Advanced Mathematics

Teacher's Guide

Senior Five

FOREWORD

Dear Teachers,

Rwanda Education Board is honoured to present the teacher's guide for Mathematics to be used in the option with Core Mathematics. This book serves as a guide to competence-based teaching and learning to ensure consistency and coherence in the learning of the mathematics content. The Rwandan educational philosophy is to ensure that learners achieve full potential at every level of education which will prepare them to be well integrated in society and exploit employment opportunities.

In line with efforts to improve the quality of education, the government of Rwanda emphasizes the importance of aligning teaching and learning materials with the syllabus to facilitate their learning process. Many factors influence what they learn, how well they learn and the competences they acquire. Those factors include the relevance of the specific content, the quality of teachers' pedagogical approaches, the assessment strategies and the instructional materials.

The ambition to develop a knowledge-based society and the growth of regional and global competition in the jobs market has necessitated the shift to a competence-based curriculum. This book provides active teaching and learning techniques that engage student teachers to develop competences.

In view of this, your role is to:

- Plan your lessons and prepare appropriate teaching materials.
- Organize group discussions for students considering the importance of social constructivism suggesting that learning occurs more effectively when the students works collaboratively with more knowledgeable and experienced people.

- Engage students through active learning methods such as inquiry methods, group discussions, research, investigative activities and group and individual work activities.
- Provide supervised opportunities for students to develop different competences by giving tasks which enhance critical thinking, problem solving, research, creativity and innovation, communication and cooperation.
- Support and facilitate the learning process by valuing students' contributions in the class activities.
- Guide students towards the harmonization of their findings.
- Encourage individual, peer and group evaluation of the work done in the classroom and use appropriate competence-based assessment approaches and methods.

To facilitate you in your teaching activities, the content of this book is self explanatory so that you can easily use it. It is divided in 3 parts:

The part I explain the structure of this book and give you the methodological guidance;

The part II gives a sample lesson plan;

The part III details the teaching guidance for concepts given in the student book.

Even though this teacher's guide contains the guidance on solutions for all activities given in the learner's book, you are requested to work through each question before judging student's findings.

I wish to sincerely express my appreciation to the people who contributed towards the development of this book, particularly, REB staff, Teachers from general education and experts from Local and international Organizations for their technical support.

Dr. NDAYAMBAJE Irénée

Director General, REB

ACKNOWLEDGEMENT

I wish to express my appreciation to the people who played a major role in the development of this teacher's guide for Senior Five Mathematics in the option with Core Mathematics as a major subject. It would not have been successful without active participation of different education stakeholders.

I owe gratitude to teachers whose efforts during the editing exercise of this book were very much valuable.

Finally, my word of gratitude goes to the Rwanda Education Board staffs who were involved in the whole process of in-house textbook production.

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Head of CTLR Department

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PART I. GENERAL INTRODUCTION

1.1. The structure of the guide

The tutor's guide of Mathematics is composed of three parts:

The Part I concerns general introduction that discusses methodological guidance on how best to teach and learn Mathematics, developing competences in teaching and learning, addressing cross-cutting issues in teaching and learning and Guidance on assessment.

Part II presents a sample lesson plan. This lesson plan serves to guide the teacher on how to prepare a lesson in Mathematics.

The Part III is about the structure of a unit and the structure of a lesson. This includes information related to the different components of the unit. This part provides information and guidelines on how to facilitate student while working on learning activities. More other, many application activities from the textbook have answers in this part.

1.2. Methodological guidance

1.2.1. Developing competences

Since 2015 Rwanda shifted from a knowledge based to a competency-based curriculum for pre-primary, primary, secondary education. This called for changing the way of learning by shifting from teacher centred to a learner centred approach. Teachers are not only responsible for knowledge transfer but also for fostering learners' learning achievement and creating safe and supportive learning environment. It implies also that learners have to demonstrate what they are able to transfer the acquired knowledge, skills, values and attitude to new situations.

The competence-based curriculum employs an approach of teaching and learning based on discrete skills rather than dwelling on only knowledge or the cognitive domain of learning. It focuses on what learner can do rather than what learner knows. Learners develop competences through subject unit with specific learning objectives broken down into knowledge, skills and attitudes through learning activities.

In addition to the competences related to Mathematics, student teachers also develop generic competences which should promote the development of the higher order thinking skills and professional skills in Mathematics teaching. Generic competences are developed throughout all units of Mathematics as follows:

Generic competences	Ways of developing generic competences
Critical thinking	All activities that require learners to calculate, convert, interpret, analyse, compare and contrast, etc have a common factor of developing critical thinking into learners
Creativity and innovation	All activities that require learners to plot a graph of a given algebraic data, to organize and interpret statistical data collected and to apply skills in solving problems of economics have a common character of developing creativity into learners
Research and problem solving	All activities that require learners to make a research and apply their knowledge to solve problems from the real-life situation have a character of developing research and problem solving into learners.
Communication	During Mathematics class, all activities that require learners to discuss either in groups or in the whole class, present findings, debate …have a common character of developing communication skills into learners.
Co-operation, interpersonal relations and life skills	All activities that require learners to work in pairs or in groups have character of developing cooperation and life skills among learners.

Lifelong learning	All activities that are connected with research have a common character of developing into learners a curiosity of applying the knowledge learnt in a range of situations. The purpose of such kind of activities is for enabling learners to become life-long learners who can adapt to the fast-changing world and the uncertain future by taking initiative to update knowledge and skills with minimum external support.

The generic competences help learners deepen their understanding of Mathematics and apply their knowledge in a range of situations. As students develop generic competences, they also acquire the set of skills that employers look for in their employees, and so the generic competences prepare students for the world of work.

1.2.2. Addressing cross cutting issues

Among the changes brought by the competence-based curriculum is the integration of cross cutting issues as an integral part of the teaching learning process-as they relate to and must be considered within all subjects to be appropriately addressed. The eight cross cutting issues identified in the national curriculum framework are: Comprehensive Sexuality Education, Environment and Sustainability, Financial Education, Genocide studies, Gender, Inclusive Education, Peace and Values Education, and Standardization Culture.

Some cross-cutting issues may seem specific to particular learning areas/ subjects but the teacher need to address all of them whenever an opportunity arises. In addition, learners should always be given an opportunity during the learning process to address these cross-cutting issues both within and out of the classroom.

Below are examples of how crosscutting issues can be addressed:

Cross-Cutting Issue	**Ways of addressing cross-cutting issues**
Comprehensive Sexuality Education: The primary goal of introducing Comprehensive Sexuality Education program in schools is to equip children, adolescents, and young people with knowledge, skills and values in an age appropriate and culturally gender sensitive manner so as to enable them to make responsible choices about their sexual and social relationships, explain and clarify feelings, values and attitudes, and promote and sustain risk reducing behaviour.	Using different charts and their interpretation, Mathematics teachers should lead students to discuss the following situation: "Alcohol abuse and unwanted pregnancies" and advise students on how they can fight those abuses. Some examples can be given when learning statistics, powers, logarithms and their properties.
Environment and Sustainability: Integration of Environment, Climate Change and Sustainability in the curriculum focuses on and advocates for the need to balance economic growth, society well-being and ecological systems. Learners need basic knowledge from the natural sciences, social sciences, and humanities to understand to interpret principles of sustainability.	Using Real life models or students' experience, Mathematics teacher should lead students to illustrate the situation of "population growth" and discuss its effects on the environment and sustainability.
Financial Education: The integration of Financial Education into the curriculum is aimed at a comprehensive Financial Education program as a precondition for achieving financial inclusion targets and improving the financial capability of Rwandans so that they can make appropriate financial decisions that best fit the circumstances of one's life.	Through different examples and calculations on interest rate problems, total revenue and total cost, Mathematics teacher can lead student to discuss how to make appropriate financial decisions.

Gender: At school, gender will be understood as family complementarities, gender roles and responsibilities, the need for gender equality and equity, gender stereotypes, gender sensitivity, etc.	Mathematics teacher should address gender as cross-cutting issue through assigning leading roles in the management of groups to both girls and boys and providing equal opportunity in the lesson participation and avoid any gender stereotype in the whole teaching and learning process.
Inclusive Education: Inclusion is based on the right of all learners to a quality and equitable education that meets their basic learning needs and understands the diversity of backgrounds and abilities as a learning opportunity.	Firstly, Mathematics teachers need to identify/recognize students with special needs. Then by using adapted teaching and learning resources while conducting a lesson and setting appropriate tasks to the level of students, they can cater for students with special education needs. They must create opportunity where student teachers can discuss how to support colleagues with special educational needs.
Peace and Values Education: Peace and Values Education (PVE) is defined as education that promotes social cohesion, positive values, including pluralism and personal responsibility, empathy, critical thinking and action in order to build a more peaceful society.	Through a given lesson, a teacher should: • Set a learning objective which is addressing positive attitudes and values, • Encourage students to develop the culture of tolerance during discussion and to be able to instil it in colleagues and cohabitants; • Encourage students to respect ideas for others.
Standardization Culture: Standardization Culture in Rwanda will be promoted through formal education and plays a vital role in terms of health improvement, economic growth, industrialization, trade and general welfare of the people through the effective implementation of Standardization, Quality Assurance, Metrology and Testing.	With different word problems related to the effective implementation of Standardization, Quality Assurance, Metrology and Testing, students can be motivated to be aware of health improvement, economic growth, industrialization, trade and general welfare of the people.

1.2.3. Guidance on how to help students with special education needs in classroom

In the classroom, students learn in different way depending to their learning pace, needs or any other special problem they might have. However, the teacher has the responsibility to know how to adopt his/her methodologies and approaches in order to meet the learning need of each student in the classroom. Also teachers need to understand that student with special needs, have to be taught differently or need some accommodations to enhance the learning environment. This will be done depending on the subject and the nature of the lesson.

In order to create a well-rounded learning atmosphere, teachers need to:

- Remember that learners learn in different ways so they have to offer a variety of activities (e.g. role-play, music and singing, word games and quizzes, and outdoor activities);
- Maintain an organized classroom and limits distraction. This will help learners with special needs to stay on track during lesson and follow instruction easily;
- Vary the pace of teaching to meet the needs of each child. Some learners process information and learn more slowly than others;
- Break down instructions into smaller, manageable tasks. Learners with special needs often have difficulty understanding long-winded or several instructions at once. It is better to use simple, concrete sentences in order to facilitate them understand what you are asking.
- Use clear consistent language to explain the meaning (and demonstrate or show pictures) if you introduce new words or concepts;
- Make full use of facial expressions, gestures and body language;
- Pair a learner who has a disability with a friend. Let them do things together and learn from each other. Make sure the friend is not over protective and does not do everything for the one with disability. Both learners will benefit from this strategy;

- Use multi-sensory strategies. As all learners learn in different ways, it is important to make every lesson as multi-sensory as possible. Learners with learning disabilities might have difficulty in one area, while they might excel in another. For example, use both visual and auditory cues.

Below are general strategies related to each main category of disabilities and how to deal with every situation that may arise in the classroom. However, the list is not exhaustive because each child is unique with different needs and that should be handled differently.

Strategy to help learners with developmental impairment:

- Use simple words and sentences when giving instructions;
- Use real objects that learners can feel and handle. Rather than just working abstractly with pen and paper;
- Break a task down into small steps or learning objectives. The learner should start with an activity that she/he can do already before moving on to something that is more difficult;
- Gradually give the learner less help;
- Let the learner with disability work in the same group with those without disability.
- Strategy to help learners with visual impairment:
- Help learners to use other senses (hearing, touch, smell and taste) and carry out activities that will promote their learning and development;
- Use simple, clear and consistent language;
- Use tactile objects to help explain a concept;
- If the learner has some sight, ask him/her what he/she can see;
- Make sure the learner has a group of friends who are helpful and who allow him/her to be as independent as possible;
- Plan activities so that learners work in pairs or groups whenever possible;

Strategy to help learners with hearing disabilities or communication difficulties

- Always get the learner's attention before you begin to speak;
- Encourage the learner to look at your face;
- Use gestures, body language and facial expressions;
- Use pictures and objects as much as possible.
- Keep background noise to a minimum.

Strategies to help learners with physical disabilities or mobility difficulties:

- Adapt activities so that learners, who use wheelchairs or other mobility aids, can participate.
- Ask parents/caregivers to assist with adapting furniture e.g. the height of a table may need to be changed to make it easier for a learner to reach it or fit their legs or wheelchair under;
- Encourage peer support when needed;
- Get advice from parents or a health professional about assistive devices if the learner has one.

Adaptation of assessment strategies:

At the end of each unit, the tutor is advised to provide additional activities to help students achieve the key unit competence. These assessment activities are for remedial, consolidation and extension designed to cater for the needs of all categories of students; slow, average and gifted students respectively. Therefore, the tutor is expected to do assessment that fits individual student.

Remedial activities After evaluation, slow students are provided with lower order thinking activities related to the concepts learnt to facilitate them in their learning.

These activities can also be given to assist deepening knowledge acquired through the learning activities for slow students.

Remedial activities	After evaluation, slow students are provided with lower order thinking activities related to the concepts learnt to facilitate them in their learning. These activities can also be given to assist deepening knowledge acquired through the learning activities for slow students.
Consolidation activities	After introduction of any concept, a range number of activities can be provided to all students to enhance/ reinforce learning.
Extended activities	After evaluation, gifted and talented students can be provided with high order thinking activities related to the concepts learnt to make them think deeply and critically. These activities can be assigned to gifted and talented students to keep them working while other students are getting up to required level of knowledge through the learning activity.

1.2.4. Guidance on assessment

Assessment is an integral part of teaching and learning process. The main purpose of assessment is for improvement of learning outcomes. Assessment for learning/ Continuous/ formative assessment intend to improve students’ learning and tutor’s teaching whereas assessment of learning/summative assessment intends to improve the entire school’s performance and education system in general.

Continuous/ formative assessment

It is an on-going process that arises during the teaching and learning process. It includes lesson evaluation and end of sub unit assessment. This formative assessment should play a big role in teaching and learning process. The teacher should encourage individual, peer and group evaluation of the work done in the classroom and uses appropriate competence-based assessment approaches and methods.

Formative assessment is used to:

- Determine the extent to which learning objectives are being achieved and competences are being acquired and to identify which students need remedial interventions, reinforcement as well as extended activities. The application activities are developed in the learner book and they are designed to be given as remedial, reinforcement, end lesson assessment, homework or assignment
- Motivate students to learn and succeed by encouraging students to read, or learn more, revise, etc.
- Check effectiveness of teaching methods in terms of variety, appropriateness, relevance, or need for new approaches and strategies. Mathematics tutors need to consider various aspects of the instructional process including appropriate language levels, meaningful examples, suitable methods and teaching aids/ materials, etc.
- Help students to take control of their own learning.

In teaching Mathematics, formative or continuous assessment should compare performance against instructional objectives. Formative assessment should measure the student's ability with respect to a criterion or standard. For this reason, it is used to determine what students can do, rather than how much they know.

Summative assessment

The assessment can serve as summative and informative depending to its purpose. The end unit assessment will be considered summative when it is done at end of unit and want to start a new one.

It will be formative assessment, when it is done in order to give information on the progress of learners and from there decide what adjustments need to be done.

The assessment done at the end of the term, end of year, is considered as summative assessment so that the teacher, school and parents are informed of the achievement of educational objective and think of improvement strategies. There is also end of level/ cycle assessment in form of national examinations.

When carrying out assessment?

Assessment should be clearly visible in lesson, unit, term and yearly plans.

- Before learning (diagnostic): At the beginning of a new unit or a section of work; assessment can be organized to find out what students already know / can do, and to check whether the students are at the same level.
- During learning (formative/continuous): When students appear to be having difficulty with some of the work, by using on-going assessment (continuous). The assessment aims at giving students support and feedback.
- After learning (summative): At the end of a section of work or a learning unit, the Mathematics Tutor has to assess after the learning. This is also known as Assessment of Learning to establish and record overall progress of students towards full achievement. Summative assessment in Rwandan schools mainly takes the form of written tests at the end of a learning unit or end of the month, and examinations at the end of a term, school year or cycle.

Instruments used in assessment.

- Observation: This is where the Mathematics teacher gathers information by watching students interacting, conversing, working, playing, etc. A teacher can use observations to collect data on behaviours that are difficult to assess by other methods such as attitudes, values, and generic competences and intellectual skills. It is very important because it is used before the lesson begins and throughout the lesson since the teacher has to continue observing each and every activity.
- Questioning

(a) Oral questioning: a process which requires a student to respond verbally to questions

(b) Class activities/ exercises: tasks that are given during the learning/ teaching process

(c) Short and informal questions usually asked during a lesson

(d) Homework and assignments: tasks assigned to students by their teacher to be completed outside of class.

Homework assignments, portfolio, project work, interview, debate, science fair, Mathematics projects and Mathematics competitions are also the different forms/instruments of assessment.

1.2.5. Teaching methods and techniques that promote active learning

The different learning styles for students can be catered for, if the teacher uses active learning whereby learners are really engaged in the learning process.

The main teaching methods used in mathematics are the following:

- **Inductive-deductive method**: Inductive method is to move from specific examples to generalization and deductive method is to move from generalization to specific examples.
- **Analytic-synthetic method**: Analytic method proceeds from unknown to known, 'Analysis' means 'breaking up' of the problem in hand so that it ultimately gets connected with something obvious or already known. Synthetic method is the opposite of the analytic method. Here one proceeds from known to unknown.
- **Skills Laboratory method**: Laboratory method is based on the maxim "learning by doing." It is a procedure for stimulating the activities of the students and to encourage them to make discoveries through practical activities.
- **Problem solving method, Project method and Seminar Method.**

The following are some active techniques to be used in Mathematics:

- Group work
- Research
- Probing questions
- Practical activities (drawing, plotting, interpreting graphs)
- Modelling
- Brainstorming

- Quiz Technique
- Discussion Technique
- Scenario building Technique

What is Active learning?

Active learning is a pedagogical approach that engages learners in doing things and thinking about the things they are doing. Learners play the key role in the active learning process. They are not empty vessels to fill but people with ideas, capacity and skills to build on for effective learning. Thus, in active learning, learners are encouraged to bring their own experience and knowledge into the learning process.

The role of the teacher in active learning	The role of learners in active learning
• The teacher engages learners through active learning methods such as inquiry methods, group discussions, research, investigative activities, group and individual work activities. • He/she encourages individual, peer and group evaluation of the work done in the classroom and uses appropriate competence-based assessment approaches and methods. • He provides supervised opportunities for learners to develop different competences by giving tasks which enhance critical thinking, problem solving, research, creativity and innovation, communication and cooperation. • Teacher supports and facilitates the learning process by valuing learners' contributions in the class activities.	**A learner engaged in active learning:** • Communicates and shares relevant information with peers through presentations, discussions, group work and other learner-centred activities (role play, case studies, project work, research and investigation); • Actively participates and takes responsibility for his/her own learning; • Develops knowledge and skills in active ways; • Carries out research/investigation by consulting print/online documents and resourceful people, and presents their findings; • Ensures the effective contribution of each group member in assigned tasks through clear explanation and arguments, critical thinking, responsibility and confidence in public speaking • Draws conclusions based on the findings from the learning activities.

Main steps for a lesson in active learning approach

All the principles and characteristics of the active learning process highlighted above are reflected in steps of a lesson as displayed below. Generally, the lesson is divided into three main parts whereby each one is divided into smaller steps to make sure that learners are involved in the learning process. Below are those main part and their small steps:

1) Introduction

Introduction is a part where the teacher makes connection between the current and previous lesson through appropriate technique. The teacher opens short discussions to encourage learners to think about the previous learning experience and connect it with the current instructional objective. The teacher reviews the prior knowledge, skills and attitudes which have a link with the new concepts to create good foundation and logical sequencings.

2) Development of the new lesson

The development of a lesson that introduces a new concept will go through the following small steps: discovery activities, presentation of learners' findings, exploitation, synthesis/summary and exercises/application activities.

- **Discovery activity**

Step 1

- The teacher discusses convincingly with learners to take responsibility of their learning
- He/she distributes the task/activity and gives instructions related to the tasks (working in groups, pairs, or individual to instigate collaborative learning, to discover knowledge to be learned).

Step 2

- The teacher let learners work collaboratively on the task;
- During this period the teacher refrains to intervene directly on the knowledge;
- He/she then monitors how the learners are progressing towards the knowledge to be learned and boosts those who are still behind (but without communicating to them the knowledge).

- Presentation of learners' findings/productions

- In this episode, the teacher invites representatives of groups to present their productions/findings.
- After three/four or an acceptable number of presentations, the teacher decides to engage the class into exploitation of learners' productions.

- Exploitation of learner's findings/ productions

- The teacher asks learners to evaluate the productions: which ones are correct, incomplete or false
- Then the teacher judges the logic of the learners' products, corrects those which are false, completes those which are incomplete, and confirms those which are correct.

- Institutionalization or harmonization (summary/conclusion/ and examples)

- The teacher summarizes the learned knowledge and gives examples which illustrate the learned content.

