{n^2 + 1}$ (5)

20. To solve $y'' - y' - 2y = 6x$ with $y(0) = y'(0) = 1$

Homogeneous equation :

$y'' - y' - 2y = 0$

Characteristic equation

$\lambda^2 - \lambda - 2 = 0 \Leftrightarrow (\lambda + 1)(\lambda - 2) = 0$

$\Leftrightarrow \lambda = -1$ or $\lambda = 2$.

General solution for solution for homogeneous equation is

$y^* = c_1 e^{-x} + c_2 e^{2x}$.

The complementary (particular) solution is given by

$y = Ax + B$.

Or $y' = A$ et $y'' = 0$.

The equation $y'' - y' - 2y = 6x$ becomes

$-A - 2Ax - 2B = 6x$

Identifying the coefficients, we get

$$\begin{cases} -A - 2B = 0 \\ -2A = 6 \end{cases} \Leftrightarrow \begin{cases} B = -\dfrac{A}{2} \\ A = -3 \end{cases} \Leftrightarrow \begin{cases} B = \dfrac{3}{2} \\ A = -3 \end{cases}$$

Thus, complementary solution is $y = -3x + \dfrac{3}{2}$

The general solution of the given equation is

$y = c_1 e^{-x} + c_2 e^{2x} - 3x + \dfrac{3}{2}$

From the initial conditions $y(0) = y'(0) = 1$, we get the values of c_1 and c_2 as follows:

$y' = -c_1 e^{-x} + 2c_2 e^{2x} - 3$

$$y(0)=y'(0)=1\Rightarrow\begin{cases}c_1+c_2+\frac{3}{2}=1\\-c_1+2c_2-3=1\end{cases}\Rightarrow\begin{cases}c_1=-\frac{5}{3}\\c_2=\frac{7}{6}\end{cases}$$

Therefore, the required solution is $y=-\frac{5}{3}e^{-x}+\frac{7}{6}e^{2x}-3x+\frac{3}{2}$

Answers for Summative Evaluation Two

1. $\begin{cases}x^2+y^2=\frac{37}{4}\\xy=\frac{3}{2}\end{cases}$

From 2nd equation, we get $x=\frac{3}{2y}$; putting this equality in 1st equation, we get

$$\left(\frac{3}{2y}\right)^2+y^2=\frac{37}{4}\Leftrightarrow\frac{9}{4y^2}+y^2=\frac{37}{4}$$

$$\Leftrightarrow 9+4y^4=37y^2\Rightarrow 4y^4-37y^2+9=0$$

$$\Delta=(-37)^2-16(9)=1369-144=1225$$

$$y^2=\frac{37+35}{8}=9 \text{ or } y^2=\frac{37-35}{8}=\frac{1}{4}$$

Solving for y, we get: $y_1=-3, y_2=3, y_3=-\frac{1}{2}, y_4=\frac{1}{2}$

Substituting y with its values in $x=\frac{3}{2y}$, we get:

$$x_1=-\frac{1}{2}, x_2=\frac{1}{2}, x_3=-3, x_4=3$$

And then, the solution set is

$$S=\left\{\left(-3,-\frac{1}{2}\right),\left(3,\frac{1}{2}\right),\left(-\frac{1}{2},-3\right),\left(\frac{1}{2},3\right)\right\}$$

2. $\ln\left(\frac{e^{\ln x}}{e^3}\right)+\ln\left(\frac{e}{x}\right)=\ln\left(\frac{e^{\ln x}}{e^3}\times\frac{e}{x}\right)$

$$=\ln\frac{e^{\ln x}}{xe^2}=\ln x\ln e-\ln xe^2$$

$$=\ln x-\left(\ln x+\ln e^2\right)=\ln x-\ln x+2\ln e=2$$

3. $\arctan x + \arctan\sqrt{3} = \dfrac{\pi}{4}$

$$\Rightarrow \arctan x + \frac{\pi}{3} = \frac{\pi}{4} \Rightarrow \arctan x = -\frac{\pi}{12} \Rightarrow x = \tan\left(-\frac{\pi}{12}\right) \Rightarrow x = -\tan\frac{\pi}{12}$$

4. $f(x) = \dfrac{\ln\left(1+x^2\right)}{e^{x^2}} \Rightarrow f'(x) = \dfrac{\left[\ln\left(1+x^2\right)\right]' \times e^{x^2} - \left(e^{x^2}\right)' \times \ln\left(1+x^2\right)}{e^{x^2}}$

$$\Rightarrow f'(x) = \frac{\dfrac{2x}{1+x^2} \times e^{x^2} - 2xe^{x^2}\ln\left(1+x^2\right)}{\left[e^{x^2}\right]^2} = \frac{2x\left[1-\left(1+x^2\right)\ln\left(1+x^2\right)\right]}{\left(1+x^2\right)e^{x^2}}$$

5. $\lim\limits_{x\to 1}\dfrac{\sqrt{x}-1}{\sqrt[3]{x}-1} = \dfrac{0}{0}\ (I.C)$

$$\lim_{x\to 1}\frac{\sqrt{x}-1}{\sqrt[3]{x}-1} = \lim_{x\to 1}\frac{\left(\sqrt{x}-1\right)\left(\sqrt{x}+1\right)\left(\sqrt[3]{x^2}+\sqrt[3]{x}+1\right)}{\left(\sqrt[3]{x}-1\right)\left(\sqrt{x}+1\right)\left(\sqrt[3]{x^2}+\sqrt[3]{x}+1\right)}$$

$$= \lim_{x\to 1}\frac{(x-1)\left(\sqrt[3]{x^2}+\sqrt[3]{x}+1\right)}{(x-1)\left(\sqrt{x}+1\right)} = \lim_{x\to 1}\frac{\sqrt[3]{x^2}+\sqrt[3]{x}+1}{\sqrt{x}+1} = \frac{3}{2}$$

6. $iz - 2 = 4i - z \rightarrow (1)$

Let $z = a + bi \rightarrow (2)$

Using (2) in (1) we get:

$i(a+bi) - 2 = 4i - (a+bi) \Rightarrow ai + bi^2 - 2 = 4i - a - bi$

$$\Rightarrow (-b-2) + ai = -a + (4-b)i \Rightarrow \begin{cases} -b-2 = -a \\ a = 4-b \end{cases} \Rightarrow \begin{cases} a-b = 2 \\ a+b = 4 \end{cases} \Rightarrow \begin{cases} a = 3 \\ b = 1 \end{cases}$$

$z = 3 + 1i$

7. $\dfrac{\sin 2x + \sin 2x}{1+\cos x + \cos 2x} = \dfrac{\sin x + 2\sin x\cos x}{1+\cos x + \cos^2 x - \sin^2 x}$

$$= \frac{\sin x + 2\sin x\cos x}{\cos x + 2\cos^2 x} = \frac{\sin x(1+2\cos x)}{\cos x(1+2\cos x)} = \frac{\sin x}{\cos x} = \tan x$$

8. $\lim\limits_{n\to+\infty}\dfrac{1+2+3+4+\ldots+n}{n^2} = \lim\limits_{n\to+\infty}\dfrac{n(n+1)}{2n^2} = \lim\limits_{n\to+\infty}\dfrac{n+1}{2n} = \dfrac{1}{2}$

9. $y = \ln(4x-11),\ x_0 = 3$

$$T \equiv y - y_0 = y'_0\left(x - x_0\right)$$

where: $\begin{cases} y_0 = y(x_0) = y(3) \\ y'_0 = y'(x_0) = y'(3) \end{cases}$

or

$y = \ln(4x-11) \Rightarrow y' = \dfrac{4}{4x-11}$

$y(3) = 0, y'(3) = 4$

Then, $T \equiv y = 4(x-3) \Rightarrow T \equiv y = 4x-12$

10. $f(x) = \ln \dfrac{x+1}{x-1}$, f is defined if: $\dfrac{x+1}{x-1} > 0$

x	$-\infty$		-1			1		$+\infty$
$x+1$		-	0	+		+	+	
$x-1$		-			-	0	+	
$\dfrac{x+1}{x-1}$		+	0		-	||	+	

$Domf =]-\infty, -1[\cup]1, +\infty[$

11. $F(x) = \dfrac{x^2}{2} + x - x\ln x \Rightarrow F'(x) = x + 1 - \left(\ln x + x \cdot \dfrac{1}{x}\right) = x - \ln x$

12. a) $e^x e^{x-1} = e \Rightarrow e^{2x-1} = e^1 \Rightarrow 2x-1 = 1 \Rightarrow 2x = 2 \Rightarrow x = 1$

$S = \{1\}$

b) $e^{2x-2} + e^{x-2} = 6e^{-2}$

$\Rightarrow e^{2x}e^{-2} + e^x e^{-2} = 6e^{-2} \Rightarrow e^{-2}(e^{2x} + e^x) = 6e^{-2} \Rightarrow e^{2x} + e^x - 6 = 0$

Let $t = e^x, (t > 0)$

$\Rightarrow t^2 + t - 6 = 0 = (t-2)(t+3) \Rightarrow t = 2 \quad or \quad t = -3$

$t = -3$ is to be rejected since $t > 0$

For $t = 2 \Rightarrow e^x = 2 \Rightarrow \ln e^x = \ln 2 \Rightarrow x = \ln 2$

$S = \{\ln 2\}$

13. $f(x) = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$

Let: $f^{-1}(x) = u(x)$ be the inverse of $f(x)$

$f(u) = x$

$\Rightarrow \dfrac{e^u - e^{-u}}{e^u + e^{-u}} = x$

$\Rightarrow e^u - e^{-u} = x\left(e^u + e^{-u}\right)$

$\Rightarrow e^u - \dfrac{1}{e^u} = xe^u + \dfrac{x}{e^u}$

$\Rightarrow \dfrac{e^{2u} - 1}{e^u} = \dfrac{xe^{2u} + x}{e^u}$

$\Rightarrow e^{2u} - 1 = xe^{2u} + x$

$\Rightarrow e^{2u}(1 - x) = x + 1$

$$\Rightarrow e^{2u} = \frac{x+1}{1-x} \Rightarrow 2u = \ln\frac{x+1}{1-x} \Rightarrow u = \frac{1}{2}\left(\ln\frac{x+1}{1-x}\right)$$

