MATHEMATICS FOR TTCs

TUTOR'S GUIDE

3

OPTIONS:

SOCIAL STUDIES EDUCATION (SSE)

FOREWORD

Dear Tutor,

Rwanda Education Board is honoured to present the tutor's guide for Year Three Mathematics in the option of Social Studies Education (SSE). This book serves as a guide to competence-based teaching and learning to ensure consistency and coherence in the learning of the mathematics content. The Rwandan educational philosophy is to ensure that student-teachers achieve full potential at every level of education which will prepare them to be well integrated in society and exploit employment opportunities.

Specifically, TTC curriculum was reviewed to train quality teachers who will confidently and efficiently implement the Competence Based Curriculum in pre-primary and primary education. The rationale of the changes is to ensure that TTC leavers are qualified for job opportunities and further studies in Higher Education in different programs under education career advancement.

In line with efforts to improve the quality of education, the government of Rwanda emphasizes the importance of aligning teaching and learning materials with the syllabus to facilitate their learning process. Many factors influence what they learn, how well they learn and the competences they acquire. Those factors include the relevance of the specific content, the quality of teachers' pedagogical approaches, the assessment strategies and the instructional materials.

The ambition to develop a knowledge-based society and the growth of regional and global competition in the jobs market has necessitated the shift to a competence-based curriculum. After a successful shift from knowledge to a competence-based curriculum in general education, TTC textbooks also were elaborated to align them to the new curriculum.

The book provides active teaching and learning techniques that engage student teachers to develop competences. In view of this, your role is to:

- Plan your lessons and prepare appropriate teaching materials.
- Organize group discussions for students considering the importance of social constructivism suggesting that learning occurs more effectively when the students works collaboratively with more knowledgeable and experienced people.
- Engage students through active learning methods such as inquiry methods, group discussions, research, investigative activities and group and individual work activities.
- Provide supervised opportunities for students to develop different competences by giving tasks which enhance critical thinking, problem

solving, research, creativity and innovation, communication and cooperation.

- Support and facilitate the learning process by valuing students' contributions in the class activities.
- Guide students towards the harmonization of their findings.
- Encourage individual, peer and group evaluation of the work done in the classroom and use appropriate competence-based assessment approaches and methods.

To facilitate you in your teaching activities, the content of this book is self explanatory so that you can easily use it. It is divided in 3 parts:

The part I explain the structure of this book and give you the methodological guidance;

The part II gives a sample lesson plan;

The part III details the teaching guidance for each concept given in the student book.

Even though this Tutor's guide contains the guidance on solutions for all activities given in the learner's book, you are requested to work through each question before judging student's findings.

I wish to sincerely express my appreciation to the people who contributed towards the development of this book, particularly, REB staff, UR-CE Lecturers, TTC Tutors, Teachers from general education and experts from Local and international Organizations for their technical support. A word of gratitude goes also to the Head Teachers and TTCs principals who availed their staff for various activities.

Dr. NDAYAMBAJE Irénée

Director General, REB

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I wish to express my appreciation to the people who played a major role in the development of this tutor`s guide for Year three Mathematics in the option of Social Studies Education (SSE). It would not have been successful without active participation of different education stakeholders.

I owe gratitude to different universities and schools in Rwanda that allowed their staff to work with REB in the in-house textbooks production initiative.

I wish to extend my sincere gratitude to lecturers and teachers whose efforts during writing exercise of this tutor`s guide was very much valuable.

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PART I. GENERAL INTRODUCTION

1.1 The structure of the guide

The tutor's guide of Mathematics is composed of three parts:

The Part I concerns general introduction that discusses methodological guidance on how best to teach and learn Mathematics, developing competences in teaching and learning, addressing cross-cutting issues in teaching and learning and Guidance on assessment.

Part II presents a sample lesson plan. This lesson plan serves to guide the tutor on how to prepare a lesson in Mathematics.

The Part III is about the structure of a unit and the structure of a lesson. This includes information related to the different components of the unit and these components are the same for all units. This part provides information and guidelines on how to facilitate student teachers while working on learning activities. More other, all application activities from the textbook have answers in this part.

1.2 Methodological guidance

1.2.1 Developing competences

Since 2015 Rwanda shifted from a knowledge based to a competency-based curriculum for pre-primary, primary, secondary education and recently the TTC curriculum. This called for changing the way of learning by shifting from teacher centred to a learner centred approach. Teachers are not only responsible for knowledge transfer but also for fostering learners' learning achievement and creating safe and supportive learning environment. It implies also that learners have to demonstrate what they are able to transfer the acquired knowledge, skills, values and attitude to new situations.

The competence-based curriculum employs an approach of teaching and learning based on discrete skills rather than dwelling on only knowledge or the cognitive domain of learning. It focuses on what learner can do rather than what learner knows. Learners develop competences through subject unit with specific learning objectives broken down into knowledge, skills and attitudes through learning activities.

In addition to the competences related to Mathematics, student teachers also develop generic competences which should promote the development of the higher order thinking skills and professional skills in Mathematics teaching.

Generic competences are developed throughout all units of Mathematics as follows:

Generic competences	Ways of developing generic competences
Critical thinking	All activities that require learners to calculate, convert, interpret, analyse, compare and contrast, etc have a common factor of developing critical thinking into learners
Creativity and innovation	All activities that require learners to plot a graph of a given algebraic data, to organize and interpret statistical data collected and to apply skills in solving problems of economics have a common character of developing creativity into learners
Research and problem solving	All activities that require learners to make a research and apply their knowledge to solve problems from the real-life situation have a character of developing research and problem solving into learners.
Communication	During Mathematics class, all activities that require learners to discuss either in groups or in the whole class, present findings, debate …have a common character of developing communication skills into learners.
Co-operation, interpersonal relations and life skills	All activities that require learners to work in pairs or in groups have character of developing cooperation and life skills among learners.
Lifelong learning	All activities that are connected with research have a common character of developing into learners a curiosity of applying the knowledge learnt in a range of situations. The purpose of such kind of activities is for enabling learners to become life-long learners who can adapt to the fast-changing world and the uncertain future by taking initiative to update knowledge and skills with minimum external support.
Professional skills	Specific instructional activities and procedures that a teacher may use in the class room to facilitate, directly or indirectly, students to be engaged in learning activities. These include a range of teaching skills: the skill of questioning, reinforcement, probing, explaining, stimulus variation, introducing a lesson; illustrating with examples, using blackboard, silence and nonverbal cues, using audio – visual aids, recognizing attending behaviour and the skill of achieving closure.

The generic competences help learners deepen their understanding of Mathematics and apply their knowledge in a range of situations. As students develop generic competences, they also acquire the set of skills that employers look for in their employees, and so the generic competences prepare students for the world of work.

1.2.2 Addressing cross cutting issues

Among the changes brought by the competence-based curriculum is the integration of cross cutting issues as an integral part of the teaching learning process-as they relate to and must be considered within all subjects to be appropriately addressed. The eight cross cutting issues identified in the national curriculum framework are: *Comprehensive Sexuality Education, Environment and Sustainability, Financial Education, Genocide studies, Gender, Inclusive Education, Peace and Values Education, and Standardization Culture*.

Some cross-cutting issues may seem specific to particular learning areas/ subjects but the teacher need to address all of them whenever an opportunity arises. In addition, learners should always be given an opportunity during the learning process to address these cross-cutting issues both within and out of the classroom.

Below are examples of how crosscutting issues can be addressed:

Cross-Cutting Issue	Ways of addressing cross-cutting issues
Comprehensive Sexuality Education: The primary goal of introducing Comprehensive Sexuality Education program in schools is to equip children, adolescents, and young people with knowledge, skills and values in an age appropriate and culturally gender sensitive manner so as to enable them to make responsible choices about their sexual and social relationships, explain and clarify feelings, values and attitudes, and promote and sustain risk reducing behaviour.	Using different charts and their interpretation, Mathematics tutor should lead students to discuss the following situations: “Alcohol abuse and unwanted pregnancies” and advise student teachers on how they can instil learners to fight those abuses. Some examples can be given when learning statistics, powers, logarithms and their properties.

Environment and Sustainability: Integration of Environment, Climate Change and Sustainability in the curriculum focuses on and advocates for the need to balance economic growth, society well-being and ecological systems. Learners need basic knowledge from the natural sciences, social sciences, and humanities to understand to interpret principles of sustainability.	Using Real life models or students' experience, Mathematics Tutor should lead student teachers to illustrate the situation of "population growth" and discuss its effects on the environment and sustainability.
Financial Education: The integration of Financial Education into the curriculum is aimed at a comprehensive Financial Education program as a precondition for achieving financial inclusion targets and improving the financial capability of Rwandans so that they can make appropriate financial decisions that best fit the circumstances of one's life.	Through different examples and calculations on interest rate problems, total revenue and total cost, Mathematics Tutor can lead student teachers to discuss how to make appropriate financial decisions.
Gender: At school, gender will be understood as family complementarities, gender roles and responsibilities, the need for gender equality and equity, gender stereotypes, gender sensitivity, etc.	Mathematics Tutor should address gender as cross-cutting issue through assigning leading roles in the management of groups to both girls and boys and providing equal opportunity in the lesson participation and avoid any gender stereotype in the whole teaching and learning process.
Inclusive Education: Inclusion is based on the right of all learners to a quality and equitable education that meets their basic learning needs and understands the diversity of backgrounds and abilities as a learning opportunity.	Firstly, Mathematics Tutors need to identify/recognize students with special needs. Then by using adapted teaching and learning resources while conducting a lesson and setting appropriate tasks to the level of students, they can cater for students with special education needs. They must create opportunity where student teachers can discuss how to cater for learners with special educational needs.

Peace and Values Education: Peace and Values Education (PVE) is defined as education that promotes social cohesion, positive values, including pluralism and personal responsibility, empathy, critical thinking and action in order to build a more peaceful society.	Through a given lesson, a tutor should: • Set a learning objective which is addressing positive attitudes and values, • Encourage students to develop the culture of tolerance during discussion and to be able to instil it in colleagues and cohabitants; • Encourage students to respect ideas for others.
Standardization Culture: Standardization Culture in Rwanda will be promoted through formal education and plays a vital role in terms of health improvement, economic growth, industrialization, trade and general welfare of the people through the effective implementation of Standardization, Quality Assurance, Metrology and Testing.	With different word problems related to the effective implementation of Standardization, Quality Assurance, Metrology and Testing, students can be motivated to be aware of health improvement, economic growth, industrialization, trade and general welfare of the people.

1.2.3 Guidance on how to help students with special education needs in classroom

In the classroom, students learn in different way depending to their learning pace, needs or any other special problem they might have. However, the teacher has the responsibility to know how to adopt his/her methodologies and approaches in order to meet the learning need of each student in the classroom. Also teachers need to understand that student with special needs, haveto be taught differently or need some accommodations to enhance the learning environment. This will be done depending onthe subject and the nature of the lesson.

In order to create a well-rounded learning atmosphere, teachers need to:

- Remember that learners learn in different ways so they have to offer a variety of activities (e.g. role-play, music and singing, word games and quizzes, and outdoor activities);
- Maintain an organized classroom and limits distraction. This will help learners with special needs to stay on track during lesson and follow instruction easily;
- Vary the pace of teaching to meet the needs of each child. Some learners process information and learn more slowly than others;

- Break down instructions into smaller, manageable tasks. Learners with special needs often have difficulty understanding long-winded or several instructions at once. It is better to use simple, concrete sentences in order to facilitate them understand what you are asking.
- Use clear consistent language to explain the meaning (and demonstrate or show pictures) if you introduce new words or concepts;
- Make full use of facial expressions, gestures and body language;
- Pair a learner who has a disability with a friend. Let them do things together and learn from each other. Make sure the friend is not over protective and does not do everything for the one with disability. Both learners will benefit from this strategy;
- Use multi-sensory strategies. As all learners learn in different ways, it is important to make every lesson as multi-sensory as possible. Learners with learning disabilities might have difficulty in one area, while they might excel in another. For example, use both visual and auditory cues.

Below are general strategies related to each main category of disabilities and how to deal with every situation that may arise in the classroom. However, the list is not exhaustive because each child is unique with different needs and that should be handled differently.

Strategy to help learners with developmental impairment:

- Use simple words and sentences when giving instructions;
- Use real objects that learners can feel and handle. Rather than just working abstractly with pen and paper;
- Break a task down into small steps or learning objectives. The learner should start with an activity that she/he can do already before moving on to something that is more difficult;
- Gradually give the learner less help;
- Let the learner with disability work in the same group with those without disability.

Strategy to help learners with visual impairment:

- Help learners to use other senses (hearing, touch, smell and taste) and carry out activities that will promote their learning and development;
- Use simple, clear and consistent language;
- Use tactile objects to help explain a concept;
- If the learner has some sight, ask him/her what he/she can see;
- Make sure the learner has a group of friends who are helpful and who allow him/her to be as independent as possible;
- Plan activities so that learners work in pairs or groups whenever possible;

Strategy to help learners with hearing disabilities or communication difficulties

- Always get the learner 's attention before you begin to speak;
- Encourage the learner to look at your face;
- Use gestures, body language and facial expressions;
- Use pictures and objects as much as possible.
- Keep background noise to a minimum.

Strategies to help learners with physical disabilities or mobility difficulties:

- Adapt activities so that learners, who use wheelchairs or other mobility aids, can participate.
- Ask parents/caregivers to assist with adapting furniture e.g. the height of a table may need to be changed to make it easier for a learner to reach it or fit their legs or wheelchair under;
- Encourage peer support when needed;
- Get advice from parents or a health professional about assistive devices if the learner has one.

Adaptation of assessment strategies:

At the end of each unit, the tutor is advised to provide additional activities to help students achieve the key unit competence. These assessment activities are for remedial, consolidation and extension designed to cater for the needs of all categories of students; slow, average and gifted students respectively. Therefore, the tutor is expected to do assessment that fits individual student.

Remedial activities	After evaluation, slow students are provided with lower order thinking activities related to the concepts learnt to facilitate them in their learning. These activities can also be given to assist deepening knowledge acquired through the learning activities for slow students.
Consolidation activities	After introduction of any concept, a range number of activities can be provided to all students to enhance/ reinforce learning.
Extended activities	After evaluation, gifted and talented students can be provided with high order thinking activities related to the concepts learnt to make them think deeply and critically. These activities can be assigned to gifted and talented students to keep them working while other students are getting up to required level of knowledge through the learning activity.

1.2.4. Guidance on assessment

Assessment is an integral part of teaching and learning process. The main purpose of assessment is for improvement of learning outcomes. Assessment for learning/ Continuous/ formative assessment intends to improve students' learning and tutor's teaching whereas assessment of learning/summative assessment intends to improve the entire school's performance and education system in general.

Continuous/ formative assessment

It is an on-going process that arises during the teaching and learning process. It includes lesson evaluation and end of sub unit assessment. This formative assessment should play a big role in teaching and learning process. The teacher should encourage individual, peer and group evaluation of the work done in the classroom and uses appropriate competence-based assessment approaches and methods.

Formative assessment is used to:

- Determine the extent to which learning objectives are being achieved and competences are being acquired and to identify which students need remedial interventions, reinforcement as well as extended activities. The application activities are developed in the learner book and they are designed to be given as remedial, reinforcement, end lesson assessment, homework or assignment
- Motivate students to learn and succeed by encouraging students to read, or learn more, revise, etc.
- Check effectiveness of teaching methods in terms of variety, appropriateness, relevance, or need for new approaches and strategies. Mathematics tutors need to consider various aspects of the instructional process including appropriate language levels, meaningful examples, suitable methods and teaching aids/ materials, etc.
- Help students to take control of their own learning.

In teaching Mathematics, formative or continuous assessment should compare performance against instructional objectives. Formative assessment should measure the student's ability with respect to a criterion or standard. For this reason, it is used to determine what students can do, rather than how much they know.

Summative assessment

The assessment can serve as summative and informative depending to its purpose. The end unit assessment will be considered summative when it is done at end of unit and want to start a new one.

It will be formative assessment, when it is done in order to give information on the progress of learners and from there decide what adjustments need to be done.

The assessment done at the end of the term, end of year, is considered as summative assessment so that the teacher, school and parents are informed of the achievement of educational objective and think of improvement strategies. There is also end of level/ cycle assessment in form of national examinations.

When carrying out assessment?

Assessment should be clearly visible in lesson, unit, term and yearly plans.

- Before learning (diagnostic): At the beginning of a new unit or a section of work; assessment can be organized to find out what students already know / can do, and to check whether the students are at the same level.
- During learning (formative/continuous): When students appear to be having difficulty with some of the work, by using on-going assessment (continuous). The assessment aims at giving students support and feedback.
- After learning (summative): At the end of a section of work or a learning unit, the Mathematics Tutor has to assess after the learning. This is also known as Assessment of Learning to establish and record overall progress of students towards full achievement. Summative assessment in Rwandan schools mainly takes the form of written tests at the end of a learning unit or end of the month, and examinations at the end of a term, school year or cycle.

Instruments used in assessment.

- **Observation:** This is where the Mathematics tutor gathers information by watching students interacting, conversing, working, playing, etc. A tutor can use observations to collect data on behaviours that are difficult to assess by other methods such as attitudes, values, and generic competences and intellectual skills. It is very important because it is used before the lesson begins and throughout the lesson since the tutor has to continue observing each and every activity.

- **Questioning**

(a) Oral questioning: a process which requires a student to respond verbally to questions

(b) Class activities/ exercises: tasks that are given during the learning/ teaching process

(c) Short and informal questions usually asked during a lesson

(d) Homework and assignments: tasks assigned to students by their tutors to be completed outside of class.

Homework assignments, portfolio, project work, interview, debate, science fair, Mathematics projects and Mathematics competitions are also the different forms/instruments of assessment.

1.2.5. Teaching methods and techniques that promote active learning

The different learning styles for students can be catered for, if the teacher uses active learning whereby learners are really engaged in the learning process.

The main teaching methods used in mathematics are the following:

- **Dogmatic method** (the teacher tells the students what to do, What to observe, How to attempt, How to conclude)
- **Inductive-deductive method:** Inductive method is to move from specific examples to generalization and deductive method is to move from generalization to specific examples.
- **Analytic-synthetic method:** Analytic method proceeds from unknown to known, 'Analysis' means 'breaking up' of the problem in hand so thatit ultimately gets connected with something obvious or already known. Synthetic method is the opposite of the analytic method. Here one proceeds from known to unknown.
- **Skills Laboratory method:** Laboratory method is based on the maxim "learning by doing." It is a procedure for stimulating the activities of the students and to encourage them to make discoveries through practical activities.
- **Problem solving method, Project method and Seminar Method.**

The following are some active techniques to be used in Mathematics:

- Group work
- Research
- Probing questions
- Practical activities (drawing, plotting, interpreting graphs)
- Modelling
- Brainstorming
- Quiz Technique
- Discussion Technique
- Scenario building Technique

What is Active learning?

Active learning is a pedagogical approach that engages learners in doing things and thinking about the things they are doing. Learners play the key role in the active learning process. They are not empty vessels to fill but people with ideas, capacity and skills to build on for effective learning. Thus, in active learning, learners are encouraged to bring their own experience and knowledge into the learning process.

The role of the teacher in active learning	The role of learners in active learning
• The teacher engages learners through active learning methods such as inquiry methods, group discussions, research, investigative activities, group and individual work activities. • He/she encourages individual, peer and group evaluation of the work done in the classroom and uses appropriate competence-based assessment approaches and methods. • He provides supervised opportunities for learners to develop different competences by giving tasks which enhance critical thinking, problem solving, research, creativity and innovation, communication and cooperation. • Teacher supports and facilitates the learning process by valuing learners' contributions in the class activities.	A learner engaged in active learning: • Communicates and shares relevant information with peers through presentations, discussions, group work and other learner-centred activities (role play, case studies, project work, research and investigation); • Actively participates and takes responsibility for his/her own learning; • Develops knowledge and skills in active ways; • Carries out research/investigation by consulting print/online documents and resourceful people, and presents their findings; • Ensures the effective contribution of each group member in assigned tasks through clear explanation and arguments, critical thinking, responsibility and confidence in public speaking • Draws conclusions based on the findings from the learning activities.

Main steps for a lesson in active learning approach

All the principles and characteristics of the active learning process highlighted above are reflected in steps of a lesson as displayed below. Generally, the lesson is divided into three main parts whereby each one is divided into smaller steps to make sure that learners are involved in the learning process. Below are those main part and their small steps:

1) Introduction

Introduction is a part where the teacher makes connection between the current and previous lesson through appropriate technique. The teacher opens short discussions to encourage learners to think about the previous learning experience and connect it with the current instructional objective. The teacher reviews the prior knowledge, skills and attitudes which have a link with the new concepts to create good foundation and logical sequencings.

2) Development of the new lesson

The development of a lesson that introduces a new concept will go through the following small steps: discovery activities, presentation of learners' findings, exploitation, synthesis/summary and exercises/application activities.

- **Discovery activity**

Step 1:

- The teacher discusses convincingly with learners to take responsibility of their learning
- He/she distributes the task/activity and gives instructions related to the tasks (working in groups, pairs, or individual to instigate collaborative learning, to discover knowledge to be learned).

Step 2:

- The teacher let learners work collaboratively on the task;
- During this period the teacher refrains to intervene directly on the knowledge;
- He/she then monitors how the learners are progressing towards the knowledge to be learned and boosts those who are still behind (but without communicating to them the knowledge).

- **Presentation of learners' findings/productions**
  - In this episode, the teacher invites representatives of groups to present their productions/findings.
  - After three/four or an acceptable number of presentations, the teacher decides to engage the class into exploitation of learners' productions.
- **Exploitation of learner's findings/ productions**
  - The teacher asks learners to evaluate the productions: which ones are correct, incomplete or false
  - Then the teacher judges the logic of the learners' products, corrects those which are false, completes those which are incomplete, and confirms those which are correct.

- **Institutionalization or harmonization (summary/conclusion/ and examples)**
  - The teacher summarizes the learned knowledge and gives examples which illustrate the learned content.
- **Application activities**
  - Exercises of applying processes and products/objects related to learned unit/sub-unit
  - Exercises in real life contexts
  - Teacher guides learners to make the connection of what they learnt to real life situations.

At this level, the role of teacher is to monitor the fixation of process and product/ object being learned.

3) Assessment

In this step the teacher asks some questions to assess achievement of instructional objective. During assessment activity, learners work individually on the task/activity. The teacher avoids intervening directly. In fact, results from this assessment inform the teacher on next steps for the whole class and individuals. In some cases, the teacher can end with a homework/ assignment. Doing this will allow learners to relay their understanding on the concepts covered that day. Teacher leads them not to wait until the last minute for doing the homework as this often results in an incomplete homework set and/or an incomplete understanding of the concept.

PART II: SAMPLE LESSON

Sample Lesson for unit 1:Logarithmic and exponential functions

School Name: ….Teacher's name: …..

Term	Date	Subject	Class	Unit N°	Lesson N°	Duration	Class size
II	…. /…./2019	Mathematics	Y3SSE	1	4 of 13	40 min	…….

Type of special educational needs to be catered for in this lesson and number of learners in each category	3 slow student-teachers and 2 low vision student-teachers
Unit title	Logarithmic and exponential functions
Key unit competency:	Apply logarithmic and exponential functions to model and solve problems about interest rates and population growth
Title of the lesson	Differentiation of logarithmic functions
Instructional Objective	Given the formula for derivative of logarithmic functions, student-teachers will be able to determine the differentiation of logarithmic functions of the given logarithmic function correctly.
Plan for this Class (location: in / outside)	The lesson is held indoors, the class is organized into groups ,3 slow student-teachers are scattered in different groups ,and 2 low vision student-teachers seat on the front desks near the blackboard in order to see and participate fully in all activities
Learning Materials (for ALL learners)	Flash cards contain formula for logarithmic functions, Textbooks and charts containing formula and examples to interpret in the course of the lesson. calculator, Manila paper, markers, pens, pencils, T-square, graph papers
References	• TTC syllabus of Mathematics of social studies education. • Student-teachers' Mathematics textbook and Teacher's guide

<table>
<tr><th>Timing for each step</th><th colspan="2">Description of teaching and learning activity</th><th>Generic competences and cross cutting issues to be addressed + a short explanation</th></tr>
<tr><td></td><td colspan="2">
• Two student-teachers chosen randomly present, one after another, their findings about activity found in their books, and they interact under the facilitation of the tutor in order to link the presentation to the lesson of the day.
• Student-teachers are organized into groups to determine differentiation of logarithmic functions by referring to activity and the examples, then presents the findings and the student-teachers interact with the guidance and harmonization of tutor.
• Finally, the student-teachers are assigned to individual tasks, and the correction is done on the chalk board.</td><td></td></tr>
<tr><td></td><td>Teacher activities</td><td>Learner activities</td><td></td></tr>
<tr><td>Introduction:

5 minutes</td><td>In groups, tutor Invites student-teachers to read and discuss in pairs the activity 1.1.4 found in their Mathematics books and share findings to their classmates

Tutor move around and guide student-teachers to harmonize the results and link the introduction to the lesson of the day.</td><td>Student-teachers read and discuss in pairs the activity 1.1.4 found in their mathematics books and share findings to their classmates

In pairs, Student-teachers harmonize their results from their readings on activity 1.1.4</td><td>Communication skills developed as student-teachers sharing ideas on the activity 1.1.4

Gender addressed when both girls and boys work together in the same group

Cooperation developed when student-teachers discuss the activity 1.1.4</td></tr>
</table>

Development of the lesson **Discovery activity:** **10 minutes**	Tutor organizes the student-teachers into groups and ask them to work out activity 1.1.4and gives instructions related to the task (organization of the group, role of each member, duration, presentation) Move around in class for facilitating student-teachers where necessary and give more clarification on how to determine the derivative of the function Guide student-teachers to work through examples in their books.	-Student-teachers follow the instructions and in small groups they work out the activity 1.1.4 Each pair analyses and discuss the given task under the guidance of the tutor Student-teachers work out through the exercises prepared in their books related to derivative/ differentiation of logarithmic function.	Cooperation and communication skills through discussions Peace and values education; Cooperation, mutual respect, tolerance through discussions with people with different views and respect one's views during discussions.
Presentation of learner's findings and exploitation: **15minutes**	• Invite one member from each group with different working steps to present their work where they must explain the working steps; • Tutor encourages student-teachers to follow attentively and guide them to answers	• Student-teachers present their findings: **Expected answers** **(**Refer to solution of activity 1.1.4, in Teacher's guide)	Cooperation and communication/ attentive listening during presentations and group discussions Critical thinking through evaluating other's findings

	• Tutor asks student-teachers to amend the presentation and to evaluate their work • Tutor harmonizes the results by highlighting the derivative of the function.	• Student-teachers follow the presentation and ask questions for more clarification. • student-teachers evaluate the findings of others • student-teachers evaluate their own findings about derivative of logarithmic functions	Problem solving developed when they evaluate their own findings about derivative of logarithmic functions
Conclusion/ **Summary** **5 minutes**	• Teacher facilitates the student-teachers to capture the main points of the derivative of logarithmic functions. • Teacher requests learners to write down the main points in their notebooks • Invite student-teachers to discuss the use of derivative of logarithmic functions in real life in advance	• student-teachers have the main points on the derivative of the function highlighting $\frac{d}{dx}\left(\ln(u(x))\right)$, $\frac{d}{dx}\left(\log_a x\right)$, $\frac{d}{dx}\left(\log_a u(x)\right)$ • student-teachers write down the main points in their notebooks	• Critical thinking and problem solving skills are developed through analysing of where derivative of logarithmic functions can be used in real life. • Lifelong learning developed as student-teachers continue to do research on where they can apply differentiation of logarithmic function in real life.

Assessment **5 minutes**	• Invite student-teachers to work individually the application activity 1.1.4	• Student-teachers discuss the use of derivative of logarithmic functions in real life by referring to the lesson of the day. Individually student-teachers work out the application activity 1.1.4 and finally they make a correction on the chalk board. **Expected answers** **(**Refer to solution in TG)	Critical thinking developed as student-teachers use the formula to find out the derivative of logarithmic functions.
Observation on lesson delivery	To be completed after receiving the feed-back from the student-teachers (what did the student-teachers liked, what challenged them,)		

PART III: UNIT DEVELOPMENT

UNIT 1

LOGARITHMIC AND EXPONENTIAL FUNCTIONS

1.1 Key unit competence:

Apply logarithmic and exponential functions to model and solve problems about interest rates and population growth

1.2 Prerequisites

Student-teachers will easily learn this unit, if they have a good background on Arithmetic (Unit 1 Year 1), and on limits of functions (unit 4 Year 1).

1.3 Cross-cutting issues to be addressed

- **Inclusive education** (promote education for all while teaching) ;
- **Peace and value Education** (respect others' view and thoughts during class discussions)
- **Gender** (provide equal opportunity to boys and girls in the lesson) ;
- **Environment and Sustainability:** During the lesson on population growth, guide Learners to discuss the effect of the high rate of population growth;
- **Financial education**: guide student-teachers to discuss how to manage the mortgage loans taken from the bank;

1.4 Guidance on introductory activity

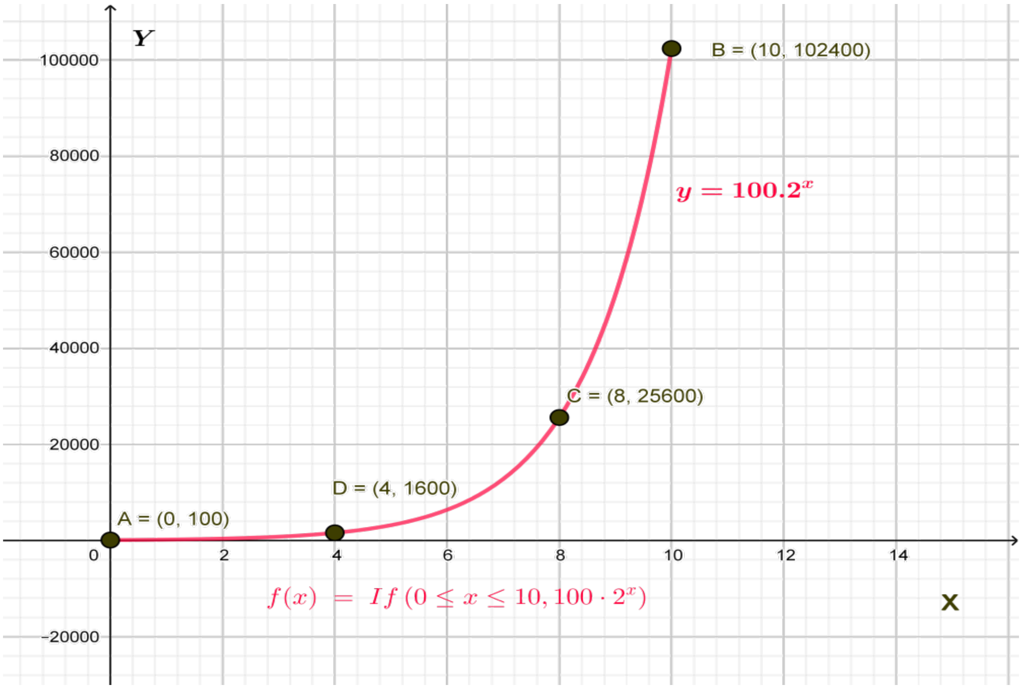
- Invite student-teachers to work in groups and give them instructions on how they can do the introductory activity found in unit 1 of student's book;
- Walk around all groups to provide pieces of advice where necessary.
- After a given time invite student-teachers to present their findings and harmonize them.
- Basing on student-teachers' experience, prior knowledge and abilities shown in answering the questions for this activity, use different questions to facilitate them to give their predictions and ensure that you arouse their curiosity on what is going to be leant in this unit.

Answer for introductory activity:

a) Student-teachers complete the table showing the money of the businessman from the 1st day up to 10th day.

Days	Amounts	USD
1st days	200USD	200
2nd day	$200x2 =$ $100x2x2 = 100x2^2$	400
3rd days	$100x2x2x2 = 100x2^3$	800
4th days	$100x2x2x2x2 = 100x2^4$	1600
10th day	$100x2x2x2x2... = 100x2^{10}$	102,400
Nth day	$100x2x2x2x2... = 100x2^n$	$100x2^n$

b) The graph plotted in a rectangular coordinate.



c) $f(n) = 100x2^n$ USD

- During the presentation let student-teachers discover the concept of exponential function $F(t)$ starting with the property of a function with powers. $F(t) = 100 \times 2^t$

- Student-teachers establish the function $Y(F)$ inverse of $F(t)$

$$Y(F) = F^{-1}(t) = \ln\left(\frac{t}{100}\right) = -\ln(100) + \ln t$$

$$Y(t) = -4.6 + \ln(t)$$

c) The economist wants to possess the money F under the same conditions, discuss how he/she can know the number of days necessary to get such money from the beginning of the business.

The economist wants to possess the money F, using the inverse function $Y(F) = -4.6 + \ln(F)$, she/he will use the equation $t = -4.6 + \ln(F)$ to calculate the number t of days required.

Conclude that $F(t)$ and $Y(t)$ are respectively exponential function and logarithmic functions that are needed to be well studied so that they may be used without problems. This unit deals with the behavior and properties of such essential functions and their application in real life situation.

1.5. List of lessons

#	Lesson title	Learning objectives	Number of periods
0.	Introductory activity	To arouse the curiosity of student teachers on the content of unit 1.	1
1	Definition and Domain of definition of Logarithmic functions	• State the restrictions on the base and the variable in a logarithmic function; • calculate the domain and range of a logarithmic function.	3
2	Limits of logarithmic functions	Calculate limit of a logarithmic function	3
3	Applications of limits of logarithmic functions to establish the continuity and asymptotes	Determine interval of continuity and equation of possible asymptotes of a logarithmic function.	3

4	Differentiation of logarithmic functions	Determine the derivative of a logarithmic function.	2
5	Application of derivative for logarithmic functions	• Apply derivative to investigate the variation of a logarithmic function. • Apply derivative of logarithmic function to remove indeterminate cases.	2
6	Definition and the domain of definition for Exponential functions	Calculate the domain and range of exponential function.	3
7	Limits of exponential functions	Calculate the limit of an exponential function	3
8	Applications of limits of exponential functions to establish the interval of continuity and equation of asymptotes	Determine interval of continuity and equation of possible asymptotes of an exponential function.	2
9	Differentiation of exponential functions	Determine the derivative of an exponential function	3
10	Application of derivative for exponential functions	• Apply derivative to investigate the variation of exponential function. • Apply derivative of exponential function to remove indeterminate cases.	2
11	Application of logarithmic and exponential functions to Interest rates problems	Apply logarithmic or exponential functions to solve problems related to interest rates.	3
12	Application of logarithmic and exponential functions to the Mortgage amount	Apply logarithmic or exponential functions to solve problems related to Mortgage amount or amortization of a loan.	2

13	Application of logarithmic and exponential functions to population growth problems	Apply logarithmic and exponential functions to solve problems related to Population growth	2
14	Application of logarithmic and exponential functions to alcohol and risk of car accident	Apply logarithmic functions to solve problems related to alcohol and risk of car accident	1
14	End unit Assessment		1
Total			**36**

Lesson 1: Definition and Domain of definition of Logarithmic functions

a) Learning objectives

- State the restrictions on the base and the variable in a logarithmic function
- Calculate the domain and range of a logarithmic function

b) Teaching resources

Learner's book and other reference books to facilitate research, calculator, Manila paper, markers, pens, pencils...

c) Prerequisites/Revision/Introduction

Student-teachers will easily learn this unit, if they have a good background on Arithmetic (Unit 1 Year 1), and on limits of functions (unit 4 Year 1).

d) Learning activities:

- Invite student-teachers to work in groups and do the activity 1.1.1 found in their Mathematics books;
- Move around in the class for facilitating student-teachers where necessary and give more clarification while discussing logarithmic functions
- Verify and identify groups with different working steps;
- Invite one member from each group with different working steps to present their work where they must explain the working steps;
- As a tutor, harmonize the findings from presentation and guide student-teachers to plot the graph of $\log_{10}(x)$ for $x > 0$
- Lead student-teachers to give observations about images found step by step for $x > 1, x = 1, 0 < x < 1$ and values $x < 0$;

- Use different probing questions and guide students to explore the content and examples given in the student-teacher's book and lead them to discover the definition of logarithmic function and determine the domain and range of a logarithmic function;
- Facilitate student-teachers to deduce the domain and the range for $f(x)=\log_{10}(x)$ then generalize for the function of type $y=\log_a(u(x))$ with $u(x)\geq 0, a\neq 0, a>1$ and then from their answers, write a short summary.
- After this step, guide student-teachers to do the application activity 1.1.1 and evaluate whether lesson objectives were achieved.