- Application activities

- Exercises of applying processes and products/objects related to learned unit/sub-unit
- Exercises in real life contexts
- Teacher guides learners to make the connection of what they learnt to real life situations. At this level, the role of teacher is to monitor the fixation of process and product/object being learned.

3) Assessment

In this step the teacher asks some questions to assess achievement of instructional objective. During assessment activity, learners work individually on the task/activity. The teacher avoids intervening directly. In fact, results from this assessment inform the teacher on next steps for the whole class and individuals. In some cases, the teacher can end with a homework/ assignment. Doing this will allow learners to relay their understanding on the concepts covered that day. Teacher leads them not to wait until the last minute for doing the homework as this often results in an incomplete homework set and/or an incomplete understanding of the concept.

PART II: EXAMPLE OF LESSON PLAN

When teaching any lesson, you can follow the following steps.

Introduction

Start by reviewing previous lesson through asking some questions to learners. If there is no previous lesson, ask them prerequisites related questions for the lesson of the day.

Lesson development

In this step, activities can be more than one (exploration activity, explanation activity and elaboration activity). For each one, give an activity to learners that will be done in groups or individually. After a while, invite one or more groups for presentation of their work to other groups. If the activity is individual, ask one or more learners to present his/her work to others. After activities, capture the main points from the presentation of the learners and guide the whole class to summarize them. After this, provide application activity in their respective groups. Request learners to correct them on chalkboard where you guide every student by addressing eventual misconception.

Evaluation

Give students an activity to be done individually as an assessment. Correct every one and provide related feedback.

Conclusion

Conclude the lesson and remember to assign a home work to students. This homework may include remedial activities, consolidation activities or extended activities depending on the feedback from the assessment. Sometimes when there is no problem in the assessment, a teacher can provide a homework which will arouse the curiosity of students for the next lesson.

See **example of a planned lesson** here bellow.

School: ……………………… Academic year: ………. Teacher's name: ……………

Term	Date	Subject	Class	Unit No	Lesson No	Duration	Class size
1	………	Mathematics	S5 MEG	1	………	80 minutes	35

Type of Special Educational Needs and number of learners

For low vision learners: to avail big printed documents and facilitate these learners. Avoid making a group of low vision only otherwise it can be considered as segregation.

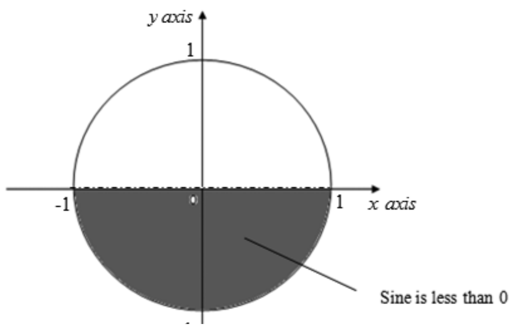
Gifted learners: to encourage them to explain, to each other and help their classmates.

Unit title	TRIGONOMETRIC FORMULAE AND EQUATIONS
Key Unit Competence:	Solve trigonometric equations, inequalities and related problems using trigonometric functions and equations.
Title of the lesson	Trigonometric inequalities
Instructional objective	Given instruments of geometry, learners should be able to represent a trigonometric inequality on trigonometric circle and find the solution accurately.
Plan for this Class	Location: Classroom Learners are organised into groups.
Learning Materials	Exercise book, pen, calculator, ruler
References	Learners' Book

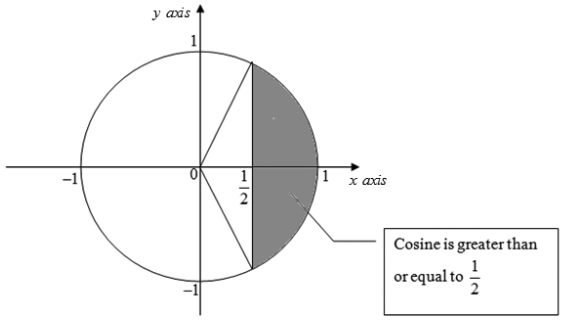
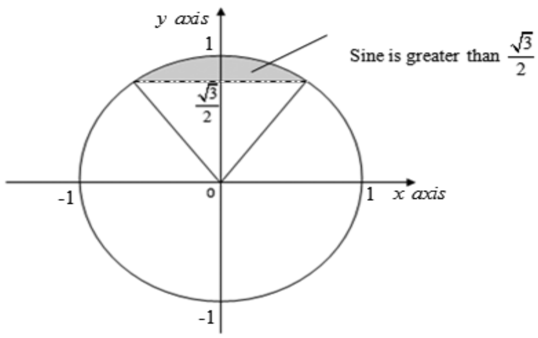
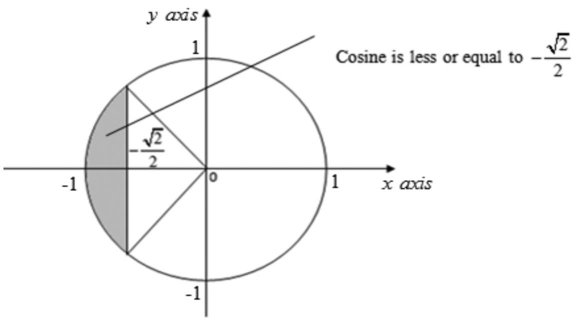
Description of teaching and learning activity

In groups, learners will do the activity 1.9 from learner's book page 23, make presentation of group findings. In conclusion, learners will do questions 1 and 2 of exercise 1.9 from learner's book page 30 in their respective groups and solve them on chalkboard. Learners will do question 3 of exercise 1.9 as individual quiz and question 4 will be an assignment. At the end of the lesson learners are also given another assignment to be discussed as an activity of the next lesson "**Application: Simple harmonic motion**".

Timing for each step	Teacher's activities	
Introduction **5 minutes**	Ask questions on previous lesson.	
Body of the lesson **15 minutes** **10 minutes**	**Step 1:** Form groups • Request the learners to do activity 1.9 from learner's book page 23 in their groups • Goes round to check the progress of the discussion, and intervenes where necessary. • Guides learners with special educational needs on how to do activity. **Step 2:** Request a reporter from each group to present the work on the chalkboard.	

	Learners' activities	Competences and cross cutting issues to be addressed
	Question: Solve the following equations $\cos x = \frac{\sqrt{2}}{2}$ $\sin 2x = \frac{1}{2}$ Respond to questions on the chalkboard Answers: $x = \pm\frac{\pi}{4} + 2k\pi,\ k \in \mathbb{Z}$ $x = \begin{cases} \frac{\pi}{12} + k\pi \\ \frac{5\pi}{12} + k\pi \end{cases},\ k \in \mathbb{Z}$	Students are developing communication skills when they are explaining and sharing ideas
	In their groups, learners will do activity 1.9 • Draw a trigonometric circle • Represent the given inequality on trigonometric circle. • Reporter represents the work. • Learners interact through questions and comments. **Answers:** sine is less than 0  	• Cooperation and interpersonal management developed through working in groups . • Communication: learners communicate and convey information and ideas through speaking when they are presenting their work. • Self confidence: learners will gain self confidence competence when they are presenting their work.

10 minutes	**Step 3:** Capture the main points from the presentation of the learners and summarise them.	

	cosine is greater or equal to $\frac{1}{2}$  sine is greater than $\frac{\sqrt{3}}{2}$  cosine is less or equal to $-\frac{\sqrt{2}}{2}$  	• In group activities, the fact of being convinced without fighting, peace and education values are developed too.

Conclusion **10 minutes**	Request learners to give the main points of the learned lesson in summary.	
15 minutes	Request learners to do application activity 1.9 in their respective groups. Goes round to check the progress of the discussion, and intervenes where necessary.	
15 minutes	Request some learners to answer to questions 1 and 2 of application activity 1.9 on chalkboard. Ensures that the learners understood the learned lesson and decide to repeat the lesson or to continue with new lesson next time	
10 minutes	Give to the learners an individual evaluation (quiz) and homework to the leaned lesson. **Lead into next lesson** Request learners to do activity 1.10 at home.	
Teacher self evaluation	Even if the objective has been achieved, some learners had no instruments of geometry and calculators. The time management has been disturbed by the fact of borrowing materials from their classmates. For this reason, next time each learner must have his/her own materials (instruments of geometry, calculator, …)	

	Summarize the learned lesson When solving inequalities, first replace the inequality sign by equal sign and then solve. Find all no equivalent angles in $[0, 2\pi]$. Place these angles on a trigonometric circle. They will divide the circle into arcs. Choose the arcs containing the angles corresponding to the given inequality. Do questions 1 and 2 of application activity 1.9, from learner's book page 30, in their respective groups. Do questions 1 and 2 of application activity 1.9, from learner's book page 30, on chalkboard. Do question 3 of application activity 1.9, from learner's book page 30, as individual quiz. Do the given quiz individually	Learners develop critical thinking through generating a summary. Through group activities, cooperation is developed. Through presentation on chalkboard, communication skills are developed

PART III: UNIT DEVELOPMENT

Unit 1 Trigonometric Formulae, Equations and Inequalities

1.1. Key unit competence

Solve trigonometric equations and related problems using trigonometric functions and equations

1.2. Objectives

After completing this unit, the learners should be able to:

- Use trigonometric formulae
- Solve trigonometric equations
- Solve trigonometric inequalities

1.3. Main materials to be used in this unit:

Exercise books, pens, instruments of geometry, calculator

1.4. Content and activities

1.4.1. Trigonometric formulae

a) Content summary

Recommended teaching periods: 14 periods

This section looks at **trigonometric formulae**

- Addition and subtraction formulae (compound formulae)

$$\cos(x+y)=\cos x\cos y-\sin x\sin y$$
$$\cos(x-y)=\cos x\cos y+\sin x\sin y$$
$$\sin(x+y)=\sin x\cos y+\cos x\sin y$$
$$\sin(x-y)=\sin x\cos y-\cos x\sin y$$

$$\tan(x+y)=\frac{\tan x+\tan y}{1-\tan x\tan y}$$

$$\tan(x-y)=\frac{\tan x-\tan y}{1+\tan x\tan y}$$

$$\cot(x+y)=\frac{\cot x\cot y-1}{\cot y+\cot x}$$

$$\cot(x-y)=\frac{\cot x\cot y+1}{\cot y-\cot x}$$

- Double angles

$\cos^2 x=1-\sin^2 x$ and $\sin^2 x=1-\cos^2 x$

$$\sin 2x=2\sin x\cos x$$

$$\cos 2x=\cos^2 x-\sin^2 x$$

$$\tan 2x=\frac{2\tan x}{1-\tan^2 x}$$

$$\cot 2x=\frac{\cot^2 x-1}{2\cot x}$$

- Half angle formulae

$$\sin\frac{x}{2}=\pm\sqrt{\frac{1-\cos x}{2}}$$

$$\cos\frac{x}{2}=\pm\sqrt{\frac{\cos x+1}{2}}$$

$\tan\frac{x}{2}=\frac{1-\cos x}{\sin x}$ or $\tan\frac{x}{2}=\frac{\sin x}{1+\cos x}$, the sign +or - is chosen depending on the quadrant in which $\frac{x}{2}$ lies

- Transformation of product in sum

$$\cos x\cos y=\frac{1}{2}\left[\cos(x+y)+\cos(x-y)\right]$$

$$\sin x\sin y=-\frac{1}{2}\left[\cos(x+y)-\cos(x-y)\right]$$

$$\sin x\cos y=\frac{1}{2}\left[\sin(x+y)+\sin(x-y)\right]$$

$$\cos x \sin y = \frac{1}{2}\left[\sin(x+y) - \sin(x-y)\right]$$

- Transformation of sum in product formulae

$$\cos p + \cos q = 2\cos\frac{p+q}{2}\cos\frac{p-q}{2}$$

$$\cos p - \cos q = -2\sin\frac{p+q}{2}\sin\frac{p-q}{2}$$

$$\sin p + \sin q = 2\sin\frac{p+q}{2}\cos\frac{p-q}{2}$$

$$\sin p - \sin q = 2\cos\frac{p+q}{2}\sin\frac{p-q}{2}$$

- *t*-Formulae

If $t = \tan\frac{A}{2}$, then $\sin A = \frac{2t}{1+t^2}, \cos A = \frac{1-t^2}{1+t^2}, \tan A = \frac{1+t^2}{1-t^2}$

b) Teaching guidelines

Let learners know basic relations in trigonometry like

$$\sec x = \frac{1}{\cos x},\ \csc x = \frac{1}{\sin x},\ \tan x = \frac{\sin x}{\cos x},\ \cot x = \frac{1}{\tan x} = \frac{\cos x}{\sin x}.$$

Help them to recall those basic relations.

- Organise class into groups. Request each group to have a group leader who will present their findings to the class.
- Request each group to do activity related to the lesson they are going to learn. The learners may need your assistance to do any activity. Help them to understand the activity. After group discussion, invite some or all groups for presentation of their work.
- After activity presentation, capture the main points from the presentation of the learners and summarise them. Guide the learners through given examples in Learner's book or through your own examples.
- Ask learners what they learned in day lesson to ensure that they understood what they have learned.

- Request learners to do exercises in their respective groups. Request learners to correct exercises on chalkboard and give them individual evaluation. Remember to give them homework.

c) Answers to activities

Activity 1.1

Materials

Exercise book, pens

Answers

a) $\sin(A+B)=\dfrac{PR}{OP}$

$$=\frac{PQ+QR}{OP}$$

$$=\frac{PQ+TS}{OP}$$

$$=\frac{PQ}{OP}+\frac{TS}{OP}$$

$$=\left(\frac{PQ}{PT}\times\frac{PT}{OP}\right)+\left(\frac{TS}{OT}\times\frac{OT}{OP}\right)$$

$$=\cos A\sin B+\sin A\cos B$$

So, $\sin(A+B)=\sin A\cos B+\cos A\sin B$.

b) Similarly, $\cos(A+B)=\dfrac{OR}{OP}$

$$=\frac{OS-RS}{OP}$$

$$=\frac{OS-QT}{OP}$$

$$=\frac{OS}{OP}-\frac{QT}{OP}$$

$$=\left(\frac{OS}{OT}\times\frac{OT}{OP}\right)-\left(\frac{QT}{PT}\times\frac{PT}{OP}\right)$$

$$=\cos A\cos B-\cos A\cos B$$

Thus, $\cos(A+B) = \cos A \cos B - \sin A \sin B$.

Now, $\tan(A+B) = \dfrac{\sin(A+B)}{\cos(A+B)}$

From the identities for $\sin(A+B)$ and $\cos(A+B)$, you have

$$\tan(A+B) = \frac{\sin A \cos B + \cos A \sin B}{\cos A \cos B - \sin A \sin B}$$

$$= \frac{\dfrac{\sin A \cancel{\cos B}}{\cos A \cancel{\cos B}} + \dfrac{\cancel{\cos A} \sin B}{\cancel{\cos A} \cos B}}{\dfrac{\cancel{\cos A}\ \cancel{\cos B}}{\cancel{\cos A}\ \cancel{\cos B}} - \dfrac{\sin A \sin B}{\cos A \cos B}}$$

$$= \frac{\dfrac{\sin A}{\cos A} + \dfrac{\sin B}{\cos B}}{1 - \dfrac{\sin A}{\cos A}\dfrac{\sin B}{\cos B}}$$

$$= \frac{\tan A + \tan B}{1 - \tan A \tan B}$$

So, $\tan(A+B) = \dfrac{\tan A + \tan B}{1 - \tan A \tan B}$

- Replacing B by $-B$ in the identity for $\sin(A+B)$ gives

  $\sin(A-B) = \sin A \cos(-B) + \cos A \sin(-B)$

  Or $\sin(A-B) = \sin A \cos B - \cos A \sin B$

- Replacing B by $-B$ in the identity for $\cos(A+B)$ gives

  $\cos(A-B) = \cos A \cos(-B) - \sin A \sin(-B)$.

  Thus,

  $\cos(A-B) = \cos A \cos B + \sin A \sin B$

- Replacing B by $-B$ in the identity for $\tan(A+B)$ yields

  $\tan(A-B) = \dfrac{\tan A + \tan(-B)}{1 - \tan A \tan(-B)}$

  Hence, $\tan(A-B) = \dfrac{\tan A - \tan B}{1 + \tan A \tan B}$

Application Activity 1.1

1. $$2\sin\theta\sin 4\theta + 2\cos\theta\cos 4\theta = 2(\sin\theta\sin 4\theta + \cos\theta\cos 4\theta)$$
$$= 2\cos(4\theta - \theta) = 2\cos 3\theta$$

2. a) $$\sin 75^0 = \sin(45^0 + 30^0)$$
$$= \sin 45^0 \cos 30^0 + \cos 45^0 \sin 30^0$$
$$= \frac{\sqrt{2}}{2}\frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2}\frac{1}{2} = \frac{\sqrt{6}+\sqrt{2}}{4}$$

b) $$\cos\frac{13\pi}{6} = \cos\left(2\pi + \frac{\pi}{6}\right)$$
$$= \cos 2\pi \cos\frac{\pi}{6} - \sin 2\pi \sin\frac{\pi}{6}$$
$$= \frac{\sqrt{3}}{2}$$

c) $$\tan 330^0 = \tan(360^0 - 30^0)$$
$$= \frac{\tan 360^0 - \tan 30^0}{1 + \tan 360^0 \tan 30^0} = \frac{0 - \frac{\sqrt{3}}{3}}{1} = -\frac{\sqrt{3}}{3}$$

5. a) $2+\sqrt{3}$ b) $\frac{\sqrt{6}+\sqrt{2}}{4}$ c) $\frac{\sqrt{3}}{2}$
d) $\frac{1}{2}$ e) -1

Activity 1.2

Materials

Exercise book, pens

Answers

1. $$\cos(x+x) = \cos x\cos x - \sin x\sin x$$
$$\Rightarrow \cos 2x = \cos^2 x - \sin^2 x$$

2. $$\cos(x-x) = \cos x\cos x + \sin x\sin x$$
$$\Leftrightarrow \cos 0 = \cos^2 x + \sin^2 x$$

$$\Leftrightarrow 1=\cos^2 x+\sin^2 x$$

$$\Rightarrow \cos^2 x+\sin^2 x=1$$

3. $\sin(x+x)=\sin x\cos x+\cos x\sin x$

$$\Rightarrow \sin 2x=2\sin x\cos x$$

4. $\tan(x+x)=\dfrac{\tan x+\tan x}{1-\tan x\tan x}$

$$\Rightarrow \tan 2x=\frac{2\tan x}{1-\tan^2 x}$$

5. $\cot(x+x)=\dfrac{\cot x\cot x-1}{\cot x+\cot x}$

$$\Rightarrow \cot 2x=\frac{\cot^2 x-1}{2\cot x}$$

Application Activity 1.2

1. $4\sin x\cos^3 x-4\cos x\sin^3 x$
2. $\cos^8 x+\sin^8 x-28\cos^2 x\sin^6 x+70\cos^4 x\sin^4 x-28\cos^6 x\sin^2 x$

$$=(1-\sin^2 x)^4+\sin^8 x-28\ (1-\sin^2 x)\sin^6 x+70\ (1-\sin^2 x)^2\sin^4 x$$

$$-28\ (1-\sin^2 x)^3\sin^2 x$$

3. $2\sin 15^0\cos 15^0=\sin\left(2\times 15^0\right)=\sin 30^0=\dfrac{1}{2}$
4. $\dfrac{1}{\sqrt{2}}$
5. a) $\dfrac{4}{5},\dfrac{3}{5},\dfrac{4}{3}$ b) $-\dfrac{4}{5},\dfrac{3}{5},-\dfrac{4}{3}$

6. We have

$$\text{LHS} = \sqrt{2+\sqrt{2+\sqrt{2+2\cos 8x}}}$$
$$= \sqrt{2+\sqrt{2+\sqrt{2(1+\cos 8x)}}}$$
$$= \sqrt{2+\sqrt{2+\sqrt{2(2\cos^2 4x)}}}$$
$$= \sqrt{2+\sqrt{2+\sqrt{4\cos^2 4x}}}$$
$$= \sqrt{2+\sqrt{2+2\cos 4x}}$$
$$= \sqrt{2+\sqrt{2(1+\cos 4x)}}$$
$$= \sqrt{2+\sqrt{2(2\cos^2 2x)}}$$
$$= \sqrt{2+2\cos 2x}$$
$$= \sqrt{2(1+\cos 2x)}$$
$$= \sqrt{2(2\cos^2 2x)}$$
$$= 2\cos x = \text{RHS(as required)}$$

Activity 1.3

Materials

Exercise book, pens

Answer

From the double angle formulae, you have

1. $\cos 2x = \cos^2 x - \sin^2 x$

$= \left(1-\sin^2 x\right) - \sin^2 x$ from $\cos^2 x + \sin^2 x = 1$

$= 1 - 2\sin^2 x$

So, $\cos 2x = 1 - 2\sin^2 x$

Letting $\theta = 2x$, $\cos 2x = 1 - 2\sin^2 x$ gives

$\cos\theta = 1 - 2\sin^2 \dfrac{\theta}{2}$

Or $2\sin^2 \dfrac{\theta}{2} = 1 - \cos\theta \Rightarrow \sin\dfrac{\theta}{2} = \pm\sqrt{\dfrac{1-\cos\theta}{2}}$