The inverse function of $f(x) = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$ is

$f^{-1}(x) = \dfrac{1}{2}\ln\dfrac{x+1}{1-x}$

14. $5\log_2 y - 3\log_2(x+4) = 2\log_2 y + 3\log_2 x$

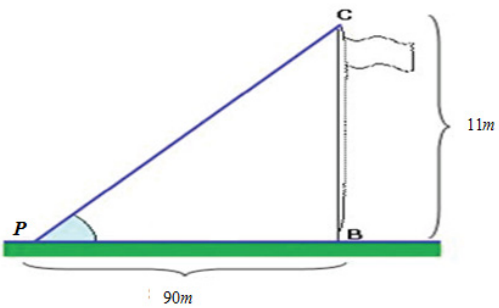
$\Rightarrow 5\log_2 y - 2\log_2 y = 3\log_2(x+4) + 3\log_2 x$

$\Rightarrow \log_2 y^5 - \log_2 y^2 = \log_2(x+4)^3 + \log_2 x^3$

$\Rightarrow \log_2 \dfrac{y^5}{y^2} = \log_2 x^3(x+4)^3$

$\Rightarrow y^3 = \left[x(x+4)\right]^3 \Rightarrow y = x(x+4)$

15. Point P is 90 m away from a vertical flagpole, which is 11 m high



From the above figure, $\tan\hat{P} = \dfrac{11}{90}$

$\hat{P} = \tan^{-1}\left(\dfrac{11}{90}\right) \approx 6.9^0$

Thus, the angle of elevation is about 6.9^0

16. Solving equations

a) $z^4-(8i-1)z^2-8i=0$ (1)

Let $z^2=y$, equation (1) can be written as

$y^2-(8i-1)y-8i=0$

$\Delta=\left[-(8i-1)\right]^2-4(-8i)$

$=-63+16i$

Finding square roots of Δ

Let $(a+bi)^2=-63+16i$

$$\Leftrightarrow\begin{cases}a^2-b^2=-63\\2ab=16\\a^2+b^2=\sqrt{(-63)^2+(16)^2}\end{cases}\quad\Leftrightarrow\begin{cases}a^2-b^2=-63\\2ab=16\\a^2+b^2=65\end{cases}$$

$$\Leftrightarrow\begin{cases}a^2-b^2=-63\\a^2+b^2=65\end{cases}$$

$2a^2=2\Rightarrow a^2=1\Rightarrow a=\pm1$

$$\Leftrightarrow\begin{cases}-a^2+b^2=63\\a^2+b^2=65\end{cases}$$

$2b^2=128\Rightarrow b^2=64\Rightarrow b=\pm8$

Square roots of Δ are $\pm(1+8i)$

$$y_1=\frac{8i-1+1+8i}{2}=8i\Rightarrow z^2=8i\Rightarrow z=\pm(2+2i)$$

$$y_2=\frac{8i-1-1-8i}{2}=-1\Rightarrow z^2=-1\Rightarrow z=\pm i$$

b) $$\begin{cases}1+\log_2(-x+2y)=\log_2\{2x-3y\}\\3^{5x+y}=\dfrac{81}{3^{-x-7y}}\end{cases}$$

$$\Rightarrow\begin{cases}\log_2 2+\log_2(-x+2y)=\log_2(2x-3y)\\3^{5x+y}=3^4\times3^{x+7y}\end{cases}$$

$$\Rightarrow\begin{cases}\log_2 2(-x+2y)=\log_2(2x-3y)\\3^{5x+y}=3^{4+x+7y}\end{cases}$$

$$\Rightarrow \begin{cases} 2(-x+2y) = 2x-3y \\ 5x+y = 4+x+7y \end{cases} \Rightarrow \begin{cases} -4x+7y=0 \\ 4x-6y=4 \end{cases}$$

$$\begin{cases} y=4 \\ x=7 \end{cases}$$

$$S=\{(7,4)\}$$

17. Given $f(x)=\dfrac{x^2-1}{x^2-4}$

a) Domain of definition

$$Domf=\{x\in\mathbb{R}: x^2-4\neq 0\}=\mathbb{R}\setminus\{-2,2\}$$

b) Limits at boundaries

$$\lim_{x\to\pm\infty}\frac{x^2-1}{x^2-4}=\lim_{x\to\pm\infty}\frac{x^2\left(1-\frac{1}{x^2}\right)}{x^2\left(1-\frac{4}{x^2}\right)}=\lim_{x\to\pm\infty}\frac{1-\frac{1}{x^2}}{1-\frac{4}{x^2}}=1$$

x	$-\infty$		-2		-1		1		2		$+\infty$
x^2-1			+		0	-	0	+			
x^2-4		+	0		-				0	+	
$\frac{x^2-1}{x^2-4}$		+	$\Vert$	-	0	+	0	-	$\Vert$	+	

$$\lim_{x\to-2^-}\frac{x^2-1}{x^2-4}=+\infty, \qquad \lim_{x\to-2^+}\frac{x^2-1}{x^2-4}=-\infty$$

$$\lim_{x\to2^-}\frac{x^2-1}{x^2-4}=-\infty, \qquad \lim_{x\to2^+}\frac{x^2-1}{x^2-4}=+\infty$$

c) Asymptotes

Vertical asymptotes: $x=-2$ and $x=2$

Horizontal asymptote: $y=1$

d) Variation table

$$f'(x)=\frac{2x(x^2-4)-2x(x^2-1)}{(x^2-4)^2}$$

$$=\frac{2x^3-8x-2x^3+2x}{(x^2-4)^2}=\frac{-6x}{(x^2-4)^2}$$

$$f'(x)=0\Rightarrow x=0$$

Variation table

x	$-\infty$		-2		0		2		$+\infty$
$f'(x)$		$+$	||	$+$	0	$-$	||	$+$	
$f(x)$	1	$\nearrow$	$+\infty$ || $-\infty$	$\nearrow$	$\frac{1}{4}$	$\searrow$	$-\infty$ || $+\infty$	$\searrow$	1

e) x intercepts: $f(x)=0 \Rightarrow x^2-1=0 \Rightarrow x=\pm 1$

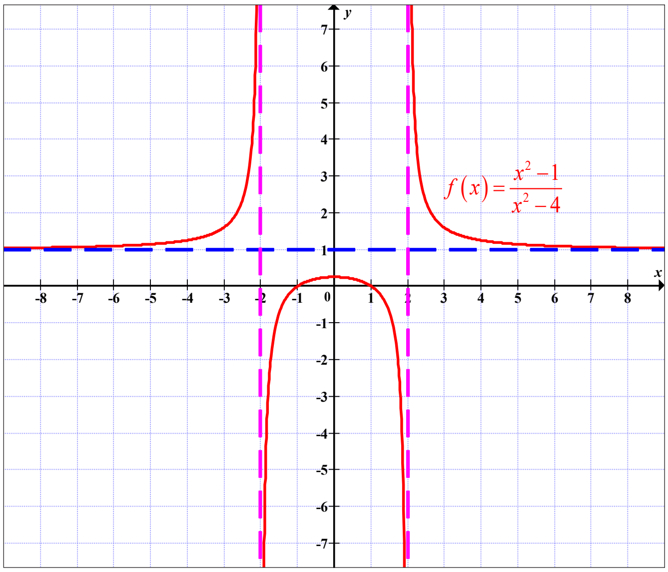
y intercept: $f(0)=\frac{1}{4}$

f) Curve

Additional points

x	-5	-4.2	-3.6	-3.4	-3.2	-3	-2.8	-2.6	-2.4	-2.2	-1.8	-1.6	-1.4	-1.2	-1	-0.8	0.6
y	1.1	1.2	1.3	1.4	1.5	1.6	1.8	2.1	2.7	4.6	-2.9	-1.1	-0.5	-0.2	0.0	0.1	0.2

x	0.8	1	1.2	1.4	1.6	1.8	2.2	2.4	2.6	2.8	3	3.2	3.4	3.6	4.2	5	
y	0.1	0.0	-0.2	-0.5	-1.1	-2.9	4.6	2.7	2.1	1.8	1.6	1.5	1.4	1.3	1.2	1.1	



18. a) 4men and 5 women

(i) Number of possible committee with no restrictions is ${}^9C_3=84$

(ii) Number of possible committee with 1 man and 2 women is ${}^4C_1 \times {}^5C_2 = 40$

(iii) Number of possible committee with 2 men and 1 woman if a certain man must be in the committee is ${}^{3}C_{1}\times{}^{5}C_{1}=15$

b) (i) There are ${}^{9}C_{3}=84$ of selecting 3 books from 9 books. There are ${}^{8}C_{2}\times{}^{1}C_{1}=28$ of selecting 1 dictionary and other 2 books.

So, required probability is $\frac{28}{84}=\frac{1}{3}$.

(iv) There are ${}^{5}C_{2}\times{}^{3}C_{1}=30$ of selecting 2 novels and 1 poem book.

So, required probability is $\frac{30}{84}=\frac{5}{14}$.

