MATHEMATICS FOR TTCs

STUDENT'S BOOK

YEAR

2

OPTION:

SOCIAL STUDIES EDUCATION (SSE)

FOREWORD

Dear Student,

Rwanda Education Board (REB) is honored to present Year Two Mathematics book for Social Studies Education (SSE) Student Teachers. This book serves as a guide to competence-based teaching and learning to ensure consistency and coherence in the learning of the Mathematics content. The Rwandan educational philosophy is to ensure that you achieve full potential at every level of education which will prepare you to be well integrated in society and exploit employment opportunities.

The government of Rwanda emphasizes the importance of aligning teaching and learning materials with the syllabus to facilitate your learning process. Many factors influence what you learn, how well you learn and the competences you acquire. Those factors include the relevance of the specific content, the quality of teachers' pedagogical approaches, the assessment strategies and the instructional materials available. In this book, we paid special attention to the activities that facilitate the learning process in which you can develop your ideas and make new discoveries during concrete activities carried out individually or with peers.

In competence-based curriculum, learning is considered as a process of active building and developing knowledge and meanings by the learner where concepts are mainly introduced by an activity, situation or scenario that helps the learner to construct knowledge, develop skills and acquire positive attitudes and values.

For efficiency use of this textbook, your role is to:

- Work on given activities which lead to the development of skills;
- Share relevant information with other learners through presentations, discussions, group work and other active learning techniques such as role play, case studies, investigation and research in the library, on internet or outside;
- Participate and take responsibility for your own learning;
- Draw conclusions based on the findings from the learning activities.

To facilitate you in doing activities, the content of this book is self explanatory so that you can easily use it yourself, acquire and assess your competences. The book is made of units as presented in the syllabus. Each unit has the following structure: the key unit competence is given and it is followed by the introductory activity before the development of mathematical concepts that are connected to real world problems or to other sciences.

The development of each concept has the following points:

- It starts by a learning activity: it is a hand on well set activity to be done by students in order to generate the concept to be learnt;
- Main elements of the content to be emphasized;
- Worked examples; and
- Application activities: those are activities to be done by the user to consolidate competences or to assess the achievement of objectives.

Even though the book has some worked examples, you will succeed on the application activities depending on your ways of reading, questioning, thinking and grappling ideas of calculus not by searching for similar-looking worked out examples.

Furthermore, to succeed in Mathematics, you are asked to keep trying; sometimes you will find concepts that need to be worked at before you completely understand. The only way to really grasp such a concept is to think about it and work related problems found in other reference books.

I wish to sincerely express my appreciation to the people who contributed towards the development of this book, particularly, REB staff, UR-CE Lecturers and TTC Tutors for their technical support. A word of gratitude goes to Head Teachers and TTCs principals who availed their staff for various activities.

Any comment or contribution would be welcome to the improvement of this text book for the next edition.

Dr. NDAYAMBAJE Irénée
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UNIT 1

SEQUENCES AND SERIES

Key unit competence: Apply arithmetic and geometric sequences to solve problems in financial mathematics.

1.0 INTRODUCTORY ACTIVITY

Suppose that an insect population is growing in such a way that each new generation is 2 times as large as the previous generation. If there are 126 insects in the first generation, on a piece of paper, write down the number of insects that will be there in second, third, fourth,…n^{th} generation.

How can we name the list of the number of insects for different generations?

1.1 Generalities on sequences

ACTIVITY 1.1

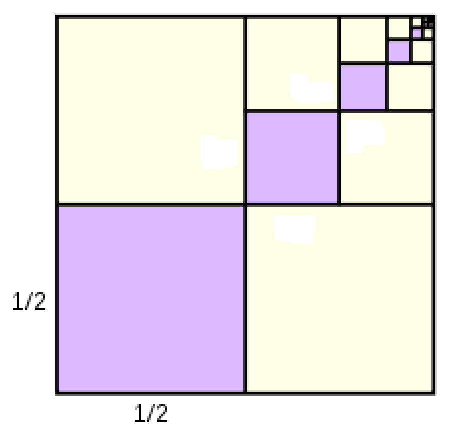
Fold once an A4 paper, what is the fraction that represents the part you are seeing?

Fold it twice, what is the fraction that represents the part you are seeing?

What is the fraction that represents the part you are seeing if you fold it ten times?

What is the fraction that represents the part you are seeing if you fold it n times?

Write a list of the fractions obtained starting from the first until the n^{th} fraction.



CONTENT SUMMARY

Let us consider the following list of numbers: $1,\frac{1}{2},\frac{1}{3},\frac{1}{4},\ldots\frac{1}{n},\ldots$. The terms of this list are compared to the images of the function $f(x)=\frac{1}{x}$. The list never ends, as the ellipsis indicates. The numbers in this ordered list are called the **terms** of the sequence. In dealing with sequences, we usually use subscripted letters, such as u_1 to represent the first term, u_2 for the second term, u_3 for the third term, and so on such as in the sequence $f(n)=u_n=\frac{1}{n}$.

However, in the sequence such as $\{u_n\}: u_n=\sqrt{n-3}$, the first term is u_3 as the previous are not possible, in the sequence $\{u_n\}: u_n=2n-5$, the first term is u_0.

Definition

A sequence is a function whose domain is the set of natural numbers.The terms of a sequence are the range elements of the function.

It is denoted by $u_1,u_2,u_3,\ldots,u_{n-1},u_n$ and shortly $\{u_n\}$. We can also write $\{u_1,u_2,u_3,\ldots,u_{n-1},u_n\}$. The dots are used to suggest that the sequence continues indefinitely, following the obvious pattern.

The numbers $u_1, u_2, u_3, \ldots, u_{n-1}, u_n$ in a sequence are called **terms of the sequence**. The natural number n is called **term number** and value u_n is called a **general term** of a sequence and the term u_1 is the **initial term** or **the first term**.

As a sequence continues indefinitely, it can be denoted as $\{u_n\}_{n=1}^{\infty}$.
The number of terms of a sequence (possibly infinite) is called the **length of the sequence**.

Notice

- Sometimes, the term number, n, starts from 0. In this case terms of a sequence are $u_0, u_1, u_2, \ldots, u_{n-1}, u_n, \ldots$ and this sequence is denoted by $\{u_n\}_{n=0}^{+\infty}$. In this case the initial term is u_0.
- A sequence can be finite, like the sequence $2, 4, 8, 16, \ldots, 256$.

The empty sequence $\{\ \}$ is included in most notions of sequences, but may be excluded depending on the context. Usually a numerical sequence is given by some formula $u_n = f(n)$, permitting to find any term of the sequence by its number n; this formula is called a **general term formula**.

A second way of defining a sequence is to assign a value to the first (or the first few) term(s) and specify the nth term by a formula or equation that involves one or more of the terms preceding it. Sequences defined this way are said to be defined **recursively,** and the rule or formula is called a **recursive formula.**

Example: The sequence $u_1 = 1, u_n = n.u_{n-1}$

Infinite and finite sequences

Consider the sequence of odd numbers less than 11: This is 1, 3,5,7,9. This is a finite sequence as the list is limited and countable. However, the sequence made by all odd numbers is:

$1, 3, 5, 7, 9, \ldots 2n+1, \ldots$ This suggests the definition that an **infinite sequence** is a sequence whose terms are infinite and its domain is the set of positive integers.

Note that it is not always possible to give the numerical sequence by a general term formula; sometimes a sequence is given by description of its terms.

Examples

1) Numerical sequences:

$1, 2, 3, 4, 5, \ldots$ *a sequence of natural numbers;*

2,4,6,8,10,... *a sequence of even numbers;*

$1.4, 1.41, 1.414, 1.4142, \ldots$ *a numerical sequence of approximate, defined more precisely values of* $\sqrt{2}$ *.*

For the last sequence it is impossible to give a general term formula, nevertheless this sequence is described completely.

2) List the first five term of the sequence $\left\{2^n\right\}_{n=1}^{+\infty}$

Solution

Here, we substitute $n = 1, 2, 3, 4, 5, \ldots$ into the formula 2^n. This gives $2^1, 2^2, 2^3, 2^4, 2^5, \ldots$

Or, equivalently, $2, 4, 8, 16, 32, \ldots$

3) Express the following sequences in general notation

a) $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots$

b) $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots$

Solution

a) $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots$

Beginning by comparing terms and term numbers:

Term number	1	2	3	4	...
Term	$\frac{1}{2}$	$\frac{2}{3}$	$\frac{3}{4}$	$\frac{4}{5}$	...

In each term, the numerator is the same as the term number, and the denominator is one greater the term number.

Thus, the n^{th} term is $\dfrac{n}{n+1}$ and the sequence may be written as $\left\{\dfrac{n}{n+1}\right\}_{n=1}^{+\infty}$.

b) $\dfrac{1}{2},\dfrac{1}{4},\dfrac{1}{8},\dfrac{1}{16},\ldots$

Beginning by comparing terms and term numbers:

Term number	1	2	3	4	$\cdots$
Term	$\dfrac{1}{2}$	$\dfrac{1}{4}$	$\dfrac{1}{8}$	$\dfrac{1}{16}$	$\cdots$

Or

Term number	1	2	3	4	$\cdots$
Term	$\dfrac{1}{2^1}$	$\dfrac{1}{2^2}$	$\dfrac{1}{2^3}$	$\dfrac{1}{2^4}$	$\cdots$

In each term, the denominator is equal to 2 powers the term number. We observe that the n^{th} term is $\dfrac{1}{2^n}$ and the sequence may be written as $\left\{\dfrac{1}{2^n}\right\}_{n=1}^{+\infty}$.

4) A sequence is defined by $\{u_n\}:\begin{cases} u_0 = 1 \\ u_{n+1} = 3u_n + 2 \end{cases}$

Determine $u_1,\ u_2\ \ and\ \ u_3$

Solution

Since $u_0 = 1$ and $u_{n+1} = 3u_n + 2$, replace n by $0,1,2$ to obtain u_1, u_2, u_3 respectively.

$$\begin{aligned} n = 0,\quad u_{0+1} = u_1 &= 3u_0 + 2 \\ &= 3\times 1 + 2 \\ &= 5 \end{aligned}$$

$$\begin{aligned} n = 1,\quad u_{1+1} = u_2 &= 3u_1 + 2 \\ &= 3\times 5 + 2 \\ &= 17 \end{aligned}$$

$n=2,\quad u_{2+1}=u_3=3u_2+2$

$$=3\times 17+2$$

$$=53$$

Thus,

$$\begin{cases} u_1=5 \\ u_2=17 \\ u_3=53 \end{cases}$$

APPLICATION ACTIVITY 1.1

1. A sequence is given by $\{u_n\}:\begin{cases} u_0=1 \\ u_n=\dfrac{2n^2}{n^2+1} \end{cases}$

   Determine u_1, u_2 and u_3

2. List the first five terms of the sequence $\left\{\sqrt{n+1}-\sqrt{n}\right\}_{n=1}^{+\infty}$

3. Express the following sequence in general notation

   1, 3, 5, 7, 9, 11, ...

1.2 Convergent or divergent sequences

ACTIVITY 1.2

Discuss the value of each of the following sequences as n approaches to $+\infty$ (plus infinity).

1. $\left\{\dfrac{3n^2-1}{n^2}\right\}$
2. $\left\{n^2\right\}$

A numerical sequence $\{u_n\}$ is said to be **convergent** if the finite limit exists. When the limit does not exist or when it is infinity the sequence is said to be **divergent.**

A number L *is called* **a limit** of a numerical sequence $\{u_n\}$ if $\lim\limits_{n\to\infty} u_n = L$

In other words, Convergent sequence is when $\lim\limits_{n\to\infty} u_n = L$ while divergent sequence is when $\lim\limits_{n\to\infty} u_n = \infty$

Examples

1) Determine whether the sequence $\left\{\dfrac{n}{2n+1}\right\}_{n=1}^{+\infty}$ converges or diverges.

Solution

First we find the limit of this sequence as n tends to infinity

$$\lim_{n\to\infty}\frac{n}{2n+1}=\lim_{n\to\infty}\frac{n}{n\left(2+\frac{1}{n}\right)}=\frac{1}{2}$$

Thus, $\left\{\dfrac{n}{2n+1}\right\}_{n=1}^{+\infty}$ converges to $\dfrac{1}{2}$.

2) Determine whether the sequence $\{8-2n\}_{n=1}^{+\infty}$ *converges or diverges.*

Solution

First we find the limit of this sequence as n tends to infinity

$$\lim_{n\to\infty}(8-2n)=8-2(+\infty)=-\infty$$

Thus, $\{8-2n\}_{n=1}^{+\infty}$ **diverges**.

APPLICATION ACTIVITY 1.2

Which of the sequences converge, and which diverge? Find the limit of each convergent sequence.

1) $\{2+(0.1)^n\}$ 2) $\left\{\dfrac{1-2n}{1+2n}\right\}$ 3) $\left\{\dfrac{1-5n^4}{n^4+8n^3}\right\}$

4) $\{-1^n\}$ 5) $\left\{\dfrac{2n}{\sqrt{3n+1}}\right\}$ 6) $\dfrac{\sqrt{7n^2+2}}{n^3+8}$

1.3 Monotonic sequences

ACTIVITY 1.3

For each of the following sequences, state whether the terms are in ascending or descending order , both or neither order.

1) 1, 2, 3, 4, 5, 6, ...

2) $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots$

3) 1, -1, 1, -1, 1, ...

4) 2, 2, 2, 2, 2, 2, ...

A sequence $\{u_n\}$ is said to be

- **Increasing** or in **ascending** order if $u_1 < u_2 < u_3 < \ldots < u_n < \ldots$
- **Non decreasing** if $u_1 \le u_2 \le u_3 \le \ldots \le u_n \le \ldots$
- **decreasing** or in **descending** order if $u_1 > u_2 > u_3 > \ldots > u_n > \ldots$
- **non increasing** $u_1 \ge u_2 \ge u_3 \ge \ldots \ge u_n \ge \ldots$

A sequence that is either non decreasing or non increasing is called **monotone**, and a sequence that is increasing or decreasing is called **strictly monotone**. Observe that a strictly monotone sequence is monotone, but not conversely.

In order, for a sequence to be **increasing**, all pairs of successive terms, u_n and u_{n+1}, must satisfy $u_n < u_{n+1}$, or equivalently, $u_n - u_{n+1} < 0$.

More generally, monotonic sequences can be classified as follows:

Difference between successive terms	Classification
$u_n - u_{n+1} < 0$	Increasing
$u_n - u_{n+1} > 0$	Decreasing
Otherwise	Non decreasing or Non increasing.

If the terms in the sequence are all positive, then we can divide both sides of the inequality $u_n < u_{n+1}$ by u_n to obtain $1 < \frac{u_{n+1}}{u_n}$ or equivalently $\frac{u_{n+1}}{u_n} > 1$

More, generally, monotonic sequences with positive terms can be classified as follows:

Difference between successive terms	Classification
$\frac{u_{n+1}}{u_n} > 1$	Increasing
$1 > \frac{u_{n+1}}{u_n}$	Decreasing
Otherwise	Non decreasing or Non increasing

Examples

1) *Prove that the sequence* $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots, \frac{n}{n+1}, \ldots$ *is an increasing sequence.*

Solution

Here, $u_n = \frac{n}{n+1}$ and

$u_{n+1} = \frac{n+1}{n+2}$

Thus, for $n \geq 1$

$$u_n - u_{n+1} = \frac{n}{n+1} - \frac{n+1}{n+2}$$
$$= \frac{n^2 + 2n - n^2 - 2n - 1}{(n+1)(n+2)}$$
$$= -\frac{1}{(n+1)(n+2)} < 0$$

This proves that the given sequence is **increasing**.

Alternative method,

We can show that the sequence $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots, \frac{n}{n+1}, \ldots$ is increasing by examining the ratio of successive terms as follows

$$\frac{u_{n+1}}{u_n} = \frac{\frac{n+1}{n+2}}{\frac{n}{n+1}} = \frac{n+1}{n+2} \times \frac{n+1}{n} = \frac{n^2 + 2n + 1}{n^2 + 2n}$$

Since the numerator exceeds the denominator, the ratio exceeds 1, that is $\frac{u_{n+1}}{u_n} > 1$ for $n \geq 1$. This proves that the sequence is increasing.

2) The sequence 4, 4, 4, 4, ... is both no decreasing no increasing.

3) The sequence -2, 2, -2, 2, -2, ... is not monotonic.

APPLICATION ACTIVITY 1.3

Which of the following sequences are in increasing, decreasing, nonincreasing, nondecreasing, not monotonic

1. 1, 2, 3, ..., *n*, ...
2. $\left\{\frac{n}{n+1}\right\}$
3. $\left\{\frac{1}{2^n}\right\}$
4. 3, 3, 3, 3, ...

5) 1, -1, 1, -1, ...

1.4 Arithmetic sequence and its general term

ACTIVITY 1.4

In each of the following sequence, each term can be found by adding a constant number to the previous. Guess that constant number.

a) Sequence $\{u_n\}$: 5,8,11,14,17,...

b) Sequence $\{v_n\}$: 26,31,36, 41, 46,...

c) Sequence $\{w_n\}$: 20,18,16,14,12, ...

Let u_1 be an initial term of a sequence. If we add d successively to the initial term to find other terms, the difference between successive terms of a sequence is always the same number and the sequence is called **arithmetic.**

This sequence has the following term u_1, $u_2 = u_1 + d$, $u_3 = u_2 + d = u_1 + 2d$, $u_4 = u_3 + d = u_1 + 3d$, ... $u_n = u_{n-1} + d = u_1 + (n-1)d$,...

An **arithmetic sequence** may be defined recursively as $u_n = u_1 + (n-1)d$ where u_1 and d are real numbers. The number u_1 is the first term, and the number d is called the **common difference.**

Examples

The following sequences are arithmetic sequences:

Sequence $\{u_n\}$: $5, 8, 11, 14, 17, \ldots$

Sequence $\{v_n\}$: $26, 31, 36, 41, 46, \ldots$

Sequence $\{w_n\}$: $20, 18, 16, 14, 12, \ldots$

Common difference

The fixed numbers that bind each sequence together are called the **common differences.** Sometimes mathematicians use the letter ***d*** when referring to these types of sequences.

d can be calculated by subtracting any two consecutive terms in an arithmetic sequence. That is $d = u_{n+1} - u_n$ or $d = u_n - u_{n-1}$.

Note: If three consecutive terms are in arithmetic sequence, the double of the middle term is equal to the sum of extreme terms. That is for an arithmetic sequence u_{n-1}, u_n, u_{n+1}, we have $2u_n = u_{n-1} + u_{n+1}$.

Proof: If u_{n-1}, u_n, u_{n+1} form an arithmetic sequence, then

$u_{n+1} = u_n + d$ and $u_{n-1} = u_n - d$

Adding two equations, you get $u_{n+1} + u_{n-1} = 2u_n$

General term of an arithmetic sequence

The n^{th} term, u_n of an arithmetic sequence $\{u_n\}$ with common difference d and initial term u_1 is given by $u_n = u_1 + (n-1)d$, which is **the general term of an arithmetic sequence.** If the initial term is u_0 then the general term of an arithmetic sequence becomes $u_n = u_0 + nd$

Generally, if u_p is any p^{th} term of a sequence then the n^{th} term is given by $u_n = u_p + (n-p)d$

Examples

1) $4, 6, 8$ are three consecutive terms of an arithmetic sequence because $2 \times 6 = 4 + 8 \Leftrightarrow 12 = 12$

2) If the first and tenth terms of an arithmetic sequence are 3 and 30, respectively, find the fiftieth term of the sequence.

Solution

$u_1 = 3$ and $u_{10} = 30$

But $u_n = u_1 + (n-1)d$,

$u_{10} = u_1 + (10-1)d$

Then $30 = 3 + (10-1)d \Leftrightarrow 30 = 3 + 9d \Rightarrow d = 3$

Now, $u_{50} = u_1 + (50-1)d = 3 + 49 \times 3 = 150$

Thus,

The fiftieth term of the sequence is 150.

3) *If the 3rd term and the 8th term of an arithmetic sequence are 5 and 15 respectively, find the common difference.*

Solution

$u_3 = 5,\ u_8 = 15$

Using the general formula: $u_n = u_p + (n-p)d$

$u_3 = u_8 + (3-8)d$

$5 = 15 - 5d$

$\Leftrightarrow 5d = 15 - 5$

$\Rightarrow 5d = 10 \Rightarrow d = 2$

The common difference is 2.

4) *Consider the sequence* $5,\ 8,\ 11,\ 14,\ 17,\ \ldots,\ 47$. *Find the number of terms in this sequence*

Solution

We see that $u_1 = 5$, $u_n = 47$ and $d = 3$.

But we know that $u_n = u_1 + (n-1)d$.

This gives

$47 = 5 + (n-1)3$

$\Leftrightarrow 42 = 3n - 3 \Rightarrow n = 15$

This means that there are 15 terms in the sequence.

5) *Consider the sequence which went from 20 to -26 with -2 as common difference. Find the number of terms.*

Solution

We have

$-26 = 20 + (n-1)(-2)$

$\Leftrightarrow -46 = -2n + 2 \Rightarrow n = 24$

This means that there are 24 terms in the sequence.

6) Show that the following sequence is arithmetic. Find the first term and the common difference. $\{s_n\}=\{3n+5\}$

Solution:

The nth term and the $(n-1)$st term of the sequence $\{s_n\}$ are $s_n=3n+5$ *and* $s_{n-1}=3(n-1)+5=3n+2$

The first term is $s_1=8$.

Their difference d is $s_n-s_{n-1}=(3n+5)-(3n+2)=3$.

Since the difference of any two successive terms is the constant 3, the sequence $\{s_n\}$ is arithmetic and the common difference is 3.

APPLICATION ACTIVITY 1.4

1) If $\frac{1}{a+b},\frac{1}{a+c},\frac{1}{b+c}$ are 3 consecutive terms of an arithmetic progression, show that it will be the same for a^2, b^2, c^2.

2) Form an arithmetic progression of three numbers such that the sum of its terms is 30 and the sum of the squares of its terms is 332.

3) Calculate x so that the squares of $1+x$, $q+x$, *and* q^2+x will be three consecutive terms of an arithmetic progression where q is any given number.

1.5.Arithmetic Means of an arithmetic sequence

ACTIVITY 1.5

Suppose that you need to form an arithmetic sequence of 7 terms such that the first term is 2 and the seventh term is 20. Write down that sequence given that those terms are $2, A, B, C, D, E, 20$.

If three or more than three numbers form an arithmetic sequence, then all terms lying between the first and the last numbers are called **arithmetic means.** If B is arithmetic mean between A and C, then $B=\dfrac{A+C}{2}$.

Let us see how to insert k terms between two terms u_1 and u_n to form an arithmetic sequence:

u_1	...	...	...	...	u_n

The terms to be inserted are called **arithmetic means** between two terms u_1 and u_n.

This requires to form an arithmetic sequence of $n=k+2$ terms whose the first term is u_1 and the last term is u_n.

While there are several methods, we will use our n^{th} term formula $u_n=u_1+(n-1)d$.

As u_1 and u_n are known, we need to find the common difference d taking $n=k+2$ where k is the number of terms to be inserted and 2 stands for the first and last terms.

Examples

1) Insert three arithmetic means between 7 and 23.

Solution

Here $k=3$ and then $n=k+2=5, u_1=7$ and $u_n=u_5=23$.

Then

$u_5=u_1+(5-1)d$

$\Leftrightarrow 23=7+4d \Rightarrow d=4$

Now, insert the terms using $d=4$, the sequence is $7, 11, 15, 19, 23$.

2) Insert five arithmetic means between 2 and 20.

Solution

Here $k=5$ and then $n=k+2=7$, $u_1=2$ and $u_n=u_7=20$.

Then

$u_7=u_1+(7-1)d$

$\Leftrightarrow 20=2+6d \Rightarrow d=3$

Now, insert the terms using $d=3$, the sequence is $2, 5, 8, 11, 14, 17, 20$.

APPLICATION ACTIVITY 1.5

1. Insert 4 arithmetic means between -3 and 7
2. Insert 9 arithmetic means between 2 and 32
3. Between 3 and 54, n terms have been inserted in such a way that the ratio of 8^{th} mean and $(n-2)^{\text{th}}$ mean is $\frac{3}{5}$. Find the value of n.
4. There are n arithmetic means between 3 and 54 terms such that 8^{th} mean is equal to $(n-1)^{\text{th}}$mean as 5 to 9. Find the value of n.
5. Find the value of n so that $\frac{a^{n+1}+b^{n+1}}{a^n+b^n}$ will be the arithmetic mean between a and b.

1.6 Arithmetic series

ACTIVITY 1.6

Consider a finite arithmetic sequence $2,5,8,11,14,\ldots$.

a) What is its first term, the common difference d and the general term?

b) Determine the sum s_6 (in function of 6 and the first term 2) of the first 6 terms for this sequence taking that for each $u_n = u_1 + (n-1)d$ where for example $u_3 = u_1 + 2d$.

c) Try to generalize your results to determine the sum s_n for the first n terms of the arithmetic sequence $\{u_n\}$.

For finite arithmetic sequence $\{u_n\} = u_1, u_2, u_3, \ldots u_n$, the sum $\sum_{n=1}^{+\infty} u_n = u_1 + u_2 + u_3 + \ldots + u_n$ is called an **arithmetic series.**

We denote the sum of the first $n$terms of the sequence by S_n.

Thus, $s_n = \sum_{n=1}^{+\infty} u_n = u_1 + u_2 + u_3 + \ldots + u_n$.

The sum of first $n$terms of a finite arithmetic sequence with initial term u_1 is given by

$$
\begin{aligned}
s_n &= u_1 + u_2 + u_3 + \ldots + u_n \\
&= u_1 + (u_1 + d) + (u_1 + 2d) + \ldots + (u_1 + (n-1)d) \\
&= (\underbrace{u_1 + u_1 + \ldots u_1}_{n\ terms}) + (d + 2d + \ldots + (n-1)d) \\
&= nu_1 + d[1 + 2 + 3 + \ldots + (n-1)] = nu_1 + d[\frac{n(n-1)}{2}]
\end{aligned}
$$

$$
\begin{aligned}
s_n &= nu_1 + \frac{n}{2}(n-1)d = \frac{n}{2}[2u_1 + (n-1)d] \\
&= \frac{n}{2}[u_1 + u_1 + (n-1)d] \\
&= \frac{n}{2}(u_1 + u_n)
\end{aligned}
$$

$$s_n = \frac{n}{2}(u_1 + u_n)$$

If the initial term is u_0, the formula becomes $S_n = \frac{n+1}{2}(u_0 + u_n)$

Examples

1) Calculate the sum of first 100 terms of the sequence $2, 4, 6, 8, \ldots$

Solution

We see that the common difference is 2 and the initial term is $u_1 = 2$. We need to find $u_n = u_{100}$.

$$\begin{aligned} u_{100} &= 2 + (100 - 1)2 \\ &= 2 + 198 \\ &= 200 \end{aligned}$$

Now,

$$\begin{aligned} S_{100} &= \frac{100}{2}(u_1 + u_{100}) \\ &= 50(2 + 200) \\ &= 10100 \end{aligned}$$

2) Find the sum of first k even integers ($k \neq 0$ *).*

Solution

$u_1 = 2$ and $d = 2$

$u_n = u_k$

$u_k = 2 + (k - 1)2$

$u_k = 2k$

$S_n = S_k$

$S_k = \frac{k}{2}(2 + 2k)$

$S_k = k(k + 1)$

3) Find the sum: 60 + 64 + 68 + 72 + ... + 120

Solution:

This is the sum s_n of an arithmetic sequence u_n whose first term is $u_1 = 60$ and whose common difference is $d = 4$. The nth term is u_n.

We have $u_n = u_1 + (n-1)d$ and

$120 = 60 + (n-1)4$

$60 = 4(n-1)$

$n = 16$

Now, the sum is

$$u_{16} = 60 + 64 + 68 + \ldots + 120 = \frac{16}{2}(60 + 120) = 1440$$

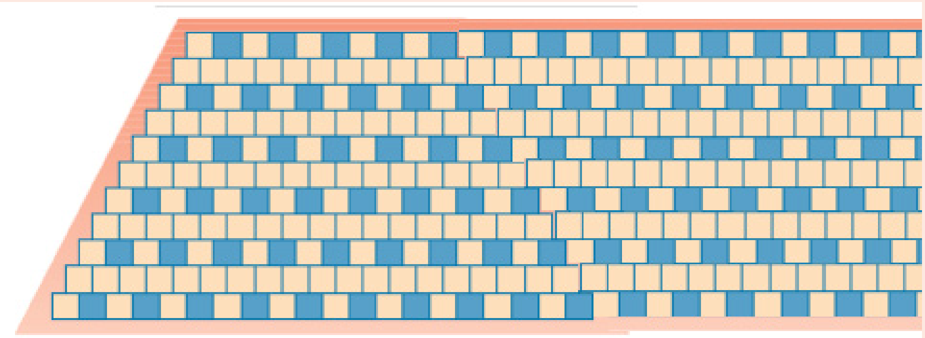
APPLICATION ACTIVITY 1.6

1)Consider the arithmetic sequence 8, 12, 16, 20, ... Find the expression for S_n

2) Sum the first twenty terms of the sequence 5, 9, 13,...

3) The sum of the terms in the sequence 1, 8, 15, ... is 396. How many terms does the sequence contain?

4) **Practical activity:** A ceramic tile floor is designed in the shape of a trapezium 10m wide at the base and 5m wide at the top as illustrated on the figure bellow:



The tiles, 10cm by 10cm, are to be placed so that each successive row contains one less tile than the preceding row. How many tiles will be required?

1.7 Harmonic sequences and its general term

ACTIVITY 1.7

Consider the following arithmetic sequence:

$2,\ 4,\ 6,\ 8,\ 10,\ 12,\ 14,\ 16,\ \ldots 2n,\ldots$

a) Form another sequence whose terms are the reciprocals of the terms of the given sequence.

b) What can you say about the new sequence? What is its first term, the third term and the general term? Is there a relationship between two consecutive terms?