So, $\sin\dfrac{\theta}{2} = \pm\sqrt{\dfrac{1-\cos\theta}{2}}$,the sign + or – is chosen depending on the quadrant in which $\dfrac{x}{2}$ lies

2. $\cos 2x = \cos^2 x - \sin^2 x$

$= \cos^2 x - \left(1 - \cos^2 x\right)$ from $\cos^2 x + \sin^2 x = 1$

$= 2\cos^2 x - 1$

So, $\cos 2x = 2\cos^2 x - 1$

Letting $\theta = 2x$, $\cos 2x = 2\cos^2 x - 1$ gives

$\cos\theta = 2\cos^2 \dfrac{\theta}{2} - 1 \Leftrightarrow 2\cos^2 \dfrac{\theta}{2} = 1 + \cos\theta$

$\cos^2 \dfrac{\theta}{2} = \dfrac{1}{2}\left(1 + \cos\theta\right) \Rightarrow \cos\dfrac{\theta}{2} = \pm\sqrt{\dfrac{1+\cos\theta}{2}}$

Thus, $\cos\dfrac{\theta}{2} = \pm\sqrt{\dfrac{1+\cos\theta}{2}}$

3. $\tan\dfrac{\theta}{2} = \dfrac{\sin\dfrac{\theta}{2}}{\cos\dfrac{\theta}{2}}$

$\Leftrightarrow \tan\dfrac{\theta}{2} = \dfrac{\pm\sqrt{\dfrac{1-\cos\theta}{2}}}{\pm\sqrt{\dfrac{1+\cos\theta}{2}}}$ $\qquad \Leftrightarrow \tan\dfrac{\theta}{2} = \dfrac{\sqrt{1-\cos\theta}}{\sqrt{1+\cos\theta}}$

By rationalizing denominator, you get

$\tan\dfrac{\theta}{2} = \dfrac{\sqrt{1-\cos\theta}}{\sqrt{1+\cos\theta}}\dfrac{\sqrt{1-\cos\theta}}{\sqrt{1-\cos\theta}}$

$$\Leftrightarrow \tan\frac{\theta}{2}=\frac{\sqrt{(1-\cos\theta)^2}}{\sqrt{1-\cos^2\theta}} \qquad \Leftrightarrow \tan\frac{\theta}{2}=\frac{|1-\cos\theta|}{\sqrt{1-\cos^2\theta}}$$

$$\Leftrightarrow \tan\frac{\theta}{2}=\frac{\pm(1-\cos\theta)}{\sqrt{1-\cos^2\theta}} \qquad \Leftrightarrow \tan\frac{\theta}{2}=\frac{\pm(1-\cos\theta)}{|\sin\theta|}$$

$$\Leftrightarrow \tan\frac{\theta}{2}=\frac{1-\cos\theta}{\sin\theta}$$

So, $\tan\frac{\theta}{2}=\frac{1-\cos\theta}{\sin\theta}$

From $\tan\frac{\theta}{2}=\frac{\sqrt{1-\cos\theta}}{\sqrt{1+\cos\theta}}$, conjugating numerator, you get

$$\tan\frac{\theta}{2}=\frac{\sqrt{1-\cos\theta}}{\sqrt{1+\cos\theta}}\frac{\sqrt{1+\cos\theta}}{\sqrt{1+\cos\theta}}$$

$$\Leftrightarrow \tan\frac{\theta}{2}=\frac{\sqrt{1-\cos^2\theta}}{\sqrt{(1+\cos\theta)^2}} \qquad \Leftrightarrow \tan\frac{\theta}{2}=\frac{\sqrt{\sin^2\theta}}{\sqrt{(1+\cos\theta)^2}}$$

$$\Leftrightarrow \tan\frac{\theta}{2}=\frac{|\sin\theta|}{|1-\cos\theta|} \qquad \Leftrightarrow \tan\frac{\theta}{2}=\frac{|\sin\theta|}{|1+\cos\theta|}$$

$$\Leftrightarrow \tan\frac{\theta}{2}=\frac{\sin\theta}{1+\cos\theta}$$

So $\tan\frac{\theta}{2}=\frac{\sin\theta}{1+\cos\theta}$

Therefore, $\tan\frac{\theta}{2}=\frac{1-\cos\theta}{\sin\theta}$ or $\tan\frac{\theta}{2}=\frac{\sin\theta}{1+\cos\theta}$

Application Activity 1.3

1. If $\cos A = -\frac{7}{25}$,

$$\sin\frac{1}{2}A = \pm\sqrt{\frac{1-\cos A}{2}} = \pm\sqrt{\frac{1-\left(-\frac{7}{25}\right)}{2}} = \pm\sqrt{\frac{\frac{32}{25}}{2}} = \pm\sqrt{\frac{16}{25}} = \pm\frac{4}{5};$$

$$\cos\frac{1}{2}A = \pm\sqrt{\frac{1+\cos A}{2}} = \pm\sqrt{\frac{1-\frac{7}{25}}{2}} = \pm\sqrt{\frac{\frac{18}{25}}{2}} = \pm\sqrt{\frac{9}{25}} = \pm\frac{3}{5};$$

$$\tan A = \pm\sqrt{\frac{1-\cos A}{1+\cos A}} = \pm\sqrt{\frac{1+\frac{7}{25}}{1-\frac{7}{25}}} = \pm\sqrt{\frac{32}{18}} = \pm\frac{4}{3}$$

Since

$$\cos A < 0,$$
$$90^0 < A < 180^0;$$
$$45^0 < \frac{A}{2} < 90^0;$$
$$\cos\frac{A}{2} > 0; \sin\frac{A}{2} > 0$$

Therefore, $\cos\frac{A}{2} = \frac{3}{5}$; $\tan\frac{A}{2} = \pm\sqrt{\frac{1-\cos A}{1+\cos A}}$

$$\sin\frac{A}{2} = \frac{4}{5}$$

Since

$$45^0 < \frac{A}{2} < 90^0;$$

$$\tan\frac{A}{2} > 0$$

Therefore,

$$\tan\frac{A}{2} = \frac{4}{3}$$

2. If $\tan 2A = \frac{7}{24}$, $0 < A < \frac{\pi}{4}$, to find $\tan A$

$$\tan 2A = \frac{2\tan A}{1 - tan^2 A}$$

$$\Rightarrow \frac{7}{24} = \frac{2\tan A}{1 - tan^2 A}$$

$$\Rightarrow 7 - 7tan^2 A = 48\tan A$$

$$\Rightarrow\ 7\tan^2 A + 48\tan A - 7 = 0$$

$$\Rightarrow (7\tan A - 1)(\tan A + 7) = 0$$

$\Rightarrow \tan A = \frac{1}{7}$ since $\tan\mathrm{A} = 7$ is impossible for $0 < A < \frac{\pi}{4}$

3. $\frac{\sqrt{2-\sqrt{2}}}{2}, \frac{\sqrt{2+\sqrt{2}}}{2}, 1 - \frac{\sqrt{2}}{2}$

4. $\frac{\sqrt{4-\sqrt{2}-\sqrt{6}}}{2\sqrt{2}}$

Activity 1.4

Materials

Exercise book, pens

Answer

1. $\sin(x+y) + \sin(x-y) = \sin x\cos y + \cancel{\cos x\sin y} + \sin x\cos y - \cancel{\cos x\sin y}$
$= 2\sin x\cos y$

2. $\sin(x+y) - \sin(x-y) = \sin x\cos y + \cos x\sin y - (\sin x\cos y - \cos x\sin y)$
$= \cancel{\sin x\cos y} + \cos x\sin y - \cancel{\sin x\cos y} + \cos x\sin y$
$= 2\cos x\sin y$

3. $\cos(x+y) + \cos(x-y) = \cos x\cos y - \cancel{\sin x\sin y} + \cos x\cos y + \cancel{\sin x\sin y}$
$= 2\cos x\cos y$

4. $\cos(x+y) - \cos(x-y) = \cos x\cos y - \sin x\sin y - (\cos x\cos y + \sin x\sin y)$
$= \cancel{\cos x\cos y} - \sin x\sin y - \cancel{\cos x\cos y} - \sin x\sin y$
$= -2\sin x\sin y$

Application Activity 1.4

1. a) $\sin x\cos 3x=\frac{1}{2}(\sin 4x-\sin 2x)$

 b) $\cos 12x\sin 9x=\frac{1}{2}(\sin 21x-\sin 3x)$

 c) $-\frac{1}{2}(\cos 20x-\cos 2x)$

 d) $\sin 8x-\sin 2x$

 e) $\frac{1}{2}(\cos 4x+\cos x)$

Activity 1.5

Materials

Exercise book, pens

Answers

The formulae for transforming product in sum are

$\cos x\cos y=\frac{1}{2}\left[\cos(x+y)+\cos(x-y)\right]$ (Equation 1)

$\sin x\sin y=-\frac{1}{2}\left[\cos(x+y)-\cos(x-y)\right]$ (Equation 2)

$\sin x\cos y=\frac{1}{2}\left[\sin(x+y)+\sin(x-y)\right]$ (Equation 3)

$\cos x\sin y=\frac{1}{2}\left[\sin(x+y)-\sin(x-y)\right]$ (Equation 4)

$\begin{cases}x+y=p\\x-y=q\end{cases}\Rightarrow\begin{cases}x=\dfrac{p+q}{2}\\[2mm] y=\dfrac{p-q}{2}\end{cases}$ (i)

From (i)

Equation (1) becomes $\cos\frac{p+q}{2}\cos\frac{p-q}{2}=\frac{1}{2}(\cos p+\cos q)$

Equation (2) becomes $\sin\frac{p+q}{2}\sin\frac{p-q}{2}=-\frac{1}{2}(\cos p-\cos q)$

Equation (3) becomes $\sin\frac{p+q}{2}\cos\frac{p-q}{2}=\frac{1}{2}(\sin p+\sin q)$

Equation (4) becomes $\cos\frac{p+q}{2}\sin\frac{p-q}{2}=\frac{1}{2}(\sin p-\sin q)$

Application Activity 1.5

1. a) $\cos x+\cos 7x=2\cos 4x\cos 3x$

   b) $\sin 4x-\sin 9x=-2\cos\frac{13x}{2}\sin\frac{5x}{2}$

   c) $\sin 3x+\sin x=2\sin 2x\cos x$

   d) $\cos 2x-\cos 4x=2\sin 3x\sin x$

1.4.2. Trigonometric equations

a) Content summary

Recommended teaching periods: 7 periods

The solutions of a trigonometric equation for which $0\le x\le 2\pi$ are called **principal solutions** while the expression (involving integer k) of solution containing all values of the unknown angle is called the **general solution** of the trigonometric equation. When the interval of solution is not given, you are required to find general solution.

When solving trigonometric equation, **note that general solution for**

- $\sin x=0$ is $x=k\pi, k\in\mathbb{Z}$
- $\cos x=0$ is $x=(2k+1)\frac{\pi}{2}, k\in\mathbb{Z}$
- $\tan x=0$ is $x=k\pi, k\in\mathbb{Z}$
- all angles having the same sine i.e. $\sin x=\sin\theta$ is $x=(-1)^k\theta+k\pi, k\in\mathbb{Z}$
- all angles having the same cosine i.e. $\cos x=\cos\theta$ is $x=\pm\theta+2k\pi, k\in\mathbb{Z}$
- all angles having the same tangent i.e. $\tan x=\tan\theta$ is $x=\theta+k\pi, k\in\mathbb{Z}$

The sum or difference of trigonometric functions containing unknown are transformed into the sum.

Remember that

$$\cos p+\cos q=2\cos\frac{p+q}{2}\cos\frac{p-q}{2}$$

$$\cos p-\cos q=-2\sin\frac{p+q}{2}\sin\frac{p-q}{2}$$

$$\sin p+\sin q=2\sin\frac{p+q}{2}\cos\frac{p-q}{2}$$

$$\sin p-\sin q=2\cos\frac{p+q}{2}\sin\frac{p-q}{2}$$

To find the general solution, the equation of the form $a\sin x+b\cos x=c$ where $a,b,\mathrm{c}\in\mathbb{Z}$ such that $|c|\le\sqrt{a^2+b^2}$

a) Divide each term by $\sqrt{a^2+b^2}$ and convert it in the form $\frac{a}{\sqrt{a^2+b^2}}\sin x+\frac{b}{\sqrt{a^2+b^2}}\cos x=\frac{c}{\sqrt{a^2+b^2}}$

b) Let $\tan\theta=\frac{b}{a}$, then $\sin\theta=\frac{b}{\sqrt{a^2+b^2}},\cos\theta=\frac{a}{\sqrt{a^2+b^2}}$.
The given equation reduces to the form
$r\cos\theta\cos x+r\sin\theta\sin x=c$ or $\cos\theta\cos x+\sin\theta\sin x=\frac{c}{r}$

c) Then, $\cos(x-\theta)=\cos\alpha$,where $\cos\alpha=\frac{c}{r}=\frac{c}{\sqrt{a^2+b^2}}$

d) Therefore, $x=\pm\alpha+\theta+2k\pi,k\in\mathbb{Z}$

Alternative method: in $a\sin x+b\cos x=c$

Using t-formula, $\sin x=\frac{2t}{1+t^2},\cos x=\frac{1-t^2}{1+t^2}$, where $t=\tan\frac{x}{2}$, gives

$$a\frac{2t}{1+t^2}+b\frac{1-t^2}{1+t^2}=c\Rightarrow 2at+b-bt^2=c\left(1+t^2\right)$$

$\Leftrightarrow(b+c)t^2-2at+c-b=0$ which is quadratic equation in t.

Remember that $t=\tan\frac{x}{2}$.

Notice

In solving the trigonometric equation, it is helpful to remember the following identities:

$\sin\alpha = \sin(\alpha + 2k\pi),\ k \in \mathbb{Z}$ $\qquad \sin\alpha = \sin(\pi - \alpha)$

$\cos\alpha = \cos(\alpha + 2k\pi),\ k \in \mathbb{Z}$ $\qquad \cos\alpha = \cos(-\alpha)$

$\tan\alpha = \tan(\alpha + k\pi),\ k \in \mathbb{Z}$ $\qquad \tan\alpha = \tan(\alpha + \pi)$

b) Teaching guidelines

Make sure that learners have scientific calculators.

- Organise class into groups. Request each group to have a group leader who will present their findings to the class.
- Request each group to do activity related to the lesson they are going to learn. The learners may need your assistance to do any activity. Help them to understand the activity. After group discussion, invite some or all groups for presentation of their work.
- After activity presentation, capture the main points from the presentation of the learners and summarise them. Guide the learners through given examples in Learner's book or through your own examples.
- Ask learners what they learned in day lesson to ensure that they understood what they have learned.
- Request learners to do exercises in their respective groups. Request learners to correct exercises on chalkboard and give them individual evaluation. Remember to give them homework.

c) Answers to activities

Activity 1.6

Materials

Exercise book, pens and calculator

Answers

1. $\begin{cases} \dfrac{\pi}{6}+2k\pi \\ \dfrac{5\pi}{6}+2k\pi \end{cases}, \quad k \in \mathbb{Z}$

2. $\pm\dfrac{\pi}{4}+2k\pi, \quad k \in \mathbb{Z}$

3. $\dfrac{\pi}{6}+k\pi, \quad k \in \mathbb{Z}$

Application Activity 1.6

1. a) $\dfrac{\pi}{3}, \dfrac{2\pi}{3}$ b) $\dfrac{2\pi}{3}, \dfrac{5\pi}{3}$ c) $\dfrac{\pi}{3}, \dfrac{5\pi}{3}$

   d) $\dfrac{\pi}{6}$ e) $\dfrac{\pi}{3}, \dfrac{2\pi}{3}$

2. a) $\dfrac{5\pi}{6}+2k\pi, k \in \mathbb{Z}$ b) $-\dfrac{\pi}{3}+k\pi, k \in \mathbb{Z}$

   c) $\dfrac{\pi}{3}+k\pi, \dfrac{\pi}{4}+k\pi, k \in \mathbb{Z}$ d) $k\pi, \pm\dfrac{\pi}{6}+k\pi, k \in \mathbb{Z}$

Activity 1.7

Materials

Exercise book, pens and calculator

Answers

a) $\cos 2x = \dfrac{1}{\sqrt{2}}$ is positive, thus, $2x$ lies in the 1st or 4th quadrant.

$$\cos 2x = \frac{1}{\sqrt{2}} = \cos\frac{\pi}{4} \quad \text{or} \quad \cos\left(2\pi - \frac{\pi}{4}\right)$$

$$\Rightarrow 2x = \frac{\pi}{4}+2k\pi \ \ or \ \ 2x = \frac{7\pi}{4}+2k\pi, k \in \mathbb{Z}$$

General solution of $\cos 2x = \dfrac{1}{\sqrt{2}}$ is $\dfrac{\pi}{8}+k\pi, \ k \in \mathbb{Z}$ or $x = \dfrac{7\pi}{8}+k\pi, \ k \in \mathbb{Z}$

b) $\sin\frac{x}{2}=-\frac{1}{2}$ is negative $\Rightarrow\frac{x}{2}$ lies in the 3rd or 4th quadrant.

Here, $\sin\frac{x}{2}=-\frac{1}{2}=\sin\left(\pi+\frac{\pi}{6}\right)$ or $\sin\left(-\frac{\pi}{6}\right)$

$\Rightarrow\frac{x}{2}=\frac{7\pi}{6}$ *or* $\frac{x}{2}=-\frac{\pi}{6}$ The general solution of $\sin\frac{x}{2}=-\frac{1}{2}$ is

$\frac{x}{2}=-\frac{\pi}{6}+2k\pi$ or $\frac{7\pi}{6}+2k\pi, k\in\mathbb{Z}$

$\Rightarrow x=-\frac{\pi}{3}+4k\pi$ or $\frac{7\pi}{3}+4k\pi\equiv\frac{\pi}{3}+4\text{k}\,\pi, k\in\mathbb{Z}$

c) $\sin mx+\sin nx=0\Leftrightarrow 2\sin\frac{(m+n)x}{2}\cos\frac{(m-n)x}{2}=0$

$\Rightarrow\sin\frac{(m+n)x}{2}=0$ *or* $\cos\frac{(m-n)x}{2}=0$

$\Rightarrow\frac{(m+n)x}{2}=k\pi$ *or* $\frac{(m-n)x}{2}=\frac{\pi}{2}+k\pi$

$\Rightarrow x=\frac{2k\pi}{m+n}$ *or* $\frac{(m-n)x}{2}=\frac{\pi}{2}(2k\pi+1)$

$\Rightarrow x=\frac{2k\pi}{m+n}$ *or* $x=\frac{(2k\pi+1)\pi}{m-n}$

General solution is $x=\frac{2k\pi}{m+n}$ *or* $x=\frac{(2k\pi+1)\pi}{m-n}$

d) $\cos 4x-\cos 2x=0\Leftrightarrow 2\sin 3x\sin x=0$

$\Rightarrow\sin 3x=0$ *or* $\sin x=0$

$\Rightarrow 3x=k\pi$ *or* $x=k\pi$

$\Rightarrow x=\frac{k\pi}{3}$ *or* $x=k\pi$

General solution is $x=\frac{k\pi}{3}$ *or* $x=k\pi$

Application Activity 1.7

1.	$\pm\frac{\pi}{12}+\frac{k\pi}{4},\ k\in\mathbb{Z}$	2.	$\left\{0,\frac{\pi}{14},\frac{\pi}{3}\right\}$
3.	$\{30^0,150^0,199.5^0,340.5^0\}$	4.	$\{170.7^0,350.7^0\}$
5.	$\frac{k\pi}{2}\ or\ \frac{\pi}{2}(2k\pi+1),\ k\in\mathbb{Z}$	6.	$(2k+1)\frac{\pi}{2},\ (2k+1)\frac{\pi}{8}$
7.	$\frac{k\pi}{3},\ \pm\frac{\pi}{12}+\frac{k\pi}{2},\ k\in\mathbb{Z}$		
8.	$(2k+1)\frac{\pi}{8},\ (2k+1)\frac{\pi}{4},\ (2k+1)\frac{\pi}{2},\ k\in\mathbb{Z}$		

Activity 1.8

Materials

Exercise book, pens and calculator

Answers

$$\sqrt{3}\cos x-\sin x=\sqrt{3}$$

1. $\Rightarrow \cos x-\frac{\sin x}{\sqrt{3}}=1$

2. $\tan\alpha=-\frac{1}{\sqrt{3}}\Rightarrow\frac{\sin\alpha}{\cos\alpha}=-\frac{1}{\sqrt{3}}\ or\ \alpha=-\frac{\pi}{6}$

3. $\cos x-\frac{\sin x}{\sqrt{3}}=1\Rightarrow\cos x-\frac{\sin\alpha}{\cos\alpha}\sin x=1$

$$\Leftrightarrow\cos\alpha\cos x-\sin\alpha\sin x=\cos\alpha$$

$$\Leftrightarrow\cos(x-\alpha)=\cos\alpha$$

$$\Leftrightarrow\cos\left(x-\left(-\frac{\pi}{6}\right)\right)=\cos\left(-\frac{\pi}{6}\right)\ \Leftrightarrow\cos\left(x+\frac{\pi}{6}\right)=\frac{\sqrt{3}}{2}$$

$$\Rightarrow\begin{array}{l}x+\frac{\pi}{6}=-\frac{\pi}{6}+2k\pi\\ \text{or}\\ x+\frac{\pi}{6}=\frac{\pi}{6}+2k\pi\end{array},\ k\in\mathbb{Z}\ \Rightarrow\begin{array}{l}x=-\frac{\pi}{3}+2k\pi\\ \text{or}\\ x=2k\pi\end{array},\ k\in\mathbb{Z}$$

4. $\cos\alpha\cos x - \sin\alpha\sin x = \cos\alpha \Leftrightarrow \cos(x-\alpha) = \cos\alpha$

$\Leftrightarrow \cos\left(x-\left(-\frac{\pi}{6}\right)\right) = \cos\left(-\frac{\pi}{6}\right)$ $\Leftrightarrow \cos\left(x+\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$

$\cos(x+\frac{\pi}{6}) = \cos\frac{\pi}{6}$

$x+\frac{\pi}{6} = \frac{\pi}{6}+2k\pi$

$x = 2k\pi$

or

$x+\frac{\pi}{6} = -\frac{\pi}{6}+2k\pi$

$x = -\frac{\pi}{3}+2k\pi$

Application Activity 1.8

1. $\left\{x=\frac{\pi}{6}+k\pi, x=\frac{\pi}{2}, k\in\mathbb{Z}\right\}$
2. $\left\{x=\frac{\pi}{4}+k\pi, k\in\mathbb{Z}\right\}$
3. $\pm\frac{3\pi}{4}-\frac{\pi}{4}+2k\pi,\ k\in\mathbb{Z}$
4. $\frac{\pi}{6}\pm\frac{\pi}{4}+2k\pi,\ k\in\mathbb{Z}$
5. $\frac{\pi}{6}\pm\frac{\pi}{3}+2k\pi,\ k\in\mathbb{Z}$
6. $-\frac{\pi}{4}+2k\pi,\ k\in\mathbb{Z}$

1.4.3. Trigonometric inequalities

a) Content summary

Recommended teaching periods: 14 periods

To solve inequalities:

- First replace the inequality sign by equal sign and then solve.
- Find all no equivalent angles in $[0, 2\pi]$.
- Place these angles on a trigonometric circle. They will divide the circle into arcs.
- Choose the arcs containing the angles corresponding to the given inequality.

b) Teaching guidelines

Help them to recall how to draw a trigonometric circle. Make sure that learners have mathematical instruments and scientific calculators.