Answer for activity 1.1.1

a) Complete the table of values for $\log_{10}(x)$

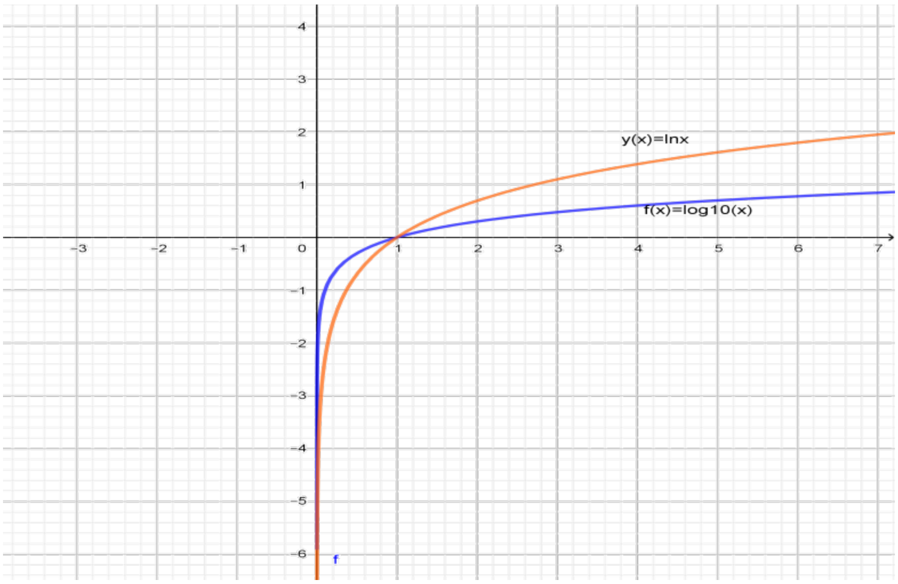
x	100	50	40	20	10	½	0.8	0.7	-5	-20	-30
$\log_{10}(x)$	2	1.69	1.6	1.30	10	-0.30	-0.09	0.15	-	-	-

b) The value of $\log_{10}(x)$ for $x<0$ do not exist in the set of real numbers.

c) Discuss the values of $\log_{10}(x)$ for $0<x<1, x=1$, and $x>1$.

$\log_{10}(1)=0$, $\log_{10}(x)<0\, for 0<x<1$ and $\log_{10}(x)>0\, for\ x>1$

d) The graph of $\log_{10}(x)$ for $x>0$



e) For $f:\mathbb{R}\to\mathbb{R}: x\mapsto \log_a x$,

$dom\ f = \{x \in \mathbb{R} : x > 0\} =]0, +\infty[= \mathbb{R}_0^+$ and range f $= \mathbb{R} =]-\infty, +\infty[$

Answer for application activity 1.1.1

1. a) $y = \log_3(x-2) + 4$ is defined for $x > 2$.

$Domf =]2, +\infty[$

To find the range we proceed as following: $y = \log_3(x-2) + 4 \Leftrightarrow y - 4 = \log_3(x-2)$ (for x in the domain) $\Leftrightarrow x - 2 = 3^{y-4} \Leftrightarrow x = 3^{y-4} + 2$

Since $3^{y-4} > 0\ \forall y \in \mathbb{R}$, we have $x = 3^{y-4} + 2 > 2$.

Thus the range is $\mathbb{R}$

b) $y = \log_5(8-2x)$ is defined only if $8 - 2x > 0 \Leftrightarrow -2x > -8 \Leftrightarrow x < 4$

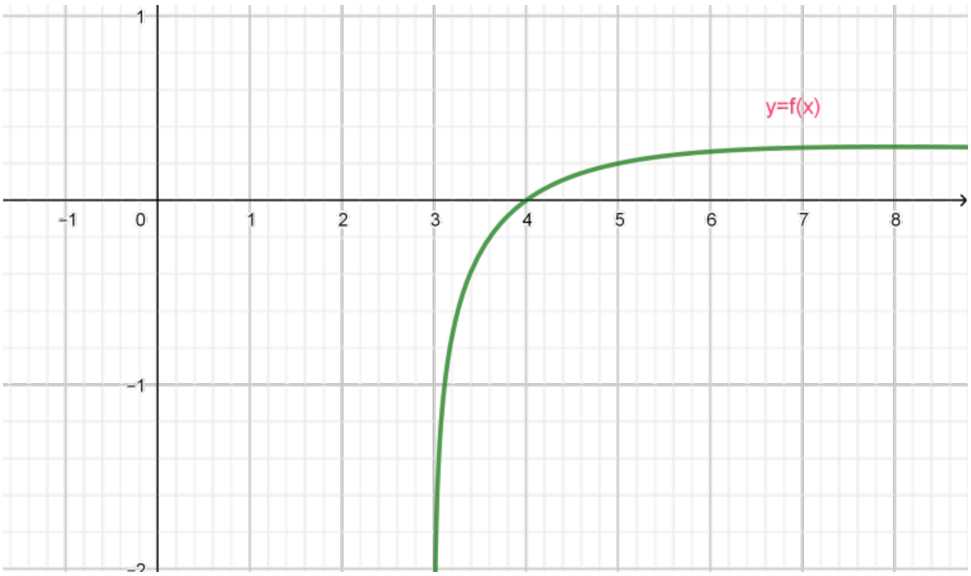
$Domf =]-\infty, 4[$

For the Range: $y = \log_5(8-2x) \Leftrightarrow 8 - 2x = 5^y \Leftrightarrow -2x = 5^y - 8 \Leftrightarrow x = 4 - 5^y$

$5^y > 0$ for all values of y implies $x = 4 - 5^y < 4$.

Thus, the range is $\mathbb{R}$.

2) From the graph



The domain of the function f is $Domf =]3, +\infty[$. The range is $\mathbb{R} =]-\infty, +\infty[$

Lesson 2: Limits of logarithmic functions

a) Learning objectives

Calculate limit of logarithmic function

b) Teaching resources

Student-teachers' book and other Reference books to facilitate research, calculator, Manila paper, markers, pens, pencils...

Textbooks, Ruler, T-square, Scientific calculators, graph papers. If possible, students may use mathematical software, such as Geogebra or Microsoft Excel to plot the graph of logarithmic functions.

c) Prerequisites/Revision/Introduction

Student-teachers will easily learn this unit, if they have a good background on Arithmetic (Unit 1 Year 1), and on limits of functions (unit 4 Year 1).

d) Learning activities:

- Invite student-teachers to work in groups and do the activity 1.1.2 found in their Mathematics books,
- Ask student-teachers to complete the given table and discuss how to determine the required limits.
- Move around to ensure all student-teachers in groups participate actively.
- Call upon groups to present their findings.
- Harmonize their findings by leading students to calculate $\lim_{x\to 0^+} \ln x$ and $\lim_{x\to +\infty} \ln x = +\infty$.
- As a tutor, harmonize the findings from presentation and guide students to use the limit of the logarithmic function at the infinity ;
- Use different probing questions and guide them to explore the content and examples given in the student's book and lead them to discover how to evaluate the limit of different logarithmic functions.
- After this step, guide students to do the application activity 1.1.2 and evaluate whether lesson objectives were achieved.

Answer for activity 1.1.2

x	0.5	0.001	0.0001	2	100	1001	10000
$y = \ln x$	-0.69	-6.90	-9.21	0.69	4.60	6.908	6.907

a) When the independent variable x takes values approaching 0 from the right, $y = \ln x$ takes the big negative values. We write $\lim_{x \to 0^+} \ln x = -\infty$. The graph of the function approaches the line of equation $x = 0$ considered as the vertical asymptote to the graph.

b) When x takes greater values, $y = \ln x$ takes also greater positive values.

Therefore, $\lim_{x \to +\infty} \ln x = +\infty$.

Answer for application activity 1.1.2

I. Evaluate the following limits

1) $\lim_{x \to +\infty} \ln(7x^3 - x^2 + 1) = +\infty$

2) $\lim_{x \to 1^+} \left(\ln \frac{1}{x-1} \right) = \lim_{x \to 1^+} \left[\ln 1 - \ln(x-1) \right] = \lim_{x \to 1^+} \ln 1 - \lim_{x \to 1^+} \ln\left(x-1\right) = 0 - (-\infty) = +\infty$

3) $\lim_{x \to 2^-} \log_5 \left(x^2 - 5x + 6\right) = -\infty$

4) $\lim_{a \to 4^+} \ln \frac{a}{\sqrt{a-4}} = \ln(\lim_{a \to 4^+} \frac{a}{\sqrt{a-4}}) = +\infty$

5) $\lim_{x \to +\infty} \ln\left(x^2 - 4x + 1\right) = +\infty$

6) $\lim_{x \to +\infty} \frac{2 + 4\log x}{x} = 0$

Lesson 3: Applications of Limits of logarithmic functions to continuity and asymptote

a) Learning objectives

Determine interval of continuity and equation of possible asymptotes of a logarithmic function

b) Teaching resources

student-teachers' book and other Reference books to facilitate research, calculator, Manila paper, markers, pens, pencils, T-square, graph papers. If possible, students may use mathematical software, such as Geogebra or Microsoft Excel to plot the graph of logarithmic functionsc)

Prerequisites/Revision/Introduction

Student-teachers will easily learn this unit, if they have a good background on Arithmetic (Unit 1 Year 1), and on limits of functions (unit 4 Year 1).

d) Learning activities:

- Invite student teachers to work in groups and do the activity 1.1.3 found in their Mathematics Student books and motivate them to complete table and deduce the continuity of a logarithmic function in a given point.
- Move around in the class for facilitating students where necessary and give more clarification
- Invite group representatives to present their findings, then help all student-teachers to conclude on the continuity and asymptote of a logarithmic function.
- As a tutor, harmonize the findings from presentation
- Use different probing questions and guide student-teachers to explore the content and examples given in the student's book and lead them to discover how to determine possible asymptotes of logarithmic function
- After this step, guide student-teachers to do individually the application activity 1.1.3 and evaluate whether lesson objectives were achieved.

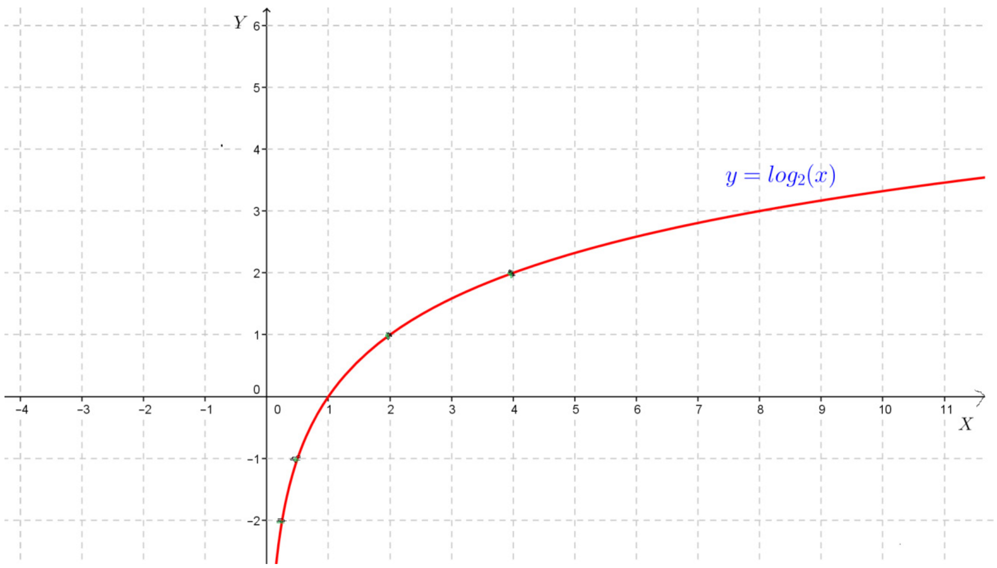
Answer for *activity 1.1.3*

1. Complete the table

$x = x_0$	$y = \log_2 x$	$\lim\limits_{x \to x_0} \log_2 x$
$\frac{1}{4}$	-2	-2
$\frac{1}{2}$	-1	-1
1	0	0
2	1	1
4	2	2

2. For all $x_0 > 0$, $\lim\limits_{x \to x_0} \log_2 x = \log_2 (x_0)$, therefore, $\log_2 x$ is continuous on $]0, +\infty[$

3. The graph of the function $y = \log_2 (x)$ can be plotted using a table of values completed using a calculator



For $a>0$, when $x \to 0, y \to -\infty$, ($\lim\limits_{x\to 0^+} \log_a x$), the line of equation $x=0$ (the y-axis) is an asymptote to the graph of $f(x)=\log_a x$.

4) The function is continuous on an interval I if $x_0 \in I, \lim\limits_{x\to x_0^+} f(x) = \lim\limits_{x\to x_0^-} f(x) = f(x_0)$

5) For $f(x)=\ln x$, $\forall x_0 \in]0,+\infty[$, $\lim\limits_{x\to x_0^+} \ln x = \lim\limits_{x\to x_0^-} \ln x = \ln(x_0)$. Therefore, $f(x)=\ln x$ is continuous on the interval $]0,+\infty[$.

$\lim\limits_{x\to 0^+} \log_2 = -\infty$, therefore, the line of equation $x=0$ (the y- axis) is an asymptote to the graph of $f(x)=\log_2 x$.

II. $\lim\limits_{x\to+\infty} \frac{\ln x}{x} = 0$, $\lim\limits_{x\to 0^+} \frac{\ln x}{x} = -\infty$, $\lim\limits_{x\to 1} \frac{\ln x}{x} = 0$, $\lim\limits_{x\to \frac{1}{5}} \left(\frac{\ln x}{x}\right) = \frac{\ln \frac{1}{5}}{\frac{1}{5}} = 5\left(\ln 5^{-1}\right) = -5\ln 5$.

Answer for application activity 1.1.3

1. Given the logarithmic function $y=-1+\ln(x+1)$

i) Vertical Asymptote has equation $x=-1$

ii) $Domf =]-1,+\infty[$

$Range = \mathbb{R} =]-\infty,+\infty[$

iii) $x - \text{int}\,ercept$

The x-intercept is obtained for $y = 0$.

$$0 = -1 + \ln(x+1)$$
$$\ln(x+1) = 1$$
$$\ln(x+1) = \ln e$$
$$x + 1 = e$$
$$x = e - 1$$

Thus, the x-intercept is $(e-1;0)$.

iv) $y - \text{int}\,ercept$

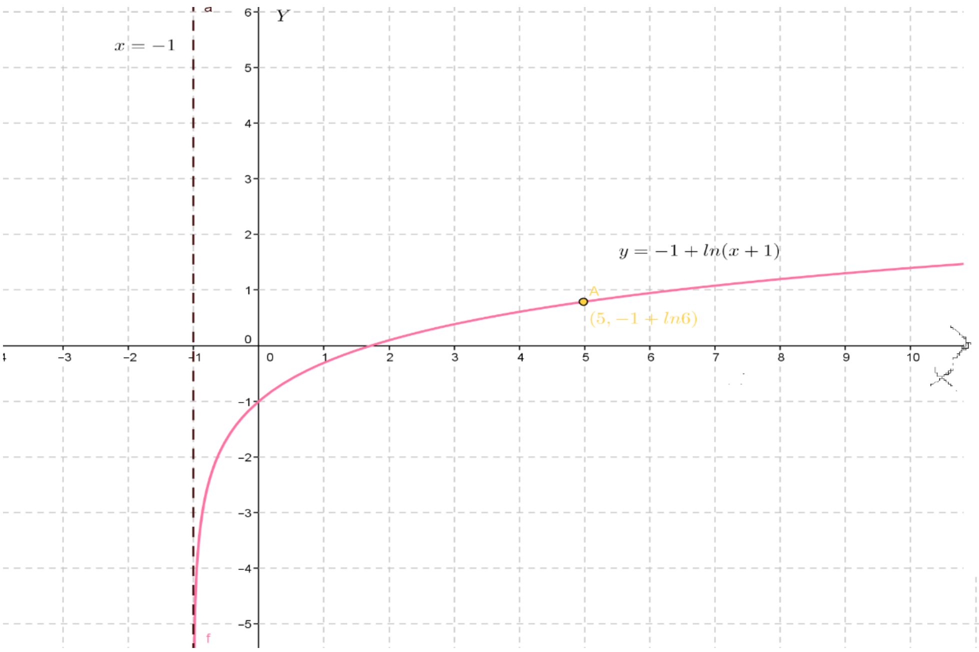
Let $x = 0$,

$$y = -1 + \ln(0+1)$$
$$y = -1 + \ln 1$$
$$y = -1 + 0$$
$$y = -1$$

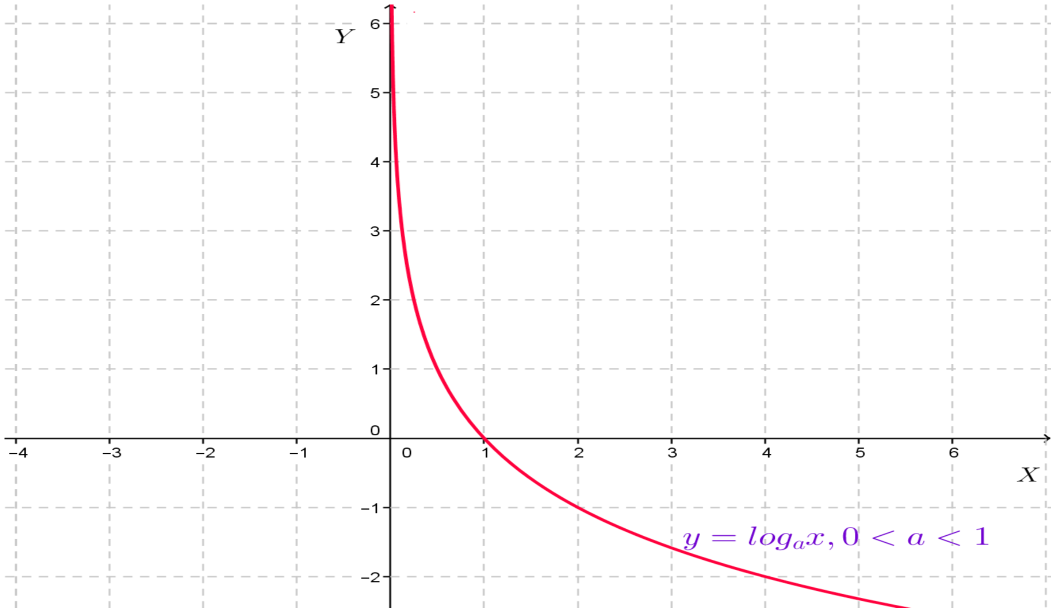
The y-intercept is (0,-1).

v) For example, when $x = 5, y = -1 + \ln(5+1) = -1 + \ln 6$ which gives the point $(5, -1 + \ln 6)$.

vi) The graph of $f(x) = y = -1 + \ln(x+1)$



The graph of the logarithmic function $f(x) = \log_a x, \quad 0 < a < 1$



Main characteristics of the logarithmic function $f(x) = \log_a x, \quad 0 < a < 1$

- The domain is $]0, +\infty[$ and $f(x)$ is continuous on this interval.
- The range is $\mathbb{R}$
- The graph intersects the $x - axis$ at $(1, 0)$

As $x \to 0, y \to +\infty$, so the line of equation $x = 0$ (the y- axis) is an asymptote to the curve of $f(x) = \log_a x, \quad 0 < a < 1$.

Lesson 4: Differentiation of logarithmic functions

a) Learning objectives

Determine the derivative of logarithmic functions.

b) Teaching resources

student-teachers' book and other Reference books to facilitate research, calculator, Manila paper, markers, pens, pencils, T-square, graph papers. If possible, students may use mathematical software, such as Geogebra or Microsoft Excel to plot the graph of logarithmic functionsc)

c) Prerequisites/Revision/Introduction

Student-teachers will easily learn this unit if they have a good background on Arithmetic (Unit 1 Year 1) and on limits of functions (unit 4 Year 1).

d) Learning activities:

- Invite student-teachers to work in groups and do the activity 1.1.4 found in their Mathematics books;
- Move around in the class for facilitating student-teachers where necessary and give more clarification on how to determine the derivative of the function $f(x)=\ln(x)$ in a point for which $x_0=2$ by the use of the definition of derivative of a function.
- Invite one member from each group with different working steps to present their work where they must explain the working steps;
- Harmonize the results by highlighting the following derivatives: $\frac{d(\ln x)}{dx}$, $\frac{d}{dx}(\ln(u(x))$, $\frac{d}{dx}(\log_a x)$ and $\frac{d}{dx}[\log_a u(x)]$.
- Guide student-teachers to work through examples in their books and work individually application activities 1.1.4 to assess the competences.

Answer for *activity 1.1.4*

h	$\frac{\ln(2+h)-\ln 2}{h}$
-0.1	0.5129329
-0.001 -0.00001	0.5001250≃ 1/2 0.5000013≃ ½
-0.0000001	0.5000000≃ ½
0.1	0.4879016
0.001	0.4998750≃ ½
0.00001	0.4999988≃ ½
0.0000001	0.50000002≃ ½

$f'(2)=\lim_{h\to 0}\frac{\ln(2+h)-\ln 2}{h}\simeq\frac{1}{2}$, these results reflect that $f'(x)=\lim_{h\to 0}\frac{\ln(x+h)-\ln x}{h}=\frac{1}{x}$

The number $f'(2)$ is the slope of the tangent line to the curve $y=f(x)=\ln x$ at the point $P(2,\ln 2)$.

Answer for application activity 1.1.4

1. $y=\ln\sqrt{\frac{1+x}{1-x}}$

Here $y=\ln\sqrt{\frac{1+x}{1-x}}=\ln\sqrt{1+x}-\ln\sqrt{1-x}=\frac{1}{2}\left[\ln(1+x)-\ln(1-x)\right]$

Therefore $\frac{dy}{dx}=\frac{1}{2}\left(\frac{1}{1+x}+\frac{1}{1-x}\right)=\frac{1}{1-x^2}$

2. To find the rate of climb (vertical velocity), we need to find the first derivative

$$\frac{d}{dt}\left[2000\ln\left(t+1\right)\right]=2000\frac{d}{dt}\ln(\text{t}+1)=\frac{2000}{t+1}$$

At $t=3$, we have $v=\frac{2000}{3+1}=\frac{2000}{4}=500$

Therefore the velocity is *500km/min*.

Lesson 5: Application of derivative of logarithmic functions

a) Learning objectives

- Apply derivative to investigate the variation of logarithmic functions.
- Apply derivative of logarithmic function to remove indeterminate cases.

b) Teaching resources

Student-teachers' books, ruler, T-square, scientific calculator.

c) Prerequisites/Revision/Introduction

Student-teachers will easily learn this unit, if they have a good background on Arithmetic (Unit 1 Year 1), and on limits of functions (unit 4 Year 1).

d) Learning activities:

- Invite student-teachers to work in groups and do the activity 1.1.5 found in their Mathematics Student books;
- Move around in the class for facilitating students where necessary and guide them to use the tables of signs for $f'(x)$ and $g'(x)$ so as to establish the variation of those functions on their domain;
- Invite one member from each group with different working steps to present their work where they must explain the working steps;

- As a tutor, harmonize the results emphasizing that the function $f\left(x\right)=\log_a x$ is strictly increasing on $\mathbb{R}_0^+$ for $a>1$ and that $f\left(x\right)=\log_a x$ is strictly decreasing on $\mathbb{R}_0^+$ for $0<a<1$.
- Guide student to use of derivative to remove indeterminate cases for logarithmic functions.
- Use different probing questions and guide them to explore the content and examples given in the student's book and lead them to discover how to study the variation of functions of the form $f(x)=\ln u(x)$ and $\mathrm{g}(x)=\log_a u(x)$ using their derivatives.
- After this step, guide students to do the application activity 1.1.5 and evaluate whether lesson objectives were achieved.

Answer for activity 1.1.5

1) $f\left(2\right)=0.693$ and $f\left(10\right)=2.303$, $g\left(2\right)=0.301$ and $g\left(10\right)=1$

Therefore, both functions $f(x)$ *and* $g(x)$ are increasing on the closed interval $\left[2,10\right]$ because $f\left(10\right)-f(2)>0$ *and* $g(10)-g(2)>0$

2) The variation table of $y=f(x)=\ln x$ and $f'(x)=\frac{1}{x}$ on the domain $]0,+\infty[$

x	0		e		$+\infty$
y'	+	+ + + + +	1/e	+ + + + +	0
y	$-\infty$	↗	1	↗	$+\infty$

The variation table of $y = g(x) = \log_{10} x$ and $g'(x) = \dfrac{1}{x\ln 10}$ on the domain $]0, +\infty[$

x	0		10		$+\infty$
y'		+ + + +	$\dfrac{1}{10(\ln 10)}$	+ + + + +	0
y	$-\infty$	$\nearrow$	1	$\nearrow$	$+\infty$

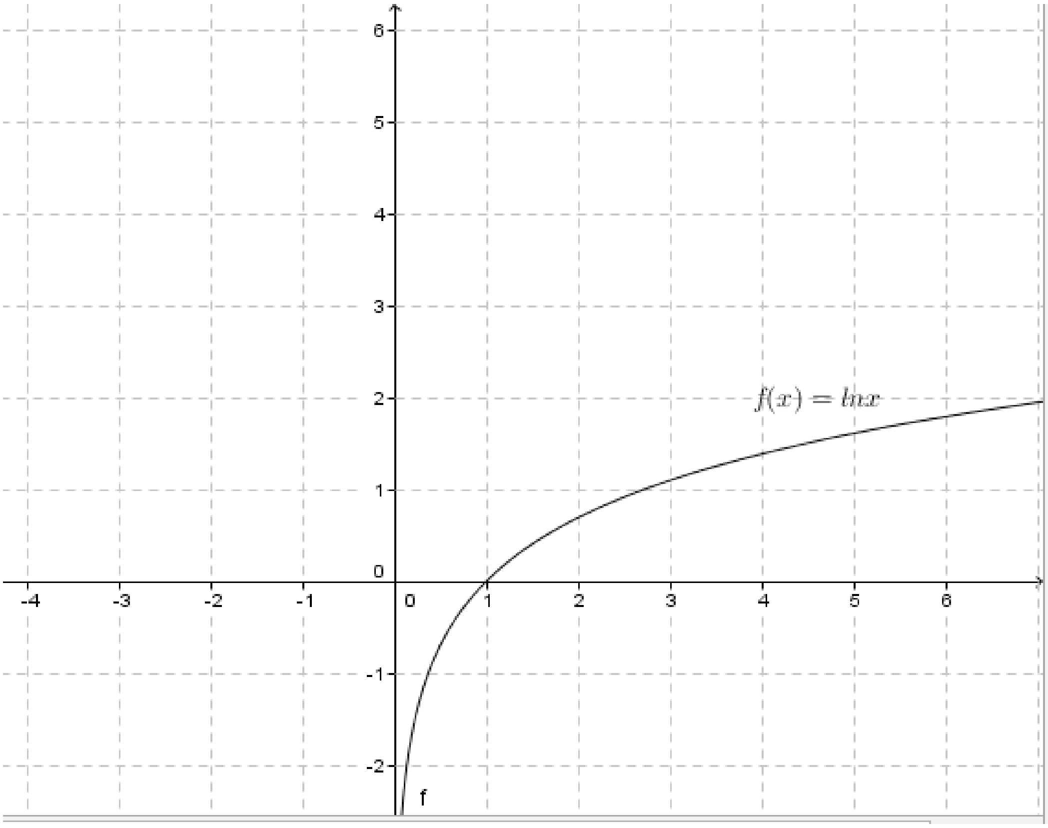
3. The difference $f(10) - f(2) = 1.61 > 0$ *and* $g(10) - g(2) = 0.699 > 0$, prove that the function f is increasing faster than g on the interval $[2,10]$.

4. $f''(x) = -\dfrac{1}{x^2}$

5.

x	0		e		$+\infty$
$f''(x) = -\dfrac{1}{x^2}$		-----------	$-\dfrac{1}{e^2}$	-----------	0
$f(x)$		∩∩∩∩∩∩	∩∩∩∩∩∩	∩∩∩∩∩∩	

The concavity of the function $f(x) = \ln x$ is turned down.



Answer for application activity 1.1.5

Variation of the function $f(x)=\frac{\ln(x-2)}{x-2}$

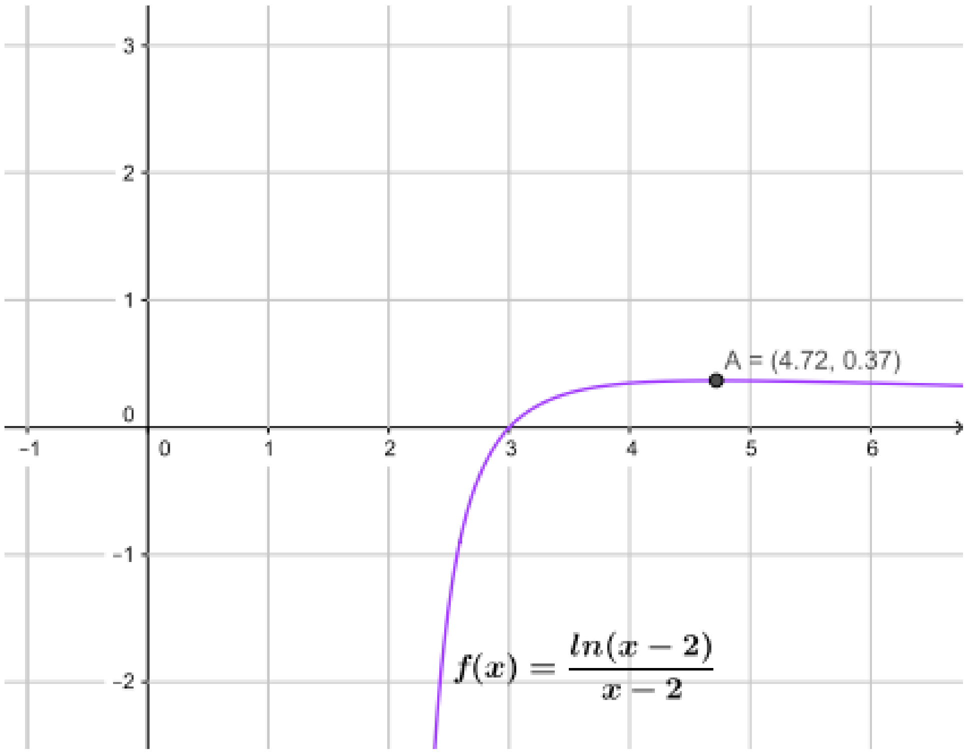
- $f(x)$ is defined $\Leftrightarrow x-2>0$. That if $x>2$.
- $Dom f=\left]2,+\infty\right[$
- $\lim\limits_{x\to 2^+}\frac{\ln(x-2)}{x-2}=-\infty$, we have a vertical asymptote with equation $x=2$
- $\lim\limits_{x\to+\infty}\frac{\ln(x-2)}{x-2}=0$, we have an horizontal asymptote with equation $y=0$
- $f'(x)=\frac{d}{dx}\left[\frac{\ln(x-2)}{x-2}\right]=\frac{\frac{1}{x-2}\times(x-2)-1\times\ln(x-2)}{(x-2)^2}=\frac{1-\ln(x-2)}{(x-2)^2}$
- $f'(x)=0\Leftrightarrow 1-\ln(x-2)=0$

$$\ln(x-2)=1$$
$$\ln(x-2)=\ln e$$
$$x-2=e$$
$$x=e+2$$

- $f(e+2)=\dfrac{\ln(e+2-2)}{e+2-2}=\dfrac{\ln e}{e}=\dfrac{1}{e}$
- Variation table of $f(x)$

x	2				$(e+2)$				$+\infty$
y'	+	+	+	+	0 -	-	-	-	-
y	$-\infty$	↗			$1/e$ Max[(e+2) ,1/e]		↘		0

Graph of f(x)



Lesson 6: Definition and the domain of definition for Exponential functions

a) Learning objectives

Calculate the domain and range of exponential function

b) Teaching resources

Student-teachers' books and other reference textbooks, ruler, T-square, scientific calculator; if possible, mathematical software and internet.

c) Prerequisites/Revision/Introduction

Student-teachers will easily learn this unit, if they have a good background on Arithmetic (Unit 1 Year 1), and on limits of functions (unit 4 Year 1).

d) Learning activities:

- Invite student-teachers to work in groups and do the activity 1.2.1 found in their Mathematics Student books;
- Move around in the class for facilitating student-teachers where necessary with the aim of establishing the domain of the function $g(x)=e^x$ inverse of $f(x)=\ln x$, the domain of $h(x)=3^x$ and their ranges.
- During group discussions, move around to each group and prompt them to discuss the domain and range of the function $p(x)=a^x$ in case $a>0, a\neq 1$ and $a=1$.
- Lead student-teachers to harmonize the results by generalizing how to find the domain and the range of the function $f(x)=a^{u(x)}$ where $u(x)$ is a function of x.
- Guide student-teachers to work through examples in their books and work individually application activities 1.2.1 to assess the competences.

Answer for activity 1.2.1

1. If f(x) = $\ln x$, let us complete the following table:

x	0	1	e	e^2	ln (3)	ln(4)
$g(x)=f^{-1}(x)$	1	e	e^e	e^{e^2}	3	4

The set of all values of $g(x)$ is composed of all positive real numbers; that is Range of $g(x)=\mathbb{R}^+=\left]0,+\infty\right[$.

2. Consider the function $h(x)=3^x$ and complete the following table

x	-10	-1	0	1	10
$h(x)=3^x$	$\frac{1}{3^{10}}$	$\frac{1}{3}$	1	3	3^{10}

a) $\forall x\in\mathbb{R}, h(x)\in\mathbb{R}^+$, the domain of $h(x)$ is $\mathbb{R}=]-\infty,+\infty[$

b) All values $h(x)$ are positive, therefore, the range of $h(x)$ is $\mathbb{R}^+=]0,+\infty[$.

Answer for application activity 1.2.1

1) $f(x)=5e^{2x}$,

$\forall x\in\mathbb{R}, f(x)x\in\mathbb{R}^+$, we realize that $domf=]-\infty,+\infty[$ and the range is the interval $]0,+\infty[$

2) $h(x)=2^{\ln x}$

$h(x)\in\mathbb{R}$ if $x>0$, therefore, $domh=]0,+\infty[$.