Harmonic sequence is a sequence of numbers in which the reciprocals of the terms are in arithmetic sequence. It is of the following form:

$\frac{1}{u},\frac{1}{u+d},\frac{1}{u+2d},\ldots\frac{1}{u+(n-1)d},\ldots$ where u is not zero, $n-1$ is a natural number.

Example of harmonic sequence is $\frac{1}{3},\frac{1}{6},\frac{1}{9},\ldots$

If you take the reciprocal of each term from the above harmonic sequence, the sequence will become 3, 6, 9, ... which is an arithmetic sequence with a common difference of 3.

Another example of harmonic sequence is 6, 3, 2. The reciprocals of each term are $\frac{1}{6},\frac{1}{3},\frac{1}{2}$ respectively which is an arithmetic sequence with a common difference of $\frac{1}{6}$.

Remark

To find the term of **harmonic sequence**, convert the sequence into arithmetic sequence then do the calculations using the arithmetic formulae. Then take the reciprocal of the answer in arithmetic sequence to get the correct term in harmonic sequence.

Example:

The 2^{nd} term of an harmonic progression is $\frac{1}{6}$ and 6^{th} term is $-\frac{1}{6}$. Find 20^{th} term and n^{th} term.

Solution

In harmonic progression, $h_2 = \frac{1}{6}$ and $h_6 = -\frac{1}{6}$.

Thus, in the corresponding arithmetic progression $a_2 = 6$ and $a_6 = -6$

Or $a_6 = a_2 + 4d \Rightarrow 6 + 4d = -6$ or $d = -3$.

Hence $a_{20} = 6 + 18(-3) = -48 \Rightarrow h_{20} = -\frac{1}{48}$

$a_n = 6 + (n-2)(-3) = 12 - 3n \Rightarrow h_n = \frac{1}{12 - 3n}$

Notice: Harmonic Means

If three terms a, b, c are in harmonic progression, the middle one is said to be Harmonic mean between the other two and $b = \frac{2ac}{a+c}$.

This can be shown as follow:If a, b, c are in harmonic progression $\frac{1}{b} = (\frac{1}{a} + \frac{1}{c}) : 2 \Leftrightarrow 2ac = bc + ba \Leftrightarrow 2ac = b(a+c) \Rightarrow \text{b} = \frac{2ac}{a+c}$

Example:

1) Insert 4 harmonic means between $\frac{2}{3}$ and $\frac{6}{19}$

Solution

Let the four harmonic means be h_1, h_2, h_3, h_4.

Then $\frac{2}{3}, h_1, h_2, h_3, h_4, \frac{6}{19}$ are in harmonic progression

$\Rightarrow \frac{3}{2}, \frac{1}{h_1}, \frac{1}{h_2}, \frac{1}{h_3}, \frac{1}{h_4}, \frac{19}{6}$ are in arithmetic progression. where $a_1 = \frac{3}{2}$ and $a_6 = \frac{19}{6}$

$a_6 = \frac{19}{6} \Leftrightarrow a_1 + 5d = \frac{19}{6}$ with d common difference.

$$\Rightarrow \frac{3}{2}+5d=\frac{19}{6} \Leftrightarrow 5d=\frac{19}{6}-\frac{3}{2} \Leftrightarrow 5d=\frac{10}{6} \Rightarrow d=\frac{1}{3}$$

$$\Rightarrow \begin{cases} \frac{1}{h_1}=\frac{3}{2}+\frac{1}{3}=\frac{11}{6} \equiv 1^{st} \text{ term of arithmetic progression} \\ \frac{1}{h_2}=\frac{3}{2}+\frac{2}{3}=\frac{13}{6} \equiv 2^{nd} \text{ term of arithmetic progression} \\ \frac{1}{h_3}=\frac{3}{2}+\frac{3}{3}=\frac{15}{6}=\frac{5}{2} \equiv 3^{rd} \text{ term of arithmetic progression} \\ \frac{1}{h_4}=\frac{3}{2}+\frac{4}{3}=\frac{17}{6} \equiv 4^{th} \text{ term of arithmetic progression} \end{cases}$$

The four harmonic means are $\frac{6}{11},\frac{6}{13},\frac{2}{5}\ and\ \frac{6}{17}$.

2) Find the nth term of the series $2\frac{1}{2}+1\frac{7}{13}+1\frac{1}{9}+\frac{20}{23}+\cdots$

Solution

The given series is $\frac{5}{2}+\frac{20}{13}+\frac{10}{9}+\frac{20}{23},\cdots$

The reciprocal of the terms are $\frac{2}{5},\frac{13}{20},\frac{9}{10},\frac{23}{20},\cdots$

There are in arithmetic progression, with the first term $\frac{2}{5}$ and the common difference $\frac{13}{20}-\frac{2}{5}=\frac{1}{4}$

The given series in arithmetic progression:

nth term of arithmetic progression: $a_n=\frac{2}{5}+(n-1)\frac{1}{4}=\frac{8+5n-5}{20}=\frac{5n+3}{20}$

Hence nth term of the given harmonic progressionis $h_n=\frac{1}{a_n}$ or $h_n=\frac{20}{5n+3}$

The n^{th} term of the series $2\frac{1}{2}+1\frac{7}{13}+1\frac{1}{9}+\frac{20}{23}+\cdots$ is $\frac{20}{5n+3}$

APPLICATION ACTIVITY 1.7

1. Find the 4^{th} **and** 8^{th} **term of the harmonic series** $6, 4, 3, \ldots$
2. Insert two harmonic means between 3 and 10.
3. If a, b, c are in harmonic progression, show that $\frac{a}{b+c}, \frac{b}{c+a}, \frac{c}{a+b}$ are also in harmonic progression.
4. Find the term number of harmonic sequence $\frac{\sqrt{5}}{3}, \frac{\sqrt{5}}{4}, \frac{1}{\sqrt{5}}, \ldots, \frac{\sqrt{5}}{13}$
5. The harmonic mean between two numbers is 3 and the arithmetic mean is 4. Find the numbers.
6. Find the n^{th} term of the series $4 + 4\frac{2}{7} + 4\frac{8}{13} + 5 + \cdots$

1. 8 Generalities on Geometric sequence and its general term

ACTIVITY 1.8

Take a piece of paper with a square shape.

1. Cut it into two equal parts.
2. Write down a fraction corresponding to one part according to the original piece of paper.
3. Take one part obtained in step 2) and repeat step 1) and then step 2)
4. Continue until you remain with a small piece of paper that you are not able to cut into two equal parts and write down the sequence of fractions obtained.
5. Observe the sequence of numbers you obtained and give the relationship between any two consecutive numbers.

Sequences of numbers that follow a pattern of multiplying a fixed number $\boldsymbol{r}$ from one term u_1 to the next are called **geometric sequences.**

The following sequences are geometric sequences:

Sequence $\{u_n\}$: $5,10,20,40,80,\ldots$

Sequence $\{v_n\}$: $2,1,\frac{1}{2},\frac{1}{4},\frac{1}{8},\ldots$

Sequence $\{w_n\}$: $1,\text{-}2,4,-8,16,\ldots$

Common ratio

We can examine these sequences to greater depth, we must know that the fixed numbers that bind each sequence together are called the **common ratios**, denoted by the letter ***r***. This means if u_1 is the first term, $u_2 = ru_1$; $u_3 = r^2u_1; u_4 = r^3u_1; \ldots u_n = r^{n-1}u_1; \ldots$

The n^{th} term or the general term of a geometric sequence becomes $u_n = r^{n-1}u_1$.

If the first term is u_0, then the general term of an geometric sequence becomes $u_n = r^n u_0$.

The common ration r can be calculated by dividing any two consecutive terms in a geometric sequence. That is

$r = \frac{u_{n+1}}{u_n}$ or $r = \frac{u_n}{u_{n-1}}$ or $u_n = ru_{n-1}$.

Generally, if u_p is the p^{th} term of the sequence, then the n^{th} term is given by $u_n = u_p r^{n-p}$.

Note that if three terms are consecutive terms of a geometric sequence, the square of the middle term is equal to the product of extreme terms. That is for a geometric sequence u_{n-1}, u_n, u_{n+1}, we have $u_n^2 = u_{n-1} \cdot u_{n+1}$.

Examples

1) $6,12,24$ *are consecutive terms of a geometric sequence because* $(12)^2 = 6\times 24 \Leftrightarrow 144 = 144$

Find b such that $8,b,18$ *will be in geometric sequence.*

Solution

$b^2 = 8 \times 18 = 144$

$b = \pm\sqrt{144} = \pm 12$

Thus, $8,12,18$ or $8,-12,18$ are in geometric sequence.

2) The product of three consecutive numbers in geometric progression is 27. The sum of the first two and nine times the third is -79. Find the numbers.

Solution

Let the three terms be $\frac{x}{a}, x, ax$.

The product of the numbers is 27. So $\frac{x}{a} xax = 27 \Rightarrow x^3 = 27 \Rightarrow x = 3$

The sum of the first two and nine times the third is -79:

$\frac{x}{a} + x + 9ax = -79 \Rightarrow \frac{3}{a} + 3 + 27a = -79$

$27a^2 + 82a + 3 = 0 \Rightarrow a = -3 \;\; or \;\; a = -\frac{1}{27}$

The numbers are: -1, 3, -9 or -81, 3, $-\frac{1}{9}$.

3) If the first and tenth terms of a geometric sequence are 1 and 4, respectively, find the nineteenth term.

Solution

$u_1 = 1$ and $u_{10} = 4$

But $u_n = u_1 r^{n-1}$, then $4 = 1r^9 \Leftrightarrow r = \sqrt[9]{4} \;\; or \;\; r = 4^{\frac{1}{9}}$

Now,

$$u_{19} = u_1 r^{19-1}$$
$$= 1\left(4^{\frac{1}{9}}\right)^{18}$$
$$= 16$$

Thus, the nineteenth term of the sequence is 16.

4) If the 2nd term and the 9th term of a geometric sequence are 2 and $-\frac{1}{64}$ *respectively, find the common ratio.*

Solution

$u_2 = 2,\ u_9 = -\frac{1}{64}$

Using the general formula: $u_n = u_p\, r^{n-p}$

$u_2 = u_9\, r^{2-9}$

$2 = -\frac{1}{64} r^{-7}$

$\Leftrightarrow 128 = -\frac{1}{r^7}$

$\Leftrightarrow r^7 = -\frac{1}{128}$

$\Leftrightarrow r = \sqrt[7]{-\frac{1}{128}} \Rightarrow r = -\frac{1}{2}$

The common ratio is $r = -\frac{1}{2}$.

5) Find the number of terms in sequence $2, 4, 8, 16, \ldots, 256$.

Solution

This sequence is geometric with common ratio 2, $u_1 = 2$ and $u_n = 256$

But $u_n = u_1 r^{n-1}$, then $256 = 2 \times 2^{n-1} \Leftrightarrow 256 = 2^n$ or $2^8 = 2^n \Rightarrow n = 8$.

Thus, the number of terms in the given sequence is 8.

APPLICATION ACTIVITY 1.8

1. If the second and fifth terms of a geometric sequence are 6 and -48, respectively, find the sixteenth term.

2. If the third term and the 8th term of a geometric sequence are $\frac{1}{2}$ and $\frac{1}{128}$ respectively, find the common ratio.

3. The 4^{th} term of a geometric sequence is square of its 2^{nd} term, and the first term is -3. Determine its 7^{th} term.

4. Find the fourth term from the end of geometric sequence $8, 4, 2, \cdots, \frac{1}{128}$

5. The fifth term of a geometric sequence is $\frac{81}{32}$ and the ratio of 3^{rd} and 4^{th} is $\frac{2}{3}$, write the geometric sequence and its 8^{th} term.

6. If p^{th} terms of two sequences $5, 10, 20, \cdots$ and $1280, 640, 320, \cdots$,are equal, find the value of p.

1.9. Geometric Means

ACTIVITY 1.9

Suppose that you need to form a geometric sequence of 6 terms such that the first term is 1 and the sixth term is 243. Given that these terms are $1, A, B, C, D, 243$. Write down that sequence.

To insert k terms called **geometric means** between two terms u_1 and u_n is to form a geometric sequence of $n = k + 2$ terms whose the first term is u_1 and the last term is u_n.

While there are several methods, we will use our n^{th} term formula $u_n = u_1 r^{n-1}$.

As u_1 and u_n are known, we need to find the common ratio r taking $n = k + 2$ where k is the number of terms to be inserted.

Example:

1) Insert three geometric means between 3 and 48.

Solution

Here $k = 3$, then $n = 5$, $u_1 = 3$ and $u_n = u_5 = 48$

$u_5 = u_1 r^{n-1} \Leftrightarrow 48 = 3r^4 \Rightarrow r = 2$

Inserting three terms using common ratio $r=2$ gives $3,6,12,24,48$

2) Insert 6 geometric means between 1 and $-\frac{1}{128}$.

Solution

Here $k=6$, then $n=8, u_1=1$ and $u_n=u_8=-\frac{1}{128}$

$$u_8=u_1r^{n-1}$$

$$\Leftrightarrow -\frac{1}{128}=1r^7$$

$$\Leftrightarrow r^7=-\frac{1}{128}$$

$$\Leftrightarrow r^7=-\frac{1}{(2)^7}$$

$$\Leftrightarrow r=\left[-\frac{1}{(2)^7}\right]^{\frac{1}{7}}=-\frac{1}{2}$$

Inserting 6 terms using common ratio $r=-\frac{1}{2}$ gives

$1,-\frac{1}{2},\frac{1}{4},-\frac{1}{8},\frac{1}{16},-\frac{1}{32},\frac{1}{64},-\frac{1}{128}$.

APPLICATION ACTIVITY 1.9

1. Insert 5 geometric means between $\frac{1}{4}$ and $\frac{1}{256}$.
2. Insert 5 geometric means between 2 and $\frac{2}{729}$
3. Find the geometric mean between

   a) 2 and 98

   b) $\frac{3}{2}$ and $\frac{27}{2}$

4. Suppose that $4, 36, 324$ are in geometric progression. Insert two more numbers in this sequence so that it again forms a geometric sequence.
5. The arithmetic mean of two numbers is 34 and their geometric mean is 16. Find the numbers.

1. 10. Geometric series

ACTIVITY 1.10

During a competition of student teachers at the district level, 5 first winners were paid an amount of money in the way that the first got 100,000Frw, the second earned the half of this money, the third got the half of the second's money, and so on until the fifth who got the half of the fourth's money.

a) Discuss and calculate the money earned by each student from the second to the fifth.

b) Determine the total amount of money for all the 5 student teachers.

c) Compare the money for the first and the fifth student and discuss the importance of winning at the best place.

d) Try to generalize the situation and guess the money for the student who passed at the n^{th} place if more students were paid. In this case, evaluate the total amount of money for n students.

A **Geometric series** is an infinite sum $\sum_{n=1}^{\infty} u_n = u_1 + u_1 r + u_1 r^2 + ... + u_1 r^{n-1} + ...$ of the terms a geometric sequence. If we have **a finite geometric sequence** $\{u_n\} = u_1, u_2, u_3, ...u_n$, the sum is $S_n = \sum_{n=1}^{n} u_n = u_1 + u_1 r + u_1 r^2 + ... + u_1 r^{n-1}$.

The sum of first *n* terms of a geometric sequence

The sum S_n of the first n terms of a geometric sequence $\{u_n\} = \{u_1 r^{n-1}\}$ is

$S_n = u_1 + r u_1 + ... + r^{n-1} u_1$ (1)

Multiply each side by r to obtain $rS_n = ru_1 + r^2u_1 + \ldots + r^n u_1$ (2)

Subtracting equation (2) from equation (1) we obtain

$$S_n - rS_n = u_1 - u_1 r^n$$

$$(1-r)S_n = u_1(1-r^n)$$

Since $r \neq 1$, we can solve for S_n and find $S_n = u_1 \dfrac{(1-r^n)}{(1-r)}$

If the initial term u_1 and common ratio r are given, the sum $s_n = \dfrac{u_1(1-r^n)}{1-r}$ with $r \neq 1$.

If the initial term is u_0, then the formula is $s_n = \dfrac{u_0(1-r^{n+1})}{1-r}$ with $r \neq 1$

If $r = 1$, $s_n = nu_1$.

Also, the product of first n terms of a geometric sequence with initial term u_1 and common ratio r is given by $P_n = (u_1)^n r^{\frac{n(n-1)}{2}}$

If the initial term is u_0 then $P_n = (u_0)^{n+1} r^{\frac{n}{2}(n+1)}$

Examples

1) *Find the sum of the first 20 terms of a geometric sequence if the first term is 1 and common ratio is 2.*

Solution

Here $u_1 = 1, r = 2, n = 20$

Then,

$$s_{20} = \frac{1(1-2^{20})}{1-2} = \frac{1-2^{20}}{-1} = 1048575$$

2) Consider the sequence $\{u_n\}$ defined by $u_0 = 0$ and $u_{n+1} = u_n + \dfrac{1}{2^n}$. Consider another sequence $\{v_n\}$ defined by $v_n = u_{n+1} - u_n$.

a) *Show that $\{v_n\}$ is a geometric sequence and find its first term and common ratio.*

b) *Express $\{v_n\}$ in term of n.*

Solution

a) $u_0 = 0, \quad v_0 = u_1 - u_0 = 1$

$$u_1 = u_0 + \frac{1}{2^0} = 1, \quad u_2 = u_1 + \frac{1}{2^1} = \frac{3}{2};$$

$$v_1 = u_2 - u_1 = \frac{1}{2}, \quad v_2 = u_3 - u_2 = \frac{1}{4}$$

$\{v_n\}$ is a geometric sequence if $v_1^2 = v_0 \cdot v_2$.

$v_1^2 = \frac{1}{4}$ and $v_0 \cdot v_2 = \frac{1}{4}$.

Thus, $\{v_n\}$ is a geometric sequence.

First term is $v_0 = 1$

Common ratio is $r = \frac{v_1}{v_0} = \frac{1}{2}$

b)General term

$$v_n = v_0 r^n$$
$$= \frac{1}{2^n}$$

Thus, $\{v_n\}$ is defined by $v_n = \frac{1}{2^n}$

3) Find the product of the first 20 terms of a geometric sequence if the first term is 1 and common ratio is 2.

Solution

$$P_n = (u_1)^n r^{\frac{n(n-1)}{2}}$$

Here $u_1 = 1, r = 2, n = 20$,

Then,

$$P_{20} = (1)^{20} 2^{\frac{20(19)}{2}} = 2^{190}$$

APPLICATION ACTIVITY 1.10

1. Find the sum of the first 8 terms of the geometric sequence 32, -16, 8, ...
2. Find the sum of the geometric sequence with the first term 0.99 and the common ration is equal to the first term.
3. Find the first term and the common ratio of the geometric sequence for which $S_n = \frac{5^n - 4^n}{4^{n-1}}$
4. Find the product of the first 10 terms of the sequence in question 1.

5. Aloys wants to begin saving money for school. He decides to deposit \$500 at the beginning of each quarter (January1, April 1, July1, and October1) in a saving account which pays an annual percentage of 6% compound interest quarterly. The interest for each quarter is posted on the last day of the quarter. Determine Aloys's account balance at the end of one year.

1.11 Infinity geometric series and its convergence

ACTIVITY 1.11

Given that the sum of n terms of a geometric sequence $\{u_n\} = \left\{5\left(\frac{1}{2}\right)^{n-1}\right\}$ is given by:

$$S_n = 5\frac{\left(1-\left(\frac{1}{2}\right)^n\right)}{\left(1-\left(\frac{1}{2}\right)\right)},$$

a) Evaluate $\lim_{n\to\infty} S_n = \lim_{n\to\infty}\left[5\frac{\left(1-\left(\frac{1}{2}\right)^n\right)}{\left(1-\left(\frac{1}{2}\right)\right)}\right]$

b) Extend your results considering the infinite geometric series $\sum_{n=1}^{\infty} u_1 r^{n-1}$ and determine $\lim_{n\to\infty} S_n = \lim_{n\to\infty} u_1 \frac{(1-r^n)}{(1-r)}$, $-1<r<1$.

The sum of a geometric series with first term u_1 and the common ratio r is the limit of this geometric sequence as n approaches ∞.

As $S_n = u_1 \frac{(1-r^n)}{(1-r)}$, this limit is $\lim_{n\to\infty} u_1 \frac{(1-r^n)}{(1-r)} = \lim_{n\to\infty} u_1 \frac{1}{1-r} - \lim_{n\to\infty} u_1 \frac{r^n}{1-r}$. This limit is a real number if $|r|<1$ as in this case $|r^n|$ approaches 0 when $n \to \infty$

Therefore, $\lim_{n\to\infty} u_1 \frac{(1-r^n)}{(1-r)} = \lim_{n\to\infty} u_1 \frac{1}{1-r} - \lim_{n\to\infty} u_1 \frac{r^n}{1-r} = u_1 \frac{1}{1-r} - 0$

Therefore, the sum of our geometric sequence becomes:

$S_\infty = \frac{u_1}{1-r}$ provided $-1<r<1$.

As a conclusion, if $|r|<1$, the infinite geometric series $\sum_{n=1}^{\infty} u_1 r^{n-1}$ converges. Its sum is $\sum_{n=1}^{\infty} u_1 r^{n-1} = \frac{u_1}{1-r}$.

Examples:

1) Given the geometric progression 16, 12, 9, ... Find the sum of terms up to infinity.

Solution

Here $u_1 = 16, r = \frac{12}{16} = \frac{3}{4}$

Thus $-1<r<1$ and hence the sum to infinity will exists

$S_\infty = \frac{u_1}{1-r} = \frac{16}{1-\frac{3}{4}} = 64$

The sum to infinity is 64.

2) Express the recurring decimal $0.\overline{32}$ as a rational number.

Solution

$0.\overline{32} = \frac{32}{10^2} + \frac{32}{10^4} + \frac{32}{10^6} + \ldots$ which is an infinite geometric series with first term $u_1 = 0.32$ and common ratio is $r = 0.01$.

Since $-1 < r < 1$, the sum to infinity exist and equal to $\frac{u_1}{1-r} = \frac{0.32}{1-0.01} = \frac{0.32}{0.99} = \frac{32}{99}$

Therefore, $0.\overline{32} = \frac{32}{99}$.

APPLICATION ACTIVITY 1.11

1) Consider the infinite geometric series $\sum_{n=1}^{\infty} 10\left(1-\frac{3x}{2}\right)^n$

   a. For what values of x does a sum to infinity exist?

   b. Find the sum of the series if $x = 1.3$

2) Show that the repeating decimal 0.9999.... equals 1.

3) Evaluate if the geometric series $\sum_{n=1}^{\infty} 2\left(\frac{2}{3}\right)^{n-1} = 2 + \frac{4}{3} + \frac{8}{9} + \ldots$ converges or diverges. If it converges, find its sum.

1.12 Application of sequences in real life

ACTIVITY 1.12

Carry out a research in the library or on internet and find out at least 3 problems or scenarios of the real life where sequences and series are applied.

There are many applications of sequences. Sequences are useful in our daily lives as well as in higher mathematics. For example; the monthly payments made to pay off an automobile or home loan with interest portion, the list of maximum daily temperatures in one area for a month are sequences. Sequences are used in calculating interest, population growth, half-life and decay in radioactivity, etc.

In economics and Finance, sequences and series can be used for example in solving problems related to:

a) Final sum, the initial sum, the time period and the interest rate for an investment.

The amount A after t years due to a principal P invested at an annual interest rate r compounded n times per year is

$$A = P.\left(1+\frac{r}{n}\right)^{n.t}$$

(i) If the compounding takes place Annually, n=1 and $A = P.\left(1+\frac{r}{1}\right)^{t}$

(ii) If the compounding takes place Monthly, n=12 and $A = P.\left(1+\frac{r}{12}\right)^{12t}$

(iii) If the compounding takes place Daily, n=365 and $A = P.\left(1+\frac{r}{365}\right)^{365t}$

In each case, **the Interest due** is $A - P$.

Note that: From the compound interest formula $A = P\left(1+\frac{r}{n}\right)^{nt}$ If we let $k = \frac{n}{r}$, then $n = kr$ and $nt = krt$, and we may write the formula as

$$A = P\left(1+\frac{1}{k}\right)^{krt} = P\left[\left(1+\frac{1}{k}\right)^{k}\right]^{rt}.$$

For continuously compound interest, we may let n (the number of interest period per year) increase without bound towards infinity ($n \to \infty$), equivalently, by $k \to \infty$.

Using the definition of e, we see that $P\left[\left(1+\frac{1}{k}\right)^{k}\right]^{rt} \to P[e]^{rt}$ as $k \to \infty$. This result gives us the following formula: $A = Pe^{rt}$ where **P=Principal or initial**

value at t = 0 ; r is the Interest rate expressed as a decimal; r is the number of years P is invested; A is the amount after t years.

The amount A after t years due to a principal P invested at an annual interest rate r compounded continuously is

$$A = P.e^{rt}.$$

b) Annual Equivalent Rate for part year investments and the nominal annual rate of return.

For loan repayments the annual equivalent rate is usually referred to as the annual percentage rate (APR). If you take out a bank loan you will usually be quoted an APR even though you will be asked to make monthly repayments.

The **corresponding AER for any given monthly rate of interest** i_m can be found using the formula

$$AER = \left(i + i_m\right)^{12} - 1.$$

The APR on loans is the same thing as the annual equivalent rate and so the same formula applies.

The relationship between the daily interest rate i_d on a deposit account and the AER can be formulated as $AER = \left(i + i_d\right)^{365} - 1.$

c) The Present value of investment

The present value P of A money to be received after t years, assuming a per annum interest rate r compounded n times per year, is $P = A.\left(1 + \frac{r}{n}\right)^{-n.t}$.

If the interest is compounded continuously, $P = A.e^{-n.t}$

d) Monthly repayments and the Annuity

Suppose P is the deposit made at the end of each payment period for an annuity paying an interest rate of i per payment period.

The amount A of the annuity after n deposits is

$$A = P.\left[\frac{(1+i)^n - 1}{i}\right]$$

Examples

1) *A child building a tower with blocks uses 15 for the bottom row. Each row has 2 fewer blocks than the previous row. Suppose that there are 8 rows in the tower.*

a) *How many blocks are used for the top row?*

b) *What is the total number of blocks in the tower?*

Solution

a) The number of blocks in each row forms an arithmetic sequence with $u_1 = 15$ and $d = -2$

$n = 8,\ u_8 = u_1 + (8-1)(-2)$. There is just one block in the top row.

b) Here we must find the sum of the terms of the arithmetic sequence formed with $u_1 = 15, n = 8, u_8 = 1$

$$S_8 = \frac{8}{2}(15+1) = 64$$

There are 64 blocks in the tower.

2) *An insect population is growing in such a way that each new generation is 1.5 times as large as the previous generation. Suppose there are 100 insects in the first generation.*

a) *How many will there be in the fifth generation?*

b) *What will be the total number of insects in the five generations?*

Solution

a) *The population can be written as a geometric sequence with* $u_1 = 100$ *as the first-generation population and common ratio* $r = 1.5$. *Then the fifth-generation population will be* $u_5 = 100(1.5)^{5-1} = 506.25$. *In the fifth-generation, the population will number about 506 insects.*

b) *The sum of the first five terms using the formula for the sum of the first n terms of a geometric sequence.*

$$S_5 = \frac{100\left(1-(1.5)^5\right)}{1-1.5} = 1318.75$$

The total population for the five generations will be about 1319 insects.

3) Find the accumulated value of \$15,000 at 5% per year for 18 years using simple interest.

Solution

$P = 15000,\ r = 0.05, t = 18$

$I = 15000(0.05)(18)$

$= 13500$

A total of \$13,500 in interest will be earned.

Hence, the accumulated value in the account will be 13,500 + 15,000 = \$28,500.

4) *Suppose 20,000Frw is deposit in a bank account that pays interest at a rate of 8% per year compound Continuously. Determine the balance in the account in 5 years.*

Solution: Applying the formula for Continuously compound

interest with $P = 20,000;\ r = 0.08;\ and\ t = 5$, we have

$A = Pe^{rt} = 20,000e^{0.08(5)} = 20,000\,\mathrm{e}^{0.4} = 29,836.49\,Frw$.

5) Find the amount of an annuity after 5 deposits if a deposit of \$100 is made each year, at 4% compounded annually. How much interest is earned?

Solution

The deposit is $P = \$100$. The number of deposits is $n = 5$ and the interest per payment period is $i = 0.04$. Using the formula, the amount A after 5 deposits is

$$A = P.\left[\frac{(1+i)^n - 1}{i}\right] = 100\left[\frac{\left((1+0.04)^5 - 1\right)}{0.04}\right] = 541.63$$

The interest earned is the amount after 5 deposits less the 5 annual payments of \$100 each:

Interest earned $= A - 500 = 543.63 - 500 = 41.63$. It is \$41.63.

6) Mary decides to put aside \$100 every month in a credit union that pays 5% compounded monthly. After making 8 deposits, how much money does Mary have?

Solution:

This is an annuity with $P = \$100, n = 8$ deposits, and interest $i = \frac{0.05}{12}$ per paymentperiod. Using the formula, the amount A after 8 deposits is

$$A = P.\left[\frac{(1+i)^n - 1}{i}\right] = 100\left[\frac{\left(\left(1+\frac{0.05}{12}\right)^5 - 1\right)}{\frac{0.05}{12}}\right] = 811.76$$

Mary has \$811.76 after making 8 deposits.

7) To save for her daughter's college education, Martha decides to put \$50 aside every month in a bank guaranteed-interest account paying 4% interest compounded monthly.

She begins this savings program when her daughter is 3 years old. How much will she have saved by the time she makes the 180th deposit? How old is her daughter at this time?

Solution:

This is an annuity with $P = \$50, \mathrm{n} = 180$ deposits, and $i = \frac{0.04}{12}$. The amount A saved is

$$A = P.\left[\frac{(1+i)^n - 1}{i}\right] = 50\left[\frac{\left(\left(1+\frac{0.04}{12}\right)^{180} - 1\right)}{\frac{0.04}{12}}\right] = 12,304.52$$

Since there are 12 deposits per year, when the 180th deposit is made $\frac{180}{12} = 15$ have passed and Martha's daughter is 18 years old.