- Organise class into groups. Request each group to have a group leader who will present their findings to the class.
- Request each group to do activity related to the lesson they are going to learn. The learners may need your assistance to do any activity. Help them to understand the activity. After group discussion, invite some or all groups for presentation of their work.
- After activity presentation, capture the main points from the presentation of the learners and summarise them. Guide the learners through given examples in Learner's book or through your own examples.
- Ask learners what they learned in day lesson to ensure that they understood what they have learned.
- Request learners to do exercises in their respective groups. Request learners to correct exercises on chalkboard and give them individual evaluation. Remember to give them homework.

c) Answers to activities

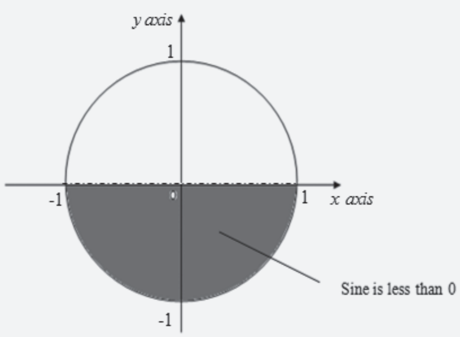
Activity 1.9

Materials

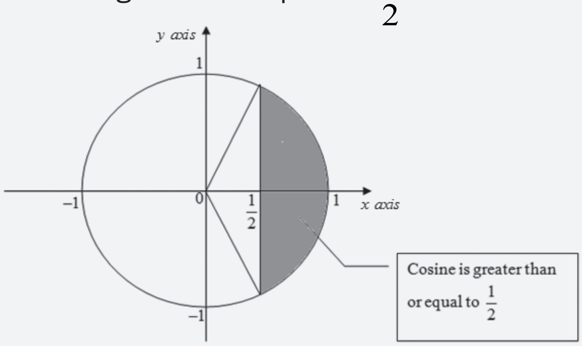
Exercise book, pens, instruments of geometry and calculator

Answers

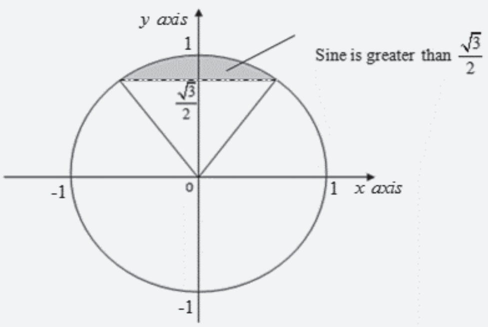
1. sine is less than 0



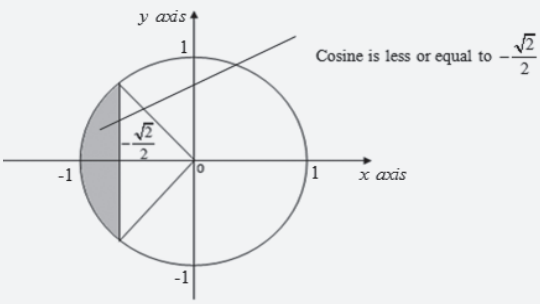
2. cosine is greater or equal to $\frac{1}{2}$



3. sine is greater than $\frac{\sqrt{3}}{2}$



4. cosine is less or equal to $-\frac{\sqrt{2}}{2}$



Application Activity 1.9

1. $\left[2k\pi, \frac{\pi}{3}+2k\pi\right] \cup \left[\frac{2\pi}{3}+2k\pi, (k+1)2\pi\right], \ k \in \mathbb{Z}$

2. $\left]2k\pi, \frac{\pi}{4}+2k\pi\right[\cup \left]\frac{3\pi}{4}+2k\pi, \pi+2k\pi\right[\cup \left]\frac{5\pi}{4}+2k\pi, \frac{7\pi}{4}+2k\pi\right[, \ k \in \mathbb{Z}$

3. $\left]\frac{7\pi}{18}+2k\pi, \frac{11\pi}{18}+2k\pi\right[\cup \left]\frac{19\pi}{18}+2k\pi, \frac{23\pi}{18}+2k\pi\right[\cup \left]\frac{31\pi}{18}\right.$

$\cup \left]\frac{31\pi}{18}+2k\pi, \frac{35\pi}{18}+2k\pi\right[, \ k \in \mathbb{Z}$

4. $k\pi, \ k \in \mathbb{Z}$

1.4.4. Applications of trigonometry

Activity 1.10

Materials

Exercise book, pens

Answers

By reading text books or accessing internet, learners will discuss on harmonic motion and how trigonometry is used in harmonic motion. An object that moves on a coordinate axis is in simple harmonic motion if its distance from the origin, d, at time t is given by either $d = a\cos\omega t$ or $d = a\sin\omega t$. The motion has amplitude $|a|$, the maximum displacement of the object from its rest position. The period of the motion is $\frac{2\pi}{\omega}$, where $\omega > 0$.

Activity 1.11

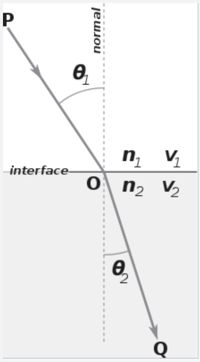
Materials

Exercise book, pens

Answers

By reading text books or accessing internet, learners will discuss on Snell's law.

The degree of bending of the light's path depends on the angle that the incident beam of light makes with the surface, and on the ratio between the refractive indices of the two media.



Snell's law state that: $n_1 \sin\theta_1 = n_2 \sin\theta_2$

1.5. Answers for the end of unit assessment

1. $\dfrac{4\tan x-4\tan^3 x}{1-6\tan^2 x+\tan^4 x}$

2. $\dfrac{\cot^4 x-6\cot^2 x+1}{4\cot^3 x-4\tan x}$

3. $\sin 2\theta=\dfrac{120}{169}$

4. $2\cot 2a=\dfrac{2}{\tan 2a}$

$$=2\left(\frac{1-\tan^2 a}{2\tan a}\right)=2\left(\frac{1}{2\tan a}-\frac{\tan^2 a}{2\tan a}\right)$$

$$=\frac{1}{\tan a}-\tan a=\cot a-\tan a$$

5. a) $\dfrac{1}{2}$ b) $\dfrac{\sqrt{2}}{2}$

c) -1 d) $\dfrac{\sqrt{3}}{2}$

6. $-\dfrac{7}{25}$

7. $-\dfrac{120}{119}$

8. a) $2\cos\dfrac{17}{2}\cos\dfrac{x}{2}$ b) $2\sin 7x\cos 4x$

9. a) $\dfrac{1}{2}(\sin 15x-\sin 7x)$ b) $\dfrac{1}{2}(\sin 16x+\sin 2x)$

10. a) $\dfrac{\pi}{6},\dfrac{\pi}{4},\dfrac{5\pi}{6}$ b) $\dfrac{\pi}{6},\dfrac{\pi}{2},\dfrac{5\pi}{6}$

11) $169.6^0, 349.6^0$

12) $60^0, 240^0$

13) $126.2^0, 306.2^0$

14) $85.9^0, 265.9^0$

15) $22.5^0, 112.5^0, 202.5^0, 292.5^0$

16) $0^0, 60^0, 180^0, 300^0, 360^0$

17) $0^0, 120^0, 180^0, 240^0, 360^0$

18) $21.5^0, 158.5^0$

19) $0^0, 60^0, 120^0, 180^0, 240^0, 300^0$

20) $19.5^0, 160.5^0, 270^0$

21) $33.7^0, 63.4^0, 213.7^0, 243.4^0$

22) $33.7^0, 153.4^0, 213.7^0, 333.4^0$

23) $30^0, 90^0, 150^0, 270^0$

24) $120^0, 240^0$

25) $15^0, 75^0, 195^0, 255^0$

26) $0^0, 180^0, 360^0, 60^0, 300^0, 120^0, 240^0$

27) $0^0, 180^0, 360^0, 45^0, 135^0, 225^0, 315^0$

28) $0^0, 14.5^0, 165.5^0, \pm 180^0$

29) $\pm 180^0$

30) $0^0, \pm 48.2^0$

31) $19.5^0, 160.5^0$

32) $\pm 45^0, \pm 135^0$

33) $-63.4^0, 0, 116.6^0, \pm 180^0$

34) $60^0, 300^0$

35) $55.9^0, 145.9^0, 235.9^0, 325.9^0$

36) $60^0, 120^0, 240^0, 300^0$

37) $120^0, 240^0$

38) $26.6^0, 135^0, 206.6^0, 315^0$

39) $68.2^0, 135^0, 248.2^0, 315^0$

40) $194.5^0, 345.5^0$

41) $26.6^0, 45^0, 206.6^0, 255^0$

42) $70.5^0, 289.5^0$

43) $\left]\frac{\pi}{5}, \frac{2\pi}{5}\right[\cup \left]\frac{3\pi}{5}, \frac{4\pi}{5}\right[\cup \left]\pi, \frac{6\pi}{5}\right[\cup \left]\frac{7\pi}{5}, \frac{8\pi}{5}\right[\cup \left]\frac{7\pi}{5}, 2\pi\right[$

44) $\left[\frac{\pi}{18}, \frac{11\pi}{18}\right] \cup \left[\frac{13\pi}{18}, \frac{23\pi}{18}\right] \cup \left[\frac{25\pi}{18}, \frac{35\pi}{18}\right]$

45) $\left]\frac{\pi}{3}, \frac{2\pi}{3}\right[\cup \left]\frac{4\pi}{3}, \frac{5\pi}{3}\right[$

46) $10cm,\ 5cm,\ -5\sqrt{2}cm$

47) $3m,\ 4m,\ -3m$

Unit 2 Sequences

2.1. Key unit competence

Understand, manipulate and use arithmetic, geometric sequences.

2.2. Objectives

After completing this unit, the learners should be able to:

- Define a sequence
- Determine whether a sequence converges or diverges
- Indicate the monotone sequences
- Determine harmonic sequences
- Identify arithmetic and geometric progressions and their properties
- Use sequences in daily life

2.3. Main materials to be used

Exercise books, pens, instruments of geometry, calculator

2.4. Content and activities

2.4.1. Generalities on sequences

a) Content summary

Recommended teaching periods: 7 periods

This section looks at the definition of a **numerical sequence**, **convergence** and **divergence** sequences, **monotonic sequences.**

A sequence is a function whose domain is either $\mathbb{N}$ or subset of the form $\{1,2,3,4,\cdots,n\}$; depending on the domain of definition, a sequence is finite or infinite.

A numerical sequence $\{u_n\}$ is said to be convergent if it has a finite limit as $n \to \infty$ otherwise it is said to be divergent.

If $\lim\limits_{n\to+\infty} u_n = L$ number L is called a limit of a numerical sequence

A sequence $\{u_n\}$ is said to be:

- **Increasing** or in ascending order if $u_1 < u_2 < u_3 < \ldots < u_n < \ldots$
- **non-decreasing** if $u_1 \le u_2 \le u_3 \le \ldots \le u_n \le \ldots$
- **decreasing** or in descending order if $u_1 > u_2 > u_3 > \ldots > u_n > \ldots$
- **non-increasing** $u_1 \ge u_2 \ge u_3 \ge \ldots \ge u_n \ge \ldots$

A sequence that is either non-decreasing or non-increasing is called **monotone**, and a sequence that is increasing or decreasing is called **strictly monotone.**

b) Teaching guidelines

Let learners know sets of numbers. You can request them to write down set of even numbers and set of odd numbers (arithmetic sequences).

- Organise the class into groups. Request each group to have a group leader who will present their findings to the class.
- Request each group to do activity related to the lesson they are going to learn. The learners may need your assistance to do any activity. Help them to understand the activity. After group discussion, invite some or all groups for presentation of their work. Let learners interact through questions and comments.
- After activity presentation, capture the main points from the presentation of the learners and summarise them. Guide the learners through given examples in Learner's book or through your own examples.
- Ask learners what they learned in day lesson to ensure that they understood what they have learned.
- Request learners to do exercises in their respective groups. Request learners to correct exercises on chalkboard and give them individual evaluation. Remember to give them homework.

c) Answers to activities

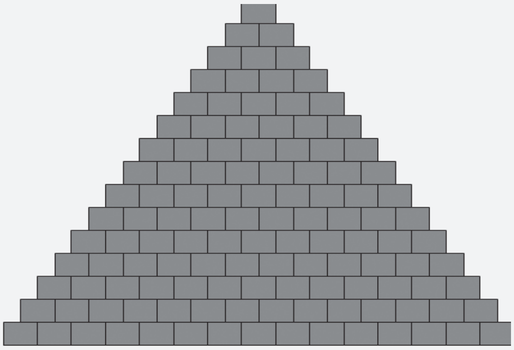
Activity 2.1

Materials

Exercise book, pens, pencil, ruler, calculator

Answers

1. a)



   b) There are 15 rows

   c) 15, 14, 13, 12, 11, 10, 9, 8, 7, 6, 5, 4, 3, 2, 1

   d) That number is -1

2. Number of insects in

   1^{st} generation $\rightarrow 126$, 2^{nd} generation $\rightarrow 252$,

   3^{rd} generation $\rightarrow 504$, 4^{th} generation $\rightarrow 1008$,

   5^{th} generation $\rightarrow 2016$, 6^{th} generation $\rightarrow 4032$,

   7^{th} generation $\rightarrow 8064$, 8^{th} generation $\rightarrow 16128$,

   9^{th} generation $\rightarrow 32256$, 10^{th} generation $\rightarrow 64512$

Application Activity 2.1

1. $1, \frac{8}{5}, \frac{18}{10}$

2. $\sqrt{2}-1, \sqrt{3}-\sqrt{2}, 2-\sqrt{3}, \sqrt{5}-2, \sqrt{6}-\sqrt{5}$

3. $\{2n-1\}_{n=1}^{+\infty}$

Activity 2.2

Materials

Exercise book, pens

Answers

1. 0	2. 0	3. $+\infty$

Application Activity 2.2

1. Converges to 2	2. Converge to -1
3. Converges to -5	4. Diverge
5. Converges to $\frac{2}{\sqrt{3}}$	6. Converges to 0

Activity 2.3

Materials

Exercise book, pens

Answers

1. Ascending	2. Descending
3. Both	4. Neither

Application Activity 2.3

1. Increasing	2. Increasing
3. Decreasing	4. Both increasing, decreasing
5. Not monotonic	

2.4.2. Arithmetic sequences and harmonic sequences

a) Content symmary

Recommended teaching periods: 7 periods

This section studies the arithmetic sequences and the harmonic sequence

Arithmetic progression

A finite or infinite sequence $a_1, a_2, a_3, \ldots, a_n$ or $a_1, a_2, a_3, \ldots, a_n, \ldots$ is said to be an Arithmetic Progression (A.P.) or Arithmetic Sequence if $a_k - a_{k-1} = d$, where d is a constant independent of k, for $k = 2, 3, \ldots, n$ or $k = 2, 3, \ldots, n, \ldots$ as the case may be.

Characteristics

If three consecutive terms, u_{n-1}, u_n, u_{n+1} are in arithmetic sequence, then,

$2u_n = u_{n-1} + u_{n+1}$

- Common difference

  In A.P., the difference between any two consecutive terms is a constant d, called **common difference**

- **General term or n^th term**

  The n^th term, u_n, of an arithmetic sequence $\{u_n\}$ with common difference d and initial term u_1 is given by

  $u_n = u_1 + (n-1)d$

  Generally, if u_p is any p^{th} term of a sequence, then the n^th term is given by $u_n = u_p + (n-p)d$

- **Arithmetic means**

  If three or more than three numbers are in arithmetic sequence, then all terms lying between the first and the last numbers are called arithmetic means. If B is arithmetic mean between A and C, then $B = \frac{A+C}{2}$.

  To insert k arithmetic means between two terms u_1 and u_n is to form an arithmetic sequence of $n = k+2$ terms whose first term is u_1 and the last term is u_n.

- **Sum of first n terms or arithmetic series**

  The sum of first n terms of a finite arithmetic sequence with initial term u_1 is given by $S_n = \frac{n}{2}(u_1 + u_n)$ which is called finite arithmetic series

Harmonic sequence

A sequence is said to be in harmonic progression if the reciprocals of its terms form an arithmetic progression.

Characteristics

If three consecutive terms, h_{n-1}, h_n, h_{n+1} are in arithmetic sequence, then,

$$\frac{2}{h_n} = \frac{h_{n-1} + h_{n+1}}{h_{n-1}\, h_{n+1}} \text{ or}$$

$$\frac{h_n}{2} = \frac{h_{n-1}h_{n+1}}{h_{n-1} + h_{n+1}} \Leftrightarrow h_n = \frac{2h_{n-1}h_{n+1}}{h_{n-1} + h_{n+1}}$$

General term or n[th] term of H.P.

Take the reciprocals of the terms of the given series; these reciprocals will be in A.P.

Find n[th] term of this A.P. using $u_n = u_1 + (n-1)d$ or $u_n = u_p + (n-p)d$

Take the reciprocal of the n[th] term of A.P., to get the required n[th] term of H.P.

