MATHEMATICS FOR TTCs

TUTOR'S GUIDE

YEAR

2

OPTION:

SOCIAL STUDIES EDUCATION (SSE)

FOREWORD

Dear Tutor,

Rwanda Education Board is honoured to present the tutor's guide for Year two Mathematics in the option of SSE. This book serves as a guide to competence-based teaching and learning to ensure consistency and coherence in the learning of the mathematics content. The Rwandan educational philosophy is to ensure that learners achieve full potential at every level of education, which will prepare them to be well integrated in society and exploit employment opportunities.

Specifically, TTC curriculum was reviewed to train quality teachers who will confidently and efficiently implement the Competence Based Curriculum in pre-primary and primary education. The rationale of the changes is to ensure that TTC leavers are qualified for job opportunities and further studies in Higher Education in different programs under education career advancement.

In line with efforts to improve the quality of education, the government of Rwanda emphasizes the importance of aligning teaching and learning materials with the syllabus to facilitate their learning process. Many factors influence what they learn, how well they learn and the competences they acquire. Those factors include the relevance of the specific content, the quality of teachers' pedagogical approaches, the assessment strategies and the instructional materials.

The ambition to develop a knowledge-based society and the growth of regional and global competition in the jobs market has necessitated the shift to a competence-based curriculum. After a successful shift from knowledge to a competence based curriculum in general education, TTC textbooks also were elaborated to align them to the new curriculum.

The book provides active teaching and learning techniques that engage student teachers to develop competences. In view of this, your role is to:

- Plan your lessons and prepare appropriate teaching materials.
- Organize group discussions for students considering the importance of social constructivism suggesting that learning occurs more effectively when the students works collaboratively with more knowledgeable and experienced people.
- Engage students through active learning methods such as inquiry methods, group discussions, research, investigative activities and group and individual work activities.
- Provide supervised opportunities for students to develop different

competences by giving tasks which enhance critical thinking, problem solving, research, creativity and innovation, communication and cooperation.

- Support and facilitate the learning process by valuing students' contributions in the class activities.
- Guide students towards the harmonization of their findings.
- Encourage individual, peer and group evaluation of the work done in the classroom and use appropriate competence-based assessment approaches and methods.

To facilitate you in your teaching activities, the content of this book is self explanatory so that you can easily use it. It is divided in 3 parts:

The part I explains the structure of this book and gives you the methodological guidance;

The part II gives the sample lesson;

The part III details the teaching guidance for each concept given in the student book.

Even though this Teacher's guide contains the guidance on solutions for all activities given in the learner's book, you are requested to work through each question before marking student's findings.

I wish to sincerely express my appreciation to the people who contributed towards the development of this book, particularly, REB staff, TTC Tutors, Teachers from general education for their technical support. A word of gratitude goes also to the Head Teachers and TTCs principals who availed their staff for various activities

Dr. NDAYAMBAJE Irénée
Director General of Rwanda Education Board

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I owe gratitude to different universities and schools in Rwanda that allowed their staff to work with REB in the in-house textbooks production initiative.

I wish to extend my sincere gratitude to lecturers and teachers whose efforts during writing exercise of this tutor`s guide was very much valuable.

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TABLE OF CONTENTS

PART I: GENERAL INTRODUCTION

The structure of the guide

The tutor's guide of Mathematics is composed of three parts:

The Part I concerns general introduction that discusses methodological guidance on how best to teach and learn Mathematics, developing competences in teaching and learning, addressing cross-cutting issues in teaching and learning and Guidance on assessment.

Part II presents a sample lesson plan. This lesson plan serves to guide the teacher on how to prepare a lesson in Mathematics.

The Part III is about the structure of a unit and the structure of a lesson. This includes information related to the different components of the unit and these components are the same for all units. This part provides information and guidelines on how to facilitate student teachers while working on learning activities. More other, all application activities from the textbook have answers in this part.

Methodological guidance

Developing competences

Since 2015 Rwanda shifted from a knowledge based to a competency-based curriculum for pre-primary, primary, secondary education and recently the TTC curriculum. This called for changing the way of learning by shifting from teacher centred to a learner centred approach. Teachers are not only responsible for knowledge transfer but also for fostering learners' learning achievement and creating safe and supportive learning environment. It implies also that learners have to demonstrate what they are able to transfer the acquired knowledge, skills, values and attitude to new situations.

The competence-based curriculum employs an approach of teaching and learning based on discrete skills rather than dwelling on only knowledge or the cognitive domain of learning. It focuses on what learner can do rather than what learner knows. Learners develop competences through subject unit with specific learning objectives broken down into knowledge, skills and attitudes through learning activities.

In addition to the competences related to Mathematics, student teachers also develop generic competences which should promote the development of the higher order thinking skills and professional skills in Mathematics teaching. Generic competences are developed throughout all units of Mathematics as follows:

Generic competences	**Ways of developing generic competences**
Critical thinking	All activities that require learners to calculate, convert, interpret, analyse, compare and contrast, etc. have a common factor of developing critical thinking into learners
Creativity and innovation	All activities that require learners to plot a graph of a given algebraic data, to organize and interpret statistical data collected and to apply skills in solving problems of economics have a common character of developing creativity into learners
Research and problem solving	All activities that require learners to make a research and apply their knowledge to solve problems from the real-life situation have a character of developing research and problem solving into learners.
Communication	During Mathematics class, all activities that require learners to discuss either in groups or in the whole class, present findings, debate …have a common character of developing communication skills into learners.
Co-operation, interpersonal relations and life skills	All activities that require learners to work in pairs or in groups have character of developing cooperation and life skills among learners.
Lifelong learning	All activities that are connected with research have a common character of developing into learners a curiosity of applying the knowledge learnt in a range of situations. The purpose of such kind of activities is for enabling learners to become life-long learners who can adapt to the fast-changing world and the uncertain future by taking initiative to update knowledge and skills with minimum external support.
Professional skills	Specific instructional activities and procedures that a teacher may use in the class room to facilitate, directly or indirectly, students to be engaged in learning activities. These include a range of teaching skills: the skill of questioning, reinforcement, probing, explaining, stimulus variation, introducing a lesson; illustrating with examples, using blackboard, silence and non verbal cues, using audio – visual aids, recognizing attending behaviour and the skill of achieving closure.

The generic competences help learners deepen their understanding of Mathematics and apply their knowledge in a range of situations. As students develop generic competences they also acquire the set of skills that employers look for in their employees, and so the generic competences prepare students for the world of work.

Addressing cross cutting issues

Among the changes brought by the competence-based curriculum is the integration of cross cutting issues as an integral part of the teaching learning process-as they relate to and must be considered within all subjects to be appropriately addressed. The eight cross cutting issues identified in the national curriculum framework are: *Comprehensive Sexuality Education, Environment and Sustainability, Financial Education, Genocide studies, Gender, Inclusive Education, Peace and Values Education, and Standardization Culture*.

Some cross-cutting issues may seem specific to particular learning areas/ subjects but the teacher need to address all of them whenever an opportunity arises. In addition, learners should always be given an opportunity during the learning process to address these cross-cutting issues both within and out of the classroom.

Below are examples of how crosscutting issues can be addressed:

Cross-Cutting Issue	**Ways of addressing cross-cutting issues**
Comprehensive Sexuality Education: The primary goal of introducing Comprehensive Sexuality Education program in schools is to equip children, adolescents, and young people with knowledge, skills and values in an age appropriate and culturally gender sensitive manner so as to enable them to make responsible choices about their sexual and social relationships, explain and clarify feelings, values and attitudes, and promote and sustain risk reducing behaviour.	Using different charts and their interpretation, Mathematics tutor should lead students to discuss the following situations: "Alcohol abuse and unwanted pregnancies" and advise student teachers on how they can instil learners to fight those abuses. Some examples can be given in powers and properties , logarithms and properties , and statistics

Environment and Sustainability: Integration of Environment, Climate Change and Sustainability in the curriculum focuses on and advocates for the need to balance economic growth, society well-being and ecological systems. Learners need basic knowledge from the natural sciences, social sciences, and humanities to understand to interpret principles of sustainability.	Using Real life models or students' experience, Mathematics Tutor should lead student teachers to illustrate the situation of "population growth" and discuss its effects on the environment and sustainability. Some examples in proportional change, logarithms, and polynomial functions
Financial Education: The integration of Financial Education into the curriculum is aimed at a comprehensive Financial Education program as a precondition for achieving financial inclusion targets and improving the financial capability of Rwandans so that they can make appropriate financial decisions that best fit the circumstances of one's life.	Through different examples and calculations on interest rate problems, total revenue and total cost, Mathematics Tutor can lead student teachers to discuss how to make appropriate financial decisions. Some examples in ratios and proportions, statistics, equations, polynomial functions
Gender: At school, gender will be understood as family complementarities, gender roles and responsibilities, the need for gender equality and equity, gender stereotypes, gender sensitivity, etc.	Mathematics Tutor should address gender as cross-cutting issue through assigning leading roles in the management of groups to both girls and boys and providing equal opportunity in the lesson participation and avoid any gender stereotype in the whole teaching and learning process.
Inclusive Education: Inclusion is based on the right of all learners to a quality and equitable education that meets their basic learning needs and understands the diversity of backgrounds and abilities as a learning opportunity.	Firstly, Mathematics Tutors need to identify/recognize students with special needs. Then by using adapted teaching and learning resources while conducting a lesson and setting appropriate tasks to the level of students, they can cater for students with special education needs. They must create opportunity where student teachers can discuss how to cater for learners with special educational needs.

Peace and Values Education: Peace and Values Education (PVE) is defined as education that promotes social cohesion, positive values, including pluralism and personal responsibility, empathy, critical thinking and action in order to build a more peaceful society.	Through a given lesson, a tutor should: - Set a learning objective which is addressing positive attitudes and values, - Encourage students to develop the culture of tolerance during discussion and to be able to instil it in colleagues and cohabitants; - Encourage students to respect ideas for others.
Standardization Culture: Standardization Culture in Rwanda will be promoted through formal education and plays a vital role in terms of health improvement, economic growth, industrialization, trade and general welfare of the people through the effective implementation of Standardization, Quality Assurance, Metrology and Testing.	With different word problems related to the effective implementation of Standardization, Quality Assurance, Metrology and Testing, students can be motivated to be aware of health improvement, economic growth, industrialization, trade and general welfare of the people.

Guidance on how to help students with special education needs in classroom

In the classroom, students learn in different way depending to their learning pace, needs or any other special problem they might have. However, the teacher has the responsibility to know how to adopt his/her methodologies and approaches in order to meet the learning need of each student in the classroom. Also teachers need to understand that student with special needs, need to be taught differently or need some accommodations to enhance the learning environment. This will be done depending to the subject and the nature of the lesson.

In order to create a well-rounded learning atmosphere, teachers need to:

- Remember that learners learn in different ways so they have to offer a variety of activities (e.g. role-play, music and singing, word games and quizzes, and outdoor activities);
- Maintain an organized classroom and limits distraction. This will help learners with special needs to stay on track during lesson and follow instruction easily;

- Vary the pace of teaching to meet the needs of each child. Some learners process information and learn more slowly than others;
- Break down instructions into smaller, manageable tasks. Learners with special needs often have difficulty understanding long-winded or several instructions at once. It is better to use simple, concrete sentences in order to facilitate them understand what you are asking.
- Use clear consistent language to explain the meaning (and demonstrate or show pictures) if you introduce new words or concepts;
- Make full use of facial expressions, gestures and body language;
- Pair a learner who has a disability with a friend. Let them do things together and learn from each other. Make sure the friend is not over protective and does not do everything for the one with disability. Both learners will benefit from this strategy;
- Use multi-sensory strategies. As all learners learn in different ways, it is important to make every lesson as multi-sensory as possible. Learners with learning disabilities might have difficulty in one area, while they might excel in another. For example, use both visual and auditory cues.
- Below are general strategies related to each main category of disabilities and how to deal with every situation that may arise in the classroom. However, the list is not exhaustive because each child is unique with different needs and that should be handled differently.

Strategy to help learners with developmental impairment:

- Use simple words and sentences when giving instructions;
- Use real objects that learners can feel and handle. Rather than just working abstractly with pen and paper;
- Break a task down into small steps or learning objectives. The learner should start with an activity that she/he can do already before moving on to something that is more difficult;
- Gradually give the learner less help;
- Let the learner with disability work in the same group with those without disability.

Strategy to help learners with visual impairment:

- Help learners to use their other senses (hearing, touch, smell and taste) and carry out activities that will promote their learning and development;
- Use simple, clear and consistent language;

- Use tactile objects to help explain a concept;
- If the learner has some sight, ask him/her what he/she can see;
- Make sure the learner has a group of friends who are helpful and who allow him/her to be as independent as possible;
- Plan activities so that learners work in pairs or groups whenever possible;

Strategy to help learners with hearing disabilities or communication difficulties

- Always get the learner's attention before you begin to speak;
- Encourage the learner to look at your face;
- Use gestures, body language and facial expressions;
- Use pictures and objects as much as possible.
- Keep background noise to a minimum.

Strategies to help learners with physical disabilities or mobility difficulties:

- Adapt activities so that learners who use wheelchairs or other mobility aids, can participate.
- Ask parents/caregivers to assist with adapting furniture e.g. the height of a table may need to be changed to make it easier for a learner to reach it or fit their legs or wheelchair under;
- Encourage peer support when needed;
- Get advice from parents or a health professional about assistive devices if the learner has one.

Adaptation of assessment strategies:

At the end of each unit, the tutor is advised to provide additional activities to help students achieve the key unit competence. These assessment activities are for remedial, consolidation and extension designed to cater for the needs of all categories of students; slow, average and gifted students respectively.

Therefore, the tutor is expected to do assessment that fits individual student.

Remedial activities	After evaluation, slow students are provided with lower order thinking activities related to the concepts learnt to facilitate them in their learning. These activities can also be given to assist deepening knowledge acquired through the learning activities for slow students.

Consolidation activities	After introduction of any concept, a range number of activities can be provided to all students to enhance/ reinforce learning.
Extended activities	After evaluation, gifted and talented students can be provided with high order thinking activities related to the concepts learnt to make them think deeply and critically. These activities can be assigned to the gifted and talented students to keep them working while other students are getting up to required level of knowledge through the learning activity.

Guidance on assessment

Assessment is an integral part of teaching and learning process. The main purpose of assessment is for improvement of learning outcomes. Assessment for learning/ Continuous/ formative assessment intends to improve students' learning and tutor's teaching whereas assessment of learning/summative assessment intends to improve the entire school's performance and education system in general.

Continuous/ formative assessment

It is an on-going process that arises during the teaching and learning process. It includes lesson evaluation and end of sub unit assessment. This formative assessment should play a big role in teaching and learning process. The teacher should encourage individual, peer and group evaluation of the work done in the classroom and uses appropriate competence-based assessment approaches and methods.

Formative assessment is used to:

- Determine the extent to which learning objectives are being achieved and competences are being acquired and to identify which students need remedial interventions, reinforcement as well as extended activities. The application activities are developed in the learner book and they are designed to be given as remedial, reinforcement, end lesson assessment, homework or assignment
- Motivate students to learn and succeed by encouraging students to read, or learn more, revise, etc.
- Check effectiveness of teaching methods in terms of variety, appropriateness, relevance, or need for new approaches and strategies. Mathematics tutors need to consider various aspects of the instructional process including appropriate language levels, meaningful examples, suitable methods and teaching aids/ materials, etc.

- Help students to take control of their own learning.

In teaching Mathematics, formative or continuous assessment should compare performance against instructional objectives. Formative assessment should measure the student's ability with respect to a criterion or standard. For this reason, it is used to determine what students can do, rather than how much they know.

Summative assessment

The assessment can serve as summative and informative depending to its purpose. The end unit assessment will be considered summative when it is done at end of unit and want to start a new one.

It will be formative assessment, when it is done in order to give information on the progress of learners and from there decide what adjustments need to be done.

The assessment done at the end of the term, end of year, is considered as summative assessment so that the teacher, school and parents are informed of the achievement of educational objective and think of improvement strategies. There is also end of level/ cycle assessment in form of national examinations.

When carrying out assessment?

Assessment should be clearly visible in lesson, unit, term and yearly plans.

- Before learning (diagnostic): At the beginning of a new unit or a section of work; assessment can be organized to find out what students already know/ can do, and to check whether the students are at the same level.
- During learning (formative/continuous): When students appear to be having difficulty with some of the work, by using on-going assessment (continuous). The assessment aims at giving students support and feedback.
- After learning (summative): At the end of a section of work or a learning unit, the Mathematics Tutor has to assess after the learning. This is also known as Assessment of Learning to establish and record overall progress of students towards full achievement. Summative assessment in Rwandan schools mainly takes the form of written tests at the end of a learning unit or end of the month, and examinations at the end of a term, school year or cycle.

Instruments used in assessment.

- **Observation:** This is where the Mathematics tutor gathers information by watching students interacting, conversing, working, playing, etc. A tutor can use observations to collect data on behaviours that are difficult to assess by other methods such as attitudes, values, and generic competences and intellectual skills. It is very important because it is used before the lesson begins and throughout the lesson since the tutor has to continue observing each and every activity.

- **Questioning**
  a) Oral questioning: a process which requires a student to respond verbally to questions
  b) Class activities/exercise: tasks that are given during the learning/teaching process
  c) Short and informal questions usually asked during a lesson
  d) Homework and assignments: tasks assigned to students by their tutors to be completed outside of class.

Homework assignments, portfolio, project work, interview, debate, science fair, Mathematics projects and Mathematics competitions are also the different forms/instruments of assessment.

Teaching methods and techniques that promote active learning

The different learning styles for students can be catered for, if the teacher uses active learning whereby learners are really engaged in the learning process.

The main teaching methods used in mathematics are the following:

- **Dogmatic method** (the teacher tells the students what to do, What to observe, How to attempt, How to conclude)
- **Inductive-deductive method:** Inductive method is to move from specific examples to generalization and deductive method is to move from generalization to specific examples.
- **Analytic-synthetic method:** Analytic method proceeds from unknown to known, 'Analysis' means 'breaking up' of the problem in hand so that it ultimately gets connected with something obvious or already known. Synthetic method is the opposite of the analytic method. Here one proceeds from known to unknown.
- **Skills Lab method:** Skills Lab method is based on the maxim "learning by doing." It is a procedure for stimulating the activities of the students and to encourage them to make discoveries through practical activities.
- **Problem solving method, Project method and Seminar Method.**

The following are some active techniques to be used in Mathematics:

- Group work
- Research
- Probing questions
- Practical activities (drawing, plotting, interpreting graphs)
- Modelling
- Brainstorming
- Quiz Technique
- Discussion Technique
- Scenario building Technique

What is Active learning?

Active learning is a pedagogical approach that engages learners in doing things and thinking about the things they are doing. Learners play the key role in the active learning process. They are not empty vessels to fill but people with ideas, capacity and skills to build on for effective learning. Thus, in active learning, learners are encouraged to bring their own experience and knowledge into the learning process.

The role of the teacher in active learning	**The role of learners in active learning**
- The teacher engages learners through active learning methods such as inquiry methods, group discussions, research, investigative activities, group and individual work activities. - He/she encourages individual, peer and group evaluation of the work done in the classroom and uses appropriate competence-based assessment approaches and methods. - He provides supervised opportunities for learners to develop different competences by giving tasks which enhance critical thinking, problem solving,	A learner engaged in active learning: - Communicates and shares relevant information with peers through presentations, discussions, group work and other learner-centred activities (role play, case studies, project work, research and investigation); - Actively participates and takes responsibility for his/her own learning; - Develops knowledge and skills in active ways; - Carries out research/investigation by consulting print/online documents and resourceful people, and presents their findings;

research, creativity and innovation, communication and cooperation. - Teacher supports and facilitates the learning process by valuing learners' contributions in the class activities.	- Ensures the effective contribution of each group member in assigned tasks through clear explanation and arguments, critical thinking, responsibility and confidence in public speaking - Draws conclusions based on the findings from the learning activities.

Main steps for a lesson in active learning approach

All the principles and characteristics of the active learning process highlighted above are reflected in steps of a lesson as displayed below. Generally, the lesson is divided into three main parts whereby each one is divided into smaller steps to make sure that learners are involved in the learning process. Below are those main part and their small steps:

1) Introduction

Introduction is a part where the teacher makes connection between the current and previous lesson through appropriate technique. The teacher opens short discussions to encourage learners to think about the previous learning experience and connect it with the current instructional objective. The teacher reviews the prior knowledge, skills and attitudes which have a link with the new concepts to create good foundation and logical sequencings.

2) Development of the new lesson

The development of a lesson that introduces a new concept will go through the following small steps: discovery activities, presentation of learners' findings, exploitation, synthesis/summary and exercises/application activities.

- **Discovery activity**

  Step 1

  - The teacher discusses convincingly with learners to take responsibility of their learning
  - He/she distributes the task/activity and gives instructions related to the tasks (working in groups, pairs, or individual to instigate collaborative learning, to discover knowledge to be learned)

  Step 2

  - The teacher let learners work collaboratively on the task;
  - During this period the teacher refrains to intervene directly on the knowledge;

- He/she then monitors how the learners are progressing towards the knowledge to be learned and boosts those who are still behind (but without communicating to them the knowledge).

- **Presentation of learners' findings/productions**
  - In this episode, the teacher invites representatives of groups to present their productions/findings.
  - After three/four or an acceptable number of presentations, the teacher decides to engage the class into exploitation of learners' productions.
- **Exploitation of learner's findings/ productions**
  - The teacher asks learners to evaluate the productions: which ones are correct, incomplete or false
  - Then the teacher judges the logic of the learners' products, corrects those which are false, completes those which are incomplete, and confirms those which are correct.
- **Institutionalization or harmonization (summary/conclusion/ and examples)**
  - The teacher summarizes the learned knowledge and gives examples which illustrate the learned content.
- **Application activities**
  - Exercises of applying processes and products/objects related to learned unit/sub-unit
  - Exercises in real life contexts
  - Teacher guides learners to make the connection of what they learnt to real life situations. At this level, the role of teacher is to monitor the fixation of process and product/object being learned.

3) Assessment

In this step, the teacher asks some questions to assess achievement of instructional objective. During assessment activity, learners work individually on the task/activity. The teacher avoids intervening directly. In fact, results from this assessment inform the teacher on next steps for the whole class and individuals. In some cases, the teacher can end with a homework/ assignment. Doing this will allow learners to relay their understanding on the concepts covered that day. Teacher leads them not to wait until the last minute for doing the homework as this often results in an incomplete homework set and/or an incomplete understanding of the concept.

PART II: SAMPLE LESSON PLAN

Lesson one

School Name: ………………………………… **Tutor's names:** ………………………

Term	Date	Subject	Class	Unit Nº	Lesson Nº	Duration	Class size
I	……. /……./2020	Mathematics	Year2 SSE	2	2 of 9	40 min	…
Type of Special Educational Needs to be catered for in this lesson and number of learners in each category				3 slow learners and 2 low vision learners:			

Unit title	**Logarithmic and exponential equations.**
Key unit competency:	Solve equations involving logarithms or exponentials and apply them to model and solve related problems.
Title of the lesson	Exponential equations
Instructional Objective	Using properties of powers, learners will be able to solve correctly exponential equations in the given time.
Plan for this Class (location: in / outside)	The lesson is held indoors, the class is organized into groups,3 slow learners are scattered in different groups ,and 2 low vision learners seat on the front desks near the blackboard in order to see and participate fully in all activities.
Learning Materials (for ALL learners)	Textbooks , the classroom in which the learners are studying
References	- TTC syllabus, - Mathematics Year2 Mathematics textbook and Teacher's guide - S5Subsidiary Mathematics textbook and Teacher's guide

<table>
<tr><th rowspan="3">Timing for each step</th><th colspan="2">Description of teaching and learning activity</th><th rowspan="3">Generic competences and cross cutting issues to be addressed + a short explanation</th></tr>
<tr><td colspan="2">- Students-Teachers work individually the activity 2.2 (question1) in the introduction, and the correction is done on the chalk board by two students-Teachers, one after another under the guidance of the tutor.
- Students-Teachers discuss in small groups the discovery activity 2.2 (question 2), followed by the presentation by a sample group, interaction of learners and harmonization of the results under the facilitation of the tutor.
- Next, they discuss in pairs the solved example and compare their results with the answer proposed in the book.
- Finally, the students-Teachers are assigned individual tasks, and the correction is done on the chalk board, and the tutor winds up the lesson.</td></tr>
<tr><td>Teacher activities</td><td>Learner activities</td></tr>
<tr><td>Introduction:

5 minutes</td><td>The tutor asks learners to work individually the activity 2.2 (the 1st question):

- The Tutor links the introduction to the lesson of the day</td><td>- Student-Teachers work individually.
- Two Student-Teachers, one after another, write the answers on the chalkboard.</td><td>Communication skills developed through the presentation and sharing ideas</td></tr>
<tr><td colspan="4" align="center">2. Development of the lesson</td></tr>
<tr><td>2.1 Discovery activity:
10 minutes</td><td>- The Tutor organizes the learners into groups
- Tutor gives learners activity 2.2 (2nd question) to discuss in groups and gives instructions related to the task</td><td>- Student-Teachers form groups
- Each group analyzes and discuss the activity 2.2(2nd question) under the direction of the task manager of the group.</td><td>- Cooperation and communication skills through discussions</td></tr>
</table>

	- Tutor goes round to monitor the work of each group and provides assistance where needed.	- Student-Teachers present to the teacher their eventual problems.	- **Peace and values education**; Cooperation , mutual respect, tolerance through discussions with people with different views and respect one's views
2.2 Presentation of learner's findings and exploitation: **15 minutes**	- Tutor invites one member of a sample group to present the findings of the group; - The Tutor encourages learners to follow attentively - Tutor takes notes on key points from student's presentation. - The Tutor asks students to amend the presentation and to evaluate their work	- The reporter presents the work on the behalf of the group. **Expected answers** **(**Refer to solution of activity 2.2, in TG) - Student-Teachers follow the presentation. - Student-Teachers evaluate the findings of other learners - Student-Teachers their own findings	- **Cooperation** and **communication/** attentive listening during presentations and group discussions - **Critical thinking** through evaluating other's findings
	- Tutor facilitates the students to elaborate the summary of the presentation. - Tutor requests students to write down the main points in their books	- The Student-Teachers come to the main point: a) For $a > 0$, and $a \neq 1$, the number a^u is positive, i.e $\forall u \in \mathbb{R}, a^u \in \mathbb{R}^+$. b) If $a^u = a^v, then\ u = v$ each side of the equality must be written with the same base.	- **Critical thinking** and **problem solving** skills are developed through analyzing and solving the application activity 2.2.