19. a) Given $f(x)=\dfrac{x^2+x+2}{x+1}$

$$ax+b+\frac{c}{x+1}=\frac{x^2+x+2}{x+1}$$

$$\Leftrightarrow \frac{ax(x+1)+b(x+1)+c}{x+1}=\frac{x^2+x+2}{x+1}$$

$$\Leftrightarrow \frac{ax^2+ax+bx+b+c}{x+1}=\frac{x^2+x+2}{x+1}$$

$$\Leftrightarrow \frac{ax^2+(a+b)x+b+c}{x+1}=\frac{x^2+x+2}{x+1}$$

$$\Rightarrow ax^2+(a+b)x+b+c=x^2+x+2$$

$$\Rightarrow \begin{cases} a=1 \\ a+b=1 \\ b+c=2 \end{cases} \Rightarrow \begin{cases} a=1 \\ b=0 \\ c=2 \end{cases}$$

Then,

$$\int f(x)\,dx=\int\left(x+\frac{2}{x+1}\right)dx=\int x\,dx+2\int\frac{1}{x+1}\,dx=\frac{x^2}{2}+2\ln|x+1|+c$$

b) $\dfrac{2y}{x}\dfrac{dy}{dx}=\dfrac{y^2}{x^2}-1 \Rightarrow \dfrac{dy}{dx}=\left(\dfrac{y^2}{x^2}-1\right)\dfrac{x}{2y} \Rightarrow \dfrac{dy}{dx}=\dfrac{y}{2x}-\dfrac{x}{2y}$

$$f(x,y)=\frac{y}{2x}-\frac{x}{2y}$$

$$f(tx,ty)=\frac{ty}{2tx}-\frac{tx}{2ty} \Rightarrow f(tx,ty)=\frac{t^0 y}{2x}-\frac{t^0 x}{2y}$$

$$\Rightarrow f(tx,ty)=t^0\left(\frac{y}{2x}-\frac{x}{2y}\right) \Rightarrow f(tx,ty)=t^0 f(x,y)$$

Then, $f(x,y)$ is homogeneous function of degree 0

To solve the given equation, put $\frac{y}{x}=z \Rightarrow y=zx$

$$\frac{dy}{dx}=z+x\frac{dz}{dx}$$

Then,

$$z+x\frac{dz}{dx}=\frac{zx}{2x}-\frac{x}{2zx} \Leftrightarrow z+x\frac{dz}{dx}=\frac{z^2x-x}{2xz}$$

$$\Leftrightarrow z+x\frac{dz}{dx}=\frac{z^2-1}{2z} \Leftrightarrow x\frac{dz}{dx}=\frac{z^2-1}{2z}-z$$

$$\Leftrightarrow x\frac{dz}{dx}=\frac{z^2-1-2z^2}{2z} \Leftrightarrow x\frac{dz}{dx}=\frac{-z^2-1}{2z}$$

$$\Leftrightarrow \frac{2z}{-z^2-1}dz=\frac{1}{x}dx \Leftrightarrow -\int\frac{2z}{z^2+1}dz=\int\frac{1}{x}dx$$

$$\Leftrightarrow -\ln\left|z^2+1\right|=\ln|x|+\ln k \Leftrightarrow \ln\left|z^2+1\right|=-\ln|kx|$$

$$\Leftrightarrow \ln\left|z^2+1\right|=\ln\left|(kx)^{-1}\right| \Leftrightarrow \left(\frac{y}{x}\right)^2+1=k^{-1}x^{-1}, \text{ since } z=\frac{y}{x}$$

$$\Leftrightarrow \frac{y^2}{x^2}+1=cx^{-1}, \quad c=k^{-1} \Leftrightarrow \frac{y^2+x^2}{x^2}=cx^{-1}$$

$$\Rightarrow y^2+x^2=cx$$

20. a) Let X represent the random variable "the number of calls between 09:00 hrs and 10:00 hrs on weekday". Then $X \sim Po(X)$ and

$$P(X=x)=e^{-\lambda}\frac{\lambda^x}{x!},\ x=0,1,2,3,\dots.$$

The probability that the office receives 6 calls between 09:00 hrs and 10:00 hrs on this Wednesday is

$$P(X=6)=e^{-5}\frac{5^6}{6!}=0.146$$

b) The average number of calls between 09:15 hrs and 09:30 hrs on weekday is 1.25. Let Y represent the random variable "the number of calls in the given 15 minutes"

Then, the probability that the office will receive exactly 3 calls between 09:15 hrs and 09:30 hrs is

$$P(Y=3)=e^{-1.25}\frac{(1.25)^3}{3!}=0.0933$$

c) The required probability is

$$^5C_2(0.09326)^2(0.90674)^3=0.0648$$

Answers for Summative Evaluation Three

1. $3-5x-x^2\geq 0$

$$\Delta=(-5)^2-4(-1)(3)=37$$

$$x_1=\frac{-5+\sqrt{37}}{2},\ x_2=\frac{-5-\sqrt{37}}{2}$$

Sign table

x	$-\infty$	$\frac{-5-\sqrt{37}}{2}$		$\frac{-5+\sqrt{37}}{2}$		$+\infty$
$3-5x-x^2$	$-$	0	$+$	0	$-$	

$$S=\left[\frac{-5-\sqrt{37}}{2},\frac{-5+\sqrt{37}}{2}\right]$$

2. Equation of a circle passing through the points $(0,1),(4,3)$ and $(1,-1)$

General equation of a circle is $x^2+y^2+ax+by+c=0$

Using the tree points, we have

$$\begin{cases}1+b+c=0\\16+9+4a+3b+c=0\\1+1+a-b+c=0\end{cases}\Leftrightarrow\begin{cases}b+c=-1\Rightarrow b=-1-c\\4a+3b+c=-25\\a-b+c=-2\end{cases}$$

$$\Leftrightarrow\begin{cases}b=-1-c\\4a+3(-1-c)+c=-25\\a-(-1-c)+c=-2\end{cases}\Leftrightarrow\begin{cases}b=-1-c\\4a-3-3c+c=-25\\a+1+c+c=-2\end{cases}$$

$$\Leftrightarrow\begin{cases}4a-2c=-22\\\underline{a+2c=-3}\end{cases}$$

$$5a=-25\Rightarrow a=-5$$

$a+2c=-3$

$\Rightarrow -5+2c=-3$

$\Rightarrow 2c=2\Rightarrow c=1$

$b=-1-c=-1-1=-2$

Then, the equation is $x^2+y^2-5x-2y+1=0$

3. $\dfrac{x^2-x+1}{x-1}=k\Leftrightarrow x^2-x+1=kx-k$

$\Leftrightarrow x^2+(-1-k)x+1+k=0$

This equation has repeated roots if the discriminant is zero; $\Delta=0$

$\Delta=(-1-k)^2-4(1+k)=1+2k+k^2-4-4k=k^2-2k-3$

$k^2-2k-3=0$

$\Delta=(-2)^2-4(-3)=4+12=16$

$k_1=\dfrac{2+4}{2}=3$ or $k_1=\dfrac{2-4}{2}=-1$

Thus, the given equation has repeated roots if $k\in\{-1,3\}$

4. Consider the following augmented matrix

$$\begin{pmatrix}1&1&-1&:&-1\\3&-2&1&:&0\\2&3&-3&:&-3\end{pmatrix}\quad\begin{matrix}r_2=r_2-3r_1\\r_3=r_3-2r_1\end{matrix}$$

$$\begin{pmatrix}1&1&-1&:&-1\\0&-5&4&:&3\\0&1&-1&:&-1\end{pmatrix}\quad r_3=5r_3+r_2$$

$$\begin{pmatrix} 1 & 1 & -1 & : & -1 \\ 0 & -5 & 4 & : & 3 \\ 0 & 0 & -1 & : & -2 \end{pmatrix}$$

The simplified system is

$$\begin{cases} x+y-z=-1 \\ -5y+4z=3 \\ \qquad -z=-2 \Rightarrow z=2 \end{cases}$$

$-5y+4z=3 \Rightarrow -5y+8=3 \Rightarrow y=1$

$x+y-z=-1 \Rightarrow x+1-2=-1 \Rightarrow x=0$

Hence, $S=\{(0,1,2)\}$

5. $\lim\limits_{x\to-\infty} \frac{\sqrt{x^2+2}}{3x-6} = \frac{\infty}{\infty}\ I.C$

$$\lim_{x\to-\infty} \frac{\sqrt{x^2+2}}{3x-6} = \lim_{x\to-\infty} \frac{-x\sqrt{1+\frac{2}{x^2}}}{x\left(3-\frac{6}{x}\right)} = \lim_{x\to-\infty} \frac{-\sqrt{1+\frac{2}{x^2}}}{3-\frac{6}{x}} = -\frac{1}{3}$$

6. $U_n = 72-6n,\ S_n = 378$

$U_1 = 72-6=66$

$$S_n = \frac{n}{2}(U_1+U_n) = \frac{n}{2}(66+72-6n) = 69n-3n^2$$

But $S_n = 378$, then $69n-3n^2=378$

$\Rightarrow 3n^2-69n+378=0 \Rightarrow n^2-23n+126=0$

$\Rightarrow (n-14)(n-9)=0$

Then, $n=9$ or $n=14$

7. $(x-1)(x-2)+(y+3)(y-4)+(z+1)(z-1)=0$

$\Leftrightarrow x^2-2x-x+2+y^2-4y+3y-12+z^2-z+z-1=0$

$\Leftrightarrow x^2-3x+2+y^2-y-12+z^2-1=0$

$\Leftrightarrow x^2-3x+\frac{9}{4}-\frac{9}{4}+2+y^2-y+\frac{1}{4}-\frac{1}{4}-12+z^2-0z-1=0$

$\Leftrightarrow \left(x^2-3x+\frac{9}{4}\right)-\frac{9}{4}+2+\left(y^2-y+\frac{1}{4}\right)-\frac{1}{4}-12+\left(z^2-0z\right)-1=0$

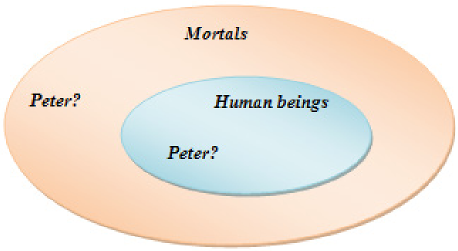
$$\Leftrightarrow\left(x-\frac{3}{2}\right)^2+\frac{-9+8}{4}+\left(y-\frac{1}{2}\right)^2+\frac{-1-48}{4}+(z-0)^2-1=0$$

$$\Leftrightarrow\left(x-\frac{3}{2}\right)^2+\left(y-\frac{1}{2}\right)^2+(z-0)^2+\frac{-9+8}{4}+\frac{-1-48}{4}-\frac{4}{4}=0$$

$$\Leftrightarrow\left(x-\frac{3}{2}\right)^2+\left(y-\frac{1}{2}\right)^2+(z-0)^2=\frac{54}{4}$$

Centre is $\left(\frac{3}{2},\frac{1}{2},0\right)$ and radius is $\sqrt{\frac{54}{4}}=\frac{3\sqrt{6}}{2}$