Thus, the n[th] term of H.P. is $\dfrac{1}{u_1 + (n-1)d}$ or $\dfrac{1}{u_p + (n-p)d}$

- **Harmonic means**

  If three or more than three numbers are in harmonic sequence, then all terms lying between the first and the last numbers are called harmonic means. If B is harmonic mean between A and C, then . $\dfrac{2}{B} = \dfrac{1}{A} + \dfrac{1}{C}$

  To insert k harmonic means between two terms h_1 and h_n is to form a harmonic sequence of $n = k + 2$ terms whose first term is h_1 and the last term is h_n .

b) Teaching guidelines

- Organise class into groups. Request each group to have a group leader who will present their findings to the class.
- Ask each group to do activity related to the lesson they are going to learn. The learners may need your assistance to do any activity. Help them to understand the activity. After group discussion, invite some or all groups for presentation of their work.
- After activity presentation, capture the main points from the presentation of the learners and summarise them. Guide the learners through given examples in Learner's book or through your own examples.
- Ask learners what they learned in day lesson to ensure that they understood what they have learned.
- Request learners to do exercises in their respective groups. Request learners to correct exercises on chalkboard and give them individual evaluation. Remember to give them homework.

c) Answers to activities

Activity 2.4

Materials

Exercise book, pens, calculator

Answers

1. 3
2. -6

Application Activity 2.4

1. $x=9$
2. No. The number added to the 4^{th} term to obtain the 5^{th} term is not equal to the one used for previous first terms.
3. Common difference is 2
4. $x=3\ or\ 7$, fourth term: 0 or 60

Activity 2.5

Materials

Exercise book, pens

Answers

$u_2 = u_1 + d$

$u_3 = u_2 + d = (u_1 + d) + d = u_1 + d$

$u_4 = u_3 + d = (u_1 + 2d) + d = u_1 + 3d$

$u_5 = u_4 + d = (u_1 + 3d) + d = u_1 + 4d$

$u_6 = u_5 + d = (u_1 + 4d) + d = u_1 + 5d$

$u_7 = u_6 + d = (u_1 + 5d) + d = u_1 + 6d$

$u_8 = u_7 + d = (u_1 + 6d) + d = u_1 + 7d$

$u_9 = u_8 + d = (u_1 + 7d) + d = u_1 + 8d$

$u_{10} = u_9 + d = (u_1 + 8d) + d = u_1 + 9d$

Generally,

$u_n = u_{n-1} + d = (u_1 + (n-2)d) + d = u_1 + (n-1)d$

Application Activity 2.5

1. 3	2. 9	4. 1
5. 336metres	6. None of these answers	

Activity 2.6

Materials

Exercise book, pens, calculator

Answers

$u_1 = 2, u_7 = 20$ $\qquad$ $u_n = u_1 + (n-1)d \Rightarrow u_7 = u_1 + 6d$

$\Rightarrow 20 = 2 + 6d$ $\qquad$ $\Rightarrow d = 3$

The sequence is $2, 5, 8, 11, 14, 17, 20$

Application Activity 2.6

1. $-3,-1,1,3,5,7$	2. $2,5,8,11,14,17,20,23,26,29,32$
3. 16	4. 14
5. 0	

Activity 2.7

Materials

Exercise book, piece of paper (or manila paper), pens

Answers

$u_1 = u_1$

$u_2 = u_1 + d$

$\vdots$

$u_{n-1} = u_1 + (n-2)d$

$u_n = u_1 + (n-1)d$

Let s_n denote the sum of these terms.

We have

$$[u_1 + d]\ s_n = u_1 + [u_1 + d] + \ldots + [u_1 + (n-2)d] + [u_1 + (n-1)d]$$

Reversing the order of the sum, we obtain

$$s_n = [u_1 + (n-1)d] + [u_1 + (n-2)d] + \ldots + [u_1 + d] + u_1$$

Adding the left sides of these two equations and corresponding elements of the right sides,

we see that:

$$\begin{aligned} 2s_n &= [2u_1 + (n-1)d] + [2u_1 + (n-1)d] + \ldots + [2u_1 + (n-1)d] \\ &= n[2u_1 + (n-1)d] \end{aligned}$$

$$\Leftrightarrow s_n = \frac{n}{2}[2u_1 + (n-1)d] = \frac{n}{2}[u_1 + u_1 + (n-1)d]$$

By replacing $u_1+(n-1)d$ with u_n, we obtain a useful formula for the sum:

$$s_n=\frac{n}{2}[u_1+u_n]$$

or $s_n=\frac{n}{2}(u_1+u_1+(n-1)d)\Rightarrow s_n=\frac{n}{2}(2u_1+(n-1)d)$

Application Activity 2.7

1. $2n(n+3)$ 2. 860 3. 11

Activity 2.8

Materials

Exercise book, piece of paper (or manila paper), pens

Answers

$\frac{1}{2},\frac{1}{4},\frac{1}{6},\frac{1}{8},\frac{1}{10},\frac{1}{12},\frac{1}{14},\frac{1}{16}$

The denominators are in arithmetic progression.

Application Activity 2.8

1. The sequence is $6,4,3,\frac{12}{5},2,\frac{12}{7},\frac{3}{2},\frac{4}{3}$. 4^{th} *term is* $\frac{12}{5}$, 8^{th} *term is* $\frac{4}{3}$

2. $3,\frac{90}{23},\frac{90}{16},10$

3. $\frac{\sqrt{5}}{3},\frac{\sqrt{5}}{4},\frac{1}{\sqrt{5}},\ldots,\frac{\sqrt{5}}{13}$

4. 6 and 2

5. $4+4\frac{2}{7}+4\frac{8}{13}+5+\cdots$

6. $\frac{60}{16-n}$

2.4.3. Geometric sequences

a) Content summary

Recommended teaching periods: 7 periods

This section studies the geometric sequences

- Definition

  A Geometric Progression (G.P.) or Geometric Sequence is a sequence in which each term is a fixed multiple of the previous term i.e. $\frac{a_k}{a_{k-1}} = r$, where r is a constant independent of k, for $k = 2,3,\ldots,n$ or $k = 2,3,\ldots,n,\ldots$.

- Characteristics

  If three consecutive terms, u_{n-1}, u_n, u_{n+1} are terms in geometric progression, then, $u_n^2 = u_{n-1}\, u_{n+1}$

- Common ratio

  In G.P., the ratio between any two consecutive terms is a constant r, called **common ratio**

- General term or n^th^ term

  The n^th^ term, u_n, of a geometric sequence $\{u_n\}$ with common ratio r and initial term u_1 is given by $u_n = u_1 r^{n-1}$

  Generally, if u_p is any p^{th} term of a sequence, then the n^th^ term is given by $u_n = u_p r^{n-p}$

Geometric means

If three or more than three numbers are in geometric sequence, then all terms lying between the first and the last numbers are called geometric means.

To insert *k* geometric **means** between two terms u_1 and u_n is to form a geometric sequence of $n = k + 2$ terms whose first term is u_1 and the last term is u_n.

- Sum of first nth terms or geometric series

  The sum of first *n* terms of a finite geometric sequence with initial term u_1 is given by $S_n = \frac{u_1(1-r^n)}{1-r}$, $r<1$ or $S_n = \frac{u_1(r^n-1)}{r-1}$, $r>1$ which is called **finite geometric series**

  If the initial term is u_0, then the formula is $s_n = \frac{u_0(1-r^{n+1})}{1-r}$ with $r \neq 1$

  If $r=1$, $s_n = nu_1$

  Also, the product of first n terms of a geometric sequence with initial term u_1 and common ratio r is given by $P_n = (u_1)^n r^{\frac{n(n-1)}{2}}$

  If the initial term is u_0 then $P_n = (u_0)^{n+1} r^{\frac{n}{2}(n+1)}$

b) Teaching guidelines

Let learners know what arithmetic sequence is. Recall that for an arithmetic sequence, we add a constant number to the term to obtain the next term.

- Organise class into groups. Request each group to have a group leader who will present their findings to the class.
- Ask each group to do activity related to the lesson they are going to learn. The learners may need your assistance to do any activity. Help them to understandthe activity. After group discussion, invite some or all groups for presentation of their work.
- After activity presentation, capture the main points from the presentation of the learners and summarise them. Guide the learners through given examples in Learner's book or through your own examples.
- Ask learners what they learned in day lesson to ensure that they understood what they have learned.

- Request learners to do exercises in their respective groups. Request learners to correct exercises on chalkboard and give them individual evaluation. Remember to give them homework.

c) Answers to activities

Activity 2.9

Materials

Exercise book, piece of paper (or manila paper), pens, calculator, scissors or blades

Answers

Learners will take a piece of paper and cut it into two equal parts. Take one part and cut it again into two equal parts. When they continue in this manner the fraction corresponding to the obtained parts according to the original piece of paper are as follows:

$\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}, \ldots$

Application Activity 2.9

1. $x = -6 \ or \ 6$
2. No, the number multiplied to the fourth term to obtain fifth term is not the same as the one used for previous terms.
3. Common ratio is -2
4. $k = -\frac{1}{2}$

Activity 2.10

Materials

Exercise book, pens

Answers

$u_2 = u_1 r$

$u_3 = u_2 r = u_1 r r = u_1 r^2$

$u_4 = u_3 r = u_1 r^2 r = u_1 r^3$

$u_5 = u_4 r = u_1 r^3 r = u_1 r^4$

$u_6 = u_5 r = u_1 r^4 r = u_1 r^5$

$u_7 = u_6 r = u_1 r^5 r = u_1 r^6$

$u_8 = u_7 r = u_1 r^6 r = u_1 r^7$

$u_9 = u_8 r = u_1 r^7 r = u_1 r^8$

$u_{10} = u_9 r = u_1 r^8 r = u_1 r^9$

Generally,

$u_n = u_{n-1} r = u_1 r^{n-2} r = u_1 r^{n-1}$

Application Activity 2.10

1. 98304
2. $\frac{\sqrt[5]{16}}{4}$
3. -21.87
4. $\frac{1}{16}$
5. $(u_n): u_n = \frac{1}{2}\left(\frac{3}{2}\right)^{n-1}$, $u_8 = \frac{2187}{256}$
6. $p = 5$

Activity 2.11

Materials

Exercise book, pens, calculator

Answers

$u_1 = 1, u_6 = 243$

$u_n = u_1 \cdot r^{n-1} \Rightarrow u_6 = u_1 \cdot r^5$

$\Rightarrow 243 = r^5$

$\Rightarrow 3^5 = r^5$

$\Rightarrow r = 3$

The sequence is $1, 3, 9, 27, 81, 243$

Application Activity 2.11

1. $\frac{1}{4},\frac{1}{8},\frac{1}{16},\frac{1}{32},\frac{1}{64},\frac{1}{128},\frac{1}{256}$ or $\frac{1}{4},-\frac{1}{8},\frac{1}{16},-\frac{1}{32},\frac{1}{64},-\frac{1}{128},\frac{1}{256}$
2. $2,\frac{2}{3},\frac{2}{9},\frac{2}{27},\frac{2}{81},\frac{2}{243},\frac{2}{729}$ or $2,-\frac{2}{3},\frac{2}{9},-\frac{2}{27},\frac{2}{81},-\frac{2}{243},\frac{2}{729}$
3. a) 14 b) $\frac{9}{2}$
4. 12 and 108
5. 64 and 4

Activity 2.12

Materials

Exercise book, pens

Answers

1. Let $s_n = u_1 + u_2 + u_3 + \ldots + u_n$

   $s_n = u_1 + u_1 r + u_1 r^2 + \ldots + u_1 r^{n-1}$ (1)

   Multiply both sides of (1) by r, we obtain

   $s_n r = u_1 r + u_1 r^2 + u_1 r^3 + \ldots + u_1 r^n$ (2)

   Subtract (2) from (1), we get

   $$\begin{array}{rl} s_n = & u_1 + u_1 r + u_1 r^2 + \ldots + u_1 r^{n-1} \\ -s_n r = & \quad -u_1 r - u_1 r^2 - u_1 r^3 - \ldots - u_1 r^n \\ \hline s_n - s_n r = & u_1 - u_1 r^n \end{array}$$

   $\Leftrightarrow s_n(1-r) = u_1(1-r^n)$ or

   $s_n = \frac{u_1(1-r^n)}{1-r}$ with $r \neq 1$

2. $P = u_1 \times u_1 r \times u_1 r^2 \times \ldots \times u_1 r^{n-1}$

   $\Leftrightarrow P = (u_1)^n (r \times r^2 \times \ldots \times r^{n-1})$

   $\Leftrightarrow P = (u_1)^n \, r^{(1+2+\ldots+n-1)}$

We need the sum $S_{n-1} = 1+2+...+n-1$

$$S_{n-1} = \frac{n-1}{2}(1+n-1) = \frac{n(n-1)}{2}$$

Then $P_n = (u_1)^n r^{\frac{n(n-1)}{2}}$

Application Activity 2.12

1) 21.25 2) 39.1 3) $1, \frac{5}{4}$ 4) -32

Activity 2.13

Materials

Exercise book, pens

Answers

If $-1 < r < 1$, $\lim\limits_{n\to\infty} r^n = 0$

thus $\lim\limits_{n\to\infty} \frac{u_1(1-r^n)}{1-r} = \frac{u_1}{1-r}$.

Application Activity 2.13

1. a) $0 < x < \frac{4}{3}$ b) $-\frac{190}{39}$
2. $115m$

2.4.4. Applications of sequences

Recommended teaching periods: 4 periods

This section studies the applications of sequences in daily life.

Teaching guidelines

Let learners now solve any problem related to sequences. Request them to read books and find out how sequences can be used in real life problems and request them to present their findings.

Activity 2.14

Materials

Exercise book, pens

Answers

By reading text books or accessing internet, learners will discuss on how sequences are used in real life.

For example; the interest portion of monthly payments made to pay off an automobile or home loan, and the list of maximum daily temperatures in one area for a month are sequences. One application in economy is calculation of interest rate. The compound interest formula: $A = P\left(1+\frac{r}{k}\right)^{kt}$ with P = principle, t = time in years, r = annual rate, and k = number of periods per year. The simple interest formula:

$I = \mathrm{P}rt$ with I = total interest, P = principle, r = annual rate, and t = time in years.

2.5. Answers for the end of unit assessment

1. a) $0,-\frac{1}{4},-\frac{2}{9},-\frac{3}{16}$ b) $1,-\frac{1}{3},\frac{1}{5},-\frac{1}{7}$ c) $1,3,1,3$
2. a) $(-1)^n$, $n=0,1,2,..$ b) $n^2-1,\ n=1,2,3,...$
   c) $4n-3,\ n=1,2,3,...$
3. a) converges to $\sqrt{2}$ b) converges to 0
   c) converges to 1
4. a) $u_{20}=78, S_{20}=800$ b) $u_{20}=23.5, S_{20}=185$
5. a) $u_n=2(n+1), S_n=n(n+3)$
   b) $u_n=20-3n, S_n=\frac{n}{2}(37-3n)$
   c) $u_n=\frac{1}{n}, S_n=\frac{n+1}{2}$
6. a) $u_8=18, S_8=88$ b) $u_1=3, S_{10}=210$
   c) $n=10, d=2$ d) $u_1=1, d=2$

7. $\frac{157}{4},\frac{79}{2},\frac{159}{4},40,\frac{161}{4},\frac{81}{2},\frac{163}{4},41,\frac{165}{4},\frac{83}{2},\frac{167}{4},42,...$

8. $5,7,9,11,13,15,17,19,21,23,25,27,29,31,...$

9. $-2,-\frac{7}{4},-\frac{6}{4},-\frac{5}{4},-1,-\frac{3}{4},-\frac{2}{4},-\frac{1}{4},0,\frac{1}{4}$

10. 8,9,10

11. 9,11,13 or 13,11,9

12. $\frac{6}{n}$

13. $1,\frac{2}{5},\frac{1}{4}$

14. $\frac{5}{37}$

15. a) $u_5 = 768, S_5 = 1023$ b) $r = -\frac{1}{2}, S_9 = \frac{513}{64}$

16. $\frac{2}{2b} = \frac{1}{b-a} + \frac{1}{b-c} \Leftrightarrow \frac{1}{b} = \frac{2b-a-c}{b^2-bc-ab+ac}$

$\Leftrightarrow b^2 + ac = 2b^2 \Rightarrow b^2 = ac$. This shows that a,b,c form a geometric progression.

17. $u_5 = \frac{81}{2}$

18. $u_8 = \frac{2187}{4}$

19. $2,2\sqrt{2},4,4\sqrt{2},8$ or $2,-2\sqrt{2},4,-4\sqrt{2},8$

20. 3,6,12

21. 128 or -972

22. 6, 6, 6 or 6, -3, -12.

23. 11,17,23

24. 5,8,11,14

25. -4,-1,2,5,8

26. 2,3

27. 6

28. £1074 million

29. $r = \frac{2}{3}$

30. $\frac{4}{5},5$

31. 2048000

32. $99.8^0 F$

33. 1800

34. 6 and 3

35. $ab\sqrt{ab}$

36. $a = b = c$

Unit 3 Logarithmic and Exponential Equations

3.1. Key unit competence

Solve equations involving logarithms or exponentials and apply them to model and solve related problems

3.2. Objectives

- Solve simple exponential equations.
- Convert a number from logarithmic form to exponential form.
- Change the base of any logarithm.
- Use the properties of logarithms to solve logarithmic and exponential equations.
- Apply logarithms or exponential to solve interest rate problems, population growth problems, radioactivity decay problems, earthquake problems,…

3.3. Materials to be used

Exercise books, pens, instruments of geometry, calculator

3.4. Contents to be used

3.4.1. Exponential and logarithmic functions

a) Content summary

Recommended teaching periods: 7 periods

This section looks at how to sketch exponential and logarithmic function in Cartesian plane.

The graph of a logarithmic function is found by reflecting the graph of the corresponding exponential function to the **first bisector** i.e. the line $y = x$.

Then the coordinates of the points for $y = a^x$ are reversed to obtain the coordinates of the points for $g(x) = \log_a x$.

b) Teaching guidelines

Let learners know how to draw linear function in 2-dimensions. Recall that to sketch a function you need a table of points.

- Organise class into groups. Request each group to have a group leader who will present their findings to the class.
- Ask each group to do activity related to the lesson they are going to learn. The learners may need your assistance to do any activity. Help them to understand the activity. After group discussion, invite some or all groups for presentation of their work.
- After activity presentation, capture the main points from the presentation of the learners and summarise them. Guide the learners through given examples in the Learner's Book or through your own examples.
- Ask learners what they learned in day lesson to ensure that they understood what they have learned.
- Request learners to do exercises in their respective groups. Request learners to correct exercises on chalkboard and give them individual evaluation. Remember to give them homework.

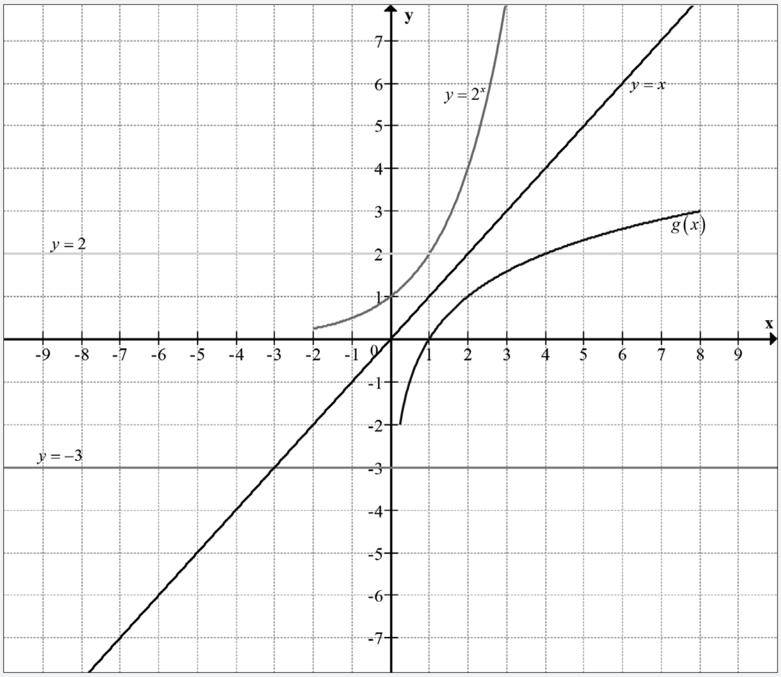
c) Answers to activities

Activity 3.1

Materials

Exercise book, pens, instruments of geometry, calculator

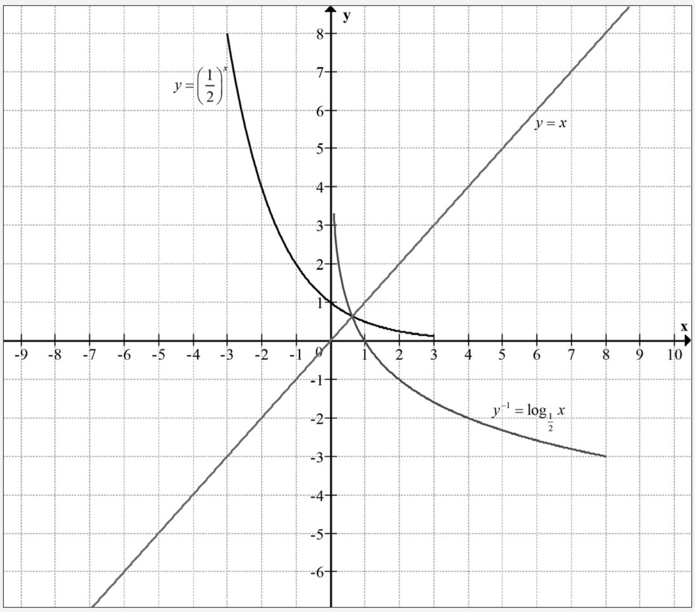
Answers

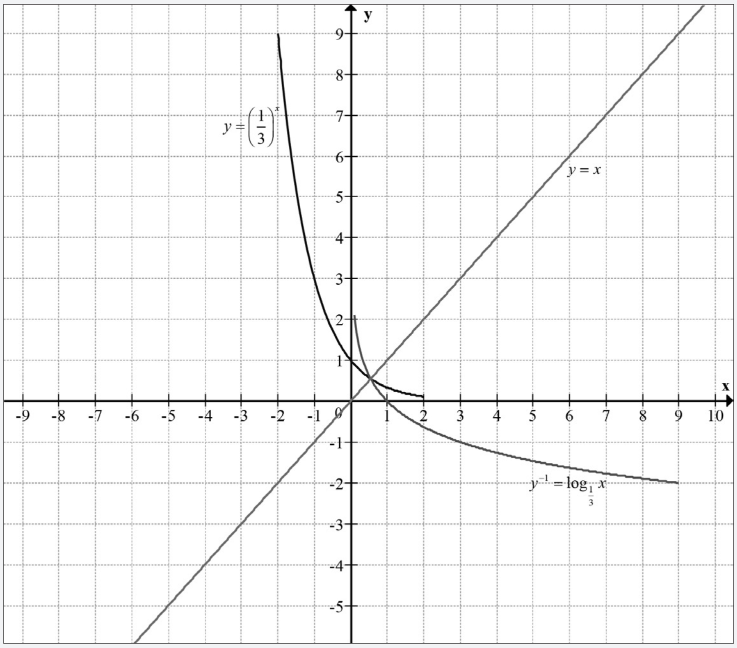


Any horizontal line crosses the curve at most once

$y = 2^x$ is one to one function (is invertible function)

Application Activity 3.1





3.4.2. Exponential and logarithmic equations

a) Content summary

Recommended teaching periods: 7 periods

This section looks at the method used to solve exponential and logarithmic equations.