APPLICATION ACTIVITY 1.12

1) If Linda deposits \$1300 in a bank at 7% interest compounded annually, how much will be in the bank 17 years later?

2) The population of a city in 1970 was 153,800. Assuming that the population increases continuously at a rate of 5% per year, predict the population of the city in the year 2000.

3) To save for retirement, Manasseh, at age 35, decides to place 2000Frw into an Individual Retirement Account (IRA) each year for the next 30 years. What will the value of the IRA be when Manasseh makes his 30th deposit? Assume that the rate of return of the IRA is 4% per annum compounded annually.

4) A private school leader received permission to issue 4,000,000Frw in bonds to build a new high school. The leader is required to make payments every 6 months into a sinking fund paying 4% compounded semiannually. At the end of 12 years the bond obligation will be retired. What should each payment be?

1.13 END UNIT ASSESSMENT

1) Find first four terms of the sequence

a) $\left\{\frac{1-n}{n^2}\right\}$ **b)** $\left\{\frac{(-1)^{n+1}}{2n-1}\right\}$ **c)** $\left\{2+(-1)^n\right\}$

2) Find the formula for the n^{th} term of the sequence

a) 1, -1, 1, -1, 1, ...

b) 0, 3, 8, 15, 24, ...

c) 1, 5, 9, 13, 17, ...

3) Which of the following sequences converge, and which diverge? Find the limit of each convergent sequence.

a) $\left\{\sqrt{\frac{2n}{n+1}}\right\}$ b) $\left\{\frac{n}{2^n}\right\}$ c) $\left\{8^{\frac{1}{n}}\right\}$

4) A mathematical child negotiates a new pocket money deal with her unsuspecting father in which she receives 1 pound on the first day of the month, 2 pounds on the second day, 4 pounds on the third day, 8 pounds on the fourth day, 16 pounds on the fifth day, ... until the end of the month. How much would the child receive during the course of a month of 30 days? (Give your answer to the nearest million pounds).

5) A culture of bacteria doubles every 2 hours. If there are 500 bacteria at the beginning, how many bacteria will there be after 24 hours?

6) You complain that the hot tub in your hotel suite is not hot enough. The hotel tells you that they will increase the temperature by 10% each hour. If the current temperature of the hot tub is 75º F, what will be the temperature of the hot tub after 3 hours, to the *nearest tenth* of a degree?

7) The sum of the interior angles of a triangle is 180º, of a quadrilateral is 360º and of a pentagon is 540º. Assuming this pattern continues, find the sum of the interior angles of a dodecagon (12 sides).

UNIT 2

LOGARITHMIC AND EXPONENTIAL EQUATIONS

Key unit competence: Solve equations involving logarithms or exponentials and apply them to model and solve related problems.

2.0 INTRODUCTORY ACTIVITY

The agriculture officer of one District presented a project of multiplying cows uplifting the local population from lower to upper level of UBUDEHE to the planning committee. The project introduced in 2005consists of increasing the number of cows given to poor families. These cows grow following a mathematical rule approximated by the formula $p(t) = p_0 e^{0.3t}$ where t is measured in years and p_0 is the total number of cows distributed to the poor families where each one got 2 cows.

Assume that you are an evaluator of the project, basing on the information of the cows' population in 2005, 2008 and 2020;

i) Show graphically the information of cow production in the years mentioned above given that 100 families were given cows at the beginning.

ii) How many cows each family should have after 15 years given that, in this process, it has to give 2 cows to another family?

iii) What is your opinion about the project? Is the number of cows increasing?

iv) What is your recommendation about the project?

2.1 Logarithmic Equations

ACTIVITY 2.1

1) Use the library, your notes for year one or the internet and summarize the properties for logarithm;
2) Use your findings and determine the value of x in the following expressions:

a) $\log x = 2$ b) $\log_2 x = \log_2 5$ c) $\ln x = \ln 10$

d) $\ln x = 3$ e) $\log(100x) = 2 + \log 4$

The **logarithmic function to the base *a*,** where $a > 0$ and $a \neq 0$ is defined by

$y = \log_a x$ if and only if $x = a^y$.

If the base is not mentioned, the logarithm is decimal (with base 10).

When the base is the number e, we have the natural logarithm or Neperian logarithm $\ln x$.

Note that $\log_a x \in \mathbb{R}$ if and only if x is positive real number.

Equations that contain logarithms are called **logarithmic equations.** Care must betaken when solving logarithmic equations algebraically. In the expression $\log_a M$ remember that a and M are positive and $a \neq 1$. Be sure to check each apparent solution in the original equation and discard any that are extraneous.

A) *Some logarithmic equations can be solved by changing the logarithmic equation to exponential form using the fact that* $y = \log_a x$ *means* $a^y = x$.

Examples:

1)Solve for x:

a) $\log_3(4x-7) = 2$

b) $\log_x 64 = 2$

Solutions:

a) We can obtain an exact solution by changing the logarithmic equation to exponential form as:

$\log_3(4x-7)=2 \Rightarrow 4x-7=3^2$ *by* applying $y=\log_a x \Leftrightarrow a^y=x$

Then, $4x-7=9 \Rightarrow 4x=16 \Rightarrow x=4$.

Checking in the initial equation to get:

$\log_3(4x-7)=\log_3(4\times4-7)=\log_3 9=2 \Leftrightarrow 3^2=9$

The Solution set is $\{4\}$.

b) We can obtain an exact solution by changing the logarithmic equation to exponential form as: $\log_x 64=2 \Rightarrow x^2=64 \Rightarrow x=\pm\sqrt{64}=\pm8$

The base of a logarithm is always positive. As result, we discard -8. We check for 8

Then, $\log_x 64=2 \Rightarrow 8^2=64$. The solution set is $\{8\}$

B)Two numbers with the same logarithm are equal; $\log_a x=\log_a y \Leftrightarrow x=y$

Example

1) Find x if $\log_2 32=x$

Solution

$\log_2 32=x \Leftrightarrow \log_2 32=\log_2 2^x$

$32=2^x$

But $32=2^5$.

So $32=2^x \Leftrightarrow 2^5=2^x \Rightarrow x=5$

2)Find the numerical value of $\log_3\sqrt[3]{9}$

Solution

Let $y = \log_3 \sqrt[3]{9}$, then $3^y = \sqrt[3]{9}$

$\Leftrightarrow 3^y = 9^{\frac{1}{3}}$

$\Leftrightarrow 3^y = 3^{2\left(\frac{1}{3}\right)}$

$\Leftrightarrow 3^y = 3^{\frac{2}{3}}$

$\Rightarrow y = \dfrac{2}{3}$

Hence, $\log_3 \sqrt[3]{9} = \dfrac{2}{3}$

3) Solve $2\log_5 x = \log_5 9$

We have that $x > 0$

$2\log_5 x = \log_5 9$

$\log_5 x^2 = \log_5 9$

$x^2 = 9$

$x = \pm 3$

Recall that the the variable $x > 0$ .Therefore, -3 is extraneous and we discard it and get $S = \{3\}$.

C) When solving logarithmic equations, you apply properties of logarithms and you study the domain of definition of logarithm.

Examples

1) Solve $\log_3 (4x - 7) = 2$

Solution

Condition $4x - 7 > 0$ should be satisfied.

$\log_3(4x-7)=2$

$\log_3(4x-7)=\log_3 3^2$

$(4x-7)=3^2$

$x=4$

2) $\log_x 64=2$

Solution

$\log_x 64=2$

$\log_x 64=\log_x x^2$

$64=x^2$

$x=\pm 8.$

The base of a logarithm is always positive. As a result, we discard -8. We check the solution 8 and find that the solution set is $S=\{8\}$

3) Solve $\ln x=\ln(x+6)-\ln(x-4)$

Solution

Condition: $x>0, x-4>0$ and $x+6>0$ should be satisfied.

As a result, the domain of the variable here is $x>4$. We begin the solution using the log of a difference property.

$\ln x=\ln(x+6)-\ln(x-4)$

$\ln x=\ln\dfrac{(x+6)}{(x-4)}$

$x=\dfrac{(x+6)}{(x-4)}$

$x^2-5x-6=0$

$$x^2 - 5x - 6 = 0$$
$$(x-6)(x+1) = 0$$
$$x = 6 \text{ or } x = -1$$

Since the domain of the variable is $x > 4$, we discard -1 as extraneous. The solution set is {6}, which you should check.

D) Use of $\log_a a = 1$ *and* $\log_a 1 = 0$

Example:

Solve the following equations: **1)** $\ln 2x = 1$ 2) $\log_5(x^2 - 9) = 0$

Solution

1) $\ln 2x = 1$

Condition $x > 0$

$$\ln 2x = \ln e$$
$$2x = e$$
$$x = \frac{e}{2}$$

2) $\log_5(x^2 - 9) = 0$

Condition $x^2 - 9 > 0$ should be satisfied.

$$\log_5(x^2 - 9) = \log_5 1$$
$$(x^2 - 9) = 1$$
$$x^2 - 10 = 0$$
$$x = \pm\sqrt{10}$$

2) In using properties of logarithms to solve logarithmic equations, avoid using the property $\log_a x^r = r \log_a x$ when r is even. The reason can be seen in this example:

Solve: $\log_3 x^2 = 4$

Solution: The domain of the variable x is all real numbers except 0.

a) $\log_3 x^2 = 4$

$$x^2 = 3^4 = 81$$
$$x = -9 \text{ or } x = 9$$

b) $\log_3 x^2 = 4$

$$2\log_3 x = 4$$
$$\log_3 x = 2$$
$$x = 9$$

However, both -9 *and* 9 are solutions of our equation (as you can verify). The solution in part (b) does notfind the solution -9 because the domain of the variable was further restricted due to the application of the property $\log_a x^r = r\log_a x$.

Note: Given any positive real numbers x, a, and b, where $a \neq 1$ and $b \neq 1$, to evaluate a non-standard log, you have to use the change of base formula:

$$\log_b x = \frac{\log_a x}{\log_a b}$$

APPLICATION ACTIVITY 2.1

1) Find the exact value of x, showing your working

a) $\log_2 8 = x$ b) $\log_x 125 = 3$ c) $\log_x 64 = 0.5$

d) $\log_4 64 = x$ e) $\log_9 x = 3\frac{1}{2}$ f) $\log_2\left(\frac{1}{2}\right) = x$

2) Solve

a) $\log_9 x + 3\log_3 x = 14$

$b)$ $\log_2 x + \log_6 x = 3$

3) Solve in the following equations in the set of real numbers:

a) $2\log_2 x + \log_x 2 = 3$

b) $\ln\left(x^2 - 1\right) = \ln\left(4x - 1\right) - 2\ln 2$

c) $\begin{cases} 2\ln x + 3\ln y = -2 \\ 3\ln x + 5\ln y = -4 \end{cases}$ d) $\begin{cases} \ln\left(xy\right) = 7 \\ \ln\dfrac{x}{y} = 1 \end{cases}$

2.2 Exponential equations

ACTIVITY 2.2

1) It is noted that for $a>0$, $a^x=a^y$ implies x = y,

   Write the right side of $3^{x+1}=81$ in the power of base 3 and solve the obtained equation.

2) Apply the same process to solve $4^{3x+1}=8^{x+3}$

Equations that involve powers as terms of their expressions are referred to as **exponential equations.** Such equations can sometimes be solved by appropriately applying the Laws of exponents and property:

a) For $a>0$, and $a\neq 1$, the number a^u is positive, i.e $\forall u\in\mathbb{R}, a^u\in\mathbb{R}^+$.

b)If $a^u=a^v$, *then* $u=v$ each side of the equality must be written with the same base.

c) Sometimes, the equation of the form $a^u=b, b\in\mathbb{R}^+$ are such that Since b cannot be written as an integer power of a, write the exponential equation as the equivalent logarithmic equation $a^u=b\Rightarrow u=\log_a b$. However, if $a^u=b$, and $b\leq 0$, the equation does not have the solution.

Examples:

1) Solve each exponential equation: $4^{2x-1}=8^{x+3}$

Solutions:

$4^{2x-1}=8^{x+3}$

$\left(2^2\right)^{(2x-1)}=\left(2^3\right)^{(x+3)}$

$2^{2(2x-1)}=2^{3(x+3)}$

$2(2x-1)=3(x+3)$ If $a^u=a^v$, *then* $u=v$

$4x-1=3x+9\Rightarrow x=11$

The solution set is $\{11\}$

2) Solve the equation $2^{3x}=3^{2x-1}$

Solution

$2^{3x}=3^{2x-1}$ taking logarithms of both sides and applying logarithmic laws give

$$
\begin{aligned}
3x\log 2=(2x-1)\log 3 \Leftrightarrow\ & 3x\log 2 = 2x\log 3-\log 3\\
\Leftrightarrow\ & 3x\log 2-2x\log 3=\log 3\\
\Leftrightarrow\ & x(3\log 2-2\log 3)=\log 3\\
\Leftrightarrow\ & x=\frac{\log 3}{3\log 2-2\log 3}\\
\Rightarrow\ & x=9.327
\end{aligned}
$$

3) Solve the equation $2\left(5^{2x}\right)-5^{x}=6$

Solution

Let $y=5^{x}, with\ y>0.$

Then $2y^2-y=6$

Or $2y^2-y-6=0$

$(2y+3)(y-2)=0$

$\Rightarrow y=-1\frac{1}{2}$ is to be excluded since $y=5^{x}$ must be positive

or $y=2$

So $y=2$ gives $5^{x}=2\Rightarrow x=\log_5 2=\frac{\log 2}{\log 5}=0.431$

4) Solve $e^{-x^2}=\left(e^{x}\right)^2.\frac{1}{e^3}$

Solution

$$e^{-x^2}=\left(e^x\right)^2 . \frac{1}{e^3}$$

$$e^{-x^2}=e^{2x}.e^{-3}=e^{2x-3}$$

$$e^{-x^2}=e^{2x-3}$$

$$-x^2=2x-3$$

$$x^2+2x-3=0$$

The solution of this equation gives $x=-3$ or $x=1$

The solution set is $S=\{-3,1\}$

5) solve the folowing equations

a) $2^x=5$ b) $8.3^x=5$ c) $5^{x-2}=3^{3x+2}$ d) $4^x-2^x-12=0$

Solution

a) $2^x=5$

$x=\log_2 5=\frac{\ln 5}{\ln 2}$

b) $8.3^x=5$

$3^x=\frac{5}{8}$

$x=\frac{\ln\left(\frac{5}{8}\right)}{\ln 3}$

c) $5^{x-2}=3^{3x+2}$

$\ln\left(5^{x-2}\right)=\ln\left(3^{3x+2}\right)$

$(x-2)\ln 5=(3x+2)\ln 3$

$x\ln 5-2\ln 5=3x\ln 3+2\ln 3$

$x\ln 5-3x\ln 3=2\ln 3+2\ln 5$

$x=\frac{2(\ln 3+\ln 5)}{\ln 5-3\ln 3}$

$S=\left\{\frac{2(\ln 3+\ln 5)}{\ln 5-3\ln 3}\right\}$

d) $4^x-2^x-12=0$

$\left(2^2\right)^x-2^2-12=0$

let $y=2^x$,

then $y^2-y-12=0$

$(y-4)(y+3)=0$

$\left(2^x-4\right)\left(2^x+3\right)=0$

$2^x=4$ or $2^x=-3$

$2^x=4$ is valid but $2^x\neq -3$

$x=2$ and S$=\{2\}$

APPLICATION ACTIVITY 2.2

Solve in the following equations in the set of real numbers:

a) $9^x-2\times 3^{x+1}=27$

b) $\frac{e^x+e^{-x}}{2}=1,\left(\text{Hint: multiply by } e^x\right)$

2.3 Application of exponential and logarithmic equations in real life

2.3.1 Application of exponential equations to estimate the Population Growth

ACTIVITY 2.3.1

1) In a laboratory, for experiment, we start with 2 cells in a dish. The number of cells in the dish doubles every 30 minutes.

   a) How many cells are in the dish after four hours?

   b) After which time are there 2^{13} cells in the dish?

2) After $10\frac{1}{2}$ hours there are 2^{22} in the dish and an experiment fluid is added which eliminates half of the cells. How many cells are left?

Exponential and logarithmic equations are very useful in solving real life problems. These problems may be related to: Population growth or decay, earthquake problems, interest rate, mortgage problems, etc.

A population whose rate of increase is proportional to the size of the population at any time obeys a law of the form $P(t) = P_0 e^{kt}$.

This is known as exponential growth. Here P_0 is the original population at the time $t = 0$ and the constant $k \neq 0$.

Examples

According to United Nation data, the world population at the beginning of 1975 was approximately 4 billion and growing at rate of about 2% per year. Assuming an exponential growth model, estimate the world population at the beginning of the year 2020.

Solution:

Let t be time (in years) elapsed from the beginning of 1975.

$P(t)$ be world population in billions.

Since the beginning of 1975 corresponds to $t=0$, it follows from the given data that $P_0 = P(0) = 4(billions)$.

Since the growth rate is 2% $(k=0.02)$, it flows that the world population at time will be $P(t) = P_0e^{kt} = 4e^{0.02t}$.

Since the beginning of the year 2020 corresponds to $t=45$ (2020-1975=45), it follows that the world population by the year 2020 will be $P(45) = 4e^{0.02(45)}(billlion)$

Or $P(45) = 4e^{0.9}(billion)$

$=4(2.459603)(billion)$

$=9.838412(billion)$

Which is a population of approximately 9.8 billion.

APPLICATION ACTIVITY 2.3.1

A colony of bacteria that grows according to the law of uninhibited growth is modeled by the function $N(t)=100e^{0.045t}$ where N is measured in grams and t is measured in days.

(a) Determine the initial amount of bacteria.

(b) What is the growth rate of the bacteria?

(c) What is the population after 5 days?

(d) How long will it take for the population to reach 140 grams?

(e) What is the doubling time for the population?

2.3.2 Application of exponential function on decay

ACTIVITY 2.3.2

The amount, $A(t)\,gram$, of radioactive material in a sample after t years is given by $A(t) = 80\left(2^{-\left(\frac{t}{100}\right)}\right)$.

a. Find the amount of material in the original sample (at $t = 0$).

b. Determine the amount of the material after 2 and 5 years and conclude whether the amount is increasing or decreasing.

c. Calculate the time taken for the material to half the amount of the original material.This time is called the half-life of the material to decay.

A population whose rate of decrease is proportional to the size of the population at any time obeys a law of the forms $P = Ae^{-kt}$ .The negative sign on exponent indicates that the population is decreasing. This is known as **exponential decay**.

If a quantity has an exponential growth model, then the time required for it to double in size is called the **doubling time**. Similarly, if a quantity has an exponential decay model, then the time required for it to reduce in value by half is called the **halving time.** For radioactive elements, halving time is called **half-life.**

Examples

1) Doubling time (T) or halving time for a quantity with an exponential growth /decay model

One can show that doubling and halving times depend only on the growth rate and not on the amount present initially.

The amount A of a radioactive material present at time t is given by $A = A_0e^{kt}$, $k < 0$

where A_0 is the original amount of radioactive material and k is a negative number that represents the rate of decay.

Doubling time for a quantity with an exponential growth model $(k>0)$ is obtained by solving the given equation where $A=2A_0$

$$A=A_0e^{kt}$$

$$2A_0=A_0e^{kt}$$

$$2=e^{kt}$$

$$kt=\ln 2$$

$$t=\frac{\ln 2}{k}$$

The doubling time $T=\frac{1}{k}\ln 2\ (k>0)$.

Repeating the same procedure that was used to calculate the doubling time, we can calculate the half-life.The halving time for a quantity with an exponential decay model $(k<0)$ is $T=-\frac{1}{k}\ln 2$

2) The radioactive element carbon-14 has a half-life of 5750 years. If 100 grams of this element are present initially, how much will be left after 1000 years?

Solution:

As $T=-\frac{1}{k}\ln 2$, the decay constant is

$$k=-\frac{1}{T}\ln 2$$

$$=-\frac{1}{5750}\ln 2$$

$$=-\frac{1}{5750}0.693147181$$

$$=-0.000120547$$

$$\simeq -0.00012$$

Radioactive decay obeys a law of the forms $P(t)=P_0e^{-kt}$.

Thus if we take $t=0$ to be the present time, then $P_0=P(0)=100$, thus the amount of carbon-14 after 1000 years will be

$$
\begin{aligned}
P(1000) &= 100e^{-0.00012(1000)} \\
&= 100e^{-0.12} \\
&\simeq 100(0.88692) \\
&\simeq 88.692
\end{aligned}
$$

Thus about 88.692grams of carbon-14 will remain.

APPLICATION ACTIVITY 2.3.2

A physicist finds that an unknown radioactive substance registers 2000 counts per minute on Geiger counter. Ten days later the substance registers 1500 counts per minute. Using calculus, it can be shown that after t days the amount of radioactive material decreases and hence the number of accounts per minute $N(t)$ is directly proportional to e^{ct} for some constant c.

Determine the half –life of the substance.

2.3.3 Application of logarithmic equations to determine the magnitude of an earthquake

ACTIVITY 2.3.3

Using internet and books, carry out a research and find out how logarithms can intervene to solve problems related to the determination of the magnitude of an earthquake.

Magnitudes of **earthquakes** are measured using the **Richter scale**. On this scale, the magnitude R of an earthquake is given by $R = \log\left(\frac{I}{I_0}\right)$ where I_0 is a fixed standard intensity used for comparison, and I is the intensity of earthquakes being measured.

a) Show that if an earthquake measures $R = 3$ on Richter scale, then its intensity is 1000 times the standard, that is , $I = 1000I_0$.

b) The San Francisco earthquake of 1906 registered $R = 8.2$ on Richter scale. Express its intensity in terms of the standard intensity.

c) How many times more intense is an earthquake measuring $R = 8$ than on measuring $R = 4$?

Solution:

a) If an earthquake measures $R = 3$ on Richter scale,

then $\log\left(\frac{I}{I_0}\right) = 3$

$$\Rightarrow \frac{I}{I_0} = 10^3$$

$$\Leftrightarrow I = 10^3 I_0$$
$$\Leftrightarrow I = 100 I_0$$
Therefore intensity is 1000 times the standard, that is , $I = 1000 I_0$.

a) The San Francisco earthquake of 1906 registered $R = 8.2$ on Richter scale.

It means that $\log\left(\frac{I}{I_0}\right) = 8.2$ or $\frac{I}{I_0} = 10^{8.2} \Leftrightarrow I = 10^{8.2} I_0$ which is express its intensity in terms of the standard intensity.

b) Let E_1, E_2 be earthquakes measuring $R = 8$ and $R = 4$ respectively.

For $E_1 : R = 8 \Rightarrow \frac{I}{I_0} = 10^8 \Leftrightarrow I = 10^8 I_0;$

For $E_2 : R = 4 \Rightarrow \frac{I}{I_0} = 10^4 \Leftrightarrow I = 10^4 I_0;$

Intensity of E_1 is $I_1 = 10^8 I_0$ (1)

Intensity of E_2 is $I_2 = 10^4 I_0$ (2)

The ratio of two above equations yields

$$\frac{I_1}{I_2} = \frac{10^8 I_0}{10^4 I_0} = 10^4 \Rightarrow I_1 = 10^4 I_2 \Leftrightarrow I_2 = 10000 I_2;$$

An earthquake measuring $R = 8$ is 10000 times more intense than one measuring $R = 4$.

APPLICATION ACTIVITY 2.3.3

Earth quake can occur in any country. Assuming that there was an earthquake I^* which occurred in Rwanda in a certain past year. Discuss how we can measure eventual earthquake which may occur in our country referring to I^* instead of referring to earthquake that happened in western countries.

2.3.4 Application of exponential equations on interest rate problems

ACTIVITY 2.3.4

Mr Cauchy has a rentable house for which he asked 20,000Frw at the first month. However, the client has pay at the beginning of every month by adding 1% of the money paid for the previous month. If the money is to be paid at the new bank account for Cauchy,

a) Calculate the money kept on Cauchy's account in the middle of the second, the third and the fourth month.

b) What is the type of sequence made by the money to be paid by Cauchy's client? Determine its general term.

c) Discuss the formula to be used to find the money Mr Cauchy will find on his account at the end of 12 months.

A bank pays interest of $r\%$ per annum compounded quarterly. If P is placed in asavings account and the quarterly interest is left in the account, how much money is inthe account after 1 year?

The amount A after t years due to a principal P invested at an annual interest rate r compounded n times per year is

$$A = P.\left(1+\frac{r}{n}\right)^{nt}$$

Example

1) Mr. John operates an account with a certain bank which pays a compound interest rate of 13.5% per annum. He opened the account at the beginning of the year with 500,000 FRW and deposits the same amount of money at the beginning of every year. Calculate how much he will receive at the end of 9 years.

After how long will the money have accumulated to 3.32 million of Rwandan Fracs?

Solution

The compound interest formula:

The 1st deposit will be

$$500000+\frac{500000\times 13.5}{100}=500000\left(1+\frac{13.5}{100}\right)$$

Or

$$500000+\frac{500000\times 13.5}{100}=500000\times 1.135$$

The 2nd deposit will grow to $500000\times(1.135)^2$

The 3rd deposit will grow to $500000\times(1.135)^3$

The nth deposit will grow to $500000\times(1.135)^n$

So the 9th deposit will grow to $500000\times(1.135)^9$

The total sum

$$S_9=500000\times(1.135)+500000\times(1.135)^2+500000\times(1.135)^3+\cdots+500000\times(1.135)^9$$

$$=500000\left[1.135+(1.135)^2+(1.135)^3+\cdots+(1.135)^9\right]$$

From, $S_n=u_1\left(\frac{1-(r^n)}{1-r}\right)$ we get

$$S_9 = 500000\left[1.135\left(\frac{1-(1.135)^9}{1-1.135}\right)\right]$$

Or $S_9 = \dfrac{500000 \times 1.135 \times 2.125811278}{0.135}$

Or $S_9 = 8,936,281$

Finding how long it will take the money to accumulate to 3,320,000 FRW

$S_n = 3320000$

$$\Rightarrow 500000\left[1.135\left(\frac{1-(1.135)^n}{1-1.135}\right)\right] = 3320000$$

$$\Rightarrow \frac{1-(1.135)^n}{1-1.135} = \frac{3320000}{500000 \times 1.135}$$

$$\Leftrightarrow \frac{1-(1.135)^n}{-0.135} = \frac{3320000}{500000 \times 1.135}$$

$$\Leftrightarrow 1-(1.135)^n = -\frac{332 \times 0.135}{50 \times 1.135}$$

$$\Leftrightarrow (1.135)^n - 1 = \frac{332 \times 0.135}{50 \times 1.135}$$

$\Leftrightarrow (1.135)^n - 1 = 0.7897$

$(1.135)^n = 0.7897 + 1$

$(1.135)^n = 1.7897$

Introducing logarithm to the base 10 on both sides gives

$n \log(1.135) = \log(1.7897)$

$n = \dfrac{\log(1.7897)}{\log(1.135)}$

$n \approx 4.6$

Hence it will take 4.6 years for the amount to accumulate to 3.32 million FRW

2) A man deposits 800,000 FRW into his savings account on which interest is 15% per annum. If he makes no withdrawals, after how many years will his balance exceed 8 million FRW?

Solution

Here, the interest rate will be compound such that amount is $P\left(1+\frac{r}{100}\right)^n$, where *n=period of time*.

$$8000000 = 800000\left(1+\frac{15}{100}\right)^n$$

$$10 = (1+0.15)^n$$

$$10 = (1.15)^n$$

$$\log 10 = \log(1.15)^n$$

$$1 = n\log(1.15)$$

$$n = \frac{1}{\log(1.15)}$$

$$n \approx 16.5 \quad years$$

APPLICATION ACTIVITY 2.3.4

What annual rate of interest compounded annually should you seek if you want to double your investment in 5 years?

2.3.5 Application of exponential equations to determine the mortgage payments

ACTIVITY 2.3.5

A loan with a fixed rate of interest is said to be **amortized** if both principal and interest are paid by a sequence of equal payments made over equal periods of time.

When a loan of money V is amortized at a rate of interest i per payment period over n payment periods, the amortization problems, request us to find the payment P that, after n payment periods gives us a present value V of annuity equal to the amount of the loan. This means that we need to find P in the formula

$$V = P\left[\frac{1-(1+i)^{-n}}{i}\right]$$

a) Solve the this equation for P

b) Use the obtained equation to determine the monthly payment necessary to pay off a loan of 8000Frw at 10% per annum

(i) in 2 years

(ii) In 3 years

Determine the total amount which is paid out for each loan.

Amortization

The payment P required to pay off a loan of V francs borrowed for n payment periods at a rate of interest i per payment period is $P = V\left[\frac{i}{1-(1+i)^{-n}}\right]$

Examples:

1) Mr. Clement has just purchased a radio of 300,000Frw and has made a down payment of 60,000Frw. He can amortize the balance (300,000Frw-60,000Frw) at 6% for 30 years.

(a) What are the monthly payments?

(b) What is his total interest payment?

(c) After 20 years, what equity does he have in his radio (that is, what is the sum of the down payment and the amount paid on the loan)?

Solution

(a) The monthly payment P needed to pay off the loan of 240,000Frw at 6% for 30 years (360 months) is:

$$P = V\left[\frac{i}{1-(1+i)^{-n}}\right] = 240,000\left[\frac{\frac{0.06}{12}}{1-\left(1+\frac{0.06}{12}\right)^{-360}}\right] = 1438.92$$

$$P = 1438.92Frw$$

(b) The total paid out for the loan is $(1438.92Frw)(360) = 518,011.20Frw$

(c) After 20 years (240 months) there remains 10 years (or 120 months) of payments. The present value of the loan is the present value of a monthly payment of 1438.92Frw for 120 months at 6%, namely,

$$V = P\left[\frac{1-(1+i)^{-n}}{i}\right] = \left[\frac{1-\left(1+\frac{0.06}{12}\right)^{-120}}{\frac{0.06}{12}}\right] = 129,608.49Frw$$

2) **W**hen Mr. Thomas Rwambikana died, he left an inheritance of 15,000Frw for his family to be paid to them over a 10-year period in equal amounts at the end of each year. If the 15,000Frw is

invested at 4% per annum, what is the annual payout to the family?