8. This argument is not valid. The conclusion is false because not only human beings are mortal. It is a converse error.



9. $f(x)=\sin^2 x\tan x$

$$f'(x)=2\sin x\cos x\tan x+\frac{\sin^2 x}{1+x^2}=2\sin x\cancel{\cos x}\frac{\sin x}{\cancel{\cos x}}+\frac{\sin^2 x}{1+x^2}$$

$$=2\sin^2 x+\frac{\sin^2 x}{1+x^2}=\sin^2 x\left(\frac{2+2x^2+1}{1+x^2}\right)$$

$$=\sin^2 x\left(\frac{2x^2+3}{1+x^2}\right)$$

10. a) Equation of the line joining the points $A(3,4,1)$ and $B(5,1,6)$

Direction vector is $\overrightarrow{AB}=(2,-3,5)$

Parametric equations

$$\begin{cases} x=3+2r \\ y=4-3r \\ z=1+5r \end{cases}$$ where r is a parameter

Or symmetric equations

$$\frac{x-3}{2}=\frac{4-y}{3}=\frac{z-1}{5}$$

b) If $z=0,\ 1+5r=0\Rightarrow r=-\frac{1}{5}$

and $\begin{cases} x=3+2\left(-\frac{1}{5}\right)=\frac{13}{5} \\ y=4-3\left(-\frac{1}{5}\right)=\frac{23}{5} \\ z=0 \end{cases}$

Then, the point is $\left(\frac{13}{5},\frac{23}{5},0\right)$

11. Given the function $f(x)=\cos 3x$

$$\begin{array}{ll} f(x)=\cos x & f(0)=1 \\ f'(x)=-3\sin 3x & f'(0)=0 \\ f''(x)=-9\cos 3x & f''(0)=-9 \\ f'''(x)=27\sin 3x & f'''(0)=0 \\ f^{(4)}(x)=81\cos 3x & f^{(4)}(0)=81 \\ f^{(5)}(x)=-243\sin 3x & f^{(4)}(0)=0 \\ f^{(6)}(x)=-729\cos 3x & f^{(5)}(0)=-729 \\ \vdots & \end{array}$$

$$\cos x=1+\frac{0}{1!}x+\frac{-9}{2!}x^2+\frac{0}{3!}x^3+\frac{81}{4!}x^4+\frac{0}{5!}x^5+\frac{-729}{6!}x^6+\ldots$$

$$=1-\frac{9x^2}{2}+\frac{27x^4}{8}-\frac{81x^6}{80}+\ldots$$

12. $\dfrac{x^2+1}{x^3+4x^2+3x}$

$$x^3+4x^2+3x=x\left(x^2+4x+3\right)=x(x+1)(x+3)$$

$$\frac{x^2+1}{x^3+4x^2+3x}=\frac{x^2+1}{x(x+1)(x+3)}$$

$$\frac{x^2+1}{x(x+1)(x+3)}=\frac{A}{x}+\frac{B}{x+1}+\frac{C}{x+3}$$

$$=\frac{A(x+1)(x+3)+Bx(x+3)+Cx(x+1)}{x(x+1)(x+3)}$$

$$x^2+1=A(x+1)(x+3)+Bx(x+3)+Cx(x+1)$$

Let $x=0$, $\Rightarrow 1=3A \Rightarrow A=\frac{1}{3}$

Let $x=-1$, $\Rightarrow 2=-2B \Rightarrow B=-1$

Let $x=-3$, $\Rightarrow 10=6C \Rightarrow C=\frac{5}{3}$

Then, $\frac{x^2+1}{x^3+4x^2+3x}=\frac{1}{3x}-\frac{1}{x+1}+\frac{5}{3x+9}$

Hence; $\int\frac{x^2+1}{x^3+4x^2+3x}dx=\int\frac{1}{3x}dx-\int\frac{1}{x+1}dx+\int\frac{5}{3x+9}dx$

$$=\frac{1}{3}\int\frac{1}{x}dx-\int\frac{1}{x+1}dx+\frac{5}{3}\int\frac{1}{x+3}dx$$

$$=\frac{1}{3}\ln|x|-\ln|x-1|+\frac{5}{3}\ln|x+3|+c$$

13. $z=\frac{2-2i}{1+i}$

$$\Rightarrow z=\frac{(2-2i)(1-i)}{(1+i)(1-i)}\Rightarrow z=\frac{2-2i-2i-2}{2}\Rightarrow z=-2i$$

$$|z|=2,\quad \arg(z)=-\frac{\pi}{2}$$

Then, $z=2\operatorname{cis}\left(-\frac{\pi}{2}\right)$

14. Given that $I=\int_0^{\ln16}\frac{e^x+3}{e^x+4}dx$ and $J=\int_0^{\ln16}\frac{dx}{e^x+4}$

$$I+J=\int_0^{\ln16}\left(\frac{e^x+3}{e^x+4}+\frac{1}{e^x+4}\right)dx=\int_0^{\ln16}\left(\frac{e^x+4}{e^x+4}\right)dx=\int_0^{\ln16}dx=[x]_0^{\ln16}=\ln16$$

$$I-3J=\int_0^{\ln16}\left(\frac{e^x+3}{e^x+4}-\frac{3}{e^x+4}\right)dx=\int_0^{\ln16}\left(\frac{e^x}{e^x+4}\right)dx$$

$$=\left[\ln\left(e^x+4\right)\right]_0^{\ln16}=\ln\left(e^{\ln16}+4\right)-\ln\left(e^0+4\right)$$

$$= \ln(16+4) - \ln 4 = \ln 20 - \ln 4$$

$$= \ln \frac{20}{4} = \ln 4$$

15. $U = \{(a,b,c,d) : b+c+d=0\}$ and
$W = \{(a,b,c,d) : a+b=0, c=2d\}$

We need to solve

$$\begin{cases} b+c+d=0 \\ a+b=0 \Rightarrow a=-b \\ c=2d \end{cases}$$

$b+c+d=0$

$\Rightarrow b+2d+d=0$

$\Rightarrow b=-3d$

$a=-b \Rightarrow a=3d$

Then $U \cap W = \{(3d,-3d,2d,d) : d \in \mathbb{R}\}$ and $\dim(U \cap W) = 1$

16. The quarterly, monthly,….rates of interest are found by dividing the nominal annual rate by 4,12,…..

Interest rate	**Number of compounding**	**Value of investment after one year in Frw**
a) Annually	1	$100,000 \times (1+0.08) = 108,000$
b) Quarterly	4	$100,000 \times \left(1+\frac{0.08}{4}\right)^4 = 100,000 \times 1.02^4 = 108,240$
c) Monthly	12	$100,000 \times \left(1+\frac{0.08}{12}\right)^{12} = 100,000 \times 1.0067^{12} = 108,300$
d) Weekly	52	$100,000 \times \left(1+\frac{0.08}{52}\right)^{52} = 100,000 \times 1.0015^{52} = 108,320$
e) Daily	365	$100,000 \times \left(1+\frac{0.08}{365}\right)^{365} = 100,000 \times 1.0002^{365} = 108,330$

17. Advertisement sports (x_i) and volume of sales in hundreds (y_i)

x_i	y_i	x_i^2	y_i^2	$x_i y_i$
1	41	1	1681	41
2	50	4	2500	100
3	54	9	2916	162
4	54	16	2916	216
5	57	25	3249	285
6	63	36	3969	378
$\sum_{i=1}^{6} x_i = 21$	$\sum_{i=1}^{6} y_i = 319$	$\sum_{i=1}^{6} x_i^2 = 91$	$\sum_{i=1}^{6} y_i^2 = 17231$	$\sum_{i=1}^{6} x_i y_i = 1182$

a) Mean: $\overline{x} = \frac{21}{6} = \frac{7}{2}$, $\overline{y} = \frac{319}{6}$

$$\sigma_x^2 = \frac{91}{6} - \left(\frac{7}{2}\right)^2 = \frac{35}{12}$$

Standard deviation for x_i is $\sigma_x = \sqrt{\frac{35}{12}} = \frac{\sqrt{105}}{6}$

$$\sigma_y^2 = \frac{17231}{6} - \left(\frac{319}{6}\right)^2 = \frac{1625}{36}$$

Standard deviation for y_i is $\sigma_x = \sqrt{\frac{1625}{36}} = \frac{5\sqrt{65}}{6}$.

b) Correlation coefficient is given by $r = \frac{\operatorname{cov}(x, y)}{\sigma_x \sigma_y}$

$$\operatorname{cov}(x, y) = \frac{\sum_{i=1}^{6} x_i y_i}{6} - \overline{x}\ \overline{y}$$

$$\operatorname{cov}(x, y) = \frac{1182}{6} - \left(\frac{21}{6}\right)\left(\frac{319}{6}\right) = \frac{7092 - 6699}{36} = \frac{393}{36} = \frac{131}{12}$$

Then,

$$r = \frac{\frac{131}{12}}{\frac{\sqrt{105}}{6} \times \frac{5\sqrt{65}}{6}} = \frac{131}{12} \times \frac{36}{5\sqrt{6825}} = \frac{393}{5\sqrt{6825}} \approx 0.95$$

c) Regression line for y on x

$$y - \overline{y} = \frac{\operatorname{cov}(x, y)}{\sigma_x^2}(x - \overline{x}) \Leftrightarrow y - \frac{319}{6} = \frac{\frac{131}{12}}{\frac{35}{12}}\left(x - \frac{21}{6}\right)$$

$$\Leftrightarrow y-\frac{319}{6}=\frac{131}{35}\left(x-\frac{21}{6}\right)\Leftrightarrow y=\frac{131}{35}x-\frac{131}{35}\times\frac{21}{6}+\frac{319}{6}$$

$$\Leftrightarrow y=\frac{131}{35}x+\frac{601}{15}$$

d) If $x=7,\ y=\frac{131}{35}\times 7+\frac{601}{15}\approx 66$

18. Given the vertices of the triangle: $A(1,2,3)$, $B(-2,1,-4)$ and $C(3,4,-2)$

a) (i) $\overrightarrow{AB}=(-3,-1,-7)\Rightarrow\left\|\overrightarrow{AB}\right\|=\sqrt{9+1+49}=\sqrt{59}$

$\overrightarrow{AC}=(2,2,-5)\Rightarrow\left\|\overrightarrow{AC}\right\|=\sqrt{4+4+25}=\sqrt{33}$