2.4.Conclusion/ Summary: **5 minutes** **Assessment** **5 minutes**	- Tutor asks students to individually work out the application activity 2.2	c) Sometimes, the equation of the form $a^u = b, b \in \mathbb{R}^+$ are such that Since b cannot be written as an integer power of a , write the exponential equation as the equivalent logarithmic equation $a^u = b \Rightarrow u = \log_a b$. However, if $a^u = b$, and $b \le 0$, the equation does not have the solution - Student-Teachers take notes in their books - Individually learners work out the application activity 2.2. and finally they make a correction on the chalk board. **Expected answers** (Refer to solution of application activity 2.2, in TG)	
Observation on lesson delivery	To be completed after receiving the feed-back from the Student-Teachers (what did the learners like, what challenged them...)		

UNIT 1

SEQUENCES AND SERIES

1.1. Key unit competence:

Apply arithmetic and geometric sequences to solve problems in financial mathematics.

1.2. Prerequisites

Student-teachers will easily learn this unit, if they have a good background on Arithmetic (Unit 1 Year 1), Equations and inequalities (unit 2 Year 1), and on limits of functions (unit 4 Year 1).

1.3. Cross-cutting issues to be addressed

Inclusive education (promote education for all while teaching)

Peace and value Education (respect others' view and thoughts during class discussions)

Gender (provide equal opportunity to boys and girls in the lesson)

1.4. Guidance on introductory activity

- Invite student-teachers to work in groups and give them instructions on how they can do the introductory activity 1.0 found in unit 1 of student's book;
- Guide students to read and analyse the questions insisting on the analysis of the given data and to determine the number of insects that will be there in second, third, fourth,…n^{th} generation.
- Invite some group members to present groups' findings, then try to harmonize their answers; try to insist on the list formed by the number of insects at any generation and the generalisation (number of insects at n^{th} generation).
- Basing on student-teachers' experience, prior knowledge and abilities shown in answering the questions for this activity, use different questions to facilitate them to give their predictions and ensure that you arouse their curiosity on what is going to be leant in this unit.

Answers for introductory activity

Number of insects in is given by:

1^{st} generation $\to 126 = 63.2^1$, 2^{nd} generation $\to 126\times 2 = 252 = 63.(2)^2$,

3^{rd} generation $\to 252\times 2 = 504 = 63.(2)^3$, 4^{th} generation $\to 1008 = 63(2)^4$,

$At\ n^{th} generation \to 126\times 2^{(n-1)} = 63.(2^n)$

5^{th} generation $\to 2016$, 6^{th} generation $\to 4032$,

7^{th} generation $\to 8064$, 8^{th} generation $\to 16128$,

9^{th} generation $\to 32256$, 10^{th} generation $\to 64512$

1.5. List of lessons

#	Lesson title	Learning objectives	Number of periods
0	Introductory activity	To arouse the curiosity of student teachers on the content of unit 1.	1
1	Generalities on sequences and series	To define sequences and series, infinite sequence and to determine terms of a sequence	2
2	Convergent and divergent sequence	Explore sequences and differentiate the convergent from a divergent sequence	1
3	Monotone sequences	Use basic concepts and formulas of sequences to identify the Monotonic sequences	1
4	Arithmetic sequence and its general term	Use basic concepts of sequences to determine terms of an arithmetic sequence	2
5	Arithmetic means	Use basic concepts of sequences to find an arithmetic mean of two numbers	1
5	Arithmetic series	Use basic concepts of sequences to calculate the sum of the first "n" terms of an arithmetic sequence	2
7	Harmonic sequences and its general term	Use basic concepts and formulas of sequences to find the terms of an harmonic sequence	1

8	Generalities on Geometric sequence and its general term	Use basic concepts and formulas of sequences to determine terms of a geometric sequence	2
9	Geometric means	Use basic concepts and formulas of sequences to find a Geometric mean of two numbers	1
10	Geometric series	Use basic concepts and formulas of sequences to calculate the sum of the first "n" term of a Geometric sequence	2
11	Infinity Geometric series	Explain infinity Geometric series and determine the their sum.	1
12	Application of sequences on Economics	Apply the concepts of sequences and series to solve problems related to finance or Economics.	3
13	End unit assessment		1
	Total		**21**

Lesson 1: Generalities on sequences and series

a) Learning objectives

To define sequences and series, infinite sequence and to determine terms of a sequence

b) Teaching resources

Learner's book and other Reference books to facilitate research, calculator, Manila paper, markers, pens, pencils...

c) Prerequisites/Revision/Introduction

Students will learn better in this lesson if they have a good background on the Arithmetic and on equations and inequalities learnt in Year 1 Unit 1& Unit 4 respectively.

d) Learning activities:

- Invite student-teachers to work in groups and do the activity 1.1 found in their Mathematics books;
- Move around in the class for facilitating students where necessary and give more clarification as they have to give the fraction that represents the part they see when they fold a paper n times;

- Verify and identify groups with different working steps;
- Invite one member from each group with different working steps to present their work where they must explain the working steps;
- As a tutor, harmonize the findings from presentation and guide students to identify the fraction that represents the part they see when they fold the paper n times;
- Use different probing questions and guide students to explore the content and examples given in the student's book and lead them to discover how to define sequences, series, infinite sequence, and to determine terms of a given sequence.
- After this step, guide students to do the application activity 1.1 and evaluate whether lesson objectives were achieved.

Answers for activity 1.1

When they fold once they see $\frac{1}{2}$; When they fold twice they see $\frac{1}{2^2}$;When they fold 3 times they see $\frac{1}{2^3}$; When they fold n times they see $\frac{1}{2^n}$;... When they fold 10 times they see $\frac{1}{2^{10}}$;...

The list of the fractions obtained is: $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots, \frac{1}{2^n}, \ldots$

Answers for application activity 1.1

Given $\{u_n\}:\begin{cases} u_0 = 1 \\ u_n = \dfrac{2n^2}{n^2+1} \end{cases}$

1) $u_1 = \frac{2\times 1^2}{1^2+1} = 1$; $u_2 = \frac{2\times 2^2}{2^2+1} = \frac{8}{5}$; $u_3 = \frac{2\times 3^2}{3^2+1} = \frac{18}{10}$

2) The five first terms of $\left\{\sqrt{n+1}-\sqrt{n}\right\}_{n=1}^{+\infty}$ are:

$$\sqrt{2}-1,\sqrt{2}-\sqrt{3},2-\sqrt{3},\sqrt{5}-2,\sqrt{6}-\sqrt{5}$$

3) $\{2n-1\}_{n=1}^{+\infty}$ or $\{2n+1\}_{n=0}^{\infty}$

Lesson 2: Convergent or divergent sequences

a) Learning objectives

To explore sequences and differentiate the convergent from a divergent sequence.

b) Teaching resources

Learner's book and other Reference books to facilitate research, calculator, manila paper, markers, pens, pencils...

c) Prerequisites/Revision/Introduction

Students will learn better in this lesson if they have a good background on the Arithmetic and on equations and inequalities learnt in Year 1 Unit 1& Unit 4.

d) Learning activities:

- Invite student-teachers to work in groups and do the activity 1.2 found in their Mathematics books, they have to determine value of a sequence as n approaches to $+\infty$;
- Move around in the class for facilitating students where necessary and give more clarification on eventual challenges they may face during their work; Verify and identify groups with different working steps;
- Invite one member from each group with different working steps to present their work where they must explain the working steps;
- As a tutor, harmonize the findings from presentation and guide students to use the limit of the sequence at the infinity ;
- Use different probing questions and guide them to explore the content and examples given in the student's book and lead them to discover how to differentiate the convergent from a divergent sequence.
- After this step, guide students to do the application activity 1.2 and evaluate whether lesson objectives were achieved.

Answers for activity 1.2

1) $\lim_{n\to\infty}\left(\frac{3n^2-1}{n^2}\right)=3$

2) $\lim_{n\to\infty} n^2=+\infty$

Answers for application activity 1.2

1) $\{2+(0.1)^n\}$ converges to 2

2) $\left\{\frac{1-2n}{1+2n}\right\}$ converges to -1

3) $\left\{\frac{1-5n^4}{n^4+8n^3}\right\}$ converges to -5

4) $\{-1^n\}$ diverges

5) $\left\{\frac{2n}{\sqrt{3}n+1}\right\}$ converges to $\frac{2}{\sqrt{3}}$

6) $\frac{\sqrt{7}n^2+2}{n^3+8}$ converges to 0.

Lesson 3: Monotonic sequences

a) Learning objectives

To use basic concepts and formulas of sequences to identify the Monotonic sequences

b) Teaching resources

Student's book and other Reference books to facilitate research, calculator, Manila paper, markers, pens, pencils...

c) Prerequisites/Revision/Introduction

Students will learn better in this lesson if they have a good background on the convergent or divergence sequence (lesson 2 of this unit).

d) Learning activities:

- Invite student teachers to work in groups and do the activity 1.3 found in their Mathematics Student books;
- Move around in the class for facilitating students where necessary and give more clarification on eventual challenges they may face during their work; Verify and identify groups with different working steps;
- Invite one member from each group with different working steps to present their work where they must explain the working steps;
- As a tutor, harmonize the findings from presentation and guide students to identify the increasing sequence, the decreasing sequence, or non increasing sequence;
- Use different probing questions and guide them to explore the content and examples given in the student's book and lead them to discover how to use basic concepts and formulas of sequences to identify the monotonic sequences as increasing sequences or the decreasing sequences.
- After this step, guide students to do the application activity 1.3 and evaluate whether lesson objectives were achieved.

Answers for activity 1.3

1) 1, 2, 3, 4, 5, 6, … is an ascending
2) $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots$ is a descending
3) 1, -1, 1, -1, 1, … it is both, not monotonic
4) 2, 2, 2, 2, 2, 2, … is neither, it is stationary.

Answers for application activity 1.3

1) 1, 2, 3, …, n, …is increasing
2) $\left\{\frac{n}{n+1} = 1 - \frac{1}{n+1}\right\}$ is decreasing.
3) $\left\{\frac{1}{2^n}\right\}$ is decreasing
4) 3, 3, 3, 3, … non increasing
5) 1, -1, 1, -1, … not monotonic.

Lesson 4: Arithmetic sequence and its general term

a) Learning objectives

To use basic concepts of sequences to determine terms of an arithmetic sequence.

b) Teaching resources

Learner's book and other Reference books to facilitate research, calculator, manila paper, markers, pens, pencil…

c) Prerequisites/Revision/Introduction

Students will learn better in this lesson if they refer to the lesson 1, lesson 2 and the lesson 3 of this unit.

d) Learning activities:

- Invite student-teachers to work in groups and do the activity 1.4 found in their Mathematics books;
- Move around in the class for facilitating students where necessary and give more clarification on eventual challenges they may face during their work; Verify and identify groups with different working steps;
- Invite one member from each group with different working steps to present their work where they must explain the working steps;
- As a tutor, harmonize the findings from presentation and guide students to identify the common difference of an arithmetic sequence;
- Use different probing questions and guide them to explore the content and examples given in the student's book and lead them to discover how to use basic concepts of sequences to determine the general term and different terms of an arithmetic sequence.
- After this step, guide students to do the application activity 1.4 and evaluate whether lesson objectives were achieved.

Answers for activity 1.4

a) $\{u_n\}$: 5, 5+3, 5+3+3, 5+3x3, …, 5+3n,…. This constant is d = 3

b) $\{v_n\} = 26; 26+5; 26+(5\times 2), 26+(5\times 3), \ldots$ This constant is d = 5

c) $\{w_n\}$: 20, 20-2, 20-2.2, 20-2.3, …20-2.n,…0.This constant is d = -2

Answers for application activity 1.4

1) If $\frac{1}{a+b},\frac{1}{a+c},\frac{1}{b+c}$ are 3 consecutive terms of an arithmetic progression, then

$$\frac{2}{a+c}=\frac{1}{a+b}+\frac{1}{b+c}$$
$$\Leftrightarrow \frac{2}{a+c}=\frac{2b+c+a}{(a+b)(b+c)}$$

$$\Leftrightarrow 2\left(ab+ac+b^2+bc\right)=(a+c)(2b+a+c)$$
$$\Leftrightarrow 2ab+2ac+2b^2+2bc=2ab+a^2+ac+2bc+ac+c^2$$
$$\Leftrightarrow 2b^2=a^2+c^2$$

Also a^2, b^2, c^2 are 3 consecutive terms of an arithmetic progression if $2b^2=a^2+c^2$.

Thus, if $\frac{1}{a+b},\frac{1}{a+c},\frac{1}{b+c}$ are 3 consecutive terms of an arithmetic progression, it will be the same for a^2, b^2, c^2.

2) Let the second term be *x*. The first term is $x-d$ and the third term is $x+d$ where *d* is the common difference.

Now, $x-d+x+x+d=30 \Rightarrow 3x=30$ or $x=10$

Also, $(x-d)^2+x^2+(x+d)^2=332$

Or $(10-d)^2+100+(10+d)^2=332$

Or $2d^2=32 \Rightarrow d=\pm 4$

Therefore, the progression is 6, 10, 14 or 14, 10, 6

3) We need to find *x* such that $(1+x)^2, (q+x)^2, \textit{and } (q^2+x)^2$ form an arithmetic progression.

$$2(q+x)^2=(1+x)^2+\left(q^2+x\right)^2$$

$$\Leftrightarrow 2\left(q^2+2qx+x^2\right)=1+2x+x^2+\ q^4+2xq^2+x^2$$
$$\Leftrightarrow 2q^2+4qx+2x^2\ =1+2x+x^2+\ q^4+2xq^2+x^2$$
$$\Leftrightarrow 2q^2+4qx=1+2x+\ q^4+2xq^2$$
$$\Leftrightarrow 4qx-2x-2xq^2\ =1-2q^2+q^4$$
$$\Leftrightarrow x\left(4q-2-2q^2\right)=\left(1-q^2\ \right)^2$$
$$\Leftrightarrow x=\frac{\left(1-q^2\ \right)^2}{-2\left(1-2q+q^2\right)}$$
$$\Leftrightarrow x=\frac{\left(1-q\ \right)^2\left(1+q\right)^2}{-2\left(1-q\right)^2}$$
$$\Leftrightarrow x=\frac{\left(1+q\right)^2}{-2}$$

Thus, $x=\dfrac{-\left(1+q\right)^2}{2}$

Lesson 5: Arithmetic Means of an arithmetic sequences

a) Learning objectives

To use basic concepts of sequences to find an arithmetic mean of two numbers.

b) Teaching resources

Learner’s book and other Reference books to facilitate research, Mathematical set, calculator, Manila paper, markers, pens, pencils...

c) Prerequisites/Revision/Introduction

Students will learn better in this lesson if they refer to the lesson 1, lesson 2, the lesson 3 and the lesson 4 of this unit.

d) Learning activities:

- Invite student-teachers to work in groups and do the activity 1.5 found in their Mathematics Student books;
- Move around in the class for facilitating students where necessary and give more clarification on eventual challenges they may face during their work; Verify and identify groups with different working steps;
- Invite one member from each group with different working steps to present their work where they must explain the working steps;
- As a tutor, harmonize the findings from presentation and guide students

to explain the arithmetic means of such sequences;

- Use different probing questions and guide them to explore the content and examples given in the student's book and lead them to discover how to use basic concepts of sequences to find arithmetic means of two terms of an arithmetic sequences.
- After this step, guide students to do the application activity 1.5 and evaluate whether lesson objectives were achieved.

Answers for activity 1.5

$u_1 = 2, u_7 = 20$

$u_n = u_1 + (n-1)d \Rightarrow u_7 = u_1 + 6d$

$\Rightarrow 20 = 2 + 6d$

$\Rightarrow d = 3$

u_1, $A = u_2 = 2+3 = 5$, $B = u_3 = 2+6 = 7$, $C = u_4 = 2+9 = 11$, $D = u_5 = 2+12 = 14$

$E = u_6 = 2+15 = 17$, $u_7 = 2+18 = 20$.

Answers for application activity 1.5

1) $-3,-1,1,3,5,7$
2) $2,5,8,11,14,17,20,23,26,29,32$
3) The arithmetic sequence has $n+2$ terms $t_1, t_2, t_3, \ldots, t_{n+2}$,

   $3; a_1, a_2, \ldots, a_8, \ldots a_{n-2}, a_{n-1}, a_n; 54$, where

   $t_1 = 3; t_2 = a_1; t_3 = a_2; \ldots; t_9 = a_8; \ldots; t_{n-1} = a_{n-2}; \ldots$

   Let d be the common difference. Then, from $t_{n+2} = t_1 + (n+1)d$, we have $54 = 3 + (n+1)d$ Solving for nd, we have: $nd = 51 - d$ (1)

   But also,

   $$\frac{a_8}{a_{n-2}} = \frac{3}{5} \Leftrightarrow 5a_8 = 3a_{n-2}$$

$$\Leftrightarrow 5(3+8d)=3[3+(n-2)d]$$

$$\Leftrightarrow 3nd = 6+46d \qquad (2)$$

Solving simultaneously (1) and (2): $\begin{cases} nd = 51-d \\ 3nd = 6+46d \end{cases}$; $d=3$ and $n=16$.

4) 14

5) 0

Lesson 6: Arithmetic Series

a) Learning objectives

To use basic concepts of sequences to calculate the sum of the first "n" terms of an arithmetic sequence.

b) Teaching resources

Learner's book and other Reference books to facilitate research, calculator, manila paper, markers, pens, pencils...

c) Prerequisites/Revision/Introduction

Students will learn better in this lesson if they refer to all previous lessons of this Unit (from the lesson 1 to lesson 5 of this unit).

d) Learning activities:

- Invite student-teachers to work in groups and do the activity 1.6 found in their Mathematics Student books;
- Move around in the class for facilitating students where necessary and give more clarification on eventual challenges they may face during their work; Verify and identify groups with different working steps;
- Invite one member from each group with different working steps to present their work;
- As a tutor, harmonize the findings from presentation and guide students to establish how to determine the sum s_n for the first n terms of the arithmetic sequence $\{u_n\}$;
- Use different probing questions and guide them to explore the content and examples given in the student's book and lead them to discover how to use basic concepts of sequences to calculate the sum of the first "n" terms of different arithmetic sequences.

- After this step, guide students to do the application activity 1.6 and evaluate whether lesson objectives were achieved.

Answers for activity 1.6

The sequence $2,5,8,11,14,....$

a) The first term is 2, $d=3$, the general term $u_n = 2+3(n-1)$

b) $S_6 = 2+(2+3)+(2+2.3)+..+(2+5.3) = 6.2+3(1+2+3+4+5) = 6.2+3\frac{6(5)}{2}$ as we apply the sum of all first positive integers not greater than n.

c) $s_n = 2n+3(1+2+3+....+n) = 2n+3\frac{n(n-1)}{2}$.

Answers for application activity 1.6

1) $2n(n+3)$
2) 860
3) 11
4) The bottom row requires 100 tiles and the top row, 50 tiles. Since each successive row requires one less tile, the total number of tiles required is $S = 100+98+96+...+(100-(n-1))+...+50$.

As $100-(n-1)=50$, we have:

$100-n+1=50$ Which gives $n=51$.

This is the sum of an arithmetic sequence; the common difference is -1. The number of terms to be added is $n=51$ with the first term $u_1 = 100$ and the last term $u_n = 50$

The sum S is $S = \frac{n}{2}(u_1+u_n) = \frac{51}{2}(100+50) = 3825$.
In all, tiles will be required.

Lesson 7: Harmonic sequences and its general terms

a) Learning objectives

To use basic concepts and formulas of sequences to find the terms of an harmonic sequence.

b) Teaching resources

Learner's book and other Reference books to facilitate research, calculator, manila paper, markers, pens, pencils...

c) Prerequisites/Revision/Introduction

Students will learn better in this lesson if they refer to all previous lessons of this unit (from the lesson 1 to lesson 6 of this unit).

d) Learning activities:

- Invite student-teachers to work in groups and do the activity 1.7 found in their Mathematics Student books;
- Move around in the class for facilitating students where necessary and give more clarification on eventual challenges they may face during their work; Verify and identify groups with different working steps;
- Invite one member from each group with different working steps to present their work where they must explain the working steps;
- As a tutor, harmonize the findings from presentation and guide students to discover that a harmonic sequence is made by the reciprocals of terms of the arithmetic sequence;
- Use different probing questions and guide them to explore the content and examples given in the student's book and lead them to discover how to find the terms of a harmonic sequence.
- After this step, guide students to do the application activity 1.7 and evaluate whether lesson objectives were achieved.

Answers for activity 1.7

a) $\frac{1}{2}, \frac{1}{4}, \frac{1}{6}, ..., \frac{1}{2n}, ...$

b) Its first term is $\frac{1}{2}$, the third is $\frac{1}{6}$, the general term is $\frac{1}{2n}$.

There is no common rules for establishing the relationship between consecutive terms of a harmonic sequence but for this sequence we have that:

The relationship between two consecutive terms is that $u_n = \left(1-\frac{1}{n}\right)u_{n+1}$

Indeed; $u_{n+1} - u_n = \frac{1}{2n+2} - \frac{1}{2n} = \frac{-1}{(2n+2)(n)}$

$$n(u_{n+1} - u_n) = u_{n+1} \Leftrightarrow u_n = \left(1-\frac{1}{n}\right)u_{n+1}$$

Answers for application activity 1.7

1) The sequence is $6,4,3,\frac{12}{5},2,\frac{12}{7},\frac{3}{2},\frac{4}{3}$. The 4^{th} *term is* $\frac{12}{5}$, 8^{th} *term is* $\frac{4}{3}$
2) $3,\frac{90}{23},\frac{90}{16},10$
3) $\frac{\sqrt{5}}{3},\frac{\sqrt{5}}{4},\frac{1}{\sqrt{5}},\ldots,\frac{\sqrt{5}}{13}$
4) $6 \text{ and } 2$
5) $\frac{60}{16-n}$

Lesson 8: Generalities on Geometric sequences and their general terms

a) Learning objectives

To use basic concepts and formulas of sequences to determine terms of a geometric sequence.

b) Teaching resources

Learner's book and other Reference books to facilitate research, calculator, manila paper, markers, pens, pencils...

c) Prerequisites/Revision/Introduction

Students will learn better in this lesson if they refer to all previous lessons of this Unit (from the lesson 1 to lesson 7 of this unit) and to the equations and inequalities learnt in Year 1 Unit 1& Unit 4 respectively.

d) Learning activities:

- Invite student-teachers to work in groups and do the activity 1.8 found in their Mathematics Student books;

- Move around in the class for facilitating students where necessary and give more clarification on eventual challenges they may face during their work; Verify and identify groups with different working steps;
- Invite one member from each group with different working steps to present their work;
- As a tutor, harmonize the findings from presentation and guide students to make a geometric sequence of numbers, its general term and the common ratio;
- Use different probing questions and guide them to explore the content and examples given in the student's book and lead them to discover how to determine terms of a geometric sequence.
- After this step, guide students to do the application activity 1.8 and evaluate whether lesson objectives were achieved.