In solving exponential or logarithmic equations, remember basic rules for exponents and/or logarithms.

Basic rules for exponents

For $a>0$ and $a\neq 1, m,n\in IR$

a) $a^m \times a^n = a^{m+n}$ b) $a^m : a^n = a^{m-n}$

c) $\left(a^m\right)^n = a^{mn}$ d) $a^{-n}=\frac{1}{a^n}$

e) $a^{\frac{1}{n}}=\sqrt[n]{a}$ f) $a^{\frac{m}{n}}=\sqrt[n]{a^m}$

g) $a^{\log_a b}=b$

Basic rules for logarithms

$\forall x, y\in \left]0,+\infty\right[, a\in \left]0,+\infty\right[\setminus \{1\}:$

a) $\log_a xy = \log_a x + \log_a y$ b) $\log_a \frac{1}{y} = -\log_a y$

c) $\log_a \frac{x}{y} = \log_a x - \log_a y$ d) $\log_a x^r = r \log_a x$, $\forall r \in \mathbb{R}$

e) $\log_a b = \frac{\log_c b}{\log_c a}$, $\forall c \in]0,+\infty[\setminus \{1\}, b > 0$

b) Teaching guidelines

Let learners know how to solve linear and quadratic equations. They should also know basic properties of powers. Help them to recall basic properties of powers.

- Organise class into groups. Request each group to have a group leader who will present their findings to the class.
- Ask each group to do activity related to the lesson they are going to learn. The learners may need your assistance to do any activity. Help them to understand the activity. After group discussion, invite some or all groups for presentation of their work.
- After activity presentation, capture the main points from the presentation of the learners and summarise them. Guide the learners through given examples in Learner's book or through your own examples.
- Ask learners what they learned in day lesson to ensure that they understood what they have learned.
- Request learners to do exercises in their respective groups. Request learners to correct exercises on chalkboard and give them individual evaluation. Remember to give them homework.

c) Answers to activities

Activity 3.2

Materials

Exercise book, pens

Answers

If $m = \log_a x, n = \log_a y$ and $z = \log_a xy$ then $x = a^m$, $y = a^n$ and $xy = a^z$.

Now, $xy = a^m a^n = a^{m+n} = a^z \Rightarrow z = m + n$.

Thus, $\log_a (xy) = \log_a x + \log_a y$.

If $m = \log_a x, n = \log_a y$ and $z = \log_a \frac{x}{y}$ then $x = a^m$, $y = a^n$ and $\frac{x}{y} = a^z$.

Now, $\frac{x}{y} = \frac{a^m}{a^n} = a^{m-n} = a^z \Rightarrow z = m - n$.

Thus, $\log_a (xy) = \log_a x - \log_a y$.

Application Activity 3.2

2\. a) $\log_4 64 = 3$ b) $\log_2 \frac{1}{8} = -3$ c) $\log_{\frac{1}{2}} y = x$

d) $\log_p q = 3$ e) $\log_8 0.5 = x$ f) $\log_5 q = p$

3\. a) $\log_2 8 = x \Leftrightarrow 8 = 2^x \Leftrightarrow 2^x = 2^3 \Rightarrow x = 3$

b) $\log_x 125 = 3 \Leftrightarrow 125 = x^3 \Rightarrow x = \sqrt[3]{125} \Leftrightarrow x = 5$

c) $\log_x 64 = 0.5 \Leftrightarrow 64 = x^{0.5} \Rightarrow x = 64^2 \Leftrightarrow x = 4096$

d) $\log_4 64 = x \Leftrightarrow 64 = 4^x \Leftrightarrow 4^x = 4^3 \Rightarrow x = 3$

e) $\log_9 x = 3\frac{1}{2} \Leftrightarrow x = 9^{3\frac{1}{2}} \Leftrightarrow x = \left(9^{\frac{1}{2}}\right)^7 \Rightarrow x = 2187$

f) $\log_2 \frac{1}{2} = x \Leftrightarrow \frac{1}{2} = 2^x \Leftrightarrow 2^x = 2^{-1} \Rightarrow x = -1$

4\. a) 5 b) 1.5 c) -3

d) -3 e) $\frac{1}{3}$ f) 1

g) 1 h) 0

Activity 3.3

Materials

Exercise book, pens

Answers

1. If $x=\log_a m$ and $z=\log_a\left(m^p\right)$ then $m=a^x$, $m^p=a^x$.

   Now, $m^p=\left(a^x\right)^p=a^{px}=a^z \Rightarrow z=px$.

   Thus, $\log_a\left(m^p\right)=p\log_a m$, as required.

2. $\log_a b=\dfrac{\log_c b}{\log_c a}$

   Let $x=\log_a b$, then $a^x=b$.

   Take logarithms in base c of both sides $\log_c a^x=\log_c b$.

   This gives $x\log_c a=\log_c b \Rightarrow x=\dfrac{\log_c b}{\log_c a}$.

   Therefore, $\log_a b=\dfrac{\log_c b}{\log_c a}$.

Application Activity 3.3

1. a) $4p$ b) $-2p$ c) $1+p$

2. $(2k\pi+1)\dfrac{\pi}{2},\ k\in\mathbb{Z}$

3. a) $\left\{\dfrac{3}{2}\right\}$ b) $\left\{\sqrt{2},2\right\}$ c) $\left\{\left(e^2,\dfrac{1}{e^2}\right)\right\}$

   d) $\left\{\left(e^4,e^3\right),\left(-e^4,-e^3\right)\right\}$ e) $\{2\}$

3.4.3. Applications of logarithmic and exponentiol equations

Activity 3.4

Materials

Exercise book, pens, calculator

Answers

$P(t)=P_0 2^{kt}$

Here $P_0=2, k=2, t\,in\,hours \Rightarrow P(t)=2^{2t+1}$

a) $P(4)=2^9=512$

b) $P(t)=2^{13} \Leftrightarrow 2^{2t+1}=2^{13} \Rightarrow 2t+1=13 \Rightarrow t=6$

c) Number of cells left is $\frac{2^{22}}{2}\, or\, 2^{21}$

Activity 3.5

Materials

Exercise book, pens, calculator

Answers

a) The original amount of material present is

$A(0)=80\left(2^0\right)=80gram$

b) For the half life, $A(t)=40$

$$40=80\left(2^{-\frac{t}{100}}\right) \Rightarrow \frac{1}{2}=2^{-\frac{t}{100}} \Rightarrow 2^{-1}=2^{-\frac{t}{100}}$$

$$1=\frac{t}{100} \Rightarrow t=100$$

Therefore the half life is 100 years

c) $A(t)=1$

$$\Rightarrow 80\left(2^{-\frac{t}{100}}\right)=1 \Rightarrow 2^{-\frac{t}{100}}=\frac{1}{80} \Rightarrow 2^{-\frac{t}{100}}=80^{-1}$$

$$\Rightarrow \log 2^{-\frac{t}{100}}=\log 80^{-1} \Rightarrow -\frac{t}{100}\log 2=-\log 80$$

$$\Rightarrow t=\frac{\log 80}{\log 2}\times 100=632$$

Therefore, it will take 632 years for material to decay to 1 gram.

Activity 3.6

Materials

Exercise book, pens

Answers

Suppose $P(t)$ has an exponential decay model so that $P(t)=P_0e^{-kt}$ $(k<0)$.

At any fixed time t_1 let $P_1=P_0e^{-kt_1}$

be the value of $P(t)$ and let T denote the amount of time required to reduce in value by half. Thus, at time t_1+T the value of $P(t)$ will be $2P_1$ so that $2P_1=P_0e^{-k(t_1+T)}=P_0e^{-kt_1}e^{-kT}$.

Since $P_1=P_0e^{-kt_1}$, $2P_1=P_0e^{-k(t_1+T)}=P_0e^{-kt_1}e^{-kT}\xrightarrow{P_1=e^{-kt_1}}2P_0e^{-kt_1}=P_0e^{-kt_1}e^{-kT}$

Or $2=e^{-kT}$, taking $\ln$ on both sides gives

$\ln 2=-kT$ or $T=-\frac{1}{k}\ln 2$ which does not depend on P_0 or t_1.

Application Activity 3.4

1. 12.5647h
2. 160.85 years
3. 866 years
4. a) a little over 95.98 b) about 66.36
5. 4,139g
6. Frw 7,557.84
7. About 14.7 years
8. 2.8147498×10^{14}
9. a) 10 years b) (i) 8 years (ii) 32.02 years.

3.5. Answers for the end of unit assessment

1.	a) $\{81\}$	b) $\{-1,6\}$	c) $\{(9,7),(7,9)\}$	d)
	$\left\{\frac{1}{5},5\right\}$	e) $\{2\}$	f) $\{(\ln 2,\ln 3)\}$	
2.	a) 5	b) 1.5	c) 1.09	
	d) 1.5	e) -3	f) 1.05	
3.	a) $\$2519.42$	b) 9 years		
4.	a) 976	b) 20		
5.	12.9^0C			
6.	a) 49.7million	b) 67.1million	c) 122.4million	
7.	a) 16.0	b) 28.7	c) 33.6	
8.	a) 20.8years	b) 138years		
9.	$P=36.4e^{0.01t}$			
10.	a) 14,400years	b) 38years		
11.	a) 69.1^0C	b) 60.2	c) $44.4\,^0C$	
12.	a) 5.0	b) 16.2	c) 26.4	
13.	c) 96min			
14.	c) After $76/77$ *years*			

Unit 4 Solving Equations by Numerical Methods

4.1. Key unit competence

To be able to use numerical methods (e.g Newton-Raphson method to approximate solution to equations)

4.2. Objectives

- Finding root by linear interpolation and extrapolation.
- Locating roots by graphical and analytical methods.
- Finding real root by Newton-Raphson method and general iterations.

4.3. Materials to be used

Exercise books, pens, instruments of geometry, calculator

4.4. Content and activities

4.4.1. Linear interpolation and extrapolation

a) Content summary

Recommended teaching periods: 4 periods

This section looks at how to use linear interpolation and extrapolation.

Linear interpolation is a process whereby the non tabulated values of the function are estimated on the assumption that the function behaves sufficiently smooth between the tabular point.

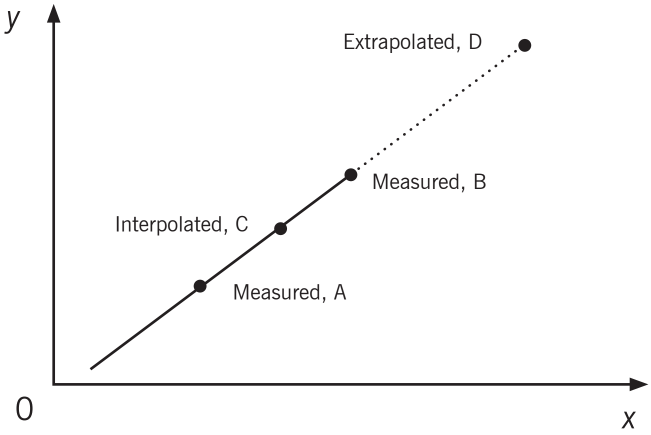
Extrapolation involves approximating the value of a function for a given value outside the given tabulated values.

The linear interpolation formula is given as

$$y = y_1 + \frac{(x - x_1)(y_2 - y_1)}{x_2 - x_1}$$

For extrapolation formula, we may also use the above formula .

An example of a linear interpolation is given in the graph shown below. Here, the line segment AB is given. The point C is interpolated; while the point D is extrapolated by extending the straight line beyond AB.



b) Teaching guidelines

Let learners know how to find equation of a line passing through given two points.

- Organise class into groups. Request each group to have a group leader who will present their findings to the class.
- Ask each group to do activity related to the lesson they are going to learn. The learners may need your assistance to do any activity. Help them to understand the activity. After group discussion, invite some or all groups for presentation of their work.
- After activity presentation, capture the main points from the presentation of the learners and summarise them. Guide the learners through given examples in Learner's book or through your own examples.

- Ask learners what they learned in day lesson to ensure that they understood what they have learned.
- Request learners to do exercises in their respective groups. Request learners to correct exercises on chalkboard and give them individual evaluation. Remember to give them homework.

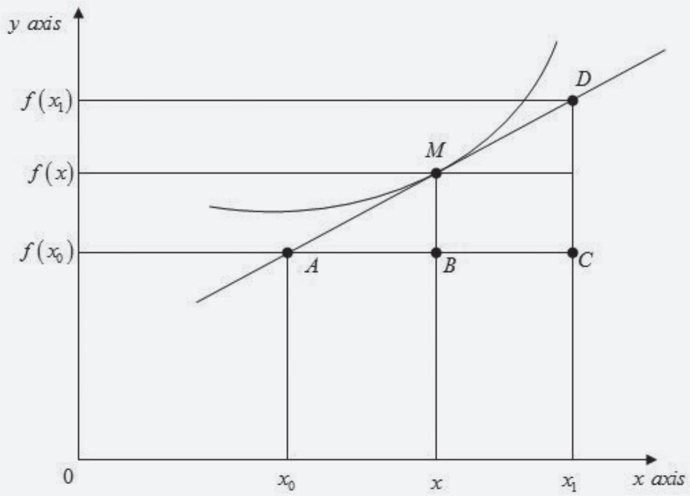
c) Answers to activities

Activity 4.1

Materials

Exercise book, pens, geometric instruments

Answers



3) $\frac{CD}{AD}=\frac{f_2-f_1}{x_2-x_1}$ $\qquad$ $\frac{BM}{AB}=\frac{f-f_1}{x-x_1}$

4) Since $\frac{CD}{AD}=\frac{BM}{AB}$, thus

$$\frac{f_2-f_1}{x_2-x_1}=\frac{f-f_1}{x-x_1}$$

Or $f-f_1=\frac{f_2-f_1}{x_2-x_1}(x-x_1)$

$$f = \frac{f_2 - f_1}{x_2 - x_1}(x - x_1) + f_1 \qquad f = \frac{x - x_1}{x_2 - x_1}(f_2 - f_1) + f_1$$

Letting $\frac{x - x_1}{x_2 - x_1} = \delta$ and $f_2 - f_1 = \Delta f_1$ gives

$f = \delta\Delta f_1 + f_1$

Application Activity 4.1

1. a) $\theta = 64.3$ when T=18s
   b) T=22.5 when 60^0C
2. a) 0.2324 b) 0.967

Activity 4.2

Materials

Exercise book, pens, calculator

Answers

Let $y = ax + b$

$a = \frac{10.5 - 10.9}{1988 - 1982} = \frac{-0.4}{6} = -0.067$

$y = -0.067x + b$

$10.5 = -0.067(1988) + b$

$\Rightarrow b = 143.7$

Then, $y = -0.067x + 143.7$

The winning time in year 2010 is estimated to be:

$y = -0.067(2010) + 143.7 = 9.03 \text{ sec}$

Unfortunately, this estimate actually is not very accurate. This example demonstrates the weakness of linear extrapolation; it uses only a couple of points, instead of using all the points like the best fit line method, so it doesn't give as accurate results when the data points follow a linear pattern.

Application Activity 4.2

1) 11.5 2) 3.33

4.4.2. Location of roots

a) Content summary

Recommended teaching periods: 8 periods

This section looks at the method used to locate root by analytical method and graphical method.

Analytical method

The root of $f(x)=0$ lies in interval $]a,b[$ if $f(a)f(b)<0$; in other words, $f(a)$ and $f(b)$ are of opposite sign.

Graphical method

To solve the equation $f(x)=0$, graphically, we draw the graph of $y=f(x)$ and read from it the value of x for which $f(x)=0$ i.e. the x-coordinates of the points where the curve $y=f(x)$ cuts the x-axis.

Alternatively, we would rearrange $f(x)=0$, in the form $h(x)=g(x)$, and find the x-coordinates of the points where the curves $y=h(x)$ and $y=g(x)$ intersect.

b) Teaching guidelines

- Organise class into groups. Request each group to have a group leader who will present their findings to the class.
- Ask each group to do activity related to the lesson they are going to learn. The learners may need your assistance to do any activity. Help them to understand the activity. After group discussion, invite some or all groups for presentation of their work.
- After activity presentation, capture the main points from the presentation of the learners and summarise them. Guide the learners through given examples in Learner's book or through your own examples.
- Ask learners what they learned in day lesson to ensure that they understood what they have learned.
- Request learners to do exercises in their respective groups. Request learners to correct exercises on chalkboard and give them individual evaluation. Remember to give them homework.

c) Answers to activities

Activity 4.3

Materials

Exercise book, pens, calculator

Answers

- Table of values for $y = x^2 - 5x + 2$

x	0	1	2	3	4	5
$y = x^2 - 5x + 2$	2	-2	-4	-4	-2	2

- The ranges of root of equation $x^2 - 5x + 2 = 0$:

  $f(0) = 2 > 0$ and $f(1) = -2 < 0$, so, a root lies between 0 and 1

  $f(4) = -2 < 0$ and $f(5) = 2 > 0$, so, a root lies between 4 and 5.