The range is the set of all $h(x)=2^{\ln x}, x\in\mathbb{R}^+$. That is range *h*= $\mathbb{R}^+=]0,+\infty[$

3. $g(x)=3^{\left(\frac{x+1}{x-2}\right)}$

Condition for the existence of $\frac{x+1}{x-2}$ in $\mathbb{R}$: $x\neq 2$.

Therefore, $Dom\ g=\mathbb{R}\setminus\{2\}=]-\infty,2[\cup]2,+\infty[$. Its range is

Lesson 7: Limits of exponential functions

a) Learning objectives

Calculate limit of exponential functions

b) Teaching resources

Student-teachers' book, calculator, ruler and T-square. If possible, mathematical software such us Geogebra, Microsoft Excel, Math lab and graphical can be used.

c) Prerequisites/Revision/Introduction

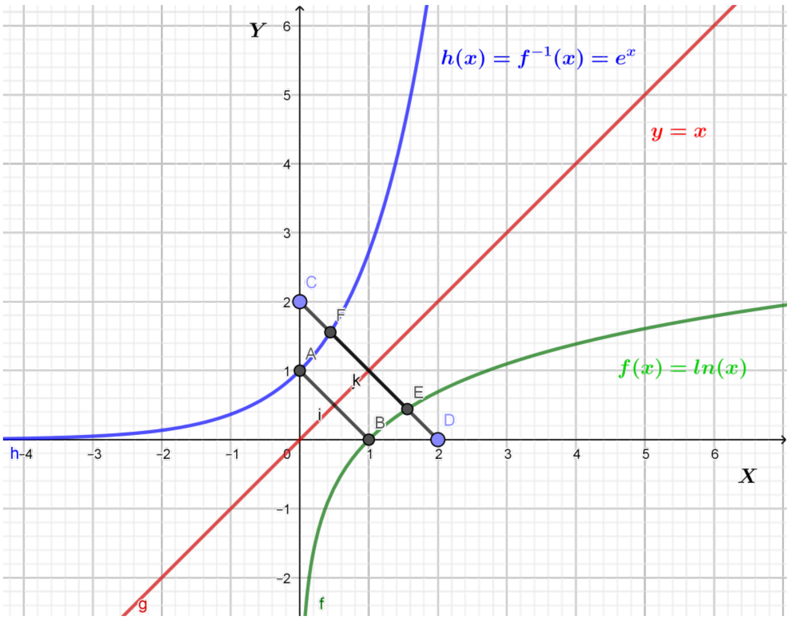
Student-teachers will easily learn this unit, if they have a good background on Arithmetic (Unit 1 Year 1), and on limits of functions (unit 4 Year 1).

d) Learning activities:

- Invite student-teachers to work in groups and do the activity 1.2.2 found in their Mathematics Student books;
- Move around in the class for facilitating students where necessary and give more clarification on on how to find the graph for the inverse of a given function and how to interpret the graph of $y = e^x$ to deduce $\lim_{x \to -\infty} e^x$ and $\lim_{x \to +\infty} e^x$.
- Invite representatives of groups to present their findings.
- Decide to engage the class into exploitation of students' findings.
- Judge the logic of the students' findings, correct those which are false, complete those which are incomplete, and confirm those which are correct and guide the students to conclude about $\lim_{x \to -\infty} a^x$ and $\lim_{x \to +\infty} a^x$ for any values of a.
- Give the summary of expected feedback based on student-teachers' answers.
- After this step, guide students to do the application activity 1.2.2 in pairs and evaluate whether lesson objectives were achieved.

Answer for activity 1.2.2

1. given the graph of $f(x)=\ln x$, the graph of its inverse $y=f^{-1}(x)=e^x$ is obtained by reflecting the graph of $f(x)=\ln x$ in the axis with equation $y=x$.



2) Based on the plotted graph above, as x decreases towards $-\infty$, the graph of e^x

approaches the line of equation $y=0$, therefore, $\lim\limits_{x\to-\infty} e^x = 0$ and the line $y=0$ is the horizontal asymptote.

As x increases towards $+\infty$, the images increase. Therefore, $\lim\limits_{x\to+\infty} e^x = +\infty$.

3. Applying properties of limits we have:

$$\lim_{x\to-\infty}\left(\frac{1}{2}\right)^x = \frac{1}{\lim\limits_{x\to-\infty}\left(2^x\right)} = +\infty$$

$$\lim_{x\to+\infty}\left(\frac{1}{2}\right)^x = \frac{1}{\lim\limits_{x\to+\infty}\left(2^x\right)} = 0$$

4. a) If a is less than one, $\lim\limits_{x\to-\infty} a^x = +\infty$ *and* $\lim\limits_{x\to+\infty} a^x = 0$

   b) If a is greater than one, $\lim\limits_{x\to-\infty} a^x = 0$ *and* $\lim\limits_{x\to+\infty} a^x = +\infty$.

   c) If a=1, the function is constant: y=1.

Answer for application activity 1.2.2

Evaluate limitof the function $f(x)$ at $+\infty$ and $-\infty$ in each of the following case.

1. $f(x)=e^{8+2x-x^3}$

$\lim\limits_{x\to+\infty} e^{8+2x-x^3}=0$ and $\lim\limits_{x\to-\infty} e^{8+2x-x^3}=+\infty$

2. $f(x)=e^{\frac{6x^2+x}{5+3x}}$

$\lim\limits_{x\to+\infty} e^{\frac{6x^2+x}{5+3x}}=e^{\lim\limits_{x\to+\infty}\frac{6x^2+x}{5+3x}}=+\infty$ and $\lim\limits_{x\to-\infty} e^{\frac{6x^2+x}{5+3x}}=e^{\lim\limits_{x\to-\infty}\frac{6x^2+x}{5+3x}}=0$

3. $f(x)=2e^{6x}-e^{-7x}-10e^{4x}$

$\lim\limits_{x\to+\infty}\left(2e^{6x}-e^{-7x}-10e^{4x}\right)=\lim\limits_{x\to+\infty} e^{6x}\left(2-e^{-13x}-10e^{-2x}\right)=+\infty$

$\lim\limits_{x\to\infty}\left(2e^{6x}-e^{-7x}-10e^{4x}\right)=\lim\limits_{x\to\infty} e^{-7x}\left(2e^{13x}-1-10e^{11x}\right)=-\infty$

4. $f(x)=3e^{-x}-8e^{-5x}-e^{10x}$

$\lim\limits_{x\to+\infty}\left(3e^{-x}-8e^{-5x}-e^{10x}\right)=\lim\limits_{x\to+\infty} e^{10x}\left(3e^{-11x}-8e^{-15x}-1\right)=-\infty$

$\lim\limits_{x\to-\infty}\left(3e^{-x}-8e^{-5x}-e^{10x}\right)=\lim\limits_{x\to-\infty} e^{-5x}\left(3e^{4x}-8-e^{15x}\right)=-\infty$

5. $f(x)=\dfrac{e^{-3x}-2e^{8x}}{9e^{8x}-7e^{-3x}}$

$\lim\limits_{x\to+\infty}\dfrac{e^{-3x}-2e^{8x}}{9e^{8x}-7e^{-3x}}=\dfrac{-2}{9}$ and

$\lim\limits_{x\to-\infty}\left(\dfrac{e^{-3x}-2e^{8x}}{9e^{8x}-7e^{-3x}}\right)=\lim\limits_{x\to-\infty}\dfrac{e^{-3x}-2e^{8x}}{9e^{8x}-7e^{-3x}}$

$=\lim\limits_{x\to-\infty}\dfrac{e^{-3x}\left(1-2e^{11x}\right)}{e^{-3x}\left(-7+9e^{11x}\right)}=-\dfrac{1}{7}$

Lesson 8: Applications of Limits to determine the continuity and asymptotes of exponential functions

a) Learning objectives

Determine interval of continuity and equation of possible asymptotes of alogarithmic function.

b) Teaching resources

Student-teachers' book, T-square, ruler, papers, if possible computers and Math draw software such as Geogebra, Microsoft Excel, Mathlab and Internet to facilitate research.

c) Prerequisites/Revision/Introduction

Student-teachers will easily learn this unit, if they have a good background on Arithmetic (Unit 1 Year 1), and on limits of functions (unit 4 Year 1).

d) Learning activities:

- Invite student-teachers to work in groups and do the activity 1.2.3 found in their Mathematics Student books;
- Move around in the class for facilitating students where necessary and give more clarification;
- Ask groups to present their findings to the whole class and then guide them to harmonize their works on the following points:
- Guide student-teachers in finding out and realizing that $f\left(x\right)$ is continuous on its domain which is the set of all real numbers ($Domf = \left]-\infty,+\infty\right[$) and that the range of $f(x)$ is $R = \left]0,+\infty\right[$.
- Invite student-teachers to find out that $\lim_{x\to-\infty} 2^{(x-2)} = 0$, $\lim_{x\to+\infty} f\left(x\right) = +\infty$ and $\lim_{x\to-\infty} \frac{f\left(x\right)}{x}$=0 and deduce that $y = 0$ is an horizontal asymptote to the graph of $f(x)$.
- Finally, invite students to find that $\lim_{x\to0^+} f\left(x\right) = \frac{1}{4}, \lim_{x\to0^-} f\left(x\right) = \frac{1}{4}, and\ f\left(0\right) = \frac{1}{4}$ and plot the graph of $f(x) = 2^{x-2}$ using the following points $x = 4; f(4) = 4; \text{and } x = -1; f(-1) = \frac{1}{8}$
- Let student-teachers work through examples in their books and find out other possible asymptotes;

- Invite students to work out individually the application activity 1.2.3 to assess their competences on the continuity and asymptotes of exponential functions.

Answer for activity 1.2.3

Let $f(x) = 2^{x-2}$

a) $Domf =]-\infty, +\infty[$ and $\operatorname{Im} f =]0, +\infty[$

b) $\lim_{x \to -\infty} 2^{x-2} = 2^{-\infty-2} = \frac{1}{2^{+\infty}} = 0$. The equation of Horizontal asymptote is $y = 0$

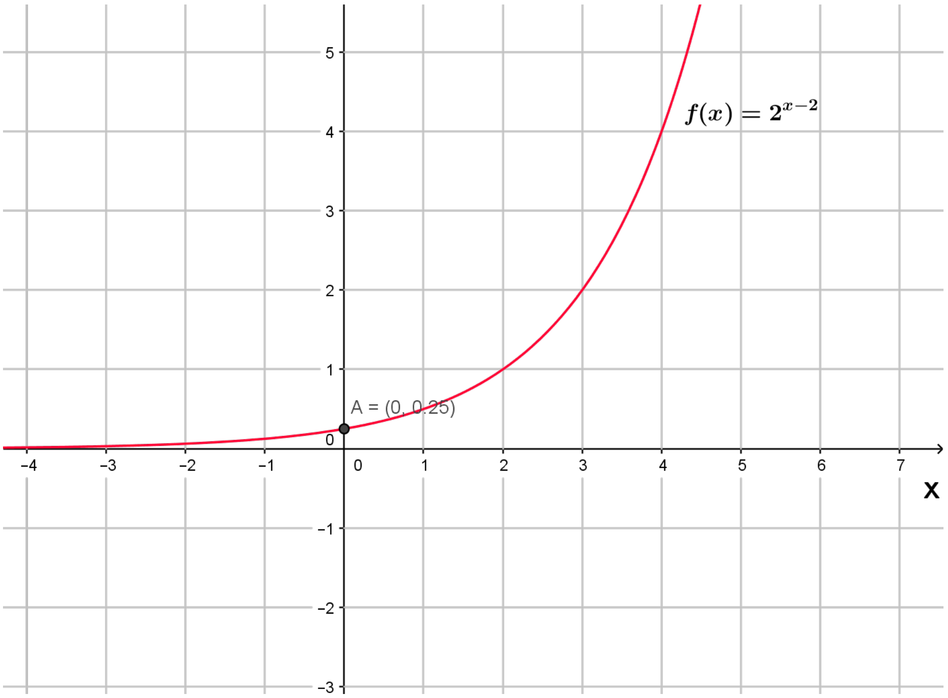
c) For $x = 0, f(x) = 2^{-2} = \frac{1}{4}$, therefore y-intercept of the graph is $(0, \frac{1}{4})$

d) $\lim_{x \to +\infty} 2^{x-2} = 2^{+\infty-2} = 2^{+\infty} = +\infty$ and $\lim_{x \to -\infty} \frac{2^{x-2}}{x} = \frac{2^{-\infty-2}}{-\infty} = \frac{1}{2^{\infty}(-\infty)} = \frac{1}{-\infty} = 0$

e) $\lim_{x \to 0^+} 2^{x-2} = 2^{0^+-2} = \frac{1}{2^2} = \frac{1}{4}$ and $\lim_{x \to 0^-} 2^{x-2} = 2^{0^--2} = \frac{1}{2^2} = \frac{1}{4}$. At $x = 0, f(0) = \frac{1}{4}$.

Since the $\lim_{x \to 0^+} f(x) = \lim_{x \to 0^-} f(x) = \frac{1}{4}$, the function has the continuity at $x = 0$

f) The graph of $f(x) = 2^{x-2}$



Answer for application activity 1.2.3

Given that f is a function given by $f(x)=2^x+1$

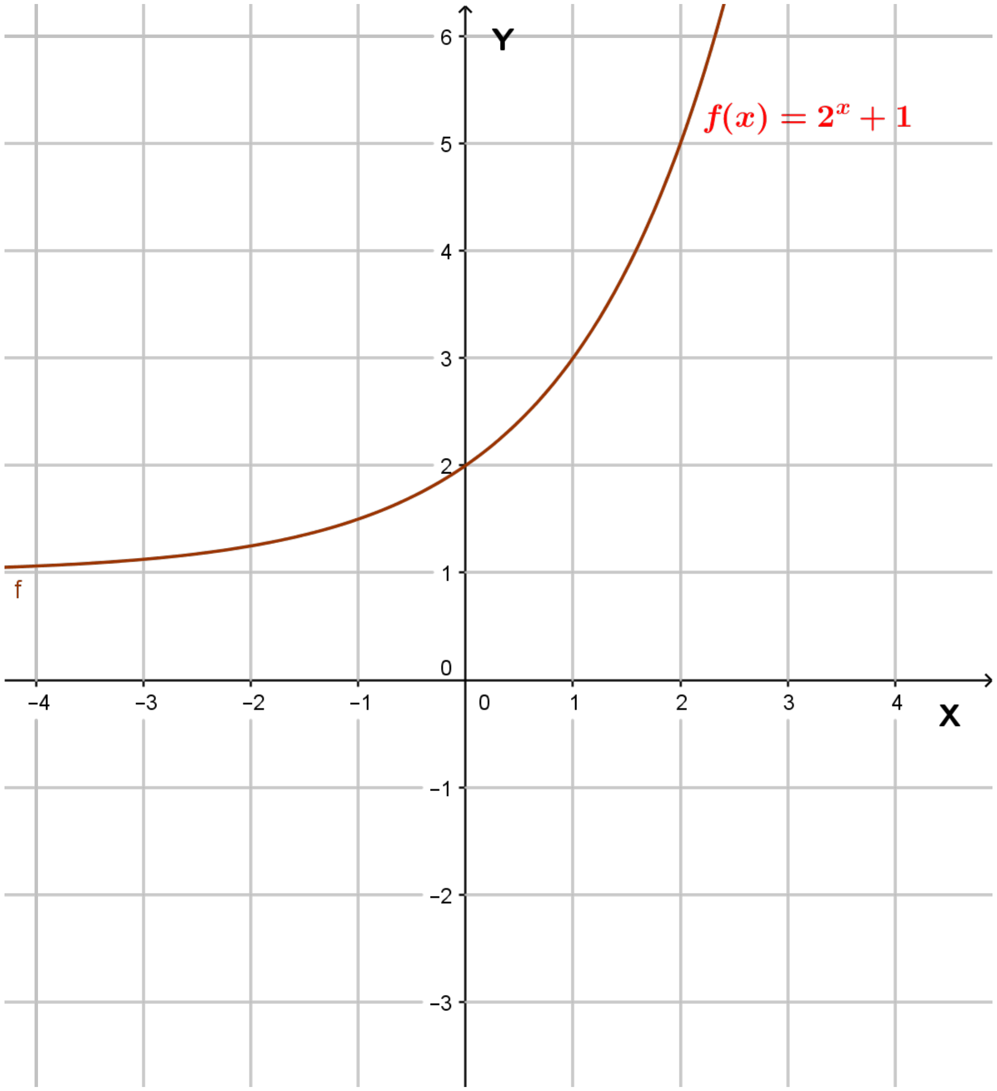
a) Dom f = $domf =]-\infty,+\infty[$ or $domf=\mathbb{R}$

b) The horizontal asymptote for the graph of $f(x)$ is the equation $y=0$,

because $\lim\limits_{x\to 0^+} f(x)=2, \lim\limits_{x\to 0^-} f(x)=2, and\ f(0)=2$

c) The $y-$intercept is $(0,2)$

d) The graph of $f(x)=2^x+1$.



Lesson 9: Differentiation of exponential functions

a) Learning objectives

To determine the derivative of logarithmic functions.

b) Teaching resources

student-teachers' book and other reference books to facilitate research, calculator, manila paper, markers, pens, pencils, T-square, ruler, if possible computers, Math draw software such as Geogebra, Microsoft Excel, Mathlab for graph sketching.

c) Prerequisites/Revision/Introduction

Student-teachers will easily learn this unit if they have a good background on Arithmetic (Unit 1 Year 1), and on limits of functions (unit 4 Year 1).

d) Learning activities:

- Invite student-teachers to work in groups and do the activity 1.2.4 found in their Mathematics books and determine the inverse of $f(x)$ and $g(x)$.
- Move around in the class for facilitating student-teachers where necessary and give more clarifications;
- In the same groups, ask student-teachers to use the derivative $p'(x)=\frac{1}{x}$ of the function $p(x)=\ln x$, $k'(x)=\frac{1}{x.\ln 2}$ of the function $k(x)=\log_2 x$ and apply the following rule $\left[f^{-1}(x)\right]'=\frac{1}{f'\left[f^{-1}(x)\right]}$ of differentiating inverse of logarithmic functions to determine that the derivative of $f'(x)=e^x$ and $g'(x)=2^x\ln 2$;
- Ask groups to present their findings to the whole class and then lead the class to harmonize their works;
- Finally , guide student-teacher to discuss the used technique of determining the derivative of $f(x)$ and $g(x)$ and to deduce the following: $\left[e^{u(x)}\right]'=u'(x)e^{u(x)}$ and $\left[a^{u(x)}\right]'=u'(x)a^{u(x)}\ln a$.
- After this step, guide students to do the application activity 1.2.4 and evaluate whether lesson objectives were achieved.

Answer for activity 1.2.4

1) Given the functions $f(x)=e^x$ and $g(x)=2^x$, their inverse are: $f^{-1}(x)=\ln x$ and $g^{-1}(x)=\log_2 x$ respectively.

2) Given that $p(x)=\ln x$ and $k(x)=\log_2 x$, it is known that $p'(x)=\frac{1}{x}$ and

$k'(x)=\frac{1}{x\ln 2}$. Then applying the rule for differentiating inverse of logarithmic

functions we find $\frac{1}{p'\left[p^{-1}(x)\right]}=\frac{1}{p'\left(e^x\right)}$. We already know that that

$p'(x)=\frac{1}{x}$, then $p'(e^x)=\frac{1}{e^x}$

Thus $\left(e^x\right)'=\left[p^{-1}(x)\right]'=\frac{1}{p'\left[p^{-1}(x)\right]}=\frac{1}{p'\left(e^x\right)}=\frac{1}{\frac{1}{e^x}}=e^x$

and $\frac{1}{k'\left[k^{-1}(x)\right]}=\frac{1}{k'\left(2^x\right)}$.

Since $k'(x)=\frac{1}{x\ln 2}$, it follows that $k'(2^x)=\frac{1}{2^x\ln 2}$ and

$g'(x)=\left(2^x\right)'=\frac{1}{k'\left[k^{-1}(x)\right]}=\frac{1}{\frac{1}{2^x\ln 2}}=2^x\ln 2$.

Answer for application activity 1.2.4

1) Given the function $f(x)=4^x$.

$f'(x)=4^x\ln 4$

2) a) $f(x)=10^{3x}\Rightarrow f'(x)=3.10^{3x}\ln 10$

b) $f(x)=xe^{x^2+1}\Rightarrow f'(x)=e^{x^2+1}(1+2x^2)$

c) $f(x)=\frac{3^{4x+2}}{x}\Rightarrow f'(x)=\frac{\left(3^{4x+2}\right)'x-3^{4x+2}(x)'}{x^2}=\frac{4x.\left(3^{4x+2}\right)\ln 3-3^{4x+2}}{x^2}=\frac{3^{4x+2}(4x\ln 3-1)}{x^2}$

Lesson 10: Application of derivative to determine the continuity and the variation of exponential functions

a) Learning objectives

- Apply derivative to investigate the variation of exponential functions.
- Apply derivative of exponential function to remove indeterminate cases.

b) Teaching resources

student-teachers' book and other reference books to facilitate research, calculator, manila paper, markers, pens, pencils...

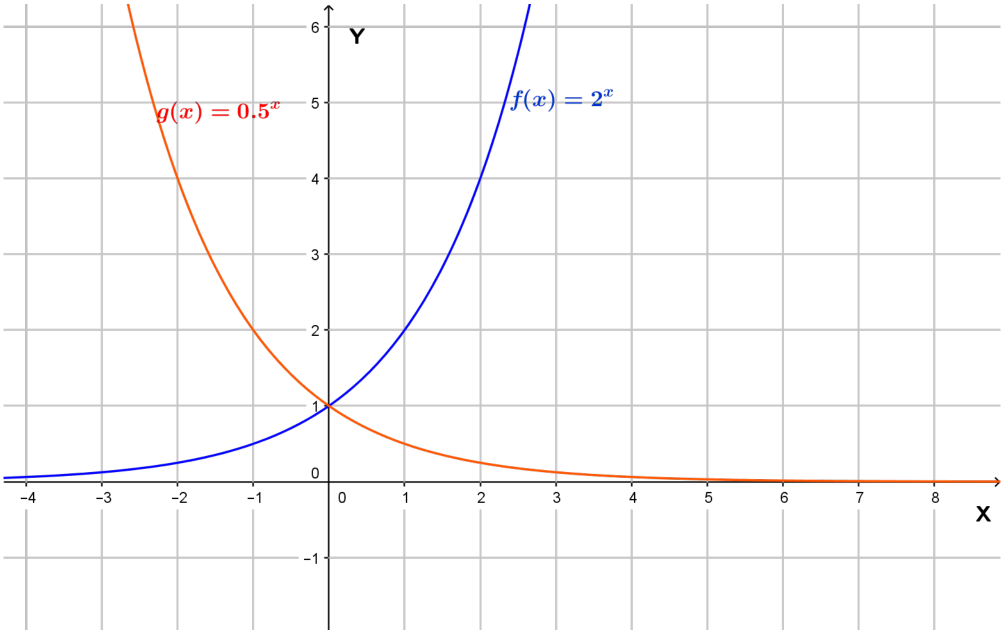
c) Prerequisites/Revision/Introduction

Student-teachers will easily learn this unit if they have a good background on Arithmetic (Unit 1 Year 1) and on limits of functions (unit 4 Year 1).

d) Learning activities:

- Invite student-teachers to work in groups and do the activity 1.2.5 found in their Mathematics books and let them find out that the functions $f(x)$ is increasing on the interval $]1,10[$, the function $g(x)=(0.5)^x$ is decreasing on the interval $]1,10[$ because $a=5$ or $5>1$; ask questions which lead students to realize and conclude that thefunction of the form $f(x)=a^x$, with $a>1$, is always increasing and that the function

  $g(x)=a^x$ with $0<a<1$, is always decreasing.
- Ask groups to present their findings to the whole class and then harmonize their works where you will guide them to highlight the following points:
- Realize that the 1st derivative of $f(x)$ and $g(x)$ are $f^{'}(x)=e^x$ and $g'(x)=5^x \ln 5$ respectively.
- Draw signs table for $f^{'}(x)$ and $g'(x)$ and note that the interval of variation of those function is $]-\infty,+\infty[$ and finally, let them plot the graphs of the functions: $f(x)$ and $g(x)$



- Guide students to emphasize that $\lim_{x\to k} e^{f(x)\ln g(x)} = e^{\lim_{x\to k} f(x)\ln g(x)}$
- Let students read through the examples in their books and invite them to work out individually the application activity 1.2.4 to improve their knowledge and skills on the variation of exponential functions.

Answer for activity 1.2.5

Given two functions $f(x)=2^x$ and $g(x)=0.5^x$,

1) $f(1)=2$ and $f(10)=2^{10}$, $f(1)<f(10)$. Thus, the function $f(x)$ is increasing on the interval $]1,10[$.

2) $g(1)=\frac{1}{2}$ and $g(10)=\frac{1}{2^{10}}$, $g(1)>g(10)$. Thus, the function $g(x)$ is decreasing on the interval $]1,10[$.

3) $f'(x)=2^x\ln 2$ and $g'(x)=(0.5)^x\ln 0.5$ or $g'(x)=\frac{1}{2^x}\ln\frac{1}{2}$

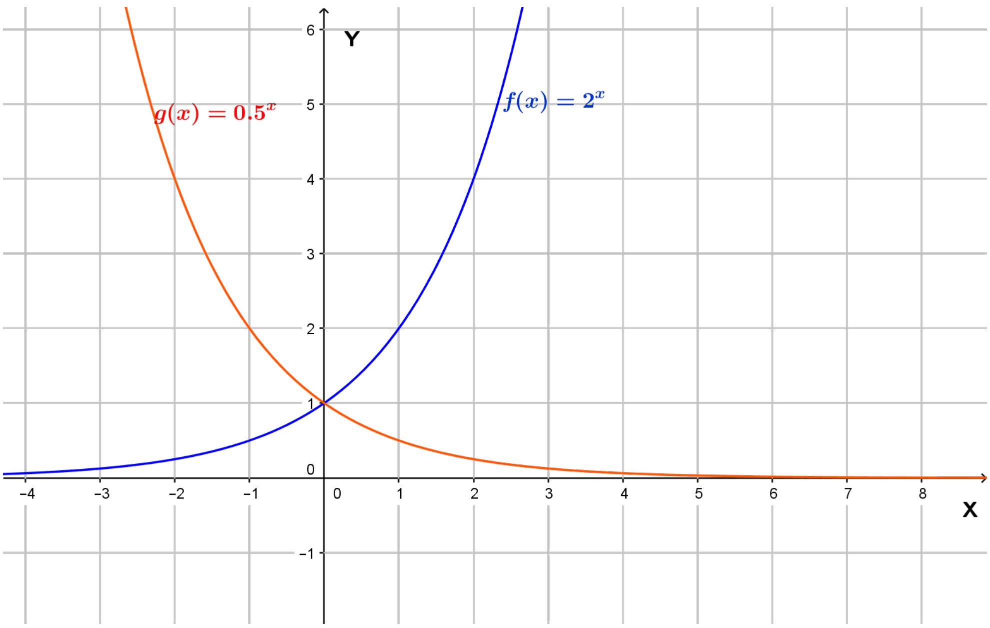
a. Table of variation of $f(x)$

x	$-\infty$		0		$+\infty$
$f'(x)$	+ + +	+	$\ln 2$	+ + + + + +	+
$f(x)$	0	$\nearrow$	1	$\nearrow$	$+\infty$

b. Table of variation of $g(x)$

x	$-\infty$		0		$+\infty$
$g'(x)$	- - - -	-	$\ln\frac{1}{2}$	- - - - - - - - -	-
$g(x)$	$+\infty$	$\searrow$	1	$\searrow$	0

1. Plot the graphs of $f(x)$ and $g(x)$



2. The exponential *function* of the form $f(x) = a^x$, with $a > 1$, is always increasing and the exponential function of the form $g(x) = a^x$ with $0 < a < 1$, is always decreasing.

Answer for application activity 1.2.5

1) Given the function $f(x) = xe^{x^2}$

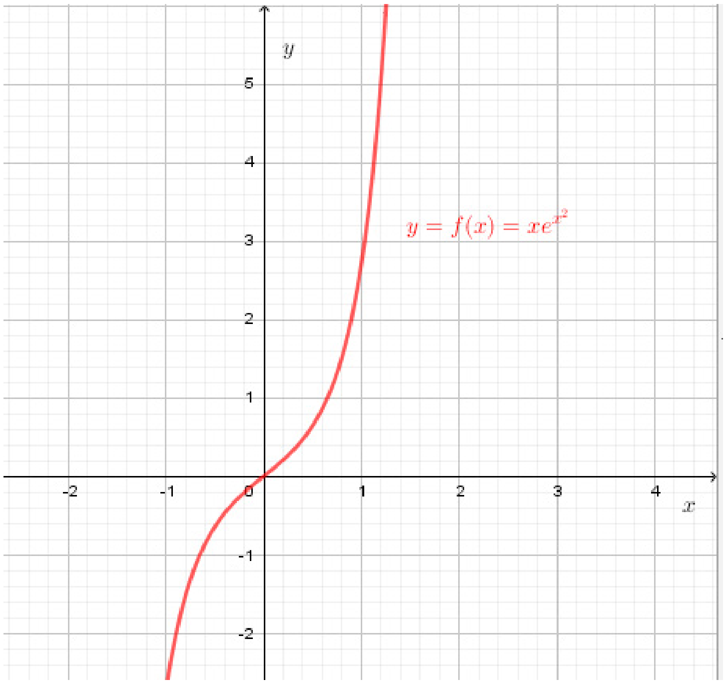
a) The derivative of $f(x)$ is $f'(x) = e^{x^2}(1 + 2x^2)$

b) The derivative of $f(x)$ has no zero in $\mathbb{R}$ and is always positive,

Table of variation of $f(x)$ is presented as follows:

$f'(x)$	+ + + + + + + + + + + + + +
$f(x)$	$-\infty$ ↗ $+\infty$

From the table above, the function $f(x) = xe^{x^2}$ is always increasing on the interval $]-\infty, +\infty[$.



2) $a)\quad \lim_{x\to\infty}\frac{e^x+1}{e^x-2}=\lim_{x\to\infty}\frac{e^\infty+1}{e^\infty-2}=\frac{\infty}{\infty}(\text{I}C)$

To remove this indeterminate case let apply Hospital's rule

$$\lim_{x\to\infty}\frac{(e^x+1)'}{(e^x-2)'}=\lim_{x\to\infty}\frac{e^x}{e^x}=\lim_{x\to\infty}1=1$$

$b)\quad \lim_{x\to-\infty}xe^x=-\infty e^{-\infty}=\frac{-\infty}{e^\infty}=\frac{-\infty}{\infty}(IC)$

Apply hospital's rule to remove this IC

$$\lim_{x\to-\infty}\frac{(x)'}{(e^{-\infty})'}=\lim_{x\to-\infty}\frac{1}{-e^{-x}}=\lim_{x\to-\infty}(-e^x)=-e^{-\infty}=\frac{1}{e^\infty}=0$$

Lesson 11: Application of logarithmic or exponential functions on the interest rate problems

a) Learning objectives

Solve problems related to economics and other social sciences involving logarithmic and exponential functions

b) Teaching resources

Student-teachers' book and other Reference books to facilitate research, calculator, manila paper, markers, pens, pencils...

c) Prerequisites/Revision/Introduction

Student-teachers will learn better the interest rate problems if they have a clear understanding of:

- Logarithmic and exponential functions(year three on previous lessons)
- The concepts: principal, interest rate and the period for investment (From Entrepreneurship)
- Sequences and series learnt in year 2.

d) Learning activities:

- Invite student-teachers to work in groups and do the activity 1.3.1 found in their Mathematics books;
- Have them discuss the terminologies used when solving an interest rate problem and predict, from the table, the total amount at the end of t years when the interest is compounded once per year;

- As they are discussing, concentrate on slow learners for further explanation and provide assistance to groups in need;
- Check how adequately the student-teachers are using calculators and how each member of the group is contributing to the discussion;
- Once the group discussion is over, ask a group, chosen randomly, to present their results while other students are following attentively.
- Have student-teachers exchange their views, in mutual respect and without confrontation to establish the main points from the presentation, and to take note.
- Ask student-teachers to work out example in their books under the guidance of tutor, and work individually application activity 1.3.1 to check the skills they have acquired.

Answer for activity 1.3.1

At the end of	The total amount
The first year	2000+0.1(2000) =2000(1+0.1)
The second year	2000(1+0.1)+0.1[2000(1+0.1)] =2000(1+0.1)^2
The third year	2000(1+01)^3
The fifth year	2000(1+0.1)^5
The t^{th} year	2000(1+0.1)^t

Answer for application activity 1.3.1

With bank I, the amount at the end of the year is $A = P\left(1+\frac{r}{n}\right)^{nt}$ =
$300000\left(1+\frac{0.1}{1}\right)^{1X10}$ Frw $= 778,122 Frw$

With bank II, the amount at the end of the year is: $A = Pe^{nt}$ =
$300000e^{0.098x10} Frw = 799336$ Frw

You should advise your aunt to invest at bank II because 999336Frw > 778122Frw.

Lesson 12: Application of logarithmic or exponential functions on the mortgage amount

a) Learning objectives

Solve problems related to economics and other social sciences involving logarithmic and exponential functions.

b) Teaching resources

Students' book, scientific calculators or Microsoft Excel, eventual other books where the content about mortgage can be found.

c) Prerequisites/Revision/Introduction

Student-teachers will learn better the mortgage problems if they have a clear understanding of:

- Logarithmic and exponential functions(previous lessons);
- The concepts: loan, principal, interest rate, payment by installment and the period for investment(From Entrepreneurship).

d) Learning activities:

- Invite student-teachers to work in groups and do the activity 1.3.2 found in their Mathematics books and let student-teachers discuss the terminologies used when solving a mortgage problem.
- Visit each group for facilitating student-teachers where necessary and give more clarification on eventual challenges they may face during their work;
- After group discussions, invite a group to present their work and request other student-teachers to follow attentively, as they are evaluating the work of their respective groups;
- As a tutor, harmonize the findings from presentation highlighting how to deal with the mortgage or the amortization of loan: The payment P required to pay off a loan of M Francs borrowed for n payment periods at a rate of interest i per payment period is $P = M\left[\frac{i}{1-(1+i)^{-n}}\right]$ where for the interest rate is ***r***, and the rate of interestper payment period $i = \frac{r}{n}$.
- Let students proceed to examples found in their books under your guidance and then request them to work individually application activity 1.3.2 to check the skills they have acquired.

Answer for activity 1.3.2

1) Verify answers provided by students on the definition of the following concepts:

the periodic payment(P), annual interest rate (r), mortgage amount (M), number t of years to cover the mortgage and the number n of payments per year.

2) $P = \frac{\frac{rM}{n}}{1-\left(1+\frac{r}{n}\right)^{-nt}} = \frac{\frac{(0,06)(20000000)}{12}}{1-\left(1+\frac{0.06}{12}\right)^{(-12)(20)}} = 143286.2$ The amount to pay per month should be 143286.2 FRW but in practice the bank will convert such amount into 143 287 FRW .

Generally bank offices round figure to the nearest greater integer. The last payment will be less amount than 143 287 FRW, as there will be an adjustment by considering the difference between the real amount and the amount to be paid per month.

The amount to pay per month is 143 287 FRW, the balance the brother will withdraw each month is 500000FRW-143287 FRW=356713 FRW.

At the end of 20 years, your brother would have paid $143287 \times 12 \times 20 Frw = 34388880 Frw$.