Answers for activity 1.8

Learners will take a piece of paper and cut it into two equal parts. Take one part and cut it again into two equal parts. When they continue in this manner the fraction corresponding to the obtained parts according to the original piece of paper are as follows:

$$\frac{1}{2},\frac{1}{4},\frac{1}{8},\frac{1}{16},\frac{1}{32},\ldots$$

Answers for application activity 1.8

1) 98304

2) $\frac{\sqrt[5]{16}}{4}$

3) -21.87

4) $\frac{1}{16}$

5) $(u_n): u_n = \frac{1}{2}\left(\frac{3}{2}\right)^{n-1}$, $u_8 = \frac{2187}{256}$

6) $p = 5$

Lesson 9: Geometric means of a geometric sequence

a) Learning objectives

To use basic concepts and formulas of sequences to find geometric means of two numbers.

b) Teaching resources

Learner's book and other reference books to facilitate research, calculator, manila paper, markers, pens, pencils...

c) Prerequisites/Revision/Introduction

Students will learn better in this lesson if they refer to all previous lessons of this Unit (from the lesson 1 to lesson 8 of this unit) and the equations and inequalities learnt in Year 1 Unit 1& Unit 4 respectively.

d) Learning activities:

- Invite student-teachers to work in groups and do the activity 1.9 found in their Mathematics books;
- Move around in the class for facilitating students where necessary and give more clarifications on eventual challenges they may face during their work; Verify and identify groups with different working steps;
- Invite one member from each group with different working steps to present their work;
- As a tutor, harmonize the findings from presentation and guide students to explain geometric means of two terms in a geometric sequence;
- Use different probing questions and guide them to explore the content and examples given in the student's book and lead them to discover how to find geometric mean of two numbers.
- After this step, guide students to do the application activity 1.9 and evaluate whether lesson objectives were achieved.

Answers for activity 1.9

$u_1 = 1, u_6 = 243$

$u_n = u_1 \cdot r^{n-1} \Rightarrow u_6 = u_1 \cdot r^5$

$\Rightarrow 243 = r^5$

$\Rightarrow 3^5 = r^5$

$\Rightarrow r = 3$

The sequence is $1, 3, 9, 27, 81, 243$

Answers for application activity 1.9

1) $\frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}, \frac{1}{64}, \frac{1}{128}, \frac{1}{256}$
2) $2, \frac{2}{3}, \frac{2}{9}, \frac{2}{27}, \frac{2}{81}, \frac{2}{243}, \frac{2}{729}$
3) a) 14 b) $\frac{9}{2}$
4) 12 and 108
5) 64 and 4

Lesson 10: Geometric Series

a) Learning objectives

To use basic concepts and formulas of sequences to calculate the sum of the first "n" terms of a Geometric sequence

b) Teaching resources

Learner's book and other reference books to facilitate research, calculator, manila paper, markers, pens, pencils...

c) Prerequisites/Revision/Introduction

Students will learn better in this lesson if they refer to all previous lessons of this Unit (from the lesson 1 to lesson 8 of this unit) and on the equations and inequalities learnt in Year 1 Unit 1& Unit 4 respectively.

d) Learning activities:

- Invite student-teachers to work in groups and do the activity 1.10 found in their Mathematics books;
- Move around in the class for facilitating students where necessary and give more clarification on eventual challenges they may face during their work; Verify and identify groups with different working steps;
- Invite one member from each group with different working steps to present their work where they must explain the working steps;
- As a tutor, harmonize the findings from presentation and guide students to determine the sum of n terms of the sequence they found;

- Use different probing questions and guide them to explore the content and examples given in the student's book and lead them to discover how to Use basic concepts and formulas of sequences to calculate the sum of the first "n" term of a Geometric sequence.
- After this step, guide students to do the application activity 1.10 and evaluate whether lesson objectives were achieved.

Answers for activity 1.10

a)

Place	Money gotten
1st	$u_1 = 100,000Frw$
2nd	$u_2 = \frac{1}{2}(100,000Frw) = 50,000Frw$
3rd	$u_3 = \frac{1}{2}(50,000Frw) = 25,000Frw$
4th	$u_4 = \frac{1}{2}(25,000Frw) = 12,500Frw$
5th	$u_5 = \frac{1}{2}(12,500Frw) = 6,250Frw$

b) The total of their money is

$$u_1 + u_2 + u_3 + u_4 + u_5$$
$$= 100,000 + \frac{1}{2}(100000) + \frac{1}{2^2}100000 + \frac{1}{2^3}100000 + \frac{1}{2^4}100000$$

c) The money for the first is $100,000Frw$, this is greater than the money for the fifth student which is $6,250Frw$. When you win at the first place, the

d) The money for the student who passed at the n^{th} place is

$$u_n = \frac{1}{2^{(n-1)}}100,000Frw$$

To determine the total amount of money for n students, students teachers will make the sum from $u_1 = 100,000Frw$ to $u_n = \frac{1}{2^{(n-1)}}100,000Frw$ and they will find

$$S_n = 100000 \times \frac{\left(1-\left(\frac{1}{2}\right)^n\right)}{\left(1-\frac{1}{2}\right)}$$

Answers for application activity 1.10

1) 21.25

2) 39.1

3) $1, \frac{5}{4}$

4) –32

5) The interest is compounded each quarter. So n=1 and the interest rate per period is $\frac{6\%}{4}$ or 1.5%. The common ratio r for the geometric series is then $(1+0.015)$ or 1.015

The first term u_1 in this series is the account balance at the end of the first quarter. Thus, $u_1 = 500(1.015)$ or 507.05.

Apply the formula for the sum of a geometric series $S_n = \frac{u_1 - u_1 r^n}{1-r}$, we have:

$$S_n = \frac{507.5 - 507.5(1.015)^4}{1-1.015}; n = 4; r = 1.015$$

$$S_4 = 2076.13$$

Aloys's account balance at the end of one year is \$2076.13.

Lesson 11: Infinity geometric series and its convergence

a) Learning objectives

To explain infinity geometric series and determine their sum.

b) Teaching resources

Student's book and other Reference books to facilitate research, calculator, manila paper, markers, pens, pencils...

c) Prerequisites/Revision/Introduction

Students will learn better in this lesson if they have a good background on all previous lessons of this Unit (from the lesson 1 to lesson 8 of this unit) and on the equations and inequalities learnt in Year 1 Unit 1& Unit 4 respectively.

d) Learning activities:

- Invite student-teachers to work in groups and do the activity 1.11 found in their Mathematics books;
- Move around in the class for facilitating students where necessary and give more clarification on eventual challenges they may face during their work;
- Invite one member from each group with different working steps to present their work;
- As a tutor, harmonize the findings from presentation and guide students to determine the limit of a given series;
- Use different probing questions and guide them to explore the content and examples given in the student's book and lead them to discover how to Explain infinity Geometric series and determine the their sum.
- After this step, guide students to do the application activity 1.11 and evaluate whether lesson objectives were achieved.

Answers for activity 1.11

a) $\lim_{n\to\infty} S_n = \lim_{n\to\infty}\left[5\frac{\left(1-\left(\frac{1}{2}\right)^n\right)}{\left(1-\left(\frac{1}{2}\right)\right)}\right] = 10$

b) If $-1<r<1, -1<r<1$, thus $\lim_{n\to\infty}\frac{u_1\left(1-r^n\right)}{1-r} = \frac{u_1}{1-r}$

Answers for application activity 1.11

1) a) $\sum_{n=1}^{\infty} 10\left(1-\frac{3x}{2}\right)^{n}$ this exists if $\left|1-\frac{3x}{2}\right|<1$. Solving this inequality we find

$0<x<\frac{4}{3}$

b) If x = 1.3, the sum can be $\frac{u_1}{1-r}=\frac{10}{1-\left(1-\frac{3(1.3)}{2}\right)}=\frac{200}{39}$

2) The decimal $0.999...=0.9+0.09+0.009+...=\frac{9}{10}+\frac{9}{100}+\frac{9}{1000}+...$ is an infinite geometric series. We will write it in the form $\sum_{k=1}^{\infty} a_1 r^{k-1}$, then

$$0.999...=\frac{9}{10}+\frac{9}{100}+\frac{9}{1000}+...=\sum_{k=1}^{\infty}\frac{9}{10^k}=\sum_{k=1}^{\infty}\frac{9}{10.10^{k-1}}=\sum_{k=1}^{\infty}\frac{9}{10}\left(\frac{1}{10}\right)^{k-1}$$

Now we can compare this series to $\sum_{k=1}^{\infty} a_1 r^{k-1}$ and conclude that $a_1=\frac{9}{10}$ and $r=\frac{1}{10}$.

Since $|r|<1$, the series converges and its sum is $0.999...=\frac{\frac{9}{10}}{1-\frac{1}{10}}=\frac{\frac{9}{10}}{\frac{9}{10}}=1$. The repeating decimal $0.999...$ equal 1.

3) $\sum_{n=1}^{\infty} 2\left(\frac{2}{3}\right)^{n-1}=2+\frac{4}{3}+\frac{8}{9}+...$ This geometric series converges as $\frac{2}{3}<1$. Its sum is its limit $\frac{2}{1-\frac{2}{3}}=6$

Lesson 12: Application of sequences in real life

a) Learning objectives

To apply the concepts of sequences and series to solve problems related to finance or Economics.

b) Teaching resources

Learner's book and other Reference books to facilitate research, calculator, manila paper, markers, pens, pencils...

c) Prerequisites/Revision/Introduction

Students will learn better in this lesson if they refer to all previous lessons of this Unit (from the lesson 1 to lesson 8 of this unit) and on the equations and inequalities learnt in Year 1 Unit 1& Unit 4 respectively.

d) Learning activities:

- Invite student-teachers to work in groups and do the activity 1.12 found in their Mathematics books;
- Visit each group for facilitating students where necessary and give more clarification on eventual challenges they may face during their work;
- Invite one member from each group to present their work;
- As a tutor, harmonize the findings from presentation;
- Use different probing questions and guide them to explore the content and examples given in the student's book and lead them to solve problems related to finance or Economics.
- After this step, guide students to do the application activity 1.12 and evaluate whether lesson objectives were achieved.

Answers for activity 1.12

Refer to the student's book to highlight some applications of sequences and series. They are also used to determine t*he monthly payments made to pay off an automobile or home loan with interest portion, the list of maximum daily temperatures in one area for a month, etc. Sequences are used in calculating interest, population growth, half-life and decay in radioactivity.*

Answers for application activity 1.12

1) $P = 1300,\ r = 7\% = 0.07,\ k = 1$

$$A = 1300\left(1+\frac{0.07}{1}\right)^{1\times 17} = 4106.46$$

The account will contain \$4,106.46.

2) We apply the formula $P = P_0 e^{rt}$ With initial population $P_0 = 153,800$ rate of growth $r = 0.05$, and time $t = 2000 - 1970 = 30 \, years$. Thus, a prediction for the population of the city in the year 2000 is $153,800e^{(0.05)(30)} = 153,800e^{1.5} \approx 689,284$.

3) This is an ordinary annuity with $\text{n} = 30$ annual deposits of $P = \$2000$. The rate of interest per payment period is $i = \frac{0.04}{1} = 0.04.$ The amount A of the annuity after 30 deposits is $A = P.\left[\frac{(1+i)^n - 1}{i}\right] = 2000\left[\frac{\left((1+0.04)^{30} - 1\right)}{0.04}\right] = 112,169.88$.

4) This is an example of a sinking fund. The payment *P* required twice a year to accumulate 4,000,000Frw in 12 years (24 payments at a rate of interest of $i = \frac{0.04}{2} = 0.02$ per payment period) obeys:

$$A = P.\left[\frac{(1+i)^n - 1}{i}\right]$$

$$4,000,000 = P\left[\frac{\left((1+0.02)^{24} - 1\right)}{0.02}\right]$$

$$4,000,000 = P(30.4218)$$

$$P = 131,484.39 \, Frw$$

The school leader will need to make a payment of \$131,484.39 every 6 months to redeem the bonds in 12 years.

1.6. Unit Summary

1) Numbers in sequence are denoted $u_1, u_2, u_3, ..., u_{n-1}, u_n, ...$ and shortly $\{u_n\}$.

   The natural number ***n*** is called **term number** and value u_n is called a **general term** of a sequence and the term u_1 is the **initial term**.

2) As a sequence continues indefinitely, it can be denoted as $\{u_n\}_{n=1}^{+\infty}$.

3) A sequence $\{u_n\}$ is said to be

   - increasing if $u_1 < u_2 < u_3 < ... < u_n < ...$
   - non decreasing if $u_1 \le u_2 \le u_3 \le ... \le u_n \le ...$
   - decreasing if $u_1 > u_2 > u_3 > ... > u_n > ...$
   - non increasing $u_1 \ge u_2 \ge u_3 \ge ... \ge u_n \ge ...$

4) A numerical sequence is said to be **convergent** if the limit exist whereas if the limit does not exist (or is infinity) the sequence is said to be **divergent.** A number L *is called* **a limit** of a numerical sequence $\{u_n\}$ if $\lim_{n\to\infty} u_n = L$

5) One of the most famous and important of all diverging series is the **harmonic series**, $\sum_{k=1}^{+\infty} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + ...$

6) Sequences of numbers that follow a pattern of adding a fixed number from one term to the next are called **arithmetic sequences** or **arithmetic progressions**.

7) For an arithmetic sequence u_{n-1}, u_n, u_{n+1}, we have $2u_n = u_{n-1} + u_{n+1}$.

8) If u_p is any p^{th} term of a sequence then the n^{th} term is given by $u_n = u_p + (n-p)d$

9) The sum of first n terms of a finite arithmetic sequence with initial term u_1 is given by $s_n = \frac{n}{2}[u_1 + u_n]$

10) Sequences of numbers that follow a pattern of multiplying a fixed number from one term to the next are called **geometric sequences**.

11) For a geometric sequence u_{n-1}, u_n, u_{n+1}, we have $u_n^2 = u_{n-1} \cdot u_{n+1}$

12) The n^{th} term, u_n, of a geometric sequence $\{u_n\}$ with common ratio r and initial term u_1 is given by $u_n = u_1 r^{n-1}$

13) The sum of first n terms of a geometric sequence with initial term u_1 and common ratio r is given by: $s_n = \frac{u_1(1-r^n)}{1-r}$ with $r \neq 1$

14) Also, the product of first n terms of a geometric sequence with initial term u_1 and common ratio r is given by $P_n = (u_1)^n r^{\frac{n(n-1)}{2}}$

15) For the formula $s_n = \frac{u_1(1-r^n)}{1-r}$

If $-1 < r < 1$, $S_\infty = \frac{u_1}{1-r}$

Sequences are used in calculating interest, population growth, half-life and decay in radioactivity.

1.7. Additional information for the teacher

Here the tutor has to emphasize the application of sequences and series in solving problems related to finance or Economics in real situation life.

1.8. End unit assessment

1)

a) $0, -\frac{1}{4}, -\frac{2}{9}, -\frac{3}{16}$ b) $1, -\frac{1}{3}, \frac{1}{5}, -\frac{1}{7}$ c) $1, 3, 1, 3$

2) a) $(-1)^n$, $n = 0, 1, 2, ..$ b) $n^2 - 1$, $n = 1, 2, 3, ...$ c) $4n - 3$, $n = 1, 2, 3, ...$

3) a) converges to $\sqrt{2}$ b) Converges to 0 c) Converges to 1

4) £11 million

5) 2048000 6) $99.8^0 F$ 7) 1800

1.9. Additional activities

1.9.1 Remedial activities

1) Find the 20th term of the following arithmetic progressions and calculate the sum of first 20 terms

   a) 2, 6, 10, 14, …

   b) -5, -3.5, -2, -0.5, …

   Solution:

   a) $u_{20} = 78, S_{20} = 800$ b) $u_{20} = 23.5, S_{20} = 185$

2) In an arithmetic progression, the sum of the 8th and 14th terms is 50. The 5th term is equal to 13. Find that progression.

   Solution:

   $$5,7,9,11,13,15,17,19,21,23,25,27,29,31,\ldots$$

1.9.2 Consolidation activities

1) Find x consecutive integers numbers known that the first number is 8 and their sum is x^3.

   Solution: $8,9,10$

2) In a geometric progression, we have

   a) $u_1 = 3, r = 4, n = 5$; find u_n and sum of terms.

   b) $u_n = \frac{3}{64}, u_1 = 12, n = 9$; find r and sum of terms.

   Solution: a. $u_5 = 768, S_5 = 341$ b. $r = \frac{1}{2}, S_n = \frac{1533}{64}$

1.9.3 Extended activities

1) In an arithmetic progression, we have

   a) $u_1 = 4, d = 2, n = 8$; find u_n and sum of terms

   b) $d = 4, u_n = 39, n = 10$; find u_1 and sum of terms

c) $u_1 = 3, u_n = 21, S_n = 120$; find n and d

b) $u_n = 199, n = 100, S_n = 10000$; find u_1 and d.

Solution:

a) $u_8 = 18, S_8 = 88$ b) $u_1 = 3, S_{10} = 210$ c) $n = 10, d = 2$ d. $u_1 = 1, d = 2$

3) A mine worker discovers an ore sample containing 500 mg of radioactive material. It is discovered that the radioactive material has a half life of 1 day. Find the amount of radioactive material in the sample at the beginning of the 7th day.

Solution:

Half life of one day means that the half of the amount remains after 1 day.

Begin of day 1:500 mg	Begin of day 2:250 mg	Begin of day 3:125 mg	...
End of day 1:250 mg	End of day 2:125 mg	End of day 3:62.5 mg	...

Decide to either work with the “beginning” of each day, or the “end” of each day, as each can yield the answer. Only the starting value and number of terms will differ. We will use “beginning”: The beginning of the 7th day corresponds to $u_7 = 500\left(\frac{1}{2}\right)^{7-1} = 7.8125mg$.

UNIT 2

LOGARITHMIC AND EXPONENTIAL EQUATIONS

2.1. Key unit competence:

Solve equations involving logarithms or exponentials and apply them to model and solve related problems.

2.2. Prerequisites

Student-teachers will easily learn this unit, if they refer to sequences and series (Unit 1 Year 2), Equations and inequalities (unit 2 Year 1) and to limits of functions (unit 4 Year 1).

2.3. Cross-cutting issues to be addressed

- Inclusive education (promote education for all while teaching).
- Peace and value Education (respect others' view and thoughts during class discussions).
- Gender (provide equal opportunity to boys and girls in the lesson).

2.4. Guidance on introductory activity

- Invite student-teachers to work in groups and give them instructions on how they can do the introductory activity 2.0 found in unit 2 of student's book;
- Guide students to read and analyse the questions insisting on the analysis of the exponential model given as the introductory activity;
- Invite some group members to present groups' findings, then try to harmonize their answers;
- Basing on student-teachers' experience, prior knowledge and abilities shown in answering the questions for this activity, use different questions to facilitate them to give their predictions and ensure that you arouse their curiosity on what is going to be leant in this unit.

Answers for introductory activity 1.0

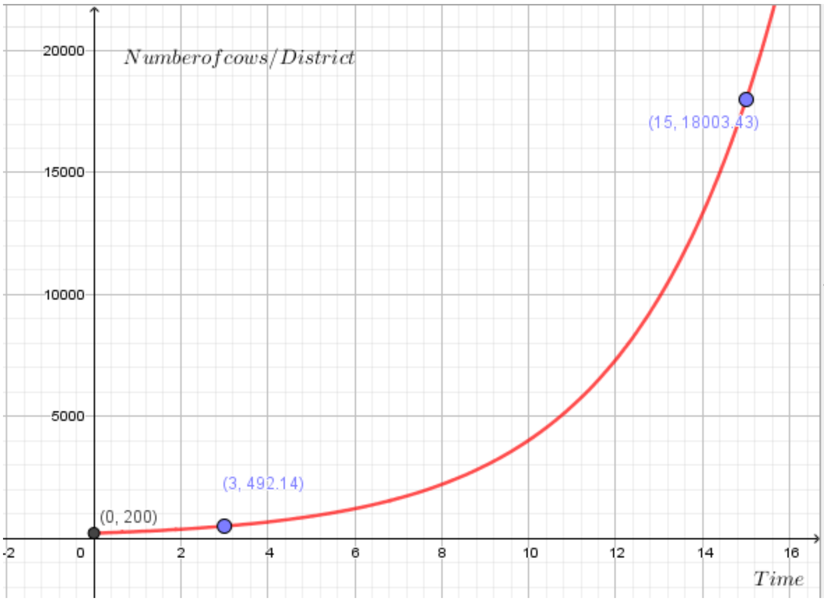
i) If you consider $p(t)=p_0e^{0.3t}$ in the whole district, $P_0=2.100$ but if you consider one family, $P_0=2$. This corresponds to the number of cows at the beginning of the project, when when $t=0$ in 2005.

Three years later, in 2008, the District will have a number of cows according to this model: $P(3)=200e^{0.9}\cong 491,9\cong 492$

In 2020, according to this model, $P(15)=200e^{15(0.3)}\cong 18,000$

Graphically, the above information is:

The function becomes $P(t)=200e^{(0.3)t}$.



This shows that the in year 2020, the district would have 18000 cows. This number cannot be observed, as people will sell some cows to gain money to be used when solving in their daily problems.

ii) After 15 years, each family would have $P(15)-2=2e^{15(0.3)}-2\cong 180-2=178$ Each family would have 178 cows. However, they can sell some of them to get money.

iii) The project would be successful but it is not possible, the production of cows cannot arrow a cow to give birth to 4 cows in one year(this is observed on the function after t = 10 years).

iv) The project must be revisited; it is not possible for cows.

2.5. List of lessons

#	Lesson title	Learning objectives	Number of periods
0	Introductory activity	To arouse the curiosity of student-teacher on the content of unit 2.	1
1	Logarithmic equations	Apply properties of logarithms to solve logarithmic equations	6
2.	Exponential equations	Apply properties of powers to solve exponential equations	6
3	Application of Exponential function on Population Growth	Use logarithmic or exponential equations to solve population growth problems	1
4	Application of Exponential function on decay problems	Use logarithmic or exponential equations to solve decay related problems	1
5	Application of logarithmic equations on Earthquake problems	To apply logarithms or exponential function on Earthquake problems	1
6	Application of Exponential equations on interest rates problems	To apply logarithms or exponential function on interest rates problems: Final or initial sum of investment,	2
7	Application of Exponential equation on the mortgage problems	To apply logarithms or exponential equations to solve mortgage related problems.	2
8	End unit assessment		1
	Total number of periods		**21**

Lesson 1: Logarithmic Equations

a) Learning objectives

Apply properties of logarithms to solve logarithmic equations.

b) Teaching resources

Learner's book and other reference books to facilitate research, Mathematical set, calculator, manila paper, markers, pens, pencils...

c) Prerequisites/Revision/Introduction

Students will learn better in this lesson if they refer to the lesson on equations and inequalities learnt in Year 1 Unit 2 and if they refer to arithmetic (Unit 1 year 1).

d) Learning activities:

- Invite student-teachers to work in groups and do the activity 2.1 found in their Mathematics books;
- Move around in the class for facilitating students where necessary and give more clarification on eventual challenges they may face during their work; Verify and identify groups with different working steps;
- Invite one member from each group with different working steps to present their work where they must explain the working steps;
- As a tutor, harmonize the findings from presentation and guide students to solve different types of logarithmic equations;
- Use different probing questions and guide them to explore the content and examples given in the student's book and lead them to discover how to apply properties of logarithms to solve logarithmic equations.
- After this step, guide students to do the application activity 2.1 and evaluate whether lesson objectives were achieved.