  The ranges of root of equation $x^2 - 5x + 2 = 0$ are $0 < x < 1$ and $4 < x < 5$

Application Activity 4.3

1. $f(x) = x^3 - 3x - 12$

   $f(2) = 2^3 - 6 - 12 = -10 < 0$

   $f(3) = 3^3 - 9 - 12 = 6 > 0$

   Since $f(2)f(3) < 0$, thus, the equation $x^3 - 3x - 12 = 0$ has a root between 2 and 3

x	2	?	3
y	-10	0	6

Hint: $y = y_1 + \dfrac{(x - x_1)(y_2 - y_1)}{x_2 - x_1} \Rightarrow x = x_1 + \dfrac{(y - y_1)(x_2 - x_1)}{y_2 - y_1}$

$x = 2.625$

2. $f(x)=3x^2+x-5$

$f(1)=3+1-5=-1<0$

$f(2)=12+1-5=8>0$

As $f(1)f(2)<0$, then, the equation $3x^2+x-5=0$ has a root between 1 and 2.

x	1	?	2
y	-1	0	9

$$x=x_1+\frac{(y-y_1)(x_2-x_1)}{y_2-y_1}$$

$x=1.1$

Activity 4.4

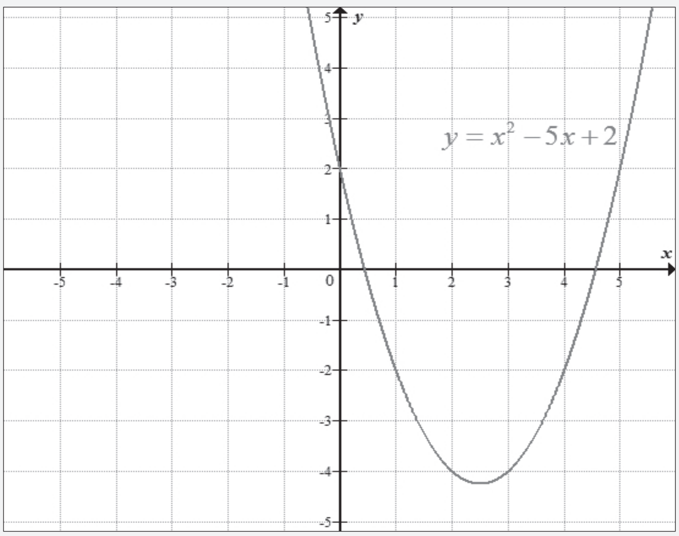
Materials

Exercise book, pens, calculator, geometrical instruments

Answers

1. Graph of $y=x^2-5x+2$

x	0	1	2	3	4	5
$y=x^2-5x+2$	2	-2	-4	-4	-2	2



2. Ranges of roots: $]0,1[$ and $]4,5[$

3. $x^2-5x+2=0 \Leftrightarrow x^2=5x-2$

   $\Rightarrow x=\sqrt{5x-2}$ or $x=-\sqrt{5x-2}$

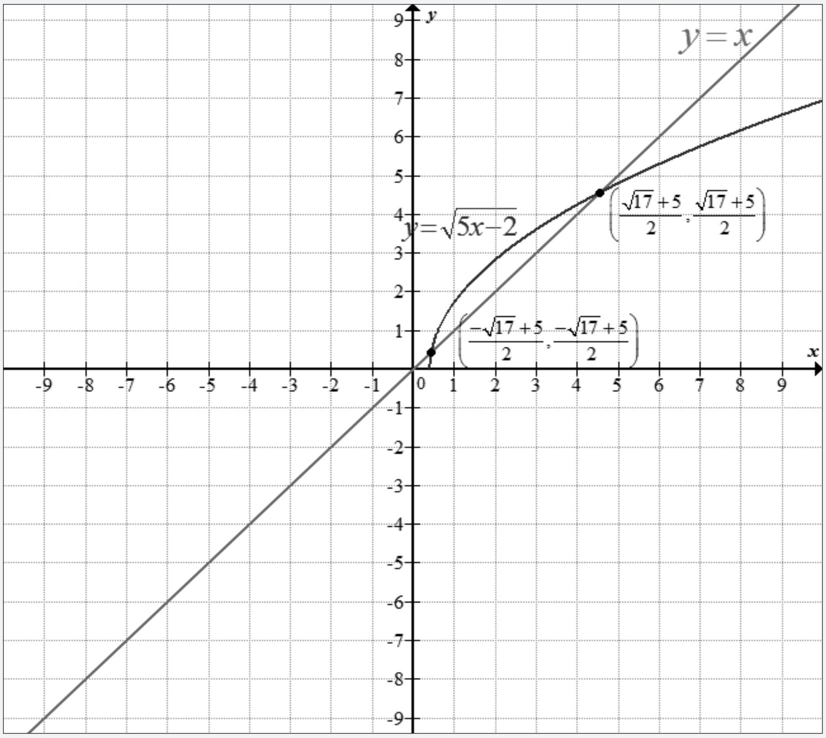
4. Taking the positive root, we get

   $y=x$ and $y=\sqrt{5x-2}$

x	0.4	1	1.5	2	2.5	3	3.5	4	4.5	5
y	0.4	1	1.5	2	2.5	3	3.5	4	4.5	5

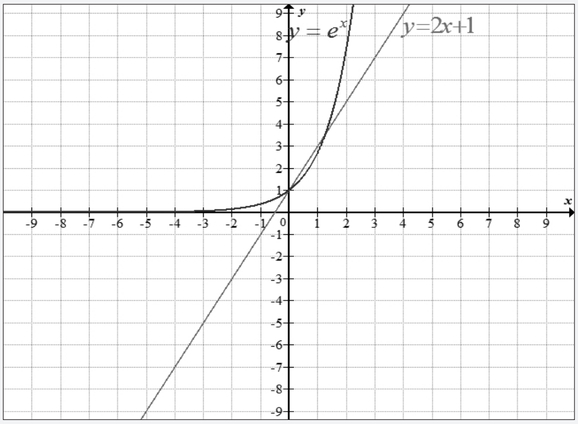
x	0.4	1	1.5	2	2.5	3	3.5	4	4.5	5
$y=\sqrt{5x-2}$	0.0	1.7	2.3	2.8	3.2	3.6	3.9	4.2	4.5	4.8

5. The graphs of $y=x$ and $y=\sqrt{5x-2}$ look like this

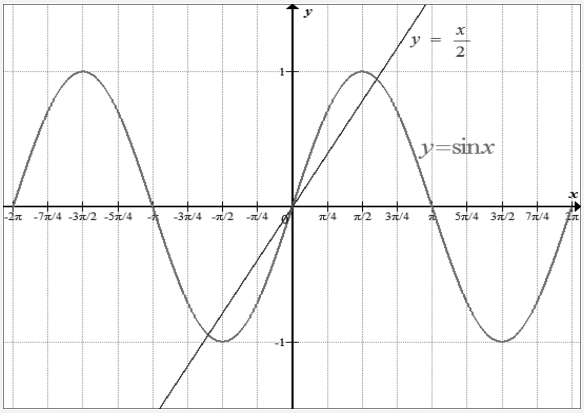


Application Activity 4.4

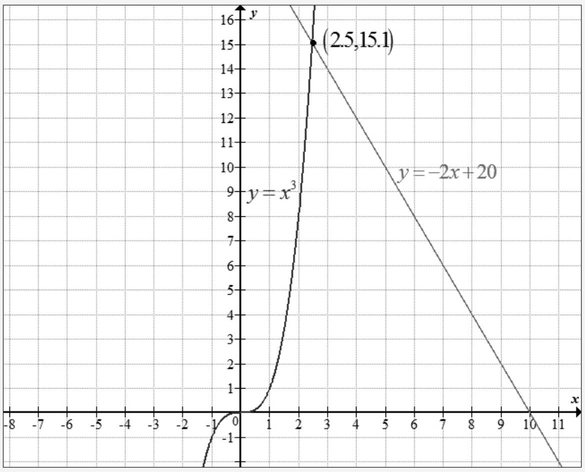
1.



2.

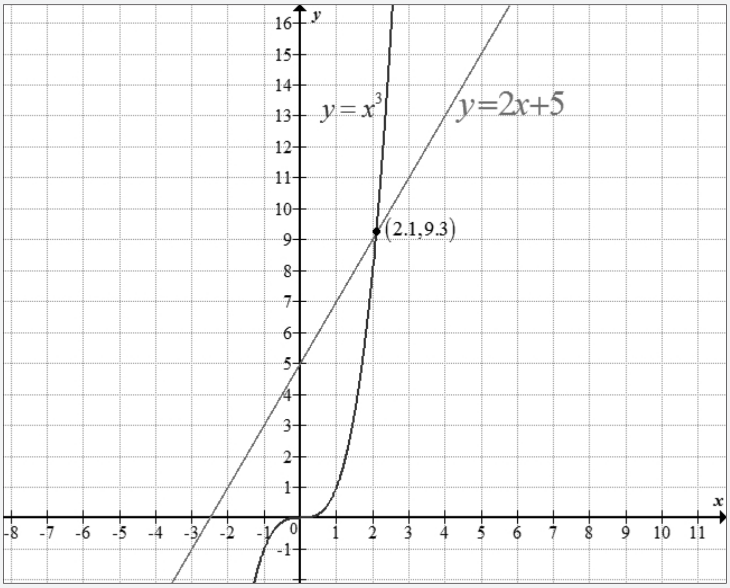


3.



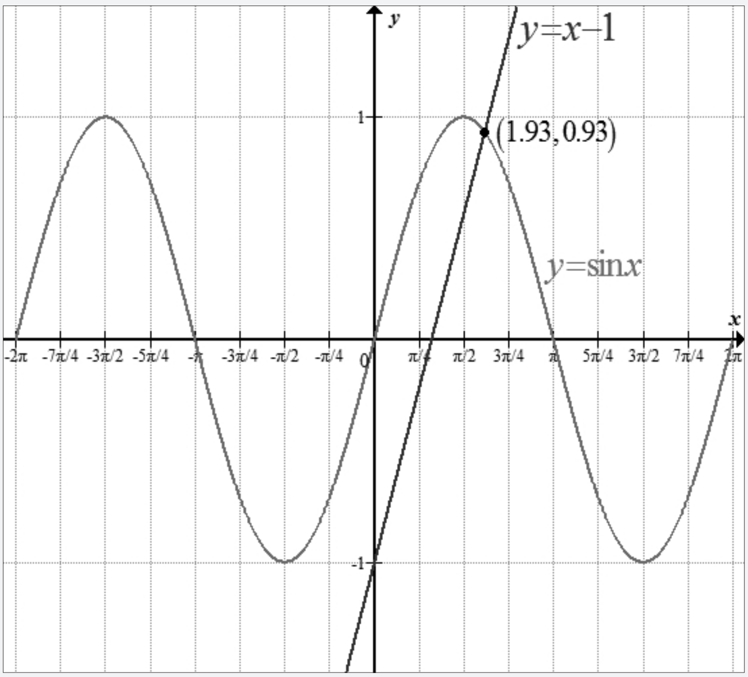
The root is 2.5

4.



The root is 2.1

5.



The root is 1.9

4.4.3. Iteration method

a) Content summary

Recommended teaching periods: 9 periods

This section looks at the method used to find roots by Newton-Raphson method and general iterations.

By this method, we get closer approximation of the root of an equation if we already know its good approximate root.

1. Guess a first approximation to a solution of the equation $f(x)=0$. A graph of $y=f(x)$ may help.
2. Use the first approximation to get a second, the second to get a third, and so on using the formula

$$x_{n+1}=x_n-\frac{f(x_n)}{f'(x_n)}, \text{ if } f'(x_n)\neq 0$$

Notice

If $f'(x_1)=0$ or nearly zero, this method fails.

b) Teaching guidelines

- Organise class into groups. Request each group to have a group leader who will present their findings to the class.
- Ask each group to do activity related to the lesson they are going to learn. The learners may need your assistance to do any activity. Help them to understand the activity. After group discussion, invite some or all groups for presentation of their work.
- After activity presentation, capture the main points from the presentation of the learners and summarise them. Guide the learners through given examples in Learner's book or through your own examples.
- Ask learners what they learned in day lesson to ensure that they understood what they have learned.
- Request learners to do exercises in their respective groups. Request learners to correct exercises on chalkboard and give them individual evaluation. Remember to give them homework.

c) Answers to activities

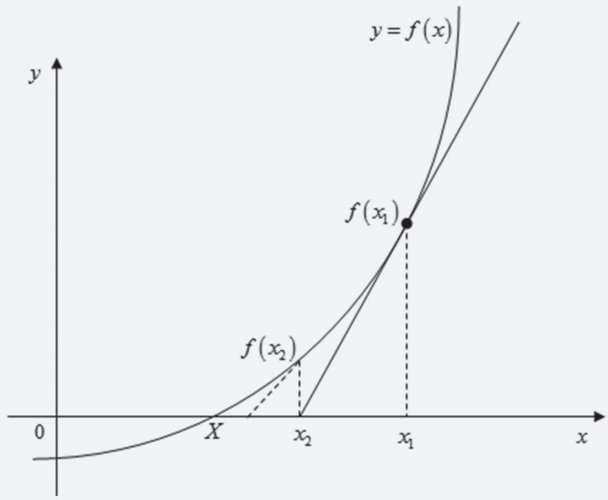
Activity 4.5

Materials

Exercise book, pens and geometric instruments

Answers

1.



Slope of the tangent line: $m = \dfrac{f(x_1) - f(x_2)}{x_1 - x_2} = f'(x)$

Equation of tangent: $f(x) - f(x_1) = f'(x_1)(x - x_1)$

2. At $X = x$, $f(x) = 0$. The tangent equation will be $y = m$

x_2 is a best approximation.

From tangent equation, we get: $-f(x_1) = f'(x_1)(x_2 - x_1)$

$$-\frac{f(x_1)}{f'(x_1)} = x_2 - x_1$$

Or

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

Application Activity 4.5

1. 1.521	2. -2.104	3. 0.79206
4. 1.224	5. 0.581	6. 0.619

Activity 4.6

Materials

Exercise book, pens, calculator

Answers

$x^3 - 3x - 5 = 0 \Leftrightarrow x^3 = 3x + 5$

$\Rightarrow x = \sqrt[3]{3x+5}$, $x_1 = 2$

$x_2 = \sqrt[3]{3\times 2+5} = \sqrt[3]{11} = 2.22398$

$x_3 = \sqrt[3]{3\times 2.22398+5} = 2.268372$

$x_4 = \sqrt[3]{3\times 2.268372+5} = 2.276967$

$x_5 = \sqrt[3]{3\times 2.276967+5} = 2.278624$

$x_6 = \sqrt[3]{3\times 2.278624+5} = 2.278943$

$x_7 = \sqrt[3]{3\times 2.278943+5} = 2.279004$

$x_8 = \sqrt[3]{3\times 2.279004+5} = 2.279016$

$x = 2.279 \ to\, 3dp$

Application Activity 4.6

1. a) $x^2 - 3x + 1 = 0$

$f(x) = x^2 - 3x + 1$

$f(0) = 0^2 - 0 + 1 = 1 > 0$

$f(1) = 1^2 - 3 + 1 = -1 < 0$

As $f(0)f(1) < 0$, then, the equation $x^2 - 3x + 1 = 0$ has a root between 0 and 1.

$f(2) = 2^2 - 6 + 1 = -1 < 0$

$f(3) = 3^2 - 9 + 1 = 1 > 0$

As $f(2)f(3) < 0$, then, the equation $x^2 - 3x + 1 = 0$ has a root between 2 and 3.

b) i) $x^2 - 3x + 1 = 0 \Leftrightarrow 3x = x^2 + 1$

$$\Leftrightarrow x = \frac{x^2+1}{3} \Rightarrow p = 1, q = 3$$

ii) $x^2 - 3x + 1 = 0 \Leftrightarrow x^2 - 3x = -1$

$$\Leftrightarrow x(x-3) = -1$$

$$\Rightarrow x - 3 = -\frac{1}{x}, x \neq 0$$

$$\Leftrightarrow x = 3 - \frac{1}{x} \Rightarrow r = 3, s = -1$$

c) $x_1 = 0.5$, $x_2 = \frac{(0.5)^2+1}{3} = 0.416667$

$$x_3 = \frac{(0.416667)^2+1}{3} = 0.391204$$

$$x_4 = \frac{(0.391204)^2+1}{3} = 0.384347$$

$$x_5 = \frac{(0.384347)^2+1}{3} = 0.382574$$

$$x_6 = \frac{(0.382574)^2+1}{3} = 0.382121$$

$$x_7 = \frac{(0.382121)^2+1}{3} = 0.382005$$

$$x_8 = \frac{(0.382005)^2+1}{3} = 0.381976$$

$$x_9 = \frac{(0.381976)^2 + 1}{3} = 0.381969$$

$$x = 0.382 \ to\, 3\, dp$$

d) $$x = 3 - \frac{1}{x}, \ x_1 = 0.5\,, \ x_2 = 3 - \frac{1}{0.5} = 1$$

$$x_3 = 3 - \frac{1}{1} = 2$$

$$x_4 = 3 - \frac{1}{2} = 2.5$$

$$x_5 = 3 - \frac{10}{26} = 2.615385$$

$$x_6 = 3 - \frac{1}{2.615385} = 2.615385$$

$$x_7 = 3 - \frac{1}{2.615385} = 2.617647$$

$$x_8 = 3 - \frac{1}{2.617647} = 2.617978$$

$$x = 2.618 \ to\, 3\, dp$$

2. $$x_{n+1} = 2 + \frac{1}{x_n^2}, \ x_0 = 2$$

$$x_1 = 2 + \frac{1}{2^2} = 2.25$$

$$x_2 = 2 + \frac{1}{(2.25)^2} = 2.197531$$

$$x_3 = 2 + \frac{1}{(2.197531)^2} = 2.207076$$

$$x_4 = 2 + \frac{1}{(2.207076)^2} = 2.205289$$

$$x_5 = 2 + \frac{1}{(2.205289)^2} = 2.205622$$

$$x_6 = 2 + \frac{1}{(2.205622)^2} = 2.20556$$

$$x_7 = 2 + \frac{1}{(2.20556)^2} = 2.205571$$

$$x_8 = 2 + \frac{1}{(2.205571)^2} = 2.205569$$

$$x_9 = 2 + \frac{1}{(2.205569)^2} = 2.205569$$

$x = 2.205569\, to\, 6\, dp$

Equation is $x^3 - 2x^2 - 1 = 0$

3. $f(x) = x^2 - \sin x$

$f(0.5) = (0.5)^2 - \sin 0.5 = -0.22943 < 0$

$f(1) = 1^2 - \sin 1 = 0.158529 > 0$

As $f(0.5)f(1) < 0$, then, the equation $x^2 - \sin x = 0$ has a root between 0.5 and 1.

$$x_{n+1} = \frac{\sin x}{x_n},\ x_0 = 0.5$$

$$x_1 = \frac{\sin 0.5}{0.5} = 0.958851$$

$$x_2 = \frac{\sin 0.958851}{0.958851} = 0.853659$$

$$x_3 = \frac{\sin 0.853659}{0.853659} = 0.882894$$

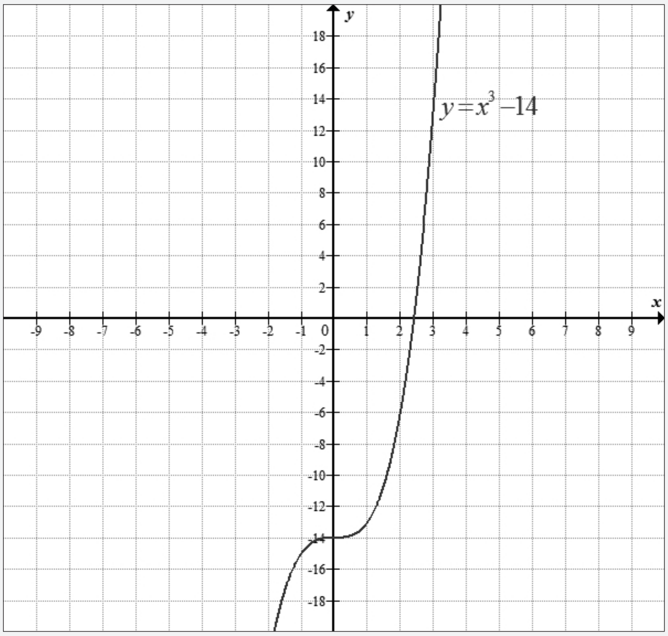
$$x_4 = \frac{\sin 0.882894}{0.882894} = 0.875054$$

$$x_5 = \frac{\sin 0.875054}{0.875054} = 0.877178$$

$x = 0.877\, to\, 3\, dp$

4.5. Answers for the end of unit assessment

1. a)



The root lies between 2 and 3

$$x=\frac{p}{x^2}+\frac{x}{2}$$

$$\Leftrightarrow x=\frac{2p+x^3}{2x^2} \qquad \Leftrightarrow 2x^3=2p+x^3$$

$$\Leftrightarrow x^3=2p$$

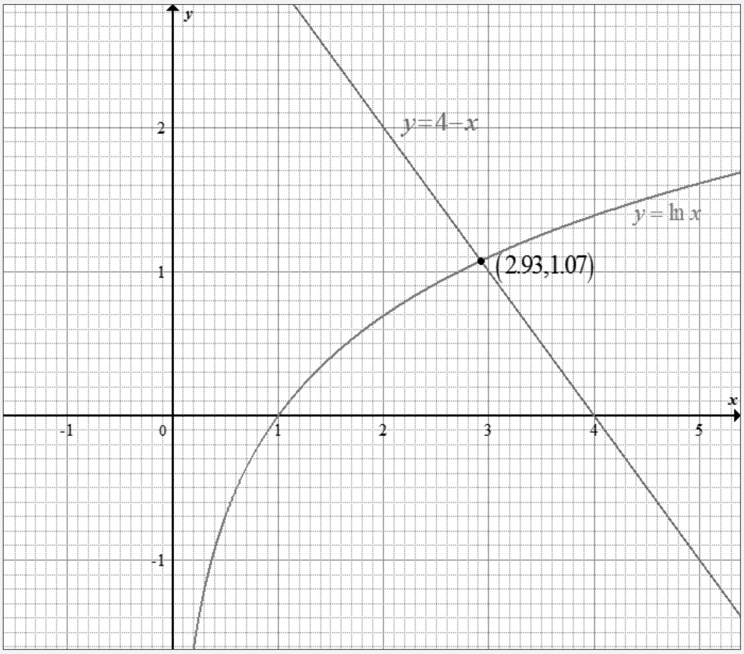
$$2p=14 \Rightarrow p=7$$

b) 2.410

2. 1.8171206

3. a) 1.67 b) 1.16 c) 1.9 d) 0.85

4. Graph



From the graph, the root lies between 2.9 and 3.

Answer to three significant figures is 2.93

5. 1.738

6. a) 1.4973 b) 3.332

7. 1.83, $x = 3^{\frac{1}{x}}$

8. a) 0.45 b) -4 is undefined c) -4.45

Unit 5 Trigonometric Functions and their Inverses

5.1. Key unit competence

Apply theorems of limits and formulas of derivatives to solve problems including trigonometric functions, optimisation, and motion.

5.2. Objectives

After completing this unit, the learners should be able to:

- Find the domain and range of trigonometric function and their inverses.
- Study the parity of trigonometric functions.
- Study the periodicity of trigonometric functions.
- Evaluate limits of trigonometric functions.
- Differentiate trigonometric functions.