Solution:

This example asks what annual payment is needed at 4% for 10 years to disperse

15,000Frw. That is, we can think of the 15,000Frw as a loan amortized at 4% for 10 years. Thepayment needed to pay off the loan is the yearly amount Mr. Rwambikana's family willreceive.

The yearly payout *P* is

$$P = V\left[\frac{i}{1-(1+i)^{-n}}\right] = \$15,000\left[\frac{0.04}{1-(1+0.04)^{-10}}\right] = \$15,000(0.1232909) = \$1849.36$$

3) Mr Unen is 20 years away from retiring and starts saving \$100 a month in an accountpaying 6% compounded monthly. When he retires, he wishes to withdraw a fixedamount each month for 25 years. What will this fixed amount be?

Solution:

After 20 years the amount A accumulated in her account is the amount of an annuity with $n = 240$ monthly payments of $P = \$100$ at an interest rate of

$i = \frac{0.06}{12}$.

$$A = P\left[\frac{(1+i)^n - 1}{i}\right] = \$100\left[\frac{\left(1+\frac{0.06}{12}\right)^{240} - 1}{\frac{0.06}{12}}\right] = \$100(462.0408952) = \$46,204.09$$

The amount W he can withdraw each month for 25 years (300 months) at 6%compounded monthly is

$$W = 46,204.09\left[\frac{\frac{0.06}{12}}{1-\left(1+\frac{0.06}{12}\right)^{-300}}\right] = \$297.69$$

APPLICATION ACTIVITY 2.3.5

A corporation is faced with a choice between two machines, both of which are designedto improve operations by saving on labor costs. Machine A costs \$8000 and will generatean annual labor savings of \$2000. Machine B costs \$6000 and will save \$1800 in laborannually. Machine A has a useful life of 7 years while machine B has a useful life of only5 years. Assuming that the time value of money (the investment opportunity rate) of the corporation is 10% per annum, which machine is preferable? (Assume annual compounding and that the savings is realized at the end of each year).

2.4 END UNIT ASSESSMENT

1) Solve the following equations for x

a) $\log_3 x = 4$

b) $\ln(x-2)(x-1) = \ln(2x+8)$

c) $\log_x 5 = \log_5 x$

d) $2^{x-1} - 2^{x-3} = 2^{3-x} - 2^{1-x}$

e) $e^{4x} - 13e^{2x} + 36 = 0$

2) A bank pays compound interest on money invested in an account. After n year a sum of $\$2000$ will rise to $\$2000 \times 1.08^n$

a) How much money is in the account after three years?

b) After how many years will the original $\$2000$ have nearly doubled

4) The population of a city, $P(n)$, n years after the population was p given is given by $P(n) = p\left(e^{\frac{n}{30}}\right)$. Find the time taken for the population to double and The time taken for the population to reach 1 million from an original population of $10\,000$.

5) The law of cooling is $\theta = Ae^{-0.02t}$ where $\theta^\circ C$ is the excess of temperature of the water over the temperature of the room temperature at time t minutes and A is a constant. Given that the constant room temperature is $20^\circ C$, and that when $t = 0$ the temperature of the water is $80^\circ C$, find the temperature of the water in Kelvin when

$t = 10$, $t = 20$, $t = 45$

UNIT 3

MATRICES OF ORDER 2 AND ORDER 3

Key unit competence: Solve problem involving the system of linear equations using matrices

3.0 INTRODUCTORY ACTIVITY

A Farmer Kalisa bought in Ruhango Market 5 Cocks and 4 Rabbits and he paid 35,000Frw, on the following day, he bought in the same Market 3 Cocks and 6 Rabbits and he paid 30,000Frw.

a)Arrange what Kalisa bought according to their types in a simple table as follows

Cocks	Rabbits	Prices

b)Discuss and explain in your own words how you can determine the cost of 1 Cocks and 1 Rabbit.

Matrices provide a means of storing large quantities of information in such a way that each piece can be easily identified and manipulated. They facilitate the solution of large systems of linear equations to be carried out in a logical and formal way so that computer implementation follows naturally. Applications of matrices extend over many areas of engineering including electrical network analysis and robotics.

3.1. Definition and order of matrix

ACTIVITY 3.1

1) One shop sold 20 cell phones and 31 computers in a particular month. Another shop sold 45 cell phones and 23 computers in the same month. Present this information as an array of rows and columns.

2) a) Observe and complete the number of students in the year two classes on one Monday.

	Boys	girls	Total
SME			
SSE			

b) If every class gets new students on Tuesday such that in SME they have 2 boys and 1 girls, in SSE they receive 1 girl and 1 boy, Complete the table for new students.

c) Complete the table for all students in an array of rows and columns.

CONTENT SUMMARY

A matrix is a rectangular arrangement of numbers or algebraic expressions which illustrate the data for a real life model in rows and columns. A matrix is denoted with a capital letter: A,B,C,…and the elements are enclosed by parenthesis

$(\ )$ *or square brackets* $[\ ]$.

Examples

$$A=\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \quad \text{B}=\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \quad \text{C}=\begin{pmatrix} 1 & 3 & 4 \\ 5 & 12 & 13 \\ 7 & 6 & 0 \end{pmatrix} \quad \text{M}=\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}$$

From matrix B, a_{32} is element of third row and second column

From matrix C, 4 is element of first row and third column.

2. The dimension or size or order of matrix

Matrix B is of order 3×3 (read "three by three") because there are 3 rows and 3 columns

Matrix M is of order $m \times n$, ("read m by n") because there are m rows and n columns

A square matrix is a matrix formed by the same number of rows and columns.

The elements of the form (a_{ij}), where the two subscripts i and j are equal, constitute the **principal diagonal** (or **leading diagonal** or **main diagonal** or **major diagonal** or **primary diagonal**).

The **secondary diagonal** (or **minor diagonal** or **anti-diagonal** or **counterdiagonal**) is formed by the elements with $i + j = n + 1$.

Square matrix of **order 2** has the form:

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

Square matrix of **order three** has the form

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

Example:

Matrix of order two

$$\begin{pmatrix} 1 & 4 \\ 3 & 11 \end{pmatrix}$$

Secondary diagonal
Leading diagonal

Matrix of order three:

$$\begin{pmatrix} 1 & -2 & 4 \\ 2 & 3 & 2 \\ 5 & 7 & 0 \end{pmatrix}$$

Secondary diagonal
Leading diagonal

Types of matrices

There are several types of matrices, but the most commonly used are

1) Row matrix: matrix formed by one row

Example: $\begin{pmatrix} 2 & 4 & 7 \end{pmatrix}$

2) Column matrix or Vector matrix: matrix formed by one column

Example: $\begin{pmatrix} 1 \\ 3 \\ 6 \end{pmatrix}$

3) Zero matrix or null matrix : all elements are zero

Example: $\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$

4) Triangular Matrix : matrix whose elements located below or above the leading diagonal are zeros. One can have

a) **Upper Triangular Matrix**

In an upper triangular matrix, the elements located below the leading diagonal are zeros.

Examples: 1) $\begin{pmatrix} 1 & 2 \\ 0 & -5 \end{pmatrix}$ 2) $\begin{pmatrix} 1 & -2 & 4 \\ 0 & 3 & 2 \\ 0 & 0 & 4 \end{pmatrix}$

b) Lower Triangular Matrix

In a lower triangular matrix, the elements above the leading diagonal are zeros.

Examples: 1) $\begin{pmatrix} 1 & 0 \\ 18 & -5 \end{pmatrix}$ 2) $\begin{pmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \\ 5 & 7 & 9 \end{pmatrix}$

5) Diagonal Matrix

In a diagonal matrix, all the elements above and below the leading diagonal are zeros.

Examples: 1) $\begin{pmatrix} 10 & 0 \\ 0 & -5 \end{pmatrix}$ 2) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{pmatrix}$

6) Scalar Matrix

A scalar matrix is a diagonal matrix in which the leading diagonal elements are equal.

Examples: 1) $\begin{pmatrix} -5 & 0 \\ 0 & -5 \end{pmatrix}$ 2) $\begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}$

7) Identity Matrix or Unity matrix

An identity matrix by multiplication of matrices (noted by **I**) is a diagonal matrix in which the leading diagonal elements are equal to 1.

Examples

1) Identity matrix of order two $I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

2) Identity matrix of order three $I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

8) Rectangular matrix: A matrix is said to be rectangular if the number of rows is not equal to the number of columns

Example: A 5x2 matrix

$$\begin{bmatrix} 1 & 2 & -3 & 4 & 5 \\ 2 & 0 & 6 & 7 & -\frac{1}{3} \end{bmatrix}$$

9) Singular matrix: a square matrix that is not invertible.

10) Nilpotent matrix A: A square matrix A satisfying $A^k = 0$ for some positive integer k

11) Invertible matrix A: A square matrix A having a multiplicative inverse, that is, a matrix B such that $AB = BA = I$

12) Idempotent matrix or projection matrix A: A matrix that is equal to its square i.e $A^2 = A \times A = A$

13) Involutory matrix: A square matrix which is its own inverse**, i.e** $AA = I$

14) Transpose matrix of A: Transpose of matrix is a matrix obtained by changing rows to columns and columns to rows. (transpose of A is denoted by A^t or A^T).

15) Orthogonal matrix: a matrix whose inverse is equal to its transpose $A^{-1} = A^t$

16) Equality of matrices

Two matrices are equal if the elements of the two matrices that occupy the same position are equal.

If $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$, then $\begin{cases} a_{11} = b_{11} \\ a_{21} = b_{21} \\ a_{12} = b_{12} \\ a_{22} = b_{22} \end{cases}$

If $\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & a_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix}$, then $\begin{matrix} a_{11} = b_{11}, a_{12} = b_{12}, a_{13} = b_{13} \\ a_{21} = b_{21}, a_{22} = b_{22}, a_{23} = b_{23} \\ a_{31} = b_{31}, a_{32} = b_{32}, a_{33} = b_{33} \end{matrix}$

APPLICATION ACTIVITY 3.1

1) Find the dimension of each matrix

$a)\begin{pmatrix} 5 & 5 & 5 \\ 1 & 1 & 1 \\ 3 & 3 & 3 \end{pmatrix}$ b) $\begin{pmatrix} 6 & -9 \end{pmatrix}$ c) $\begin{pmatrix} 2 \end{pmatrix}$ d) $\begin{pmatrix} 0 & 0 & 0 & 2 & 6 \\ 1 & 2 & 3 & 4 & 5 \end{pmatrix}$ e) $\begin{pmatrix} 1 & 1 \\ 6 & 1 \end{pmatrix}$

2) Name the following matrices

a) $\begin{pmatrix} a & \text{b} & \text{c} \end{pmatrix}$ b) $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ c) $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ c) $\begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}$ d) $\begin{pmatrix} a & b & c \\ 0 & b & d \\ 0 & 0 & e \end{pmatrix}$ e) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

f) $\begin{pmatrix} a & 0 & 0 \\ c & b & 0 \\ d & 0 & c \end{pmatrix}$ g) $\begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}$

3) *If* $A=\begin{pmatrix}3y+2 & 2\\ 2x+1 & 6\end{pmatrix}$ *and* $B=\begin{pmatrix}y-3 & 2\\ 5 & 6\end{pmatrix}$ *are equal.*

Find the value of x and y

3.2. Operations on matrices

3.2.1 Addition and subtraction of matrices

ACTIVITY 3.2.1

1) In a survey of 900 people, the following information was obtained:

200 males thought federal defense spending was too high,150 males thought federal defense spending was too low, 45 males had no opinion, 315 females thought federal defense spending was too high

125 females thought federal defense spending was too low, 65 females had no opinion.

Discuss and arrange these data in a rectangular array as follows:

	Too high	Too Low	No opinion
Male			
Female			

Then, form a matrix from the data of this table.

2) Consider the matrix A formed by present students in two classes where students of one class make one row such that $A=\begin{bmatrix}23 & 2\\ 20 & 4\end{bmatrix}$ *and B the matrix formed by students who got absent for they went to participate in a competition* $B=\begin{pmatrix}2 & 1\\ 3 & 2\end{pmatrix}$ *on that day.*

Form one matrix C representing students for the two classes when they are all present.

CONTENT SUMMARY

Given two matrices of the same dimension, $A = (a_{ij})$ and $B = (b_{ij})$, the matrix sum is defined as: $A + B = (a_{ij} + b_{ij})$. That is, the resultant matrix elements are obtained by adding the elements of the two matrices that occupy the same position.

i) If $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ and $B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$, then

$$A + B = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} + \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = \begin{pmatrix} a_{11}+b_{11} & a_{12}+b_{12} \\ a_{21}+b_{21} & a_{22}+b_{22} \end{pmatrix}$$ and

$$A - B = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} - \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = \begin{pmatrix} a_{11}-b_{11} & a_{12}-b_{12} \\ a_{21}-b_{21} & a_{22}-b_{22} \end{pmatrix}$$

ii) If $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$ and $B = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix}$, then

$$A + B = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} + \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix} = \begin{pmatrix} a_{11}+b_{11} & a_{12}+b_{12} & a_{13}+b_{13} \\ a_{21}+b_{21} & a_{22}+b_{22} & a_{23}+b_{23} \\ a_{31}+b_{31} & a_{32}+b_{32} & a_{33}+b_{33} \end{pmatrix}$$

$$A - B = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} - \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix} = \begin{pmatrix} a_{11}-b_{11} & a_{12}-b_{12} & a_{13}-b_{13} \\ a_{21}-b_{21} & a_{22}-b_{22} & a_{23}-b_{23} \\ a_{31}-b_{31} & a_{32}-b_{32} & a_{33}-b_{33} \end{pmatrix}$$

Example:

1) If $A = \begin{pmatrix} 3 & 2 \\ -1 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 2 \\ 4 & 6 \end{pmatrix}$, find the sum $A + B$ and the difference $A - B$

Solution

$$A+B=\begin{pmatrix}3 & 2\\ -1 & 1\end{pmatrix}+\begin{pmatrix}1 & 2\\ 4 & 6\end{pmatrix}=\begin{pmatrix}4 & 4\\ 3 & 7\end{pmatrix}$$

$$A-B=\begin{pmatrix}3 & 2\\ -1 & 1\end{pmatrix}-\begin{pmatrix}1 & 2\\ 4 & 6\end{pmatrix}=\begin{pmatrix}2 & 0\\ -5 & -5\end{pmatrix}$$

2) Consider the matrices $A=\begin{pmatrix}2 & 0 & 1\\ 3 & 0 & 0\\ 5 & 1 & 1\end{pmatrix}$ and $B=\begin{pmatrix}1 & 0 & 1\\ 1 & 2 & 1\\ 1 & 1 & 0\end{pmatrix}$, find $A+B$ and $A-B$

Solution

$$A+B=\begin{pmatrix}2+1 & 0+0 & 1+1\\ 3+1 & 0+2 & 0+1\\ 5+1 & 1+1 & 1+0\end{pmatrix}=\begin{pmatrix}3 & 0 & 2\\ 4 & 2 & 1\\ 6 & 2 & 1\end{pmatrix}$$

$$A-B=\begin{pmatrix}2-1 & 0-0 & 1-1\\ 3-1 & 0-2 & 0-1\\ 5-1 & 1-1 & 1-0\end{pmatrix}=\begin{pmatrix}1 & 0 & 0\\ 2 & -2 & -1\\ 4 & 0 & 1\end{pmatrix}$$

Properties

Note: The properties are the same for matrices of order two or three

1) Closure

The sum of two matrices of order two or three is another matrix of order two or three.

2) Associative

$$A+(B+C)=(A+B)+C$$

3) Additive identity

$A+0=A$, where 0 is the zero-matrix of the same dimension.

4) Additive inverse

$$A+(-A)=O$$

The opposite matrix of A is $-A$.

Example: $A=\begin{pmatrix}-2 & 1\\ 3 & 2\end{pmatrix}$, $\quad -A=\begin{pmatrix}2 & -1\\ -3 & -2\end{pmatrix}$

5) Commutative

$A + B = B + A$

b) Scalar multiplication

Given a matrix, $A = \left(a_{ij}\right)$, and a real number, $k \in IR$, the product of a real number by a matrix is a matrix of the same dimension as **A**, and each element is multiplied by **k.**

$$k \cdot A = \left(k\ a_{ij}\right)$$

If $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$ **, then** $kA = \begin{pmatrix} ka_{11} & ka_{12} & ka_{13} \\ ka_{21} & ka_{22} & ka_{23} \\ ka_{31} & ka_{32} & ka_{33} \end{pmatrix}$

Example:

1) If $A = \begin{pmatrix} -3 & 6 \\ 5 & 2 \end{pmatrix}$, find $2A$

Solution

$$2A = 2\begin{pmatrix} -3 & 6 \\ 5 & 2 \end{pmatrix} = \begin{pmatrix} -6 & 12 \\ 10 & 4 \end{pmatrix}$$

2)Consider the matrix $A = \begin{pmatrix} 2 & 0 & 1 \\ 3 & 0 & 0 \\ 5 & 1 & 1 \end{pmatrix}$, find $2A$

***Solution*: ** $2A = 2\begin{pmatrix} 2 & 0 & 1 \\ 3 & 0 & 0 \\ 5 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 4 & 0 & 2 \\ 6 & 0 & 0 \\ 10 & 2 & 2 \end{pmatrix}$

Properties:

1) $\alpha\left(\beta A\right) = \left(\alpha\beta\right)A, \quad A \in M_{m\times n}, \alpha, \beta \in IR$

2) $\alpha\left(A + B\right) = \alpha A + \alpha B, \quad A, B \in M_{m\times n}, \alpha \in IR$

3) $(\alpha+\beta)A=\alpha A+\beta A,\quad A\in M_{m\times n},\alpha,\beta\in IR$

4) $1A=A,\quad A\in M_{m\times n}$

APPLICATION ACTIVITY 3.2.1

1) Consider the matrices $A=\begin{pmatrix}13 & 4\\ 6 & 10\end{pmatrix}$ and $B=\begin{pmatrix}7 & 10\\ 3 & 4\end{pmatrix}$,

Find a) $A+3B$ b) $2A-B$

2) If $A=\begin{pmatrix}1 & -2 & 6\\ 2 & 2 & -5\\ 0 & 4 & 8\end{pmatrix}$, $B=\begin{pmatrix}0 & 8 & 9\\ 14 & 4 & 6\\ 0 & 6 & 7\end{pmatrix}$ and $C=\begin{pmatrix}13 & -2 & 0\\ 10 & 0 & 3\\ 9 & 1 & -5\end{pmatrix}$.

Evaluate:

a) $A-B$ b) $A+B-2C$ c) $2A-B+C$

3.3.2 Multiplying matrices

ACTIVITY 3.2.2

1) A clothing store sells men's shirts for \$40, silk ties for \$20, and wool suits for \$400.

Last month, the store had sales consisting of 100 shirts, 200 ties, and 50 suits.

Using matrix, discuss and explain in your own words how to determine the total revenue due to these sales.

2) Considering that

$$A \times B = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \times \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix}$$

$$= \begin{pmatrix} a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31} & a_{11}b_{12} + a_{12}b_{22} + a_{13}b_{32} & a_{11}b_{13} + a_{12}b_{23} + a_{13}b_{33} \\ a_{21}b_{11} + a_{22}b_{21} + a_{23}b_{31} & a_{21}b_{12} + a_{22}b_{22} + a_{23}b_{32} & a_{21}b_{13} + a_{22}b_{23} + a_{23}b_{33} \\ a_{31}b_{11} + a_{32}b_{21} + a_{33}b_{31} & a_{31}b_{12} + a_{32}b_{22} + a_{33}b_{32} & a_{31}b_{13} + a_{32}b_{23} + a_{33}b_{33} \end{pmatrix}$$

Evaluate $A \times B$ for $A = \begin{pmatrix} -1 & 1 & 1 \\ 2 & 2 & 5 \\ 0 & 3 & 4 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & 1 & -1 \\ 3 & 2 & 3 \\ -1 & 6 & 4 \end{pmatrix}$

CONTENT SUMMARY

Two matrices A and B can be multiplied together if and only if the number of columns of the first matrix A is equal to the number of rows of the second matrix B.

$$A_{m\times n} \times B_{m\times p} = M_{m\times p}$$

The element, c_{ij} , of the product matrix is obtained by multiplying every element in row ***i*** of matrix *A* by each element of column **j** of matrix *B* and then adding them together. This multiplication is called **ROCO** (row, column).

If $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$ and $B = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix}$, then

$$A \times B = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \times \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix}$$

$$= \begin{pmatrix} a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31} & a_{11}b_{12} + a_{12}b_{22} + a_{13}b_{32} & a_{11}b_{13} + a_{12}b_{23} + a_{13}b_{33} \\ a_{21}b_{11} + a_{22}b_{21} + a_{23}b_{31} & a_{21}b_{12} + a_{22}b_{22} + a_{23}b_{32} & a_{21}b_{13} + a_{22}b_{23} + a_{23}b_{33} \\ a_{31}b_{11} + a_{32}b_{21} + a_{33}b_{31} & a_{31}b_{12} + a_{32}b_{22} + a_{33}b_{32} & a_{31}b_{13} + a_{32}b_{23} + a_{33}b_{33} \end{pmatrix}$$

Examples

1) If $A=\begin{pmatrix}1 & 3\\ 2 & 5\end{pmatrix}$ and $B=\begin{pmatrix}2 & 0\\ 1 & 1\end{pmatrix}$, find the product $A\cdot B$

Solution

$$A\cdot B=\begin{pmatrix}1 & 3\\ 2 & 5\end{pmatrix}\cdot\begin{pmatrix}2 & 0\\ 1 & 1\end{pmatrix}$$

$$=\begin{pmatrix}1\cdot 2+3\cdot 1 & 1\cdot 0+3\cdot 1\\ 2\cdot 2+5\cdot 1 & 2\cdot 0+5\cdot 1\end{pmatrix}$$

$$=\begin{pmatrix}5 & 3\\ 9 & 5\end{pmatrix}$$

2) Consider matrices $A=\begin{pmatrix}2 & 0 & 1\\ 3 & 0 & 0\\ 5 & 1 & 1\end{pmatrix}$ and $B=\begin{pmatrix}1 & 0 & 1\\ 1 & 2 & 1\\ 1 & 1 & 0\end{pmatrix}$, find $A\times B$

Solution

$$A\times B=\begin{pmatrix}2 & 0 & 1\\ 3 & 0 & 0\\ 5 & 1 & 1\end{pmatrix}\times\begin{pmatrix}1 & 0 & 1\\ 1 & 2 & 1\\ 1 & 1 & 0\end{pmatrix}$$

$$=\begin{pmatrix}2\times 1+0\times 1+1\times 1 & 2\times 0+0\times 2+1\times 1 & 2\times 1+0\times 1+1\times 0\\ 3\times 1+0\times 1+0\times 1 & 3\times 0+0\times 2+0\times 1 & 3\times 1+0\times 1+0\times 0\\ 5\times 1+1\times 1+1\times 1 & 5\times 0+1\times 2+1\times 1 & 5\times 1+1\times 1+1\times 0\end{pmatrix}$$

$$=\begin{pmatrix}3 & 1 & 2\\ 3 & 0 & 3\\ 7 & 3 & 6\end{pmatrix}$$

APPLICATION ACTIVITY 3.2.2

If $A=\begin{pmatrix}1 & -2 & 6\\ 2 & 2 & -5\\ 0 & 4 & 8\end{pmatrix}$, $B=\begin{pmatrix}0 & 8 & 9\\ 14 & 4 & 6\\ 0 & 6 & 7\end{pmatrix}$ and $C=\begin{pmatrix}13 & -2 & 0\\ 10 & 0 & 3\\ 9 & 1 & -5\end{pmatrix}$.

Evaluate: a) $A\times B$ b) $A\times C$ c) $B\times C$

3.2.3 Properties of Matrices Multiplication

ACTIVITY 3.2.3

Consider the matrices $A=\begin{pmatrix}3 & 0 & 1\\ 0 & -1 & 1\\ -2 & 3 & 2\end{pmatrix}$ $B=\begin{pmatrix}0 & 1 & -1\\ 1 & 1 & -2\\ -1 & 0 & 1\end{pmatrix}$ and $C=\begin{pmatrix}1 & 1 & -1\\ 1 & -2 & 1\\ -1 & 1 & 0\end{pmatrix}$

find and comment on:

1. $A\times B$ and $B\times A$
2. $A\times(B+C)$ and $A\times B+A\times C$

CONTENT SUMMARY

Let A, B, C be matrices of order two or three

1) **Associative**

$A\times(B\times C)=(A\times B)\times C$

2) **Multiplicative Identity**

$A \times I = A$, where ***I*** is the identity matrix with the same order as matrix *A*.

3) **Not Commutative**

$A \times B \neq B \times A$

4) **Distributive**

$A \times (B + C) = (A \times B) + (A \times C)$

Example: Given the matrices:

$$A = \begin{pmatrix} 2 & 0 & 1 \\ 3 & 0 & 0 \\ 5 & 1 & 1 \end{pmatrix} \text{ And } B = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

Find

a) The product $A \times B$

b) The product $B \times A$

c) Conclude about the commutativity of multiplication of matrices

Solution

a)

$$A \times B = \begin{pmatrix} 2 & 0 & 1 \\ 3 & 0 & 0 \\ 5 & 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 2\times1+0\times1+1\times1 & 2\times0+0\times2+1\times1 & 2\times1+0\times1+1\times0 \\ 3\times1+0\times1+0\times1 & 3\times0+0\times2+0\times1 & 3\times1+0\times1+0\times0 \\ 5\times1+1\times1+1\times1 & 5\times0+1\times2+1\times1 & 5\times1+1\times1+1\times0 \end{pmatrix}$$

$$= \begin{pmatrix} 3 & 1 & 2 \\ 3 & 0 & 3 \\ 7 & 3 & 6 \end{pmatrix}$$

b)

$$B\times A=\begin{pmatrix}1&0&1\\1&2&1\\1&1&0\end{pmatrix}\begin{pmatrix}2&0&1\\3&0&0\\5&1&1\end{pmatrix}$$

$$=\begin{pmatrix}1\times2+0\times3+1\times5 & 1\times0+0\times0+1\times1 & 1\times1+0\times0+1\times1\\1\times2+2\times3+1\times5 & 1\times0+2\times0+1\times1 & 1\times1+2\times0+1\times1\\1\times2+1\times3+0\times5 & 1\times0+1\times0+0\times1 & 1\times1+1\times0+0\times1\end{pmatrix}$$

$$=\begin{pmatrix}7&1&2\\13&1&2\\5&0&1\end{pmatrix}$$

c) Since $B\times A\neq A\times B$, then commutativity of multiplication of matrices is not verified.

Examples:

1) Given matrices $A=\begin{pmatrix}1&-1&1\\-3&2&-1\\-2&1&0\end{pmatrix}$ and $B=\begin{pmatrix}1&2&3\\2&4&6\\1&2&3\end{pmatrix}$.

Find the product AB. What is your observation?

Solution

$$AB=\begin{pmatrix}1&-1&1\\-3&2&-1\\-2&1&0\end{pmatrix}\begin{pmatrix}1&2&3\\2&4&6\\1&2&3\end{pmatrix}$$

$$=\begin{pmatrix}1-2+1 & 2-4+2 & 3-6+3\\-3+4-1 & -6+8-2 & -9+12-3\\-2+2+0 & -4+4+0 & -6+6+0\end{pmatrix}$$

$$=\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}$$

Observation: If $AB = 0$, it does not necessarily follow that $A = 0$ or $B = 0$.

2) Given matrices $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 2 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & 4 & 0 \\ 3 & 1 & 0 \\ -1 & -4 & 1 \end{pmatrix}$. Find the product AB and BA. What is your observation?

Solution

$$AB = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 2 \end{pmatrix}\begin{pmatrix} 2 & 4 & 0 \\ 3 & 1 & 0 \\ -1 & -4 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 4 & 0 \\ 3 & 1 & 0 \\ 0 & -4 & 2 \end{pmatrix}$$

$$BA = \begin{pmatrix} 2 & 4 & 0 \\ 3 & 1 & 0 \\ -1 & -4 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 2 \end{pmatrix} = \begin{pmatrix} 2 & 4 & 0 \\ 3 & 1 & 0 \\ 0 & -4 & 2 \end{pmatrix}$$

$\Rightarrow AB = BA$

Observation: The given matrices commute in multiplication.

Notice

- If $AB = 0$, it does not necessarily follow that $A = 0$ or $B = 0$.
- **Commuting matrices in multiplication:**In general the multiplication of matrices is not commutative, i.e, $AB \neq BA$, but we can have the case where two matrices A and B satisfy $AB = BA$. In this case A and B are said to be **commuting**.

Trace of matrix

The sum of the entries on the leading diagonal of a square matrix, *A*, is known as the **trace** of that matrix, noted $tr(A)$.

Example

1. Trace of $\begin{pmatrix} 1 & -2 & 4 \\ 2 & 3 & 2 \\ 5 & 7 & 2 \end{pmatrix} = 1+3+2=6$

2. Trace of $\begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix} = 1+1=2$

Properties of trace of matrix

1) $tr(A+B)=tr(A)+tr(B)$

2) $tr(\alpha A)=\alpha\, tr(A)$

3) $tr(AB)=tr(BA)$

4) $tr(ABC)=tr(BCA)=tr(CAB)$, cyclic property.

5) $tr(ABC)\neq tr(ACB)$, arbitrary permutations are not allowed.

APPLICATION ACTIVITY 3.2.3

Consider the matrices $A=\begin{pmatrix} 1 & 0 & 1 \\ 1 & -1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ $B=\begin{pmatrix} -2 & 0 & 1 \\ 1 & 0 & 2 \\ -1 & 0 & 1 \end{pmatrix}$ and $C=\begin{pmatrix} 1 & 0 & -1 \\ -2 & 2 & 0 \\ -1 & 1 & 0 \end{pmatrix}$

find

a) $A\times B$ and $B\times A$

b) $tr(AB)$

3.3. Transpose of Matrix

ACTIVITY 3.3

Consider the matrices $A=\begin{pmatrix}1 & 3 & 1\\ 1 & -2 & 2\\ 3 & -2 & 0\end{pmatrix}$ and $B=\begin{pmatrix}12 & 3 & -1\\ 3 & -2 & 0\\ -4 & -1 & 0\end{pmatrix}$

1. Interchange the rows and columns of matrix A and B
2. Add two matrices obtained in 1
3. Add A and B
4. Interchange the rows and columns of matrix obtained in 3
5. What can you say about result in 2 and 4?