Answers for activity 2.1

1)

i) $\log_a(xy) = \log_a x + \log_a y$.

ii) $\log_a\left(\dfrac{x}{y}\right) = \log_a(x) - \log_a(y)$

iii) $\log_a(x^y) = y\log_a(x)$

iv) $\log_a(a) = 1$

v) $\log_a(1) = 0$

2) a) $\log x = 2 \Leftrightarrow x = 10^2 \Rightarrow x = 100$.

b) $\log_2 x = \log_2 5 \Leftrightarrow x = 5$.

c) $\ln x = \ln 10 \Leftrightarrow x = 10$.

d) $\ln x = 3 \Leftrightarrow \ln x = \ln e^3 \Rightarrow x = e^3$.

e) $\log(100x) = 2 + \log 4 \Leftrightarrow \log(100x) = \log 10^2 + \log 4$

$\Leftrightarrow \log(100x) = \log(10^2 \times 4) \Rightarrow 100x = 400 \Rightarrow x = 4$

Answers for application activity 2.1

1) a) $\log_2 8 = x \Leftrightarrow 8 = 2^x \Leftrightarrow 2^x = 2^3 \Rightarrow x = 3$

b) $\log_x 125 = 3 \Leftrightarrow 125 = x^3 \Rightarrow x = \sqrt[3]{125} \Leftrightarrow x = 5$

c) $\log_x 64 = 0.5 \Leftrightarrow 64 = x^{0.5} \Rightarrow x = 64^2 \Leftrightarrow x = 4096$

d) $\log_4 64 = x \Leftrightarrow 64 = 4^x \Leftrightarrow 4^x = 4^3 \Rightarrow x = 3$

e) $\log_9 x = 3\frac{1}{2} \Leftrightarrow x = 9^{3\frac{1}{2}} \Leftrightarrow x = \left(9^{\frac{1}{2}}\right)^7 \Rightarrow x = 2187$

f) $\log_2 \frac{1}{2} = x \Leftrightarrow \frac{1}{2} = 2^x \Leftrightarrow 2^x = 2^{-1} \Rightarrow x = -1$

2) a) $\log_9 x + 3\log_3 x = 14 \Leftrightarrow \frac{1}{\log_3 9}\log_3 x + 3\log_3 x = 14$

$\frac{1}{2}\log_3 x + 3\log_3 x = 14 \Leftrightarrow \frac{7}{2}\log_3 x = 14 \Leftrightarrow x = 3^4 = 81$

b) $\log_2 x + \log_6 x = 3 \Leftrightarrow \log_2 x + \frac{1}{\log_2 6}\log_2 x = 3$

$(\log_2 6)(\log_2 x) + \log_2 x = 3\log_2 6 \Leftrightarrow (1 + \log_2 6)\log_2 x = 3\log_2 6$

$\log_2 x = \frac{3\log_2 6}{1 + \log_2 6} \Leftrightarrow x = 2^{\frac{3\log_2 6}{1+\log_2 6}}$

3) a)

$2\log_2 x + \log_x 2 = 3 \Leftrightarrow 2\log_2 x + \frac{1}{\log_2 x}\log_2 x = 3 \Leftrightarrow 2(\log_2 x)^2 + 1 = 3 \Leftrightarrow S = \{\sqrt{2}, 2\}$

b) $\ln(x^2 - 1) = \ln(4x - 1) - 2\ln 2 \Leftrightarrow \ln(x^2 - 1) = \ln\frac{(4x-1)}{4} \Leftrightarrow x = \left\{\frac{3}{2}\right\}$

because $x=-\frac{1}{2}$ is rejected.

c) $\begin{cases} 2\ln x+3\ln y=-2 \\ 3\ln x+5\ln y=-4 \end{cases}$ $by\ u\sin g\ e\lim ination\ method;\ S=\left\{\left(e^2,\frac{1}{e^2}\right)\right\}$

d) $\begin{cases} \ln(xy)=7 \\ \ln\frac{x}{y}=1 \end{cases}$ $by\ u\sin g\ e\lim ination\ method;\ S=\left\{\left(e^4,e^3\right),\left(-e^4,-e^3\right)\right\}$

Lesson 2: Exponential equations

a) Learning objectives

To apply properties of powers to solve exponential equations.

b) Teaching resources

Learner's book and other reference books to facilitate research, Mathematical set, calculator, manila paper, markers, pens, pencils...

c) Prerequisites/Revision/Introduction

Students will learn better in this lesson if they refer to equations and inequalities learnt in Year 1 Unit 2 and the previous lesson(lesson one of this unit).

d) Learning activities:

- Invite student-teachers to work in groups and do the activity 2.2 found in their Mathematics Student books;
- Move around in the class for facilitating students where necessary and give more clarification on eventual challenges they may face during their work; Verify and identify groups with different working steps;
- Invite one member from each group with different working steps to present their work;
- As a tutor, harmonize the findings from presentation and guide students to solve different types of exponential equations.
- Use different probing questions and guide them to explore the content and examples given in the student's book and lead them to discover how to apply properties of powers to solve exponential different types of equations.
- After this step, guide students to do the application activity 2.2 and evaluate whether lesson objectives were achieved.

Answers for activity 2.2

1) Since $81=3^4$, write the equation as $3^{x+1}=81=3^4$. As we have the same base, 3, on each side, then the exponents are equal : $x+1=4 \Rightarrow x=3$

   The solution set is {3}.

2) $4^{2x-1}=8^{x+3} \Rightarrow \left(2^2\right)^{(2x-1)}=\left(2^3\right)^{(x+3)} \Rightarrow 2^{2(2x-1)}=2^{3(x+3)} \Rightarrow 2(2x-1)=3(x+3)$

   If $a^u=a^v, then\ u=v$

   $4x-1=3x+9 \Rightarrow x=11$; The solution set is $\{11\}$.

Answers for application activity 2.2

a) $9^x-2\times 3^{x+1}=27 \Leftrightarrow \left(3^2\right)^x-2\times 3^x\times 3=3^3 \Leftrightarrow t^2-6t=27 \Leftrightarrow t=2$

b) $\dfrac{e^x+e^{-x}}{2}=1 \Leftrightarrow e^x\left(\dfrac{e^x+e^{-x}}{2}\right)=1\times e^x \Rightarrow \dfrac{e^{2x}+1}{2}=e^x \Leftrightarrow x=0$

Lesson 3: Application of Exponential equations to estimate the Population Growth

a) Learning objectives

Use logarithmic or exponential equations to solve population growth problems

b) Teaching resources

Learner's book and other Reference books to facilitate research, Mathematical set, calculator, manila paper, markers, pens, pencils...

c) Prerequisites/Revision/Introduction

Students will learn better in this lesson if they refer to the previous lessons of this unit and to the equations and inequalities learnt in Year 1 Unit 2 and to the lesson 1 & 2 of this Unit.

d) Learning activities:

- Invite student-teachers to work in groups and do the activity 2.3.1 found in their Mathematics Student books;
- Move around in the class for facilitating students where necessary and give more clarification on eventual challenges they may face during their work; Verify and identify groups with different working steps;

- Invite one member from each group with different working steps to present their work where they must explain the working steps;
- As a tutor, harmonize the findings from presentation;
- Use different probing questions and guide them to explore the content and examples given in the student's book and lead them to apply logarithmic and exponential equations to solve population growth problems.
- After this step, guide students to do the application activity 2.3.1 and evaluate whether lesson objectives were achieved.

Answers for activity 2.3.1

1) $P(t) = P_0 2^{kt}$

Here $P_0 = 2, k = 2, t\ in\ hours \Rightarrow P(t) = 2^{2t+1}$

a) $P(4) = 2^9 = 512$

b) $P(t) = 2^{13} \Leftrightarrow 2^{2t+1} = 2^{13}$

$\Rightarrow 2t + 1 = 13 \Rightarrow t = 6$

2) Number of cells left is $\frac{2^{22}}{2}\ or\ 2^{21}$

Answers for application activity 2.3.1

a) $N(0) = 100e^{0.045\times 0} \Rightarrow N(0) = 100$. The initial amount of bacteria is 100

b) The rate growth is 0.045 which is 4.5%

c) The population after 5 years is $N(5) = 100\,\mathrm{e}^{0.045\times 5} = 125.232 \approx 125$. After 5years , we have 125 bacteria

d)

$$140 = 100\,\mathrm{e}^{0.045t} \Leftrightarrow \frac{140}{100} = e^{0.045t} \Leftrightarrow 1.4 = e^{0.045t} \Leftrightarrow 0.045t = \ln 1.4 \Rightarrow t = \frac{1}{0.045}\ln 1.4 = 7.477 \approx 7.5$$

It will take approximately 8 days for the population to reach 140 grams.

e) $2N(0) = 100e^{0.045t} \Leftrightarrow 200 = 100e^{0.045t} \Leftrightarrow t = \frac{\ln 2}{0.045} = 15.4$.

The doubling time for the population is approximately to 16 days.

Lesson 4: Application of exponential function on decay

a) Learning objectives

Apply logarithmic and exponential equations to solve problems on decay.

b) Teaching resources

Learner's book and other Reference books to facilitate research, Mathematical set, calculator, manila paper, markers, pens, pencils...

c) Prerequisites/Revision/Introduction

Students will learn better in this lesson if they have a good background on the equations and inequalities learnt in Year 1 Unit 2 and on the lesson 1 & 2 of this Unit.

d) Learning activities:

- Invite student-teachers to work in groups and do the activity 2.3.2 found in their Mathematics Student books;
- Move around in the class for facilitating students where necessary and give more clarification on eventual challenges they may face during their work; Verify and identify groups with different working steps;
- Invite one member from each group with different working steps to present their work where they must explain the working steps;
- As a tutor, harmonize the findings from presentation and guide students to discover that the decay means the diminution in umbers;
- Use different probing questions and guide them to explore the content and examples given in the student's book and lead them to apply logarithmic and exponential equations to solve problems on decay.
- After this step, guide students to do the application activity 2.3.2 and evaluate whether lesson objectives were achieved.

Answers for activity 2.3.2

a) The original amount of material present is $A(0)=80\left(2^{0}\right)=80gram$

b) $A(2)=5\left(2^{\frac{2}{3}}\right)$ and $A(5)=5\left(2^{\frac{1}{5}}\right)$

b) For the half life, $A(t)=40$; $40=80\left(2^{-\frac{t}{100}}\right)\Rightarrow\frac{1}{2}=2^{-\frac{t}{100}}\Rightarrow 2^{-1}=2^{-\frac{t}{100}}$

$$1=\frac{t}{100}\Rightarrow t=100$$

Therefore the half life is 100 years

Answers for application activity 2.3.2

Since $N(t)$ is directly proportional to e^{ct},

$N(t)=ke^{ct}$, where k is a constant, for $t=0$, at the beginning $N(0)=2000$, we obtain $2000=ke^{c\times 0}=k\times 1=k$;

Hence, the formula for $N(t)$ may be written $N(t)=2000e^{ct.}$.

Since $N(10)=1500$ (*for* 10 *days*); we may determine c as follows:

$$1500=2000e^{c\times 10}$$

$$\frac{3}{4}=e^{10c}\Rightarrow 10c=\ln\frac{3}{4}\Rightarrow c=\frac{1}{10}\ln\frac{3}{4}.$$

Finally, Since the half-life corresponds to the time t at which $N(t)$ is equal to 1000, we have the following:

$$1000=2000e^{c\times 10}$$

$$\frac{1}{2}=e^{ct}\Rightarrow ct=\ln\frac{1}{2}\Rightarrow t=\frac{1}{c}\ln\frac{1}{2}$$

As $c=\frac{1}{10}\ln\frac{3}{4}$ we obtain $t=\frac{1}{\frac{1}{10}\ln\frac{3}{4}}\ln\frac{1}{2}\approx 24days$.

Lesson 5: Application of logarithmic equations to determine the magnitude of an earthquake

a) Learning objectives

To apply logarithmic and exponential equations to determine the magnitude of an earthquake;

b) Teaching resources

Learner’s book and other Reference books to facilitate research, Mathematical set, calculator, manila paper, markers, pens, pencils...

c) Prerequisites/Revision/Introduction

Students will learn better in this lesson if they have a good background on the equations and inequalities learnt in Year 1 Unit 2 and on the lesson 1 & 2 of this Unit.

d) Learning activities:

- Invite student-teachers to work in groups and do the activity 2.3.3 found in their Mathematics books;
- Visit every group to facilitate students where necessary and give more clarification on eventual challenges they may face during their work;
- Invite one member from each group to present their work;
- As a tutor, harmonize the findings from presentation and guide them to establish how they can determine the magnitude of an earthquake basing on a well identified earthquake;
- Use different probing questions and guide them to explore the content and examples given in the student's book and lead them to apply logarithmic and exponential equations to determine the magnitude of an earthquake;
- After this step, guide students to do the application activity 2.3.3 and evaluate whether lesson objectives were achieved.

Answers for activity 2.3.3

Magnitudes of **earthquakes** are measured using the **Richter scale.** On this scale, the magnitude R of an earthquake is given by $R = \log\left(\frac{I}{I_0}\right)$ where I_0 is a fixed and known standard intensity used for comparison, and I is the intensity of earthquake being measured.

Answers for application activity 2.3.3

As on the Richter scale, the magnitude R of an earthquakes is given by

$$R = \log\left(\frac{I}{I_0}\right)$$

Wit I_0 is a certain known intensity. Here in Rwanda we can use I^* instead of I_0 assuming that there was a well known earthquake I^* which occurred in

Rwanda in certain past years. Then,

$$R = \log\left(\frac{I}{I^*}\right) \Leftrightarrow \frac{I}{I^*} = 10^R \Rightarrow I = I^* 10^R.$$

Lesson 6: Application of Exponential equations on interest rates problems

a) Learning objectives

To apply logarithmic and exponential equations to solve financial problems related to interest rates.

b) Teaching resources

Learner's book and other reference books to facilitate research, calculator, manila paper, markers, pens, pencils, etc.

c) Prerequisites/Revision/Introduction

Students will learn better in this lesson if they have a good background on the equations and inequalities learnt in Year 1 Unit 2 and on the lesson 1 & 2 of this Unit.

d) Learning activities:

- Invite student-teachers to work in groups and do the activity 2.3.4 found in their Mathematics books;
- Move around in the class for facilitating students where necessary and give more clarification on eventual challenges they may face during their work; Verify and identify groups with different working steps;
- Invite one member from each group with different working steps to present their work where they must explain the working steps;
- As a tutor, harmonize the findings from presentation and guide students to determine the money saved or invested;
- Use different probing questions and guide them to explore the content and examples given in the student's book and lead them to apply logarithmic and exponential equations to solve interest rate problems;
- After this step, guide students to do the application activity 2.3.4 and evaluate whether lesson objectives were achieved.

Answers for activity 2.3.4

a) For the first month he has to pay20000Frw, For the second month 20000(1+0.01); For the third month 20000(1+0.01)(1+0.01) and for the fourth months 20000(1+0.01)(1+0.01)(1+0.01)

b) It is geometric sequences, which has the general term $u_n = 20000(1+0.01)^{n-1}$

c) After discussion tutor helps students-teacher to guess the following formula $A = P.(1+r)^t$ $, as\ n = 1$

The sum of money is ...

Answers for application activity 2.3.4

If P is the principal and we want P to double, the amount A will be 2P.

Use the compound Interest Formula $A = P\left(1+\frac{r}{n}\right)^{nt}$ with n=1 and t=0 to find r.

$2P = P(1+r)^5$; here A=2P, n=1,t=5 then,

$$2 = (1+r)^5 \Leftrightarrow \sqrt[5]{2} = 1+r \Rightarrow r = \sqrt[5]{2} - 1 \approx 1.148698 - 1 = 0.148698.$$

The annual rate of interest needed to double the principal in 5 years is 14.87%

Lesson 7: Application of exponential equations to determine the mortgage payments

a) Learning objectives

Use logarithmic or exponential equations to determine the mortgage payments

b) Teaching resources

Learner's book and other Reference books to facilitate research, calculator, manila paper, markers, pens, pencils...

c) Prerequisites/Revision/Introduction

Students will learn better in this lesson if they refer to exponential equations learnt in this unit.

d) Learning activities:

- Invite student-teachers to work in groups and do the activity 2.3.5 found in their Mathematics books;

- Move around in the class for facilitating students where necessary and give more clarification on eventual challenges they may face during their work; Verify and identify groups with different working steps;
- Invite one member from each group with different working steps to present their work where they must explain the working steps;
- As a tutor, harmonize the findings from presentation;
- Use different probing questions and guide them to explore the content and examples given in the student's book and lead them to apply logarithmic and exponential equations to to determine the mortgage payments.
- After this step, guide students to do the application activity 2.3.5 and evaluate whether lesson objectives were achieved.

Answers for activity 2.3.5

a) $P=\dfrac{V}{\left[\dfrac{1-(1+i)^{-n}}{i}\right]}$

b) i) In two years $P=\dfrac{8000}{\left[\dfrac{1-(1+0.1)^{-2}}{0.1}\right]}=4444.44...$

ii) In three years $P=\dfrac{8000}{\left[\dfrac{1-(1+0.1)^{-3}}{0.1}\right]}=3200Frw$

Answers for application activity 2.3.5

The company can invest money and receive 10% per annum. Machine A costs \$8000 and has a useful life of 7 years. The cost per year of Machine A is the annual payment *P* required to amortize a loan of \$8000 for 7 years at 10% per annum. As $V=\$8000;\ i=0.10;\ and\ n=7$; We use the following Formula:

$$P=V\left[\frac{i}{1-(1+i)^{-n}}\right]=800\left[\frac{0.1}{1-(1+0.1)^{-7}}\right]=8000(0.205405)=\$1643.34$$

The annual cost of Machine A to the company is $1643.24.

Similarly, Machine B, with a cost of $6000 and a useful life of 5 years, will cost the company each year an amount equal to the annual payment P required to amortize a loan of $6000 for 5 years at 10% per annum. As $V = \$6000;\ i = 0.1;\ and\ n = 5$; We use also the following formula:

$$P = V\left[\frac{i}{1-(1+i)^{-n}}\right] = 6000\left[\frac{0.1}{1-(1+0.1)^{-5}}\right] = 6000(0.263797) = \$1582.78$$

The annual cost of Machine B is $1582.78.

Machine A will generate more annual savings to the company than Machine B.

Machine A is preferable to Machine B.

Now we take into account the savings the company in labor generated by each machine.

	A	B
Annual labor savings	$2000.00	$1800.00
Annual cost	$1643.24	$1582.78
Net Savings	$356.76	$217.12

2.6. Unit Summary

1) Equations that contain logarithms are called **logarithmic equations.** Care must be taken when solving logarithmic equations algebraically. In the expression $\log_a M$ remember that a and M are positive and $a \neq 1$. Be sure to check each apparent solution in the original equation and discard any that are extraneous.

2) Equations that involve powers as terms of their expressions are referred to as **exponential equations.** Such equations can sometimes be solved by appropriately applying the Laws of exponents and property:

   a) For $a > 0$, and $a \neq 1$, the number a^u is positive, i.e $\forall u \in \mathbb{R}, a^u \in \mathbb{R}^+$

   b) If $a^u = a^v, then\ u = v$ each side of the equality must be written with the same base.

   c) Sometimes, the equation of the form $a^u = b, b \in \mathbb{R}^+$ are such that Since b cannot be written as an integer power of a ,write the exponential equation as the equivalent logarithmic equation $a^u = b \Rightarrow u = \log_a b$.

3) Exponential and logarithmic functions are used in population growth, half life, and decay in radioactivity. Logarithmic functions are also used to find interest rate problems

A quantity is said to have an **exponential growth (decay) model** if at each instant of time its rate of increase (decrease) is proportional to the amount of the quantity present

Exponential growth function is given by $P(t)=P_oe^{kt}$ where k is a positive number and t the time.

Exponential decay function is given by $P(t)=P_oe^{-kt}$ where k is a positive number t the time.

For exponential growth model, the time required for it to double in size is called the **doubling time.** Similarly, for exponential decay model, the time required for it to reduce in value by half is called the **halving time.** For radioactive elements, halving time is called **half-life.**

2.7. Additional information for the tutor

The tutor has to emphasize the application of logarithmic and exponential equations to estimate the population growth, to calculate the interest rate problems, to determine the magnitude of earthquake, to calculate the mortgage payment, etc.

2.8. End Unit assessment

1) a) $\log_3 x=4\Leftrightarrow x=3^4\Rightarrow S=\{81\}$

b) $\ln(x-2)(x-1)=\ln(2x+8)\Leftrightarrow(x-2)(x-1)=2x+8\Leftrightarrow S=\{-1,6\}$

c) $\log_x 5=\log_5 x\Leftrightarrow\dfrac{1}{\log_5 x}\log_5 x=\log_5 x\Leftrightarrow\log_5 x=\log_5 x^2\Leftrightarrow S=\{2\}$

d) $2^{x-1}-2^{x-3}=2^{3-x}-2^{1-x}\Leftrightarrow s=\left\{\dfrac{1}{5},5\right\}$

e) $e^{4x}-13e^{2x}+36=0\Leftrightarrow\left(e^{2x}\right)^2-13e^{2x}+36=0\Leftrightarrow s=\{(\ln 2,\ln 3)\}$

2) a) $2000(1.08)^3=2519\$$ b) $2000(1.08)^n=4000\Leftrightarrow n=\dfrac{\log 2}{\log 1.08}=9\,years$

3) a) $1000000\left(2^{-10}\right)=977\,germs$

b) $1000000\left(2^{-n}\right)<1 \Leftrightarrow n>\dfrac{6}{\log 2}=20 \min utes$

4) $Pe^{\frac{n}{30}}=2P \Leftrightarrow n=30\ln 2=20.8 years$

$10000e^{\frac{n}{30}}=1000000 \Leftrightarrow n=60\ln 10=138 years$

5) For a temperature of $80^0 C, when\ t=0$, the excess to $20^0 C$ is $\theta=Ae^{-0.02(0)}=A=60$,

Then, $\theta(t)=60e^{-0.02t}$,

At $t=10 \Rightarrow \theta=60e^{-0.02(10)}=41^0 C=322^0 K$,

At $t=20 \Rightarrow \theta=60e^{-0.02(20)}=40.2^0 C=313.2^0 K$,

At $t=45 \Rightarrow \theta=60e^{-0.02(45)}=24.4^0 C=297.4^0 K$

2.9. Additional activities

2.9.1 Remedial activities

1) Solve for x the equation: $3^{5x-8}=9^{x+2}$

Solution:

$3^{5x-8}=9^{x+2}$

$3^{5x-8}=\left(3^2\right)^{x+2} \Rightarrow 3^{5x-8}=3^{2x+4}\ (law\ of\ \exp onents)$

$5x-8=2x+4 \Rightarrow 3x=12 \Rightarrow x=4$. The solution set is $\{4\}$

2) Solve for x the equation: $\log_4(5+x)=3$

Solution:

$\log_4(5+x)=3 \Rightarrow 5+x=4^3 \Rightarrow x=59$. The solution set is $\{59\}$

2.9.2 Consolidation activities

1) Assume that a population is growing continuously at a rate of 4% per year . Approximate the amount of time it takes for the population to double its.

Solution:

Note that an initial population size is not given.

This fact does not present a problem, since we wish only to determine the time needed to obtain a population size relative to the initial population size. Using the growth formula: $q = q_0 e^{rt}$ with $r = 0.04$ give us $2q_0 = q_0 e^{rt}$ with $q = 2q_0$, then $2 = e^{0.04t} \Leftrightarrow 0.04t = \ln 2 \Leftrightarrow t = 25\ln 2 \approx 17.3\,years$

The fact that q_0 did not have any effect on the answer indicates that the doubling for a population of 1000 is the same as the doubling time for a population of 1,000,000, or any other reasonable initial population.

2.9.3 Extended activities

1) On January 1, 2010, \$2000 is placed in an Individual Retirement Account (IRA) that will pay interest of 3% per annum compounded continuously.

   a) What will the IRA be worth on January 1, 2030?

   b) What is the effective annual rate of interest?

Solution:

a) On January, 1, 2030, the initial principal of \$2000 will have earned interest of 3% compounded continuously for 20 years. The amount *A* after 20 years is

$$A = Pe^{rt} = \$2000e^{(0.03)(20)} = \$3644.24$$

b) First, compute the interest earned on \$2000 at $r = 3\%$ compounded continuously after 1 year. $A = \$2000e^{0.03(1)} = \$2060.91$

So the interest earned after 1 year is $\$2060.91 - \$2000.00 = \$60.91$ To find the effective rate of interest *R*, use the simple interest formula $I = PRt$ with $I = \$60.91,\ P = \$2000, t = 1$

$$I = \$60.91 = \$2000R \Rightarrow R = \frac{\$60.91}{\$2000} = 0.0305$$

The effective annual rate of interest *R* is 3.05%.

UNIT 3

MATRICES OF ORDER 2 AND ORDER 3

3.1. Key unit competence:

Solve problem involving the system of linear equations using matrices

3.2. Prerequisites

Student-teachers will easily learn this unit, if they have a good background on Arithmetic (Unit 1, Year 1) and on Equations and inequalities (unit 2 Year 1).

3.3. Cross-cutting issues to be addressed

- Inclusive education (promote education for all while teaching)
- Peace and value Education (respect others' view and thoughts during class discussions)
- Gender (provide equal opportunity to boys and girls in the lesson)

3.4. Guidance on introductory activity

- Invite student-teachers to work in groups and give them instructions on how they can do the introductory activity 3.0 found in unit 3 of student's book;
- Guide students to read and analyse the questions insisting on solving problems involving the system of linear equations using matrices
- Invite some group members to present groups' findings, then try to harmonize their answers;
- Basing on student-teachers' experience, prior knowledge and abilities shown in answering the questions for this activity, use different questions to facilitate them to give their predictions and ensure that you arouse their curiosity on what is going to be leant in this unit.

Answers for introductory activity 3.0

a)

Cocks	Rabbits	Prices
5	4	35,000
3	6	30,000

Let x be the cost of one cock and y be the cost of one rabbit, then,

$$\begin{cases} 5x+4y=35,000 \\ 3x+6y=30,000 \end{cases}$$

$$\begin{cases} 5x+4y=35,000 & \times(3) \\ 3x+6y=30,000 & \times(-5) \end{cases} \Rightarrow \begin{cases} 15x+12y=105,000 \\ -15x-30y=-150,000 \end{cases}$$

$$\Rightarrow -18y=-45,000 \Rightarrow y=2,500$$

If we replace y in the first equation we obtain

$$5x+4(2500)=35,000 \Rightarrow 5x=25,000 \Rightarrow x=5,000$$

Thus the cost of 1 cock is 5,000Frw and the cost of one rabbit is 2,500Frw.