The interest the bank will realize is 34 388 880 FRW-20 000 000 FRW=14 338 880FRW

Answer for application activity 1.3. 2

The periodic payment is P=200 000, the annual rate r=10%=0.01, the number of payments per year n=12(since the payment is monthly), the number of years to cover the mortgage is t=20.,

Solving in M $P = \frac{\frac{rM}{n}}{1-(1+\frac{r}{n})^{-nt}}$ yields to $M = \frac{P\left[1-(1+\frac{r}{n})^{-nt}\right]}{r}n$.

Replacing each quantity by its value in the formula, we have

$$M = \frac{200000\times 12\times\left[1-(1+\frac{0.1}{12})^{-12(20)}\right]}{0.1}$$

Calculations give M=20 808 156.36. So, the mortgage is 20 808 156 Frw.

Lesson 13: Application of logarithmic or exponential functions on the Population growth problems

a) Learning objectives

Solve problems related to economics and other social sciences involving logarithmic and exponential functions.

b) Teaching resources

Students' book, scientific calculators or Microsoft Excel, eventual other books where the content about mortgage can be found.

c) Prerequisites/Revision/Introduction

Student-teachers will learn better the population growth problems if they have a clear understanding of:

- Logarithmic and exponential functions(previous lessons)
- Graphical interpretation of functions(previous lessons)

d) Learning activities:

- Invite student-teachers to work in groups and do the activity 1.3.3 found in their Mathematics books and let them analyze the graph and answer the questions related to it;
- Move around in groups to provide facilitation to the groups in need;
- Once discussions are over, invite groups to present their work and ask student-teachers to give constructive remarks in order to obtain an improved information to be written by all members.
- As a tutor, harmonize the findings from presentation highlighting that the number of cells after the time ***t*** can be calculated using an exponential function of ***t***.
- Let students proceed to examples found in their books and then request them to work individually application activity 1.3.3 to check the skills they have acquired.

Answer for activity 1.3.3

a)

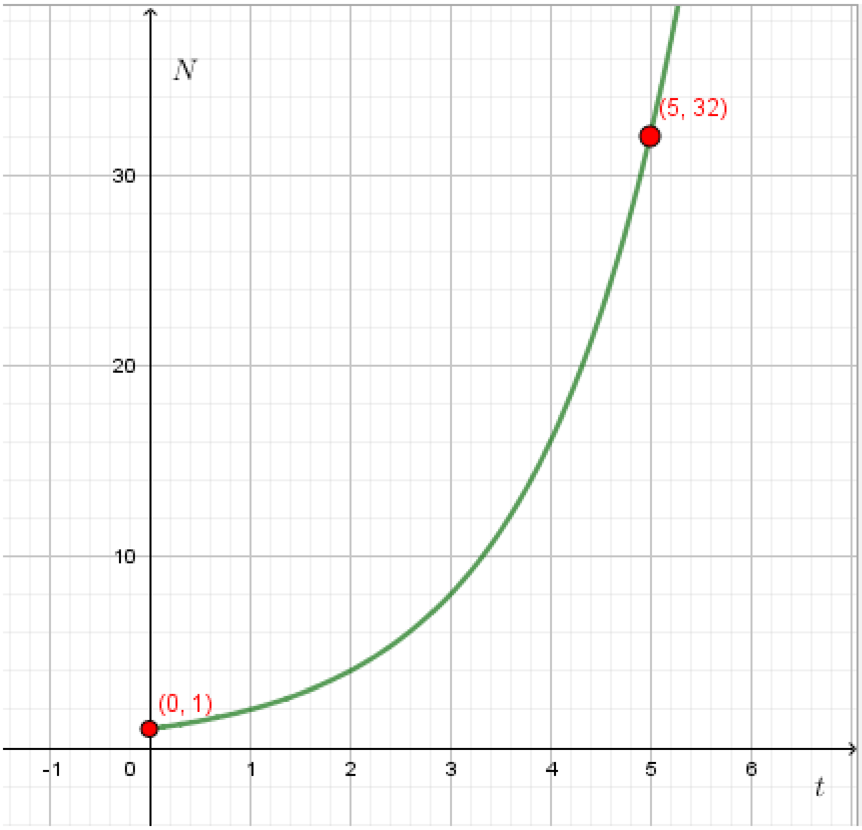
Time t(minutes)	0	1	2	3	4
Number of cells	1	2	4	8	16

b) The number of cells will be $N(\mathrm{t}) = 2^t$

If $N(\mathrm{t}) = N_0 e^{kt}$, then $N_0 = 1$,since it is independent of the base , and then $e^{kt} = 2^t$

Making k the subject of the formula and by applying natural logarithm on both sides of the equation $e^{kt} = 2^t$ and then $k = \ln 2$,we find that $N(\mathrm{t}) = e^{(\ln 2)t}$

After 5 minutes, the number of cells is $N(5) = e^{5(\ln 2)} = e^{\ln 32} = 32$.



From the graph, it is clear that at $t = 0$, $N = 1$ and at $t = 5$, $N = 32$ and as the time becomes larger and larger, the number of cells grows exponentially following $N(\mathrm{t}) = e^{(\ln 2)t}$. The number of cell is growing as $k = \ln 2 > 0$.

Answer for application activity 1.3. 3

1) $N(t) = N_0 e^{tk}$. Substituting for $N_0 = 1000000$ and $N(5) = 2N_0$,we obtain

$k = \frac{1}{5}\ln 2.$

The population, in 10 years would be $N(10) = 1000000e^{10(\frac{1}{5}\ln 2)} = 4000000$

2) $A(t) = A_0 e^{tk} = 56\times 10^9 e^{(0.025)(1.75)} = 58.504,384$

3) For t=17, $13000 = 11000e^{17k} \Rightarrow k = \frac{1}{17}\ln\left(\frac{13}{11}\right) = 0.1\%$.

Lesson 14: Application of logarithmic or exponential functions on problems related to alcohol and risk of car accident

a) Learning objectives

Apply logarithmic functions to solve problems related to alcohol and risk of car accident.

b) Prerequisites/Revision/introduction:

Learners will learn better "problems about alcohol and risk of car accident" if they have a clear understanding of:

- Logarithmic and exponential functions(previous lessons)
- Graphical interpretation of functions(previous lessons)

c) Teaching resources:

Learner's book, scientific calculators, charts and eventual other books where the content can be found, and, if possible computer with the software Geogebra.

d) Learning activities:

- Organize students into groups to discuss the activity 1.3.4;
- Distribute the tasks and give clear instructions on the duration and the internal organization of each group;
- Let students discuss the relationship between the alcohol concentration in the blood of a driver and the risk of car accident.
- When the students are on task, provide support to the groups in need;
- Once discussions are over, choose a group to present his work when other students are following attentively;
- Ask students to give constructive remarks and complements, in order to obtain a conclusion to be noted by all students and guide them to conclude that the risk R (given as a present) of having an accident while

driving can be modeled by an equation of the type $R(x)=R_0e^{kx}$ where x is the variable concentration of alcohol in the blood and k is a constant.

- Let students proceed to example under your guidance, and check their working against the solution proposed in the book,
- Invite students to discuss how to avoid alcoholism to avoid car accident as a crosscutting issue;
- Ask them to work individually application activity 1.3.4 to check the skills they have acquired.

Answer/solution to activity 1.3.4

- a) Excess of alcohol taken by the driver can yield to car accident.
- b) i) The risk when there is no alcohol in the driver's blood is 1; it is not zero because the car accident is not due only to excess of alcohol in the driver's blood. There are other factors.
- ii) More the concentration of alcohol in the driver's blood, more the risk of accident.
- c) Since the risk grows exponentially, the equation is of the type $R(x)=R_0e^{kx}$
- From $R(0)=1$ and $R(4)=5$, we determine the value of k and get . Then,

  $R(x)=e^{(\frac{1}{4}\ln 5)x}$

Answer/solution to application activity 1.3.4

a) For the concentration of alcohol in the blood of 0.05 and a risk of 8%, we have:

$$8=4e^{k(0.05)} \Leftrightarrow e^{0.05k}=2 \Leftrightarrow 0.05k=\ln 2 \Leftrightarrow k=\frac{\ln 2}{0.05}=13.86$$

b) Using k=13.86 and x=0.18, we have: $R=4e^{(13.86)(0.18)}=48.477$

For a concentration of alcohol in the blood of 0.18, the risk of accident is about 48.5%.

c) $100=4e^{13.86x} \Leftrightarrow 13.86x=\ln 25 \Leftrightarrow x=0.2442$: for a concentration of alcohol of 0.24, the risk of accident is 100%.

1.6 Unit summary

I. Logarithmic functions

- Definition: $\log_a x = y \Leftrightarrow a^y = x$, where $a > 0, a \neq 1, x > 0$;this definition is used to determine the domain and the range.

- Formula for changing the base, from base a to base e: $\log_a x = \frac{\ln x}{\ln a}$
- Limits of a logarithmic function:

For $f(x) = \log_a x$ the domain is $]0, +\infty[$, If $x_0 \in]0, +\infty[$, then $\lim\limits_{x \to x_0} \log_a x = \log_a x_0$

$\lim\limits_{x \to 0^+} \log_a x = \begin{cases} -\infty; & ;a > 1 \\ +\infty; & 0 < a < 1 \end{cases}$; in particular, $\lim\limits_{x \to 0^+} \ln x = -\infty$.

$\lim\limits_{x \to +\infty} \log_a x = \begin{cases} +\infty; & ;a > 1 \\ -\infty; & 0 < a < 1 \end{cases}$; in particular, $\lim\limits_{x \to +\infty} \ln x = +\infty$

Indeterminate cases $\frac{0}{0}; \frac{\infty}{\infty}$: the indeterminate can be removed by applying Hospital's rule

Indeterminate cases $\infty - \infty; 0 \times \infty; 1^\infty; 0^0; \infty^0$:Re write the limit to obtain $\frac{0}{0}$ or $\frac{\infty}{\infty}$

and then apply Hospital's rule

- **Derivative of a logarithmic function:**

$(\log_a \text{u})' = \frac{u'}{\text{u} \ln a}$; $(\ln \text{u})' = \frac{u'}{\text{u}}$, where u is function of variable x;

For more elaborated functions, such as product, power, quotient, etc, containing

Logarithms, the rules for differentiation still apply

- Variations and graphs of logarithmic functions: either graph the function, using software, such as Geogebra, and then analyse the graph to draw the conclusion about maximum, minimum, increasing, decreasing, concavity, inflection point or, study the sign of the first derivative (eventually the second derivative) and draw conclusion about the variations, then graph the function.

II. Exponential functions

- Definition: $f(x) = a^x$, where $a > 0, a \neq 1,$;this definition is used to determine the domain and the range of an exponential function
- Limits of an exponential function

For $f(x) = a^x$ the domain is $]-\infty, +\infty[$

- If $x_0 \in]-\infty,+\infty[$, then $\lim_{x\to x_0} a^x = a^{x_0}$
  - $\lim_{x\to-\infty} a^x = \begin{cases} 0 \ ; a>1 \\ +\infty; \ 0<a<1 \end{cases}$; in particular, $\lim_{x\to-\infty} e^x = 0$
  - $\lim_{x\to+\infty} a^x = \begin{cases} +\infty; \ ; a>1 \\ 0; \ 0<a<1 \end{cases}$; in particular, $\lim_{x\to+\infty} e^x = +\infty$

Indeterminate cases $\frac{0}{0};\frac{\infty}{\infty}$: the indeterminate form can be removed by applying Hospital's rule Indeterminate cases $\infty-\infty; 0\times\infty; 1^{\infty}; 0^0; \infty^0$:Re write the limit to obtain $\frac{0}{0}$ or $\frac{\infty}{\infty}$ and then apply Hospital's rule

- **Derivative of an exponential function:**

$(a^u)' = u'a^u \ln a$;

$(e^u)' = u'e^u$, where u is function of variable x

For more elaborated functions, such as product, power, quotient, etc, containing logarithms, the rules for differentiation still apply

- Variations and graphs of exponential functions: either graph the function, using software, such as Geogebra, and then analyse the graph to draw the conclusion about maximum, minimum, increasing, decreasing, concavity, inflection point or, study the sign of the first derivative (eventually the second derivative) and draw conclusion about the variations, then graph the function.

III. Applications of logarithmic and exponential functions

- Interest compounded n times per year, r: rate of annual interest, P: Principal

$$A = P\left(1+\frac{r}{n}\right)^{nt}$$

- Interest compounded continuously: $A = Pe^{rt}$
- Formula connecting the quantities involved in a mortgage problem:

$$P = \frac{\frac{rM}{n}}{1-\left(1+\frac{r}{n}\right)^{-nt}}$$

where P: Amount after t years, M: Mortgage, r: Annual rate interest, n: Number of payment per year, t: Number of years to cover the Mortgage

- **Law of exponential growth or decay**

$P(t) = P_0 e^{kt}$ where P: Population at time t, P_0 :Initial population, $k > 0 \ or \ k < 0$, t: time

1.7. Additional information for the teacher

Given that the knowledge of the teacher must be wider than the one of the learners, the following information is useful for the teacher, though not stated in the learner's book:

- The study of logarithmic and exponential functions can follow the study of integrals. In this case, the natural logarithm of x is defined as $\ln x = \int_1^x \frac{dt}{t}$,where $x > 0$
- The concepts of logarithmic functions and exponential functions can be taught interchangeably. Exponential functions can be taught first: $f(x) = a^x$, where $a > 0$ and $a \neq 1$
- The roots of some equations involving logarithms or exponentials can be approximated using Taylor's expansion. Similarly, in the calculation of limits, some indeterminate cases can be removed by approximating the function involved in the limit, using Taylor's expansion.

  The Taylor's expansion of f(x) at x=0,or the Maclaurin's expansion of $f(x) = e^x$ and for $f(x) = \ln(1+x)$ are given below:

  $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$ For any value of x

$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots$ for values of x in the neighborhood of 0

Examples:

a) Approximate e^x by a quadratic function and approximate the roots of the equation

$e^x - x^2 = 0$

b) Approximate e^x and $\ln(1+x)$ by quadratic functions then calculate

$$\lim_{x\to 0}\frac{x^3e^x}{xe^{-x}-\ln(x+1)}$$

Solution:

a)

$$e^x-x^2=0\Leftrightarrow (1+x+\frac{x^2}{2})-x^2=0\Leftrightarrow -\frac{1}{2}x^2+x+1=0\Leftrightarrow x_1=-1+\sqrt{3};x_2=-1-\sqrt{3}$$

b) Substituting x in the limit, we obtain the indeterminate case $\frac{0}{0}$

Then $\lim_{x\to 0}\frac{x^2e^x}{xe^{-x}-\ln(x+1)}=\lim_{x\to 0}\frac{x^2(1+x+\frac{x^2}{2})}{x(1-x+\frac{x^2}{2})-(x-\frac{x^2}{2})}=-2$

1.8 End unit assessment

This part provides the answers of end unit assessment activities designed in integrative approach to assess the key unit competence with cross reference to the textbook.

Answers for question one

a) $f(x)=\log_2(3x-2)$

$f(x)=\log_2(3x-2)$ is defined if and only if $(3x-2)>0\Leftrightarrow 3x>2\Leftrightarrow x>\frac{2}{3}$

Thus, $Domf=\left]\frac{2}{3},+\infty\right[$

From the function, $0<3x-2<+\infty$.

Then $-\infty<\log_2(3x-2)<+\infty$

Therefore, the range is $\mathbb{R}=\left]-\infty,+\infty\right[$

b) $f(x)=\ln\left(x^2-1\right)$

$f(x)=\ln\left(x^2-1\right)$ is defined if and only if $x^2-1>0\Leftrightarrow(x-1)(x+1)>0$.

x	$-\infty$		-1		1		$+\infty$
$x-1$	$- - - - - - - - -$		$- - - - -$		0	+ + +	
$x+1$	$- - - - - - - - -$	0	+ + + + + + +			+ + + + + +	
$(x-1)(x+1)$	+ + + + + + +		$- - - - - -$			+ + + + + + + +	

$Domf =]-\infty,-1[\cup]1,+\infty[$

From the function, $0 < x^2 - 1 < +\infty$.

Then $-\infty < \ln(x^2 - 1) < +\infty$

Therefore, the range is $\mathbb{R} =]-\infty,+\infty[$

c) $f(x) = 2e^{3x+1}$

$Domf = \mathbb{R} =]-\infty,+\infty[$

From the function, $-\infty < 3x+1 < +\infty$;

Then $0 < e^{3x+1} < +\infty$;

$0 < 2e^{3x+1} < +\infty$;

Therefore, the range is $]0,+\infty[$

d) $f(t) = 4^{\sqrt{3t+1}}$

$f(t) = 4^{\sqrt{3t+1}}$ is defined if and only if $3t+1 \geq 0 \Leftrightarrow t \geq \frac{-1}{3}$.

$Domf = \left[\frac{-1}{3},+\infty\right[$

From the function, $0 \leq 3t+1 < +\infty$;

Then $0 \leq \sqrt{3t+1} < +\infty$;

$1 \leq 4^{\sqrt{3t+1}} < +\infty$;

Therefore, the range is $[1,+\infty[$

Answers for question two

a) $\lim\limits_{x\to 0^+} \frac{\ln x}{x} = -\infty$

We have a vertical asymptote which has equation $x = 0$

b) $\lim\limits_{x\to +\infty} \left(3 + x^2 \ln x\right) = +\infty$

No horizontal asymptote

Answers for question three

1) a)

$$f(x) = \log_2 \sqrt{\frac{x^2-4}{x+2}} = \frac{1}{2}\left[\log_2(x-2) + \log_2(x+2) - \log_2(x+2)\right] = \frac{1}{2}\log_2(x-2)$$

$$\frac{d}{dx} f\left(x\right) = \frac{1}{(2\ln 2)(x-2)}$$

b) $\frac{d}{dx} h\left(x\right) = \frac{d}{dx}[\frac{1}{3}\left(4^{2x+5}\right)] = \frac{2}{3}(4^{2x+5})\ln 4$

Answers for question four

Five applications of logarithmic or exponential functions:

- In Geography: the magnitude of an earthquake is found using logarithms
- In Entrepreneurship: the interest rate problems can be solved using logarithm and exponential functions
- In Chemistry: the radioactive decay problems are solved using logarithmic and exponential functions
- In History: archaeology involves carbon dating whose principles are based on the use of logarithms
- In Social studies: logarithms and exponentials are used in the determination of risk corresponding to a given concentration of alcohol

Answers for question five

Assuming exponential growth, $N(t) = 5.7e^{0.02t}$

The population will reach 114 billion after t years such that $114 = 5.7e^{0.02t}$ .solving for t,

$$t=\frac{1}{0.02}\ln\frac{114}{5.7}=26.278 \text{ years.}$$

The population will reach 114 billion in the year 2021.

Answers for question six

Answers will vary accordingly on how this unit inspired for every student-teacher in relation to learning other subjects or to his/her future.

Help them to harmonize their answers referring to the importance of this unit in real life by emphasizing the use of logarithmic and exponential functions in economics.

1.9. Additional activities

1.9.1 Remedial activities

1. Find the domain and the range of the function;

   a) $f(x)=\log(x-1)$

   b) $f(x)=4^{\sqrt{6x}}$

2. Calculate

   a) $\lim_{x\to+\infty}\left(2-\frac{3}{\ln x}+e^{-x}\right)$

3. Given the logarithmic function $f(x)=\log_2(x-5)$

   a) What is the equation of the asymptote line?

   b) If $x=7$ find y

4) Let $f(x)=3^{-x}-1$,

   a) Determine $\lim_{x\to+\infty} f(x)$ and deduce the related asymptote to the curve of $f(x)$

   b) Give the derivative of $f(x)$ and use it to determine the interval where the function is decreasing.

c) Use a table of values to sketch the graph of $f(x)$.

Solutions:

1\. a) Domain: $]1,+\infty[$

Range: $\mathbb{R} =]-\infty,+\infty[$

b) Domain: $[0,+\infty[$

Range: $\mathbb{R} = [1,+\infty[$

2\. $\lim\limits_{x\to+\infty}\left(2-\dfrac{3}{\ln x}+e^{-x}\right)=2-\dfrac{3}{\lim\limits_{x\to+\infty}\ln x}+\lim\limits_{x\to+\infty}e^{-x}=2-\dfrac{3}{+\infty}+e^{-\infty}=2-0+0=2$

3\. a) Vertical asymptote: $x=5$, since $\lim\limits_{x\to5^+}\log_2(x-5)=-\infty$

b) $y=f(7)=\log_2(7-5)=\log_2 2=1$.

4) a) $domf=\mathbb{R}$ and $\lim\limits_{x\to\infty}f(x)=\lim\limits_{x\to\infty}\left(3^{-x}-1\right)=-1$

Therefore, the line of equation $y=-1$ is an horizontal asymptote of the graph of $f(x)$

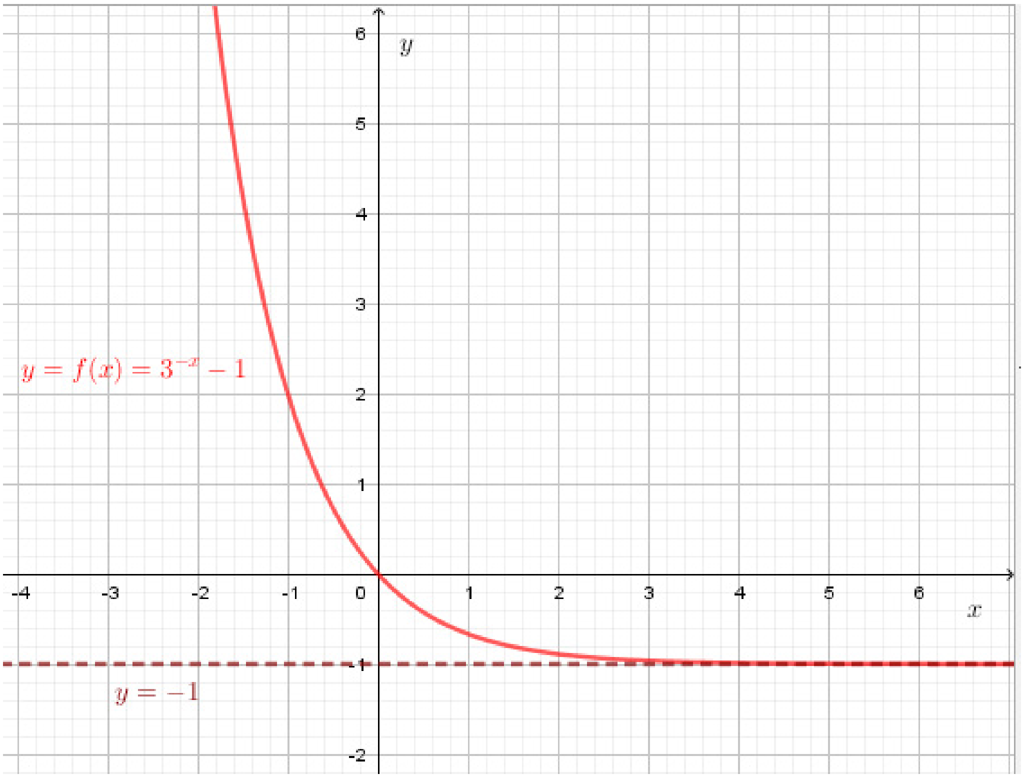
b) The derivative of $f(x)$ is $f'(x)=-3^{-x}\ln 3$.

x	$-\infty$		0		$+\infty$
$f'(x)$	- -		- ln(3)	- - -	
	$+\infty$	↘	0	↘	-1

The derivative of $f(x)$ shows that this function is decreasing on $]-\infty,+\infty[$.

c) We can use this table of values to sketch the graph of $f(x)$.

x	-2	-1	0	1	2
$f(x)$	8	2	0	$-\dfrac{2}{3}$	$-\dfrac{8}{9}$



The graph shows that the function $f(x)$ is decreasing and the line of equation $y = -1$ is an horizontal asymptote of the graph of $f(x)$.

1.9.2 Consolidation activities

Suggestion of questions and answers for deep development of competences.

1. Consider the function $f(x) = 6^{x-2}$

   a) Determine $f'(x)$

   b) Find the equation of the tangent to the graph of the function at the point where $x = 3$

   c) Graph the function and its tangent

2. Suppose the function $f(x) = 2x - \ln x$

   a) State the domain and range

   b) Find the 1st derivative.

   c) Solve for $f'(x) = 0$

   d) Determine a point through which the graph passes

   e) Draw the variation table of $f(x)$, and find the stationary point and its nature.

f) Sketch the graph

3. An amount of 1, 000 000 FRW is invested at a bank that pays an interest rate of 10% compounded annually.

   a) How much will the owner have at the end of 15 years, in each of the following alternatives?

   The interest rate is compounded:

   i) Once a year.

   ii) Twice a year

   b) Compare the two types of compounding, and explain which one is the best

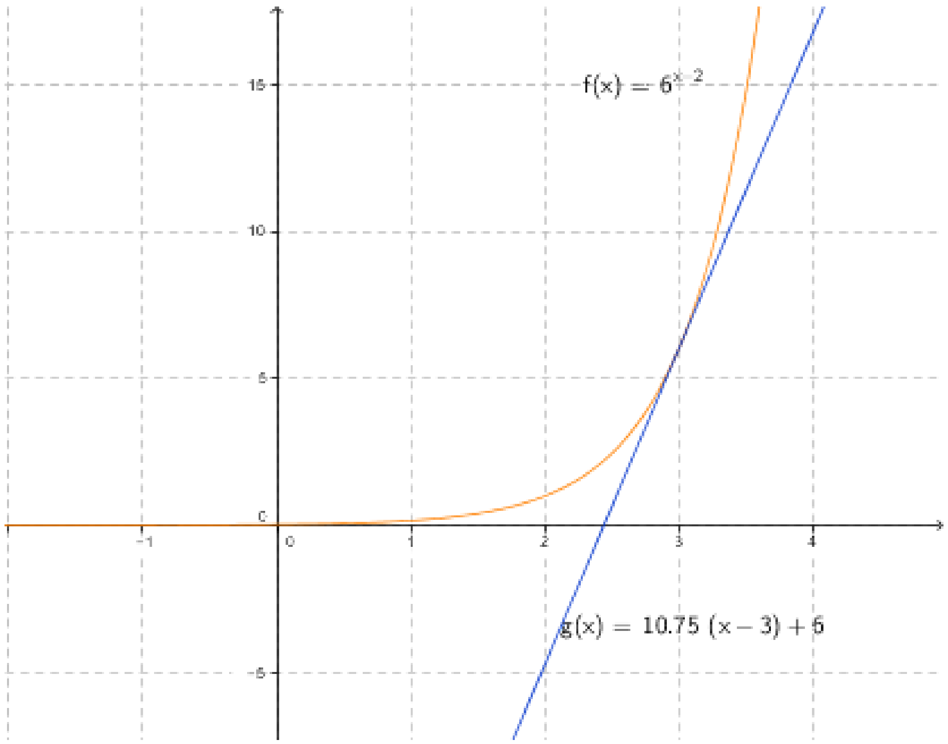
4. How long will it take money to double at 5% interest when compounded quarterly?

Solutions

1. a) $f'(x) = (6^{x-2})' = 6^{x-2}\ln 6$

   b) $f'(3) == 6^{3-2}\ln 6 = 6\ln 6$ and $f(3) = 6^{3-2} = 6$

   c) the equation of the tangent is then $y - 6 = (6\ln 6)(x - 3)$



2. Function $f(x) = 2x - \ln x$

a) Domain: $]0, +\infty[$

The range is $\mathbb{R} =]-\infty, +\infty[$

b)The first derivative: $f'(x) = 2 - \frac{1}{x}$

c) Calculate $f'(\text{x}) = 0 \Leftrightarrow 2 - \frac{1}{x} = 0$

$$\Leftrightarrow \frac{1}{x} = 2 \Leftrightarrow 2x = 1$$

$$\Leftrightarrow x = \frac{1}{2}$$

d) An example of a point through which the graph of $f(x)$ passes is $\text{A}(1, 2)$

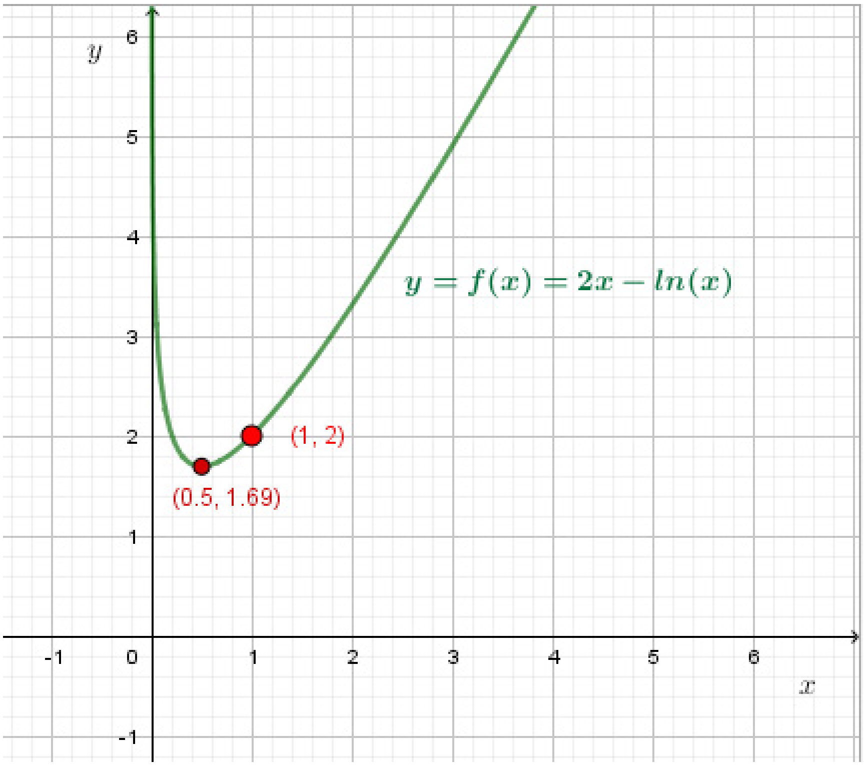
e) Variation table of $f(x)$

x	0	$\frac{1}{2}$	$+\infty$
$f'(x)$	$------$	0	$+++$
$f(x)$	$+\infty$ ↘	$1 + \ln 2$	↗ $+\infty$

Function f is decreasing on interval $]0, \frac{1}{2}[$ and increasing on $]\frac{1}{2}, +\infty[$.

The graph has a minimum at point $(\frac{1}{2}, 1 + \ln 2)$

f) The graph of f(x)



3. a). i. For once a year, at the end of 15 years the owner will have

$$A = P(1+r)^t = 1,000,000(1+0.10)^{15}$$
$$= 1,000,000(1.10)^{15} = 4,177,248.16\,frw$$

ii) For twice a year, at the end of 15 years , the owner will have

$$A = P(1+\frac{r}{2})^{2t} = 1,000,000(1+\frac{0.10}{2})^{2(15)}$$
$$= 1,000,000(1.05)^{30} = 4,321,942.37\,Frw$$

b) Conclusion: since $4,321,942.3 > 4,177,248.1$, compounding many times per year is better.

4. $A = P(1+\frac{0.05}{4})^{4t} = 2P \Leftrightarrow (1.0125)^{4t} = 2$

$t = 13.95$ years.

1.9.3 Extended activities

Suggestion of Questions and Answers for gifted and talented student-teachers.

1) The revenue R obtained by selling x units of a certain item at price p per unit is $R = xp$.

If x and p are related by $p(x)=8.25e^{-0.02x}$ .Find the price and the number of units to sell for the revenue to be maximized

2) Organic waste is dumped into a pond. As the waste material oxidizes, the level of oxygen in the pond is given by $f(t)=\frac{t-e^{-t}}{t}$, where t is the time in weeks.

Find the level of oxygen in the pond as the time gets larger and larger (express the answer in percentage).

Solution

1) The revenue is $\mathrm{R}(x)=8.25xe^{-0.02x}$

The revenue is maximum if $\mathrm{R}'(x)=(8.25xe^{-0.02x})'=0$

$\Leftrightarrow (8.25e^{-0.02x}-0.165\,\mathrm{x}\mathrm{e}^{-0.02x})'=0$

$\Leftrightarrow 1-0.02x=0 \Leftrightarrow x=50$;

$p(50)=8.25e^{-0.02(50)}=\frac{8.25}{e}=3.035$

2) $\lim_{t\to+\infty}\frac{t-e^{-t}}{t}=1-\lim_{t\to+\infty}\frac{e^{-t}}{t}=1-0=1$

As the time gets larger and larger, the level of oxygen approaches 1, that is 100%

UNIT 2

INTEGRATION

2.1 Key unit competence:

Use integration as the inverse of differentiation to solve problems related to marginal and total cost.

2.2 Prerequisites

Student-teachers will easily learn this unit if they are well skilled in:

- Logarithmic and exponential equations (Unit 2 Year 2)
- Logarithmic and exponential functions (unit 1 Year 3)
- Derivative of functions and their applications (unit 5 year 1)

2.3 Cross-cutting issues to be addressed

- Inclusive education (promote education for all while teaching).
- Peace and value Education (respect others' view and thoughts during class discussions).
- Gender (provide equal opportunity to boys and girls in the lesson).
- Financial education (mortgage, marginal cost, demand function, marginal demand,)
- Environment and sustainability (population growth and its effect on physical environment).

2.4 Guidance on introductory activity

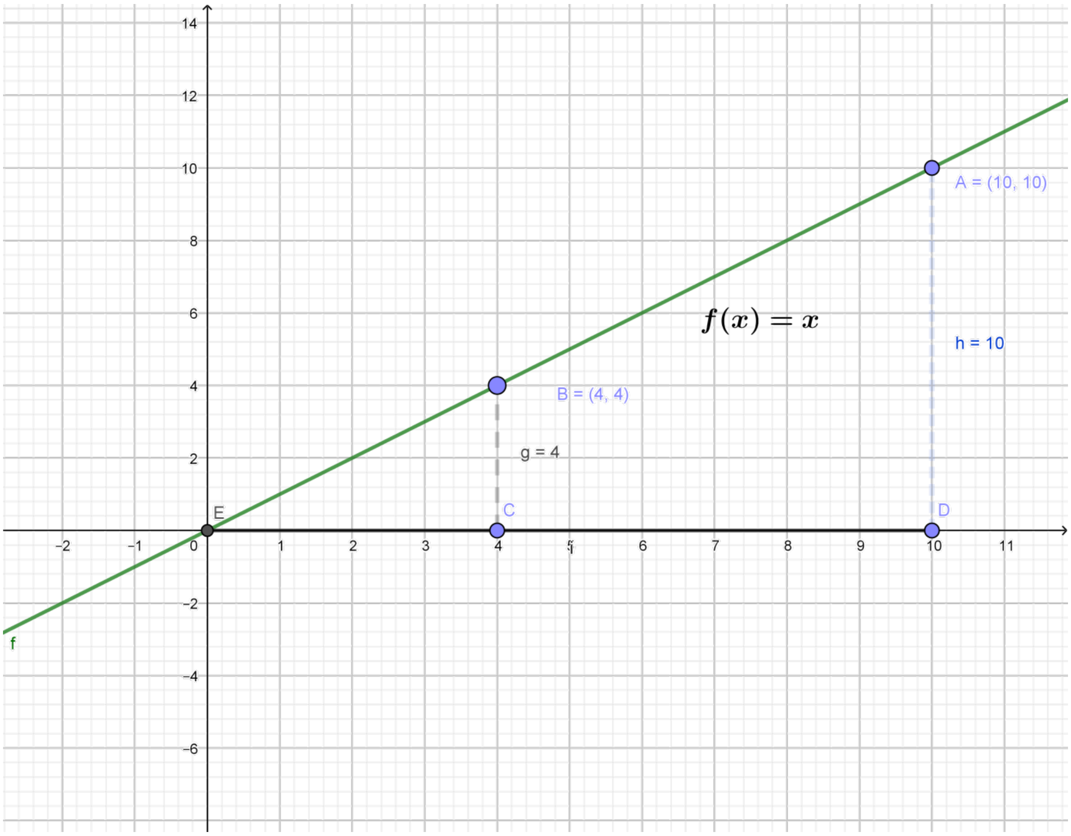
- Invite student-teachers to work in groups and give them instructions on how they can do the introductory activity 2 found in unit 2 of student's book ;
- Guide student-teachers to understand the concept of the anti- derivative; using integration as the inverse of differentiation through analysis of introductory activity 2.
- Invite group members to present groups' findings, then try to harmonize their answers;

- Basing on student-teachers' experience, prior knowledge and abilities shown in answering the questions for this activity, use different questions to facilitate them to give their predictions and ensure that you arouse their curiosity on what is going to be leant in this unit.