Interchange the rows and columns of matrix A twice. What can you conclude?

CONTENT SUMMARY

Given matrix A, **the transpose of matrix A**, noted A^t, is another matrix where the elements in the columns and rows have interchanged. In other words, the rows become the columns and the columns become the rows.

If $A=\begin{pmatrix}a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{pmatrix}$, **then** $A^t=\begin{pmatrix}a_{11} & a_{21} & a_{31}\\ a_{12} & a_{22} & a_{32}\\ a_{13} & a_{23} & a_{33}\end{pmatrix}$

Example:

$$A=\begin{pmatrix}1 & 3 & 6\\ 0 & 2 & 0\\ 3 & 5 & 8\end{pmatrix} \qquad A^t=\begin{pmatrix}1 & 0 & 3\\ 3 & 2 & 5\\ 6 & 0 & 8\end{pmatrix}$$

Properties of transpose of matrices

Let A, B be matrices of order two or three

1) $\left(A^t\right)^t = A$

2) $\left(A + B\right)^t = A^t + B^t$

3) $\left(\alpha \times A\right)^t = \alpha \times A^t, \alpha \in \mathbb{R}$

APPLICATION ACTIVITY 3.3

1) If $A = \begin{pmatrix} x-1 & -3 \\ 4z+x & 3y+4 \end{pmatrix}$ and $B = \begin{pmatrix} 6x+5 & -3 \\ 1 & 1 \end{pmatrix}$

If $A = B$, find the value of x, y and z and hence find

a. A

b. A^t

2) Consider matrices $A = \begin{pmatrix} 0 & 4 & 2 \\ 1 & 3 & 6 \\ 3 & -2 & 8 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 2 & 1 \\ -4 & 0 & 3 \\ 6 & 2 & 5 \end{pmatrix}$.

Evaluate: a) $\left(A+B\right)^t$ b) $3A^t + B$ c) $\left(-3B+4A\right)^t$

3) Find the value of x in $M = \begin{pmatrix} 1 & 2 & x^2 \\ 4 & 1 & 0 \\ 1 & x+3 & 8 \end{pmatrix}$ if $M^t = \begin{pmatrix} 1 & 4 & 1 \\ 2 & 1 & 1 \\ 4 & 0 & 8 \end{pmatrix}$

3.4. Determinants and inverse of a matrix of order two and three

3.4.1. Determinant of order two or three

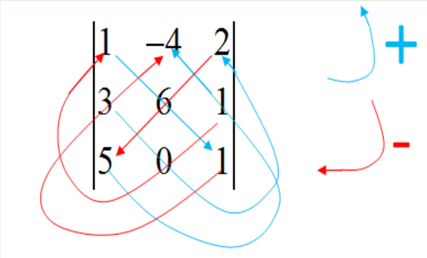
ACTIVITY 3.4.1

1) Given that $\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{21}a_{12}$

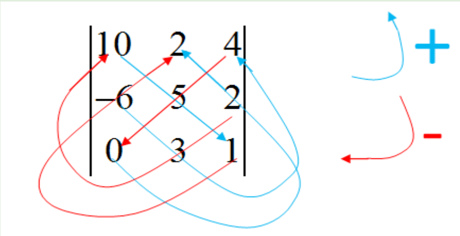
Determine: a) $\begin{vmatrix} 1 & 0 \\ 2 & 1 \end{vmatrix}$ b) $\begin{vmatrix} -2 & -4 \\ 3 & 6 \end{vmatrix}$ c) $\begin{vmatrix} 3 & 1 \\ 6 & 8 \end{vmatrix}$ d) $\begin{vmatrix} 12 & 3 \\ -2 & 9 \end{vmatrix}$

2) Evaluate the following operations by considering the direction of arrows(sum of the blue products minus sum of the red products)

a)



b)



CONTENT SUMMARY

Consider two matrices, one of order two and another one of order three:

$$M = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \quad \text{and } A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}.$$

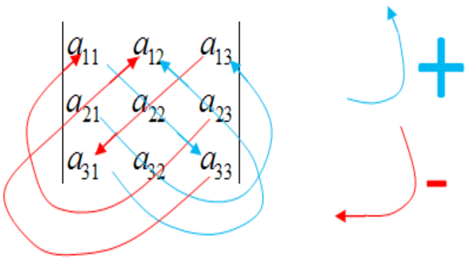
The determinant of A is calculated by SARRUS rule:

The terms with a **positive sign** are formed by the elements of the **principal diagonal** and those of the **parallel diagonals** with its corresponding **opposite vertex.**

The terms with a **negative sign** are formed by the elements of the **secondary diagonal** and those of the **parallel diagonals** with its corresponding **opposite vertex.**

$$\det M = |M| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \text{ or } |M| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{21}a_{12}$$

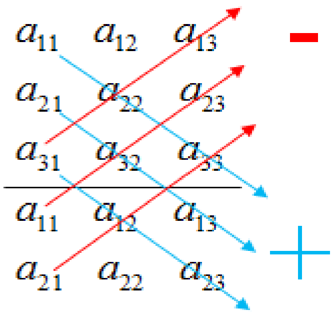
$$\det A = |A| = \begin{vmatrix} a_{11} & a_{12} & a_{3} \\ a_{21} & a_{22} & a_{3} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} =$$



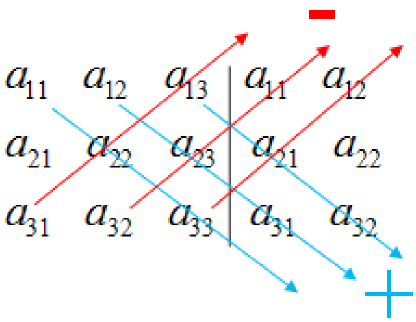
$$\det = a_{11}a_{22}a_{33} + a_{21}a_{32}a_{13} + a_{31}a_{23}a_{12} - a_{13}a_{22}a_{31} - a_{23}a_{32}a_{11} - a_{33}a_{21}a_{12}$$

Or we can work as follow:

To calculate the 3x3 determinant we rewrite the first two rows below the determinant (or first two columns to the right of the determinant).



$$\det = a_{11}a_{22}a_{33} + a_{21}a_{32}a_{13} + a_{31}a_{12}a_{23} - a_{31}a_{22}a_{13} - a_{11}a_{32}a_{23} - a_{21}a_{12}a_{33}$$



Or

$$\det = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{31}a_{22}a_{13} - a_{32}a_{23}a_{11} - a_{33}a_{21}a_{12}$$

As multiplication of real numbers is commutative, the three are the same.

Examples

1) Determine detN given that

$$N = \begin{pmatrix} 12 & 6 \\ 5 & 4 \end{pmatrix} \Rightarrow |N| = \begin{vmatrix} 12 & 6 \\ 5 & 4 \end{vmatrix} = 12\times 4 - 5\times 6 = 48 - 30 = 18$$

2) Determine detQ if $Q = \begin{pmatrix} 3 & 2 & 1 \\ 0 & 2 & -5 \\ -2 & 1 & 4 \end{pmatrix}$.

Solution:

Det Q =

$$\begin{vmatrix} 3 & 2 & 1 \\ 0 & 2 & -5 \\ -2 & 1 & 4 \end{vmatrix} = 3\times2\times4+0\times1\times1+(-2)\times(-5)\times2-1\times2\times(-2)-(-5)\times1\times3-4\times0\times2$$

$$= 24+0+20+4+15-0$$

$$= 63$$

Determinant of $n\times n$ matrices by method of minors and cofactors

General method of finding the determinant of matrix with $n\times n$ dimension $(2\times2,\ 3\times3,\ 4\times4,\ 5\times5,\ldots)$ is the use of cofactors.

Minor

An element, a_{ij}, to the value of the determinant of order $n-1$, obtained by deleting the row i and the column j in the matrix is called a **minor**.

$$\begin{vmatrix} 1 & 2 & 1 \\ 2 & [5] & 4 \\ 3 & 6 & 2 \end{vmatrix} \rightarrow \begin{vmatrix} 1 & 1 \\ 3 & 2 \end{vmatrix}$$

Cofactor

For an n by n determinant A, the cofactor of entry a_{ij}, denoted by A_{ij}, is given

$$A_{ij} = (-1)^{i+j} M_{ij}$$

where M_{ij} is the minor of entry a_{ij}.

The exponent of $(-1)^{i+j}$ is the sum of the row and column of the entry a_{ij} so if $i+j$

is even, $(-1)^{i+j}$ will equal 1, and if $i+j$ is odd, will equal -1

To find the value of a determinant, multiply each entry in any row or column byits cofactor and sum the results.

This process is referred to as **expanding across a rowor column.**

For example, the value of the 3 by 3 determinant; if we choose to expand down column 2, we obtain

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = (-1)^{1+2} a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + (-1)^{2+2} a_{22} \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} + (-1)^{3+2} a_{32} \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix}$$

Expand down column 2.

If we choose to expand across row 3, we obtain

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = (-1)^{3+1} a_{31} \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} + (-1)^{3+2} a_{32} \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} + (-1)^{3+3} a_{33} \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

Expand across row 3.

It can be shown that the value of a determinant does not depend on the choice of the row or column used in the expansion. However, expanding across a row or column that has an entry equal to 0 reduces the amount of work needed to compute the value of the determinant.

Example

$$\begin{vmatrix} 3 & 2 & 1 \\ 0 & 2 & -5 \\ -2 & 1 & 4 \end{vmatrix} = 3\begin{vmatrix} 2 & -5 \\ 1 & 4 \end{vmatrix} - 2\begin{vmatrix} 0 & -5 \\ -2 & 4 \end{vmatrix} + 1\begin{vmatrix} 0 & 2 \\ -2 & 1 \end{vmatrix}$$

$$= 3(8+5) - 2(0-10) + 1(0+4)$$
$$= 39 + 20 + 4$$
$$= 63$$

Note that we choose only one line (row or column).

APPLICATION ACTIVITY 3.4.1

Find the determinant of the following matrices

1. $A = \begin{pmatrix} 1 & 3 & 1 \\ -4 & 5 & -2 \\ -3 & 1 & 3 \end{pmatrix}$

2. $B=\begin{pmatrix}1 & 4 & -2\\ 0 & 1 & 2\\ 1 & 2 & 4\end{pmatrix}$

3. $C=\begin{pmatrix}2 & -1\\ 2 & -5\end{pmatrix}$

3.4.2. Properties of determinant

ACTIVITY 3.4.2

Consider the matrices $A=\begin{pmatrix}1 & 0 & 1\\ 1 & -1 & 1\\ 0 & 0 & 0\end{pmatrix}$, $B=\begin{pmatrix}-2 & 1 & -1\\ 1 & 3 & 4\\ -1 & 4 & 3\end{pmatrix}$,

$$C=\begin{pmatrix}1 & 0 & -1\\ 0 & 2 & 1\\ 0 & 0 & 3\end{pmatrix} \text{ and } D=\begin{pmatrix}1 & 3 & -1\\ -1 & 2 & 2\\ 0 & 1 & -1\end{pmatrix}$$

find

1. $|A|$ and $|B|$
2. $|C\cdot D|$ and $|C|\cdot|D|$. What can you conclude?
3. The product of elements for the leading diagonal of C and $|C|$. What can you conclude?

CONTENT SUMMARY

1) Matrix A and its transpose A^t have the same determinant.

$$|A^t| = |A|$$

Example

1) $A = \begin{pmatrix} 2 & 3 & 0 \\ 3 & 2 & 7 \\ 2 & 1 & 6 \end{pmatrix}$, $A^t = \begin{pmatrix} 2 & 3 & 2 \\ 3 & 2 & 1 \\ 0 & 1 & 6 \end{pmatrix}$, $|A| = |A^t| = -2$

2) $|A| = 0$ in the following cases:

a) If matrix A has two equal rows or lines

Example

$$|A| = \begin{vmatrix} 2 & 3 & 2 \\ 3 & 2 & 3 \\ 2 & 3 & 2 \end{vmatrix} = 0$$

b) If all elements of a row or column are zero.

Example $|A| = \begin{vmatrix} 2 & 3 & 2 \\ 3 & 2 & 3 \\ 0 & 0 & 0 \end{vmatrix} = 0$

c) If the elements of a line are a linear combination of the others. (say line means row or column)

Example $|A| = \begin{vmatrix} 2 & 3 & 2 \\ 1 & 2 & 4 \\ 3 & 5 & 6 \end{vmatrix} = 0$

$\mathbf{r}_3 = \mathbf{r}_1 + \mathbf{r}_2$

3) A triangular matrix 's determinant is the product of the leading diagonal elements.

Example: $|A| = \begin{vmatrix} 2 & 0 & 0 \\ 1 & 2 & 0 \\ 3 & 5 & 6 \end{vmatrix} = 2 \times 2 \times 6 = 24$

4) If a determinant switches two parallel lines its determinant changes sign.

Example: $|A| = \begin{vmatrix} 2 & 1 & 2 \\ 1 & 2 & 0 \\ 3 & 5 & 6 \end{vmatrix} = -\begin{vmatrix} 1 & 2 & 0 \\ 2 & 1 & 2 \\ 3 & 5 & 6 \end{vmatrix}$

5) If the elements of a line are added to the elements of another parallel line previously multiplied by a real number, the value of the determinant is unchanged.

Example $\begin{vmatrix} 2 & 1 & 2 \\ 1 & 2 & 0 \\ 3 & 5 & 6 \end{vmatrix} = 16 \quad c_3 = 2c_1 + c_2 + c_3 \quad \begin{vmatrix} 2 & 1 & 7 \\ 1 & 2 & 4 \\ 3 & 5 & 17 \end{vmatrix} = 16$

6) If a determinant is multiplied by a real number, any line can be multiplied by the above mentioned number, but only one.

Example: $2 \times \begin{vmatrix} 2 & 1 & 2 \\ 1 & 2 & 0 \\ 3 & 5 & 6 \end{vmatrix} = \begin{vmatrix} 2\times 2 & 1 & 2 \\ 2\times 1 & 2 & 0 \\ 2\times 3 & 5 & 6 \end{vmatrix} = \begin{vmatrix} 4 & 1 & 2 \\ 2 & 2 & 0 \\ 6 & 5 & 6 \end{vmatrix} = 32$

$$2 \times \begin{vmatrix} 2 & 1 & 2 \\ 1 & 2 & 0 \\ 3 & 5 & 6 \end{vmatrix} = 2 \times 16 = 32$$

7) If all the elements of a line are formed by two addends, the above mentioned determinant decomposes in the sum of two determinants.

Example: $\begin{vmatrix} 2 & 1 & 2 \\ a+b & a+c & a+d \\ 3 & 5 & 6 \end{vmatrix} = \begin{vmatrix} 2 & 1 & 2 \\ a & a & a \\ 3 & 5 & 6 \end{vmatrix} + \begin{vmatrix} 2 & 1 & 2 \\ b & c & d \\ 3 & 5 & 6 \end{vmatrix}$

8) The determinant of a product equals the product of the determinants.

$$|A \times B| = |A| \times |B|$$

Example

Let $A=\begin{pmatrix}1&2&1\\2&4&8\\3&2&3\end{pmatrix}$, $B=\begin{pmatrix}3&-1&1\\0&4&1\\3&2&2\end{pmatrix}$ then $A\times B=\begin{pmatrix}6&9&5\\30&30&22\\18&11&11\end{pmatrix}$

$$|A\times B|=\begin{vmatrix}6&9&5\\30&30&22\\18&11&11\end{vmatrix}=72$$

$$|A|=\begin{vmatrix}1&2&1\\2&4&8\\3&2&3\end{vmatrix}=24,\ |B|=\begin{vmatrix}3&-1&1\\0&4&1\\3&2&2\end{vmatrix}=3$$

APPLICATION ACTIVITY 3.4.2

Consider the following matrices:

$$A=\begin{pmatrix}12&0&1\\34&0&2\\-3&0&3\end{pmatrix},\ B=\begin{pmatrix}1&4&5\\2&6&8\\3&2&5\end{pmatrix},\ C=\begin{pmatrix}6&7&6\\2&4&8\\1&3&9\end{pmatrix},\ D=\begin{pmatrix}-3&5&1\\2&10&1\\1&8&1\end{pmatrix}$$

Find $|A|$, $|B|$, $|C|$ and $|D|$

3.4.3 Inverse of matrices of order two or three

ACTIVITY 3.4.3

Given the Matrix $A=\begin{pmatrix}1 & -2 & 0\\ 3 & 1 & 5\\ -1 & 2 & 3\end{pmatrix}$ and the matrix $B=\begin{pmatrix}\frac{-7}{21} & \frac{6}{21} & \frac{-10}{21}\\ \frac{-14}{21} & \frac{3}{21} & \frac{-5}{21}\\ \frac{7}{21} & 0 & \frac{7}{21}\end{pmatrix}$

a) Determine the matrix $A\times B$ and conclude on the relationship between the matrix A and the matrix B.

b) Using books or internet, make a research and find how to find the inverse of a given matrix of order 2 and the inverse for the matrix of order 3.

CONTENT SUMMARY

Calculating matrix inverse of matrix *A*, is to find matrix A^{-1} such that,

$$A\cdot A^{-1}=A^{-1}\cdot A=I$$

where *I* is identity matrix.

The matrix inverse of matrix *A* is equal to the inverse value of its determinant multiplied by the**adjoint oradjugate matrix**.

$$A^{-1}=\frac{1}{|A|}.adj(A)$$

Where $adj(A)$ is the **adjoint** (also called **adjugate**) matrix which is the transpose of the cofactormatrix. The cofactor matrix is found by replacing every element in matrix *A* by its cofactor.

Notice:

If det $A = 0$ (i.e the determinant is zero), the matrix has no inverse and is said to be a singularmatrix.

Example

Find the inverse of the following matrix

$$A = \begin{pmatrix} 2 & 0 & 1 \\ 3 & 0 & 0 \\ 5 & 1 & 1 \end{pmatrix}$$

Solution

We find its inverse as follow: $|A| = 3$

Cofactor of each element:

$$c(2) = \begin{vmatrix} 0 & 0 \\ 1 & 1 \end{vmatrix} = 0 \quad c(0) = -\begin{vmatrix} 3 & 0 \\ 5 & 1 \end{vmatrix} = -3 \quad c(1) = \begin{vmatrix} 3 & 0 \\ 5 & 1 \end{vmatrix} = 3$$

$$c(3) = -\begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix} = 1 \quad c(0) = \begin{vmatrix} 2 & 1 \\ 5 & 1 \end{vmatrix} = -3 \quad c(0) = -\begin{vmatrix} 2 & 0 \\ 5 & 1 \end{vmatrix} = -2$$

$$c(5) = \begin{vmatrix} 0 & 0 \\ 0 & 1 \end{vmatrix} = 0 \quad c(1) = -\begin{vmatrix} 2 & 1 \\ 3 & 0 \end{vmatrix} = 3 \quad c(1) = \begin{vmatrix} 2 & 0 \\ 3 & 0 \end{vmatrix} = 0$$

The cofactor matrix is

$$\begin{pmatrix} 0 & -3 & 3 \\ 1 & -3 & -2 \\ 0 & 3 & 0 \end{pmatrix}, \text{ and then } \quad adj(A) = \begin{pmatrix} 0 & 1 & 0 \\ -3 & -3 & 3 \\ 3 & -2 & 0 \end{pmatrix}$$

Therefore, the matrix inverse of A is $A^{-1} = \frac{1}{|A|} adj(A) = \frac{1}{3}\begin{pmatrix} 0 & 1 & 0 \\ -3 & -3 & 3 \\ 3 & -2 & 0 \end{pmatrix}$

$$A^{-1}=\begin{pmatrix}0 & \frac{1}{3} & 0\\ -1 & -1 & 1\\ 1 & \frac{-2}{3} & 0\end{pmatrix}$$

NB: Let A be a matrix of order two, $A=\begin{pmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{pmatrix}$, the inverse of A is given by

$$A^{-1}=\frac{1}{\det A}\begin{pmatrix}a_{22} & -a_{12}\\ -a_{21} & a_{11}\end{pmatrix}$$

Examples:

1) Find the inverse of $A=\begin{pmatrix}1 & 0\\ 2 & 1\end{pmatrix}$

Solution

$\det A = 1-0=1$

$$A^{-1}=\frac{1}{1}\begin{pmatrix}1 & 0\\ -2 & 1\end{pmatrix}=\begin{pmatrix}1 & 0\\ -2 & 1\end{pmatrix}$$

2) Find the inverse of $A=\begin{pmatrix}1 & 3\\ 2 & 6\end{pmatrix}$

Solution

$\det A = 6-6=0$

Since the determinant is zero, the given matrix has no inverse.

APPLICATION ACTIVITY 3.4.3

Find the inverse of the following matrices

1) $A=\begin{pmatrix}1 & 4 & 5\\ 2 & 6 & 8\\ 3 & 2 & 5\end{pmatrix}$ 2) $B=\begin{pmatrix}11 & -8 & 1\\ 0 & -6 & 2\\ 3 & 2 & 7\end{pmatrix}$

3) $C=\begin{pmatrix}6 & 7 & 6\\ 2 & 4 & 8\\ 1 & 3 & 9\end{pmatrix}$ 4) $D=\begin{pmatrix}-3 & 5 & 1\\ 2 & 10 & 1\\ 1 & 8 & 1\end{pmatrix}$

3.4.4 Properties of the Inverse Matrix

ACTIVITY 3.4.4

Consider the matrices $A=\begin{pmatrix}1 & 0 & 1\\ 1 & -1 & 1\\ 0 & 0 & 1\end{pmatrix}$ and $B=\begin{pmatrix}2 & 1 & 1\\ 1 & 1 & 1\\ -1 & 1 & 0\end{pmatrix}$, find

1. $(AB)^{-1}$, $B^{-1}A^{-1}$ and compare them.
2. $(A^{-1})^{-1}$ and compare $(A^{-1})^{-1}$ and A.
3. $(4A)^{-1}$, $\frac{1}{4}A^{-1}$ and compare them.
4. $(A^{t})^{-1}$ and $(A^{-1})^{t}$

What can you conclude for each result?

CONTENT SUMMARY

For two invertible matrices A and B

1) $(A \cdot B)^{-1} = B^{-1} \cdot A^{-1}$

2) $\left(A^{-1}\right)^{-1} = A$

3) $(\alpha \cdot A)^{-1} = \alpha^{-1} \cdot A^{-1}$

4) $\left(A^{t}\right)^{-1} = \left(A^{-1}\right)^{t}$

Examples

Consider matrix $A = \begin{pmatrix} 1 & 1 & -2 \\ 0 & 1 & -1 \\ 3 & 0 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 1 & 0 \\ 2 & -2 & 0 \\ 1 & 1 & 1 \end{pmatrix}$, find

a) A^{-1} and B^{-1}

b) $(AB)^{-1}$

c) $(3B)^{-1}$

d) $\left(B^{t}\right)^{-1}$

Solution

a)

$$|A| = 3, \quad Adj(A) = \begin{pmatrix} 0 & 0 & 1 \\ -3 & 6 & 1 \\ -3 & 3 & 1 \end{pmatrix}, \quad A^{-1} = \begin{pmatrix} 0 & 0 & \frac{1}{3} \\ -1 & 2 & \frac{1}{3} \\ -1 & 1 & \frac{1}{3} \end{pmatrix}$$

$$|B| = -2, \quad Adj(B) = \begin{pmatrix} -2 & -1 & 0 \\ -2 & 0 & 0 \\ 4 & 1 & -2 \end{pmatrix}, \quad B^{-1} = \begin{pmatrix} 1 & \frac{1}{2} & 0 \\ 1 & 0 & 0 \\ -2 & -\frac{1}{2} & 1 \end{pmatrix}$$

b)

$$\left(AB\right)^{-1}=B^{-1}A^{-1}=\begin{pmatrix}1 & \frac{1}{2} & 0\\ 1 & 0 & 0\\ -2 & -\frac{1}{2} & 1\end{pmatrix}\begin{pmatrix}0 & 0 & \frac{1}{3}\\ -1 & 2 & \frac{1}{3}\\ -1 & 1 & \frac{1}{3}\end{pmatrix}=\begin{pmatrix}-\frac{1}{2} & 1 & \frac{1}{2}\\ 0 & 0 & \frac{1}{3}\\ -\frac{1}{2} & 0 & -\frac{1}{2}\end{pmatrix}$$

c)

$$\left(3B\right)^{-1}=\frac{1}{3}B^{-1}=\frac{1}{3}\begin{pmatrix}1 & \frac{1}{2} & 0\\ 1 & 0 & 0\\ -2 & -\frac{1}{2} & 1\end{pmatrix}=\begin{pmatrix}\frac{1}{3} & \frac{1}{6} & 0\\ \frac{1}{3} & 0 & 0\\ -\frac{2}{3} & -\frac{1}{6} & \frac{1}{3}\end{pmatrix}$$

d)

$$\left(B^{t}\right)^{-1}=\left(B^{-1}\right)^{t}=\begin{pmatrix}1 & \frac{1}{2} & 0\\ 1 & 0 & 0\\ -2 & -\frac{1}{2} & 1\end{pmatrix}^{t}=\begin{pmatrix}1 & 1 & 2\\ \frac{1}{2} & 0 & -\frac{1}{2}\\ 0 & 0 & 1\end{pmatrix}$$

APPLICATION ACTIVITY 3.4.4

1)If $A=\begin{pmatrix}-1 & -3\\ 1 & 1\end{pmatrix}$ and $B=\begin{pmatrix}6 & 3\\ 2 & 1\end{pmatrix}$, find

a) A^{-1} b) B^{-1} c) $\left(AB\right)^{-1}$ d) $\left(A^{t}\right)^{-1}$ e) $\left(4B\right)^{-1}$

2) Consider of the following matrices $A=\begin{pmatrix}3 & 7 & 2\\ 1 & 0 & 2\\ 0 & 1 & 1\end{pmatrix}$ and $B=\begin{pmatrix}2 & -2 & 1\\ 3 & 1 & 6\\ -1 & 1 & 1\end{pmatrix}$

find :

a) A^{-1} and B^{-1} b) $\left(A^{-1}\right)^{-1}$ c) $(10A)^{-1}$ d) $\left(A^{t}\right)^{-1}$.

3.5. Applications of matrices and determinants

3.5.1 Solving System of linear equations using inverse matrix

ACTIVITY 3.5.1

A Farmer Kalisa bought in Ruhango Market 5 Cocks and 4 Rabbits and he paid 35,000Frw, on the following day, he bought in the same Market 3 Cocks and 6 Rabbits and he paid 30,000Frw.

a) Considering x as the cost for one cock and y the cost of one Rabbit, formulate equations that illustrate the activity of Kalisa;

b) Make a matrix A indicating the number of cocks and rabbits

c) If B is a matrix column made by the money paid by Kalisa, ie $B=\begin{pmatrix}35000\\ 30000\end{pmatrix}$, write the equation $A\begin{pmatrix}x\\ y\end{pmatrix}=B$

d) Discuss and explain in your own words how you can determine

$\begin{pmatrix}x\\ y\end{pmatrix}$ the cost of 1 Cocks and 1 Rabbit.

Let us explore how to find the solution of the system of 3 linear equations in 3 unknowns.

$$\begin{cases} a_{11}x + a_{12}y + a_{13}z = b_1 \\ a_{21}x + a_{22}y + a_{23}z = b_2 \\ a_{31}x + a_{32}y + a_{33}z = b_3 \end{cases} \qquad (1)$$

This equation can be written in the following:

$$\begin{pmatrix} a_{11}a_{12} & a_{13} \\ a_{21}a_{22} & a_{23} \\ a_{31}a_{32} & a_{33} \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \text{ or } AX = B, \quad \text{where } A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \text{ and } B = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}.$$

It is clear that $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = A^{-1}B$, provided that A^{-1}.

Notice

- If at least, one of **b_i** is different of zero the system is said to be **non-homogeneous** and if all **b_i** are zero the system is said to be **homogeneous.**
- The set of values of x, y, z that satisfy all the equations of system (1) is called **solution of the system.**
- For the homogeneous system, the solution $x = y = z = 0$ is **called trivial solution**. Other solutions are **non-trivial solutions**.

Non- homogeneous system cannot have a trivial solution as at least one of x, y, z is not zero.

Example

1) Solve the system of equations

$$\begin{cases} x+y+z=6 \\ 2x+y-z=1 \\ 3x+2y+z=10 \end{cases}$$

Solution:

$$\begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ 3 & 2 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}=\begin{pmatrix} 6 \\ 1 \\ 10 \end{pmatrix}$$

$$A=\begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ 3 & 2 & 1 \end{pmatrix},\ X=\begin{pmatrix} x \\ y \\ z \end{pmatrix},\ B=\begin{pmatrix} 6 \\ 1 \\ 10 \end{pmatrix}$$

We find the inverse of A.

A is invertible if its determinant is not zero.

$\det(A)=\begin{vmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ 3 & 2 & 1 \end{vmatrix}=1+4-3-3+2-2=-1\neq 0$, then A has inverse.

We have seen that the adjugate matrix and determinant of a matrix are used to

find its inverse.

Let use another useful method.