3.5. List of lessons

No	Lesson title	Learning objectives	Number of periods
0	Introductory activity	To arouse the curiosity of student teachers on the content of unit 3	1
1	Definition, size and types of matrices	Define and differentiate types of matrices	1
2	Addition and subtraction of matrices	Perform operations (addition, subtraction) on matrices of order 2 and order 3.	1
3	Multiplication of matrices	Perform operations of multiplication on matrices of order 2 and order 3.	2
4.	Properties of Matrices Multiplication	Perform on properties of multiplication on matrices of order 2 and order 3	2

5	Transpose of a matrix	Construct a transpose of a given matrix	1
6	Determinant of matrices of order 2 and 3 and their properties	Calculate the determinant of a matrix of order 2 and 3.	2
7	Properties of determinant	Calculate the determinant of a matrix of order 2 and 3.	1
8	Inverse of matrix of order 2 and 3	Determine the inverse of a matrix of order 2 or 3.	2
9	Properties of the Inverse Matrix	Determine the inverse of a matrix of order 2 or 3.	1
10	Solving Simultaneous equations with 2 or 3 unknowns using the inverse of a matrix of order 2 and order 3	solve system of 3 linear equations using the inverse of a matrix of order 2 and order 3	1
11	Solving Simultaneous equations with 2 or 3 unknowns using Cramer's rule	Use matrices (Cramer's rule) to solve Simultaneous equations with 2 or 3 unknowns.	1
12	Solving Simultaneous equations with 2 or 3 unknowns Gaussian elimination	Use matrices (Gaussian elimination) to solve Simultaneous equations with 2 or 3 unknowns.	1
13	End unit assessment		1
Total number of periods in this unit.			**18**

Lesson1: Definition, size and types of matrices

a) Learning objectives

To define and differentiate types of matrices.

b) Teaching resources

Learner's book and other Reference books to facilitate research, calculator, manila paper, markers, pens, pencils...

c) Prerequisites/Revision/Introduction

Students will learn easily in this lesson, if they have a good background on Arithmetic (Unit 1,Year 1) and on Equations and inequalities(unit 2 Year 1).

d) Learning activities:

- Invite student-teachers to work in groups and do the activity 3.1 found in their Mathematics Student books;
- Move around in the class for facilitating students where necessary and give more clarification on eventual challenges they may face during their work; Verify and identify groups with different working steps;
- Invite one member from each group with different working steps to present their work where they must explain the working steps;
- As a tutor, harmonize the findings from presentation on definition and types of matrices;
- Use different probing questions and guide them to explore the content and examples given in the student's book and lead them to discover how to define and differentiate types of matrices.
- After this step, guide students to do the application activity 3.1 and evaluate whether lesson objectives were achieved.

Answers for activity 3.1

a) $\begin{pmatrix} 20 & 31 \\ 45 & 23 \end{pmatrix}$

b) Each tutor will respond according to his/her class size.

Answers for application activity 3.1

1) a) Matrix square of order 3 or 3×3

b) Matrix of one row and two column 1×2

c) Matrix of one row and one column 1 *or* 1×1

d) Matrix of two rows and five column 2×5

e) Matrix square of order 2 or 2×2

2) $a)\begin{pmatrix} a & b & c \end{pmatrix}$ is a row matrix of order 3x1;

b) $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ is a Zero matrix of order 2x2;

c) $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ is a column matrix of order 1x3 ;

d) $\begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}$ is a diagonal matrix of order 3x3;

e) $\begin{pmatrix} a & b & c \\ 0 & b & d \\ 0 & 0 & e \end{pmatrix}$ is aan upper triangular matrix of order 3x3 ;

f) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ is an identity matrix of order 3x3;

g) $\begin{pmatrix} a & 0 & 0 \\ c & b & 0 \\ d & 0 & c \end{pmatrix}$ is a lower triangular matrix of order 3x3;

h) $\begin{pmatrix} b & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & b \end{pmatrix}$ is a scalar matrix of order 3x3.

3)

$$\begin{cases} 3y+2=y-3 \\ 2x+1=5 \end{cases} \Rightarrow \begin{cases} y=-\dfrac{5}{2} \\ x=2 \end{cases}$$

Lesson 2: Addition and subtraction of matrices

a) Learning objectives

To Perform operations (addition, subtraction) on matrices of order 2 and order 3.

b) Teaching resources

Learner’s book and other Reference books to facilitate research, calculator, manila paper, markers, pens, pencils...

c) Prerequisites/Revision/Introduction

Students will learn easily in this lesson, if they have a good background on Arithmetic (Unit 1,Year 1) and on Equations and inequalities(unit 2 Year 1).

d) Learning activities:

- Invite student-teachers to work in groups and do the activity 3.2.1 found in their Mathematics Student books;
- Move around in the class for facilitating students where necessary and give more clarification on eventual challenges they may face during their work; Verify and identify groups with different working steps;
- Invite one member from each group with different working steps to present their work where they must explain the working steps;
- As a tutor, harmonize the findings from presentation on addition and subtraction of matrices;
- Use different probing questions and guide them to explore the content and examples given in the student’s book and lead them to discover how to perform operations (addition, subtraction) on matrices of order 2 and order 3.
- After this step, guide students to do the application activity 3.2.1 and evaluate whether lesson objectives were achieved.

Answers for activity 3.2.1

1)

	Too high	Too Low	No opinion
Male	200	150	45
Female	315	125	65

Then, in matrix form: $\begin{pmatrix} 200 & 150 & 45 \\ 315 & 125 & 65 \end{pmatrix}$

2) The matrix C representing students who were present is

$$C = \begin{pmatrix} 23-2 & 2-1 \\ 20-3 & 4-2 \end{pmatrix} = \begin{pmatrix} 21 & 1 \\ 17 & 2 \end{pmatrix}$$

Answers for application activity 3.2.1

1) a) $\begin{pmatrix} 13 & 4 \\ 6 & 10 \end{pmatrix} + \begin{pmatrix} 21 & 30 \\ 9 & 12 \end{pmatrix} = \begin{pmatrix} 34 & 34 \\ 15 & 22 \end{pmatrix}$

b) $\begin{pmatrix} 26 & 8 \\ 12 & 20 \end{pmatrix} - \begin{pmatrix} 7 & 10 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 19 & -2 \\ 9 & 16 \end{pmatrix}$

2) a) $\begin{pmatrix} 1 & -10 & -3 \\ -12 & -2 & -11 \\ 0 & -2 & 1 \end{pmatrix}$

b) $\begin{pmatrix} -25 & 10 & 15 \\ -4 & 6 & -5 \\ -18 & 8 & 25 \end{pmatrix}$

c) $\begin{pmatrix} 15 & -14 & 3 \\ 0 & 0 & -13 \\ 9 & 3 & 4 \end{pmatrix}$

Lesson 3: Multiplication of matrices

a) Learning objectives

To perform operations of multiplication on matrices of order 2 and order 3.

b) Teaching resources

Learner's book and other Reference books to facilitate research, calculator, manila paper, markers, pens, pencils...

c) Prerequisites/Revision/Introduction

Students will well perform in this lesson, if they have a good background on

Arithmetic (Unit 1,Year 1) , on Equations and inequalities (unit 2 Year 1) , on lesson 1 and lesson 2 of this Unit.

d) Learning activities:

- Invite student-teachers to work in groups and do the activity 3.2.2 found in their Mathematics books;
- Move around in the class for facilitating students where necessary and give more clarification on eventual challenges they may face during their work; Verify and identify groups with different working steps;
- Invite one member from each group with different working steps to present their work where they must explain the working steps;
- As a tutor, harmonize the findings from presentation on multiplication of matrices of order 2 and order 3;
- Use different probing questions and guide them to explore the content and examples given in the student's book and lead them to discover how to perform operations of multiplication on matrices of order 2 and order 3.
- After this step, guide students to do the application activity 3.2.2 and evaluate whether lesson objectives were achieved.

Answers for activity 3.2.2

1) We can represent prices of each item as P = $\begin{pmatrix} 40\$ \\ 20\$ \\ 400\$ \end{pmatrix}$.

The total number of each item in the store is $N = \begin{pmatrix} 100 & 200 & 50 \end{pmatrix}$

Then, the total revenue due to these sales is given by

$$N \times P = \begin{pmatrix} 100 & 200 & 50 \end{pmatrix} \begin{pmatrix} 40\$ \\ 20\$ \\ 400\$ \end{pmatrix} = 28000\$$$

$$A \times B = \begin{pmatrix} -2+3-1 & -1+2+6 & 1+3+4 \\ 4+6-5 & 2+4+30 & -2+6+20 \\ 0+9-4 & 0+6+24 & 0+9+15 \end{pmatrix} = \begin{pmatrix} 0 & 7 & 8 \\ 5 & 36 & 24 \\ 5 & 30 & 25 \end{pmatrix}$$

Answers for application activity 3.2.2

1) $A \times B = \begin{pmatrix} -28 & 36 & 39 \\ 28 & -6 & -5 \\ 56 & 64 & 80 \end{pmatrix}$

2) $A \times C = \begin{pmatrix} 47 & 4 & -36 \\ 1 & -9 & 31 \\ 112 & 8 & -28 \end{pmatrix}$

3) $B \times C = \begin{pmatrix} 161 & 9 & -21 \\ 276 & -22 & -18 \\ 123 & 7 & -17 \end{pmatrix}$

Lesson 4: Properties of Matrices Multiplication

a) Learning objectives

To perform on properties of multiplication on matrices of order 2 and order 3.

b) Teaching resources

Learner's book and other Reference books to facilitate research, calculator, manila paper, markers, pens, pencils...

c) Prerequisites/Revision/Introduction

Students will well perform in this lesson, if they have a good background on multiplying matrices (lesson 3 of this Unit).

d) Learning activities:

- Invite student-teachers to work in groups and do the activity 3.2.3 found in their Mathematics Student books;
- Move around in the class for facilitating students where necessary and give more clarification on eventual challenges they may face during their work; Verify and identify groups with different working steps;
- Invite one member from each group with different working steps to present their work where they must explain the working steps;
- As a tutor, harmonize the findings from presentation on properties of multiplication of matrices;
- Use different probing questions and guide them to explore the content

and examples given in the student's book and lead them to discover how to perform on properties of multiplication on matrices of order 2 and order 3.

- After this step, guide students to do the application activity 3.2.3 and evaluate whether lesson objectives were achieved.

Answers for activity 3.2.3

1) $A\times B=\begin{pmatrix}-1 & 3 & -2\\ -2 & -1 & 3\\ 1 & 1 & -2\end{pmatrix}$, $B\times A=\begin{pmatrix}2 & -4 & -1\\ 7 & -7 & -2\\ -5 & 3 & 1\end{pmatrix}$

$A\times B\neq B\times A$. Multiplication of matrices is not commutative

$$A\times(B+C)=\begin{pmatrix}3 & 0 & 1\\ 0 & -1 & 1\\ -2 & 3 & 2\end{pmatrix}\begin{pmatrix}1 & 2 & -2\\ 2 & -1 & -1\\ -2 & 1 & 1\end{pmatrix}=\begin{pmatrix}1 & 7 & -5\\ -4 & 2 & 2\\ 0 & -5 & 3\end{pmatrix},$$

$$A\times B+A\times C=\begin{pmatrix}-1 & 3 & -2\\ -2 & -1 & 3\\ 1 & 1 & -2\end{pmatrix}+\begin{pmatrix}2 & 4 & -3\\ -2 & 3 & -1\\ -1 & -6 & 5\end{pmatrix}=\begin{pmatrix}1 & 7 & -5\\ -4 & 2 & 2\\ 0 & -5 & 3\end{pmatrix}$$

$A\times(B+C)=A\times B+A\times C$. Multiplication of matrices is distributive over addition

Answers for application activity 3.2.3

a) $A\times B=\begin{pmatrix}-3 & 0 & 2\\ -4 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}$, $B\times A=\begin{pmatrix}-2 & 0 & -2\\ 1 & 0 & 1\\ -1 & 0 & -1\end{pmatrix}$

b) $tr(A\times B)=tr\begin{pmatrix}-3 & 0 & 2\\ -4 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}=-3$

Lesson 5: Transpose of Matrix

a) Learning objectives

To Construct a transpose of a given matrix of order 2 and order 3.

b) Teaching resources

Learner's book and other Reference books to facilitate research, calculator, manila paper, markers, pens, pencils...

c) Prerequisites/Revision/Introduction

Students will learn easily in this lesson, if they well perform on operation (Addition, subtraction and multiplication)on matrices.

d) Learning activities:

- Invite student-teachers to work in groups and do the activity 3.3 found in their Mathematics Student books;
- Move around in the class for facilitating students where necessary and give more clarification on eventual challenges they may face during their work; Verify and identify groups with different working steps;
- Invite one member from each group with different working steps to present their work where they must explain the working steps;
- As a tutor, harmonize the findings from presentation on transpose of a given matrix of order 2 and order 3;
- Use different probing questions and guide them to explore the content and examples given in the student's book and lead them to discover how to construct a transpose of a given matrix of order 2 and order 3.
- After this step, guide students to do the application activity 3.3 and evaluate whether lesson objectives were achieved.

Answers for activity 3.3

1) $\begin{pmatrix} 1 & 1 & 3 \\ 3 & -2 & -2 \\ 1 & 2 & 0 \end{pmatrix}$ and $\begin{pmatrix} 12 & 3 & -4 \\ 3 & -2 & -1 \\ -1 & 0 & 0 \end{pmatrix}$

2) $\begin{pmatrix} 13 & 4 & -1 \\ 6 & -4 & -3 \\ 0 & 2 & 0 \end{pmatrix}$

3) $\begin{pmatrix} 13 & 6 & 0 \\ 4 & -4 & 2 \\ -1 & -3 & 0 \end{pmatrix}$

4) $\begin{pmatrix} 13 & 4 & -1 \\ 6 & -4 & -3 \\ 0 & 2 & 0 \end{pmatrix}$

5) Matrix obtained in 2 is equal to the matrix obtained in 4

6) $A = \begin{pmatrix} 1 & 3 & 1 \\ 1 & -2 & 2 \\ 3 & -2 & 0 \end{pmatrix}$

Interchanging the rows and columns of matrix A once we get the new matrix $\begin{pmatrix} 1 & 1 & 3 \\ 3 & -2 & -2 \\ 1 & 2 & 0 \end{pmatrix}$

Interchanging the rows and columns of matrix A twice we get the new matrix $\begin{pmatrix} 1 & 3 & 1 \\ 1 & -2 & 2 \\ 3 & -2 & 0 \end{pmatrix}$

The last matrix is equal to matrix A

Answers for application activity 3.3

1) $\begin{pmatrix} 1 & 1 & 3 \\ 3 & -2 & -2 \\ 1 & 2 & 0 \end{pmatrix}$ and $\begin{pmatrix} 12 & 3 & -4 \\ 3 & -2 & -1 \\ -1 & 0 & 0 \end{pmatrix}$

2) $\begin{pmatrix} 13 & 4 & -1 \\ 6 & -4 & -3 \\ 0 & 2 & 0 \end{pmatrix}$

3) $\begin{pmatrix} 13 & 6 & 0 \\ 4 & -4 & 2 \\ -1 & -3 & 0 \end{pmatrix}$

4) $\begin{pmatrix} 13 & 4 & -1 \\ 6 & -4 & -3 \\ 0 & 2 & 0 \end{pmatrix}$

5) Matrix obtained in 2 is equal to the matrix obtained in 4

6) $A = \begin{pmatrix} 1 & 3 & 1 \\ 1 & -2 & 2 \\ 3 & -2 & 0 \end{pmatrix}$

Interchanging the rows and columns of matrix A once we get the new matrix $\begin{pmatrix} 1 & 1 & 3 \\ 3 & -2 & -2 \\ 1 & 2 & 0 \end{pmatrix}$

Interchanging the rows and columns of matrix A twice we get the new matrix $\begin{pmatrix} 1 & 3 & 1 \\ 1 & -2 & 2 \\ 3 & -2 & 0 \end{pmatrix}$

The last matrix is equal to matrix A

Lesson 6: Determinant of order two or three

a) Learning objectives

To calculate the determinant of a matrix of order 2 and order 3.

b) Teaching resources

Learner's book and other Reference books to facilitate research, calculator, manila paper, markers, pens, pencils...

c) Prerequisites/Revision/Introduction

Students will learn easily in this lesson, if they performed well on operation (Addition, subtraction and multiplication)on matrices.

d) Learning activities:

- Invite student-teachers to work in groups and do the activity 3.4.1 found in their Mathematics Student books;
- Move around in the class for facilitating students where necessary and give more clarification on eventual challenges they may face during their work; Verify and identify groups with different working steps;

- Invite one member from each group with different working steps to present their work where they must explain the working steps;
- As a tutor, harmonize the findings from presentation on determinant of a matrix of order 2 and order 3.
- Use different probing questions and guide them to explore the content and examples given in the student's book and lead them to discover how to calculate the determinant of a matrix of order 2 and order 3.
- After this step, guide students to do the application activity 3.4.1 and evaluate whether lesson objectives were achieved.

Answers for activity 3.4.1

1) $(1\times 6\times 1)+(3\times 0\times 2)+(5\times 1\times(-4))-(2\times 6\times 5)-(1\times 0\times 1)-(1\times 3\times(-4))=-62$

2) $(10\times 2\times 4)+((-6)\times 5\times 2)+(0\times 3\times 1)-(4\times 5\times 0)-(2\times 3\times 10)-(1\times(-6)\times 2)=-70$

Answers for application activity 3.4.1

1) 82 2) 10 3) -8

Lesson 7: Properties of determinant

a) Learning objectives

To perform on the properties of determinant of a matrix of order 2 and order 3.

b) Teaching resources

Learner's book and other Reference books to facilitate research, calculator, manila paper, markers, pens, pencils...

c) Prerequisites/Revision/Introduction

Students will learn easily in this lesson, if they well perform on operation (Addition, subtraction and multiplication)on matrices.

d) Learning activities:

- Invite student-teachers to work in groups and do the activity 3.4.2 found in their Mathematics Student books;
- Move around in the class for facilitating students where necessary and give more clarification on eventual challenges they may face during their work; Verify and identify groups with different working steps;

- Invite one member from each group with different working steps to present their work where they must explain the working steps;
- As a tutor, harmonize the findings from presentation on properties of determinant of a matrix of order 2 and order 3
- Use different probing questions and guide them to explore the content and examples given in the student's book and lead them to discover how to perform on the properties of determinant of a matrix of order 2 and order 3
- After this step, guide students to do the application activity 3.4.2 and evaluate whether lesson objectives were achieved.

Answers for activity 3.4.2

1) $|A| = 0, |B| = 0$
2) $|C \cdot D| = -36, |C| \cdot |D| = 6 \times (-6) = -36$

   $|C \cdot D| = |C| \cdot |D|$

   Determinant of product is equal to the product of determinants
3) $tr(C) = 6, \ |C| = 6$

   $tr(C) = |C|$

   Determinant of a matrix is equal to the trace of that matrix

Answers for application activity 3.4.2

1) $|A| = 0, |B| = 0, \ |C| = 14, |D| = -5$

Lesson 8: Inverse of matrices of order two or three

a) Learning objectives

To determine the inverse of a matrix of order 2 and order 3.

b) Teaching resources

Learner's book and other Reference books to facilitate research, calculator, manila paper, markers, pens, pencils...

c) Prerequisites/Revision/Introduction

Students will learn easily in this lesson, if they well perform on operation (Addition, subtraction and multiplication)on matrices.

d) Learning activities:

- Invite student-teachers to work in groups and do the activity 3.4.3 found in their Mathematics Student books;
- Move around in the class for facilitating students where necessary and give more clarification on eventual challenges they may face during their work; Verify and identify groups with different working steps;
- Invite one member from each group with different working steps to present their work where they must explain the working steps;
- As a tutor, harmonize the findings from presentation on the inverse of a matrix of order 2 and order 3.
- Use different probing questions and guide them to explore the content and examples given in the student's book and lead them to discover how to determine the inverse of a matrix of order 2 and order 3.
- After this step, guide students to do the application activity 3.4.3 and evaluate whether lesson objectives were achieved.

Answers for activity 3.4.3

1) a) $\begin{pmatrix} 1 & -2 & 0 \\ 3 & 1 & 5 \\ -1 & 2 & 3 \end{pmatrix} \times \begin{pmatrix} \frac{-7}{21} & \frac{6}{21} & \frac{-10}{21} \\ \frac{-14}{21} & \frac{3}{21} & \frac{-5}{21} \\ \frac{7}{21} & 0 & \frac{7}{21} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$

Matrix B is the inverse of Matrix A

b) After the research in library or internet they will find the following:

$|A| = -1$

Cofactor of each element:

$cofactor(1) = 3,\quad cofactor(1) = -5,\quad cofactor(1) = 1$

$cofactor(2) = 1,\quad cofactor(1) = -2,\quad cofactor(-1) = 1$

$cofactor(3) = -2,\quad cofactor(2) = 3,\quad cofactor(1) = -1$

Cofactor matrix

$$\begin{pmatrix} 3 & -5 & 1 \\ 1 & -2 & 1 \\ -2 & 3 & -1 \end{pmatrix}$$

Transpose of cofactor matrix is

$$\begin{pmatrix} 3 & 1 & -2 \\ -5 & -2 & 3 \\ 1 & 1 & -1 \end{pmatrix}$$

$$\frac{1}{-1}\begin{pmatrix} 3 & 1 & -2 \\ -5 & -2 & 3 \\ 1 & 1 & -1 \end{pmatrix} = \begin{pmatrix} -3 & -1 & 2 \\ 5 & 2 & -3 \\ -1 & -1 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ 3 & 2 & 1 \end{pmatrix} \times \begin{pmatrix} -3 & -1 & 2 \\ 5 & 2 & -3 \\ -1 & -1 & 1 \end{pmatrix} = \begin{pmatrix} -3+5-1` & -1+2-1 & 2-3+1 \\ -6+5+1 & -2+2-1 & 4-3-1 \\ -9+10+1 & -3+4-1 & 6-6+1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I$$

The product of these two matrices is a unity (identity) matrix I .

Answers for application activity 3.4.3

1) Since determinant is 0, then, there is no inverse

2) $\begin{pmatrix} \frac{23}{268} & -\frac{29}{268} & \frac{5}{268} \\ -\frac{3}{268} & -\frac{37}{268} & \frac{11}{268} \\ -\frac{9}{268} & \frac{23}{268} & \frac{33}{268} \end{pmatrix}$

3) $\begin{pmatrix} \frac{6}{7} & -\frac{45}{14} & \frac{16}{7} \\ -\frac{5}{7} & \frac{24}{7} & -\frac{18}{7} \\ \frac{1}{7} & -\frac{11}{14} & \frac{5}{7} \end{pmatrix}$

4) $\begin{pmatrix} -\frac{2}{5} & -\frac{3}{5} & 1 \\ \frac{1}{5} & \frac{4}{5} & -1 \\ -\frac{6}{5} & -\frac{29}{5} & 8 \end{pmatrix}$

Lesson 9: Properties of the Inverse Matrix

a) Learning objectives

To perform on properties of the inverse of a matrix of order 2 and order 3.

b) Teaching resources

Learner's book and other Reference books to facilitate research, calculator, manila paper, markers, pens, pencils...

c) Prerequisites/Revision/Introduction

Students will learn easily in this lesson, if they show a good performance on operation (Addition, subtraction and multiplication) of matrices.

d) Learning activities:

- Invite student-teachers to work in groups and do the activity 3.4.4 found in their Mathematics Student books;
- Move around in the class for facilitating students where necessary and give more clarification on eventual challenges they may face during their work; Verify and identify groups with different working steps;
- Invite one member from each group with different working steps to present their work where they must explain the working steps;
- As a tutor, harmonize the findings from presentation on properties of the inverse of a matrix of order 2 and order 3.
- Use different probing questions and guide them to explore the content and examples given in the student's book and lead them to discover how to perform on properties of the inverse of a matrix of order 2 and order 3.
- After this step, guide students to do the application activity 3.4.4 and evaluate whether lesson objectives were achieved.