Answers for the introductory activity

a) One unit stands for one meter:

Figure: The quadrilateral field



1) The area A found by the first group is $A_1 = area(\Delta EDA) - area(\Delta ECB)$

Calculate the area of $area(\Delta EDA) = \frac{10x10}{2} = 50m^2$

Calculate the area of $area(\Delta ECB) = \frac{4X4}{2} = 8m^2$

Therefore $A_1 = (50-8) = 42m^2$

2) The second group with high critical thinking skills used a function $F(x)$ that was differentiated to find $f(x) = x$ (which means $F'(x) = f(x)$) and the x-coordinate d of D and the x- coordinate c of C in the following way:

$A_2 = F(d) - F(c)$.

Therefore $F(x) = \frac{x^2}{2} + c$, where c is a given constant

$F(x)$ Is said to be an integral of $f(x)$ or anti-derivative of $f(x)$, because

$F'(x) = f(x) = x$ in this case.

3) The area A_2 found by the second group using $F(x)$ is $A_2 = F(d) - F(c)$

$= F(10) - F(4)$

Then $F(10) = \frac{10^2}{2} = 50$ and $F(4) = \frac{4^2}{2} = 8$. Therefore

$A_2 = F(10) - F(4) \Leftrightarrow 50 - 8 = 42m^2$

4) We realize that A_1 and A_2 are equal (see results found in (1) and in (2).

Therefore, referring to the graph of the function $f(x)$ on the figure 2.0, you can find the area bounded by a function $f(x)$ and x-axis and the lines of equations are $x = x_1$ and $x = x_2$.

If $F'(x) = f(x)$, the area is calculated using $Area = F'(x_2) - F'(x_1)$.

2.5. List of lessons

#	Lesson title	Learning objectives	Number of periods
0.	Introductory activity	To arouse the curiosity of student teachers on the content of unit 2.	2
1	Increment and differential of a function	Calculate the differential of a given function	3
2	Definition of Indefinite integrals	Define indefinite integral State and clarify the relationship between derivative and anti-derivative of a function.	3
3	Properties of Indefinite integrals	Use properties of integrals to simplify the calculation of indefinite integrals	4
4	Basic integration formulae	Use properties and formulas of integrals to simplify the calculation of indefinite integrals	3

5	Integration by change of variables	Determine the integral of functions using the change of variables as a simple technique.	3
6	Integration by Parts	Determine the integral of functions by parts as a technique of integration.	3
7	Definition of definite integral	Define definite integral using Riemann's sum and the fundamental theorem of calculus	2
8	Properties of definite integrals	Use properties and formulas of integrals to simplify the calculation of indefinite integrals	2
9	Techniques of integration	Select the appropriate techniques of integration.	3
10	Application of definite integrals in economics and finance	Use of integration to model the marginal and total cost, consumer and producer surplus.	3
11	Application of definite integrals on Present and Future Values of an Income Stream.	Use integrals to solve different problems related to present and future values of an income stream	2
12	Application of integrals on the Population Growth Rate	Use integrals to solve different problems related to the population growth rate.	2
13	End unit assessment		1
Total			**36**

Lesson 1: Increment and differential of a function

a) Learning objectives

To calculate the differential of a given function

b) Teaching resources

Student-teachers' book, Reference books, pens, pencils, Mathematical set, scientific calculators. If possible computers with mathematical software such as Geogebra, Microsoft Excel, Math-lab and internet can be used.

c) Prerequisites/Revision/Introduction

Student-teachers will easily learn this unit, if they are well skilled in: 2

- Logarithmic and exponential equations (Unit 2 Year 2)
- Logarithmic and exponential functions (unit 1 Year 3)
- Derivative of functions and their applications (unit 5 year 1)

d) Learning activities:

- Invite student-teachers to work in groups and do the activity 2.1.1 found in their Mathematics books;
- Move around in the class for facilitating student-teachers where necessary and give more clarification on eventual challenges they may face during their work; Verify and identify groups with different working steps;
- Invite one member from each group with different working steps to present their work where they must explain the working steps;
- As a tutor, harmonize the findings from presentation of different groups and guide them to highlight the increment $\Delta y = f(x+\Delta x) - f(x)$ or $\Delta y = f'(x)\Delta x$ of the function $f(x)$ and the differential $dy = f'(x)dx$ for this function.
- Use different probing questions and guide them to explore the content and examples given in the student-teacher's book and lead them to calculate differentials;
- After this step, guide student-teachers to do the application activity 2.1.1 and evaluate whether lesson objectives were achieved and student-teachers develop competences.

Answer for activity 2.1.1

a) $y = f(x) = 4 + 0.5x + 0.1\sqrt{x}$;

The consumption at x= 2 is $f(2) = 4 + 0.5(2) + 0.1\sqrt{2} = 5 + 0.1\sqrt{2}$

The consumption at x=10 is $f(10) = 4 + 0.5(10) + 0.1\sqrt{10} = 9 + 0.1\sqrt{10}$

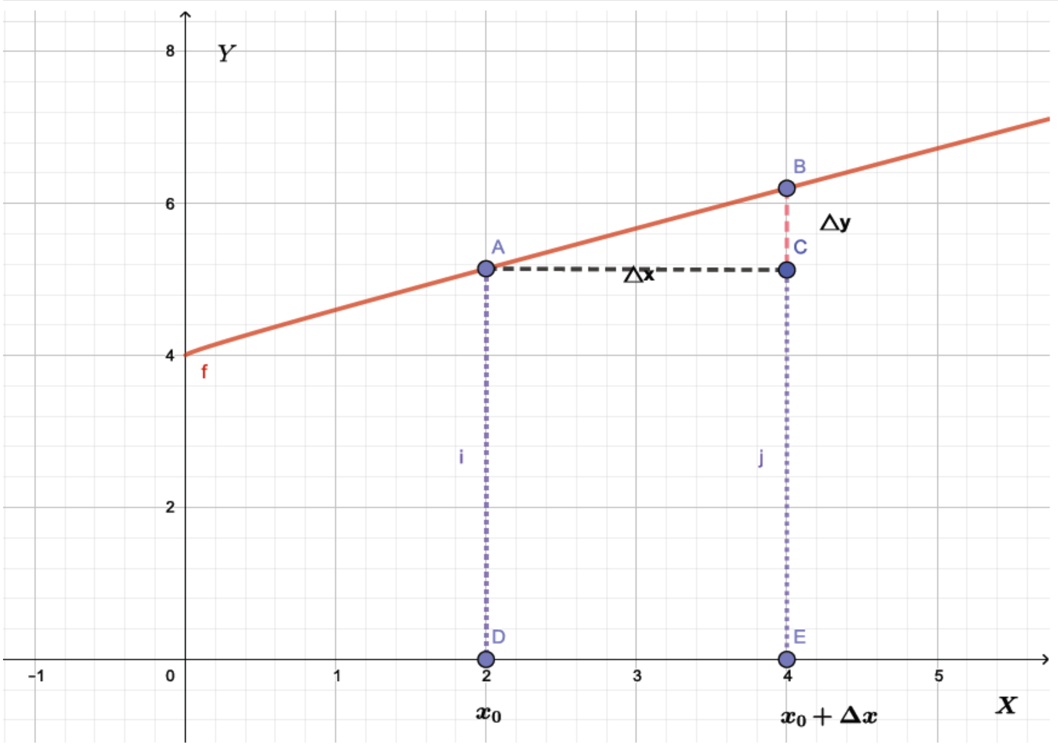
b) The corresponding increment of y is

$f(10) - f(2) = (9 + 0.1\sqrt{10}) - (5 + 0.1\sqrt{2}) \approx 4.175$

c) If x changes from x_0 to x_1 where $(x_1 > x_0)$, then $f(x)$ changes from $f(x_0)$ to $f(x_1)$ the increment of x is $\Delta x = x_1 - x_0$ and the change in y is $\Delta y = f(x_1) - f(x_0) = \ldots$

d) If Δx is very small, then we have $\lim_{\Delta x \to 0} \Delta y = \lim_{\Delta x \to 0} f'(x_0)\Delta x = dy = f'(x_0)dx$.

e) Graphical interpretation:



In this regard, $dy = 0.5 + \frac{0.1}{2\sqrt{x}} dx$.

Answer for application 2.1.1

1) a) $d(x^2e^x) = (2x + x^2)e^x dx$

b) $d(\frac{\ln x}{x}) = (\frac{1 - \ln x}{x^2})dx$

2) Let x be the edge of the tank in the form of the shape of cube and V its volume. We have $x = 4m$ and $dx = 0.02m$. However $V = x^3$ and $dv = 3x^2dx$

Therefore, $dv = 3x^2dx \Leftrightarrow dv = 3(4)^2 \times 0.02 = 0.96m^3$,

The capacity of the container in liters is $V = x^3 = (4m)^3 = 64m^3 = 64000l$

Thus, the approximation error on the measurement of the volume is $0.96m^3$

Lesson 2: Definition of Indefinite integrals

a) Learning objectives

- Define indefinite integral
- State and clarify the relationship between derivative and anti-derivative of a function.

b) Teaching resources

Student-teacher's book and other reference books to facilitate research, Mathematical set, calculator, Manila paper, markers, pens, pencils...

c) Prerequisites/Revision/Introduction

Student-teachers will get a better understanding of the content of this lesson if they refer dderivative of functions and their applications (unit 5 year 1) and previous lessons

d) Learning activities:

- Invite student-teachers to work in groups and do the activity 2.1.2 found in their Mathematics Student books;
- As they are working, move around to ensure that they are performing the task effectively and cooperatively.
- Provide assistance where necessary by facilitating student-teachers to determine the positions of the caterpillars, to draw the graphs and to guess a function differentiated to find a given derivative.
- Invite every group to present their findings to the whole class and then,
- As a tutor harmonize the findings from presentations and guide students to highlight that the function $F(x)$, whose derivative is $f(x)$, is called anti-derivative of $f(x)$;
- Use different probing questions and guide student-teachers to explore the content and examples given in the student-teachers' book and guide them to discover the definition of indefinite integral.
- After this step, guide student-teachers to work individually the application activity 2.1.2 and evaluate whether lesson objectives were achieved.

Answer for activity 2.1.2

1) If V is the constant velocity, the position of a moving body at time t is given by

$e(t) = vt + e_0$ where e_0 is the initial position.

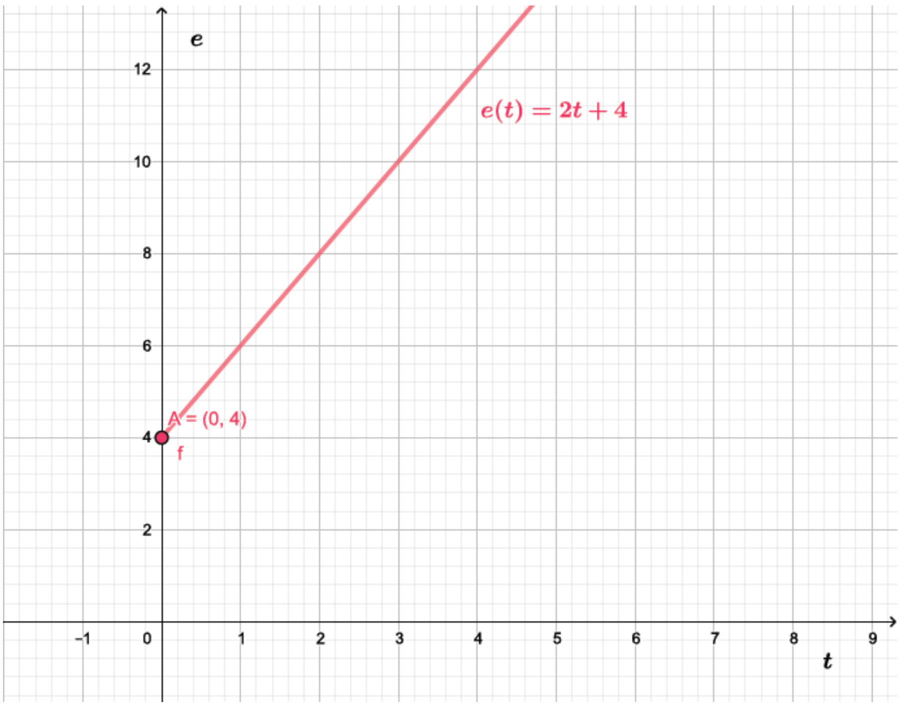
i) $e(t) = 2t + 1$

ii) $e(t) = 2t + 2$

iii) $e(t) = 2t + 4$

2) For the third caterpillar, $e(t) = 2t + 4$.

Therefore, $e'(t) = 2$ m/min which is the velocity.



$e'(t) = 4 = v(t)$

3. i) $F(x) = 2\left(\frac{x^2}{2}\right) + c$ where C is a constant, since

$F'(x) = \left[2\left(\frac{x^2}{2}\right) + c\right] = \frac{4x}{2} = 2x$

ii) There are infinitely many possibilities for $F(x)$ because C can take different values in the set of real numbers.

iii) They all differ by a constant.

Answer for application activity 2.1.2

a) $\int xdx = \frac{x^2}{2} + c$

b) $\int 3xdx = \frac{3x^2}{2} + c$

b) $\int x^2 dx = \frac{x^3}{3} + c$

Lesson 3: Properties of Indefinite integrals

a) Learning objectives

Use properties of integrals to simplify the calculation of indefinite integrals

b) Teaching resources

Student-teacher's book and other Reference books to facilitate research, Mathematical set, calculator, Manila paper, markers, pens, pencils. if possible computers with mathematical software such as Geogebra, Microsoft Excel, Math-lab or graph-application and internet can be used...

c) Prerequisites/Revision/Introduction

Student-teachers will get a better understanding of the content of this lesson if they refer derivative of functions and their applications (unit 5 year 1) and previous lessons of this unit.

d) Learning activities:

- Invite student-teachers to work in groups and do the activity 2.1.3 in their Mathematics books;
- Move around in the class for facilitating student teachers where necessary.
- Verify and identify groups with different working steps;
- Invite one member from each group with different working steps to present their work where they must explain the working steps;
- As a tutor, harmonize student-teachers' findings from the presentation.
- Use different probing questions and guide student teachers to explore the content and examples given in the student- teachers' book and lead them to discover the properties of indefinite integrals
- After this step, guide student-teachers to do the application activity 2.1.3 and evaluate whether lesson objectives were achieved.

Answer for activity 2.1.3

a) i) $I_1 = \int f(x)\,dx = \int 5dx = 5x + c$

ii) $I_2 = \int g(x)\,dx = \int \frac{dx}{x} = \ln|x| + c$

b) $I = \int (f+g)(x)\,dx = \int \left(5 + \frac{1}{x}\right)dx = 5x + \ln|x| + c$

c) $I = I_1 + I_2 = 5x + \ln|x| + c$, yes there are the same

Answer for application activity 2.1.3

1. a. $\int (x^3 + 3\sqrt{x} - 7)dx = \frac{1}{4}x^4 + 2\sqrt{x^3} - 7x + C$

   b. $\int (4\mathrm{x} - 12\mathrm{x}^2 + 8\mathrm{x} - 9)dx = 2x^2 - 4x^3 + 4x^2 - 9x + C$

   c. $\int (\frac{1}{x^2} + \mathrm{e}^{-x} - \frac{2}{x})dx = -\frac{1}{x} - e^{-x} - 2\ln x + C$

2. It is not correct because the integral of a quotient is not the quotient of integrals.

$$\int \frac{x^3 - 2}{x^3}dx = \int (1 - 2x^{-3})dx = x + \frac{1}{x^2} + C$$

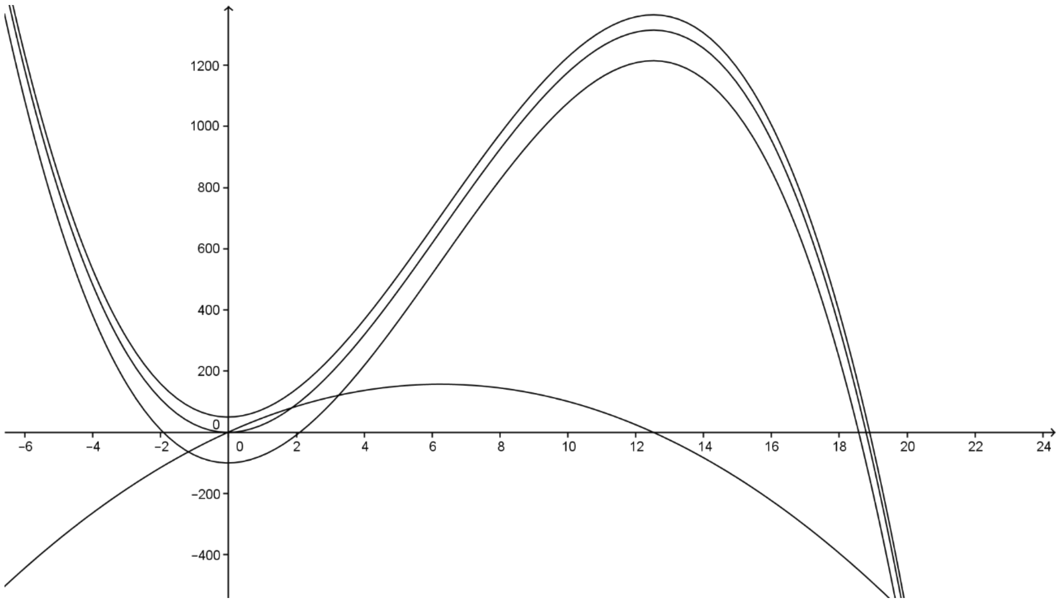
3. As $\int \frac{dy}{dx} = y$, Then $y = \int \frac{x^3 - 5}{x^2}dx = \int \left(x - 5x^{-2}\right)dx = \frac{1}{2}x^2 + \frac{5}{x} + c$

   Thus, $y = f(x) = \frac{1}{2}x^2 + \frac{5}{x} + c$; $f(1) = \frac{1}{2} + 5 + c \Leftrightarrow c = -5$

Therefore, $f(x) = \frac{1}{2}x^2 + \frac{5}{x} - 5$

4. $f(x) = \int (1 + 50x - 4x^2)dx = x + 25x^2 - \frac{4}{3}x^3 + C$

Figure: graph of the marginal cost and three of its corresponding possible total costs



Lesson 4: Basic integration formulae

a) Learning objectives

Use properties and formulas of integrals to simplify the calculation of indefinite integrals

b) Teaching resources

Textbooks and if possible the internet to facilitate research

c) Prerequisites/Revision/Introduction

Student-teachers will get a better understanding of the content of this lesson if they refer dderivative of functions and their applications (unit 5 year 1) and previous lessons

d) Learning activities:

- Invite student-teachers to work in groups and do the activity 2.2.1 found in their Mathematics Student books;
- Move around in the class for facilitating students where necessary and give more clarification on eventual challenges they may face during their work; Verify and identify groups with different working steps;
- Invite one member from each group with different working steps to present their work where they must explain the working steps;

- As a tutor, harmonize the findings from presentation and guide student-teachers to discover that the decay means the diminution in umbers by

  encouraging student-teachers to realize that $\int a^x dx = \frac{a^x}{\ln a} + c$ using anti-derivatives properties from the given data.
- Help them realize that indefinite integral of a given function is a set of all anti-derivatives of that function and that any anti-derivatives F of the function f, every possible anti-derivatives of f can be written in the form of $F(\text{x}) + c$,where c is any constant
- Invite student-teachers to explore and discuss the list of basic integration formula of indefinite integral in their textbook.
- Invite student-teachers to workout individually application activities 2.2.1 to improve their skills in calculating indefinite integral of functions by using definition and basic integration formula

Answer for activity 2.2.1

a) $\int 5x^2 dx = 5\int x^2 dx = 5\frac{x^3}{3} + c$ (used of basic integration formula)

b) $\int \frac{2x+1}{x^2+x+4} dx = \ln\left|x^2+x+4\right| + c$ (used basic integration formula)

Answer for application activity 2.2.1

1. $\int e^{3x+1} dx = \frac{e^{3x+1}}{3} + C$

2. $\int 3^x dx = \frac{3^x}{\ln 3} + C$

3) $\int (8 - x^5) dx = \int 8dx - \int x^5 dx = 8x - \frac{x^6}{6} + C$

4) $\int \frac{dx}{\sqrt{x^2+9}} = \ln\left|x + \sqrt{x^2+9}\right| + c$

Lesson 5: Integration by change of variables

a) Learning objectives

Use properties and integration by change of variables as a technique of integrals to simplify the calculation of indefinite integrals

b) Teaching resources

T-square, ruler and textbooks. If possible mathematical software such as geogebra, Math-lab and internet.

c) Prerequisites/Revision/Introduction

Student-teachers will get a better understanding of the content of this lesson if they refer dderivative of functions and their applications (unit 5 year 1) and previous lessons

d) Learning activities:

- Invite student-teachers to work in groups and do the activity 2.2.2 found in their Mathematics books;
- Follow up the working steps of different group to give support where necessary and motivate student-teachers to mention any difficulty met when integrating $\int 2x(x^2+4)^5 dx$ if any.
- Stimulate student-teachers to let $u = x^2+4$, find 1st derivative of $u = x^2+4$ and through group discussion, allow them to determine $\int 2x(x^2+4)^5 dx$ using expression of u.
- Invite each group to present their findings.
- Harmonize the student-teachers 'works and help them realize that it is not easy to determine integral of the form $\int 2x(x^2+4)^5 dx$ by using basic formula but the task becomes very simple if we change the variable to obtain a new expression. It means that if we cannot integrate $\int h(x)dx$ directly, we should find a new variable u and function $f(u)$ for which $\int h(x)dx = \int f(u(x))\frac{du}{dx}dx = \int f(u)du$. The method is the integration by changing variables or integration by substitution.
- After this step, guide student-teachers to do individually the application activity 2.2.2 for enhancing their skills in calculating integral of functions by changing variables or by substitution.

Answer for activity 2.2.2

1. $i)\int x^5 dx = \frac{x^{5+1}}{5+1} + C = \frac{x^6}{6} + C$

   $i) \therefore \int x^5 dx = \frac{x^6}{6} + C$

   ii) Cannot be integrated using basic formulae.

2. $\int 2x(x^2+4)^5 dx$

To integrate this immediately is more difficult because you can't find the integration formula which can be applicable immediately.

Therefore substitution method or changing variable can help as follows:

Let $u = x^2 + 4$

$\frac{du}{dx} = 2x$

By multiplying all sides by dx we get $du = 2xdx$

Replacing variables u by $x^2 + 4$ and du by $2xdx$,gives

$\int 2x(x^2+4)^5 dx = \int u^5 du$

To integrate $\int u^5 du$ is easy by using basic integration formula seen previously.

$\int u^5 du = \frac{u^6}{6} + C$

By substituting u by $x^2 + 4$,

$\frac{u^6}{6} + C = \frac{(x+4)^6}{6} + C$

Therefore $\int 2x(x^2+4)^5 dx = \frac{(x+4)^6}{6} + C$

Application activity 2.2.2

Determine the following integrals

$1)\int xe^{x2}dx$

Let $u = x^2$

$\Rightarrow du = 2xdx$

$\frac{1}{2}du = xdx$

Therefore

$\int xe^{x2}dx = \int \frac{1}{2}e^{u}du = \frac{1}{2}\int e^{u}du$

$\Rightarrow \frac{1}{2}e^{u} + C \Rightarrow \frac{1}{2}e^{x2} + C$

Therefore

$\int xe^{x2}dx = \frac{1}{2}e^{x2} + C$

$2)\int \frac{dx}{(1-2x)^2}$

$u = 1-2x$, $du = -2dx \Rightarrow dx = \frac{-1}{2}du$

$$\int \frac{dx}{(1-2x)^2} = \frac{-1}{2}\int \frac{du}{u^2} = \frac{-1}{2}\int u^{-2}du = \frac{-1}{2}\left(\frac{u^{-2+1}}{-2+1}\right) + C = \frac{-1}{2}(-u^{-1}) + C = \frac{1}{2}(u^{-1}) + C$$

Substituting u by $1-2x$, we get $\int \frac{dx}{(1-2x)^2} = \frac{1}{2}[(1-2\text{x})^{-1}] + C$

$$3)\int \frac{x+x^2}{(-3x^2+4-2x^3)^3}dx =$$

Let: $u = 4-3x^2-2x^3$

$$du = (-6x-6x^2)dx \Rightarrow dx = \frac{du}{-6(\mathrm{x}+\mathrm{x}^2)}$$

Substituting u by $4-3x^2-2x^3$ and dx by $\frac{du}{-6(\mathrm{x}+\mathrm{x}^2)}$

$$3\int \frac{x+x^2}{(-3x^2+4-2x^3)^3}dx = \frac{-1}{6}\int \frac{du}{u^3} = \frac{1}{12u^2}+c$$

Coming back to the variable x

we get

$$\int \frac{x+x^2}{(-3x^2+4-2x^3)^3}dx = \frac{1}{12(4-3x^2-2x^3)^2}+C$$

$$4)\int \frac{x}{(1-2x^2)^{\frac{1}{3}}}dx =$$

Let

$u = 1-2x^2$

$$du = -4xdx \Rightarrow dx = \frac{du}{-4x}$$

$$\Rightarrow \int \frac{x}{(1-2x^2)^{\frac{1}{3}}}dx = \int -\frac{1}{4u^{\frac{1}{3}}}du = -\frac{1}{4}\int u^{\frac{-1}{3}}du = -\frac{1}{4}(\frac{\mathrm{u}^{\frac{2}{3}}}{\frac{2}{3}})+\mathrm{C} = -\frac{3}{8}(u^{\frac{2}{3}})+C$$

Finally

$$\int \frac{x}{(1-2x^2)^{\frac{1}{3}}}dx = -\frac{3}{8}(1-2x^2)^{\frac{2}{3}} + C$$

$$5)\int x\sqrt{-1+x^2}\,dx =$$

$$u = -1 + x^2$$

$$du = 2xdx \Rightarrow dx = \frac{du}{2x}$$

$$\int x\sqrt{-1+x^2}\,dx = \int \frac{1}{2}\sqrt{u}\,du = \frac{2}{6}u^{\frac{3}{2}} + C$$

By substitution we get $\int x\sqrt{-1+x^2}\,dx = = \frac{2}{6}(-1+\mathrm{x}^2)^{\frac{3}{2}} + C$

Lesson 6: Integration by Parts

a) Learning objectives

Use properties and integration by parts as a technique of integrals to simplify the calculation of indefinite integrals

b) Teaching resources

Textbooks, mathematics softwares and Internet if available.

c) Prerequisites/Revision/Introduction

Student-teachers will perform well in this lesson if they refer to: Differential of a function, anti-derivatives of a function, basic integration formula for integration and Integration by substitution seen in previous lessons (year three, unit2).

d) Learning activities:

- Invite student-teachers to work in groups and do the activity 2.2.3 found in their Mathematics books;
- Move around in the class for facilitating students where necessary and give more clarification on eventual challenges they may face during their work; Verify and identify groups with different working steps;
- Invite one member from each group with different working steps to present their work where they must explain the working steps

- Through harmonization guide learners to find that there are some integrals that cannot be found using the substitution method, to overcome this situation, we use other methods such as the integration by parts where the integration of the product of two functions u and dv can be obtained by using the formula $\int udv = u \times v - \int vdu$ which is deduced from the derivative of a product of two functions u and v.
- Let student-teachers go through the example from their books, and individually work out application activities 2.2.3 to develop their skills in calculating integral of functions by integration by parts.

Answer for activity 2.2.3

1. $\int 3x^2(x^3+1)dx =$

Let $u = x^3 + 1$

$$du = 3x^2 dx \Rightarrow dx = \frac{du}{3x^2}$$

$$\int 3x^2(x^3+1)dx = \int \frac{1}{3}udu = \frac{1}{3}\frac{u^2}{2} + C = \frac{1}{6}u^2 + C$$

$$\int 3x^2(x^3+1)dx = \frac{1}{6}(x^3+1) + C$$

2. $\int xe^x dx =$

i) It is not easy to integrate this integral by substitution.

ii) Let $u = x$, then, $du = dx$, $dv = e^x dx \Rightarrow v = \int dv = \int e^x dx = e^x$

iii) $\int udv = \int xe^x dx$ (are equals)

iv) $\int udv = uv - \int vdu = xe^x - \int e^x dx = xe^x - e^x + c$

Therefore, $\int xe^x dx = xe^x - e^x + c$

Answer for application activity 2.2.3

1. $\int 3xe^{-x}dx = 3\int xe^{-x}dx$

Integrate this by integration by parts method: let $u = x$ then $du = dx$

$dv = e^{-x}dx \Rightarrow v = \int e^{-x}dx = -e^{-x}$

Thus $\int udv = u.v - \int vdu = x(-e^{-x}) - \int(-e^{-x})dx = -xe^{-x} - e^{-x} + C$

$\int 3xe^{-x}dx = 3(-xe^{-x} - e^{-x}) + C$

2.

$\int x^2 \ln xdx =$ Let use integration by parts.

Set $\begin{cases} u = \ln x \\ dv = x^2dx \end{cases}$, then $\begin{cases} du = \frac{1}{x}dx \\ v = \frac{x^3}{3} \end{cases}$

$\int udv = u.v - \int vdu$

$\int udv = (\frac{x^3}{3})\ln x - \int \frac{x^3}{3}\frac{dx}{x} = \frac{x^3}{3}\ln x - \frac{1}{3}\int x^2dx = \frac{x^3}{3}\ln x - \frac{1}{9}x^3 + C$

$\int x^2 \ln xdx = \frac{x^3}{3}\ln x - \frac{1}{9}x^3 + C = \frac{x^3}{3}(\ln x - \frac{1}{3}) + c$

3. $\int x\sqrt{x+5}dx$

Integration by parts $\begin{cases} u = x \\ dv = \sqrt{x+5} \end{cases} \Rightarrow \begin{cases} du = dx \\ v = \frac{2}{3}(x+5)^{\frac{3}{2}} \end{cases}$

$\int x\sqrt{x+5}dx \Leftrightarrow \frac{2}{3}x(x+5)^{\frac{3}{2}} - \int \frac{2}{3}(x+5)^{\frac{3}{2}}dx$

Therefore $\int x\sqrt{x+5}dx = \frac{2}{3}x(x+5)^{\frac{3}{2}} - \frac{4}{15}(x+5)^{\frac{5}{2}} + C$

Lesson 7: Definition of definite integral

a) Learning objectives

Define definite integral

b) Teaching resources

- Scientific calculators, ruler, textbooks, graph papers.
- Internet to facilitate research
- If possible, the use of Mathematical software such as Geogebra to plot graphs of functions is essential.

c) Prerequisites/Revision/Introduction

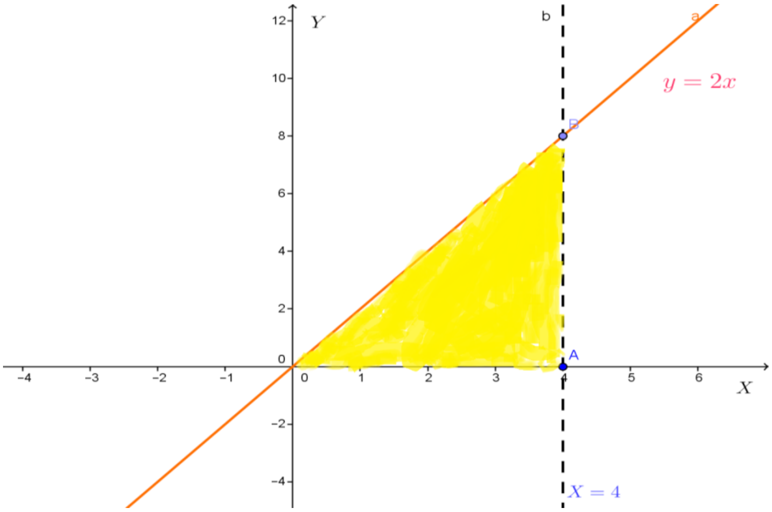
Student-teachers will perform well in this lesson if they refer to: Differential of a function, anti-derivatives of a function and basic integration formula for integration seen in previous lessons (year three, unit2). Linear functions, equations and inequalities (Senior 1, unit 3) and Linear and quadratic functions (Senior 3,Unit 5)

d) Learning activities:

- Invite student-teachers to work in groups and do the activity 2.3.1 found in their Mathematics books;
- Ask student-teachers to plot in a Cartesian plane the following: $f(x)=2x$, $y=0$, $x=0$, and $x=4$ and let student-teachers identify and name the shape made by the lines $y=2x$, $y=0$, $x=0$, and $x=4$;
- In the same groups, ask student-teachers to determine the area of the obtained triangle using the formula and then request student-teachers to find the anti-derivative $F(x)$ of $f(x)=2x$, then calculate $F(4)-F(0)$ and then compare the area of the shape and the result of $F(4)-F(0)$.
- Move around in the class for facilitating student-teachers where necessary and give more clarification on eventual challenges they may face during their work; Verify and identify groups with different working steps;
- Invite randomly some groups to present their findings to the whole class and after presentation facilitate student-teachers to have a lesson summary
- As a tutor, harmonize the findings from presentation;
- Guide student-teachers to explore the content given in the student-teacher's book and lead them to the definition of definite integral.
- Individually, invite student-teachers to read through examples prepared in their books and then work out the application activities 2.3.1 to enhance their knowledge and skills about definite integrals.

Answer for activity 2.3.1

a)



The shape obtained is the Triangle that has three vertices $O(0,0)$, $A(4,0)$, and $B(4,8)$

b) Area of Triangle $= \frac{BH}{2}$

The base $B = 4$ *units of length*, and the height $H = 8$ *units of length*

$$Area = \frac{4\times 8}{2} = 16 \text{ square units}$$

The area of the triangle is 16 *square units*

c) The anti-derivative of $f(x) = 2x$ is $F(x) = x^2 + c$

$$F(4) - F(0) = \left[(4^2 + c) - (0^2 + c)\right] = 16 + c - c = 16$$

Comparison shows that the findings are the same.