$\overrightarrow{BC}=(5,3,2)\Rightarrow\left\|\overrightarrow{BC}\right\|=\sqrt{25+9+4}=\sqrt{38}$

The perimeter is $\sqrt{59}+\sqrt{33}+\sqrt{38}$ units of length

b) Centre of gravity $\frac{1}{3}(A+B+C)=\frac{1}{3}(2,7,-3)=\left(\frac{2}{3},\frac{7}{3},-1\right)$

c) $\measuredangle\left(\overrightarrow{AB},\overrightarrow{AC}\right)=\cos^{-1}\left(\frac{-6-2+35}{\sqrt{59\times 33}}\right)$

$$=\cos^{-1}\left(\frac{-32}{\sqrt{2242}}\right)=132.5^0=\cos^{-1}\left(\frac{27}{\sqrt{1947}}\right)=52.3^0$$

Thus, $\theta_1=52.3^0$

$\measuredangle\left(\overrightarrow{AB},\overrightarrow{BC}\right)=\cos^{-1}\left(\frac{-15-3-14}{\sqrt{59\times 38}}\right)$

$$=\cos^{-1}\left(\frac{-32}{\sqrt{2242}}\right)=132.5^0$$

$$=\cos^{-1}\left(\frac{-32}{\sqrt{2242}}\right)=132.5^0$$

Therefore, $\theta_2=47.5$

$\measuredangle\left(\overrightarrow{AC},\overrightarrow{BC}\right)=\cos^{-1}\left(\frac{10+6-10}{\sqrt{33\times 38}}\right)=\cos^{-1}\left(\frac{10}{\sqrt{1254}}\right)=80.2^0$

Therefore, $\theta_3=80.2^0$

(or $\theta_3=180^0-52.3^0-47.5^0=80.2^0$).

d) The area of triangle ABC is given by $\frac{1}{2}\left\|\overrightarrow{AB}\times\overrightarrow{AC}\right\|$

$$\frac{1}{2}\left\|\overrightarrow{AB}\times\overrightarrow{AC}\right\| = \frac{1}{2}\left\|\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ -3 & -1 & -7\\ 2 & 2 & -5\end{vmatrix}\right\|$$

$$= \frac{1}{2}\left\|(5+14)\vec{i}-(15+14)\vec{j}+(-6+2)\vec{k}\right\|$$

$$= \frac{1}{2}\left\|19\vec{i}-29\vec{j}-4\vec{k}\right\| = \frac{\sqrt{361+841+16}}{2} = \frac{\sqrt{1218}}{2} \text{ sq. units}$$

19. $f(x) = x + |x| + 1 - \frac{1}{x+2}$

a) Domain of definition

Existence condition: $x+2\neq 0 \Leftrightarrow x\neq -2$

Then, $Domf = \mathbb{R}\setminus\{-2\}$ or $Domf =]-\infty,-2[\cup]-2,+\infty[$

b) $f(x)$ without the symbol of absolute value

$$f(x) = \begin{cases} x+x+1-\frac{1}{x+2}, x\geq 0 \\ x-x+1-\frac{1}{x+2}, x<0 \text{ or } x\neq -2\end{cases}$$

$$\Leftrightarrow f(x) = \begin{cases} 2x+1-\frac{1}{x+2}, x\geq 0 \\ 1-\frac{1}{x+2}, x<0 \text{ or } x\neq -2\end{cases}$$

Or $f(x) = \begin{cases} 2x+1-\frac{1}{x+2}, x\in[0,+\infty[\\ 1-\frac{1}{x+2}, x\in]-\infty,-2[\cup]-2,0[\end{cases}$

c) Limits on boundaries of domain of definition and asymptotes

$$\lim_{x\to -2} f(x) = \lim_{x\to -2} 1-\frac{1}{x+2} = \infty$$

Thus, $V.A. \equiv x = -2$

Table of sign for determining sided limits:

x	$-\infty$	-2	$+\infty$
$x+2$	$-$	0	$+$
$-\frac{1}{x+2}$	$+$		$-$

From table of sign, we deduce that

$\lim_{x\to -2^-} f(x) = +\infty$ and $\lim_{x\to -2^+} f(x) = -\infty$

$$\lim_{x\to -\infty} f(x) = \lim_{x\to -\infty} 1 - \frac{1}{x+2} = 0$$

Hence, for $x \to -\infty$, there is $H.A. \equiv y = 1$

$$\lim_{x\to +\infty} f(x) = \lim_{x\to +\infty}\left(2x+1-\frac{1}{x+2}\right) = +\infty$$

Thus, for $x \to +\infty$, there is no horizontal asymptote.

Let us check if there is an oblique asymptote

For $x \to +\infty$, $f(x) = 2x+1-\dfrac{1}{x+2}$;

As $\lim_{x\to +\infty} \dfrac{1}{x+2} = 0$, $y = 2x+1$ is oblique asymptote.

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0}\left(2x+1-\frac{1}{x+2}\right) = \frac{1}{2} = \lim_{x\to 0^-} f(x) = \lim_{x\to 0}\left(1-\frac{1}{x+2}\right)$$

Therefore, f is continuous at $x = 0$

d) Interval of increasing

$$f(x) = \begin{cases} 2x+1-\dfrac{1}{x+2}, x\in [0,+\infty[\\ 1-\dfrac{1}{x+2}, x\in]-\infty,-2[\cup]-2,0[\end{cases} \Rightarrow f'(x) = \begin{cases} 2+\dfrac{1}{(x+2)^2}, x\in [0,+\infty[\\ \dfrac{1}{(x+2)^2}, x\in]-\infty,-2[\cup]-2,0[\end{cases}$$

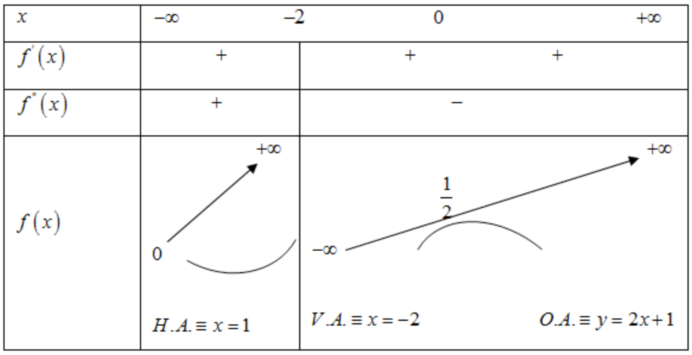
As $f'(x) > 0, \forall \in Domf$, f is increasing on its domain of definition.

e) Concavity

$$f'(x) = \begin{cases} 2+\dfrac{1}{(x+2)^2}, x\in [0,+\infty[\\ \dfrac{1}{(x+2)^2}, x\in]-\infty,-2[\cup]-2,0[\end{cases}$$

$$\Rightarrow f''(x) = -\frac{x+2}{(x+2)^4} = -\frac{1}{(x+2)^3}, \forall x \in Domf$$

f) Table of variation



g) Curve sketching

Additional points:

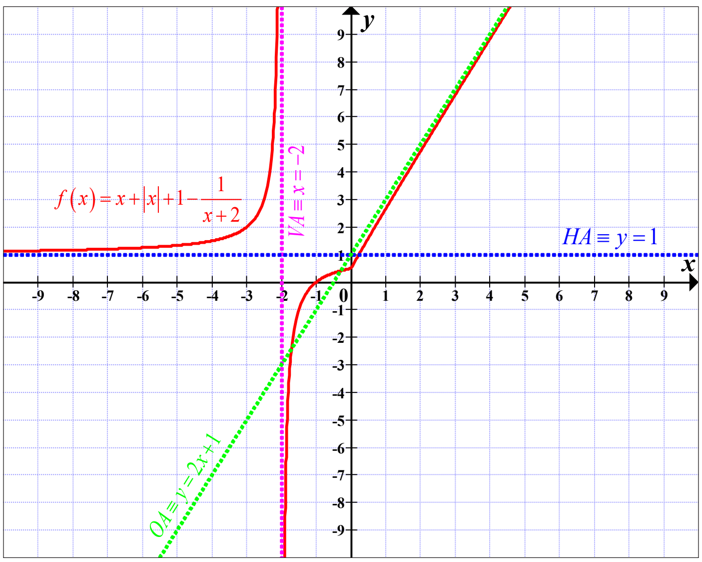
For $x < -2$

x	-5	-4.5	-4	-3.5	-3	-2.5	-2.2
y	1.3	1.4	1.5	1.7	2	3	6

For $x > -2$

x	-1.8	-1.5	-1	-0.5	0	0.5	1	1.5	2	2.5	3
y	-4	-1	0	-0.7	0.5	1.6	2.7	3.7	4.8	5.8	6.8

Curve



20. a) Let $P(x, y)$ be any point on the parabola using focus-directrix $(F - M)$ property of the parabola: $\overline{FP} = \overline{PM}$.

Therefore, $\sqrt{(x+1)^2+(y+2)^2}=\dfrac{|x-2y+3|}{\sqrt{1^2+(-2)^2}}$

$\Leftrightarrow (x+1)^2+(y+2)^2=\dfrac{(x-2y+3)^2}{5}$

$\Leftrightarrow x^2+2x+1+y^2+4y+4=\dfrac{x^2+4y^2+9-4xy+6x-12y}{5}$

$\Leftrightarrow 5x^2+10x+5y^2+20y+25=x^2+4y^2+9-4xy+6x-12y$

$\Leftrightarrow 4x^2+y^2+4xy+4x+32y+16=0$ which is the required equation of the parabola.

b) Let $P(x,y)$ be any point of focus and the given point $(0,4)$ be dented by A.

Then, $PA=\frac{2}{3}\times$ distance of P from the line $y=9$.

$\Leftrightarrow \sqrt{x^2+(y-4)^2}=\dfrac{2}{3}\times\dfrac{|y-9|}{\sqrt{0^2+1^2}} \Leftrightarrow x^2+(y-4)^2=\dfrac{4}{9}\times(y-9)^2$

$\Leftrightarrow x^2+y^2-8y+16=\dfrac{4}{9}\left(y^2-18y+81\right)$

$\Leftrightarrow 9x^2+9y^2-72y+144=4y^2-72y+324$

$\Leftrightarrow 9x^2+5y^2-180=0$ which is the required equation of locus.

c) The equation of hyperbola is $x^2-4y^2=4 \Leftrightarrow \dfrac{x^2}{4}-\dfrac{y^2}{1}=1$

Here, $a^2=4,\ b^2=1$ then, $a=2,\ b=1$.