Answers for activity 3.4.4

1) $(AB)^{-1}=\begin{pmatrix}0&1&-1\\0&1&0\\1&-3&1\end{pmatrix}$, $B^{-1}A^{-1}=\begin{pmatrix}1&-1&0\\1&-1&1\\-2&3&-1\end{pmatrix}\begin{pmatrix}1&0&-1\\1&-1&0\\0&0&1\end{pmatrix}=\begin{pmatrix}0&1&-1\\0&1&0\\1&-3&1\end{pmatrix}$

$(AB)^{-1}=B^{-1}A^{-1}$

2) $(A^{-1})^{-1}=\begin{pmatrix}1&0&1\\1&-1&1\\0&0&1\end{pmatrix}$

$(A^{-1})^{-1}=A$

3) $(4A)^{-1}=\frac{1}{4}\begin{pmatrix}1&0&-1\\1&-1&0\\0&0&1\end{pmatrix}$, $\frac{1}{4}A^{-1}=\frac{1}{4}\begin{pmatrix}1&0&-1\\1&-1&0\\0&0&1\end{pmatrix}$

$(kA)^{-1}=\frac{1}{k}A^{-1},\ k\neq 0$

4) $(A^{t})^{-1}=\begin{pmatrix}1&1&0\\0&-1&0\\-1&0&1\end{pmatrix}$, $(A^{-1})^{t}=\begin{pmatrix}1&0&-1\\1&-1&0\\0&0&1\end{pmatrix}^{t}=\begin{pmatrix}1&1&0\\0&-1&0\\-1&0&1\end{pmatrix}$

$(A^{t})^{-1}=(A^{-1})^{t}$

Answers for application activity 3.4.4

1) $A^{-1}=\frac{1}{2}\begin{pmatrix}1&3\\-1&-1\end{pmatrix}$, B^{-1} doesn't exist, $(AB)^{-1}$ doesn't exist,

$(A^{t})^{-1}=\frac{1}{2}\begin{pmatrix}1&-1\\3&-1\end{pmatrix}$, $(4B)^{-1}$ doesn't exist

2) $A^{-1}=\frac{1}{11}\begin{pmatrix}2&5&-14\\1&-3&4\\-1&3&7\end{pmatrix}$, $B^{-1}=\begin{pmatrix}-\frac{5}{12}&\frac{1}{4}&-\frac{13}{12}\\-\frac{3}{4}&\frac{1}{4}&-\frac{3}{4}\\\frac{1}{3}&0&\frac{2}{3}\end{pmatrix}$

$$\left(A^{-1}\right)^{-1}=\begin{pmatrix}3&7&2\\1&0&2\\0&1&1\end{pmatrix}$$

$$(10A)^{-1}=\frac{1}{110}\begin{pmatrix}2&5&-14\\1&-3&4\\-1&3&7\end{pmatrix}$$

$$\left(A^{t}\right)^{-1}=\frac{1}{11}\begin{pmatrix}2&1&-1\\5&-3&3\\-14&4&7\end{pmatrix}$$

Lesson 10: Solving a System of linear equations using inverse matrix

a) Learning objectives

To solve a system of linear equations using the inverse matrix of order 2 or order 3.

b) Teaching resources

Learner's book and other Reference books to facilitate research, calculator, manila paper, markers, pens, pencils...

c) Prerequisites/Revision/Introduction

Students will learn easily in this lesson, if they performed well operation (Addition, subtraction and multiplication) of matrices.

d) Learning activities:

- Invite student-teachers to work in groups and do the activity 3.5.1 found in their Mathematics Student books;
- Move around in the class for facilitating students where necessary and give more clarification on eventual challenges they may face during their work; Verify and identify groups with different working steps;
- Invite one member from each group with different working steps to present their work where they must explain the working steps;
- As a tutor, harmonize the findings from presentation on solving the system of 3 linear equations using the inverse of a matrix of order 2 and order 3;
- Use different probing questions and guide them to explore the content and examples given in the student's book and lead them to discover how

to solve system of 3 linear equations using the inverse of a matrix of order 2 and order 3.

- After this step, guide students to do the application activity 3.5.1 and evaluate whether lesson objectives were achieved.

Answers for activity 3.5.1

a) Let consider x as the cost for one cock and y the cost of one Rabbit, the equations that illustrate the activity of Kalisa is $\begin{cases} 5x+4y=35,000 \\ 3x+6y=30,000 \end{cases}$

b) The matrix A indicating the number of cocks and rabbits is $A=\begin{pmatrix} 5 & 4 \\ 3 & 6 \end{pmatrix}$

c) In matrix form the activity of Kalisa is written as $\begin{pmatrix} 5 & 4 \\ 3 & 6 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} 35,000 \\ 30,000 \end{pmatrix}$

d) After discussion student-teacher discover that the cost of one cock is 5000Frw and the cost of one rabbit is 2500Frw.

Answers for application activity 3.5.1

$S=\{(1,2,0)\}$

Lesson 11: Solving System of linear equations using Cramer method

a) Learning objectives

To solve system of of linear equations using Cramer method.

b) Teaching resources

Learner’s book and other Reference books to facilitate research, calculator, manila paper, markers, pens, pencils...

c) Prerequisites/Revision/Introduction

Students will learn easily in this lesson if they refer to operation (Addition, subtraction and multiplication) of matrices, determinant of matrices and the inverse of a matrix.

d) Learning activities:

- Invite student-teachers to work in groups and do the activity 3.5.2 found in their Mathematics Student books;
- Move around in the class for facilitating students where necessary and give more clarification on eventual challenges they may face during their work; Verify and identify groups with different working steps;
- Invite one member from each group with different working steps to present their work where they must explain the working steps;
- As a tutor, harmonize the findings from presentation on solving system of of linear equations using Cramer method;
- Use different probing questions and guide them to explore the content and examples given in the student's book and lead them to discover how to solve system of of linear equations using Cramer method.
- After this step, guide students to do the application activity 3.5.2 and evaluate whether lesson objectives were achieved.

Answers for activity 3.5

$$\begin{cases} x+2y-3z=0 \\ 3x+3y-z=5 \\ x-2y+2z=1 \end{cases}$$

$$a)\begin{bmatrix} 1 & 2 & -3 \\ 3 & 3 & -1 \\ 1 & -2 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 0 \\ 5 \\ 1 \end{bmatrix}$$

$$b)\ \Delta=\begin{vmatrix} 1 & 2 & -3 \\ 3 & 3 & -1 \\ 1 & -2 & 2 \end{vmatrix}=[6\ -2+18]-[-9+2+12]=17$$

$$c)\ \Delta_x=\begin{vmatrix} 0 & 2 & -3 \\ 5 & 3 & -1 \\ 1 & -2 & 2 \end{vmatrix}=0-2+30-(-9+0+20)=17$$

$$d)\ \Delta_y=\begin{vmatrix} 1 & 0 & -3 \\ 3 & 5 & -1 \\ 1 & 1 & 2 \end{vmatrix}=10+0-9-(-15-1+0)=17$$

$$e)\ \Delta_z=\begin{vmatrix} 1 & 2 & 0 \\ 3 & 3 & 5 \\ 1 & -2 & 1 \end{vmatrix}=3+10+0-(0-10+6)=17$$

$$f)\ \begin{cases} x=\dfrac{\Delta_x}{\Delta}=\dfrac{17}{17}=1 \\ y=\dfrac{\Delta_y}{\Delta}=\dfrac{17}{17}=1 \\ z=\dfrac{\Delta_z}{\Delta}=\dfrac{17}{17}=1 \end{cases}$$

$$\begin{cases} x+2y-3z=0 \\ 3x+3y-z=5 \\ x-2y+2z=1 \end{cases}$$

$$a)\begin{bmatrix} 1 & 2 & -3 \\ 3 & 3 & -1 \\ 1 & -2 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 0 \\ 5 \\ 1 \end{bmatrix}$$

$$b)\ \Delta=\begin{vmatrix} 1 & 2 & -3 \\ 3 & 3 & -1 \\ 1 & -2 & 2 \end{vmatrix}=[6\ -2+18]-[-9+2+12]=17$$

$$c)\ \Delta_x=\begin{vmatrix} 0 & 2 & -3 \\ 5 & 3 & -1 \\ 1 & -2 & 2 \end{vmatrix}=0-2+30-(-9+0+20)=17$$

$$d)\ \Delta_y=\begin{vmatrix} 1 & 0 & -3 \\ 3 & 5 & -1 \\ 1 & 1 & 2 \end{vmatrix}=10+0-9-(-15-1+0)=17$$

$$e)\ \Delta_z=\begin{vmatrix} 1 & 2 & 0 \\ 3 & 3 & 5 \\ 1 & -2 & 1 \end{vmatrix}=3+10+0-(0-10+6)=17$$

$$f)\ \begin{cases} x=\dfrac{\Delta_x}{\Delta}=\dfrac{17}{17}=1 \\ y=\dfrac{\Delta_y}{\Delta}=\dfrac{17}{17}=1 \\ z=\dfrac{\Delta_z}{\Delta}=\dfrac{17}{17}=1 \end{cases}$$

$S=\{(1,1,1)\}$

g) Using other method ,we get the same solution

Answers for application activity 3.5.2

1) $S=\{(0,0,0)\}$
2) $S=\{\ \}$
3) $S=\{(1,2,0)\}$

Lesson 12: Solving system of linear equations using Gaussian method (elimination of Gauss)

a) Learning objectives

To solve system of linear equations using Gaussian method (elimination of Gauss).

b) Teaching resources

Learner's book and other Reference books to facilitate research, calculator, manila paper, markers, pens, pencils...

c) Prerequisites/Revision/Introduction

Students will learn easily in this lesson, if they refer to operation (Addition, subtraction and multiplication) of matrices, determinant of matrices and inverse of matrices

d) Learning activities:

- Invite student-teachers to work in groups and do the activity 3.5.3 found in their Mathematics Student books;
- Move around in the class for facilitating students where necessary and give more clarification on eventual challenges they may face during their work; Verify and identify groups with different working steps;
- Invite one member from each group with different working steps to present their work where they must explain the working steps;
- As a tutor, harmonize the findings from presentation on solving system of linear equations using Gaussian method;
- Use different probing questions and guide them to explore the content and examples given in the student's book and lead them to discover how to solve system of linear equations using Gaussian method (elimination of Gauss).
- After this step, guide students to do the application activity 3.5.3 and evaluate whether lesson objectives were achieved.

Answers for activity 3.5.3

$$\left[\begin{array}{ccc|c} 1 & 2 & -2 & 1 \\ 2 & 1 & -4 & -1 \\ 4 & -3 & 1 & 1 \end{array}\right] \begin{array}{l} r_1 \\ r_2 = 2r_1 - r_2 \\ r_3 = 4r_1 - r_3 \end{array} \sim \left[\begin{array}{ccc|c} 1 & 2 & -2 & 1 \\ 0 & 3 & 0 & 3 \\ 0 & 1 & -9 & -7 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 2 & -2 & 1 \\ 0 & 3 & 0 & 3 \\ 0 & 1 & -9 & -7 \end{array}\right]\begin{bmatrix} x \\ y \\ z \end{bmatrix} \Rightarrow \begin{cases} x+2y-2z=1 \\ 3y=3 \\ 1\ y-9z=-7 \end{cases}$$

$$\begin{cases} x=3 \\ y=1 \\ z=2 \end{cases}$$

Answers for application activity 3.5.3

1) $S=\left\{\left(\frac{17}{14},-\frac{9}{7}\right)\right\}$

2) $S=\{(4,3,-1)\}$

3.6. Unit summary

1) Square matrix of order three has the form:

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

2) In an upper triangular matrix, the elements located below the leading diagonal are zeros

3) In a lower triangular matrix, the elements above the leading diagonal are zeros.

4) In a diagonal matrix, all the elements above and below the leading diagonal are zeros.

5) A scalar matrix is a diagonal matrix in which the leading diagonal elements are equal.

6) An identity matrix (noted by **I**) is a diagonal matrix in which the leading diagonal elements are equal to 1.

7) If $\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}=\begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & a_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix}$, then $\begin{matrix} a_{11}=b_{11},\ a_{12}=b_{12},\ a_{13}=b_{13} \\ a_{21}=b_{21},\ a_{22}=b_{22},\ a_{23}=b_{23} \\ a_{31}=b_{31},\ a_{32}=b_{32},\ a_{33}=b_{33} \end{matrix}$

8) If $A=\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$ and $B=\begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix}$, then

$$A+B=\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}+\begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix}=\begin{pmatrix} a_{11}+b_{11} & a_{12}+b_{12} & a_{13}+b_{13} \\ a_{21}+b_{21} & a_{22}+b_{22} & a_{23}+b_{23} \\ a_{31}+b_{31} & a_{32}+b_{32} & a_{33}+b_{33} \end{pmatrix}$$

$$A-B=\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}-\begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix}=\begin{pmatrix} a_{11}-b_{11} & a_{12}-b_{12} & a_{13}-b_{13} \\ a_{21}-b_{21} & a_{22}-b_{22} & a_{23}-b_{23} \\ a_{31}-b_{31} & a_{32}-b_{32} & a_{33}-b_{33} \end{pmatrix}$$

9) If $A=\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$, then $kA=\begin{pmatrix} ka_{11} & ka_{12} & ka_{13} \\ ka_{21} & ka_{22} & ka_{23} \\ ka_{31} & ka_{32} & ka_{33} \end{pmatrix}$

10) If $A=\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$, then $A^t=\begin{pmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{pmatrix}$

11) If $A=\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$ and $B=\begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix}$, then

$$A\cdot B=\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}\cdot\begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix}$$

$$=\begin{pmatrix} a_{11}b_{11}+a_{12}b_{21}+a_{13}b_{31} & a_{11}b_{12}+a_{12}b_{22}+a_{13}b_{32} & a_{11}b_{13}+a_{12}b_{23}+a_{13}b_{33} \\ a_{21}b_{11}+a_{22}b_{21}+a_{23}b_{31} & a_{21}b_{12}+a_{22}b_{22}+a_{23}b_{32} & a_{21}b_{13}+a_{22}b_{23}+a_{23}b_{33} \\ a_{31}b_{11}+a_{32}b_{21}+a_{33}b_{31} & a_{31}b_{12}+a_{32}b_{22}+a_{33}b_{32} & a_{31}b_{13}+a_{32}b_{23}+a_{33}b_{33} \end{pmatrix}$$

12) The sum of the entries on the leading diagonal of a square matrix, A, is known as the trace of that matrix, noted $tr(A)$.

13) Consider an arbitrary 3x3 matrix, $A=\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$.

The determinant of A is defined as follows:

$$|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} - a_{12}a_{21}a_{33} - a_{11}a_{23}a_{32}$$

Steps to Calculate the Inverse Matrix

Calculate the determinant of A, $|A|$. If the determinant is zero the matrix has no inverse.

Find the cofactor matrix which is found by replacing every element in matrix A by its cofactor.

Find the **adjugate** (or **classical adjoint**) matrix, noted $adj(A)$, which is the transpose of the cofactor matrix.

The matrix inverse is equal to the inverse value of its determinant multiplied by the adjugate matrix.

Consider the following system

$$\begin{cases} a_{11}x + a_{12}y + a_{13}z = c_1 \\ a_{21}x + a_{22}y + a_{23}z = c_2 \\ a_{31}x + a_{32}y + a_{33}z = c_3 \end{cases} \qquad (1)$$

The system (1) can be written in the form

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$$ or $AX = C$ where $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$, $X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

and $C = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$ and the solution of system (1) is given by

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}^{-1}\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$$, provided that $\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}^{-1}$ exists or $X = A^{-1}C$

provided that A^{-1} exists.

Or we can use Cramer's rule as follows

$$\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$\Delta_x = \begin{vmatrix} c_1 & a_{12} & a_{13} \\ c_2 & a_{22} & a_{23} \\ c_3 & a_{32} & a_{33} \end{vmatrix}$$

$$\Delta_y = \begin{vmatrix} a_{11} & c_1 & a_{13} \\ a_{21} & c_2 & a_{23} \\ a_{31} & c_3 & a_{33} \end{vmatrix}$$

$$\Delta_z = \begin{vmatrix} a_{11} & a_{12} & c_1 \\ a_{21} & a_{22} & c_2 \\ a_{31} & a_{32} & c_3 \end{vmatrix}$$

$x = \frac{\Delta_x}{\Delta}$, $y = \frac{\Delta_y}{\Delta}$ and $z = \frac{\Delta_z}{\Delta}$

3.7. Additional information for the tutor

Here the tutor has to emphasize on solving system of linear equations using inverse matrix, Cramer method and using Gaussian method.

3.8. End Unit assessment

1) $3\begin{pmatrix} x & y-1 \\ 4 & 3z \end{pmatrix} = \begin{pmatrix} 15 & 6 \\ 6z & 3x+y \end{pmatrix} \Leftrightarrow \begin{pmatrix} 3x & 3y-3 \\ 12 & 9z \end{pmatrix} = \begin{pmatrix} 15 & 6 \\ 6z & 3x+y \end{pmatrix}$

$\begin{cases} 3x = 15 \Rightarrow x = 5 \\ 12 = 6z \Rightarrow z = 2 \end{cases}$ and $3y = 9 \Rightarrow y = 3$

2) a) $\begin{pmatrix} -7 & -3 & 0 \\ 0 & 4 & -12 \\ 0 & -10 & -2 \end{pmatrix}$ b) $\begin{pmatrix} -9 & -23 & 6 \\ 0 & -4 & -16 \\ -4 & 2 & -8 \end{pmatrix}$ c) $\begin{pmatrix} 7 & 8 & 3 \\ 2 & 4 & -10 \\ 2 & -14 & 7 \end{pmatrix}$

d) $\begin{pmatrix} 29 & 28 & 15 \\ 10 & -34 & -21 \\ -4 & 28 & -16 \end{pmatrix}$

3) a) A row matrix which represents Kamana's portfolio is $\begin{pmatrix} 42 & 59 & 21 & 18 \end{pmatrix}$

b) Kamana's portfolio for september is $\begin{pmatrix} 42 & 59 & 21 & 18 \end{pmatrix}\begin{pmatrix} 33.81 \\ 15.06 \\ 54 \\ 52.06 \end{pmatrix} = 4,379.64$,

Kamana's portfolio for October is $\begin{pmatrix} 42 & 59 & 21 & 18 \end{pmatrix}\begin{pmatrix} 30.91 \\ 13.25 \\ 54 \\ 44.69 \end{pmatrix} = 2,884.39$,

Kamana's portfolio for November is $\begin{pmatrix} 42 & 59 & 21 & 18 \end{pmatrix}\begin{pmatrix} 27.25 \\ 8.75 \\ 46.44 \\ 34.38 \end{pmatrix} = 3,254.83$

4) $\begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}$

5) $\{(0,0,0)\}$

3.9. Additional activities

3.9.1 Remedial activities

1) Find $C-D$,if $C = \begin{bmatrix} 9 & 4 \\ -1 & 3 \\ 0 & -4 \end{bmatrix}$ and $C = \begin{bmatrix} 8 & -2 \\ -6 & 1 \\ 5 & -5 \end{bmatrix}$

Solution: $C-D=C+(-D)=\begin{bmatrix}9 & 4\\ -1 & 3\\ 0 & -4\end{bmatrix}+\begin{bmatrix}-8 & 2\\ 6 & -1\\ -5 & 5\end{bmatrix}=\begin{bmatrix}1 & 6\\ 5 & 2\\ -5 & 1\end{bmatrix}$

2) Find the inverse of matrix $\begin{bmatrix}2 & -3\\ 4 & 4\end{bmatrix}$

Solution:

First, find the determinant of $\begin{bmatrix}2 & -3\\ 4 & 4\end{bmatrix}$ which is $\begin{vmatrix}2 & -3\\ 4 & 4\end{vmatrix}=2(4)-4(-3)=20$

The inverse is $\frac{1}{20}\begin{bmatrix}4 & 3\\ -4 & 2\end{bmatrix}=\begin{bmatrix}\frac{1}{5} & \frac{3}{20}\\ -\frac{1}{5} & \frac{1}{10}\end{bmatrix}$

3.9.2 Consolidation activities

1) Find the value of $\begin{vmatrix}-4 & -6 & 2\\ 5 & -1 & 3\\ -2 & 4 & -3\end{vmatrix}$

Solution:

$$\begin{vmatrix}-4 & -6 & 2\\ 5 & -1 & 3\\ -2 & 4 & -3\end{vmatrix}=-4\begin{vmatrix}-1 & 3\\ 4 & -3\end{vmatrix}-(-6)\begin{vmatrix}5 & 3\\ -2 & -3\end{vmatrix}+2\begin{vmatrix}5 & -1\\ -2 & 4\end{vmatrix}$$

$$-4(-9)+6(-9)+2(18)=18$$

2) Solve the system of equations by using matrix inverse

$$\begin{cases}2x+3y=-17\\ x-y=4\end{cases}$$

Solution:

Write the system as a matrix equation:

$$\begin{bmatrix} 2 & 3 \\ 1 & -1 \end{bmatrix} \times \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -17 \\ 4 \end{bmatrix}$$

To solve the matrix equation, first find the inverse of the coefficient matrix

$$\frac{1}{\begin{vmatrix} 2 & 3 \\ 1 & -1 \end{vmatrix}} \begin{bmatrix} -1 & -3 \\ -1 & 2 \end{bmatrix} = -\frac{1}{5} \begin{bmatrix} -1 & -3 \\ -1 & 2 \end{bmatrix}$$

Now multiply each side of the matrix equation by the inverse and solve.

$$\frac{-1}{5} \begin{bmatrix} -1 & -3 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 1 & -1 \end{bmatrix} . \begin{bmatrix} x \\ y \end{bmatrix} = \frac{-1}{5} \begin{bmatrix} -1 & -3 \\ -1 & 2 \end{bmatrix} . \begin{bmatrix} -17 \\ 4 \end{bmatrix} \Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -1 \\ -5 \end{bmatrix}$$

The solution is (-1,-5)

3.9.3 Extended activities

1) The National Endowment for the Arts exists to broaden public access to the arts. In 1992 and 2002, it performed studies to find what types of arts were most popular and how they could attract more people. The matrix below represent their findings.

Percent of people listening or watching Performances

1992 2002

TV Radio Recording TV Recording

$$\begin{array}{c} Classical \\ Jazz \\ Opera \\ Musical \end{array} \begin{bmatrix} 26 & 31 & 24 \\ 22 & 28 & 21 \\ 12 & 9 & 7 \\ 17 & 4 & 6 \end{bmatrix} \qquad \begin{array}{c} Classical \\ Jazz \\ Opera \\ Musical \end{array} \begin{bmatrix} 18 & 24 & 19 \\ 16 & 24 & 17 \\ 6 & 6 & 6 \\ 12 & 2 & 4 \end{bmatrix}$$

Source: National Endowment for the Arts

b) Find the difference in arts patronage from 1992 t0 2002. Express your answer as matrix.

c) Which area saw the greatest decrease in this time?

Solution:

a) $\begin{bmatrix} 26 & 31 & 24 \\ 22 & 28 & 21 \\ 12 & 9 & 7 \\ 17 & 4 & 6 \end{bmatrix} - \begin{bmatrix} 18 & 24 & 19 \\ 16 & 24 & 17 \\ 6 & 6 & 6 \\ 12 & 2 & 4 \end{bmatrix} = \begin{bmatrix} 8 & 7 & 5 \\ 6 & 4 & 4 \\ 6 & 3 & 1 \\ 5 & 2 & 2 \end{bmatrix}$

b) The Opera is the greatest decrease in this time

UNIT 4

BIVARIATE STATISTICS

4.1. Key unit Competence:

Extend understanding, analysis and interpretation of bivariate data to correlation coefficients and regression lines.

4.2. Prerequisites

Student-teachers will easily learn this unit, if they have a good background on descriptive statistics (Unit 6, Year 1) and on Arithmetic (Unit 1, Year 1).

4.3. Cross-cutting issues to be addressed

Inclusive education (promote education for all while teaching)

Peace and value Education (respect others' view and thoughts during class discussions)

Gender (provide equal opportunity to boys and girls in the lesson)

4.4. Guidance on introductory activity

- Invite student-teachers to work in groups and give them instructions on how they can do the introductory activity 4.0 found in unit 4 of the student's book;
- Guide students to read and analyse the questions insisting on the analysis of statistical data with two variables (x, y) and how they can interpret the bivariate data using correlation.
- Invite some group members to present groups' findings, then try to harmonize their answers;
- Basing on student-teachers' experience, prior knowledge and abilities shown in answering the questions for this activity, use different questions to facilitate them to give their predictions and ensure that you arouse their curiosity on what is going to be leant in this unit.