Thus, the area of the triangle = $F(4) - F(0) = 16$ *square units*

Answer for application activity 2.3.1

1) $\int_0^3 xdx = \left[\frac{x^2}{2}\right]_0^3 = \left(\frac{3^2}{2} - \frac{0^2}{2}\right) = \frac{9}{2} - \frac{0}{2} = \frac{9}{2}$

2) $\int_0^3 (x+1)dx = \int_0^3 xdx + \int_0^3 1dx = \left[\frac{x^2}{2}\right]_0^3 - \left[\frac{0^2}{2}\right]_0^3 + (1\times 3 - 1\times 0)$

$= \frac{3^2}{2} - \frac{0}{2} + 3 = \frac{15}{2}$

Lesson 8: Properties of definite integral

a) Learning objectives

Use properties and formulas of integrals to simplify the calculation of indefinite integrals

b) Teaching resources

- Scientific calculators, ruler, textbooks, graph papers.
- Internet to facilitate research
- If possible, the use of Mathematical software such as Geogebra to plot graphs of functions is essential.

c) Prerequisites/Revision/Introduction

Student-teachers will perform well in this lesson if they refer to: Differential of a function, anti-derivatives of a function and basic integration formula for integration seen in previous lessons (year three, unit2)

d) Learning activities:

- Invite student-teachers to work in groups and do the activity 2.3.2 in their Mathematics books;
- Move around in the class for facilitating student teachers where necessary.
- Verify and identify groups with different working steps;
- Invite one member from each group with different working steps to present their work where they must explain the working steps;
- As a tutor, harmonize their findings from the presentation.;
- Use different probing questions and guide student teachers to explore the content and examples given in the student teachers' book and lead them to discover the properties of definite integral
- After this step, guide students to do the application activity 2.3.2 and evaluate whether lesson objectives were achieved.

Answers of activity 2.3.2

1) a) $\int_1^2 f(x)dx = \frac{27}{4}$; b) $\int_1^2 g(x)dx = -\frac{14}{3}$; c) $\int_1^2 (f+g)(x)dx = \frac{25}{12}$

2) $\int_1^2 (f+g)(x)dx = \int_1^2 f(x)dx + \int_1^2 g(x)dx = \frac{27}{4} + (-\frac{14}{3}) = \frac{25}{12}$ they are the same

and we find $\int_a^b (f+g)(x)dx$ as $\int_a^b f(x)dx + \int_a^b g(x)dx = [F(x)]_a^b + [G(x)]_a^b$.

Answers for application activity 2.3.2

1. $\int_1^2 (4x^2 - 3x)dx =$

$$\int_1^2 (4x^2 - 3x)dx = \left[\frac{4x^3}{3} - \frac{3x^2}{2}\right]_1^2 = \left[\left(\frac{4\times 2^3}{3} - \frac{3\times 2^2}{2}\right) - \left(\frac{4\times 1^3}{3} - \frac{3\times 1^2}{2}\right)\right]$$

2. Given $f(x) = y = 30 - 2x - x^2$

a) If $xo = 3,\ y_0 = 30 - 2\times 3 - 3^2 = 30 - 6 - 9 = 30 - 15 = 15$

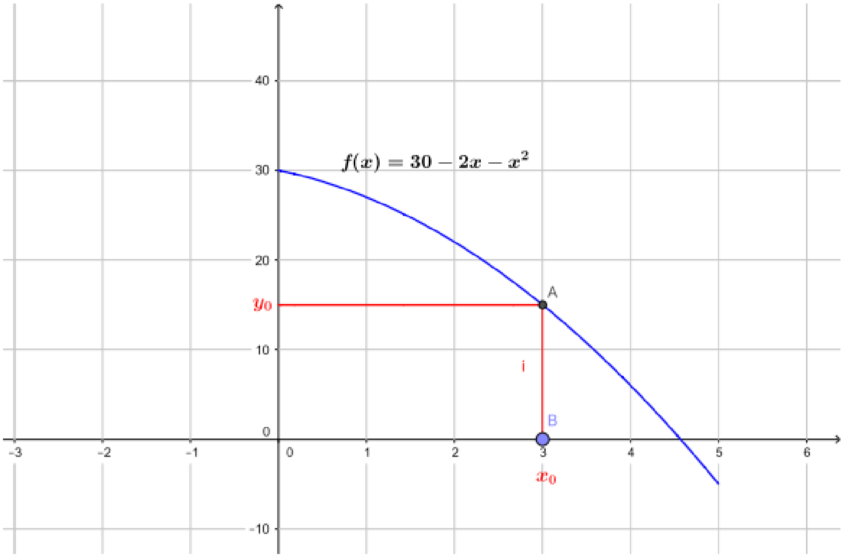
The consumer's surplus $= \int_0^{x_0} f(x)dx - x_0 y_0$

$$= \left[30x - 2\frac{x^2}{2} - \frac{x^3}{3}\right]_0^3 - (3\times 15)$$

$$= [(30\times 3) - 9 - 9] - 45$$

$$= 90 - 18 - 45 = 27$$

b) The graph of $f(x) = y = 30 - 2x - x^2$



Lesson 9: Techniques of integration

a) Learning objectives

Select and use the appropriate techniques of integration.

b) Teaching resources

student teacher's book and other Reference books to facilitate research, scientific calculator, Manila paper, markers, pens, pencils, if possible computers with mathematical software such as Geogebra, Microsoft Excel, Math-lab or graph-application and internet can be used...

c) Prerequisites/Revision/Introduction

Student-teachers will perform well in this lesson if they refer to: Differential of a function, anti-derivatives of a function and basic integration formula for integration seen in previous lessons (year three, unit2).

d) Learning activities:

- Invite student-teachers to work in groups and do the activity 2.3.3 in their Mathematics books;
- Move around in the class for facilitating student teachers where necessary.
- Verify and identify groups with different working steps;
- Invite one member from each group with different working steps to present their work where they must explain the working steps;
- As a tutor, harmonize their findings from the presentation.;

- Use different probing questions and guide student teachers to explore the content and examples given in the student teachers' book and lead them to discover and use techniques of integration to calculate integrals.
- After this step, guide students to do the application activity 2.3.3 and evaluate whether lesson objectives were achieved.

Answer for activity 2.3.3

1) $f(x) = e^{x^2}$

i) $t = x^2$, then $x = 0 \Rightarrow t = 0$ and $x = 2 \Rightarrow t = 2^2 = 4$

ii) $t = x^2 \Rightarrow dt = 2xdx$ and $dx = \frac{dt}{2x}$

iii) $\int_0^2 2xe^{x^2} dx \Rightarrow \int_0^4 2xe^t \frac{dt}{2x} = \int_0^4 e^t dt = \quad e^t \Big|_0^4$

iv) It is clear that when we apply the substitution method, we also substitute boundaries to keep integral the same.

2) Evaluation of $\int_1^e \text{x}^2 \ln xdx$

Let evaluate this integral by parts.

Let $u = \ln x$ and $dv = x^2 dx$

$du = \frac{dx}{x}$ and $v = \int x^2 dx = \frac{x^3}{3}$

$$\int_1^e \text{x}^2 \ln xdx = \frac{x^3}{3} \ln x\Big|_0^e - \int_0^e \frac{x^3}{3}\frac{dx}{x} = = \frac{x^3}{3} \ln x\Big|_0^e - \frac{1}{9}x^3\Big|_0^e = \frac{e^3}{3} - \frac{e^3}{9} = \frac{2e^3}{9}$$

Answers for application activity 2.3.3

1. $\int_0^1 \ln(1+x)dx$ using integration by parts

Let $u = \ln(1+x) \Rightarrow du = \frac{1}{1+x}$ $and \quad dv = dx \Rightarrow v = x + c$

$I = \int_a^b udv = uv - \int_a^b vdu \quad becomes$

$$I=\int_0^1 \ln(1+x)=\left[x\ln(1+x)\right]_0^1-\int_0^1 \frac{x}{1+x}dx$$

$$=\ln 2-\int_0^1\left(1-\frac{1}{1+x}\right)dx=\ln 2-\left[x-\ln(1+x)\right]_0^1=\ln 2-1+\ln 2=-1+2\ln 2=-1+\ln 4$$

2. $\int_1^2 x^2\ln x dx$; Let $u=\ln x$ and $dv=x^2dx$

$du=\frac{dx}{x}$ and $v=\int x^2dx=\frac{x^3}{3}$

$$\int_1^e \text{x}^2\ln x dx=\frac{x^3}{3}\ln x\Big|_0^e-\int_0^e\frac{x^3}{3}\frac{dx}{x}==\frac{x^3}{3}\ln x\Big|_0^e-\frac{1}{9}x^3\Big|_0^e=\frac{e^3}{3}-\frac{e^3}{9}=\frac{2e^3}{9}$$

3. $\int_0^2 3x^2e^{-x}dx$; $Let\ u=x^2\Rightarrow du=2dx\quad and\ dv=e^{-x}dx\Rightarrow v=-e^{-x}$

$Thus, \int_0^2 3x^2e^{-x}dx=\left[-x^2e^{-x}\right]_0^2+2\int_0^2 xe^{-x}dx$

$For \int_0^2 xe^{-x}dx \quad Let\ u=x\Rightarrow du=dx\ and\ dv=e^{-x}dx\Rightarrow v=-e^{-x}$

$Therefore, \int_0^2 3x^2e^{-x}dx=3\left[-x^2e^{-x}+2(-xe^{-x}-e^{-x}\right]_0^2=\frac{6\times e^2-30}{e^2}=\frac{6(\text{e}^2-5)}{e^2}$

Lesson 10: Applications of definite integrals: marginal and total cost, consumer and producer surplus

a) Learning objectives

Illustrate the use of basic interpretation properties, techniques and formulas to model economics functions

b) Teaching resources

student teacher's book and other Reference books to facilitate research, scientific calculator, Manila paper, markers, pens, pencils, if possible computers with mathematical software such as Geogebra, Microsoft Excel, Math-lab or graph-application and internet can be used...

c) Prerequisites/Revision/Introduction

Student-teachers will perform well in this lesson if they refer to unit 2 of year three (previous lessons of this unit)

d) Learning activities:

- Invite student-teachers to work in groups and do the activity 2.4.1 in their Mathematics books; Move around in the class for facilitating student teachers where necessary.
- Verify and identify groups with different working steps;
- Invite one member from each group with different working steps to present their work where they must explain the working steps;
- As a tutor, harmonize their findings from the presentation.;
- Use different probing questions and guide student teachers to explore the content and examples given in the student teachers' book and guide them in the application of skills they have about integration to solve problems involving marginal and total cost, consumer and producer surplus
- After this step, guide students to do the application activity 2.4.1 and evaluate whether lesson objectives were achieved.

Answer for activity 2.4.1

The given marginal cost is $3(q-4)^2$

$$\text{The total manufacturing cost} = \int_6^{10} 3(q-4)^2\,dq$$

$$\int_6^{10} 3(q-4)^2\,dq = 3\int_6^{10}(q^2-8q+16)dq$$

$$3\int_6^{10}(q^2-8q+16)dq = \left[\frac{3q^3}{3}-\frac{3\times 8q}{2}+48q\right]_6^{10}$$

$$\left[\frac{3q^3}{3}-\frac{3\times 8q}{2}+48q\right]_6^{10} = (1000-120+480)-(216-72+288)$$

$$=1360-432$$
$$=928$$

The total manufacturing cost is 928 dollars.

Answers for application activity 2.4.1

1. $P = 1800 - 0.6q^2$
$MR = 1800 - 1.8q^2$

i) TR when q is 10

$$\int_0^{10} MRdq = \int_0^{10} (1800 - 1.8q^2)dq = \left[1800q - 0.6q^3\right]_0^{10}$$

$\Rightarrow 1800 - 600 = 17400$ Therefore, $TR = 17400$

ii) The change in TR when q increases from $10\ to\ 20$ will be

$$\int_{10}^{20} MRdq = \int_{10}^{20} (1800 - 1.8q^2)dq = \left[1800q - 0.6q^3\right]_{10}^{20}$$

$$= (36000 - 4800) - (18000 - 600)$$
$$= 13800$$

Therefore the change in TR when q increases from $10\ to\ 20$ will be 13800

iii) Consumer surplus when q is 10 will be the definite integral of the demand function minus the total revenue actually spent by consumers

$$\int_0^{10} (1800 - 0.6q^2)dq = \left[1800q - 0.2q^3\right]_0^{10}$$

$$= 1800 - 200 = 17800$$

$$TR = Pq = 1800q - 0.6q^3$$
$$= 18000 - 600 = 17400$$

Therefore, consumer surplus is $17800 - 17400 = 400$

2. Given $\frac{dP}{dx}=40-3\sqrt{x}$ thus The change in profit is

$$P=\int_{100}^{121}\left(40-3\sqrt{x}\right)dx=\left[40x-3\frac{x^{3/2}}{3/2}\right]_{100}^{121}$$

$$=\left(40\times121-2\times121^{3/2}\right)-\left(40\times100-2\times100^{3/2}\right)$$

$$=\left(4840-4000\right)-2\left(1331-1000\right)=840-662=178.$$

Lesson 11: Applications of definite integrals: Present, Future Values of an Income Stream

a) Learning objectives

- Use integrals to solve different problems in economics(present, future values of an income stream and population growth rates)
- Extend the concept of indefinite integral to definite integrals to solve problems involving present, future values of an income stream and population growth rates

b) Teaching resources

student teacher's book and other Reference books to facilitate research, scientific calculator, Manila paper, markers, pens, pencils, if possible computers with mathematical software such as Geogebra, Microsoft Excel, Math-lab or graph-application and internet can be used...

c) Prerequisites/Revision/Introduction

Student-teachers will perform well in this lesson if they refer to unit 1 and unit 2 of year three (previous lessons)

d) Learning activities:

- Invite student-teachers to work in groups and do the activity 2.4.2 in their Mathematics books;
- Move around in the class for facilitating student teachers where necessary.
- Verify and identify groups with different working steps;
- Invite one member from each group with different working steps to present their work where they must explain the working steps;
- As a tutor, harmonize their findings from the presentation.;
- Use different probing questions and guide student teachers to explore the content and examples given in the student teachers' book and guide them

in the application of skills they have about integration to solve problems related to Present, Future Values of an Income Stream and Population Growth Rates

- After this step, guide students to do the application activity 2.4.2 and evaluate whether lesson objectives were achieved.

Answers for activity 2.4.2

a) If you deposit a lump sum of $P is the present value of $50,000. So, using $B = Pe^{rt}$ with $B = 50000$ and $r = 0.02$, $t = 8$

$$50000 = Pe^{0.02(8)}$$

$$P = \frac{50000}{e^{0.02(8)}} = 42,607$$

If you deposit $42,607 into the account now, you will have $50,000 in 8 years' time

b) Suppose you deposit money at a constant rate of $S per year. Then,

$$Present\ value\ of\ deposit = \int_0^8 Se^{-0.02t}dt$$

Since S is a constant, we can take it out in front of the integral sign

$$Present\ value = S\int_0^8 e^{-0.02t}dt \approx S(7.3928)$$

But the present value of the continuous deposit must be the same as the present value of the lump sum deposit that is $42607

So, $42607 = S(7.3928)$

$$S \approx 5763$$

To meet your goal of $\$50000$ you need to deposit money at a continuous rate of $\$5,673$ per year or about $\$480$ per month.

Answers for application activity 2.4.2

Using $s(t)=1,000$ and $r=0.06$ we have

$$\text{Present value}=\int_0^{20} 1000e^{-0.06t}dt = \$11,647$$

We can get the future value, B from the present value P, using $B=Pe^{rt}$, so

$$\text{Future value}=11,647e^{0.06(20)}=\$38,669$$

Notice that since money was deposit at rate of 6% a year for 20 years, the total amount deposit was $1000. The future value is $38669, so the money has almost doubled because of the interest.

Lesson 12: Applications of definite integrals on Population Growth Rates

a) Learning objectives

- Use integrals to solve different problems related to the population growth rates
- Extend the concept of indefinite integral to definite integrals to solve problems involving population growth rates

b) Teaching resources

student teacher's book and other Reference books to facilitate research, scientific calculator, Manila paper, markers, pens, pencils, if possible computers with mathematical software such as Geogebra, Microsoft Excel, Math-lab or graph-application and internet can be used...

c) Prerequisites/Revision/Introduction

Student-teachers will perform well in this lesson if they refer to unit 1 and unit 2 of year three (previous lessons)

d) Learning activities:

- Invite student-teachers to work in groups and do the activity 2.4.3 in their Mathematics books;
- Move around in the class for facilitating student teachers where necessary.
- Verify and identify groups with different working steps;
- Invite one member from each group with different working steps to present their work where they must explain the working steps;

- As a tutor, harmonize their findings from the presentation.;
- Use different probing questions and guide student teachers to explore the content and examples given in the student teachers' book and guide them in the application of skills they have about integration to solve problems related to the Population Growth Rates
- After this step, guide students to do the application activity 2.4.3 and evaluate whether lesson objectives were achieved.

Answers for activity 2.4.3

As rate of change (in time) of this population $\frac{dP}{dt}$ is proportional to the population P present with a constant k of proportionality which is $\frac{dP}{P} = kdt$.

2) Integrating both sides we get: $\int \frac{dP}{P} = K\int dt \Rightarrow \ln P = Kt + c \Rightarrow P = ce^{Kt}$

If the initial population at time $t = 0 \; is \; P_0$, and $K = 0.05 \; then \; P_0 = ce^{(0.05\times 0)} = c$

Therefore, $P_0 = c$ and we have $P = P_0 e^{0.05t}$

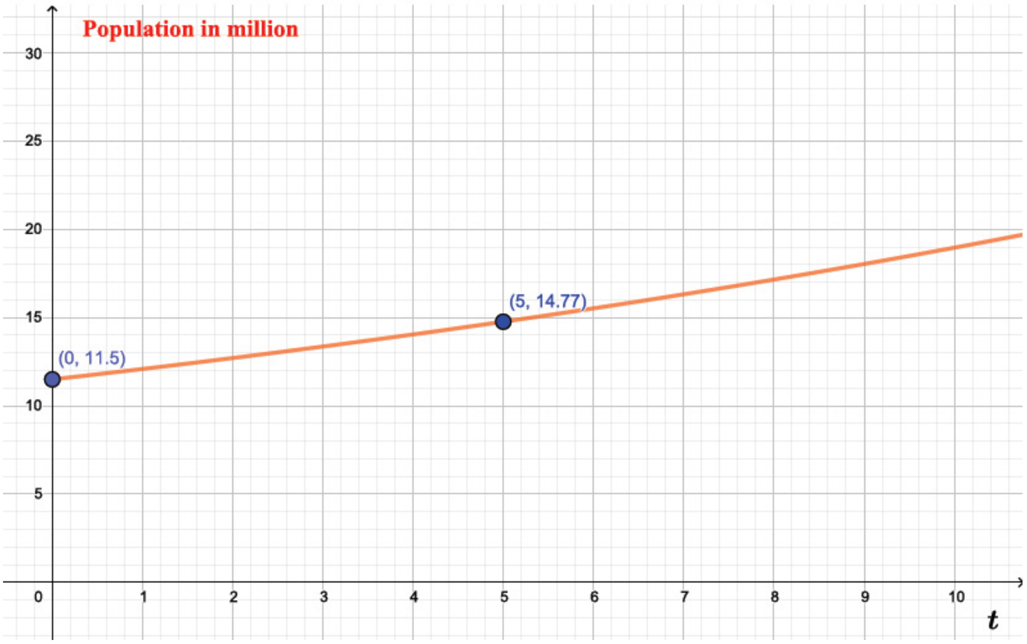
1) If the population $P_0 = 11,500,000$; respects the same variation $P = P_0 e^{0.05t}$ then

$P = 11,500,000e^{0.05t}$.

After 5 years, this population will be
$P = 11,500,000e^{0.05\times 5} = 14,766,292.29 \; people$

that is 14,766,292 people.

Graph showing the population growth



The population is exponentially increasing and the policy makers of that town should think about family planning, environment protection, etc

Answers for application activity 2.4.3

1) Let P be the number of population; then, we have $\frac{dP}{P} = \lambda dt$, integrating both sides: $\int \frac{dP}{P} = \lambda \int dt \Rightarrow \ln P = \lambda t + c$

$\Rightarrow P = e^{\lambda t + c} \Leftrightarrow P = e^c e^{\lambda t} \Leftrightarrow P = \alpha e^{\lambda t}$.

For $t = 0, P_0 = \alpha e^0 \Rightarrow P_0 = \alpha$; Hence $P = P_0 e^{\lambda t}$ because $\alpha = P_0$

$P = 2P_0$ in $t = 20$ years with $P = P_0 e^{\lambda t}$,

Thus $2P_0 = P_0 e^{20\lambda} \Rightarrow 2 = e^{20\lambda} \Rightarrow 20\lambda = \ln 2 \Rightarrow \lambda = \frac{\ln 2}{20}$

If P_0 is the initial population, then $t = ?$ if we have $3P_0 = P \Rightarrow P = 3P_0$ then $t = ?$

with $P = P_0 e^{\frac{\ln 2}{20} t} \Rightarrow 3 = e^{\frac{\ln 2}{20} t} \Rightarrow \ln 3 = \frac{\ln 2}{20} t \Rightarrow t = \frac{20 \ln 3}{20} = 31.69925001$ years

The population will triple after approximately 32, it means years; t=31 years; 8Months; 11days; 17h;31min and 12s.

2) Let P be the population of bacteria $\frac{dP}{P}=\alpha dt \Rightarrow \int \frac{dP}{P}=\alpha\int dt$ where α is the proportionality coefficient

$\ln P=\alpha t+c$ where $A=e^{c}$

The number of bacteria is increasing from 1000 to 3000 in 10 hours

αt

For t=0h; P=1000; for t=10h, P=3000

. $P=1000; P=Ae^{\alpha t} \Rightarrow 1000=Ae^{0} \Rightarrow A=1000$; hence $P=1000e^{\alpha t}$; for P=3000

and t=10h $\Rightarrow 3000=1000e^{10\alpha} \Rightarrow \alpha=\frac{\ln 3}{10}=0.109861228$

If A=1000 and $\alpha=0.109861228$; then $P=1000e^{0.109861228t}$

b) if t=5h; p=? $P=1,732.05807$ bacteria.

Hence, after 5 hours the number of bacteria is about 1732.

2.6 UNIT SUMMARY

1. Increment and differential of a function

The rate of change $\frac{\Delta y}{\Delta x}=\frac{f(x+h)-f(x)}{h}$ means that $\Delta y=f'(x)\Delta x$, When Δx becomes very small, the change in y can be approximated by the differential of y, that is, $\Delta y \approx dy \text{ and } \Delta x=dx$

Therefore, $dy=f'(x)dx$. The differential of a function $f(x)$ is the approximated increment of that function when the variation in x becomes very small. It is given by $dy=f'(x)dx$. f ′ (x) {\displaystyle f'(x)}

2. Definition of Indefinite integrals

Let $y=f(x)$ be a continuous function of variable x. An anti-derivative of $f(x)$ is any function $F(x)$ such that $F'(x)=f(x)$.

For any arbitrary C, $F(x)+C$ is also an anti-derivative of $f(x)$ because

$\left(F(x)+c\right)'=F'(x)$

3. Indefinite integral

Let $y = f(x)$ be a continuous function of variable x. The **indefinite integral** of $f(x)$ is the set of all its anti-derivatives. If $F(x)$ is any anti-derivative of function $f(x)$, then the indefinite integral of $f(x)$ is denoted and defined as follows: $\int f(x)dx = F(x) + c$ Where c an arbitrary constant is called the **constant of integration**.

Thus, $\int f(x)dx = F(x) + c$ if and only if $F(x) = f(x)$. The process of finding the indefinite integral of a function is called **integration.** The symbol $\int$ is the sign of integration while $f(x)$ is **the integrand.** Note that the integrand is a differential, dx shows that one is integrating with respect to variable x.

- **Properties of indefinite integral**

Let $y = f(x)$ and $y = g(x)$ be continuous functions and k a constant. Integration obeys the following properties:

1) $\int kf(x)dx = k\int f(x)dx$: The integral of the product of a constant by a function is equal to the product of the constant by the integral of the function.

2) $\int [f(x) \pm g(x)]dx = \int f(x)dx \pm \int g(x)dx$: The indefinite integral of the algebraic sum or difference of two functions is equal to the algebraic sum or difference of the indefinite integrals of those functions.

3) The derivative of the indefinite integral is equal to the function to be integrated. $\frac{d}{dx}\int f(x)dx = f(x)$

- **Basic integration formulae**

List of basic integration formula

1. If k is constant, $\int kdx = kx + c$

2. $\int x^n dx = \frac{1}{n+1}x^{n+1} + c$,where $n \neq -1$, n is a constant

3. If $b \neq -1$, and u a differentiable function , $\int u^b du = \frac{u^{b+1}}{b+1} + c$

4. By definition, $\int x^{-1}dx = \int \frac{1}{x}dx = \ln|x| + c$ for x nonzero

5. $\int e^x dx = e^x + c$, the integral of exponential function of base e

6. If $a > 0$ *and* $a \neq 1, \int a^x dx = \frac{a^x}{\ln a} + c$

7. $\int a^{nx+b} dx = \frac{1}{n}\frac{a^{nx+b}}{\ln a} + c$

8. $\int \frac{f'(x)}{f(x)} dx = \ln|f(x)| + c$

9. If $a \neq 0, \frac{1}{a}\int \frac{dx}{ax+b} = \frac{1}{a}\ln|ax+b| + c$

10. If $a \neq 0$ *and* $n \neq -1, \int (ax+b)^n dx = \frac{(ax+b)^{n+1}}{a(n+1)} + c$

11. $\int \frac{1}{(ax+b)^n} dx = \frac{1}{a(-n+1)(ax+b)^{n-1}} + c$, *where* $n \neq -1$

12. $\int \frac{dx}{x^2 - a^2} = \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right| + c$

13. $\int \frac{dx}{a^2 - x^2} = \frac{1}{2a}\ln\left|\frac{a+x}{a-x}\right| + c$

14. $\int \frac{dx}{\sqrt{x^2 \pm a^2}} = \ln\left|x + \sqrt{x^2 \pm a^2}\right| + c$

15. $\int f'(x)[f(x)]^n dx = \frac{1}{n+1}[f(x)]^{n+1} + c$, where $n \neq -1$

4. Techniques of integration of indefinite integrals

a) Integration by substitution

It is the method in which the original variables are expressed as functions of other variables.

Generally if we cannot integrate $\int h(x)dx$ directly, it is possible to find a new variable u and function $f(u)$ for which $\int h(x)dx = \int f(u(x))dx = \int f(u)du$

b) Integration by parts

If u and v are two functions of x, the product rule for differentiation can be used to integrate the product udv or vdu in the following way. Since

$d(uv) = udv + vdu$ *it comes that* $\int d(uv) = \int udv + \int vdu$ this leads to:

$uv = \int udv + \int vdu$

Thus $\int udv = uv - \int vdu$.

When using integration by parts, keep in mind that you are splitting up the integrand into two parts. One of these parts, corresponding to u will be differentiated and the other, corresponding to dv, will be integrated. Since you can differentiate easily both parts, you should choose a dv for which you know an anti-derivative to make easier the integration.

5. Definite integrals

Let f be a continuous function defined on a close interval $[a,b]$ and F be an anti-derivative of f for any anti-derivative $F(x)$ of $f(x)$ on $[a,b]$ the difference $F(b) - F(a)$ has a unique value.

This value is defined as a definite integral of $f(x)$ for $a \le x \le b$. We write,

$\int_a^b f(x)dx = F(b) - F(a)$ thus, if $F(x)$ is an anti-derivative of $f(x)$ then,

$$\int_a^b f(x)dx = [F(x)+c]_a^b = [(F(b)+c)-(F(a)+c)] = [F(b)+\text{c}-\text{F(a)}-\text{c}] = F(b)-F(a)$$

$\int_a^b f(x)dx$ is read as the "integral from a to b of $f(x)$, a is called **lower limit** and b is called **upper limit**. The interval $[a,b]$ is called **the range of integration**.

a. Fundamental theorem of integral calculus:

Let $F(x)$ and $f(x)$ be functions defined on an interval $[a,b]$. If $f(x)$ is continuous and $F'(x) = f(x)$, *then* $\int_a^b f(x)dx = [F(x)]_a^b = F(b) - F(a)$.

b. Properties of definite integral

If $f(x)$ *and* $g(x)$ are continuous functions on a closed interval $[a,b]$ then:

1. $\int_a^b 0dx = 0$

2. $\int_a^b f(x)dx = -\int_b^a f(x)dx$ (Permutation of bounds)

3. $\int_a^b [\alpha f(x) \pm \beta g(x)]dx = \alpha\int_a^b f(x)dx \pm \beta\int_a^b g(x)dx, \alpha \text{ and } \beta \in \mathbb{R} \ (linearity)$

4. $\int_a^b f(x)dx = 0 \ (bounds\ are\ equal, a = b)$

5. $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx \text{ with } a < c < b$ (Chasles relation)

6. $\forall x \in [a,b], f(x) \le g(x) \Rightarrow \int_a^b f(x)dx \le \int_a^b g(x)dx$ it follows that

   $fx(x) \ge 0 \Rightarrow \int_a^b f(x) \ge 0$ and

   $\left|\int_a^b f(x)dx\right| \le \int_a^b |f(x)|dx$ (Positivity)

7. $\int_{-a}^a f(x)dx = \begin{cases} 2\int_0^a f(x)dx, \text{ if } f(x) \text{ is even function} \\ 0, \text{ if } f(x) \text{ is odd function} \end{cases}$

c. Techniques of integration of definite integrals

c.1 Integration by substitution

The method in which we change the variable to some other variable is called "**Integration by substitution".** When definite integral is to be found by substitution then change the lower and upper limits of integration. If substitution is $\varphi(x)$ and lower limit of integration is a and upper limit is b then new lower and upper limits will be $\varphi(\text{a})$ and $\varphi(\text{b})$ respectively.

c.2 Integration by parts

To compute the definite integral of the form $\int_a^b f(x)g(x)dx$ using integration by parts, simply set $u = f(x) \text{ and } dv = g(x)dx$. Then $du = f'(x)dx$ and $v = G(x)$, anti-derivative of $g(x)$ so that the integration by parts becomes:

$$\int_a^b udv = [uv]_a^b - \int_a^b vdu$$

6. Application of definite integrals

- **Determination of cost function in Economics**

In economics, the marginal function is obtained by differentiating the total function. Now, when marginal function is given and initial values are given, the total function can be obtained using integration. If C denotes the total cost and $M(x) = \frac{dC}{dx}$ is the marginal cost, we can write $C = C(x) = \int M(x)dx + K$, where the constant of integration K represents the fixed cost.

- **Marginal cost and change in total cost**

The area under marginal cost curve between $q = 0 \text{ and } q = b$ is the total increase in cost between a producing of 0 and a production of b. This is called Total variable cost. Adding this to fixed cost gives the total cost to produce b units.

Cost to increase production from a units to b units is given $C(b) - C(a) = \int_a^b C'(q)dq$

$$Total\ \text{variable cost to produce } b \text{ units} = \int_0^b C'(q)dq$$

$$Total\ \text{cost of producing } b \text{ units} = Fixed\ \text{cost} + Total\ \text{variable cost}$$

$$= C(0) + \int_0^b C'(q)\,dq$$

- **Consumer and Producer Surplus**

The **consumer surplus** measures the consumers' gain from trade. It is the total amount gained by consumers by buying the item at the current price rather than at the price they would have been willing to pay.

The **producer surplus** measures the suppliers' gain from trade. It is the total amount gained by producers by selling at the current price, rather than at the price they would have been willing to accept.

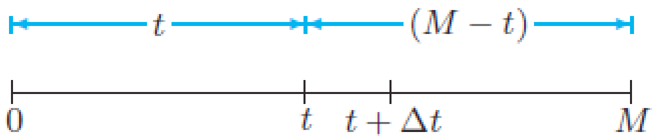
$$consumer\ expenditure = \int_0^{q^*} f(q)dq = area\ under\ demand\ curve\ from\ 0\ to\ q^*$$

- **Present and Future Values of an Income Stream**

Just as we can find the present and future values of a single payment, so we can find the present and future values of a stream of payments

When we are working with a continuous income stream, we will assume that interest is compounded continuously. If the interest rate is r, the present value P of a deposit B made t years in the future is $P = Be^{-rt}$

In order to use what we know about single deposits to calculate the present value of an income stream, we divide the stream into many small deposits, and imagine each deposited at one instant. Dividing the interval $0 \le t \le M$ into subintervals of length Δt



Assuming Δt is small, the rate, $S(t)$ at which deposits are being made does not vary much within one subinterval. Thus, between t and $t + \Delta t$:

$$Amount\ paid \approx Rate\ of\ deposits \times Time$$

$$= S(t)\Delta t\ dollars$$

The deposit of $S(t)\Delta t$ is made t years in the future. Thus, assuming a continuous interest rate

$present\ value \approx S(t)\Delta t e^{-rt}$. Deposit in interval t to $t + \Delta t$

$$Total\ present\ value \approx \sum S(t)e^{-rt}\Delta t$$

In the limit as $\Delta t \to 0$, we get the following integral:

$$\text{Present value} = \int_0^M S(t)e^{-rt}\,dt$$

$$\text{Future value} = \text{Present value} \times e^{rM}$$

- **Population Growth Rates**

Population growth rates are often given by the relative rate change $(1/P)(dP/dt)$, rather than by derivative. We can use the relative growth rate to find the percentage change in a population. Recall that

$\text{Relative growth rate} = \frac{1}{P}\frac{dP}{dt} = \frac{d}{dt}(\ln P)$. By the fundamental theorem of calculus, the integral of the relative growth rate gives the total change in $\ln(P)$:

$$\int_a^b \frac{P'(t)}{dt}\,dt = \int_a^b \frac{d}{dt}\left[\ln P(t)\right]dt = \ln(P(b)) - \ln(\mathrm{P(a)}) = \ln\left(\frac{P(b)}{P(a)}\right).$$

2.7. Additional information for the teacher

Tutor should have a broad knowledge of the topic. The following may be useful:

Improper integrals

A definite integral with infinity for either an upper or lower limit of integration or a function that is discontinuous on the integration interval is called an improper integral.

$\int_a^{\infty} f(x)dx$, $\int^b f(x)dx$ and $\int_1^5 \frac{dx}{x-3}$ are improper integrals. For the two first integrals, $\mp\infty$ is not a number and cannot be substituted for in $\mathrm{D}(x)$. For the third, the integrand function is discontinuous on [1,3]. However, they can be defined as limits of other integrals, as shown below:

$$\int_a^{\infty} f(x)dx = \lim_{x\to\infty}\int_a^b f(x)dx,\ ,\ \int_1^5 \frac{dx}{x-3} = \lim_{p\to 3^-}\int_1^p \frac{dx}{x-3} + \lim_{p\to 3^+}\int_p^5 \frac{dx}{x-3}$$

If the limit exists, the improper integral is said to converge, and their integral has a definite value. If the limit does not exist, the improper integral diverges and is meaningless.

Example

Evaluate the following improper integrals:

$$\int_1^{\infty} 3x^{-2}dx$$

$$\int_1^{\infty} 3x^{-2}dx = \lim_{x\to\infty}\int_1^{b} 3x^{2}dx$$ if limit exits

$$\lim_{x\to\infty}\int_1^{b} 3x^{-2}dx = \lim_{x\to\infty}\left[\frac{-3}{x}\right]_1^b = \lim_{x\to\infty}\left[\frac{-3}{x}\right]_1^b = \lim_{b\to\infty}\left[\frac{-3}{b}-\frac{(-3)}{1}\right] = \lim_{x\to\infty}\left[\frac{-3}{b}+3\right] = 3$$

Therefore $\int_1^{\infty} 3x^{-2}dx = 3$.

2.8 End Unit assessment

This part provides the answers of end unit assessment activities designed in integrative approach to assess the key unit competence with cross reference to the textbook.