Therefore, axes are 4 and 2.

$b^2=a^2\left(e^2-1\right)$ or $1=4\left(e^2-1\right)$ which gives $e=\dfrac{\sqrt{5}}{2}$

Thus, $Eccentricity=\dfrac{\sqrt{5}}{2}$

Since, coordinates of foci are given by $(\pm ae,0)$, then they are

$\left(\pm 2\times\dfrac{\sqrt{5}}{2},0\right)$ or $\left(\pm\sqrt{5},0\right)$

Length of latus rectum is $\dfrac{2b^2}{a}=\dfrac{2\times 1}{2}=1$

Alternative method:

From $\dfrac{x^2}{4}-\dfrac{y^2}{1}=1$, $a=2,\ b=1$;

For hyperbola $c^2 = a^2 + b^2$

Here, $c^2 = 4 + 1 = 5$ or $c = \sqrt{5}$

- **A** Axes are $2a = 4$ and $2b = 2$
- **A** Coordinates of foci $F(\pm c, 0) = (\pm\sqrt{5}, 0)$
- **A** Eccentricity $e = \frac{c}{a} = \frac{\sqrt{5}}{2}$
- **A** Length of latus rectum is equal to $\frac{2b^2}{a} = \frac{2\times 1}{2} = 1$

Answers for Summative Evaluation Four

1. $\left(x + \frac{1}{x}\right)^{20} = \left(x + x^{-1}\right)^{20}$

$$= {}^{20}C_r x^r \left(x^{-1}\right)^{20-r} = {}^{20}C_r x^r x^{-20+r} = {}^{20}C_r x^{2r-20}$$

For the independent term, $2r - 20 = 0$ or $r = 10$

Then, the independent term is ${}^{20}C_{10} = \frac{20!}{10!10!} = 184756$

2. If $6x^3 + 7x^2 + ax + b$ is divisible by $x - 2$, the remainder is

$6(2)^3 + 7(2)^2 + 2a + b = 72$

$48 + 28 + 2a + b = 72$

$\Rightarrow 2a + b = -4$

Also, $6x^3 + 7x^2 + ax + b$ is exactly divisible by $x + 1$ then

$6(-1)^3 + 7(-1)^2 - a + b = 0$

$-6 + 7 - a + b = 0$

$\Rightarrow -a + b = -1$

$$\begin{cases} 2a + b = -4 \\ -a + b = -1 \end{cases} \Rightarrow \begin{cases} 2a + b = -4 \\ \underline{a - b = 1} \end{cases}$$

$$3a = -3 \Rightarrow a = -1$$

$-a + b = -1 \Rightarrow 1 + b = -1 \Rightarrow b = -2$ $\quad \begin{cases} a = -1 \\ b = -2 \end{cases}$

3. $\sin x + \sqrt{3}\cos x = 1$

Let $\sqrt{3} = \tan\alpha \Rightarrow \alpha = \frac{\pi}{3}$

$$\sin x+\tan\alpha\cos x=1\Rightarrow \sin x+\frac{\sin\alpha}{\cos\alpha}\cos x=1$$

$$\Rightarrow \sin x\cos\alpha+\sin\alpha\cos x=\cos\alpha$$

$$\Rightarrow \sin(x+\alpha)=\cos\alpha\Rightarrow \sin\left(x+\frac{\pi}{3}\right)=\cos\frac{\pi}{3}$$

$$\Rightarrow \sin\left(x+\frac{\pi}{3}\right)=\frac{1}{2}$$

$$x+\frac{\pi}{3}=\begin{cases}\frac{\pi}{6}+2k\pi\\ \frac{5\pi}{6}+2k\pi\end{cases},\quad k\in\mathbb{Z}\qquad x=\begin{cases}-\frac{\pi}{6}+2k\pi\\ \frac{\pi}{2}+2k\pi\end{cases}$$

$$S=\left\{-\frac{\pi}{6}+2k\pi,\frac{\pi}{2}+2k\pi\right\},\ k\in\mathbb{Z}$$

4. A matrix has no inverse if its determinant is zero

$$\begin{vmatrix}11-x & 2 & 8\\ 2 & 2-x & -10\\ 8 & -10 & 5-x\end{vmatrix}=0$$

$$\Rightarrow (11-x)(2-x)(5-x)-160-160-64(2-x)-100(11-x)-4(5-x)=0$$

$$\Rightarrow 110-65x+5x^2-22x+13x^2-x^3-320-128+64x-1100+100x-20+4x=0$$

$$\Rightarrow -x^3+18x^2+81x-1458=0$$

$$\Rightarrow x^3-18x^2-81x+1458=0$$

9 is one of the roots

	1	-18	-81	1458
9		9	-81	-1458
	1	-9	-162	0

$$x^3-18x^2-81x+1458=(x-9)\left(x^2-9x-162\right)$$

$$x^2-9x-162=0\Rightarrow (x+9)(x-18)=0\Rightarrow x=-9\ \ or\ \ x=18$$

Thus, the given matrix is singular if $x\in\{-9,9,18\}$

5. $\begin{cases}\log(x+y)=1\\ \log_2 x+2\log_4 y=4\end{cases}$

$$\log(x+y)=1\Leftrightarrow \log(x+y)=\log 10\Rightarrow x+y=10$$

$\log_2 x + 2\log_4 y = 4$

$\Leftrightarrow \log_2 x + 2\dfrac{\log_2 y}{\log_2 4} = 4$

$\Leftrightarrow \log_2 x + 2\dfrac{\log_2 y}{2} = 4$

$\Leftrightarrow \log_2 x + \log_2 y = 4\log_2 2$

$\Leftrightarrow \log_2 xy = \log_2 2^4$

$\Rightarrow xy = 16$

Now,

$\begin{cases} x + y = 10 \Rightarrow x = 10 - y \\ xy = 16 \end{cases}$

$(10 - y)y = 16 \Rightarrow 10y - y^2 - 16 = 0 \Rightarrow y^2 - 10y + 16 = 0$

$(y-2)(y-8) = 0$

$y - 2 = 0 \Rightarrow y = 2 \Rightarrow x = 10 - y = 10 - 2 = 8$

$y - 8 = 0 \Rightarrow y = 8$

$\Rightarrow x = 10 - 8 = 2$

6. $x^2 - x - 3 = 0$

We know that for the equation of the form $ax^2 + bx + c = 0$,

if α and β are the roots, then, $\alpha + \beta = -\dfrac{b}{a}$ and $\alpha\beta = \dfrac{c}{a}$

Here, $\alpha + \beta = 1$ and $\alpha\beta = -3$

$(\alpha + \beta)^3 = \alpha^3 + 3\alpha^2\beta + 3\alpha\beta^2 + \beta^3$

$\Leftrightarrow (\alpha + \beta)^3 - 3\alpha^2\beta - 3\alpha\beta^2 = \alpha^3 + \beta^3$

$\Leftrightarrow \alpha^3 + \beta^3 = (\alpha + \beta)^3 - 3\alpha\beta(\alpha + \beta)$

$\Leftrightarrow \alpha^3 + \beta^3 = (1)^3 - 3(-3)(1) = 10$

Then, $\alpha^3 + \beta^3 = 10$

7. $x^2 + 4y^2 - 4x + 8y + 4 = 0$

$\Leftrightarrow x^2 - 4x + 4y^2 + 8y + 4 = 0 \Leftrightarrow x^2 - 4x + 4 + 4(y^2 + 2y) = 0$

$\Leftrightarrow (x-2)^2 + 4\left[(y+1)^2 - 1\right] = 0 \Leftrightarrow (x-2)^2 + 4(y+1)^2 - 4 = 0$

$\Leftrightarrow (x-2)^2 + 4(y+1)^2 = 4 \Leftrightarrow \dfrac{(x-2)^2}{4} + \dfrac{(y+1)^2}{1} = 1$

The centre is $(2, -1)$

$a^2 = 4,\ b^2 = 1$

The eccentricity is $e=\sqrt{\frac{a^2-b^2}{a^2}}=\sqrt{\frac{4-1}{4}}=\frac{\sqrt{3}}{2}$

Foci are $(2e+2,-1)$ and $(-2e+2,-1)$

Or $(\sqrt{3}+2,-1)$ and $(-\sqrt{3}+2,-1)$

8. Tangent and normal line to the curve $3x^2-xy-2y^2+12=0$ at point $(2,3)$

$(3x^2-xy-2y^2+12)'=6x-(y+xy')-4yy'$

$6x-y-xy'-4yy'=0 \Rightarrow xy'+4yy'=6x-y$

$\Rightarrow y'(x+4y)=6x-y \Rightarrow y'=\frac{6x-y}{x+4y}$

$y'_{(2,3)}=\frac{6(2)-3}{2+4(3)}=\frac{9}{14}$

$T \equiv y-3=\frac{9}{14}(x-2)$ $\qquad N \equiv y-3=-\frac{14}{9}(x-2)$

$\equiv y=\frac{9}{14}x-\frac{9}{7}+3$ $\qquad \equiv y=-\frac{14}{9}x+\frac{28}{9}+3$

$\equiv y=\frac{9}{14}x+\frac{12}{7}$ $\qquad \equiv y=-\frac{14}{9}x+\frac{55}{9}$

9. $\int_0^1 \frac{1}{(2x+k)^2}dx=\frac{1}{3}$

$\int_0^1 \frac{1}{(2x+k)^2}dx=-\frac{1}{2}\left[\frac{1}{2x+k}\right]_0^1=-\frac{1}{2}\left(\frac{1}{2+k}-\frac{1}{k}\right)=-\frac{1}{4+2k}+\frac{1}{2k}$

$\Rightarrow -\frac{1}{4+2k}+\frac{1}{2k}=\frac{1}{3}$

$\Rightarrow \frac{-2k+4+2k}{2k(4+2k)}=\frac{1}{3} \Rightarrow \frac{4}{2k(4+2k)}=\frac{1}{3}$