We have $A=\begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ 3 & 2 & 1 \end{pmatrix}$, to find its inverse, suppose that its inverse is given by

$$A^{-1}=\begin{pmatrix} a & d & g \\ b & e & h \\ c & f & i \end{pmatrix}$$

We know that $AA^{-1}=I$, then

$$\begin{pmatrix}1 & 1 & 1\\ 2 & 1 & -1\\ 3 & 2 & 1\end{pmatrix}\begin{pmatrix}a & d & g\\ b & e & h\\ c & f & i\end{pmatrix}=\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}\Rightarrow\begin{cases}\begin{cases}a+b+c=1\\ 2a+b-c=0\\ 3a+2b+c=0\end{cases} & (1)\\ \begin{cases}d+e+f=0\\ 2d+e-f=1\\ 3d+2e+f=0\end{cases} & (2)\\ \begin{cases}g+h+i=0\\ 2g+h-i=0\\ 3g+2h+i=1\end{cases} & (3)\end{cases}$$

We solve these three systems to find value of $a,\ b,\ c,\ d,\ e,\ f,\ g,\ h,$ and i .

$$\begin{cases}a+b+c=1\\ 2a+b-c=0\\ 3a+2b+c=0\end{cases}\quad (1)\Rightarrow\begin{cases}a=-3\\ b=5\\ c=-1\end{cases}$$

$$\begin{cases}d+e+f=0\\ 2d+e-f=1\\ 3d+2e+f=0\end{cases}\quad (2)\Rightarrow\begin{cases}d=-1\\ e=2\\ f=-1\end{cases}$$

$$\begin{cases}g+h+i=0\\ 2g+h-i=0\\ 3g+2h+i=1\end{cases}\quad (3)\Rightarrow\begin{cases}g=2\\ h=-3\\ i=1\end{cases}$$

Then,

$$A^{-1}=\begin{pmatrix}-3 & -1 & 2\\ 5 & 2 & -3\\ -1 & -1 & 1\end{pmatrix}$$

$$X=\begin{pmatrix}x\\ y\\ z\end{pmatrix}=\begin{pmatrix}-3 & -1 & 2\\ 5 & 2 & -3\\ -1 & -1 & 1\end{pmatrix}\begin{pmatrix}6\\ 1\\ 10\end{pmatrix}=\begin{pmatrix}1\\ 2\\ 3\end{pmatrix}$$

Therefore, $S=\{(1,2,3)\}$.

APPLICATION ACTIVITY 3.5.1

Solve the following system using the inverse of a matrix

$$\begin{cases} 5x+15y+56z=35 \\ -4x-11y-41z=-26 \\ -x-3y-11z=-7 \end{cases}$$

3.5.2 Solving System of linear equations using Cramer method

ACTIVITY 3.5.2

Consider the following system of 3 linear equations with 3 unknowns.

$$\begin{cases} x+2y-3z=0 \\ 3x+3y-z=5 \\ x-2y+2z=1 \end{cases}$$

a) Rewrite this system in matrix form $A\begin{pmatrix} x \\ y \\ z \end{pmatrix}=\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$

b) Calculate $\Delta = \det A$, determinant of coefficients of unknowns

c) Calculate Δ_x , determinant of coefficients of unknowns where the column of x was replaced by the column of independent terms $\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$

d) Calculate Δ_y , determinant of coefficients of unknowns where the column of y was replaced by the column of independent terms.

e) Calculate Δ_z , determinant of coefficients of unknowns where the column of z was replaced by the column of independent terms.

f) Find the $x=\frac{\Delta_x}{\Delta}$, y $=\frac{\Delta_y}{\Delta}$, z $=\frac{\Delta_z}{\Delta}$,

g) Compare the obtained values to solutions of the given system,solved by using other methods seen in previous levels.

CONTENT SUMMARY

Consider the system

$$\begin{cases} a_{11}x+a_{12}y+a_{13}z=b_1 \\ a_{21}x+a_{22}y+a_{23}z=b_2 \\ a_{31}x+a_{32}y+a_{33}z=b_3 \end{cases}$$

We use Cramer's rule as follows

$$\Delta=\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$\Delta_x=\begin{vmatrix} c_1 & a_{12} & a_{13} \\ c_2 & a_{22} & a_{23} \\ c_3 & a_{32} & a_{33} \end{vmatrix}$$

$$\Delta_y=\begin{vmatrix} a_{11} & c_1 & a_{13} \\ a_{21} & c_2 & a_{23} \\ a_{31} & c_3 & a_{33} \end{vmatrix}$$

$$\Delta_z=\begin{vmatrix} a_{11} & a_{12} & c_1 \\ a_{21} & a_{22} & c_2 \\ a_{31} & a_{32} & c_3 \end{vmatrix}$$

$x=\frac{\Delta_x}{\Delta}$, $y=\frac{\Delta_y}{\Delta}$ and $z=\frac{\Delta_z}{\Delta}$

Notice:

- If $\Delta=0$, $\Delta_x\neq0$, $\Delta_y\neq0$,$\Delta_z\neq0$, then the system has no solution. In this case, the system is said to be inconsistent.
- If $\Delta=0$, $\Delta_x=0$, $\Delta_y=0$,$\Delta_z=0$, then the system has infinitely many solutions. In this case, the system is said to be consistent.
- If $\Delta\neq0$, $\Delta_x\neq0$, $\Delta_y\neq0$,$\Delta_z\neq0$, then the system has a unique solution. . In this case, the system is said to be consistent.

Example

Use Cramer's method tosolve the following system

$$\begin{cases} x+y+z=6 \\ 2x+y-z=1 \\ 3x+2y+z=10 \end{cases}$$

Solution:

$$\Delta=\begin{vmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ 3 & 2 & 1 \end{vmatrix}=-1$$

$$\Delta_x=\begin{vmatrix} 6 & 1 & 1 \\ 1 & 1 & -1 \\ 10 & 2 & 1 \end{vmatrix}=-1$$

$$\Delta_y=\begin{vmatrix} 1 & 6 & 1 \\ 2 & 1 & -1 \\ 3 & 10 & 1 \end{vmatrix}=-2$$

$$\Delta_z=\begin{vmatrix} 1 & 1 & 6 \\ 2 & 1 & 1 \\ 3 & 2 & 10 \end{vmatrix}=-3$$

$$x=\frac{\Delta_x}{\Delta}=\frac{-1}{-1}=1,\quad y=\frac{\Delta_y}{\Delta}=\frac{-2}{-1}=2,\quad z=\frac{\Delta_z}{\Delta}=\frac{-3}{-1}=3$$

Therefore, $S=\{(1,2,3)\}$

APPLICATION ACTIVITY 3.5.2

Use matrix inverse method to solve the following systems

1. $\begin{cases} 3x+y+z=0 \\ 2x-y+2x=0 \\ 7x+y-3z=0 \end{cases}$

2. $\begin{cases} 4x+y-z=1 \\ x-3y+z=2 \\ 5x-2y=4 \end{cases}$

3. $\begin{cases} x+y-z=3 \\ 3x-y+z=1 \\ -2x+y+z=0 \end{cases}$

3.5.3 Solving system of linear equations using Gaussian method (elimination of Gauss)

ACTIVITY 3.5.3

Consider the following system of 3 linear equations with 3 unknowns.

$$\begin{cases} x+2y-2z=1 \\ 2x+y-4z=-1 \\ 4x-3y+z=11 \end{cases}$$

equivalent to

$$\begin{bmatrix} 1 & 2 & -2 \\ 2 & 1 & -4 \\ 4 & -3 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 1 \\ -1 \\ 11 \end{bmatrix}$$

From the given matrix, complete the augmented martix of the form:

$$\left[\begin{array}{ccc|c} 1 & 2 & -2 & 1 \\ 2 & 1 & -4 & -1 \\ 4 & -3 & 1 & 11 \end{array}\right] \quad \begin{array}{c} r_1 \\ r_2 = 2r_1 - r_2 \\ r_3 = 4r_1 - r_3 \end{array} \sim \left[\begin{array}{ccc|c} \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \end{array}\right].$$

Multiplythe equivalent matrixobtained by the matix of unknowns and determine the values of x, y *and* z (using the following form).

$$\left(\begin{array}{ccc|c} . & . & . & . \\ . & . & . & . \\ . & . & . & . \end{array}\right)\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

CONTENT SUMMARY

Resolution of systems of linear equation by **Gauss's method**

Let system:

$$(1)\begin{cases} a_{11}x_1 + a_{11}x_2 \dots\dots\dots\dots\dots\dots\dots a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 \dots\dots\dots\dots\dots\dots\dots a_{2n}x_n = b_2 \\ \dots \\ a_{n1}x_1 + a_{n2}x_2 \dots\dots\dots\dots\dots\dots\dots a_{nn}x_n = b_n \end{cases}$$

The system is equivalent to the system:

$$\begin{pmatrix} a_{11}a_{12} \dots\dots a_{1n} \\ \dots \\ a_{n1}a_{n2} \dots\dots a_{nn} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ . \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ . \\ b_n \end{pmatrix}$$

A simply $Ax = B$ where $A = \left(a_{ij}\right)$ and called matrix of coefficients $B = \left(b_{ij}\right)$ and called matrix of constants $X = \left(x_{ij}\right)$ and called matrix of unknowns

$$\left(\begin{array}{ccc|c} a_{11} & \cdots & a_{1n} & b_1 \\ \vdots & \ddots & \vdots & . \\ a_{m1} & \cdots & a_{mn} & b_n \end{array}\right)$$

From the matrix called **augmented matrix**

We manipulate the **elimination of Gauss**, for getting the equivalent matrix if we perform one of the following operations:

1) Exchange the rows L_i and L_j
2) Replace the row L_i by L_j
3) Replace the row L_i by $L_i + kL_j$
4) Replace the row L_i by $L_i + \sum_{j\neq 1} kL_j$

Examples

1) Solve by elimination method of Gauss $\begin{cases} x+y-z=0 \\ x+2y+3z=14 \\ 2x+y+4z=16 \end{cases}$

Solution

$\rightarrow$ Augmented matrix: $\left(\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 1 & 2 & 3 & 14 \\ 2 & 1 & 4 & 6 \end{array}\right) L_2 \sim L_2 - L_1$

$$\rightarrow \left(\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 0 & 1 & 4 & 14 \\ 2 & 1 & 4 & 6 \end{array}\right) L_3 \sim L_3 - 2L_1$$

$$\rightarrow \left(\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 0 & 1 & 4 & 14 \\ 0 & 0 & 10 & 30 \end{array}\right);$$

from the last matrix we have $10z = 30 \Rightarrow z = 3$

$\Rightarrow y + 4z = 14 \Rightarrow y + 12 = 14 \Rightarrow y = 2$

$\Rightarrow x + y - z = 0 \Rightarrow x + 2 - 3 = 0 \Rightarrow x = 1$

$S = \{(1,2,3)\}$

2)Solve by using Gauss's method

$$1)\ \begin{cases} 3x+2y+4z=-1 \\ 2x-y+2z=-2 \\ -x+y+2z=2 \end{cases}$$

$$2)\ \begin{cases} -x+2y=5 \\ 2x+3y=4 \\ 3x-6y=-15 \end{cases}$$

Solution

$$1)\ \left(\begin{array}{ccc|c} 3 & 2 & 4 & -1 \\ 2 & -1 & 2 & -1 \\ -1 & 1 & 1 & 2 \end{array}\right) L_2 \sim 3L_2 - 2L_1$$

$$\Rightarrow \left(\begin{array}{ccc|c} 3 & 2 & 4 & -1 \\ 0 & -7 & -2 & -2 \\ -1 & 1 & 1 & 2 \end{array}\right) L_3 \sim 3L_3 - L_1$$

$$\Rightarrow \left(\begin{array}{ccc|c} 3 & 2 & 4 & -1 \\ 0 & -7 & -2 & -4 \\ 0 & 5 & 10 & 5 \end{array}\right) L_3 \sim 5L_2 + 7L_3$$

$$\Rightarrow \begin{pmatrix} 3 & 2 & 4 & -1 \\ 0 & -7 & -2 & -4 \\ 0 & 0 & 60 & 15 \end{pmatrix}$$

$$\Rightarrow 60z = 15 \Rightarrow \boldsymbol{z} = \frac{1}{4}$$

$$3x+2y+4z=-1 \Rightarrow 3x+1+1=-1 \Rightarrow 3x=-3 \Rightarrow \boldsymbol{x}=-1$$

$$\Rightarrow -7y-2z=-4 \Rightarrow -7y-\frac{2}{4}=-4 \Rightarrow \boldsymbol{y}=\frac{1}{2}$$

$$S=\left\{-1;\ \frac{1}{2};\ \frac{1}{4}\right\}$$

2) $\begin{cases} -x+2y=5 \\ 2x+3y=4 \\ 3x-6y=-15 \end{cases}$

$$\Rightarrow [A/B]=\left[\begin{array}{cc|c} -1 & 2 & 5 \\ 2 & 3 & 4 \\ 3 & -6 & -15 \end{array}\right] L_2 \sim L_2+2L_1$$

$$\Rightarrow [A/B]=\left[\begin{array}{cc|c} -1 & 2 & 5 \\ 0 & 7 & 14 \\ 3 & -6 & -15 \end{array}\right] L_3 \sim L_3+3L_1$$

$$\Rightarrow \left[\frac{A}{B}\right]=\left[\begin{array}{cc|c} -1 & 2 & 5 \\ 0 & 7 & 14 \\ 0 & 0 & 0 \end{array}\right] L_2 \sim \frac{1}{7}L_2 \Rightarrow [A/B]=\left[\begin{array}{cc|c} -1 & 2 & 5 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore

$y=2 \Rightarrow -x+4=5$ and $x=-1$

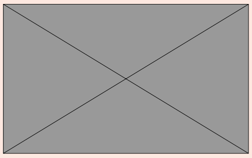
This gives the set of solution $S=\{(-1,2)\}$.

Note: The method of **GAUSS** helps us to solve the above system where **CRAMER'S** method cannot.

APPLICATION ACTIVITY 3.5.3

Solve by using elimination of Gauss

1) $\begin{cases} 2x-2y=5 \\ 4x+3y=1 \end{cases}$

2) 

3.6 END UNIT ASSESSMENT

1) Find the values of x, y and z for $3\begin{pmatrix} x & y-1 \\ 4 & 3z \end{pmatrix} = \begin{pmatrix} 15 & 6 \\ 6z & 3x+y \end{pmatrix}$

2) If $A = \begin{pmatrix} 3 & -1 & 3 \\ 1 & 0 & -6 \\ 0 & -4 & 2 \end{pmatrix}$, $B = \begin{pmatrix} 10 & 2 & 3 \\ 1 & -4 & 6 \\ 0 & 6 & 4 \end{pmatrix}$ and $C = \begin{pmatrix} 11 & 12 & 0 \\ 1 & 0 & 8 \\ 2 & 0 & 7 \end{pmatrix}$.

Evaluate: a) $A-B$ b) $A+B-2C$ c) $2A-B+C$ d) $A\times B$

3) Investors choose different stocks to keep their goods. The matrix below shows the prices (in US dollars) of one share of each of several stocks on the first business day of September, October and November.

$$\begin{array}{l} \\ Stock\ A \\ Stock\ B \\ Stock\ C \\ Stock\ D \end{array} \begin{pmatrix} Sept & Oct & Nov \\ 33.81 & 30.91 & 27.25 \\ 15.06 & 13.25 & 8.75 \\ 54 & 54 & 46.44 \\ 52.06 & 44.69 & 34.38 \end{pmatrix}$$

a) Kamana owns 42 shares of stock A, 59 shares of stock B, 21 shares of stock C, and 18 shares of stock D. Write a row matrix to represent Kamana's portfolio.

b) Use matrix multiplication to find the total value of Kamana's portfolio for each month to the nearest cent.

4) Find the inverse of $A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$

5) Use matrix inverse method to solve $\begin{cases} x+3y+3z=0 \\ 3x+4y-z=0 \\ -3x-9y+z=0 \end{cases}$

UNIT 4

BIVARIATE STATISTICS

Key unit Competence: Extend understanding, analysis and interpretation of bivariate data to correlation coefficients and regression lines

4.0 INTRODUCTORY ACTIVITY

In Kabeza village, after her 9 observations about farming, UMULISA saw that in every house observed, where there is a cow (X) if there is also domestic duck (Y), then she got the following results:

$(1,4),(2,8),(3,4),(4,12),(5,10)$
$(6,14),(7,16),(8,6),(9,18)$

a. Represent this information graphically in $(x,y)-coordinates$.

b. Find the equation of line joining any two points of the graph and guess the name of this line.

c. According to your observation from (a), explain in your own words if there is any relationship between the variation of Cows (X) and the variation of domestic duck (Y).

4.1 Bivariate data, scatter diagram and types of correlation

ACTIVITY 4.1

Consider the situation in which the mass, y (g), of a chemical is related to the time , x minutes, for which the chemical reaction has been taking place ,according to the table.

Time, x min	5	7	12	16	20
Mass,y g	4	12	18	21	24

a) Plot the above information in (x, y) coordinates.

b) Explain in your own words the relationship between x *and* y

In statistics, **bivariate** or **double series** includes technique of analyzing data in two variables, when focus on the relationship between a dependent variable-y and an independent variable-x.

For example, between age and weight, weight and height, years of education and salary, amount of daily exercise and cholesterol level, etc. As with data for a single variable, we can describe bivariate data both graphically and numerically. In both cases we will be primarily concerned with determining whether there is a *linear* relationship between the two variables under consideration or not.

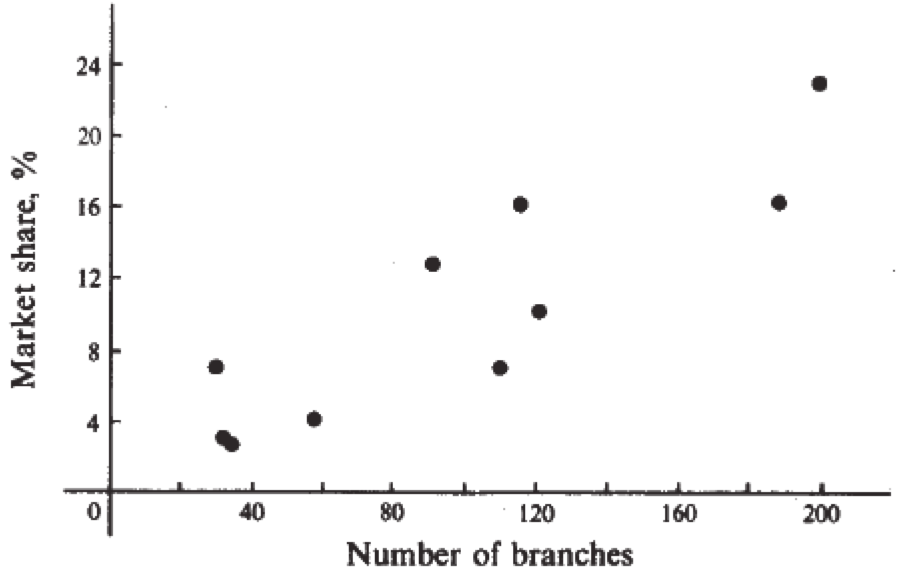
It should be kept in mind that a statistical relationship between two variables does not necessarily imply a *causal* relationship between them. For example, a strong relationship between weight and height does not imply that either variable causes the other.

Scatter plots or Scatter diagram and types of correlation

Consider the following data which relate *x*, the respective number of branches that 10 different banks have in a given common market, with *y*, the corresponding market share of total deposits held by the banks:

x	198	186	116	89	120	109	28	58	34	31
y	22.7	16.6	15.9	12.5	10.2	6.8	6.8	4.0	2.7	2.8

If each point (x, y) of the data is plotted in an x, y *coordinate* plane, **the *scatter plot or Scatter diagram*** is obtained.

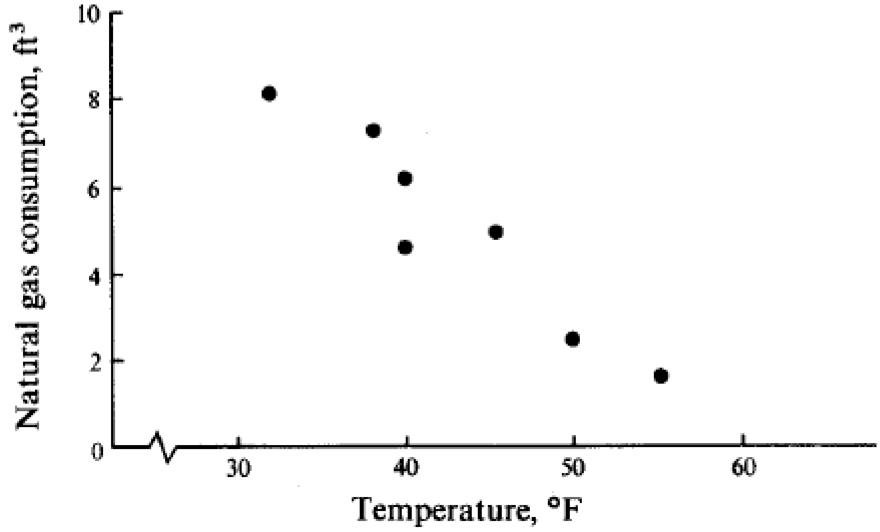


The scatter plot or scatter diagram (in the figure above) indicates that, roughly speaking, the market share increases as the number of branches increases. We say that x and y have a ***positive correlation.***

On the other hand, consider the data below, which relate average daily temperature x, in degrees Fahrenheit, and daily natural gas consumption y, in cubic metre.

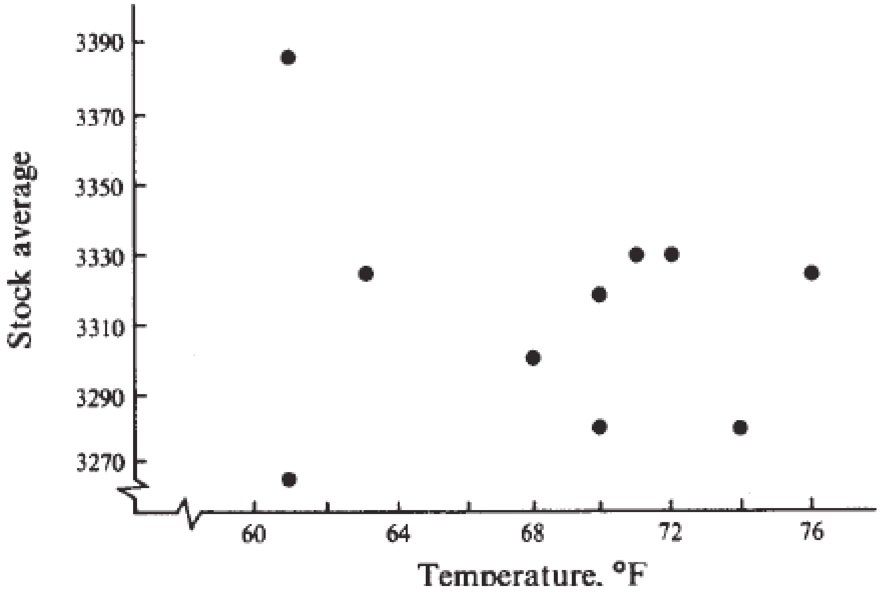
x, ° F	50	45	40	38	32	40	55
y, ft^3	2.5	5.0	6.2	7.4	8.3	4.7	1.8

Finally, consider the data items (x, y) below, which relate daily temperature x over a 10-day period to the Dow Jones stock average y.



We see that y tends to decrease as x *increases*. Here, x and y have a ***negative correlation***.

Finally, consider the data items (x, y) below, which relate daily temperature x over a 10-day period to the Dow Jones stock average y: (63, 3385); (72, 3330); (76, 3325); (70, 3320); (71, 3330); (65, 3325); (70, 3280); (74, 3280) ;(68, 3300); (61, 3265).



There is **no apparent relationship between *x* and *y*** (***no correlation*** or ***Weak correlation.***

APPLICATION ACTIVITY 4.1

One measure of personal fitness is the time taken for an individual's pulse rate to return to normal after strenuous exercise, the greater the fitness, the shorter the time. Following a short program of strenuous exercise Norman recorded his pulse rates P at time t minutes after he had stopped exercising. Norman's results are given in the table below.

t	0.5	1.0	1.5	2.0	3.0	4.0	5.0
P	125	113	102	94	81	83	71

a) Draw a scatter diagram to represent this information in

(x, y) *coordinates*

b) Explain the relationship between Norman's pulse P and time t.

4.2 Covariance

ACTIVITY 4.2

Complete the following table

i	x_i	y_i	$x_i - \bar{x}$	$y_i - \bar{y}$	$(x_i - \bar{x})(y_i - \bar{y})$
1	3	6			
2	5	9			
3	7	12			
4	3	10			
5	2	7			
6	6	8			
	$\sum_{i=1}^{6} x_i = \ldots$ $\bar{x} = \ldots..$	$\sum_{i=1}^{6} y_i = \ldots$ $\bar{y} = \ldots..$			$\sum_{i=1}^{6} (x_i - \bar{x})(y_i - \bar{y}) = \ldots$

What can you get from the following expressions?

1) $\sum_{i=1}^{k}\left(x_i-\bar{x}\right)\left(x_i-\bar{x}\right)$

2) $\sum_{i=1}^{k}\left(x_i-\bar{x}\right)\left(y_i-\bar{y}\right)$

In case of two variables, say x and y, there is another important result called **covariance of x and y**, denoted $\operatorname{cov}(x, y)$.

The **covariance of variables x and y** is a measure of how these two variables change together. If the greater values of one variable mainly correspond with the greater values of the other variable, and the same holds for the smaller values, i.e. the variables tend to show similar behavior, the covariance is positive. In the opposite case, when the greater values of one variable mainly correspond to the smaller values of the other, i.e. the variables tend to show opposite behavior, the covariance is negative. If covariance is zero the variables are said to be **uncorrelated,** itmeans that there is no linear relationship between them.

Therefore, the sign of covariance shows the tendency in the linear relationship between the variables. The magnitude of covariance is not easy to interpret.

Covariance of variables x and y, where the summation of frequencies $\sum_{i=1}^{k} f_i = n$ are equal for both variables, is defined to be

$$\operatorname{cov}(x, y)=\frac{1}{n}\sum_{i=1}^{k} f_i\left(x_i-\bar{x}\right)\left(y_i-\bar{y}\right)$$

Developing this formula we have

$$\begin{aligned}
\operatorname{cov}(x, y)&=\frac{1}{n}\sum_{i=1}^{k} f_i\left(x_i y_i-x_i\,\bar{y}-\bar{x}\,y_i+\bar{x}\,\bar{y}\right)\\
&=\frac{1}{n}\sum_{i=1}^{k} f_i x_i y_i-\frac{1}{n}\sum_{i=1}^{k} f_i x_i\,\bar{y}-\frac{1}{n}\sum_{i=1}^{k} f_i\,\bar{x}\,y_i+\frac{1}{n}\sum_{i=1}^{k} f_i\,\bar{x}\,\bar{y}\\
&=\frac{1}{n}\sum_{i=1}^{k} f_i x_i y_i-\frac{1}{n}\bar{y}\sum_{i=1}^{k} f_i x_i-\frac{1}{n}\bar{x}\sum_{i=1}^{k} f_i\,y_i+\bar{x}\,\bar{y}\frac{1}{n}\sum_{i=1}^{k} f_i \qquad \left[\frac{1}{n}\sum_{i=1}^{k} f_i=\frac{1}{n}\times n=1\right]\\
&=\frac{1}{n}\sum_{i=1}^{k} f_i x_i y_i-\bar{x}\,\bar{y}-\bar{x}\,\bar{y}+\bar{x}\,\bar{y}\\
&=\frac{1}{n}\sum_{i=1}^{k} f_i x_i y_i-\bar{x}\,\bar{y}
\end{aligned}$$

Thus, the covariance is also given by

$$\operatorname{cov}(x, y)=\frac{1}{n} \sum_{i=1}^{k} f_i x_i y_i-\bar{x}\, \bar{y}$$

Examples

1) Find the covariance of x and y in following data sets

x	3	5	6	8	9	11
y	2	3	4	6	5	8

Solution

We have

x_i	y_i	$x_i-\bar{x}$	$y_i-\bar{y}$	$(x_i-\bar{x})(y_i-\bar{y})$
3	2	-4	-2.6	10.4
5	3	-2	-1.6	3.2
6	4	-1	-0.6	0.6
8	6	1	1.4	1.4
9	5	2	0.4	0.8
11	8	4	3.4	13.6
$\sum_{i=1}^{6} x_i=42$	$\sum_{i=1}^{6} y_i=28$			
$\bar{x}=7$	$\bar{y}=4.6$			$\sum_{i=1}^{6}(x_i-\bar{x})(y_i-\bar{y})=30$

Thus, as *f = 1*,

$$\operatorname{cov}(x, y)=\frac{1}{6} \sum_{i=1}^{6} f_i(x_i-\bar{x})(y_i-\bar{y}) \quad \text{As f=1,}$$

$$=\frac{1}{6}(30)$$

$$=5$$

2) Find the covariance of the following distribution

y \ x	0	2	4
1	2	1	3
2	1	4	2
3	2	5	0

Solution

Convert the double entry into a simple table and compute the arithmetic means

x_i	y_i	f_i	$x_i f_i$	$y_i f_i$	$x_i y_i f_i$
0	1	2	0	2	0
0	2	1	0	2	0
0	3	2	0	6	0
2	1	1	2	1	2
2	2	4	8	8	16
2	3	5	10	15	30
4	1	3	12	3	12
4	2	2	8	4	16
4	3	0	0	0	0
		$\sum_{i=1}^{9} f_i = 20$	$\sum_{i=1}^{9} x_i f_i = 40$	$\sum_{i=1}^{9} y_i f_i = 41$	$\sum_{i=1}^{9} x_i y_i f_i = 76$

$$\bar{x} = \frac{40}{20} = 2, \quad \bar{y} = \frac{41}{20} = 2.05$$

$$\operatorname{cov}(x, y) = \frac{76}{20} - 2 \times 2.05 = -0.3$$

Alternative method

$$\operatorname{cov}(x, y) = \frac{1}{n} \sum_{i=1}^{k} f_i x_i y_i - \bar{x}\,\bar{y}$$

$$\bar{x} = \frac{1}{n} \sum_{i=1}^{k} x_i f_i, \quad \bar{y} = \frac{1}{n} \sum_{i=1}^{k} y_i f_i$$

$y \backslash x$	0	2	4	Total
1	2	1	3	6
2	1	4	2	7
3	2	5	0	7
Total	5	10	5	20

$$\bar{x} = \frac{1}{20}(0\times5+2\times10+4\times5)$$

$$= \frac{40}{20} = 2$$

$$\bar{y} = \frac{1}{20}(1\times6+2\times7+3\times7)$$

$$= \frac{41}{20} = 2.05$$

$$\operatorname{cov}(x,y) = \frac{1}{20}\begin{pmatrix}0\times1\times2+0\times2\times1+0\times3\times2+2\times1\times1+2\times2\times4\\ +2\times3\times5+4\times1\times3+4\times2\times2+4\times3\times0\end{pmatrix} - 2\times2.05$$

$$= \frac{1}{20}(0+0+0+2+16+30+12+16+0)-4.1$$

$$= \frac{76}{20} - 4.1 = -0.3.$$

APPLICATION ACTIVITY 4.2

1. The scores of 12 students in their mathematics and physics classes are

Mathematics	2	3	4	4	5	6	6	7	7	8	10	10
Physics	1	3	2	4	4	4	6	4	6	7	9	10

Find the covariance of the distribution

2. The values of two variables x and y are distributed according to the following table

y \ x	100	50	25
14	1	1	0
18	2	3	0
22	0	1	2

Calculate the covariance

4.3 Coefficient of correlation

Pearson's coefficient of correlation (or Product moment coefficient of correlation)

ACTIVITY 4.3

Consider the following table

x	y
3	6
5	9
7	12
3	10
2	7
6	8

Find σ_x, σ_y

1. Find $\text{cov}(x, y)$
2. Calculate the ratio $\dfrac{Cov(x,y)}{\sigma_x . \sigma_y}$

The Pearson's coefficient of correlation (or Product moment coefficient of correlation or simply coefficient of correlation), denoted by r, is a measure of the strength of linear relationship between two variables.