Solution

1. a. $\int (9x^7 + \frac{1}{x-1} - \frac{1}{2}e^x)dx = \frac{9x^8}{8} + \ln|x-1| - \frac{1}{2}e^x + c$

b. $\int \frac{x}{\sqrt{x+3}}dx$

Set $\sqrt{x+3} = t$ or equivalently $x+3 = t^2$. This implies $x = t^2 - 3,$ then $dx = 2tdt.$

$$\int \frac{x}{\sqrt{x+3}}dx = \int \frac{\left(t^2-3\right)\times 2tdt}{t} = 2\int \left(t^2-3\right)dt = 2\frac{t^3}{3} - 6t + c$$

Since $t = \sqrt{x+3}$, we have $\int \frac{x}{\sqrt{x+3}}dx = 2\frac{\left(\sqrt{x+3}\right)^3}{3} - 6\sqrt{x+3} + c$

2. a. i) Total Cost Function $=\int \frac{x}{\sqrt{x^2+1600}}dx$

Set $\sqrt{x^2+1600}=t$ or equivalently $x^2+1600=t^2$. Thus $xdx=tdt$.

Total Cost Function $=\int \frac{x}{\sqrt{x^2+1600}}dx=\int \frac{tdt}{t}=\int dt=t+c$

Replacing t by $\sqrt{x^2+1600}$,

gives the Total Cost Function $C(x)=\int \frac{x}{\sqrt{x^2+1600}}dx=\sqrt{x^2+1600}+c$.

Given that the Fixed Cost is $500FRW$ we get:

$$500=\sqrt{0^2+1600}+c$$
$$500=400+c$$
$$c=100$$

Total Cost Function $C(x)=\sqrt{x^2+1600}+100$

ii) An Average Cost $AC=\frac{C(x)}{x}=\frac{\sqrt{x^2+1600}+100}{x}=\sqrt{1+\frac{1600}{x^2}}+\frac{100}{x}$

b. $f(x)=4-\sqrt{x}$

The $y-$intercept is the point with abscissa $x=0$. We have $y=4-\sqrt{0}=4\Rightarrow A(0,4)$

The $x-$intercept, $y=0$. We have $0=4-\sqrt{x}\Rightarrow -\sqrt{x}=-4\Rightarrow x=16\Rightarrow B(16,0)$

The graph

The shaded area in terms of a definite integral is expressed by

$$A = \int_0^{16} f(x)dx \Rightarrow A = \int_0^{16} \left(4 - \sqrt{x}\right)dx$$

$$A = \int_0^{16} \left(4 - \sqrt{x}\right)dx = \left[4x - \frac{x^{3/2}}{\frac{3}{2}}\right]_0^{16} = \left[(4 \times 16) - \frac{16^{3/2}}{\frac{3}{2}}\right]$$

$$= 64 - \frac{2}{3}\sqrt{4096} = 64 - \frac{2}{3} \times 64 = 64\left(1 - \frac{2}{3}\right) = 64\left(\frac{3-2}{3}\right) = \frac{64}{3} \quad .$$

The area of the shaded region is $\frac{64}{3}$ *Square unit*.

3. Discussions on how this unit inspired student-teachers in relation of learning other subjects or to their future will vary accordingly, as they refer to what they learnt through the whole unit.

2.9 Additional activities

2.9.1 Remedial activities

Suggestion of Questions and Answers for remedial activities for slow learners.

Evaluate the following:

1. $\int (4x^5 + x + 1)dx$

2. $\int_1^4 \left(e^x - x^{\frac{1}{2}} \right) dx$

3) $\int (3x^2 - 1)x dx$

4) $\int_1^2 (e^{3x} + 3x^2) dx$

Answers for remedial activities

1) $\int (4x^5 + x + 1)dx$

$$\int (4x^5 + x + 1)dx = \int 4x^5 dx + \int x dx + \int dx$$

$$= \frac{4x^6}{6} + \frac{x^2}{2} + x + C$$

2) $\int_1^4 \left(e^x - x^{\frac{1}{2}} \right) dx =$

$$\int_1^4 \left(e^x - x^{\frac{1}{2}} \right) dx = \int_1^4 e^x dx - \int_1^4 x^{\frac{1}{2}} dx = \left[e^x \right]_1^4 - \left[\frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1} \right]_1^4 = \left(e^4 - e^1 \right) - \left(\frac{4^{\frac{3}{2}}}{\frac{3}{2}} - \frac{1^{\frac{3}{2}}}{\frac{3}{2}} \right)$$

$$= \left(e^4 - e \right) - \left(\frac{16}{3} - \frac{2}{3} \right)$$

$$= e^4 - e - \frac{14}{3}$$

3) $\int (3x^2-1)xdx$

Set $u=3x^2-1$

$$du=6xdx$$

$$\frac{1}{6}du=xdx$$

$$\int (3x^2-1)dx=\frac{1}{6}\int udu=\frac{1}{6}\frac{u^2}{2}+C=\frac{1}{12}u^2+C$$

By substituting u by $3x^2-1$ we get

$$\int (3x^2-1)xdx=\frac{1}{12}(3x^2-1)^2+C$$

4) $\int_1^2 (e^{3x}+3x^2)dx$

$$\int_1^2 (e^{3x}+3x^2)dx=\left[\frac{e^{3x}}{3}+x^3\right]_1^2$$

$$\frac{e^6-e^3}{3}+7.$$

2.9.2 Consolidation activities

Suggestion of questions and answers for deep development of competences.

1) $\int_0^3 x\sqrt{10-x^2}dx$

$$\int_0^3 x\sqrt{10-x^2}dx$$

$$10-x^2=t,\ \text{then}-2xdx=dt,\ or\ xdx=-\frac{1}{2}dt$$

$$when\ x=0,\ t=10,\ when\ x=3,\ t=10-9=1$$

$$\int_0^3 x\sqrt{10-x^2}\,dx = \int_{10}^{1} -\sqrt{t}\,\frac{dt}{2} = \frac{1}{2}\int_1^{10}\sqrt{t}\,dt$$

$$= \frac{1}{2}\int_1^{10} t^{\frac{1}{2}}dt = \frac{1}{2}\left[\frac{t^{\frac{1}{2}+1}}{\frac{1}{2}+1}\right]_1^{10} = \frac{1}{2}\left[\frac{t^{\frac{3}{2}}}{\frac{3}{2}}\right]_1^{10} = \frac{1}{2}\times\frac{2}{3}\left[10^{\frac{3}{2}} - 1^{\frac{3}{2}}\right] =$$

$$\Rightarrow \frac{1}{3}\left(\sqrt{10^3} - 1\right)$$

$$\Rightarrow \frac{1}{3}\sqrt{1000} - 1$$

$$\therefore \int_0^3 x\sqrt{10-x^2}\,dx = \frac{1}{3}\sqrt{1000} - 1$$

1) The demand and supply functions for a product are modeled by $Demand : P = -0.36x + 9$ and $Supply : P = 0.14x + 2$ where x is the number of units (in millions) and the price is $3.96 per unit. Find the consumer and producer surplus for this function

Solution

By equating the demand and supply functions, you determine that the point of equilibrium occurs when $x = 14 millions$ and the price is #3.96 per unit.

$$consumer\ surplus = \int_0^{14} (demand\ \ function - price)dx$$

$$\int_0^{14} \left[(-0.36x + 9) - 3.96\right]dx$$

$$\left[-018x^2 + 5.04x\right]_0^{14} = 35.28$$

$$\text{Pr}\,oducer\ surplus = \int_0^{14} (price - \text{supply function})\,\text{dx}$$

$$= \int_0^{14} \left[3.96 - (0.14x + 2)\right]dx$$

$$\Rightarrow \int_0^{14} \left[0.07x^2 + 1.96x\right] dx = 13.72$$

Therefore, consumer surplus is 35.28 and producer surplus is 13.72.

2.9.3 Extended activities

Suggestion of Questions and Answers for gifted and talented student-teachers.

1) Suppose that the consumers' demand function for a certain commodity is $D(\mathrm{q}) = 4(25 - q^2)$ dollars per unit.

   a. Find the total amount of the money consumers are willing to spend to get 3 units of commodity.

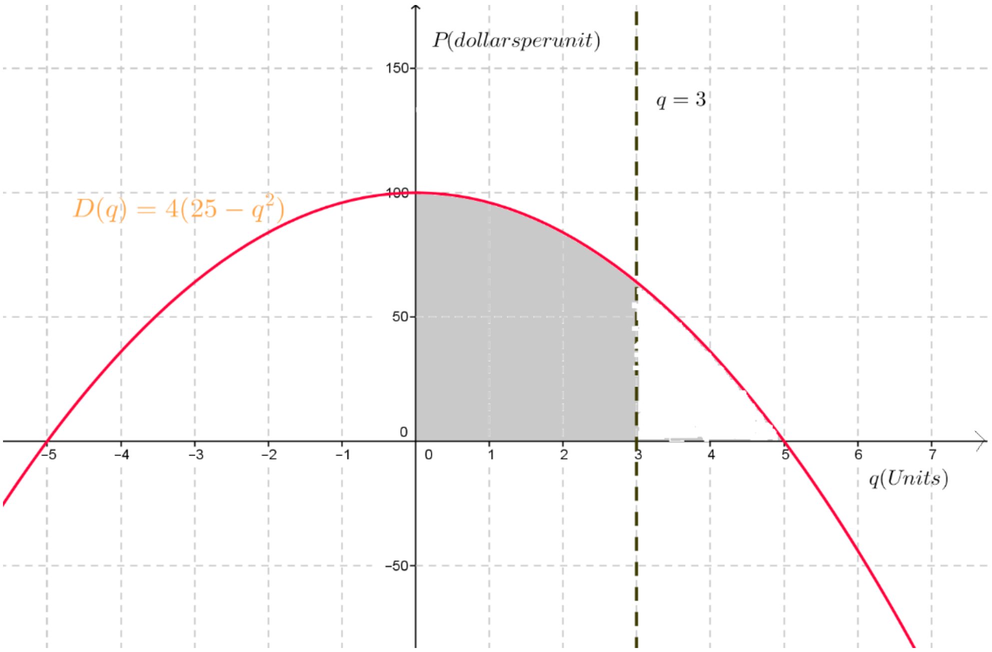
   b. Sketch the demand curve showing the willingness to spend for 3 unitsas an area.

Solution

1) The total amount of money

a) $T = \int_0^3 D(q)\,\mathrm{dq}$

$T = \int_0^3 D(q)\,\mathrm{dq} = \int 4(25 - q^2)dq$, $T = 4(25q - \frac{1}{3}q^3)\Big|_0^3$, Thus $T = \$264$



2) The marginal cost for producing x units of a product is modeled by $\frac{dC}{dx} = 32 - 0.04x$ it costs $50 to produce one unit. Find the total cost of producing 200 units.

Solution

To find the cost function, integrate the marginal cost function.

$$C = \int (32 - 0.04x)dx$$

$$= 32x - 0.04\left(\frac{x^2}{2}\right) + k$$ Where k is constant

$$= 32x - 0.02x^2 + k$$ (Is cost function)

To solve for k, use the initial condition that $C = 50$ when $x = 1$

$$50 = 32(1) - 0.02(1^2) + k$$

$$18.02 = k$$

So, the total cost function is given by $C = 32x - 0.02x^2 + 18.02$ (Is cost function)

Which implies that the cost of producing 200 units is

$$C = 32(200) - 0.02(200)^2 + 18.02$$

$$C = \$5618.02$$

The total cost of producing 200 units is $C = \$5618.02$.

UNIT 3

ORDINARY DIFFERENTIAL EQUATIONS

3.1 Key unit competence:

Use ordinary differential equations of first order to model and solve related problems in Economics.

3.2 Prerequisites

Student-teachers will easily learn this unit, if they have a good background on integration (Unit 2 Year 3).

3.3 Cross-cutting issues to be addressed

- **Inclusive education**: promote the participation of all student-teachers while teaching.
- **Peace and value Education**: During group activities, the tutor will encourage student-teachers to help each other and to respect opinions of colleagues.
- **Gender**: During group activities try to form heterogeneous groups (with boys and girls) or when student-teachers start to present their findings encourage both (boys and girls) to present.
- **Environment and Sustainability:** During the lesson on population growth, guide student-teachers to discuss the effects of the high rate of population growth on the environment and sustainability.

3.4 Guidance on introductory activity

- Invite student-teachers to form groups and let them to work independently on introductory activity to understand the concept of differential equation.
- Walk around to provide various pieces of advice where necessary.
- After a given time, invite student-teachers to present their findings and through their works help them to have an idea on the differential equation.
- Harmonize their works and emphasize that they had a differential equation representing a situation of the population for a country and that it can be solved to obtain the formula for estimating the population of that country

at any time t.

- Invite student-teachers to discuss positive measures that should be taken to address the problem of exponential growth of the population.
- Ask student-teachers to discuss the importance of studying how to solve differential equations.

Answer for introductory activity

The quantity $y(t)$ satisfies the exponential growth model: $\frac{dy}{y} = kdt$.

To find $y(t)$, let us integrate both sides of the equation: $\frac{dy}{y} = kdt$;

$\int \frac{dy}{y} = \int kdt \Leftrightarrow \ln y = kt + c$ where c is an arbitrary constant. We have the function $y = e^c e^{kt}$.

Taking the constant $\mathrm{C} = e^c$, we find $y(t) = Ce^{kt}$.

This is an exponential function that is increasing when the constant k is positive.

Assuming an exponential growth model of the population y and constant growth rate k, at initial time $(t = 0)$, the population is $y(0) = Ce^{k0} = Ce^0 = c_0$.

If the population of a country is C_0 at time $t = 0$, this population with the growth rate k will be $y(t) = c_0 e^{kt}$ after the time *t.*

Therefore, given that the size of the Rwandan population is now (in the year 2018) estimated to

$C_0 = 12,089,721$ with a growth rate of about $k = 2,37\% = 0.0237$ comparatively to the year 2017, the equation of Rwandan population becomes $y(t) = 12089721e^{0.0237t}$.

1) The national population at the beginning of the year 2020, 2030, 2040 and 2050:

From now in 2018 taken as initial time, in 2020 the time $t = 2$, in 2030 the time $t = 12$, in 2040 the time $t = 22$, in 2050 the time $t = 32$.

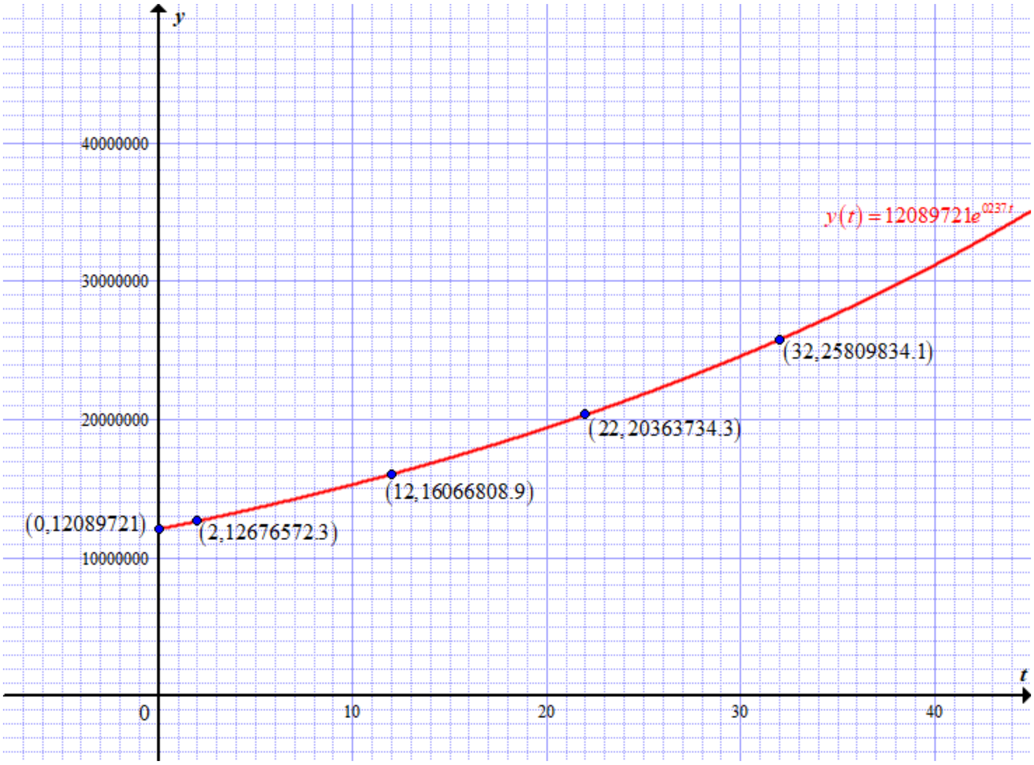
- Hence, the national population at the beginning of the year 2020 will be $y(2) = 12089721e^{(0.0237)2} = 12089721(0.04854134) = 12,676,572.3$ people.
- The national population at the beginning of the year 2030 will be $y(12) = 12089721e^{(0.0237)12} = 12089721(1.32896441) = 16,066,808.9$ people.
- The national population at the beginning of the year 2040 will be

$y(22) = 12089721e^{(0.0237)22} = 20,363734.3$ people.

- The national population at the beginning of the year 2050 will be

  $y(32) = 12089721e^{(0.0237)32} = 25,809,834.1$ people.

2) Graph representing the increasing population $y(t) = 12089721e^{0.0237t}$



3) Our observation is that the national population is increasing with time but the surface where to live remains constant and we are not sure that the economy of the country is going to increase exponentially.

4) Pieces of advice: Police makers should adopt the family planning policy and sensitize the population as well as integrate the family planning programs into school curricula at all levels of education. The tutor should encourage student-teachers to provide more ideas.

3.5. List of lessons

Key unit competence: Use ordinary differential equations of first order to model and solve related problems in Economics			
#	**Lesson title**	**Learning objectives**	**Number of periods**
0.	Introductory activity	To arouse the curiosity of student teachers on the content of unit 2.	2
1	Definition and classification of ODE	Define differential equation and classify differential equations Give examples of 1^{st} order differential equations related to economics State the order and the degree of an ordinary differential equation. Identify the characteristics of differential equations (first order)	3
2	Differential equations with separable variables	Determine whether an ordinary differential equation of 1^{st} order is with separable variables, homogeneous or linear.	6
3	Linear differential equations	Solve an ordinary linear differential equation of first order by variation of constant and by integrating factor. Use appropriate method to solve an ordinary differential equations of first order related to economics and other social sciences.	5
4	Differential equation solutions to predict values in basic market and macroeconomic models	Find the solution of the first order differential equations related to economics to predict values in basic market and macroeconomic model. Predict the form the particular solution of an ordinary linear differential equation of first order.	6

5	Stability of economic models where growth is continuous	Discuss the stability of economic models where growth is continuous using solutions of differential equations.	6
6	Continuously compounded interest	Discuss the stability of economic models where growth is continuously compounded using solutions of differential equations.	3
7	Differential equations and the quantity of a drug in the body	Use differential equations to solve problems related to the quantity of a drug in the body.	3
12	End unit assessment		2
Total			**36**

Lesson 1: Definition and classification of ODE

a) Learning objectives

- Define differential equation and classify differential equations
- Give examples of 1^{st} order differential equations related to economics
- State the order and the degree of an ordinary differential equation.
- Identify the characteristics of differential equations (first order)

b Teaching resources

Student-teachers' book and other Reference books to facilitate research, calculator, Manila paper, markers, pens, pencils, Internet to facilitate research.

c) Prerequisites/Revision/Introduction

Student-teachers will perform better in this lesson if they refer to techniques of derivatives (year two, unit 5: Derivative of functions and their application) and they refer to previous content of year three unit 2: Integration)

d) Learning activities:

- Invite student-teachers to work in groups and do the activity 3.1 found in their Mathematics books;
- Move around in the class for facilitating student-teachers where necessary and give more clarification while discuss and write down the highest order of the derivative that occurs in the given equations in activity 3.1
- Guide student-teachers to derivate and to guess the power of the highest

derivative.

- Invite one member from each group with different working steps to present their work where they must explain the working steps;
- As a tutor, harmonize the findings from presentation and guide student-teachers to conclude on the new concept
- Use different probing questions and guide students to explore the content and examples given in the student's book and lead them to discover definition and classification of ODE
- After this step, guide student-teachers to do the application activity 3.1 and evaluate whether lesson objectives were achieved.

Answer for activity 3.1

1) On differentiation $\frac{dy}{dx}=4k \Rightarrow k=\frac{dy}{4dx}$

The given equation becomes $y=\frac{dy}{dx}x \ or \ y=y'x$ Order of the derivative is 1.

2) $y=kx+bx^2$

Differentiate to get $\frac{dy}{dx}=k+2bx$. Solving for k yields to $k=y'-2bx$

Differentiating again: $\frac{d}{dx}\left(\frac{dy}{dx}\right)=\frac{d^2y}{dx^2}=2b$ then $b=\frac{d^2y}{2dx^2} \ or \ b=\frac{y''}{2}$

Replace k and b by their values in $y=kx+bx^2$ to get:

$y=\left[\frac{dy}{dx}-2\left(\frac{d^2y}{2dx^2}x\right)\right]x+\frac{1}{2}\frac{d^2y}{dx^2}x^2 \ or \ y=y'x-\frac{1}{2}y''x^2$ This is a differential equation of 2nd order.

Answer for application 3.1

a) $\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^4-4x+y=1 \Rightarrow$ This is differential equation of the 2^{nd} order and degree one

b)

$(y'')^3+(y')-2y=x \Rightarrow$ This is a differential equation of the 2^{nd} order and degree 3.

c)

$$x^2\left(\frac{d^2y}{dx^2}\right)^4+y\left(\frac{dy}{dx}\right)+y^4=0\Rightarrow$$ This is a differential equation of the 2^{nd} order and degree 4.

Notice:

Ask student-teachers to explain in words their answers based on the definition of order and degree of differential equations.

Lesson 2: Differential equations with separable variables

a) Learning objectives

Determine whether an ordinary differential equation of 1^{st} order is with separable variables, homogeneous or linear.

b) Teaching resources

Student-teachers' book and other Reference books to facilitate research, calculator, Manila paper, markers, pens, pencils,graph papers and Internet to facilitate research.

c) Prerequisites/Revision/Introduction

Student-teachers will perform better in this lesson if they refer to techniques of derivatives (year two, unit 5: Derivative of functions and their application) and they refer to previous content of year three unit 2: Integration).

d) Learning activities:

- Invite student-teachers to work in groups and do the activity 3.2 found in their Mathematics books;
- Let student-teachers work independently. Thereafter, as they are working, walk around to ensure that they are performing the task effectively and cooperatively.
- Provide assistance where necessary by facilitating student-teachers to separate variables and integrate both sides of the obtained equality after separation.
- Call upon groups to present their findings.
- Ask student-teachers to follow attentively their classmates' presentations, interact with them about their findings and write a short summary with complementary ideas.
- Harmonize student-teachers' findings by leading them to solve differential equations with separable variables.
- Let student-teachers go through the examples prepared in their books

under the guidance of tutor, and work individually application activities 3.2 to reinforce their skills in solving 1st order differential equations with separable variables.

Answer for activity 3.2

1. $4y'-2x=0$

a) Solve for y' to get y' : $4\frac{dy}{dx}=2x \Rightarrow 4dy=2xdx \Rightarrow dy=\frac{x}{2}dx$

Integrate both sides to deduce the value of the dependent variable y :

b) $\int dy=\frac{1}{2}\int xdx \Rightarrow y=\frac{x^2}{4}+c$ Knowing that $y'=\frac{dy}{dx}$, it is clear that $4\frac{dy}{dx}-2x=0$ *and* $4y'-2x=0$ are the same.

c) Replace the value of y in the given equation to get:

$4\left(\frac{x^2}{4}+c\right)'-2x=0 \Leftrightarrow 4\left(\frac{2x}{4}\right)-2x=0$ and it is clear that the equality remain correct.

2) a) $x\frac{dy}{dx}=1$

Separating variables yields to $dy=\frac{1}{x}dx \Leftrightarrow dy=\frac{dx}{x}$

Integrating both sides:

$$\int dy=\int\frac{dx}{x} \Rightarrow y=\ln|x|+c$$

3) To solve $f(y)\frac{dy}{dx}=g(x)$ we separate variables to both sides of the equation and then integrate both sides to deduce the value of the dependent variable y.

Answer for application activity 3.2

1) The general solution for $x\frac{dy}{dx}=2-4x^3$

Rearranging $x\frac{dy}{dx}=2-4x^3$ gives $\frac{dy}{dx}=\frac{2-4x^3}{x}=\frac{2}{x}-4x^2$.

Integrating both sides gives:

$$y=\int\left(\frac{2}{x}-4x^2\right)dx=2\ln x-\frac{4}{3}x^3+c$$

Thus the required solution is $y=2\ln x-\frac{4}{3}x^3+c$

2) $(x+1)\frac{dy}{dx}=x$, $y(0)=0$

$$(x+1)\,\mathrm{dy}=xdx$$

$$dy=\frac{x}{x+1}dx$$

Integrating on both sides to get:

$$\int dy=\int\frac{x}{x+1}dx$$

$$y=\int(1-\frac{1}{x+1})dx$$

$$y=x-\ln|x+1|+c$$

$$y(0)=0-\ln|0+1|+c\Rightarrow c=0$$

3) (a) Let $y=y(x)$ be the required function and $P(x,y)$ any point on the curve. The line OP has a slope $\frac{y}{x}$. The tangent to the curve at $P(x,y)$ that is perpendicular to the line OP has therefore the slope $\frac{-x}{y}$ since $\frac{y}{x}\left(\frac{-x}{y}\right)=-1$

We know that the slope of the tangent line to the curve $y=y(x)$ at $P(x,y)$

is defined by $\frac{dy}{dx}$. Therefore, $\frac{dy}{dx}=\frac{-x}{y}$ is the required. The initial value problem is $\frac{dy}{dx}=\frac{-x}{y};\ y(0)=1$

(b) The differential $\frac{dy}{dx}=\frac{-x}{y}$ is equivalent to $ydy=-xdx$ that is separable.

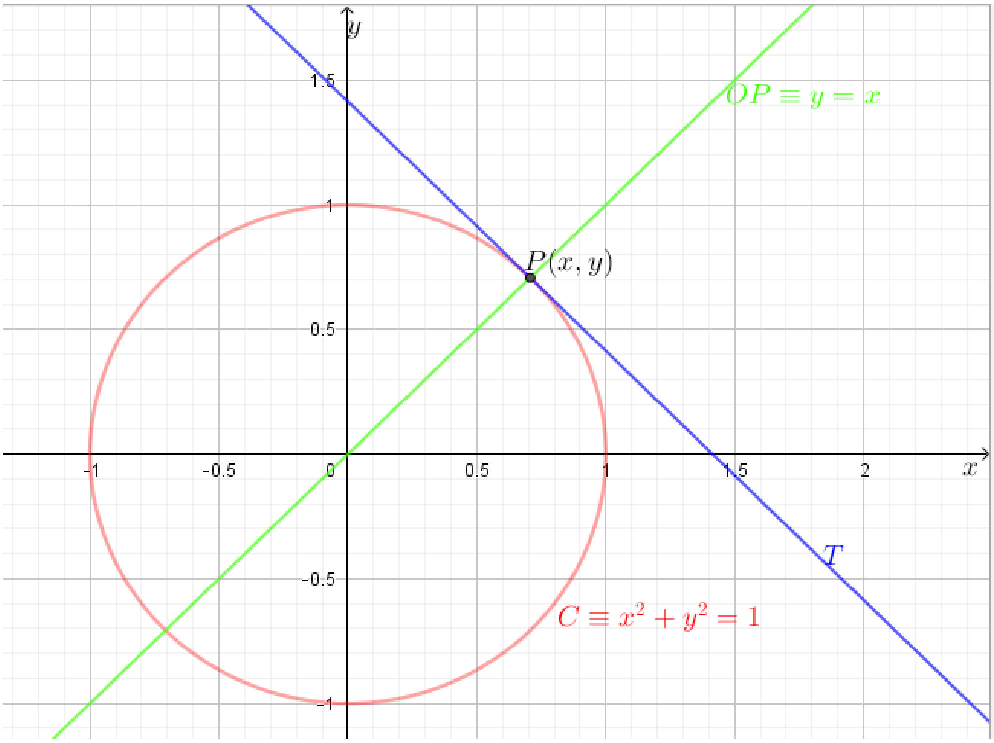
Let's solve the problem $ydy=-xdx \Rightarrow \int ydy=-\int xdx$

Simple integration gives: $\frac{y^2}{2}+\frac{x^2}{2}=c$.

Replace the point $(0,1)$ to get $c=\frac{1}{2}$. So, $y^2+x^2=1$ is the final solution.

The curve representing this solution is the circle centered at $(0,0)$ and radius 1.

Graph of the differentiable function $y^2+x^2=1$



This confirms the theorem of geometry that states that a tangent to a circle forms a right angle with the circle's radius, at the point of contact of the tangent

4) Rearranging the given equation, $\frac{y^2-1}{3y}dy = dt \Leftrightarrow \left(\frac{y}{3}-\frac{1}{3y}\right)dy = dt$

Direct integration yields

$\frac{y^2}{6}-\frac{\ln y}{3} = t+c$ or $t = \frac{y^2}{6}-\frac{\ln y}{3}-c$ that is general equation

Given that $y=1$ when $t = 2\frac{1}{6}$, we have:

$$2\frac{1}{6} = \frac{(1)^2}{6}-\frac{\ln 1}{3}-c \Leftrightarrow 2\frac{1}{6} = \frac{1}{6}-c \Rightarrow c = \frac{1-13}{6} = -2$$

Hence particular solution is $t = \frac{y^2}{6}-\frac{\ln y}{3}+2$

5) Separating the variables gives:

$$\frac{dy}{y} = \frac{x}{1+x^2}dx$$

Integrating both sides gives

$$\ln y = \frac{1}{2}\ln\left(1+x^2\right)+c$$

Given that $y=1$ when $x=0$, $\ln 1 = \frac{1}{2}\ln 1 + c \Rightarrow c = 0$

The particular solution is $\ln y = \frac{1}{2}\ln\left(1+x^2\right)$ or $y = \sqrt{1+x^2}$.

Lesson 3: Linear differential equations

a) Learning objectives

- Solve an ordinary linear differential equation of first order by variation of constant and by integrating factor.
- Use appropriate method to solve an ordinary differential equations of first order related to economics and other social sciences.

b) Teaching resources

Student-teachers' book and other Reference books to facilitate research, calculator, Manila paper, markers, pens, pencils,graph papers and Internet to facilitate research.

c) Prerequisites/Revision/Introduction

Student-teachers will perform better in this lesson if they refer to techniques of derivatives (year two, unit 5: Derivative of functions and their application) and they refer to previous content of (techniques of integration), year three unit 2.

d) Learning activities:

- Invite student-teachers to work in groups and do the activity 3.3 found in their Mathematics books;
- Let student-teachers work independently. Thereafter, as they are working, walk around to ensure that they are performing the task effectively and cooperatively.
- Provide assistance where necessary by facilitating student-teachers in their work.
- Invite randomly some groups to present their findings to the whole class and after presentation facilitate student-teachers to have a lesson summary through harmonization of their findings
- Harmonize student-teachers' findings by leading them to be aware of linear differential equations of the first order.
- Individually, invite student-teachers to read through examples prepared in their books and then work out the application activities 3.3 to enhance their knowledge and skills about linear differential equations of the first order.

Answer for activity 3.3

Given the differential equation $\frac{dy}{dx}+2xy=x\ or\quad y'+2xy=x$ (1)

1) $\frac{dy}{dx}+2xy=x$ (1)

$I(x)=e^{\int 2xdx}=e^{x^2}$. For convenience, we set the integration constant to 0.

2) Multiplying both sides in the differential equation (1) by $I(x)=e^{x^2}$ to get:

$e^{x^2}(y'+2xy)=xe^{x^2}$

Or $\frac{d}{dx}\left(e^{x^2}y(x)\right)=xe^{x^2}$

3) Integrating both sides and divide by integrating factor $I(x)$ to get:

$y(x)e^{x^2}=\frac{1}{2}e^{x^2}+c.$.

4) Solve for $y(x)$ to get $y(x)=\frac{1}{2}+ce^{-x^2}$.

5) Replacing $y(x)=\frac{1}{2}+ce^{-x^2}$ and $y'=-2cxe^{-x^2}$ in (1) we find that $y(x)$ is a solution of (1)

$$y'+2xy=-2cxe^{-x^2}+2x(\frac{1}{2}+ce^{-x^2})=-2cxe^{-x^2}+x+2cxe^{-x^2}=x$$

Answer for application activity 3.3

1. a) $y'+\frac{y}{x}=1$ This is the form of $\frac{dy}{dx}+py=q$; $p=\frac{1}{x},q=1$

$y=uv$

$u=\int qe^{\int pdx}dx$; $v=e^{-\int pdx}$

$$u=\int e^{\int \frac{1}{x}dx}dx=\int e^{\ln x}dx=\int xdx=\frac{x^2}{2}+c$$

$\Rightarrow u=\frac{x^2}{2}+c$

$v=e^{-\int\frac{1}{x}dx}=e^{-\ln x}=\frac{1}{e^{\ln x}}=\frac{1}{x}$

$y(x)=uv=(\frac{x^2}{2}+c)\frac{1}{x}=\frac{x}{2}+\frac{c}{x}$, where c is a constant

b) $y'+xy=x$, This is the form of $\frac{dy}{dx}+py=q$, $p=x, q=x$

$y=uv$,

$u=\int qe^{\int pdx}dx=\int xe^{\int xdx}dx=\int xe^{\frac{x^2}{2}}dx$, let $t=\frac{x^2}{2}\Rightarrow dt=xdx$, then, $\int e^t dt=e^t+c$,
$u=e^{\frac{x^2}{2}}+c$ and $v=e^{-\int pdx}=e^{-\int xdx}=e^{-\frac{x^2}{2}}$.

$y(x)=uv=\left(e^{\frac{x^2}{2}}+c\right)e^{-\frac{x^2}{2}}=1+ce^{-\frac{x^2}{2}}$, where c is a constant

c) $y'+\frac{y}{x}=x$ This is the form of $\frac{dy}{dx}+py=q$,where $p=\frac{1}{x}, q=x$

$y=uv$;

$u=\int qe^{\int pdx}dx \Leftrightarrow u=\int xe^{\int\frac{1}{x}dx}dx=\int xe^{\ln x}dx=\int x^2dx=\frac{x^3}{3}+c\Leftrightarrow u=\frac{x^3}{3}+c$

$v=e^{-\int pdx}=e^{-\int\frac{1}{x}dx}=e^{-\ln x}=e^{\ln x^{-1}}=\frac{1}{x}$.

Then, $y=uv=(\frac{x^3}{3}+c)\frac{1}{x}=\frac{x^2}{3}+\frac{c}{x}$, where c is a constant

d) $y'+2y=e^x$, This is the form of $\frac{dy}{dx}+py=q$, where $p=2, q=e^x$

$y=uv$, $u=\int qe^{\int pdx}dx$

$u=\int e^xe^{\int 2dx}dx=\int e^xe^{2x}dx=\int e^{3x}dx=\frac{e^{3x}}{3}+c$

$v=e^{-\int pdx}=e^{-\int 2dx}=e^{-2x}=\frac{1}{e^{2x}}$

$y = uv = (\frac{e^{3x}}{3} + c)\frac{1}{e^{2x}} = \frac{e^{3x}}{3e^{2x}} + \frac{c}{e^{2x}} = \frac{e^{x}}{3} + \frac{c}{e^{2x}}$, where c is a constant

e) $y' - 2xy = e^{x^2}$; $p(x) = -2x$.