Answers for introductory activity

	X_i	Y_i	X_i^2	$X_i\,Y_i$
	1	4	1	4
	2	8	4	16
	3	2	9	6
	4	12	16	48
	5	10	25	50
	6	14	36	84
	7	16	49	112
	8	6	64	48
	9	18	81	162
Σ	45	90		

$\boldsymbol{X} = \frac{1}{n}\Sigma X_i = \frac{45}{9} = 5$, $\boldsymbol{Y} = \frac{1}{n}\Sigma Y_i = \frac{90}{9} = 10$

$$\delta_{X,Y} = \frac{\Sigma X_i \Sigma Y_i - \frac{1}{n}\Sigma X_i \Sigma Y_i}{\Sigma X^2 - \frac{1}{n}(\Sigma X)^2} = \frac{530 - \frac{1}{9}(45)(90)}{285 - \frac{1}{9}(45)^2} = \frac{4}{3} = 1.33$$

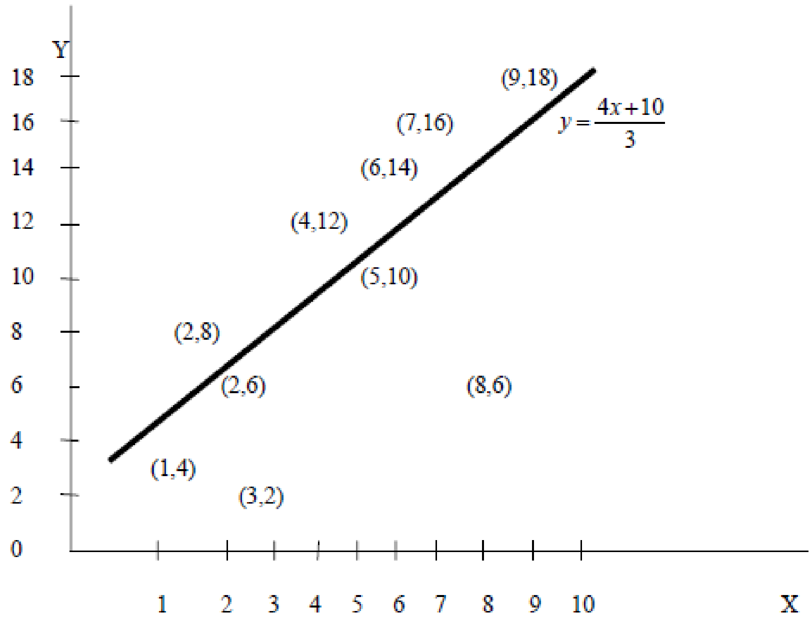
The equation of regression line of $Y - Y_i = \delta_{X,Y}(X - X_I)$

$$Y - 10 = 1.33(X - 5)$$

$$Y = 10 + 1.33X - 6.65$$

Scatter diagram: plotting the 9 sample points (1,4),(2,8),(3,4),(4,12),(5,10),(6, 14),(7,16), (8,6),(9,18).

The first point on the line is . Another point on the line is so the regression line of y on x passes through the two points and plot these points and join them the required line of regression of is obtained.



4.5. List of lessons

#	Lesson title	Learning objectives	Number of periods
0	Introductory activity	To arouse the curiosity of student-teachers on the content of unit 4.	1
1	Bivariate data ,scatter diagram and types of correlation	To define bivariate statistics data and plot the scatter diagram and differentiate types of correlation.	2
2	Covariance	Explain and determine the covariance for bivariate statistics data	3
3	Coefficient of correlation	To define and calculate the Coefficient of correlation for Bivariate statistics data. Appreciate the importance of using the Coefficient of correlation to interpret data and to infer conclusion	4
4	Regression line	To define the regression line and establish its equation, Appreciate the importance of using regression line to interpret data.	3

5	Interpretation of statistical data	Analyze, interpretation and predict bivariate statistical data from various areas (Business, Geography, Demography …)	6
6	End unit assessment		1
	Total		**21**

Lesson 1: Bivariate data, scatter diagram and types of correlation

a) Learning objectives

To define bivariate statistics data, plot the scatter diagram and differentiate types of correlation

b) Teaching resources

Student's book and other Reference books to facilitate research, Mathematical set, calculator, manila paper, markers, pens, pencils...

c) Prerequisites/Revision/Introduction

Students will learn better in this lesson if they refer to descriptive statistics (Unit 6,Year 1) and to Arithmetic (Unit 1,Year 1).

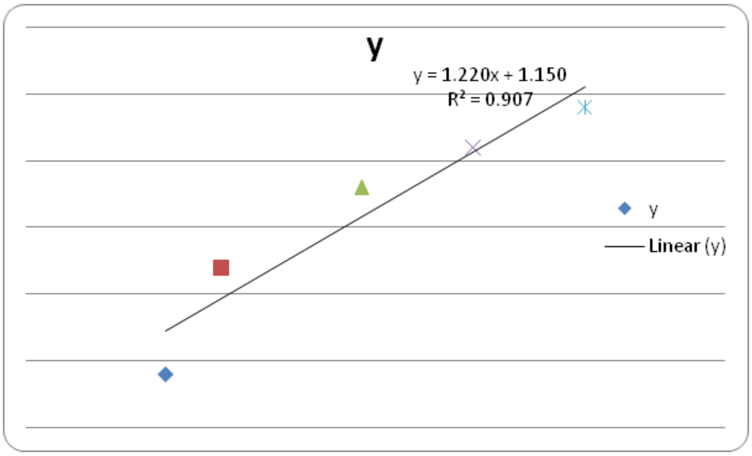
d) Learning activities:

- Invite student-teachers to work in groups and do the activity 4.1 found in their Mathematics Student books;
- Move around in the class for facilitating students where necessary and give more clarification on eventual challenges they may face during their work; Verify and identify groups with different working steps;
- Invite one member from each group with different working steps to present their work where they must explain the working steps;
- As a tutor, harmonize the findings from presentation and guide them to decide whether they can guess if there is a relationship between two variables;
- Use different probing questions and guide them to explore the content and examples given in the student's book and lead them to discover how to define bivariate statistics data, plot the scatter diagram and to distinguish different types of correlation between two variables.

- After this step, guide students to do the application activity 4.1 and evaluate whether lesson objectives were achieved.

Answers for activity 4.1

a)



b) As time is increasing the chemical reaction also is increasing.

Answers for application activity 4.1

t	P	t^2	P^2	tP
0.5	125	0.25	15625	62.5
1	113	1	12769	113
1.5	102	2.25	10404	153
2	94	4	8836	188
3	81	9	6561	243
4	83	16	6889	332
5	71	25	5041	355
$\sum_{i=1}^{7} t_i = 17$	$\sum_{i=1}^{7} P_i = 669$	$\sum_{i=1}^{7} t_i^2 = 57.5$	$\sum_{i=1}^{7} P_i^2 = 66125$	$\sum_{i=1}^{7} t_i P_i = 1446.5$

We need the line $P = at + b$

Use the formula

$$\begin{cases}\sum_{i=1}^{7} P_i = a\sum_{i=1}^{7} t_i + bn \\ \sum_{i=1}^{7} t_i P_i = a\sum_{i=1}^{7} t_i^2 + b\sum_{i=1}^{7} t_i\end{cases}$$

We have

$$\begin{cases}669 = 17a + 7b \\ 1446.5 = 57.5a + 17b\end{cases}$$

Solving we have

$$\begin{cases}a = -11 \\ b = 122.3\end{cases}$$

Then $P = -11t + 122.3$

So, Norman's pulse rate 2.5 minutes after stopping the exercise program is estimated to be $P = -11(2.5) + 122.3$ or 94.8.

Lesson 2: Covariance

a) Learning objectives

To explain and determine the covariance for bivariate statistics data.

b) Teaching resources

Student's book and other Reference books to facilitate research, Mathematical set, calculator, manila paper, markers, pens, pencils...

c) Prerequisites/Revision/Introduction

Students will learn better in this lesson if they refer to descriptive statistics (Unit 6,Year 1) , and to arithmetic (Unit 1,Year 1) and on lesson 1 of this Unit.

d) Learning activities:

- Invite student-teachers to work in groups and do the activity 4.2 found in their Mathematics Student books;
- Move around in the class for facilitating students where necessary and give more clarification on eventual challenges they may face during their work; Verify and identify groups with different working steps;
- Invite one member from each group with different working steps to

present their work;

- As a tutor, harmonize the findings from presentation and guide students to organise a table of values and to decide how to use these values;
- Use different probing questions and guide them to explore the content and examples given in the student's book and lead them to discover how to determine and explain the covariance for bivariate statistics data.
- After this step, guide students to do the application activity 4.2 and evaluate whether lesson objectives were achieved.

Answers for activity 4.2

x	y	$x-\overline{x}$	$y-\overline{y}$	$\left(x-\overline{x}\right)\left(y-\overline{y}\right)$
3	6	-1.3	-2.6	-3.9
5	9	0.7	0.4	1.1
7	12	2.7	3.4	6.1
3	10	-1.3	1.4	0.1
2	7	-2.3	-1.6	-3.9
6	8	1.7	-0.6	1.1
$\sum_{i=1}^{6} x_i = 26$	$\sum_{i=1}^{6} y_i = 52$			$\sum_{i=1}^{6}\left(x_i-\overline{x}\right)\left(y_i-\overline{y}\right)=0.6$
$\overline{x}=4.3$	$\overline{y}=8.6$			

1) If you divide by total frequency you get **variance**
2) If you divide by total frequency you get **covariance**

Answers for application activity 4.2

1) $\operatorname{cov}\left(x,y\right)=\frac{71}{12}$

2) $\operatorname{cov}\left(x,y\right)=98.75$

Lesson 3: Coefficient of correlation

a) Learning objectives

To define and calculate the Coefficient of correlation for Bivariate statistics data.

b) Teaching resources

Learner's book and other Reference books to facilitate research, Mathematical set, calculator, manila paper, markers, pens, pencils...

c) Prerequisites/Revision/Introduction

Students will learn better in this lesson if they refer to lesson 1 & lesson 2 of this Unit.

d) Learning activities:

- Invite student-teachers to work in groups and do the activity 4.3 found in their Mathematics books;
- Move around in the class for facilitating students where necessary and give more clarification on eventual challenges they may face during their work; Verify and identify groups with different working steps;
- Invite one member from each group with different working steps to present their work;
- As a tutor, harmonize the findings from presentation;
- Use different probing questions and guide them to explore the content and examples given in the student's book and lead them to discover how to define and calculate the Coefficient of correlation for Bivariate statistics data.
- Guide students to differentiate the spearman's coefficient of correlation and **Pearson's coefficient of correlation** and when to use each one.
- After this step, guide students to do the application activity 4.3 and evaluate whether lesson objectives were achieved.

Answers for activity 4.3

1) $\sigma_x = 1.8,\ \sigma_y = 1.97$

2) $\operatorname{cov}(x, y) = \dfrac{41}{18}$

3) $\dfrac{\operatorname{cov}(x, y)}{\sigma_x \sigma_y} = 0.64$

Answers for application activity 4.3

1) $r = 0.94$. As the correlation coefficient is very close to 1, the correlation is very strong.

2) $r = -0.26$. As the correlation coefficient is very close to zero, the correlation is very weak.

3) $\sigma = 0.14$. There is a week positive correlation between the English and Mathematics rankings.

Lesson 4: Regression line

a) Learning objectives

To define the regression line and establish its equation

b) Teaching resources

Student's book and other reference books to facilitate research, Mathematical set, calculator, manila paper, markers, pens, pencils...

c) Prerequisites/Revision/Introduction

Students will learn better in this lesson if they refer to the lesson 1 , lesson 2 and lesson 3 of this Unit.

d) Learning activities:

- Invite student-teachers to work in groups and do the activity 4.4 found in their Mathematics books;
- Move around in the class for facilitating students where necessary and give more clarification on eventual challenges they may face during their work; Verify and identify groups with different working steps;
- Invite one member from each group with different working steps to present their work where they must explain the working steps;
- As a tutor, harmonize the findings from presentation and guide students to use covariance and the predetermined means to establish the equation of a regression line;
- Use different probing questions and guide them to explore the content and examples given in the student's book and lead them to discover how to define the equation of a regression line, establish its equation and appreciate the importance of using regression line to interpret data;
- After this step, guide students to do the application activity 4.4 and evaluate whether lesson objectives were achieved.

Answers for activity 4.4

1) $D_b^{/} = 2\sum_{i=1}^{k}(y_i - ax_i - b)(-1)$ or $D_b^{/} = -2\sum_{i=1}^{k}(y_i - ax_i - b)$

2) $\sum_{i=1}^{k}(y_i - ax_i - b) = 0$ or $\sum_{i=1}^{k} y_i - \sum_{i=1}^{k} ax_i - \sum_{i=1}^{k} b = 0$ or $\sum_{i=1}^{k} b = \sum_{i=1}^{k} y_i - \sum_{i=1}^{k} ax_i$

Dividing both sides by n gives

$$\frac{1}{n}\sum_{i=1}^{k} b = \frac{1}{n}\sum_{i=1}^{k} y_i - \frac{1}{n}\sum_{i=1}^{k} ax_i \text{ or } \frac{b}{n}\sum_{i=1}^{k} 1 = \frac{1}{n}\sum_{i=1}^{k} y_i - \frac{a}{n}\sum_{i=1}^{k} x_i \text{ or } b = \overline{y} - a\overline{x}$$

3) $\sum_{i=1}^{k}(y_i - ax_i - b)^2 = \sum_{i=1}^{k}\left(y_i - ax_i - \overline{y} + a\overline{x}\right)^2$

Or

$$\sum_{i=1}^{k}(y_i - ax_i - b)^2 = \sum_{i=1}^{k}\left[\left(y_i - \overline{y}\right) - a\left(x_i - \overline{x}\right)\right]^2$$

Differentiation with respect to a and equating to zero:

$$\sum_{i=1}^{k} 2\left[\left(y_i - \overline{y}\right) - a\left(x_i - \overline{x}\right)\right]\left[-\left(x_i - \overline{x}\right)\right] = 0$$

$$-2\sum_{i=1}^{k}\left[\left(y_i - \overline{y}\right) - a\left(x_i - \overline{x}\right)\right]\left(x_i - \overline{x}\right) = 0$$

$$\Leftrightarrow \sum_{i=1}^{k}\left[\left(y_i - \overline{y}\right) - a\left(x_i - \overline{x}\right)\right]\left(x_i - \overline{x}\right) = 0$$

$$\Leftrightarrow \sum_{i=1}^{k}\left[\left(x_i - \overline{x}\right)\left(y_i - \overline{y}\right) - a\left(x_i - \overline{x}\right)^2\right] = 0$$

$$\Leftrightarrow \sum_{i=1}^{k}\left(x_i - \overline{x}\right)\left(y_i - \overline{y}\right) - \sum_{i=1}^{k} a\left(x_i - \overline{x}\right)^2 = 0$$

$$\Leftrightarrow \sum_{i=1}^{k} a\left(x_i - \overline{x}\right)^2 = \sum_{i=1}^{k}\left(x_i - \overline{x}\right)\left(y_i - \overline{y}\right)$$

$$\Leftrightarrow a\sum_{i=1}^{k}\left(x_i-\overline{x}\right)^2=\sum_{i=1}^{k}\left(x_i-\overline{x}\right)\left(y_i-\overline{y}\right)$$

Dividing both sides by n gives

$$\Leftrightarrow \frac{a}{n}\sum_{i=1}^{k}\left(x_i-\overline{x}\right)^2=\frac{1}{n}\sum_{i=1}^{k}\left(x_i-\overline{x}\right)\left(y_i-\overline{y}\right)$$

$$\Rightarrow a=\frac{\frac{1}{n}\sum_{i=1}^{k}\left(x_i-\overline{x}\right)\left(y_i-\overline{y}\right)}{\frac{1}{n}\sum_{i=1}^{k}\left(x_i-\overline{x}\right)^2}$$

4) The variance for variable x is $\sigma_x^2=\frac{1}{n}\sum_{i=1}^{k}\left(x_i-\overline{x}\right)^2$ and the variance for variable y is $\sigma_y^2=\frac{1}{n}\sum_{i=1}^{k}\left(y_i-\overline{y}\right)^2$ and the covariance of these two variables is

$$\operatorname{cov}(x,y)=\frac{1}{n}\sum_{i=1}^{k}\left(x_i-\overline{x}\right)\left(y_i-\overline{y}\right)$$

Then $a=\dfrac{\operatorname{cov}(x,y)}{\sigma_x^2}$

5) Now, we have that the regression line y on x is $y=ax+b$, where

$$\begin{cases} a=\dfrac{\operatorname{cov}(x,y)}{\sigma_x^2} \\ b=\overline{y}-a\overline{x} \end{cases}$$

Or

$$y=\frac{\operatorname{cov}(x,y)}{\sigma_x^2}x+\left(\overline{y}-\frac{\operatorname{cov}(x,y)}{\sigma_x^2}\overline{x}\right)$$

Answers for application activity 4.4

1) a) $y = 0.19x - 8.098$ b) $y = 4.06$

2) $x = -5.6y + 163.3$, $y = -0.06x + 21.8$

Lesson 5: Interpretation of statistical data

a) Learning objectives

To analyze, interpret and predict bivariate statistical data from various areas (Business, Geography, Demography …)

b) Teaching resources

Student's book and other Reference books to facilitate research, Mathematical set, calculator, manila paper, markers, pens, pencils...

c) Prerequisites/Revision/Introduction

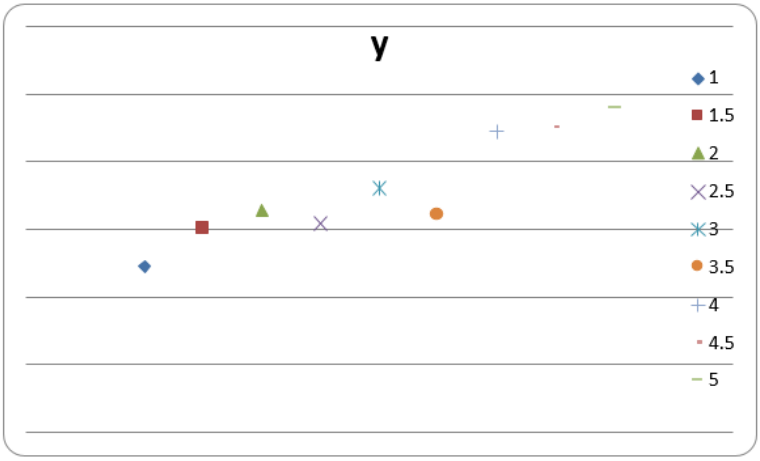
Students will learn better in this lesson if they refer to the lesson 1, lesson 2, lesson 3 and lesson 4 of this Unit.

d) Learning activities:

- Invite student-teachers to work in groups and do the activity 4.5 found in their Mathematics books;
- Visit each group for facilitating students where necessary and give more clarification on eventual challenges they may face during their work;
- Invite one member from each group to present their work;
- As a tutor, harmonize the findings from presentation and guide students to explore real life situations where bivariate statistics is applied;
- Use different probing questions and guide them to explore the content and examples given in the student's book and lead them to discover different real life situations where bivariate statistics is applied;
- After this step, guide students to do the application activity 4.5 and evaluate whether lesson objectives were achieved.

Answers for application activity 4.5

a) Scatter diagram



b) Sample F was damaged.

c) $\bar{x}=\frac{\sum x}{n}=\frac{23.5}{8}=2.9375$ and $\bar{y}=\frac{\sum y}{n}=\frac{584}{8}=73$

To calculate r: $s_{xy}=\frac{1}{n}\sum xy-\bar{x}\,\bar{y}=\frac{1}{8}\times 1883-2.9375\times 73=20.9375$

$$s_{xx}=\frac{1}{n}\sum x^2-\left(\bar{x}\right)^2=\frac{1}{8}\times 83.75-(2.9375)^2=1.839...$$

$$s_{yy}=\frac{1}{n}\sum y^2-\left(\bar{y}\right)^2=\frac{1}{8}\times 44622-(73)^2=248.75$$

$$r=\frac{s_{xy}}{s_x s_y}=\frac{20.9375}{\sqrt{1.839...}\sqrt{248.75}}=0.9787...$$, Then, $r=0.98\,(2\,s.f.)$

d) Yes it is sensible to conclude that x *and* y are related. Since $r=0.98\,(2\,s.f.)$ is very closed to 1, it would appear to indicate a very strong position linear correlation.

e) For the regression line $y=a+bx, a=\bar{y}-b\bar{x}$ and $b=\frac{s_{xy}}{s_{xx}}=\frac{20.9375}{1.839...}=11.38.$;

$b=\frac{s_{xy}}{s_{xx}}=\frac{20.9375}{1.839...}=11.38...$ Then, $a=\bar{y}-b\bar{x}=73-11.38...\times 2.9375=39.57...$

$y=39.6+11.4x\ (3\,s.f.)$

f) When $x=3.5, y=38.57...+11.38...\times 3.5=79\,(2\,s.f.)$ The constrast index would have been 79.

g) No, it would not be sensible to use the regression equation when $x=0$ since this is outside the range of data. Extrapolating outside the data is unreliable.

4.6. Unit summary

1) **Bivariate** or **double series** includes technique of analyzing data in two variables.

2) If each point (x, y) of the data is plotted in an x, y *coordinate* plane, we say that we have **the *scatter plot or Scatter diagram***

3) If $x-coordonates$ increases as $y-coordonates$ increases also; We say that *x* and *y* have a ***positive correlation.*** When *y* tends to decrease as *x increases*, then *x* and *y* have a ***negative correlation***.

4) The **covariance of variables *x* and *y*** is a measure of how these two variables change together.

Ifcovarianceiszerothevariablesaresaidtobe**uncorrelated,**meansthatthereis no linear relationship between them. Then, $\text{cov}(x,y)=\frac{1}{n}\sum_{i=1}^{k} f_i\left(x_i-\bar{x}\right)\left(y_i-\bar{y}\right)$

5) The **Pearson's coefficient of correlation** (or **Product moment coefficient of correlation** or simply **coefficient of correlation**), denoted by *r*, is a measure of the strength of linear relationship between two variables.

The coefficient of correlation between two variables *x* and *y* is given by

$$r=\frac{\text{cov}(x,y)}{\sigma_x\sigma_y}$$

Where, $\text{cov}(x,y)$ is covariance of x and y

σ_x is the standard deviation for x

σ_y is the standard deviation for y

6) We use the regression line to **predict** a value of *y* for any given value of *x* and vice versa. The "best" line would make the best predictions: the observed *y*-values should stray as little as possible from the line. This straight line is the regression line from which we can adjust its algebraic expressions and it is written as $y = ax + b$, where a is the gradient and c is the y-intercept.

   The regression line y on x is written as $y = \frac{\operatorname{cov}(x,y)}{\sigma_x^2} x + \left(\bar{y} - \frac{\operatorname{cov}(x,y)}{\sigma_x^2} \bar{x} \right)$

   We may write

$$L_{y/x} \equiv y - \bar{y} = \frac{\operatorname{cov}(x,y)}{\sigma_x^2} \left(x - \bar{x} \right)$$

4.7. Additional information for the tutor

Make a research on the difference between the spearman's coefficient of correlation and the Pearson's coefficient of correlation.

SCC is used when data are ordinal and ranked variables. The PCC is used for numerical data.

In addition, the tutor has to emphasize how to analyze, interpret and predict bi-variate statistical data using regression line and coefficient of correlation from various areas (Business, Geography, Demography …) observed in newspapers or from national statistics related to the people's development.

4.8. End Unit assessment

1) $y = 0.611x + 10.5, x = 1.478y - 1.143, y = 28.83$

2) $y = -8 + 1.2x$

3) $\bar{x} = 13, \bar{y} = 17, y = 0.8x + 6.6, x = 0.45y - 5.35, r = 0.6, \sigma_y = 4$

4) 0.26

5) a) i. -0.976 ii. $-0.292 (or\ 0.292)$

   b) The transport manager's order is more profitable for the seller, saleswomen is unlikely to try to dissuade.

   c) i. No, maximum value is 1

   ii. Yes, higher performing cars generally do less mileage to the gallon.

iii. No, the higher the engine capacity, the dearer the car.

d) When only the ranking is known; when relationship is non-linear.

4.9. Additional activities

4.9.1 Remedial activities

1) For a given set of data it is known that $\bar{x}=10 \, and \, \bar{y}=4$. The gradient of the regression line y on x is 0.6. Find the equation of this regression line and estimate y when $x=12$

Solution:

The equation of the regression line is $y=a+bx, where \, b=0.6$;

Then $y=a+0.6x$

The regression line goes through $\left(\bar{x},\bar{y}\right)$, so $\bar{y}=a+0.6\bar{x}$

$4=a+0.6\times10 \Rightarrow a=-2$; Thus, the equation of the regression line is

$y=-2+0.06x$; When $x=12, \, y=-2+0.6\times12=5.2$

4.9.2 Consolidation activities

1) Find the regression line of x on y if the line goes through $(1,4)$ and has gradient 2.

Solution:

Equation of regression line x on y is $x=c+dy$; rearranging

$$dy=x-c \Leftrightarrow y=\frac{1}{d}x-\frac{c}{d}$$

Gradient $=\frac{1}{d}$, then $2=\frac{1}{d} \Leftrightarrow d=0.5$; So $x=c+0.5y$

You are given that $(1,4)$ lies on the line $1=c+0.5\times4 \Leftrightarrow c=-1$.

The equation of regression line x on y is $x=-1+0.5y$

4.9.3 Extended activities

1) A student found the following data for the female life expectancy, x years, and the Gross Domestic Production (GDP) per head, $\$y$, in six countries in South Asia in 1988.

Country	x	Y
Afghanistan	42	143
Bangladesh	50	179
Bhutan	47	197
India	58	335
Pakistan	57	384
Sri Lanka	73	423

$$\left[n=6, \sum x=327, \sum y=1661, \sum x^2=18415, \sum y^2=529909, \sum xy=96412\right]$$

a) It is required to estimate the value of x for Nepal, where the value of y was 160. (i) Find the equation of a suitable line of regression. Simplify your answer as far as possible, giving the constant s correct to three significant figures. (ii)Use your equation to obtain the required estimate.

b) Use your equation to estimate the value of x for North Korea, where the value of y was 858. Comment on your answer.