The coefficient of correlation between two variables x and y is given by

$$r = \frac{\text{cov}(x, y)}{\sigma_x \sigma_y}$$

Where, $\text{cov}(x, y)$ is covariance of x and y

σ_x is the standard deviation for x

σ_y is the standard deviation for y

Properties of the coefficient of correlation

a) The coefficient of correlation does not change the measurement scale. That is, if the height is expressed in meters or feet, the coefficient of correlation does not change.

b) The sign of the coefficient of correlation is the same as the covariance.

c) The square of the coefficient of correlation is equal to the product of the gradient of the regression line of y on x, and the gradient of the regression line of x on y.

In fact, $r = \frac{\text{cov}(x, y)}{\sigma_x \sigma_y}$. Squaring both sides gives

$$r^2 = \left[\frac{\text{cov}(x, y)}{\sigma_x \sigma_y}\right]^2$$

$$= \frac{\text{cov}^2(x, y)}{\sigma_x^2 \sigma_y^2}$$

$$= \frac{\text{cov}(x, y)}{\sigma_x^2} \times \frac{\text{cov}(x, y)}{\sigma_y^2}$$

$= ac$, where $y = ax + b$ is the equation of the regression line of y on x, and $x = cy + d$ is the equation of the regression line of x on y

d) If the coefficient of correlation is known, it can be used to find the gradients or slopes of two regression lines.

We know that the gradient of the regression line of y on x is $\frac{\operatorname{cov}(x,y)}{\sigma_x^2}$.

From this we have,
$$\frac{\operatorname{cov}(x,y)}{\sigma_x^2} = \frac{\operatorname{cov}(x,y)}{\sigma_x\sigma_x} \times \frac{\sigma_y}{\sigma_y}$$
$$= \frac{\operatorname{cov}(x,y)}{\sigma_x\sigma_y} \times \frac{\sigma_y}{\sigma_x}$$
$$= r\frac{\sigma_y}{\sigma_x}$$

We know that the gradient of the regression line of x on y is $\frac{\operatorname{cov}(x,y)}{\sigma_y^2}$. From this we have,
$$\frac{\operatorname{cov}(x,y)}{\sigma_y^2} = \frac{\operatorname{cov}(x,y)}{\sigma_y\sigma_y} \times \frac{\sigma_x}{\sigma_x}$$
$$= \frac{\operatorname{cov}(x,y)}{\sigma_x\sigma_y} \times \frac{\sigma_x}{\sigma_y}$$
$$= r\frac{\sigma_x}{\sigma_y}$$

Thus, the gradient of the regression line y on x is given by $r\frac{\sigma_y}{\sigma_x}$ and the gradient of the regression line of x on y is given by $r\frac{\sigma_x}{\sigma_y}$.

e) Cauchy Inequality: $\operatorname{cov}^2(x,y) \le \sigma_x^2\sigma_y^2$

In fact, $r = \frac{\operatorname{cov}(x,y)}{\sigma_x\sigma_y} \Leftrightarrow \operatorname{cov}(x,y) = r\sigma_x\sigma_y$.

Squaring both sides gives $\operatorname{cov}^2(x,y) = r^2\sigma_x^2\sigma_y^2$

Or $\operatorname{cov}^2(x,y) \le \sigma_x^2\sigma_y^2$

f) The coefficient of correlation takes value ranging between -1 and +1. That is $-1 \le r \le 1$

In fact, from Cauchy Inequality we have,

$$\operatorname{cov}^2(x,y) \le \sigma_x^2\sigma_y^2$$

$$\Leftrightarrow \frac{\operatorname{cov}^2(x,y)}{\sigma_x^2\sigma_y^2} \le 1$$

$$\Leftrightarrow \left[\frac{\operatorname{cov}(x,y)}{\sigma_x\sigma_y}\right]^2 \le 1$$

$$\Leftrightarrow r^2 \le 1.$$

Taking square roots both side $\Leftrightarrow \sqrt{r^2} \le 1 \quad \Leftrightarrow |r| \le 1$ since $\sqrt{x^2} = |x|$

$|r| \le 1$ is equivalent to $-1 \le r \le 1$.

Thus, $-1 \le r \le 1$.

g) If the linear coefficient of correlation takes values closer to **−1**, the **correlation** is **strong and negative**, and will become stronger the closer ***r*** approaches −1.

h) If the linear coefficient of correlation takes values close to **1** the **correlation** is **strong and positive**, and will become stronger the closer ***r*** approaches 1

i) If the linear coefficient of correlation takes values close to **0**, the **correlation** is **weak**.

j) If ***r = 1*** or ***r = −1***, there is **perfect correlation** and the line on the scatter plot is increasing or decreasing respectively.

k) If ***r = 0***, there is **no linear correlation**.

Examples:

1) A test is made over 200 families on number of children (x) and number of beds y per family. Results are collected in the table below

2)

y \ x	0	1	2	3	4	5	6	7	8	9	10
1	0	2	7	5	2	0	0	0	0	0	0
2	2	2	10	8	15	1	0	0	0	0	0
3	1	3	5	6	8	6	1	0	0	0	0
4	0	2	8	2	6	12	10	8	0	0	0
5	0	1	0	2	5	6	10	5	7	3	3
6	0	0	0	2	4	5	5	2	3	3	2

a) What is the average number for children and beds per a family?

b) Find the covariance.

c) Can we confirm that there is a high linear correlation between the number of children and number of beds per family?

Solution

a) Average number of children per family:

Contingency table:

y \ x	0	1	2	3	4	5	6	7	8	9	10	Total
1	0	2	7	5	2	0	0	0	0	0	0	16
2	2	2	10	8	15	1	0	0	0	0	0	38
3	1	3	5	6	8	6	1	0	0	0	0	30
4	0	2	8	2	6	12	10	8	0	0	0	48
5	0	1	0	2	5	6	10	5	7	3	3	42
6	0	0	0	2	4	5	5	2	3	3	2	26
Total	**3**	**10**	**30**	**25**	**40**	**30**	**26**	**15**	**10**	**6**	**5**	**200**

Marginal series:

x_i	0	1	2	3	4	5	6	7	8	9	10	Total
f_j	3	10	30	25	40	30	26	15	10	6	5	$\sum f_i = 200$
$f_i x_i$	0	10	60	75	160	150	156	105	80	54	50	$\sum f_i x_i = 900$

y_j	1	2	3	4	5	6	Total
f_j	16	38	30	48	42	26	$\sum f_j = 200$
$f_j y_j$	16	76	90	192	210	156	$\sum f_j y_j = 740$

$$\bar{x} = \frac{1}{n}\sum_{i=1}^{k} f_i x_i,\quad \bar{y} = \frac{1}{n}\sum_{i=1}^{k} f_i y_i$$

The means are

$$\bar{x} = \frac{1}{n}\sum_{i=1}^{11} f_i x_i = \frac{900}{200} = 4.5$$

And $\bar{y} = \frac{1}{n}\sum_{j=1}^{6} f_j y_j = \frac{740}{200} = 3.7$

There are about 5 children per family and about 4 beds per family.

b) The covariance is calculated as follow:

$\operatorname{cov}(x,y) = \left(\frac{1}{n}\sum_{(i,j)=(1,1)}^{(p,q)=(11,6)} f_{ij} x_i y_j\right) - \overline{x}\,\overline{y}$ where i assumes values from 1 to $p = 11$, and j assumes values from 1 to $q = 6$, or

$\operatorname{cov}(x,y) = \frac{1}{200}\sum_{i=1}^{66} f_i x_i y_i - \overline{x}\,\overline{y}$ where $\bar{y} = 3.7$ and $\bar{x} = 4.5$

$$= \frac{1}{200}\begin{pmatrix} 0+2+14+15+8+0+4+40+48+120+10+0 \\ +9+30+54+96+90+18+0+8+64+24+96 \\ +240+240+224+0+5+0+30+100+150 \\ +300+175+280+135+150+0+36+96+150 \\ +180+84+144+162+120 \end{pmatrix} - 4.5\times 3.7$$

$$= \frac{3751}{200} - 16.65$$

$$= 18.7555 - 16.65$$

$$= 2.105$$

c) Correlation coefficient is given by $\frac{\operatorname{cov}(x,y)}{\sigma_x \sigma_y}$

$$\sigma_x^2 = \frac{1}{200}\sum_{i=1}^{11} f_i x_i^2 - \left(\bar{x}\right)^2$$

$$= \frac{1}{200}\begin{pmatrix} 0+10+4\times 30+9\times 25+40\times 16+30\times 25+ \\ 36\times 26+49\times 15+64\times 10+81\times 6+100\times 5 \end{pmatrix} - (4.5)^2$$

$$= 25.21 - 20.25 = 4.96$$

$$\sigma_y^2 = \frac{1}{200}\sum_{i=1}^{6} f_i y_i^2 - \left(\bar{y}\right)^2$$

$$= \frac{1}{200}(16+38\times 4+30\times 9+48\times 16+42\times 25+26\times 36) - (3.7)^2$$

$$= 15.96 - 13.69$$

$$= 2.27$$

Therefore the correlation coefficient is

$$r = \frac{2.105}{\sqrt{4.96}\sqrt{2.27}} \approx 0.63$$

There is a high correlation.

NOTICE:

Spearman's coefficient of rank correlation

A Spearman coefficient of rank correlation or Spearman's rho is measure of statistical dependence between two variables. It assesses how well the relationship between two variables can be described using a monotonic function. The Spearman's coefficient of rank correlation is denoted and defined by

$$\rho = 1 - \frac{6\sum_{i=1}^{k} d_i^2}{n\left(n^2 - 1\right)}$$

Where, d refers to the difference of ranks between paired items in two series and n is the number of observations. It is much easier to calculate the Spearman's coefficient of rank correlation than to calculate the Pearson's coefficient of correlation as there is far less working involved. However, in general, the Pearson's coefficient of correlation is a more accurate measure of correlation when data are numerical.

Method of ranking

Ranking can be done in ascending order or descending order.

Examples:

1) Suppose that we have the marks, x, of seven students in this order:

12, 18, 10, 13, 15, 16, 9

We assign the rank 1, 2, 3, 4, 5, 6, 7 such that the smallest value of x will be ranked 1.

That is

x	12	18	10	13	15	16	9
$Rank(x)$	3	7	2	4	5	6	1

If we have two or more equal values we proceed as follow:

Consider the following series

x	66	65	66	67	66	64	68	68

To assign the rank to this series we do the following:

$x = 64$ will take rank 1, since it is the smallest value of x

$x = 65$ will be ranked 2.

$x = 66$ appears 3 times, since the previous value was ranked 2 here 66 would be ranked 3, another 66 would be ranked 4 and another 5 but since there are three 66's we need to find the average of those ranks which is $\frac{3+4+5}{3} = 4$ so that each 66 will be ranked 4.

$x = 67$ will be ranked 6 since we are on the 6th position

$x = 68$ appears 2 times, since the previous value was ranked 6 here 68 would be ranked 7, and another 66 would be ranked 8 but since there are two 68's we need to find the average of those ranks which is $\frac{7+8}{2} = 7.5$ so that each 68 will be ranked 7.5

Thus we have the following

x	66	65	66	67	66	64	68	68
$Rank(x)$	4	2	4	6	4	1	7.5	7.5

2) Calculate the Spearman's coefficient of rank correlation for the series

x	12	8	16	12	7	10	12	16	12	9
y	6	5	7	7	4	6	8	13	10	10

Solution

x	y	$Rank(x)$	$Rank(y)$	$Rank(x) - Rank(y) = d$	d^2
12	6	6.5	3.5	3	9
8	5	2	2	0	0
16	7	9.5	5.5	4	16
12	7	6.5	5.5	1	1
7	4	1	1	0	0
10	6	4	3.5	0.5	0.25
12	8	6.5	7	0.5	0.25
16	13	9.5	10	0.5	0.25
12	10	6.5	8.5	2	4
9	10	3	8.5	5.5	30.25
					$\sum_{i=1}^{10} d_i^2 = 61$

Then

$$\rho = 1 - \frac{6 \times 61}{10(100-1)}$$

$$\Leftrightarrow \rho = 1 - \frac{366}{990}$$

$$\Leftrightarrow \rho = \frac{990-366}{990}$$

Or

$$\rho = 0.63$$

APPLICATION ACTIVITY 4.3

1) The scores of 12 students in their mathematics and physics classes are

Mathematics	2	3	4	4	5	6	6	7	7	8	10	10
Physics	1	3	2	4	4	4	6	4	6	7	9	10

Find the correlation coefficient distribution and interpret it.

2) The values of the two variables X and Y are distributed according to the following table:

X \ Y	0	2	4
1	2	1	3
2	1	4	2
3	2	5	0

Calculate the correlation coefficient.

2) The marks of eight candidates in English and Mathematics are

Candidate	1	2	3	4	5	6	7	8
English (x)	50	58	35	86	76	43	40	60
Mathematics (y)	65	72	54	82	32	74	40	53

Rank the results and hence find Spearman's rank correlation coefficient between the two sets of marks. Comment on the value obtained.

4.4 Regression lines

ACTIVITY 4.4

Given the data in the table below:

x	5	7	12	16	20
y	4	12	18	21	24

Determine:

a) Variance of x

b) Variance of y

c) Covariance $of\ (x, y)$

d) The value given by $\frac{\operatorname{cov}(x,y)}{\operatorname{var} x} = \frac{S_{x,y}}{S_{x,x}}$

e) The value given by $b = \overline{Y} - a\overline{X}$

f) Establish the equation of the line $y = ax + b$

g) In the Cartesian plane, represent the data (x, y) in the table above and the line $y = ax + b$ found in (f).

h) Discuss the position of the points of coordinate (x, y) with respect to the line $y = ax + b$.

CONTENT SUMMARY

We use the regression line of y on x to **predict** a value of y for any given value of x and vice versa, we use the regression line of x *on* y, to **predict** a value of x for a given value of *y.* The "best" line would make the best predictions: the observed y-values should stray as little as possible from the line. This straight line is the regression line from which we can adjust its algebraic expressions and it is written as $y = ax + b$, where a *is* the gradient and b is the y-intercept.

The regression line y on x is written as $y = \frac{\operatorname{cov}(x, y)}{\sigma_x^2} x + \left(\bar{y} - \frac{\operatorname{cov}(x, y)}{\sigma_x^2} \bar{x} \right)$

We may write

$$L_{y/x} \equiv y - \bar{y} = \frac{\operatorname{cov}(x, y)}{\sigma_x^2} (x - \bar{x})$$

Note that the regression line x on y is $x = cy + d$, where c *is* the gradient of the line and d is the x-intercept, it is given by

$$x - \bar{x} = \frac{\operatorname{cov}(x, y)}{\sigma_y^2} (y - \bar{y})$$

This line is written as

$$L_{x/y} \equiv x - \bar{x} = \frac{\operatorname{cov}(x, y)}{\sigma_y^2} (y - \bar{y})$$

Shortcut method of finding regression line

To abbreviate the calculations, the two regression lines can be determined as follow:

a) The equation of the regression line of y on x is $L_{y/x} \equiv y = ax + b$ and the values of a and b are found by solving the simultaneous equations:

$$\begin{cases} \sum_{i=1}^{k} f_i y_i = a \sum_{i=1}^{k} f_i x_i + b n \\ \sum_{i=1}^{k} f_i x_i y_i = a \sum_{i=1}^{k} f_i x_i^2 + b \sum_{i=1}^{k} f_i x_i \end{cases}$$

These equations are called the **normal equations** for y on x.

b) The equation of the regression line of x on y is $L_{x/y} \equiv x = cy + d$ and the values of c and d are found by solving the simultaneous equations:

$$\begin{cases} \sum_{i=1}^{k} f_i x_i = c\sum_{i=1}^{k} f_i y_i + d\,n \\ \sum_{i=1}^{k} f_i x_i y_i = c\sum_{i=1}^{k} f_i y_i^2 + d\sum_{i=1}^{k} f_i y_i \end{cases}$$

These equations are called the **normal equations** for x on y.

Examples:

1) Find the equation of the regression line of y on x , and the equation of the regression line of x on y for the following data and estimate the value of y for $x = 4, x = 7, x = 16$ *and the value of x for* $y = 7, y = 9, y = 16$.

x	3	5	6	8	9	11
y	2	3	4	6	5	8

Solution

x	y	$x-\bar{x}$	$y-\bar{y}$	$(x-\bar{x})^2$	$(y-\bar{y})^2$	$(x-\bar{x})(y-\bar{y})$
3	2	-4	-2.6	16	6.76	10.4
5	3	-2	-1.6	4	2.56	3.2
6	4	-1	-0.6	1	0.36	0.6
8	6	1	1.4	1	1.96	1.4
9	5	2	0.4	4	0.16	0.8
11	8	4	3.4	16	11.56	13.6
$\sum_{i=1}^{6} x_i = 42$	$\sum_{i=1}^{6} y_i = 28$			$\sum_{i=1}^{6}(x_i-\bar{x})^2 = 42$	$\sum_{i=1}^{6}(y_i-\bar{y})^2 = 23.36$	$\sum_{i=1}^{6}(x_i-\bar{x})(y_i-\bar{y}) = 30$

$$\bar{x} = \frac{42}{6} = 7\,,\ \bar{y} = \frac{28}{6} = 4.7$$

$$\operatorname{cov}(x, y) = \frac{1}{n}\sum(x-\bar{x})(y-\bar{y}) = \frac{30}{6} = 5$$

$$\sigma_x^2 = \frac{42}{6} = 7\,,\ \sigma_y^2 = \frac{23.36}{6} = 3.89$$

$$L_{y/x} \equiv y - \overline{y} = \frac{\operatorname{cov}(x,y)}{\sigma_x^2}\left(x - \overline{x}\right)$$

$$L_{y/x} \equiv y - 4.7 = \frac{5}{7}(x - 7)$$

Finally, equation of the regression line of y on x is $L_{y/x} \equiv y = \frac{5}{7}x - 0.3$

And

$$L_{x/y} \equiv x - \overline{x} = \frac{\operatorname{cov}(x,y)}{\sigma_y^2}\left(y - \overline{y}\right)$$

$$L_{x/y} \equiv x - 7 = \frac{5}{3.89}(y - 4.7)$$

Finally, equation of the regression lineof x on y is $L_{x/y} \equiv y = 1.3x + 1$.

Alternative method

x	y	x^2	y^2	xy
3	2	9	4	6
5	3	25	9	15
6	4	36	16	24
8	6	64	36	48
9	5	81	25	45
11	8	121	64	88
$\sum_{i=1}^{6} x_i = 42$	$\sum_{i=1}^{6} y_i = 28$	$\sum_{i=1}^{6} x_i^2 = 336$	$\sum_{i=1}^{6} y_i^2 = 154$	$\sum_{i=1}^{6} x_i y_i = 226$

$$L_{y/x} \equiv y = ax + b$$

$$\begin{cases} \sum_{i=1}^{k} f_i y_i = a\sum_{i=1}^{k} f_i x_i + b\,n \\ \sum_{i=1}^{k} f_i x_i y_i = a\sum_{i=1}^{k} f_i x_i^2 + b\sum_{i=1}^{k} f_i x_i \end{cases}$$

$$\begin{cases} 28 = 42a + 6b \\ 226 = 336a + 42b \end{cases} \Leftrightarrow \begin{cases} a = \dfrac{5}{7} \\ b = -0.3 \end{cases}$$

Thus, the line of y on x is $L_{y/x} \equiv y = \dfrac{5}{7}x - 0.3$

$x = 4 \Rightarrow y = 2.5$

$x = 7 \Rightarrow y = 4.7$

$x = 16 \Rightarrow y = 11.1$

$L_{x/y} \equiv x = cy + d$

$$\begin{cases} \sum_{i=1}^{k} f_i x_i = c\sum_{i=1}^{k} f_i y_i + d\, n \\ \sum_{i=1}^{k} f_i x_i y_i = c\sum_{i=1}^{k} f_i y_i^2 + d\sum_{i=1}^{k} f_i y_i \end{cases}$$

$$\begin{cases} 42 = 28c + 6d \\ 226 = 154c + 28d \end{cases} \Leftrightarrow \begin{cases} c = 1.3 \\ d = 1 \end{cases}$$

Thus, the line of x on y is $L_{x/y} \equiv x = 1.3y + 1$

$y = 7 \Rightarrow x = 10.1$

$y = 9 \Rightarrow x = 12.7$

$y = 16 \Rightarrow x = 21.8$

APPLICATION ACTIVITY 4.4

1. Consider the following table

x	y
60	3.1
61	3.6
62	3.8
63	4
65	4.1

Find the regression line of y on x

a) Calculate the approximate y value for the variable $x = 64$

2. The values of two variables x and y are distributed according to the following table

y \ x	100	50	25
14	1	1	0
18	2	3	0
22	0	1	2

Find the regression lines

4.5 Interpretation of statistical data (Application)

ACTIVITY 4.5

Explain in your own words how statistics, especially bivariate statistics, can be used in our daily life.

Bivariate statistics can help in prediction of a value for one variable if we know the value of the other.

Examples:

1. *One measure of personal fitness is the time taken for an individual's pulse rate to return to normal after strenuous exercise, the greater the fitness, the shorter the time. Following a short program of strenuous exercise Norman recorded his pulse rates* P *at time* t *minutes after he had stopped exercising. Norman's results are given in the table below.*

t	0.5	1.0	1.5	2.0	3.0	4.0	5.0
P	125	113	102	94	81	83	71

Estimate Norman's pulse rate 2.5 minutes after stopping the exercise program.

Solution

t	P	t^2	P^2	tP
0.5	125	0.25	15625	62.5
1	113	1	12769	113
1.5	102	2.25	10404	153
2	94	4	8836	188
3	81	9	6561	243
4	83	16	6889	332
5	71	25	5041	355
$\sum_{i=1}^{7} t_i = 17$	$\sum_{i=1}^{7} P_i = 669$	$\sum_{i=1}^{7} t_i^2 = 57.5$	$\sum_{i=1}^{7} P_i^2 = 66125$	$\sum_{i=1}^{7} t_i P_i = 1446.5$

We need the line $P = at + b$

Use the formula

$$\begin{cases} \sum_{i=1}^{7} P_i = a\sum_{i=1}^{7} t_i + bn \\ \sum_{i=1}^{7} t_i P_i = a\sum_{i=1}^{7} t_i^2 + b\sum_{i=1}^{7} t_i \end{cases}$$

We have

$$\begin{cases} 669 = 17a + 7b \\ 1446.5 = 57.5a + 17b \end{cases}$$

Solving we have

$$\begin{cases} a = -11 \\ b = 122.3 \end{cases}$$

Then $P = -11t + 122.3$

So, the Norman's pulse rate 2.5 minutes after stopping the exercise program is estimated to be $P = -11(2.5) + 122.3$ or 94.8.

APPLICATION ACTIVITY 4.5

1) An old film is treated with a chemical in order to improve the contrast. Preliminary tests on nine samples drawn from a segment of the film produced the following results.

Sample	A	B	C	D	E	F	G	H	I
x	1.0	1.5	2.0	2.5	3.0	3.5	4.0	4.5	5.0
y	49	60	66	62	72	64	89	90	90

The quantity is a measure of the amount of chemical applied, and is the constant index, which takes values between 0 (no contrasts) and 100(maximum contrast).

i). Plot a scatter diagram to illustrate the data.

ii).It is subsequently discovered that one of the samples of film was damaged and produced an incorrect result. State which sample you think this was.

In all subsequent calculations this incorrect sample was ignored. The remaining data can be summarized as follows: $\sum x = 23.5$, $\sum y = 584$, $\sum x^2 = 83.75$, $\sum y^2 = 44622$, $\sum xy = 1883$ $n = 8$.

iii. Calculate the product moment correlation coefficient,

iv. State with a reason whether it is sensible to conclude from your answer to part(iii) that and are linearly related.

v. The line of regression of on x has equation $y = ax + b$. Calculate the value of a and b each correct to three significant figures.

vi. Use your regression line to estimate what the contrast index corresponding to the damaged piece of film would have been if the piece has been undamaged.

vii. State with a reason, whether it would be sensible to use your regression equation to estimate the contrast index when the quantity of chemical applied to the film is zero.

4.6 END UNIT ASSESSMENT

1) The following results were obtained from lineups in Mathematics and Physics examinations:

	Mathematics (x)	*Physics* (y)
Mean	475	39.5
Standard deviation	16.8	10.8

$r = 0.95$

Find both equations of the regression lines. Also estimate the value of y for $x = 30$.

2) For a set of 20 pairs of observation s of the variables x and y, it is known that $\sum_{i=1}^{k} f_i x_i = 250$, $\sum_{i=1}^{k} f_i y_i = 140$, and that the regression line of y on x passes through $(15,10)$. Find the equation of that regression line and use it to estimate y when $x = 10$.

3) In a partially destroyed laboratory record of an analysis of correlation data, the following results only are legible: Variance of x is 9

Equationsofregressionlines: $8x - 10y + 66 = 0$ and $40x - 18y - 214 = 0$

What were

a) the mean values of x and y

b) the standard deviation of y, and

c) the coefficient of correlation between x and y.

4) The table below shows the marks awarded to six students in a competition:

Student	A	B	C	D	E	F
Judge 1	6.8	7.3	8.1	9.8	7.1	9.2
Judge 2	7.8	9.4	7.9	9.6	8.9	6.9

Calculate a coefficient of rank correlation.

5) A company is to replace its fleet of cars. Eight possible models are considered and the transport manager is asked to rank them, from 1 to 8, in order of preference. A saleswoman is asked to use each type of car for a week and grade them according to their suitability for the job (*A*-very suitable to *E*-unsuitable).

The price is also recorded:

Model	Transport manager's ranking	Saleswoman's grade	Price (£10s)
S	5	B	611
T	1	B+	811
U	7	D-	591
V	2	C	792
W	8	B+	520
X	6	D	573
Y	4	C+	683
Z	3	A-	716

a. Calculate the Spearman's coefficient of rank correlation between

  i. price and transport manager's rankings,

  ii. price and saleswoman's grades.

b. Based on the result of a. state, giving a reason, whether it would be necessary to use all three different methods of assessing the cars.

c. A new employee is asked to collect further data and to do some calculations. He produces the following results:

   The coefficient of correlationbetween

   i. price and boot capacity is 1.2,

   ii. maximum speed and fuel consumption in miles per gallons is -0.7,

   iii. price and engine capacity is -0.9

   For each of his results say, giving a reason, whether you think it is reasonable.

d. Suggest two sets of circumstances where Spearman's coefficient of rank correlation would be preferred to the Pearson's coefficient of correlation as a measure of association.

UNIT 5

CONDITIONAL PROBABILITY AND BAYES THEOREM

Key unit Competence: Apply rules of probability to solve problems related to dependent and independent events.

5.0 INTRODUCTORY ACTIVITY

1) Consider a machine which manufactures electronic components. These must meet certain specification. The quality control departmentregularly samples the components.

Suppose, on average, 92 out of 100 components meet the specification. Imaginethata

Componentisselected atrandom and let A be the outcome that a component meets the specification; let B be the outcome that a component does not meet the specification.

a) Explain in your own words and determine the probability that a components meet the specification.

b) Explain in your own words and determine the probability that a component does not meet the specification.

2) A box contains 4 whitechalks and 3 black chalks. One chalk is drawn at random; its color is noted but not replaced in the box.

a) What is the probability of selecting2 white chalks?

b) Determine the probability of selecting 3 white and 2 black chalks.

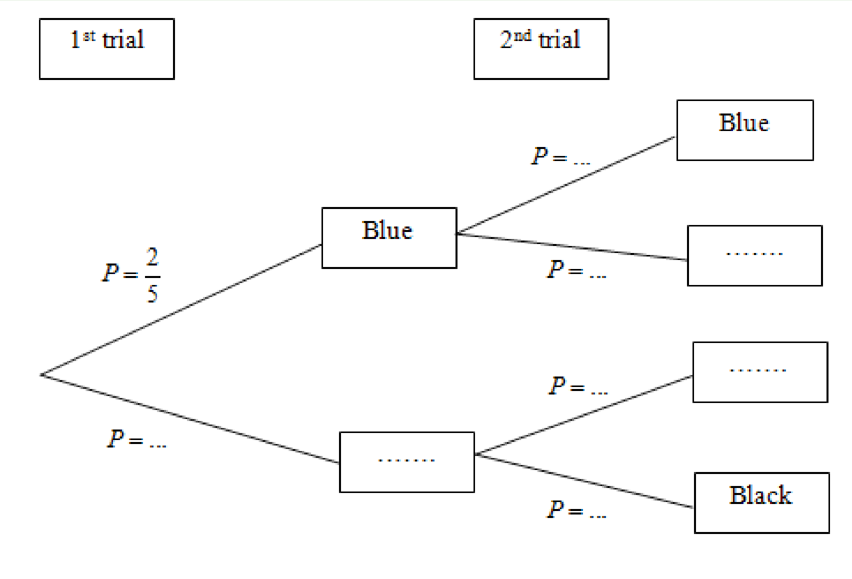
Probability is a measure of the likelihood of the occurrence of a particular outcome.