The integrating factor is $I(x) = e^{-\int 2xdx} = e^{-x^2}$. Multiplying each side of the differential equation by integrating factor $I(x) = e^{-x^2}$ to get: $e^{-x^2}(y' - 2xy) = 1$

Lesson 4: Differential equation solutions to predict values in basic market and macroeconomic models

a) Learning objectives

- Find the solution of the first order differential equations related to economics to predict values in basic market and macroeconomic model.
- Predict the form the particular solution of an ordinary linear differential equation of first order.

b) Teaching resources

Student-teachers' book and other Reference books to facilitate research, calculator, Manila paper, markers, pens, pencils,graph papers and Internet to facilitate research.

c) Prerequisites/Revision/Introduction

Student-teachers will perform better in this lesson if they refer to techniques of derivatives (year two, unit 5: Derivative of functions and their application) and they refer to previous content of (techniques of integration), year three unit 2. Differential equations of 1^{st} order with separable variables and linear differential equations of 1^{st} order seen in previous lessons.

d) Learning activities:

- Invite student-teachers to work in groups and do the activity 3.4 found in their Mathematics books;
- Let student-teachers work independently. Thereafter, as they are working, walk around to ensure that they are performing the task effectively and cooperatively.
- Provide assistance where necessary by facilitating student-teachers in their work.
- Invite randomly some groups to present their findings to the whole class and after presentation facilitate student-teachers to have a lesson summary through harmonization of their findings
- Harmonize student-teachers' findings by leading them to predict values in basic market and macroeconomic models using differential equation solutions.

- Individually, invite student-teachers to read through examples prepared in their books and then work out the application activities 3.4 to enhance their knowledge and skills about prediction of the values in basic market and macroeconomic models using differential equation solutions.

Answer for activity 3.4.1

$\frac{dy}{dt} = 5y$ Then one possible solution is $y = e^{5t}$ as gives $\frac{dy}{dt} = 5e^{5t} = 5y$

However, there are other possible solutions,

If $y = 3e^{5t}$ *then* $\frac{dy}{dt} = 5(3e^{5t}) = 5y$

If $y = 7e^{5t}$ *then* $\frac{dy}{dt} = 5(7e^{5t}) = 5y$

For $t = 0$ as any number taken to the power zero is the number itself. For example $\frac{dy}{dt} = 5y$ will be $y_t = Ae^{5t}$ where y has been given the subscript t to denote the time period that it corresponds to. If it is known that when $t = 0$ then $y_0 = 12$ then by substituting these values into (1) we get $y_0 = 12 = Ae^0$ as we know that $e^0 = 1$ then $12 = A$, substituting this value into **general solution** we get **definite solution** $y_t = 12e^{5t}$. *This definite solution can now be used to predict* y_t *for any value* t**.**

For example when $t = 3$ then, $y_3 = 12e^{5(3)} = 12e^{15} = 12(3,269,017.4) = 39,228,208$

Answers for application activity 3.4.1

The differential equation $\frac{dy}{dt} = 1.5y$ if the value of y is 34 when $t = 0$, predict the value of y when $t = 7$.

The general solution to this differential equation is $y_t = Ae^{1.5t}$

When $t = 0$ then $y_0 = 34 = Ae^0$

Therefore, $34 = A$

The definite solution is thus $y_t = 34e^{1.5t}$

When $t = 7$ then using this definite solution we can predict

$$y_7 = 34e^{1.5(7)} = 34e^{10.5} = 34(36,315.5) = 1,234,727 .$$

Lesson 5: Stability of economic models where growth is continuous

a) Learning objectives

Discuss the stability of economic models where growth is continuous using solutions of differential equations.

b) Teaching resources

Student-teachers' book and other Reference books to facilitate research, calculator, Manila paper, markers, pens, pencils,graph papers and Internet to facilitate research.

c) Prerequisites/Revision/Introduction

Student-teachers will perform better in this lesson if they refer to techniques of derivatives (year two, unit 5: Derivative of functions and their application) and they refer to previous content of (techniques of integration), year three unit 2. Differential equations of 1st order with separable variables and linear differential equations of 1st order seen in previous lessons.

d) Learning activities:

- Invite student-teachers to work in groups the activity 3.4.2 found in their Mathematics books;
- Let student-teachers work independently. Thereafter, as they are working, walk around to ensure that they are performing the task effectively and cooperatively.
- Provide assistance where necessary by facilitating student-teachers in their work.
- Invite randomly some groups to present their findings to the whole class and after presentation facilitate student-teachers to have a lesson summary through harmonization of their findings
- Harmonize student-teachers' findings by leading them to solve problems involving Stability of economic models where growth is continuous.
- Individually, invite student-teachers to read through examples prepared in

their books and then work out the application activities 3.4.2 to enhance their knowledge and skills about Stability of economic models where growth is continuous

Answer for activity 3.4.2

1) In general we have $\frac{dP}{dt} = k\left(Q_d - Q_s\right)$, if $k = 0.08$ Then our equation becomes $\frac{dP}{dt} = 0.08\left(Q_d - Q_s\right)$

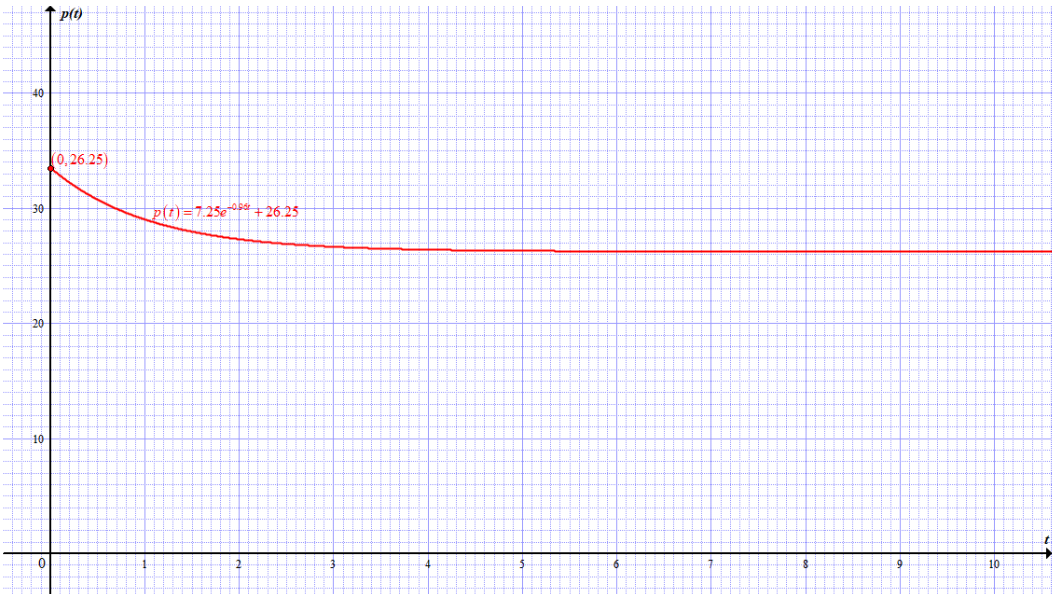
2) $\frac{dP}{dt} = k\left(Q_d - Q_s\right)$

$\Rightarrow \frac{dP}{dt} = 0.08\left(280 - 4P - \left(-35 + 8P\right)\right)$ $\frac{dP}{dt} = 0.96P + 26.25$. Which is a linear first-order differential equation.

3) Considering the initial condition $P(0) = 19$, we find $A = 7.25$

Therefore, $P\left(t\right) = 7.25e^{-0.96t} + 26.25$

Graph of $P\left(t\right) = 7.25e^{-0.96t} + 26.25$.



4) At $t=1, P(1)=7.25e^{-0.96}+26.25=33.5$ and

$$\lim_{t\to\infty} p(t)=\lim_{t\to\infty}\left(7.25e^{-0.96t}+26.25\right)=26.25.$$

If you compare those two situations, you find that the price is decreasing and tends to 26.25 as t gets larger.

Answer for application activity 3.4.2

Substituting the demand and supply functions into the rate of change function

gives $\frac{dP}{dt}=0.4\left[(50-0.2P)-(-10+0.3P)\right]=0.4(-0.2-0.3)P+0.4(50+10)$

$\frac{dP}{dt}=-0.2P+24$

The reduced equation without the constant term is $\frac{dP}{dt}=-0.2P$

The complementary function will be therefore be $P_t = Ae^{-0.2t}$

To find the particular solution we assume P is equal to a constant K at the equilibrium price and so, $\frac{dP}{dt}=-0.2K+24=0, \Rightarrow K=120$

The general solution is given by $P_t = Ae^{-0.2t}+120$

As price is 100 in time period $t=0$ then $P_0=100=Ae^{0}+120$

$-20=A$, and so the definite solution to this differential equation is
$P_t=-20e^{-0.2t}+120$

We can tell that this market is stable as the coefficient of t in the exponential function is the negative number -0.2. However, the sample values calculated below show that the convergence of P_t on its equilibrium value of 120 is relatively slow.

Lesson 6: Continuously compounded interest

a) Learning objectives

Discuss the stability of economic models where growth is continuously compounded using solutions of differential equations.

b) Teaching resources

Student-teachers' book and other Reference books to facilitate research, calculator, Manila paper, markers, pens, pencils,graph papers and Internet to facilitate research.

c) Prerequisites/Revision/Introduction

Student-teachers will perform better in this lesson if they refer to techniques of derivatives (year two, unit 5: Derivative of functions and their application) and they refer to previous content of (techniques of integration), year three unit 2. Differential equations of 1^{st} order with separable variables and linear differential equations of 1^{st} order seen in previous lessons.

d) Learning activities:

- Invite student-teachers to work in groups and do the activity 3.4.3 found in their Mathematics books;
- Let student-teachers work independently. Thereafter, as they are working, walk around to ensure that they are performing the task effectively and cooperatively.
- Provide assistance where necessary by facilitating student-teachers in their work.
- Invite randomly some groups to present their findings to the whole class and after presentation facilitate student-teachers to have a lesson summary through harmonization of their findings
- Harmonize student-teachers' findings by leading them to solve problems related to continuously compounded interest
- Individually, invite student-teachers to read through examples prepared in their books and then work out the application activities 3.4.3 to enhance their knowledge and skills about continuously compounded interest

Answer for activity 3.4.3

(a) We are looking for S, the balance in the account in dollars, as a function of t, time in years.

Interest is being added continuously to the account at a rate of 6% of the balance at that moment, So, Rate at which balance is increasing 6%=0.06 of current balance.

Thus, a differential equation that describes the process is $\frac{dS}{dt} = 0.06S$

It does not involve the $4000, the initial condition, because the initial deposit does not affect the process by which interest is earned.

(b) Since $S_0 = 4000$ is the initial value of S, the solution to this differential equation is $S = S_0 e^{0.06t} = 4000e^{0.06t}$.

Answer for applicationactivity 3.4.3

1) $P = 1000 \ rate = 0.08 \ t = 25 \ years$, Then

$P = P_0 e^{(0.08\times 25)} = 1000\left(e^{0.08\times 25}\right) = 7389.06$

Thus, the accrued amount after 25 years is $\$7389.06$

2) a) From the equation $\frac{dP}{dt} = rP$

$\frac{dP}{dt} = rP \Rightarrow \frac{dP}{P} = rdt$,

Integrating both sides yields, $\int \frac{dP}{P} = \int rdt$

$\ln|P| = rt + c$

Solve for $P = ce^{rt}$

Substituting $P_0 = P(0)$ and put $t = 0$ we have $P_0 = ce^0 = c$, so the formula for principal becomes $P(t) = P_0 e^{rt}$

b) For $t = 20$, The interest rate (r) is 4% and $P_0 = \$500$

$P(t) = P_0 e^{rt}$

$P(20) = 500e^{(0.04)(20)} = 1112.77$

So, the accrued amount after investment for 20 years is $\$1112.77$

Lesson 7: Differential equations and the quantity of a drug in the body

a) Learning objectives

Use differential equations to solve problems related to the quantity of a drug in the body.

b) Teaching resources

Student-teachers' book and other Reference books to facilitate research, Scientific calculators, Manila paper, markers, pens, pencils,graph papers, GeoGebra for graph drawing and Internet to facilitate research.

c) Prerequisites/Revision/Introduction

Student-teachers will perform better in this lesson if they refer to techniques of derivatives (year two, unit 5: Derivative of functions and their application) and they refer to previous content of (techniques of integration), year three unit 2. Differential equations of 1^{st} order with separable variables and linear differential equations of 1^{st} order seen in previous lessons.

d) Learning activities:

- Invite student-teachers to work in groups and do the activity 3.4.4 found in their Mathematics books;
- Let student-teachers work independently. Thereafter, as they are working, walk around to ensure that they are performing the task effectively and cooperatively.
- Follow up the working steps of different groups to give support where necessary and motivate learners to mention any difficulty met when modeling and solving the differential equation related to the given situation "the quantity of a drug in the body".
- Facilitate each group in presentation of their findings.
- Harmonize the student-teachers' works and help them to find the quantity of drug $Q(t)$ left in the body at the time t.
- Invite student-teachers to take decision after discussing their points of view on what happens when the patient does not respect the dose of medicine as prescribed by the Doctor.
- Individually, invite student-teachers to read through examples prepared in their books and then work out the application activities 3.4.4 to enhance their knowledge and skills in the application of differential equations of 1^{st} order

Answer for activity 3.4.4

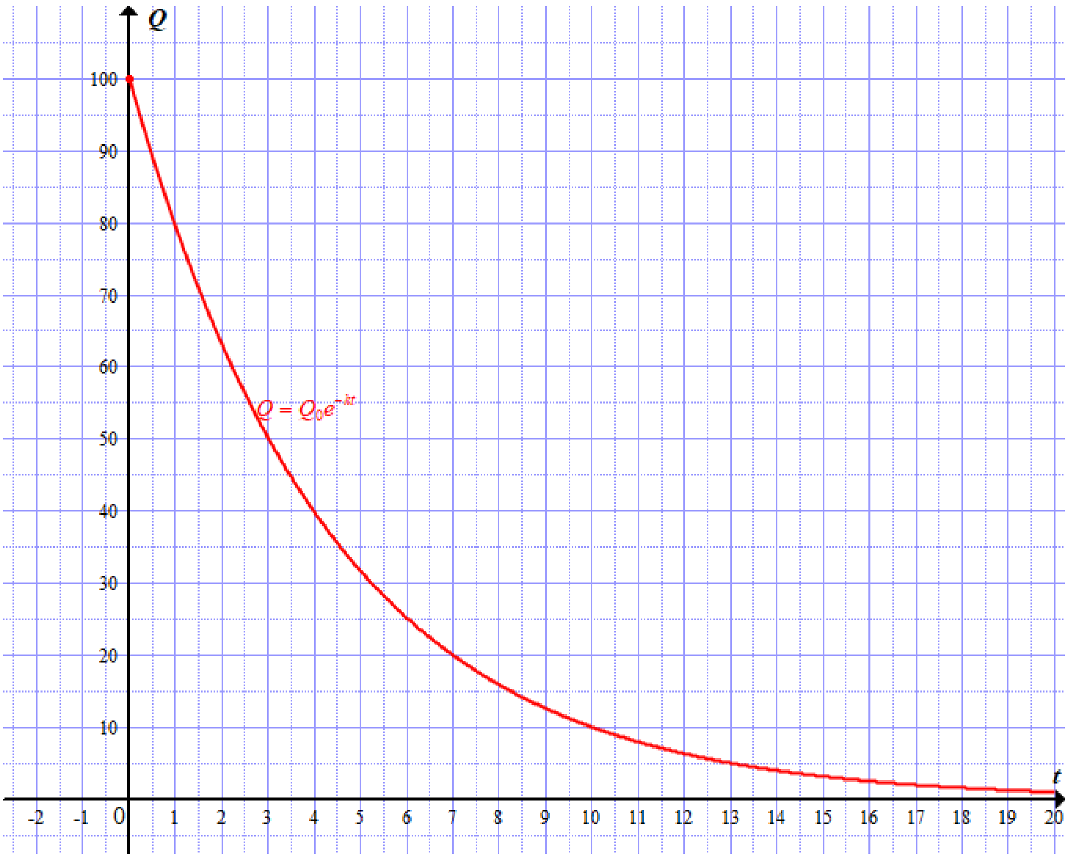
1) The equation for modeling the situation is $\frac{dQ}{dt} = -kQ$

2) $\frac{dQ}{Q} = -kdt \Rightarrow \int \frac{dQ}{Q} = -\int Kdt \Rightarrow \ln Q = -Kt$

The solution of this equation is $Q = Q_0 e^{-kt}$

3) When t = 0, the drug provided was $Q_0 = 100mg$ then $Q = 100e^{-kt}$

The graph of the equation $Q = 100e^{-kt}$



The graphs shows that the drug in the human body decreases to 0 when the time of taking medicine increases.

4) When a patient does not respect the doctor's prescriptions he/she could suffer from the effects of medicine.

Answer for application activity 3.4.4

(a) Since the half-life is 15 hours, we know that the quantity remaining $Q=\frac{1}{2}Q_0$ when t = 15. We substitute into the solution to the differential equation, $Q=Q_0e^{-kt}$, and solve for k: $Q=Q_0e^{-kt}$

$0.5Q_0 = Q_0e^{-k(15)}$

$0.5=e^{-15k}$ (Divide by Q_0)

$\ln 0.5=-15k$ (Take the natural logarithm of both sides)

$k=\frac{-\ln 0.5}{15}=0.0462$

(b) To find the time when 10% of the original dose remains in the body, we substitute $0.10Q_0$

For the quantity remaining, Q; and solve for the time, t.

$0.10Q_0 = Q_0e^{-0.0462t}$

$0.10=e^{-0.0462t}$

$\ln 0.10=-0.0462t$

$t=\frac{\ln 0.10}{-0.0462}=49.84$

There will be 10% of the drug still in the body at $t=49.84$, or after about 50 hours.

3.6 Unit summary

3.6.1. Definition and classification of differential equations

An ordinary differential equation (ODE) for a dependent variable *y* (unknown)in terms of an independent variable *x* is any equation which involves first or higher order derivatives of *y* with respect to *x*, and possibly *x* and *y*.

The general differential equation of the 1st order is $F\left(x, y, \frac{dy}{dx}\right) = 0$ or $\frac{dy}{dx} = f(x, y)$

The order of a differential equation is the highest derivative present in the differential equation.

The degree of an ordinary differential equation is the algebraic degree of its highest ordered derivative after simplification.

3.6.2. First order of Differential equations with separable variables

A separable differential equation is an equation of the form $\frac{dy}{dx} = f(x)h(y)$.

Before integrating both sides, such equation can be rewritten so that all terms involving y are on one side of the equation and all terms involving x are on the other. That is

$\frac{dy}{h(y)} = f(x)dx$ and $\int \frac{dy}{h(y)} = \int f(x)dx + c$.

3.6.3. Linear differential equations of the first order

If p and q are functions in x or constants the general linear equation of first order can take the form $\frac{dy}{dx} + py = q$

To solve such equation, determine an integrating factor $I(x) = e^{\int pdx}$ taking the integrating constant $c = 0$ and then find $y(x) = \frac{\int I(x)q(x) + C}{I(x)}$.

Or let $y = uv$ where u and v are functions in x to be determined in the following ways:

$v = e^{-\int pdx}$ By taking the constant $c = 0$ and $u = \int q\, e^{\int pdx}\, dx$

The solution of the equation $\frac{dy}{dx}+py=q$ becomes $y=uv$ where $u=\int q\,e^{\int pdx}\,dx$ and $v=e^{-\int pdx}$.

3.6.4. Application of differential equations of first order

a. Differential equation solutions to predict values in basic market and macroeconomic models.

If $y=e^t$ *then* $\frac{dy}{dt}=e^t$ thus, using the chain rule for differentiation, for any constant b,

If $y=e^t$ *then* $\frac{dy}{dt}=be^{bt}$. Therefore, if the differential equation to be solved has no constant term and has the format $\frac{dy}{dt}=by$ then a possible solution is $y=e^{bt}$ because this would give $\frac{dy}{dt}=be^{bt}=by$. With definite solution can now be used to predict y_t for any value t.

b. Stability of economic models where growth is continuous.

Assume that the demand and supply functions are Q_d and Q_s,

$Q_d=a+bP$ *and* $Q_s=c+dP$

With the parameters $a,d>0$ *and* $b,c<0$. If r represents the rate of adjustment of P in proportion to excess demand then we can write $\frac{dP}{dt}=r(Q_d-Q_s)$

Substituting the demand and supply functions for Q_d *and* Q_s gives

$$\frac{dP}{dt}=r\left[(a+bP)-(c+dP)\right]$$
$$=r(a-c+bP-dP)$$
$$=r(\mathrm{b-d})\,\mathrm{P}+r(-\mathrm{c})$$

As r, a, b, c and d d are all constant parameters this is effectively a first-order linear differential equation with one term in *P* plus a constant term.

c. Continuously compounded interest

If t represents time, then the rate of change of the initial deposit is $\frac{dS}{dt}$ and assuming that the initial deposit is compounded continuously, then we have that: $\frac{dS}{dt} = rs$, We can further set up an initial value problem to this differential equation. Suppose that the initial deposit is S_0 then $S(0) = S_0$. The solution to this initial value with the differential equation and initial condition will give us a function S which give us amount in the individuals account at time t:

$$\frac{dS}{dt} = rS, \frac{1}{S} dS = rdt$$

$$\int \frac{1}{S} dS = \int rdt$$

$$\ln S = rt + k$$

$$S = ce^{rt}$$

Using the initial condition that $S(0) = S_0$ and we have that $c = S_0$. Therefore the solution to this initial value is $S(t) = S_0 e^{rt}$ where S_0 is the principal invested, r is rate and $S(t)$ tells us the amount at any time t.

d. Differential equations are applied to determine the quantity of a drug in the body

The quantity of drug Q in the body of a patient in the time t is modeled by

$\frac{dQ}{dt} = -kQ$ where k is a constant that depends on the specific type of drug.

3. 7. Additional information for the teacher

3.7.1 Integrating factor $I(x)$ of a linear first order differential equation

We are given to solve a linear first order differential equation $\frac{dy}{dx} + py = q$ (1) where p and q are functions in x or constants. Such equation is solved by the use of the integrant factor $I(x)$ which is a function we assume it exists.

Let us multiply both sides of the equation (1) by $I(x)$.

We get $I(x)\frac{dy}{dx}+I(x)p(x)y=I(x)q(x)$ (2)

The function $I(x)$ plays a role such that $I(x)p(x)=I'(x)$ (3)

Therefore, substituting (3) into (2) we get $I(x)\frac{dy}{dx}+I'(x)y=I(x)q(x)$ (4)

The objective is to write the first side of (4) as a derivative for the product of $I(x)$ by $y(x)$.

That is $I(x)\frac{dy}{dx}+I'(x)y=\left(I(x)y(x)\right)'=I(x)q(x)$

This implies that $\int\left(I(x)y(x)\right)'dx=\int I(x)q(x)dx \Leftrightarrow I(x)y(x)+c=\int I(x)q(x)dx$

It is necessary to include the constant c because if it is left out, you will get the wrong answer every time.

The final step is then some algebra to solve for the solution $y(x)$:

$$I(x)y(x)=\int I(x)q(x)dx-c$$

$$\Leftrightarrow y(x)=\frac{\int I(x)q(x)dx-c}{I(x)}$$

Given that the constant of integration c is an arbitrary constant, to make our life easier we will consider a positive constant C and this will not affect the final answer for the solution.

Therefore, $y(x)=\frac{\int I(x)q(x)dx+C}{I(x)}$

This is a general solution of the equation (1) but we need to determine the function $I(x)$,

Let us start with the equality (3):

$I(x)p(x)=I'(x)$,

Separating $I(x)$ from $p(x)$, we get $\frac{I'(x)}{I(x)} = p(x)$

The first side is the derivative of $\ln(I(x))$ because $\left(\ln(I(x))\right)' = \frac{I'(x)}{I(x)}$

Our equality $\frac{I'(x)}{I(x)} = p(x)$ implies that $\left(\ln I(x)\right)' = p(t)$

The integrating both sides gives $\ln I(x) = \int p(t)dt + k$ and

$I(x) = Ke^{\int p(t)dt}$ where K is a constant.

The solution $y(x) = \frac{\int I(x)q(x)dx + C}{I(x)} = \frac{\int e^{\int pdx} q(x)dx + \frac{C}{K}}{e^{\int pdx}}$

$y(x) = \frac{\int e^{\int pdx} q(x)dx + C_1}{e^{\int pdx}}$ where C_1 is an arbitrary constant. This is the formula that was previously given in the unit summary.

3.8 End unit assessment

This part provides the answers of end unit assessment activities designed in integrative approach to assess the key unit competence with cross reference to the textbook.

1) $\frac{d^2y}{dx^2} + \frac{dy}{dx} - 6y = 0,\ y(0) = 10, y'(0) = 0$

The Differential equation is of order 2 and degree one.

2) We have $\frac{dp}{dt} = 0.2(Q_d - Q_s) \Rightarrow \frac{dp}{dt} = 0.2\left(35 - 5P - (-23 + 6P)\right)$

a) $\frac{dP}{dt} = 0.2(58 - 11P) \Leftrightarrow \frac{dP}{dt} + 2.2P = 11.6$ Which is a first order linear differential equation $p = 2.2;\quad q = 11.6$ and the initial condition is $P(0) = 100$.

Let us determine an integrating factor $I(t) = e^{\int pdx} = e^{\int 2.2dt} = e^{2.2t}$

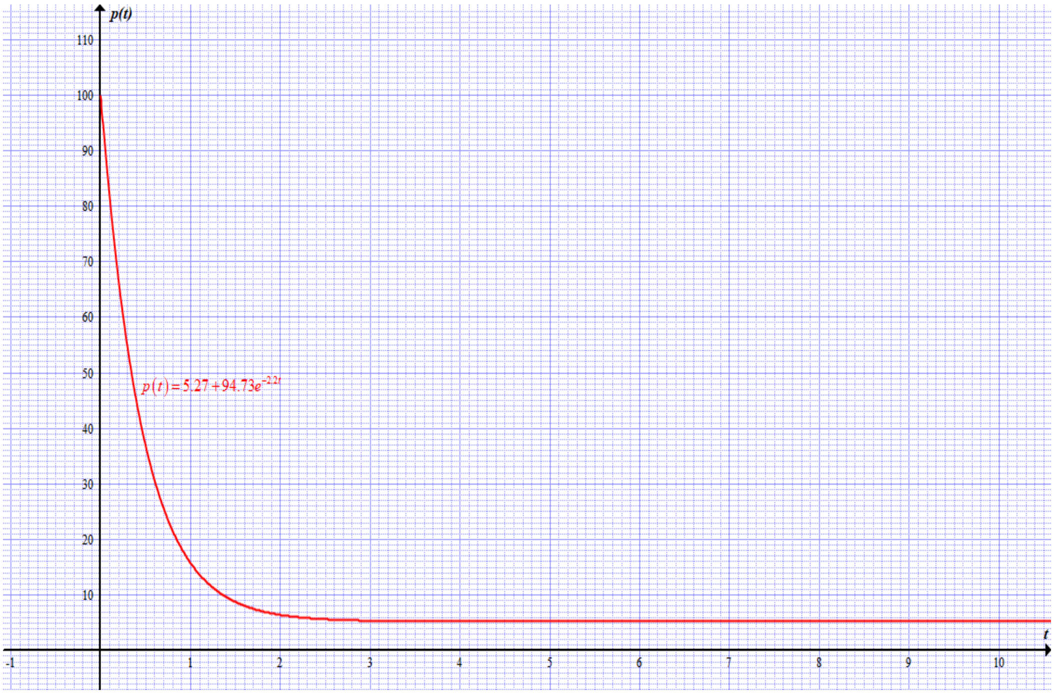
$$P(t) = \frac{\int I(t)q(t) + C}{I(t)} = \frac{\int e^{2.2t}11.6 + C}{e^{2.2t}} = \frac{11.6}{2.2} + Ce^{-2.2t} = 5.27 + Ce^{-2.2t}$$

.

Applying the initial condition is $P(0)=100 \Leftrightarrow 5.27+C=100 \Leftrightarrow C=94.73$,

$P(t)=5.27+94.73e^{-2.2t}$.

b) The graph of $P(t)=5.27+94.73e^{-2.2t}$



This market is stable to the price of 5.27 when t becomes larger.

3) Discussions on how this unit inspired student-teachers in relation to learning other subjects or to their future vary according to their expectations. As tutor harmonize their answers referring to the application of exponential function to real life.

4) $\frac{dy}{dt}=1.5$, the value of $y=34$ when $t=0$

The general solution to this differential equation is $y_t = Ae^{1.5t}$

When $t=0$ then $y_0=34=Ae^0$

Therefore, $34=A$

The definite solution is thus $y_t=34e^{1.5t}$

When $t=7$ then using this definite solution we can predict

$y_7=34e^{1.5(7)}=34e^{10.5}=34(36,315.5)=1,234,727$

3.9. Additional activities

3.9.1 Remedial activities

1. Solve the differential equation $\frac{dy}{dx} = x$

Solution

$$\int dy = \int x dx$$

$y = \frac{x^2}{2} + c$, where c is constant of integration hence, the general solution to $\frac{dy}{dx} = x$ is $y = \frac{x^2}{2} + c$

2) Given that $x = 0$ when $y = 2$, solve $y\frac{dy}{dx} = 2x(1+y)$

Solution

By separating variables we have $\frac{y}{1+y}dy = 2xdx$

$$\int\left(1 - \frac{1}{1+y}\right)dy = \int 2xdx$$

$$y - \ln(1+y) = x^2 + k$$

At $y(0) = 2, \Rightarrow 2 - \ln(1+2) = 0^2 + k$

$$k = 2 - \ln 3$$

Therefore, $y - \ln(1+y) = x^2 + 2 - \ln 3$ this is a particular solution equation $y\frac{dy}{dx} = 2x(1+y)$

3.9.2 Consolidation activities

Activity 1

Solve $x^2\frac{dy}{dx}+xy=x^2e^x$ given that $y(0)=0$

Solution

$$x^2\frac{dy}{dx}+xy=x^2e^x \Rightarrow \frac{dy}{dx}+\frac{1}{x}y=e^x$$

$$\int I(x)=e^{\int\frac{1}{x}dx}, \quad I(x)=e^{\ln x}=x$$

$$x\frac{dy}{dx}+x\left(\frac{1}{x}y\right)=xe^x$$

$$\frac{dy}{dx}(xy)=xe^x$$

$$\int d(xy)=\int xe^x dx$$

Let $u=x,\Rightarrow\frac{du}{dx}=1$

$$\frac{dv}{dx}=e^x,\Rightarrow v=e^x$$

$$xy=xe^x-\int e^x dx$$

$$xy=xe^x-e^x+c$$

At $y(0)=0$

$$0=0-e^0+c$$
$$c=0$$

$$\Rightarrow xy=xe^x+1-e^x$$
$$y=e^x+\frac{1}{x}\left(1-e^x\right)$$

Activity 2:

Solve the equation $(x-2)\frac{dy}{dx}+3y=\frac{2}{x-2}$ given that $y=-\frac{1}{2}, x=0$

Solution

$$\frac{dy}{dx}+\frac{3}{x-2}y=\frac{2}{(x-2)^2}$$

$$I(x)=e^{\int\frac{3}{x-2}dx}=e^{3\ln(x-2)}=e^{\ln(x-2)^3}$$

$$I(x)=(x-2)^3$$

$$(x-2)^3\frac{dy}{dx}+3(x-2)^2y=2(x-2)$$

$$\frac{d}{dx}\left(y(x-2)^3\right)=2(x-2)$$

$$\int d\left[y(x-2)^3\right]=\int 2(x-2)dx$$

$$y(x-2)^3=x^2-4x+c$$

At $y(0)=-\frac{1}{2}$

$$-\frac{1}{2}(-2)^3=0-0+c, \Rightarrow c=4$$

$$y(x-2)^3=(x-2)^2, \Rightarrow y=\frac{1}{x-2}$$

Activity 3:

Find the solution y to the initial value problem $y'=-3y+1,\ y(0)$

Solution

Write the differential equation as $y'+3y=1$ and multiply the equation by the integrating factor $I=e^{3t}$ which will convert the left-hand side above into total derivative,

$e^{3t}y' + 3e^{3t}y = e^{3t} \Leftrightarrow e^{3t}y' + \left(e^{3t}\right)' y = e^{3t}$

This is key idea because $\left(e^{3t}y\right)' = e^{3t}$

The exponent e^{3t} is an integrating factor. Integrate both sides of equation yields

$e^{3t}y = \frac{1}{3}e^{3t} + c$

So, every solution of the differential equation above is given by

$y(t) = ce^{-3t} + \frac{1}{3},\ c \in \mathbb{R}$

The initial condition $y(0) = 1$

$1 = c + \frac{1}{3} \Rightarrow c = \frac{2}{3}$

We get the solution $y(t) = \frac{1}{3}e^{-3t} + \frac{1}{3}$

3.9.3 Extended activities

Activity 1:

The size S of the population at time t satisfies approximately the differential equation $\frac{dS}{dt} = kS$ where k is positive constant.

i. Show that $S = Ce^{kt}$ where C is constant

ii. If the population number was 32,000 in 1910 and increased to 48,000 in 1980. Estimate what its size in the year 2010

Solution

i. $\int \frac{dS}{dt} = kS$

$\int \frac{dS}{S} = \int kdt$

$S = kt + c$

$S - C = kt$

$\ln(S-C)=\ln kt$

$\ln\left(\frac{S}{C}\right)=kt$

$S=Ce^{kt}$

At $t=0$ *and* $32,000$

$\ln 32,000=0+\ln C$, $\Rightarrow C=32,000$

$S=32,000e^{kt}$

At $t=70\,(\mathit{from}\ 1910\ \mathit{to}\ 1980),\ S=48,000$

$48,000=32,000e^{k(70)}$

$1.5=e^{70k},\Rightarrow k=\frac{1}{70}\ln 1.5$

At $t=100\,(\mathit{from}\ 1910\ \mathit{to}\ 2010),\ S=?$

$S=32,000e^{\frac{1}{70}[\ln 1.5]100}=32,000e^{\frac{10}{70}\ln(1.5)}$

$S=57109.75$

$S\approx 57110$ *people*

Activity 2:

If $e^{-2x}\frac{dy}{dx}=2y+1$ when $y(0)=0$ show that $\frac{1}{2}\left[e^{\left(e^{2x}-1\right)}-1\right]$

Solution

$\int\frac{dy}{1+2y}=\frac{dx}{e^{-2x}}=\int e^{2x}dx$

$\frac{1}{2}\ln(1+2y)=\frac{1}{2}e^{2x}+c$

If $y(0)=0$

$$\frac{1}{2}\ln(1)=\frac{1}{2}e^{0}+c,\ \Rightarrow c=-\frac{1}{2}$$

$$\frac{1}{2}\ln(1+2y)=\frac{1}{2}e^{2x}-\frac{1}{2}$$

$$\ln(1+2y)=e^{2x}-1$$

$$1+2y=e^{\left(e^{2x}-1\right)}$$

$y=\frac{1}{2}\left[e^{\left(e^{2x}-1\right)}-1\right]$. Therefore $e^{-2x}\frac{dy}{dx}=2y+1$ *is equivalent* $\frac{1}{2}\left[e^{\left(e^{2x}-1\right)}-1\right]$

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