$\Rightarrow 4k^2+8k=12 \Rightarrow 4k^2+8k-12=0 \Rightarrow k^2+2k-3=0$

$\Rightarrow (k+3)(k-1)=0 \Rightarrow k=-3 \ \ or \ \ k=1$

10. $z^6=1$

$z_k=\text{cis}\frac{2k\pi}{6}=\text{cis}\frac{k\pi}{3},\ k=0,1,2,3,4,5$

$z_0 = \text{cis}\,0 = 1$

$z_1 = \text{cis}\frac{\pi}{3} = \frac{1}{2} + i\frac{\sqrt{3}}{2}$ $\qquad$ $z_2 = \text{cis}\frac{2\pi}{3} = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$

$z_3 = \text{cis}\frac{3\pi}{3} = -1$ $\qquad$ $z_4 = \text{cis}\frac{4\pi}{3} = -\frac{1}{2} - i\frac{\sqrt{3}}{2}$

$z_2 = \text{cis}\frac{5\pi}{3} = \frac{1}{2} - i\frac{\sqrt{3}}{2}$

$$S = \left\{1, \frac{1}{2} + i\frac{\sqrt{3}}{2}, -\frac{1}{2} + i\frac{\sqrt{3}}{2}, -1, -\frac{1}{2} - i\frac{\sqrt{3}}{2}, \frac{1}{2} - i\frac{\sqrt{3}}{2}\right\}$$

11. u_1 and d be the first term and the common difference respectively

$$u_1 + (u_1 + d) + (u_1 + 2d) + (u_1 + 3d) + (u_1 + 4d) + (u_1 + 5d) = 72$$

$$\Rightarrow 6u_1 + 15d = 72$$

But $u_2 = 7u_5$ or

$$u_1 + d = 7(u_1 + 4d) \Rightarrow u_1 + d - 7u_1 - 28d = 0 \Rightarrow -6u_1 - 27d = 0$$

$$\begin{cases} 6u_1 + 15d = 72 \\ \underline{-6u_1 - 27d = 0} \end{cases}$$

$$-12d = 72 \Rightarrow d = -6$$

$$6u_1 + 15d = 72 \Rightarrow 6u_1 - 90 = 72 \Rightarrow u_1 = 27$$

12. $\lim\limits_{x\to\frac{\pi}{2}} (\tan x)^{\cos x} = \lim\limits_{x\to\frac{\pi}{2}} e^{\ln(\tan x)^{\cos x}} = \lim\limits_{x\to\frac{\pi}{2}} e^{\cos x \ln(\tan x)} = e^{\lim\limits_{x\to\frac{\pi}{2}} \cos x \ln(\tan x)}$

$$\lim_{x\to\frac{\pi}{2}} \cos x \ln(\tan x) = 0 \times \infty \ IC$$

$$\lim_{x\to\frac{\pi}{2}} \cos x \ln(\tan x) = \lim_{x\to\frac{\pi}{2}} \frac{\ln(\tan x)}{\frac{1}{\cos x}} = \lim_{x\to\frac{\pi}{2}} \frac{\frac{(\tan x)'}{\tan x}}{\frac{\sin x}{\cos^2 x}} \quad \text{L'H}\hat{\text{o}}\text{pital's rule}$$

$$= \lim_{x\to\frac{\pi}{2}} \frac{\cos^2 x}{\cos^2 x \tan x \sin x}$$

$$= \lim_{x\to\frac{\pi}{2}} \frac{1}{\tan x \sin x} = \frac{1}{\tan\frac{\pi}{2} \sin\frac{\pi}{2}} = 0$$

Then, $\lim_{x\to\frac{\pi}{2}}\left(\tan x\right)^{\cos x}=e^{0}=1$

13. $\overline{x}=6.2,\ \sigma_x=3.03315,\ \overline{y}=2.04,\ \sigma_y=0.461519$ and $r_{xy}=0.957241$

The regression line of y on x is $y-\overline{y}=\dfrac{\operatorname{cov}(x,y)}{\sigma_x^2}\left(x-\overline{x}\right)$

But $r_{xy}=\dfrac{\operatorname{cov}(x,y)}{\sigma_x\sigma_y}\Leftrightarrow \operatorname{cov}(x,y)=r_{xy}\sigma_x\sigma_y$

Then, $y-\overline{y}=\dfrac{r_{xy}\sigma_x\sigma_y}{\sigma_x^2}\left(x-\overline{x}\right)$ or

$$y-\overline{y}=\frac{r_{xy}\sigma_y}{\sigma_x}\left(x-\overline{x}\right)\Leftrightarrow y=\frac{r_{xy}\sigma_y}{\sigma_x}\left(x-\overline{x}\right)+\overline{y}$$

$$\Rightarrow y=\frac{0.957241\times 0.461519}{3.03315}\left(x-6.2\right)+2.04$$

$$\Rightarrow y=0.145x+1.141$$

14. Let S be the sample space, then $n(S)={}^{52}C_2=1326$

Let E be the event of getting two Kings.

$n(E)={}^{4}C_2=6$

Then, $P(E)=\dfrac{6}{1326}=\dfrac{1}{221}$

15. Let θ be the angle between vectors (2, 5) and (−1, 3)

$$\cos\theta=\frac{(2,5)\cdot(-1,3)}{\|(2,5)\|\|(-1,3)\|}=\frac{-2+15}{\sqrt{4+25}\sqrt{1+9}}=\frac{13\sqrt{290}}{290}$$

$$\theta=\cos^{-1}\left(\frac{13\sqrt{290}}{290}\right)=40.24\deg=0.70\,rad$$

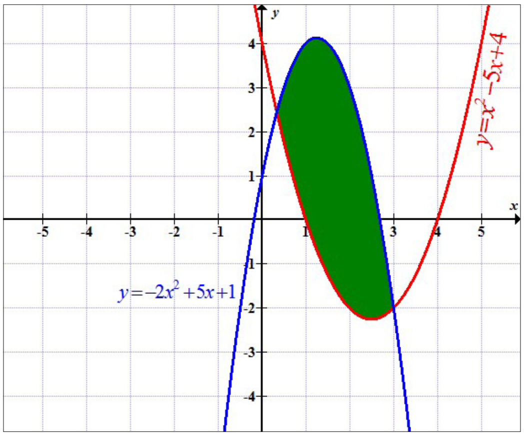
16. $y=x^2-5x+4$ and $y=-2x^2+5x+1$

Intersection:

$$x^2-5x+4=-2x^2+5x+1\Rightarrow 3x^2-10x+3=0$$

$$\Rightarrow (x-3)(3x-1)=0\Rightarrow x=3\ or\ \ x=\frac{1}{3}$$

The curves intersect at $x=3$ and $x=\dfrac{1}{3}$



The area of the region enclosed between the two curves is:

$$\int_{1/3}^{3}\left(-2x^2+5x+1-x^2+5x-4\right)dx=\int_{1/3}^{3}\left(-3x^2+10x-3\right)dx$$

$$=\left[-x^3+5x^2-3x\right]_{1/3}^{3}$$

$$=-3^3+5(3)^2-3(3)+\left(\frac{1}{3}\right)^3-5\left(\frac{1}{3}\right)^2+3\left(\frac{1}{3}\right)$$

$$=-27=45-9+\frac{1}{27}-\frac{5}{9}+1$$

$$=\frac{256}{27}\ sq.\ units$$

17. a) i) $P(A\cup C)=P(A)+P(C)-P(A\cap C)$

But $P(A\cap C)=0$ since A and C are mutually exclusive events

$$P(A\cup C)=P(A)+P(C)=\frac{2}{3}+\frac{1}{5}=\frac{13}{15}$$

Since A and B are independent events,

$$P(A\cap B)=P(A)P(B)\Rightarrow P(B)=\frac{P(A\cap B)}{P(A)}$$

$$P(A\cup B)=P(A)+P(B)-P(A\cap B)$$

$$P(A\cup B)=P(A)+\frac{P(A\cap B)}{P(A)}-P(A\cap B)$$

$$\frac{4}{5}=\frac{2}{3}+\frac{P(A\cup B)}{2/3}-P(A\cup B)$$

$$\Rightarrow \frac{4}{5}-\frac{2}{3}=\frac{3}{2}P(A\cup B)-P(A\cup B)$$

$$\Rightarrow \frac{2}{15}=\frac{P(A\cup B)}{2}$$

$$\Rightarrow P(A\cup B)=\frac{4}{15}$$

$$P(B)=\frac{P(A\cap B)}{P(A)}=\frac{\frac{4}{15}}{\frac{2}{3}}=\frac{2}{5}$$

ii) B and C are independent events if

$$P(B\cap C)=P(B)P(C)$$

$$P(B)P(C)=\frac{2}{5}\times\frac{1}{5}=\frac{2}{25}$$

$$P(B\cap C)=P(B)+P(C)-P(B\cup C)$$

$$=\frac{2}{5}+\frac{1}{5}-\frac{13}{25}=\frac{2}{25}$$

$$P(B\cap C)=P(B)P(C)$$

Thus, B and C are independent events.

b) Let $X_i, i=1,2,3$ be the event "patient have the virus" and let D be the vent "selected patient recovers".