5.1 Tree diagram

ACTIVITY 5.1

A box contains 4 blue pens and 6 black pens. One pen is drawn at random, its color is noted and the pen is replaced in the box. A pen is again drawn from the box and its color is noted.

1) For the 1st trial, what is the probability of choosing a blue pen and probability of choosing a black pen?
2) For the 2nd trial, what is the probability of choosing a blue pen and probability of choosing a black pen? Remember that after the 1st trial the pen is replaced in the box.
3) In the following figure complete the missing colors and probabilities



CONTENT SUMMARY

A **tree diagram** is one way of illustrating the probabilities of certain **outcomes** occurring when two or more **trials** take place in succession by use of arrows in the form of a tree and branches. A tree diagram has branches and sub-branches which help us to see the sequence of events and all the possible outcomes at each stage.

The **outcome** is written at the end of the branch and the fraction on the branch gives the probability of the outcome occurring.

For each **trial** the number of branches is equal to the number of possible outcomes of that trial.

Examples:

1) A bag contains 8 balls of which 3 are red and 5 are green. One ball is drawn at random, its colour is noted and the ball replaced in the bag. A ball is again drawn from the bag and its colour is noted. Find the probability that the 2 balls drawn will be

a) red followed by green,

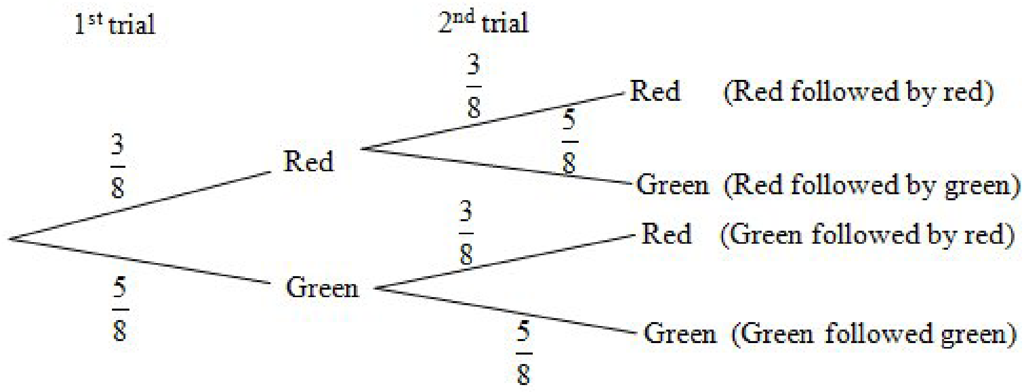
b) red and green in any order,

c) of the same colour.

Solution

Since there are 3 red balls and 5 green balls, for the 1st trial, the probability of choosing a red ball is $\frac{3}{8}$ and probability of choosing a green ball is $\frac{5}{8}$ and since after the 1st trial the ball is replaced in the bag, for the second trial the probabilities are the same as in the first trial.

Draw a tree diagram showing the probabilities of each outcome of the two trials.



Applying the multiplication rule , we have:

a) $P(\text{Red followed by green}) = \frac{3}{8} \times \frac{5}{8} = \frac{15}{64}$

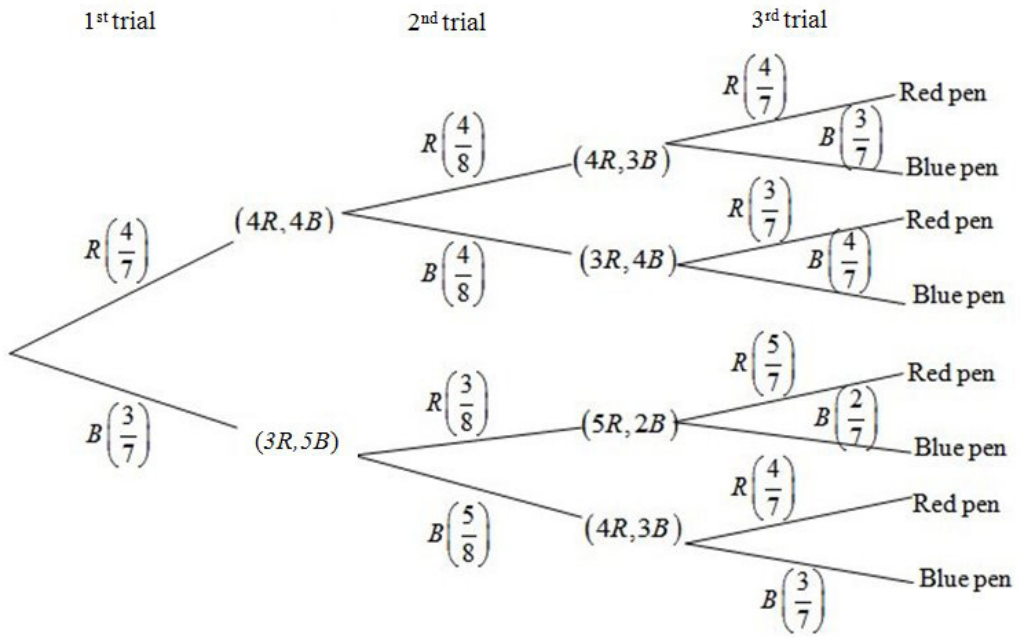
b) $P(\text{Red and green in any order}) = \frac{3}{8} \times \frac{5}{8} + \frac{5}{8} \times \frac{3}{8} = \frac{15}{32}$

c) $P(\text{both of the same colors}) = \frac{3}{8} \times \frac{3}{8} + \frac{5}{8} \times \frac{5}{8} = \frac{17}{32}$

2) A bag (1) contains 4 red pens and 3 blue pens. Another bag (2) contains 3 red pens and 4 blue pens. A pen is taken from the first bag (1) and placed into the second bag (2). The second bag (2) is shaken and a pen is taken from it and placed in the first bag (1). If now a pen is taken from the first bag, use the tree diagram to find the probability that it is a red pen.

Solution

Tree diagram is given below:



From tree diagram, the probability to have a red pen is

$$P(R) = \frac{4}{7} \times \frac{4}{8} \times \frac{4}{7} + \frac{4}{7} \times \frac{4}{8} \times \frac{3}{7} + \frac{3}{7} \times \frac{3}{8} \times \frac{5}{7} + \frac{3}{7} \times \frac{5}{8} \times \frac{4}{7}$$

$$= \frac{64}{392} + \frac{48}{392} + \frac{45}{392} + \frac{60}{392}$$

$$= \frac{31}{56}$$

APPLICATION ACTIVITY 5.1

1. Calculate the probability of three coins landing on: Three heads.
2. A class consists of six girls and 10 boys. If a committee of three is chosen at random, find the probability of:
   a) Three boys being chosen.
   b) Exactly two boys and a girl being chosen.
   c) Exactly two girls and a boy being chosen.
   d) Three girls being chosen.
3. A bag contains 7 discs, 2 of which are red and 5 are green. Two discs are removed at random and their colors noted. The first disk is not replaced before the second is selected. Find the probability that the discs will be:
   a) both red   b) of different colors   c) the same colors.
4. Three discs are chosen at random, and without replacement, from a bag containing 3 red, 8 blue and 7 white discs. Find the probability that the discs chosen will be
   a) all red   b) all blue   c) one of each color.

5.2 The Addition law of probability

ACTIVITY 5.2

Consider a machine which manufactures car components. Suppose each component falls into one of four categories: **top quality, standard, substandard, reject**

After many samples have been taken and tested, it is found that under certain specific conditions the probability that a component falls into a category is as shown in the following table.

The probability of a car component falling into one of four categories.

Category	*Probability*
Top quality	*0.18*
Standard	*0.65*
Substandard	*0.12*
Reject	*0.05*

The four categories cover all possibilities and so the probabilities must sumto 1. If 100 samples are taken, then on average 18 will be top quality, 65 of standardquality, 12 substandard and 5 will be rejected.

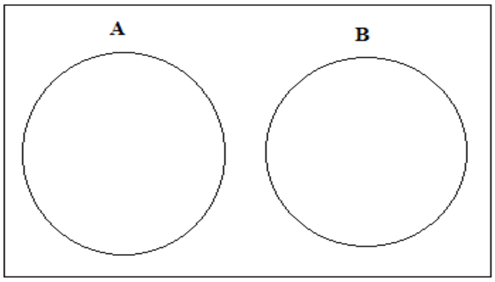
Using the data in table determine the probability that a component selected at random is either standard or top quality.

If the occurrence of either of events E_i *or* E_j excludes the occurrence of the other then E_i *and* E_j are said to be mutually exclusive events. In other words, *Mutually exclusive Events* are events that cannot happen together. If E_i *or* E_j are mutually exclusive we denote this by $E_i \cap E_j = \phi$. We use ϕ to denote the empty set or a set with no elements.

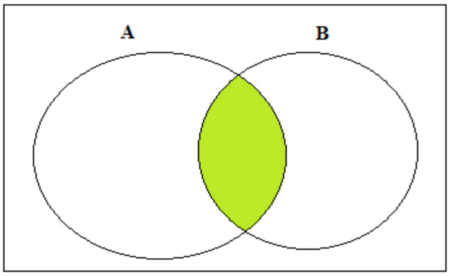
In effect, we are stating that the compound event $E_i \cap E_j$ is an **impossible event** and so will never occur.

On Venn diagram A and B shownas mutually exclusive (**disjoint sets) events** and shown as nonmutually exclusive.

Mutually exclusive (disjoint) events A and B



Non-mutually exclusive events A and B



Suppose that $E_1, E_2, E_3, \ldots, E_n$ are n events and that in a single trial only one of these events can occur. The occurrence of any event, E_i ,excludes the occurrence of all other events. Such events are mutually exclusive.

For mutually exclusive events the addition law of probability applies:

$$P(E_1 \; or \; E_2 \; or \ldots or \; E_n) = P(E_1 \cup E_2 \cup E_3 \ldots \cup E_n) = P(E_1) + P(E_2) + P(E_3) + \ldots + P(E_n)$$

For two events A and B,we have:

$$P(A \; U \; B) \; = P(A) + P(B)$$

Examples:

1) In a competition in which there are no dead heats, the probability that John wins is 0.3,the probability that Mike wins is 0.2 and the probability that Putin wins is 0.4. Find the probability that :

(a) John or Mike wins

(b) John or Putin or Mike wins,

(c) Someone else wins.

Solution:

Since only one person wins, the events are mutually exclusive.

(a) $P(John \; or \; Mike \, win) = 0.3 + 0.2 = 0.5$

(b) $P(\; John \; or \; Putin \; or \, Mike \; win) = 0.3 + 0.2 + 0.4 = 0.9$

c) $P(Someone \; else \; wins) = 1 - 0.9 = 0.1$

2) Machines A and B make components. Machine A makes 60% of the Components. The probability that a component is acceptable is 0.93 when made by machine A and 0.95 when made by machine B. A component is picked at random. Calculate the probability that it is:

a) Made by machine A and is acceptable.

b) Made by machine B and is acceptable.

c) Acceptable.

Solution:

(a)We know that 60% of the components are made by machine A and 93% of these are acceptable. Converting these percentages to decimal numbers we have

$$P(\text{component is made by machine A and is acceptable}) = \frac{60}{100} \times \frac{93}{100} = 0.60 \times 0.93 = 0.558$$

(b)We know that 40% of the components are made by machine B and 95% of these are acceptable.

$$P(\text{component is made by machine B and is acceptable}) = \frac{40}{100} \times \frac{95}{100} = 0.40 \times 0.95 = 0.38$$

(c) Note that the events described in (a) and (b) are mutually exclusive and so the addition law can be applied.

P(component is acceptable) = *P*(component is made by machine A and is acceptable)

+*P*(component is made by machine B and is acceptable) = 0.558 + 0.38 = 0.938

APPLICATION ACTIVITY 5.2

1. Each of the eight possible blood types is listed in Table below along with the percent of the U.S. population having that type.

Blood Types
O Positive—39%
A Positive—31%
B Positive—9%
O Negative—9%
A Negative—6%
AB Positive—3%
B Negative—2%
AB Negative—1%

Source: AABB Facts about Blood, 2010

(a)What is the probability that a randomly selectedperson in the United States has a blood type that is Rh-negative?

(b) What is the probability a person chosen at random is type O or type A?

5.3 Independent events

ACTIVITY 5.3

A box contains 3 red pens, 4 green pens and 5 blue pens. One pen is taken from the box and then replaced. Another pen is taken from the box. Let A be the event "the first pen is red" and B be the event the second pen is blue."

Is the occurrence of event B affected by the occurrence of event A? Explain.

Events A and B in a probability space S are said to be ***independent*** if the occurrence of one of them does not influence the occurrence of the other.

If probability of event B is not affected by the occurrence of event A, then we use the **multiplication law** of probability $P(A\cap B)=P(A)\times P(B)$.

Examples:

1) A die is thrown twice. Find the probability of obtaining a 4 on the first throw and an odd number on the second throw.

Solution

Let A be the event: "a 4 is obtained on the first throw", then $P(A)=\frac{1}{6}$.

That is $A=\{4\}$

B be the event: "an odd number is obtained on the second throw". That is $B=\{1,3,5\}$

Since the result on the second throw is not affected by the result on the first throw, A and B are independent events.

There are 3 odd numbers, then

$$P(B)=\frac{3}{6}=\frac{1}{2}$$

Therefore,

$$\begin{aligned} P(A\cap B)&=P(A)P(B)\\ &=\frac{1}{6}\times\frac{1}{2}\\ &=\frac{1}{12} \end{aligned}$$

2)A factory runs two machines. The first machine operates for 80% of the time while the second machine operates for 60% of the time and at least one machine operates for 92% of the time. Do these two machines operate independently?

Solution

Let the first machine be M_1 and the second machine be M_2, then
$P(M_1)=80\%=0.8$, $P(M_2)=60\%=0.6$ and $P(M_1 \cup M_2)=92\%=0.92$

Now,

$$P(M_1 \cup M_2)=P(M_1)+P(M_2)-P(M_1 \cap M_2)$$
$$\begin{aligned} P(M_1 \cap M_2) &= P(M_1)+P(M_2)-P(M_1 \cup M_2) \\ &= 0.8+0.6-0.92 \\ &= 0.48 \\ &= 0.8\times 0.6 \\ &= P(M_1)\times P(M_2) \end{aligned}$$

Thus, the two machines operate independently.

2) A coin is weighted so that heads is three times as likely to appear as tails. Find $P(H)$ and $P(T)$.

Solution

Let $P(T)=p_1$, then $P(H)=3p_1$.

But $P(H)+P(T)=1$

Therefore $p_1+3p_1=1 \Leftrightarrow 4p_1=1 \Rightarrow p_1=\dfrac{1}{4}$

Thus, $P(H)=\dfrac{3}{4}$ and $P(T)=\dfrac{1}{4}$.

APPLICATION ACTIVITY 5.3

1. A dresser drawer contains one pair of socks with each of the following colors: blue, brown, red, white and black. Each pair is folded together in a matching set. You reach into the sock drawer and choose a pair of socks without looking. You replace this pair and then choose another pair of socks. What is the probability that you will choose the red pair of socks both times?
2. A coin is tossed and a single 6-sided die is rolled. Find the probability of landing on the head side of the coin and rolling a 3 on the die.
3. A husband and a wife appear in an interview for two vacancies in the same post. The probability of husband's selection is $\frac{1}{7}$ and that of wife's selection is $\frac{1}{5}$. What is the probability that
   a) both of them will be selected
   b) only one of them will be selected,
   c) none of them will be selected?

5.4.Dependent events

ACTIVITY 5.4

Suppose that you have a deck of 52 cards. You can draw a card from that deck , continue without replacing it, and then draw a second card .

a) What is the sample space for each event?

b) Suppose you select successively two cards, what is the probability of selecting two red cards?

c) Explain if there is any relationship (Independence or dependence) between those two events considering the sample space. Does the selection of the first card affect the selection of the second card?

When the outcome or occurrence of the first event affects the outcome or occurrence of the second event in such a way that the probability is changed, the events are said to be ***dependent***.

Examples:

1)Suppose a card is drawn from a deck and not replaced, and then the second card is drawn. What is the probability of selecting an ace on the first card and a king on the second card?

Solution:

The probability of selecting an ace on the first draw is $\frac{4}{52}$. But since that card is not replaced, the probability of selecting a king on the second card is $\frac{4}{51}$,since there are 51 cards remaining.

The outcomes of the first draw has affected the outcome of the second. By multiplication rule , the probability of both events occurring is : .

$$\frac{4}{52}\times\frac{4}{51}=\frac{16}{2652}=\frac{4}{663}=0.006$$

Note that:

The event of getting a king on the second draw given that an ace was drawn the first time is called a **conditional probability**.

APPLICATION ACTIVITY 5.4

The world wide Insurance Company found that 53% of the residents of a city had home owner's Insurance with its company of the clients, 27% also had automobile Insurance with the company. If a resident is selected at random, find the probability that the resident has both home owner's and automobile Insurance with the world wide Insurance Company.

5.5 Conditional probability

ACTIVITY 5.5

A box contains 3 red pens, 4 green pens and 5 blue pens. One pen is taken from the box and is not replaced. Another pen is taken from the box. Let A be the event "the first pen is red" and B be the event "the second pen is blue."

Is the occurrence of event B affected by the occurrence of event A? Explain.

The probability of an event B given that event A has occurred is called the **conditional probability** of B given A and it is written $P(B|A)$.This notation does not mean that B is divided by A; rather, it means the probability that event B occurs given that event A has already occurred.

In this case $P(B|A)$ is the probability that B occurs considering A as the sample space, and since the subset of A in which B occurs is $A \cap B$, then

$$P(B|A) = \frac{P(B \cap A)}{P(A)}$$

From this result, we have general statement of the multiplication law:

$$P(A \cap B) = P(A) \times P(B|A)$$

This shows us that the probability that two events will both occur is the product of the probability that one will occur and the conditional probability that the other will occur given that the first has occurred. We can also write $P(A \cap B) = P(B) \times P(A|B)$. Since A and B are interchangeable.

If A and B are **independent**, then the probability of B is not affected by the occurrence of A and the probability of A is not affected by the occurrence of B. That is, $P(B|A) = P(B)$ and respectively and giving $P(A \cap B) = P(A) \times P(B)$

Examples:

1) A die is tossed. Find the probability that the number obtained is a 4 given that the number is greater than 2.

Solution

Let A be the event: "the number is a 4", then $A=\{4\}$

B be the event: "the number is greater than 2", then $B=\{3,4,5,6\}$ and $P(B)=\frac{4}{6}=\frac{2}{3}$

But $A\cap B=\{4\}$ and $P(A\cap B)=\frac{1}{6}$

Therefore,

$$P(A|B)=\frac{P(A\cap B)}{P(B)}$$

$$P(A|B)=\frac{\frac{1}{6}}{\frac{2}{3}}$$

$$P(A|B)=\frac{1}{6}\times\frac{3}{2}$$

$$=\frac{1}{4}$$

2) At a middle school, 18% of all students play football and basketball, and 32% of all students play football. What is the probability that a student who plays football also plays basketball?

Solution

Let A be a set of students who play football and B a set of students who play basketball then the set of students who play both games is $A\cap B$. We have $P(A)=32\%=0.32$, $P(A\cap B)=18\%=0.18$. We need the probability of B known that A has occurred.

Therefore,

$$P(B|A)=\frac{P(A\cap B)}{P(A)}$$

$$=\frac{0.18}{0.32}$$

$$=0.5625$$

$$=56\%$$

Notice:

Contingency table

Contingency table (or **Two-Way table**) provides a different way of calculating probabilities. The table helps in determining conditional probabilities quite easily. The table displays sample values in relation to two different variables that may be dependent or contingent on one another.

Below, the contingency table shows the favorite leisure activities for 50 adults, 20 men and 30 women. Because entries in the table are frequency counts, the table is a **frequency table.**

	Dance	**Sports**	**TV**	**Total**
Men	2	10	8	20
Women	16	6	8	30
Total	18	16	16	50

Entries in the total row and total column are called **marginal frequencies** or the **marginal distribution**. Entries in the body of the table are called **joint frequencies**.

Examples:

1.Suppose a study of speeding violations and drivers who use car phones produced the following fictional data:

	Speeding violation in the last year	No speeding violation in the last year	Total
Car phone user	25	280	305
Not a car phone user	45	405	450
Total	70	685	755

Calculate the following probabilities using the table:

a) P(person is a car phone user)

b) P(person had no violation in the last year)

c) P(person had no violation in the last year AND was a car phone user)

d) P(person is a car phone user OR person had no violation in the last year)

e) P(person is a car phone user GIVEN person had a violation in the last year)

f) P(person had no violation last year GIVEN person was not a car phone user)

Solution

a) $P(\text{person is a car phone user}) = \dfrac{\text{number of car phone users}}{\text{total number in study}} = \dfrac{305}{755}$

b) $P(\text{person had no violation in the last year}) = \dfrac{\text{number that had no violation}}{\text{total number in study}} = \dfrac{685}{755}$

c) $P(\text{person had no violation in the last year AND was a car phone user}) = \dfrac{280}{755}$

P(person is a car phone user OR person had no violation in the last year)

d) $= \left(\dfrac{305}{755} + \dfrac{685}{755}\right) - \dfrac{280}{755} = \dfrac{710}{755}$

e) The sample space is reduced to the number of persons who had a violation. Then .

$P(\text{person is a car phone user GIVEN person had a violation in the last year}) = \dfrac{25}{70}$

f) The sample space is reduced to the number of persons who were not car phone users. Then

$P(\text{person had no violation last year GIVEN person was not a car phone user}) = \dfrac{405}{450}$

2) A recent survey asked 100 people if they thought women in the armed forces should be permitted to participate in combat. The results of the survey are shown.

Gender	Yes	No	Total
Male	32	18	50
Female	8	42	50
Total	40	60	100

Find these probabilities.

a. The respondent answered yes, given that the respondent was a female.

b. The respondent was a male, given that the respondent answered no.

Solution:

Let M = respondent was a male Y= respondent answered Yes

F =respondent was a female N= respondent answered No

a) $P(\mathrm{Y}/F) = \dfrac{\mathrm{P}(F \cap Y)}{P(F)}$

The probability $P(F$ and $Y)$ is the number of females who responded yes, divided by the total number of respondents:

$$\mathrm{P}(F \cap Y) = \frac{8}{100}$$

The probability $P(F)$ is the probability of selecting a female: $\mathrm{P(F)} = \dfrac{50}{100}$

Then $P(\mathrm{Y}/F) = \dfrac{\mathrm{P}(F \cap Y)}{P(F)} = \dfrac{\frac{8}{100}}{\frac{50}{100}} = \dfrac{4}{25}$

b) The problem is to find $P(\mathrm{M}/\mathrm{N})$

$$P(\mathrm{M}/\mathrm{N}) = \frac{\mathrm{P}(M \cap N)}{P(\mathrm{N})} = \frac{\frac{8}{100}}{\frac{60}{100}} = \frac{3}{10}$$

3) A game is played by drawing 4 cards from an ordinary deck and replacing each card after it is drawn. Find the probability that at least 1 ace is drawn.

Solution

It is much easier to find the probability that no aces are drawn (i.e., losing) and then subtract that value from 1 than to find the solution directly, because that would involve finding the probability of getting 1 ace, 2 aces, 3 aces, and 4 aces and then adding the results.

Let E=at least 1 ace is drawn and $\bar{E}$ no aces drawn. Then

$$P(\bar{E}) = \frac{48}{52} \cdot \frac{48}{52} \cdot \frac{48}{52} \cdot \frac{48}{52} = \frac{20,736}{28,561}$$

Hence, $P(\mathrm{E}) = 1 - P\left(\bar{E}\right) = 0,27$ or a hand with at least 1 ace will win about 27% of the time.

APPLICATION ACTIVITY 5.5

The world wide Insurance Company found that 53% of the residents of a city had home owner's Insurance with its company of the clients, 27% also had automobile Insurance with the company. If a resident is selected at random, find the probability that the resident has both home owner's and automobile Insurance with the world wide Insurance Company.

1. A jar contains black and white marbles. Two marbles are chosen without replacement. The probability of selecting a black marble and then a white marble is 0.34, and the probability of selecting a black marble on the first draw is 0.47. What is the probability of selecting a white marble on the second draw, given that the first marble drawn was black?

2. A bag contains five discs, three of which are red. A box is contains six discs, four of which are red. A card is selected at random from a normal pack of 52 cards, if the card is a club a disc is removed from the bag and if the card is not a club a disc is removed from the box. Find the probability that, if the removed disc is red it came from the bag.

3. The probability that Gerald parks in a no-parking zone *and* gets a parking ticket is 0.06, and the probability that Gerald cannot find a legal parking space and has to park in the no-parking zone is 0.20. On Tuesday, Gerald arrives at Headquarter office and has to park in a no-parking zone. Find the probability that he will get a parking ticket.

5.6 Bayes theorem and its applications

ACTIVITY 5.6

Suppose that entire output of a factory is produced on three machines. Let B_1 denote the event that a randomly chosen item was made by machine 1, B_2 denote the event that a randomly chosen item was made by machine 2 and B_3 denote the event that a randomly chosen item was made by machine 3. Let A denote the event that a randomly chosen item is defective.

1) Use conditional probability formula and give the relation should be used to find the probability that the chosen item is defective, $P(A)$, given that it is made by machine 1 or machine 2 or machine 3.

2) If we need the probability that the chosen item is produced by machine 1 given that is found to be defective, i.e $P(B_1 | A)$, give the formula for this conditional probability. Recall that $P(B_i \cap A)$ can be written as $P(A | B_i)P(B_i)$. Do the same if the item is produced by machine 2 and by machine 3. Give the general formula if the item is produced by machine i (i from 1 to 3)

Let $B_1, B_2, B_3, ..., B_n$ be incompatible and exhaustive events and let A be an arbitrary event.

We have:

$$P(B_i | A) = \frac{P(B_i \cap A)}{P(A)} = \frac{P(A | B_i)P(B_i)}{\sum_{i=1}^{n} P(A | B_i)P(B_i)}$$

This formula is called **Bayes' formula**.

Remark

We also have **(Bayes' rule)**

$$P(B \mid A)=\frac{P(A \mid B)P(B)}{P(A)}$$

Examples:

1) *Suppose that machines* M_1, M_2, and M_3 *produce, respectively, 500, 1000, and 1500 parts per day, of which 5%, 6%, and 7% are defective. A part produced by one of these machines is taken at random, at the end of a given workday, and it is found to be defective. What is the probability that it was produced by machine* M_3*?*

Solution

Let A_i be the event "the part taken at random was produced by machine M_i,"

for $i=1,2,3$; and let D be "the part taken at random is defective."

Using Bayes' formula, we seek

$$P(A_3 \mid D)=\frac{P(D \mid A_3)P(A_3)}{\sum_{i=1}^{3} P(D \mid A_i)P(A_i)}$$

$$=\frac{(0.07)\left(\frac{1500}{3000}\right)}{(0.05)\left(\frac{1}{6}\right)+(0.06)\left(\frac{1}{3}\right)+(0.07)\left(\frac{1}{2}\right)}$$

$$=\frac{105}{190}$$

$$=\frac{21}{38}$$

2) Two machines A and B produce 60% and 40% respectively of total output of a factory. Of the parts produced by machine A, 3% are defective and of the parts produced by machine B, 5% are defective. A part is selected at random from a day's production and found to be defective. What is the probability that it came from machine A?

Solution

Let E be the event that the part came from machine A

C be the event that the part came from machine B and

D be the event that the part is defective.

We require $P(E|D)$.

Now, $P(E)\times P(D|E) = 0.6\times 0.03 = 0.018$ and

$P(D) = P(E\cap D) + P(C\cap D)$

$= 0.018 + 0.4\times 0.05$

$= 0.038$

Therefore, the required probability is $\frac{0.018}{0.038} = \frac{9}{19}$

APPLICATION ACTIVITY 5.6

1. 20% of a company's employees are engineers and 20% are economists. 75% of the engineers and 50% of the economists hold a managerial position, while only 20% of non-engineers and non-economists have a similar position. What is the probability that an employee selected at random will be both an engineer and a manager?

2. The probability of having an accident in a factory that triggers an alarm is 0.1. The probability of its sounding after the event of an incident is 0.97 and the probability of it sounding after no incident has occurred is 0.02. In an event where the alarm has been triggered, what is the probability that there has been no accident?

5.7 END UNIT ASSESSMENT

1) The probability that it is Friday and that a student is absent is 0.03. Since there are 5 school days in a week, the probability that it is Friday is 0.2. What is the probability that a student is absent given that today is Friday?

2) Dr. Richard is conducting a survey of families with 3 children. If a family is selected at random, what is the probability that the family will have exactly 2 boys if the second child is a boy? Assume that the probability of giving birth to a boy is equal to the probability of giving birth to a girl.

3) A 12-sided die (dodecahedron) has the numerals 1 through 12 on its faces. The die is rolled once, and the number on the top face is recorded. What is the probability that the number is a multiple of 4 if it is known that it is even?

4) At Kennedy Middle School, the probability that a student takes Technology and Spanish is 0.087. The probability that a student takes Technology is 0.68. What is the probability that a student takes Spanish given that the student is taking Technology?

5) A bag contains 2 red, 3 green and 2 blue balls. Two balls are drawn at random. What is the probability that none of the balls drawn is blue?

6) In a certain college, 5% of the men and 1% of the women are taller than 180 *cm*. Also, 60% of the students are women. If a student is selected at random and found to be taller than 180 *cm*, what is the probability that this student is a woman?

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