Solution:

a) Neither variable has been controlled in the given data and since you are required to estimate the life expectancy, x years, When the Gross Domestic Product per head, \$y is \$160, it is sensible to use the regression line of x on y.

The least squares regression line of x on y. Has the equation $x = c + dy$

Where $c = \bar{x} - d\,\bar{y}$ and $d = \dfrac{s_{xy}}{s_{yy}}$. $\bar{x} = \dfrac{\sum x}{n} = \dfrac{327}{6}$ and $\bar{y} = \dfrac{\sum y}{n} = \dfrac{1661}{6} =$,

Then $s_{xy} = \dfrac{1}{n}\sum xy - \bar{x}\,\bar{y} = \dfrac{1}{6}\times 96412 - \dfrac{327}{6}\times\dfrac{1661}{6} = 981.25$

$$s_{xy} = \frac{1}{n}\sum y^2 - \left(\bar{y}\right)^2 = \frac{1}{6}\times 529909 - \left(\frac{1661}{6}\right)^2 = 11681.47\ldots$$

$$d = \frac{s_{xy}}{s_{yy}} = \frac{981.25}{11681.47} = 0.084000\ldots$$

$$c = \bar{x} - d\,\bar{y} \Leftrightarrow c = \frac{327}{6} - 0.08400\ldots\times\frac{1661}{6} = 31.24\ldots$$

The equation of regression line of x on y is $x = 31.2 + 0.084y\,(3s.f.)$

ii) When $y = 160,\ x = 31.2 + 0.0840 \times 160 = 45\ (2s.f.)$

The estimated value of the life expectancy in Nepal is 45 years.

b) From the equation, when $y = 858$,

$y = 160,\ x = 31.2 + 0.0840 \times 858 = 103\ (3s.f.)$

This would give the life expectancy in North Korea as 103 years, which is clearly not sensible. The value of $y = 858$ is a long way outside the range of the data, and should not be used to estimate a value of x.

UNIT 5

CONDITIONAL PROBABILITY AND BAYES THEOREM

5.1. Key unit Competence:

Apply rules of probability to solve problems related to dependent and independent events.

5.2. Prerequisites

Student-teachers will perform well in this unit if they make a short revision on the elementary probability learnt in Year 1 unit 7.

5.3. Cross-cutting issues to be addressed

Inclusive education (promote education for all while teaching)

Peace and value Education (respect others' view and thoughts during class discussions)

Gender (provide equal opportunity to boys and girls in the lesson)

5.4. Guidance on introductory activity

- Invite student-teachers to work in groups and give them instructions on how they can do the introductory activity 5.0 found in unit 5 of student's book;
- Guide students to read and analyse the questions insisting on the analysis of the given data and apply rules of probability to solve problems related to dependent and independent events.
- Invite some group members to present groups' findings, then try to harmonize their answers on solving problems related to dependent and independent events ;
- Basing on student-teachers' experience, prior knowledge and abilities shown in answering the questions for this activity, use different questions to facilitate them to give their predictions and ensure that you arouse their curiosity on what is going to be leant in this unit.

Answers for introductory activity

1) We say the probability of A occurring is $\frac{92}{100} = 0.92$ and the probability of B occurring is $\frac{8}{100} = 0.08$. The probability is thus a measure of the likelihood of the occurrence of a particular outcome. We write P(A) = probability of A occurring = 0.92

   P (B) = probability of B occurring = 0.08

2) a) The probability of selecting 2 white chalks is $\frac{2}{7}$

   b) The probability of selecting 3 white and 2 black chalks is $\frac{3}{7} \times \frac{2}{7}$

5.5. List of lessons

#	Lesson title	Learning objectives	Number of periods
0	Introductory activity	To arouse the curiosity of student-teacher on the content of unit 5.	1
1	Tree diagram	Determine the probability of some events using tree diagrams	3
2	Addition rules for probability	Determine the probability for mutually or non-mutually exclusive events	3
3	Independent events	To define Independent events and determine the probability for independent events	6
4	Dependent events	To define Dependent events and determine the probability for dependent events	6
5	Conditional probability	To Compute the probability of an event B occurring when event A has already taken place.	4
6	Bayes theorem and its Applications	Extend the Conditional probability to Bayes theorem.	2
7	End unit assessment		2
	Total		**27**

Lesson 1: Tree diagram

a) Learning objectives

To determine the probability of some events using tree diagrams.

b) Teaching resources

Learner's book and other Reference books to facilitate research, calculator, manila paper, markers, pens, pencils...

c) Prerequisites/Revision/Introduction

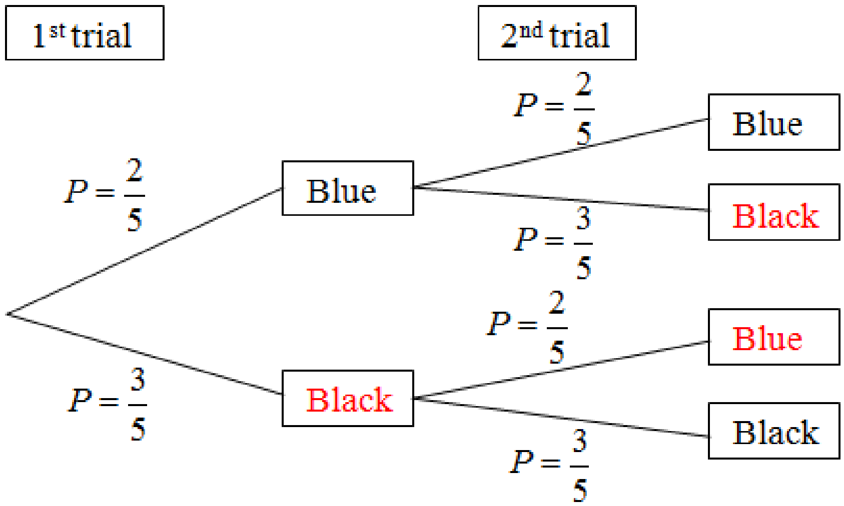
Students will learn better in this lesson if they have a good background on the elementary probability learnt in Year 1 unit 7.

d) Learning activities:

- Invite student-teachers to work in groups and do the activity 5.1 found in their Mathematics Student books;
- Move around in the class for facilitating students where necessary and give more clarification on eventual challenges they may face during their work; Verify and identify groups with different working steps;
- Invite one member from each group with different working steps to present their work where they must explain the working steps;
- As a tutor, harmonize the findings from presentation on the probability of some events using tree diagrams;
- Use different probing questions and guide them to explore the content and examples given in the student's book and lead them to discover how to determine the probability of some events using tree diagrams.
- After this step, guide students to do the application activity 5.1 and evaluate whether lesson objectives were achieved.

Answers for activity 5.1

1) Probability of choosing a blue pen is $\frac{4}{10}=\frac{2}{5}$ and probability of choosing a black pen is $\frac{6}{10}=\frac{3}{5}$.
2) Probabilities on the second trial are equal to the probabilities on the first trial since after the 1st trial the pen is replaced in the box.
3) Complete figure



Answers for application activity 5.1

1) $\frac{1}{2}\times\frac{1}{2}\times\frac{1}{2}=\frac{1}{8}$

2) a) $P(3\,\text{boys})=\frac{10}{16}\times\frac{9}{15}\times\frac{8}{14}=0.214$

b) $P(2\text{ boys and }1\text{ girl})=\frac{10}{16}\times\frac{9}{15}\times\frac{6}{14}+\frac{10}{16}\times\frac{6}{15}\times\frac{9}{14}+\frac{6}{16}\times\frac{10}{15}\times\frac{9}{14}=0.482$

c) $P(2\text{ girls and }1\text{ boy})=\frac{10}{16}\times\frac{6}{15}\times\frac{5}{14}+\frac{6}{16}\times\frac{10}{15}\times\frac{5}{14}+\frac{6}{16}\times\frac{5}{15}\times\frac{10}{14}=0.268$

d) $P(3\,\text{girls})=\frac{6}{16}\times\frac{5}{15}\times\frac{4}{14}=0.0357$

3) a) $\frac{1}{21}$ b) $\frac{10}{21}$ c) $\frac{11}{21}$

4) a) $\frac{1}{816}$ b) $\frac{7}{102}$ c) $\frac{7}{34}$

Lesson 2: The Addition law of probability

a) Learning objectives

To determine the probability for mutually or non-mutually exclusive events.

b) Teaching resources

Learner's book and other Reference books to facilitate research, calculator, manila paper, markers, pens, pencils...

c) Prerequisites/Revision/Introduction

Students will learn better in this lesson if they have a good background on the elementary probability learnt in Year 1 unit 7 and on Tree diagram (lesson 1 of this unit).

d) Learning activities:

- Invite student-teachers to work in groups and do the activity 5.2 found in their Mathematics Student books;
- Move around in the class for facilitating students where necessary and give more clarification on eventual challenges they may face during their work; Verify and identify groups with different working steps;
- Invite one member from each group with different working steps to present their work where they must explain the working steps;
- As a tutor, harmonize the findings from presentation on the probability for mutually or non-mutually exclusive events;
- Use different probing questions and guide them to explore the content and examples given in the student's book and lead them to discover how to determine the probability for mutually or non-mutually exclusive events
- After this step, guide students to do the application activity 5.2 and evaluate whether lesson objectives were achieved.

Answers for activity 5.2

On average 18 out of 100 components are top quality and 65 out of 100 are standard quality. So, 83 out of 100 are either top quality or standard quality.

Hence the probability that a component is either top quality or standard quality is 0.83. The solution may be expressed more formally as follows.

Let A be the event that a component is top quality.

Let B be the event that a component is standard quality.

$P(A) = 0.18$; $P(B) = 0.65$

Then,

$$P(A \cup B) = 0.18 + 0.65 = 0.83$$

Note that in this activity: $P(A \cup B) = P(A) + P(B)$

Answers for application activity 5.2

The sample space S consists of the eight blood types, and the percents, when given as decimals, are the probability assignments for each element of the sample space.

For example, the simple event "O positive" is assigned the probability 0.39.

Let E be the event that a randomly selected person in the United States has a blood

Type that is Rh-negative. We seek $P(E)$. Since E is the union of the simple events:

$\{O\,negative\},\{A\,negative\},\{AB\,negative\}$ *and* $\{B\,negative\}$, we have

$$E=\{O\,negative, A\,negative, AB\,negative, B\,negative\}$$

The probabilities assigned each of these simple events are: $P(O\,negative)=0.09$

$P(A\,negative)=0.06$; $P(B\,negative)=0.02$; $P(AB\,negative)=0.01$

Then,

$$P(E)=P\{O\,negative\}+P\{A\,negative\}+P\{AB\,negative\}+P\{B\,negative\}$$

$$=0.09+0.06+0.01+0.02=0.18$$

The probability is 0.18, or 18%, that a randomly selected person in the United States has Rh-negative blood.

b) Define the events E and F as: *E:* person selected is type O; *F:* person selected is type A

Then; $E=\{O\,positive, O\,negative\}$; $F=\{A\,positive, A\,negative\}$

Now; $P(E)=P(O\,positive)+P(O\,negative)=0.39+0.09=0.48$

$$P(\mathrm{F})=P(\mathrm{A}\,positive)+P(\mathrm{A}\,negative)=0.31+0.06=0.37$$

Since $E\cap F=\phi$, the events E and F are mutually exclusive. Based on Formula $P(\mathrm{E}\cup\mathrm{F})=\mathrm{P(E)}+\mathrm{P(F)}-\mathrm{P(E}\cap\mathrm{F)}$, the probability of E or F, namely $P(\mathrm{E}\cup\mathrm{F})$, is $P(\mathrm{E}\cup\mathrm{F})=\mathrm{P(E)}+\mathrm{P(F)}=0.48+0.37=0.85$ Because $\mathrm{P(E}\cap\mathrm{F)}=0$

So, 85% of the U.S. population is either type O or type A.

Lesson 3: Independent events

a) Learning objectives

To define Independent events and determine the probability for independent events.

b) Teaching resources

Learner’s book and other Reference books to facilitate research, calculator, manila paper, markers, pens, pencils...

c) Prerequisites/Revision/Introduction

Students will learn better in this lesson if they have a good background on the elementary probability learnt in Year 1 unit 7 , on Tree diagram and on addition law of probability (lesson 1 & r lesson 2of this unit).

d) Learning activities:

- Invite student-teachers to work in groups and do the activity 5.3 found in their Mathematics Student books;
- Move around in the class for facilitating students where necessary and give more clarification on eventual challenges they may face during their work; Verify and identify groups with different working steps;
- Invite one member from each group with different working steps to present their work where they must explain the working steps;
- As a tutor, harmonize the findings from presentation on Independent events and determine the probability for independent events ;
- Use different probing questions and guide them to explore the content and examples given in the student’s book and lead them to discover how to define Independent events and determine the probability for independent events.
- After this step, guide students to do the application activity 5.3 and evaluate whether lesson objectives were achieved.

Answers for activity 5.3

The occurrence of event B does not affected by occurrence of event A because after the first trial the pen is replaced in the box. It means that the sample space does not change.

Answers for application activity 5.3

1) $P(\text{red and red}) = \frac{1}{5} \times \frac{1}{5} = \frac{1}{25}$

2) $P(\text{head and 3}) = \frac{1}{2} \times \frac{1}{6} = \frac{1}{12}$

3) a) $\frac{1}{35}$ b) $\frac{2}{7}$ c) $\frac{24}{35}$

Lesson 4: Dependent events

a) Learning objectives

To define Dependent events and determine the probability for dependent events.

b) Teaching resources

Learner's book and other Reference books to facilitate research, calculator, manila paper, markers, pens, pencils...

c) Prerequisites/Revision/Introduction

Students will learn better in this lesson if they have a good background on the elementary probability learnt in Year 1 unit 7, on Tree diagram, on addition law of probability and on independent events (lesson 1, lesson 2 and lesson 3 of this unit).

d) Learning activities:

- Invite student-teachers to work in groups and do the activity 5.4 found in their Mathematics Student books;
- Move around in the class for facilitating students where necessary and give more clarification on eventual challenges they may face during their work; Verify and identify groups with different working steps;
- Invite one member from each group with different working steps to present their work where they must explain the working steps;
- As a tutor, harmonize the findings from presentation on dependent events;
- Use different probing questions and guide them to explore the content and examples given in the student's book and lead them to discover how to define Dependent events and determine the probability for dependent events.
- After this step, guide students to do the application activity 5.4 and evaluate whether lesson objectives were achieved.

Answers for activity 5.4

a) Sample space for the first drawing is $\Omega = 52$, But for the second drawing the sample space is $\Omega = 51$. (b) The outcomes of the first draw has affected the outcome of the second i.e. the outcome of the second dependent on the first drawing.

Answers for application activity 5.4

$$P\left(H \text{ and } A\right) = P(H) \times P(A \setminus H) = 0.53 \times 0.27 = 0.1431$$

Lesson 5: Conditional probability

a) Learning objectives

To Compute the probability of an event B occurring when event A has already taken place.

b) Teaching resources

Learner's book and other Reference books to facilitate research, calculator, manila paper, markers, pens, pencils...

c) Prerequisites/Revision/Introduction

Students will learn better in this lesson if they have a good background on the elementary probability learnt in Year 1 unit 7 , on Tree diagram , on addition law of probability , on independent events and on Dependent events (lesson 1 , lesson 2 , lesson 3 and on lesson 4 of this unit).

d) Learning activities:

- Invite student-teachers to work in groups and do the activity 5.5 found in their Mathematics Student books;
- Move around in the class for facilitating students where necessary and give more clarification on eventual challenges they may face during their work; Verify and identify groups with different working steps;
- Invite one member from each group with different working steps to present their work where they must explain the working steps;
- As a tutor, harmonize the findings from presentation on conditional probability;
- Use different probing questions and guide them to explore the content and examples given in the student's book and lead them to discover how

to Compute the probability of an event B occurring when event A has already taken place

- After this step, guide students to do the application activity 5.5 and evaluate whether lesson objectives were achieved.

Answers for activity 5.5

The occurrence of event B is affected by occurrence of event A because after the first trial the pen is not replaced in the box. It means that the sample space will be changed for the second trial.

Lesson 6: Bayes theorem and its applications

a) Learning objectives

To Extend the Conditional probability to Bayes theorem.

b) Teaching resources

Learner's book and other Reference books to facilitate research, calculator, manila paper, markers, pens, pencils...

c) Prerequisites/Revision/Introduction

Students will learn better in this lesson if they have a good background on Conditional probability

d) Learning activities:

- Invite student-teachers to work in groups and do the activity 5.6 found in their Mathematics Student books;
- Move around in the class for facilitating students where necessary and give more clarification on eventual challenges they may face during their work; Verify and identify groups with different working steps;
- Invite one member from each group with different working steps to present their work where they must explain the working steps;
- As a tutor, harmonize the findings from presentation on Bayes theorem and its application;
- Use different probing questions and guide them to explore the content and examples given in the student's book and lead them to extend the Conditional probability to Bayes theorem.
- After this step, guide students to do the application activity 5.6 and evaluate whether lesson objectives were achieved.

Answers for activity 5.6

1) $P(A)=P(A|B_1)P(B_1)+P(A|B_2)P(B_2)+P(A|B_3)P(B_3)$

2)

$$P(B_1|A)=\frac{P(B_1\cap A)}{P(A)}=\frac{P(A|B_1)P(B_1)}{P(A|B_1)P(B_1)+P(A|B_2)P(B_2)+P(A|B_3)P(B_3)}$$

$$P(B_2|A)=\frac{P(B_2\cap A)}{P(A)}=\frac{P(A|B_2)P(B_2)}{P(A|B_1)P(B_1)+P(A|B_2)P(B_2)+P(A|B_3)P(B_3)}$$

$$P(B_3|A)=\frac{P(B_3\cap A)}{P(A)}=\frac{P(A|B_3)P(B_3)}{P(A|B_1)P(B_1)+P(A|B_2)P(B_2)+P(A|B_3)P(B_3)}$$

Generally,

$$P(B_i|A)=\frac{P(B_i\cap A)}{P(A)}=\frac{P(A|B_i)P(B_i)}{\sum_{i=1}^{3}P(A|B_i)P(B_i)}$$

Answers for application activity 5.6

1) $P(engineer\,|\,managerial)=\dfrac{0.2\times0.75}{0.2\times0.75+0.2\times0.5+0.6\times0.2}=0.405$

2) $P(No\ accident\,|\,Triggered\ alarm)=\dfrac{0.9\times0.02}{0.1\times0.97+0.9\times0.02}=0.157$

5.6. Unit summary

1) A **tree diagram** is diagram that can be used to show the probabilities of certain outcomes occurring when two or more trials take place in succession. The outcome is written at the end of the branch and the fraction on the branch gives the probability of the outcome occurring. For each trial the number of branches is equal to the number of possible outcomes of that trial. In the diagram there are two possible outcomes, *A* and *B*, of each trial.

2) Events *A* and *B* are said to be **independent** if and only if

   $P(A\cap B)=P(A)\times P(B)$

3) The probability of an event *B* given that event *A* has occurred is called the **conditional probability** of *B* given *A* and is written $P(B|A)$. In

this case $P(B|A)$ is the probability that B occurs considering A as the sample space, and since the subset of A in which B occurs is $A\cap B$, then

$$P(B|A)=\frac{P(B\cap A)}{P(A)}.$$

4) Let $B_1, B_2, B_3, \ldots, B_n$ be incompatible and exhaustive events and A an arbitrary event. The Bayes' formula says that

$$P(B_i|A)=\frac{P(B_i\cap A)}{P(A)}=\frac{P(A|B_i)P(B_i)}{\sum_{i=1}^{n}P(A|B_i)P(B_i)}$$

5.7. Additional information for the tutor

Here the tutor has to provide more examples to emphasize the application of conditional probability in solving real –life problems.

5.8. End Unit assessment

1) 0.15

2) The sample space is $s=\{BBB, BBG, BGB, BGG, GBB, GBG, GGB, GGG\}$ and includes all of the possible outcomes for a family with three children. Determine the reduced sample spaces that satisfy the given conditions that there are exactly 2 boys and that the second child is a boy. The condition that there are exactly 2 boys reduces the sample space to exclude the outcomes where there are 1,3,or no boys. Let x represents the event that there are two boys.

$$x=\{BBG, BGB, GBB\};\ P(x)=\frac{3}{8}$$

The condition that the second child is a boy reduces the sample space to exclude the outcomes where the second child is a girl. Let y represents the the event that the second child is a boy. $y=\{BBB, BBG, GBB, GBG\}$; $P(y)=\frac{4}{8}\,or\,\frac{1}{2}$ $(x\,and\,y)$ is the intersection of x and y. $(x\,and\,y)=\{BBG, GBB\}$, so,

$$P(x\,and\,y)=\frac{2}{8}\,or\,\frac{1}{4}. \text{ Then, } P(x|y)=\frac{P(x\,and\,y)}{P(y)}=\frac{\frac{1}{4}}{\frac{1}{2}}=\frac{1}{4}\times\frac{2}{1}=\frac{1}{2}.$$

The probability that a family with 3 children selected at random will have exactly 2 boys, given that the second is a boy, is $\frac{1}{2}$.

3) Let A represents the event that the number is a multiple of 4. Thus

$$A=\{4,8,12\}.\Rightarrow P(A)=\frac{3}{12}\, or\, \frac{1}{4}.$$

Let B represents the event that the number is even. So

$$B=\{2,4,6,8,10,12\};\ P(B)=\frac{6}{12}=\frac{1}{2}.$$

In this situation, A is a subset of B.

$$P(A\, and\, B)=P(A)=\frac{1}{4};\ P(B)=\frac{1}{2}.$$

Then, $P(A\,|\,B)=\frac{P(A)}{P(B)}=\frac{\frac{1}{4}}{\frac{1}{2}}=\frac{1}{2}$. The probability that a multiple of 4 is rolled, given that the number is even, is $\frac{1}{2}$

4) 0.13

5) $\frac{10}{21}$

6) $\frac{3}{13}$

5.9. Additional activities

5.9.1 Remedial activities

1) Agnes tosses two coins. What is the probability that she has tossed 2 heads, given that she has tossed at least 1 head?

Solution:

Let event A be that the two coins come up heads. Let event B be that there is at least one head.

$P(B)=\frac{3}{4}$ (Three of the four outcomes have at least one head)

$P(A\, and\, B)=\frac{1}{4}$ (One of the four outcomes has two heads)

$$P(\mathrm{A}\mid B)=\frac{P(A\,and\,B)}{P(B)}=\frac{\frac{1}{4}}{\frac{3}{4}}=\frac{1}{4}\times\frac{4}{3}=\frac{1}{3}$$

The probability of tossing two heads, given that at least one toss was a head is $\frac{1}{3}$

5.9.2 Consolidation activities

1) Suppose a population of 1000 people includes 70 accountants and 520 females. There are 40 females who are accountants. A person is chosen at random, find the probability that the person chosen is female.

Solution:

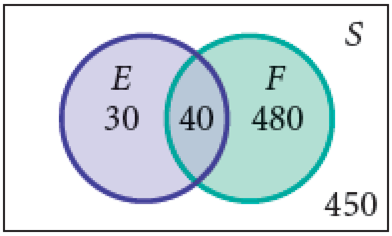
Since there are 520 females, 40 of whom are accountants, the probability that the person is an accountant, given that the person is female, is $\frac{40}{520}$.

That is, the *conditional probability* of the event *E* (accountant) assuming the event *F* (the person chosen isfemale) is $\frac{40}{520}=\frac{1}{13}$

If ***E*** is the event "***A person chosen at random is an accountant***" and ***F*** is the event **"A person chosen at random is female,"** we write

$$P(E\mid F)=\frac{40}{520}=\frac{1}{13}$$

Or we may use Venn diagram as follow:



$$P(E\mid F)=\frac{40}{520}=\frac{n(E\cap F)}{n(F)}$$

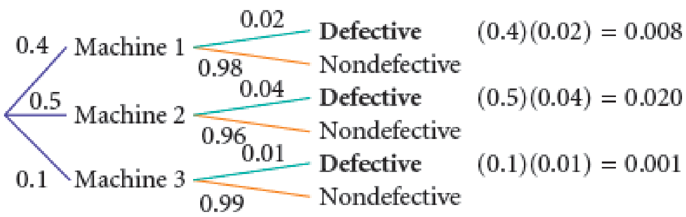
$$P(E|F)=\frac{n(E\cap F)}{n(F)}=\frac{\frac{n(E\cap F)}{n(s)}}{\frac{n(F)}{n(s)}}=\frac{P(E\cap F)}{P(F)}=\frac{40}{520}=\frac{1}{13}$$

5.9.3 Extended activities

1) Three machines, I, II, and III, manufacture 0.4, 0.5, and 0.1 of the total production in a plant respectively. The percentage of defective items produced by I, II, and III is 2%, 4%, and 1%, respectively. For an item chosen at random, what is the probability that it is defective?

Solution:

Using tree diagram to get:



Probability for defective $= 0.008 + 0.020 + 0.001 = 0.029$

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