We need $P(X_3 \mid D)$

$$P(X_3 \mid D)=\frac{P(D\mid X_3)P(X_3)}{\sum_{i=1}^{3}P(D\mid X_i)}=\frac{\frac{1}{8}\times\frac{1}{8}}{\frac{1}{2}\times\frac{1}{2}+\frac{3}{8}\times\frac{3}{8}+\frac{1}{8}\times\frac{1}{8}}=\frac{1}{26}$$

18. a) Given the points $A(2,-3,-1), B(3,-4,2)$ and $C(4,-5,2)$

(i) $\overrightarrow{AB}=(1,-1,3),\ \overrightarrow{AC}=(2,-2,3)$

$$\overrightarrow{AB}\times\overrightarrow{AC}=\begin{vmatrix}\vec{i} & 1 & 2\\ \vec{j} & -1 & -2\\ \vec{k} & 3 & 3\end{vmatrix}$$

$$=\vec{i}\begin{vmatrix}-1 & -2\\ 3 & 3\end{vmatrix}-\vec{j}\begin{vmatrix}1 & 2\\ 3 & 3\end{vmatrix}+\vec{k}\begin{vmatrix}1 & 2\\ -1 & -2\end{vmatrix}$$

$$=\vec{i}(-3+6)-\vec{j}(3-6)+\vec{k}(-2+2)$$

$$=3\vec{i}+3\vec{j}$$

(ii) The area of triangle ABC is given by $\frac{1}{2}\left\|\overrightarrow{AB}\times\overrightarrow{AC}\right\|$

$$\frac{1}{2}\left\|\overrightarrow{AB}\times\overrightarrow{AC}\right\|=\frac{1}{2}\left\|3\vec{i}+3\vec{j}\right\|=\frac{1}{2}\sqrt{9+9}=\frac{3\sqrt{2}}{2}\ sq.\ units$$

b) We have points $A(2,1,1)$, $B(0,5,3)$

(i) Direction vector is $\overrightarrow{AB}=(-2,4,2)$

Parametric equations:

$$\begin{cases}x=2-2r\\ y=1+4r\\ z=1+2r\end{cases},\quad r \text{ is a parameter}$$

(ii) Given $C(5,-4,2)$

Vectors $\overrightarrow{CD}$ is perpendicular to vector $\overrightarrow{AB}$ if $\overrightarrow{CD}\cdot\overrightarrow{AB}=0$

Let $D(x,y,z)$ be the point on line AB then

$\overrightarrow{CD}=(x-5,y+4,z-2)$ and

$$\overrightarrow{CD}\cdot\overrightarrow{AB}=-2(x-5)+4(y+4)+2(z-2)$$

$$=-2x+10+4y+16+2z-4$$

$$=-2x+4y+2z+22$$

But

$$\begin{cases}x=2-2r\\ y=1+4r\\ z=1+2r\end{cases}$$

Then

$$-2(2-2r)+4(1+4r)+2(1+2r)+22=0$$
$$\Rightarrow -4+4r+4+16r+2+4r+22=0$$
$$\Rightarrow 24r+24=0$$
$$\Rightarrow r=-1$$

$$\begin{cases} x=4 \\ y=-3 \\ z=-1 \end{cases}$$

And point D has coordinates $(4,-3,-1)$

(iii) If the plane π contains the line AB, the vector $\overrightarrow{CD}$ is perpendicular to the plane π since this vector is also perpendicular to the line AB. So this is a contradiction, no plane can contain the line AB and be parallel to CD.

19. $U=\text{cis}\frac{2\pi}{5}=\cos\frac{2\pi}{5}+i\sin\frac{2\pi}{5}$

a) $\frac{1}{U}=\frac{1}{\cos\frac{2\pi}{5}+i\sin\frac{2\pi}{5}}=\cos\frac{2\pi}{5}-i\sin\frac{2\pi}{5}$

$$\frac{1}{2}\left(U+\frac{1}{U}\right)=\frac{1}{2}\left(\cos\frac{2\pi}{5}+i\sin\frac{2\pi}{5}+\cos\frac{2\pi}{5}-i\sin\frac{2\pi}{5}\right)$$
$$=\frac{1}{2}\left(2\cos\frac{2\pi}{5}\right)=\cos\frac{2\pi}{5} \quad \textit{as required}$$

b) $U^5=\left(\cos\frac{2\pi}{5}+i\sin\frac{2\pi}{5}\right)^5$

$$=\cos\frac{5\times2\pi}{5}+i\sin\frac{5\times2\pi}{5}=\cos2\pi+i\sin2\pi=1$$

c) Since $U=\text{cis}\frac{2\pi}{5}$

$$U^0=\text{cis}\frac{0\times2\pi}{5},\ U^1=\text{cis}\frac{1\times2\pi}{5}=1,\ U^2=\text{cis}\frac{2\times2\pi}{5}=\text{cis}\frac{4\pi}{5},$$
$$U^3=\text{cis}\frac{3\times2\pi}{5}=\text{cis}\frac{6\pi}{5},U^4=\text{cis}\frac{4\times2\pi}{5}=\text{cis}\frac{8\pi}{5}$$

These are five fifth roots of unit. Then, their sum must be zero. Hence, $U^4+U^3+U^2+U+1=0$

d) $U^4+U^3+U^2+U+1=\text{cis}\frac{8\pi}{5}+\text{cis}\frac{6\pi}{5}+\text{cis}\frac{4\pi}{5}+\text{cis}\frac{2\pi}{5}+1$

$$U^4+U^3+U^2+U+1=\cos\frac{8\pi}{5}+\cos\frac{6\pi}{5}+\cos\frac{4\pi}{5}$$
$$+\cos\frac{2\pi}{5}+1+i\left(\sin\frac{8\pi}{5}+\sin\frac{6\pi}{5}+\sin\frac{4\pi}{5}+\sin\frac{2\pi}{5}\right)$$

Take the real party

$$\cos\frac{8\pi}{5}+\cos\frac{6\pi}{5}+\cos\frac{4\pi}{5}+\cos\frac{2\pi}{5}+1=0$$

We know that $\cos\alpha=\cos(2\pi-\alpha)$, then

$$\cos\frac{8\pi}{5}=\cos\left(2\pi-\frac{8\pi}{5}\right)=\cos\frac{2\pi}{5}$$
$$\cos\frac{6\pi}{5}=\cos\left(2\pi-\frac{6\pi}{5}\right)=\cos\frac{4\pi}{5}$$

Also, $\cos 2\alpha=2\cos^2\alpha-1\Rightarrow\cos\frac{4\pi}{5}=2\cos^2\frac{2\pi}{5}-1$

$$\cos\frac{2\pi}{5}+2\cos^2\frac{2\pi}{5}-1+2\cos^2\frac{2\pi}{5}-1+\cos\frac{2\pi}{5}+1=0$$
$$\Rightarrow 4\cos^2\frac{2\pi}{5}+2\cos\frac{2\pi}{5}-1=0$$

But $x=U+\frac{1}{U}=2\cos\frac{2\pi}{5}$

Then, $4\cos^2\frac{2\pi}{5}+2\cos\frac{2\pi}{5}-1=x^2+x-1$ or $x^2+x-1=0$

e) $x^2+x-1=0$

$\Delta=1+4=5$

$x_1=\frac{-1+\sqrt{5}}{2}$, $x_2=\frac{-1-\sqrt{5}}{2}$

$x_2=\frac{-1-\sqrt{5}}{2}$ is to be rejected.

For $x_1=\frac{-1+\sqrt{5}}{2}$, $2\cos\frac{2\pi}{5}=\frac{-1+\sqrt{5}}{2}\Rightarrow\cos\frac{2\pi}{5}=\frac{-1+\sqrt{5}}{4}$

20. a) Taking 5 men together, 4 women together, 3 children together we have $3!$ ways.

But 5 men can be permuted among them in $5!$ ways, 4 women can be permuted among them in $4!$ ways and 3 children can be permuted among them in $3!$ ways

The total ways is $3!5!4!3! = 103680$ ways

b) $\begin{cases} {}^{x}C_{y} = {}^{x}C_{y+1} \\ 4\left({}^{x}C_{y}\right) = 5\left({}^{x}C_{y-1}\right) \end{cases}$

$${}^{x}C_{y} = {}^{x}C_{y+1} \Leftrightarrow \frac{x!}{y!(x-y)!} = \frac{x!}{(y+1)!(x-y-1)!}$$

$$\Rightarrow (y+1)!(x-y-1)! = y!(x-y)!$$

$$\Rightarrow (y+1)y!(x-y-1)! = y!(x-y)(x-y-1)!$$

$$\Rightarrow y+1 = x-y$$

$$\Rightarrow -x+2y = -1$$

$$4\left({}^{x}C_{y}\right) = 5\left({}^{x}C_{y-1}\right) \Leftrightarrow 4\frac{x!}{y!(x-y)!} = 5\frac{x!}{(y-1)!(x-y+1)!}$$

$$\Rightarrow 4(y-1)!(x-y+1)! = 5y!(x-y)!$$

$$\Rightarrow 4(y-1)!(x-y+1)(x-y)! = 5y(y-1)!(x-y)!$$

$$\Rightarrow 4(x-y+1) = 5y$$

$$\Rightarrow 4x-4y+4 = 5y$$

$$\Rightarrow 4x-9y = -4$$

$$\begin{cases} -x+2y=-1 \\ 4x-9y=-4 \end{cases} \Leftrightarrow \begin{cases} -4x+8y=-4 \\ \underline{4x-9y=-4} \end{cases}$$

$$-y = -8 \Rightarrow y = 8$$

$$-x+2y = -1$$

$$\Rightarrow -x+16 = -1$$

$$\Rightarrow x = 17$$

Thus, $S = \{(17,8)\}$

c) ${}^{n-2}C_{m} + 2\left({}^{n-2}C_{m-1}\right) + {}^{n-2}C_{m-2} = {}^{n}C_{m}$

$$\frac{(n-2)!}{m!(n-m-2)!} + \frac{2(n-2)!}{(m-1)!(n-m-1)!} + \frac{(n-2)!}{(m-2)!(n-m)!}$$

$$= \frac{\frac{n!}{n(n-1)}}{\frac{m!(n-m)!}{(n-m)(n-m-1)}} + \frac{2\frac{n!}{n(n-1)}}{\frac{m!}{m}\frac{(n-m)!}{n-m}} + \frac{\frac{n!}{n(n-1)}}{\frac{m!(n-m)!}{m(m-1)}}$$

$$= \frac{n!(n-m)(n-m-1)}{m!(n-m)!n(n-1)} + \frac{2n!m(n-m)}{m!(n-m)!n(n-1)} + \frac{n!m(m-1)}{m!(n-m)!n(n-1)}$$

$$= \frac{n!}{m!(n-m)!}\left[\frac{(n-m)(n-m-1)+2m(n-m)+m(m-1)}{n(n-1)}\right]$$

$$= \frac{n!}{m!(n-m)!}\left[\frac{n^2-nm-n-mn+m^2+m+2mn-2m^2+m^2-m}{n(n-1)}\right]$$

$$= \frac{n!}{m!(n-m)!}\left(\frac{n^2-n}{n^2-n}\right) = \frac{n!}{m!(n-m)!}$$

$$= {}^nC_m \qquad \textit{as required}$$

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