5.3. Materials to be used

Exercise books, pens, instruments of geometry, calculator

5.4. Content and activities

5.4.1. Generalities on trigonometric functions and their inverses

a) Content summary

Recommended teaching periods: 14 periods

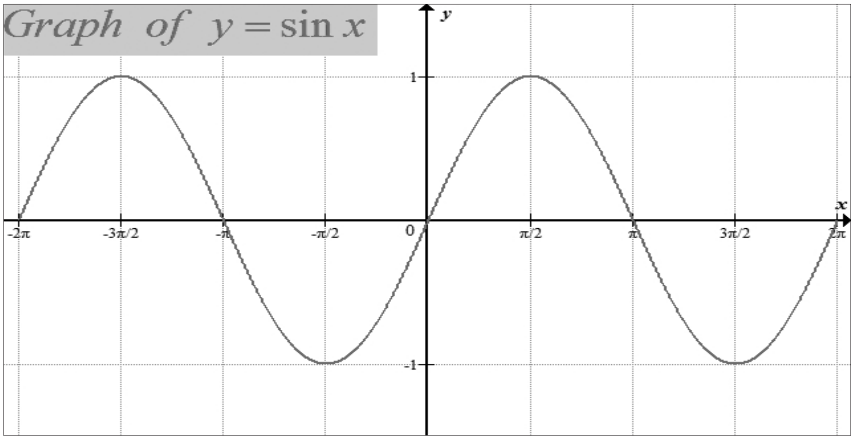
This section looks at the **domain and range of trigonometric functions** and **their inverses**. It also looks at the parity and periodicity of trigonometric functions.

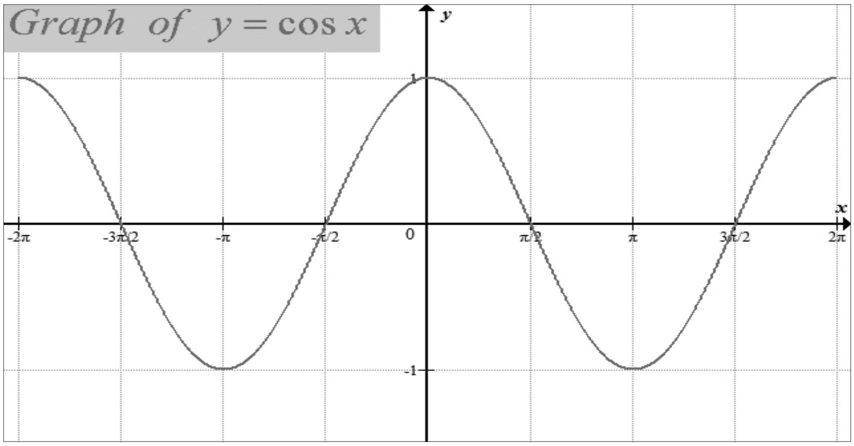
Domain and range of trigonometric functions

Cosine and sine

The domain of $\sin x$ and $\cos x$ is the set of real numbers.

The range of $\sin x$ and $\cos x$ is $[-1,1]$.

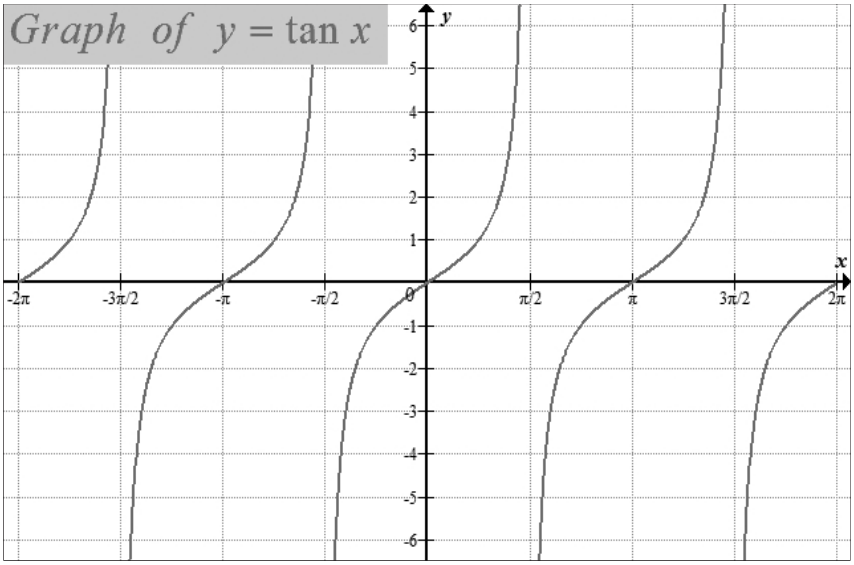


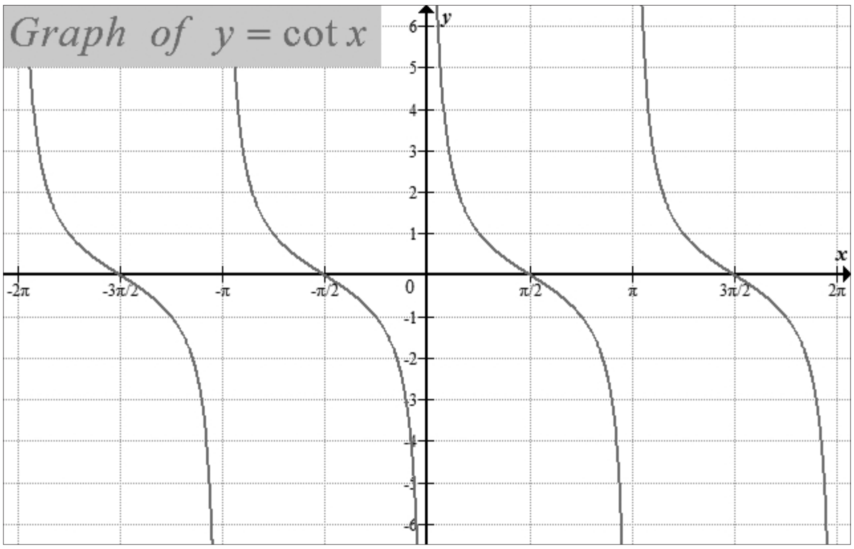


Tangent and cotangent

The domain of $\tan x$ is $\mathbb{R}\setminus\left\{\frac{\pi}{2}+k\pi\right\}, k\in\mathbb{Z}$. The range of $\tan x$ is the set of real numbers.

The domain of $\cot x$ is $\mathbb{R}\setminus\{k\pi\}, k\in\mathbb{Z}$. The range of $\cot x$ is the set of real numbers.

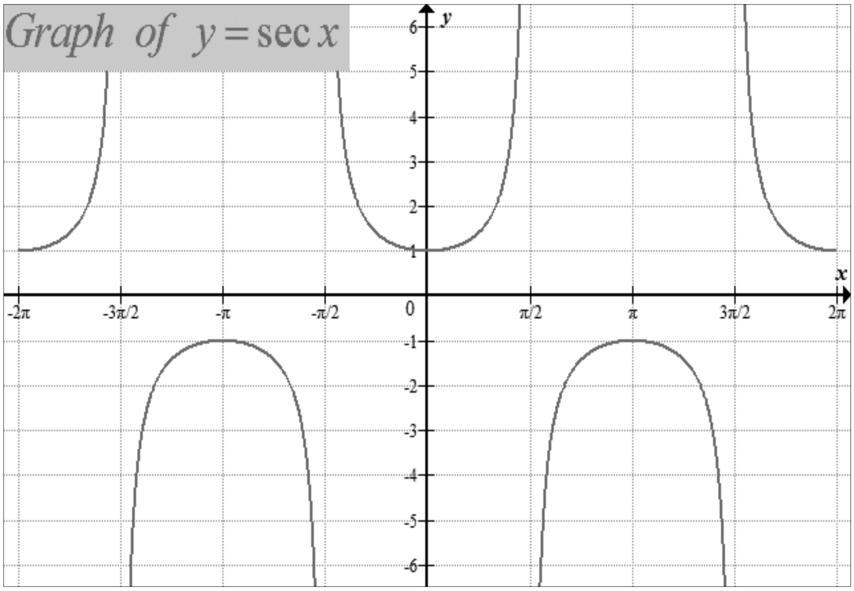


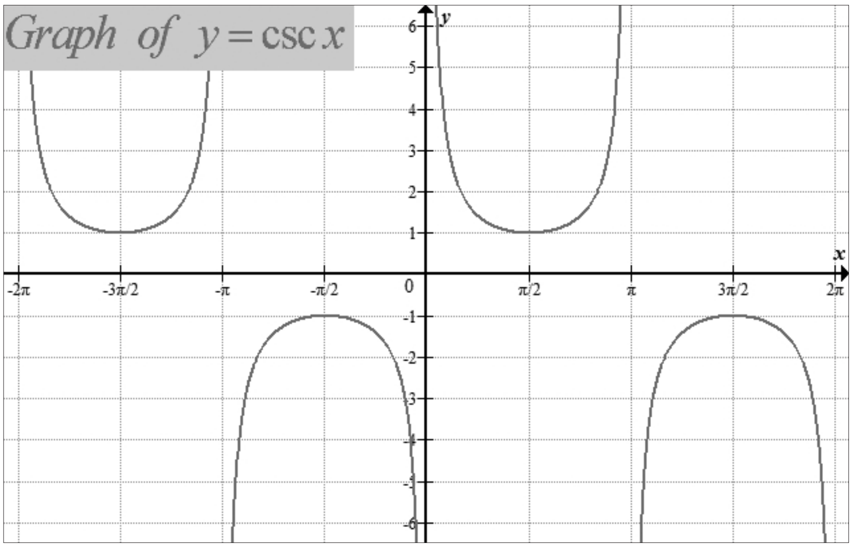


Secant and cosecant

The domain of $\sec x$ is $\mathbb{R} \setminus \left\{ \frac{\pi}{2} + k\pi \right\}, k \in \mathbb{Z}$. Since $\sec x = \frac{1}{\cos x}$ and range of cosine is $[-1,1]$, $\frac{1}{\cos x}$ will vary from negative infinity to -1 or from 1 to plus infinity. Thus, the range of $\sec x$ is $]-\infty, -1] \cup [1, +\infty[$

The domain of $\csc x$ is $\mathbb{R}\backslash\{k\pi\}, k\in\mathbb{Z}$. Since $\csc x = \frac{1}{\sin x}$ and range of sine is $[-1,1]$, $\frac{1}{\sin x}$ will vary from negative infinity to -1 or from 1 to plus infinity. Thus, the range of $\csc x$ is $]-\infty,-1]\cup[1,+\infty[$



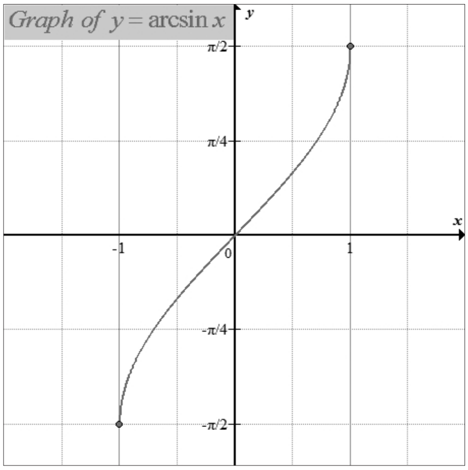


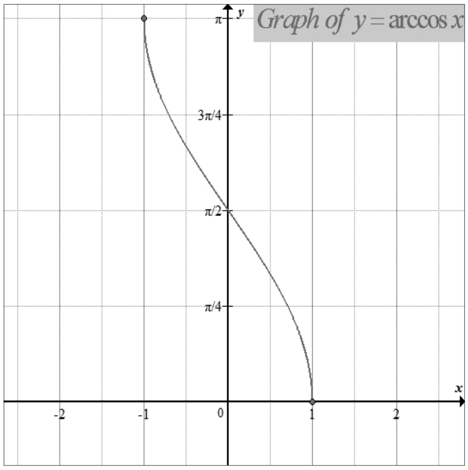
Inverse trigonometric functions

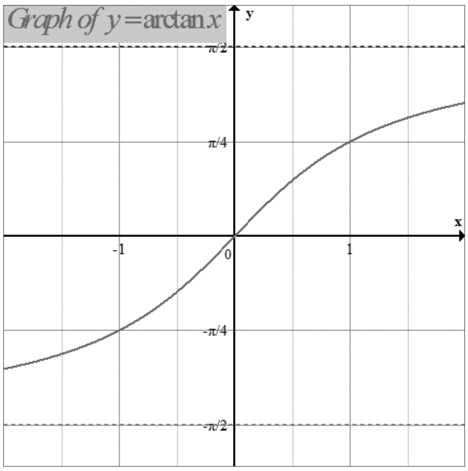
$\sin x$ and $\cos x$ have the inverses called inverse sine and inverse cosine denoted by $\sin^{-1} x$ and $\cos^{-1} x$ respectively.

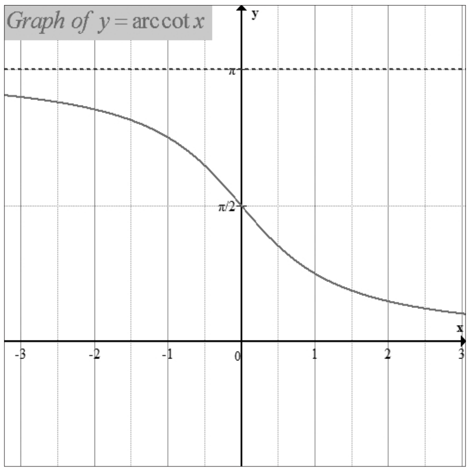
Note that the symbols $\sin^{-1} x$ and $\cos^{-1} x$ are never used to denote $\frac{1}{\sin x}$ and $\frac{1}{\cos x}$ respectively. If desired, $\frac{1}{\sin x}$ and $\frac{1}{\cos x}$ can be written as $(\sin x)^{-1}$ and $(\cos x)^{-1}$ (or $\csc x$ and $\sec x$) respectively.

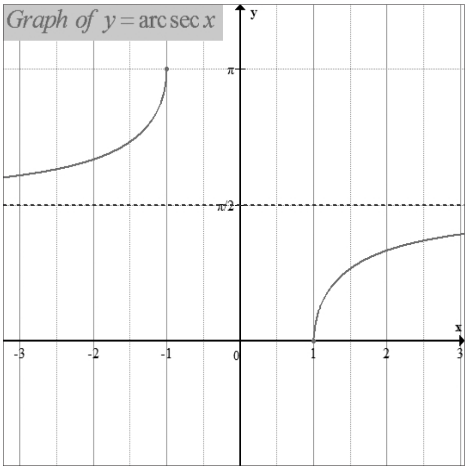
$\sin^{-1} x$ and $\cos^{-1} x$ are also called arcsine of x and arccosine of x and they are denoted by $arc\sin x$ and $arc\cos x$ respectively.

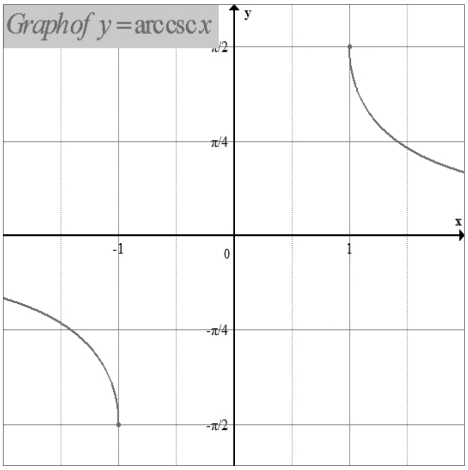












Domain restrictions that make the trigonometric functions one to one

Function	Domain restriction	Range
Sine	$[-\frac{\pi}{2}, \frac{\pi}{2}]$	[-1,1]
Cosine	$[0,\pi]$	[-1,1]
Tangent	$\left]-\frac{\pi}{2}, \frac{\pi}{2}\right[$	$\mathbb{R}$
Cotangent	$]0,\pi[$	$\mathbb{R}$
Secant	$\left[0, \frac{\pi}{2}\right[\cup \left]\frac{\pi}{2}, \pi\right]$	$\left]-\infty, -1\right] \cup \left[1, +\infty\right[$
Cosecant	$\left[-\frac{\pi}{2}, 0\right[\cup \left]0, \frac{\pi}{2}\right]$	$\left]-\infty, -1\right] \cup \left[1, +\infty\right[$

Because $\sin x$ (restricted) and $\sin^{-1} x$; $\cos x$ (restricted) and $\cos^{-1} x$ are inverses to each other, it follows that:

- $\sin^{-1}(\sin y) = y$ if $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$; $\sin(\sin^{-1} x) = x$ if $-1 \le x \le 1$
- $\cos^{-1}(\cos y) = y$ if $0 \le y \le \pi$; $\cos(\cos^{-1} x) = x$ if $-1 \le x \le 1$

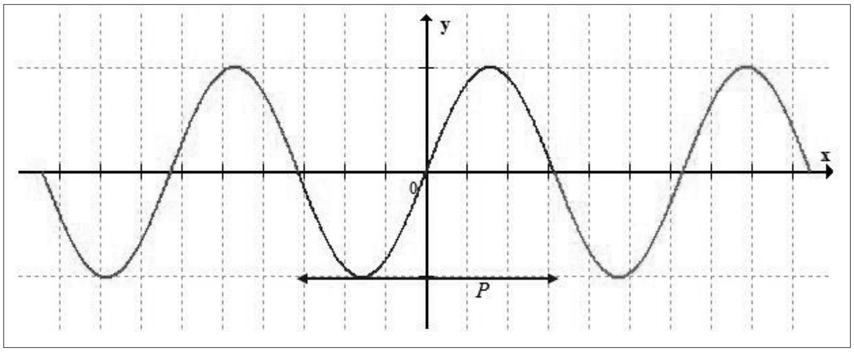
From these relations, we obtain the following important result:

Theorem 1

- If $-1 \le x \le 1$ and $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$, then $y = \sin^{-1} x$ and $\sin y = x$ are equivalent.
- If $-1 \le x \le 1$ and $0 \le y \le \pi$, then $y = \cos^{-1} x$ and $\cos y = x$ are equivalent.

Periodic functions

A function f is called periodic if there is a positive number P such that $f(x+P) = f(x)$ whenever x and $x+P$ lie in the domain of f.



Any function which is not periodic is called **aperiodic.**

The period of sum, difference or product of trigonometric function is given by the ***Lowest Common Multiple (LCM)*** of the periods of each term or factor.

b) Teaching guidelines

Let learners know how to find domain of definition of a polynomial, rational and irrational functions. Recall that the domain of definition of a function is the set of elements where the function is defined.

- Organise class into groups. Request each group to have a group leader who will present their findings to the class.
- Ask each group to do activity related to the lesson they are going to learn. The learners may need your assistance to do any activity. Help them to understand the activity. After group discussion, invite some or all groups for presentation of their work.
- After activity presentation, capture the main points from the presentation of the learners and summarise them. Guide the learners through given examples in Learner's book or through your own examples.

- Ask learners what they learned in day lesson to ensure that they understood what they have learned.
- Request learners to do exercises in their respective groups. Request learners to correct exercises on chalkboard and give them individual evaluation. Remember to give them homework.

c) Answers to activities

Activity 5.1

Materials

Exercise book, pens, calculator

Answers

1. No real number for x
2. No real number for x
3. $x=\frac{\pi}{2}+k\pi, k\in\mathbb{Z}$
4. $x=k\pi, k\in\mathbb{Z}$
5. $x=\frac{\pi}{2}+k\pi,\ k\in\mathbb{Z}$
6. $x=k\pi,\ k\in\mathbb{Z}$

Application Activity 5.1

1. $\mathbb{R}$
2. $\mathbb{R}\setminus\{0\}$
3. $\mathbb{R}\setminus\{0\}$
4. $\mathbb{R}\setminus\{0\}$
5. $\mathbb{R}\setminus\left\{\frac{\pi}{2}+k\pi\right\}, k\in\mathbb{Z}$
6. $]0,+\infty[$

Activity 5.2

Materials

Exercise book, pens, calculator

Answers

1. $]-\infty,-1[\cup]1,+\infty[$	2. $]-\infty,-1[\cup]1,+\infty[$
3. No real number for x	4. No real number for x
5. $]-1,1[$	6. $]-1,1[$

Application Activity 5.2

1. $\left[-\frac{1}{2},0\right[\cup\left]0,\frac{1}{2}\right]$	2. $[-1,1]$
3. $]0,1]$	4. $]-\infty,-1]\cup[1,+\infty[$

Activity 5.3

Materials

Exercise book, pens, calculator

Answers

1. $f(-x)=\frac{\sin x}{x},\ -f(x)=-\frac{\sin x}{x}, f(-x)\neq -f(x), f(-x)=f(x)$
2. $g(-x)=-\frac{\cos x}{x},\ -g(x)=-\frac{\cos x}{x}, g(-x)=-g(x), g(-x)\neq g(x)$
3. $h(-x)=-\sin x+\cos x,\ -h(x)=-\sin x-\cos x, h(-x)\neq -h(x), h(-x)\neq h(x)$

Application Activity 5.3

1. Even 2. Odd 3. Odd
4. Neither even nor odd, 1 is in domain but -1 is not in domain.

Activity 5.4

Materials

Exercise book, pens, calculator

Answers

1. $2k\pi,\ k\in\mathbb{Z}$ 2. $2k\pi,\ k\in\mathbb{Z}$ 3. $k\pi,\ k\in\mathbb{Z}$

Application Activity 5.4

1. π	2. 3π	3. $\frac{\pi}{3}$
4. 2π	5. $\frac{2\pi}{\lvert w\rvert}; w \neq 0$	6. $\frac{\pi}{2}$

Activity 5.5

Materials

Exercise book, pens, calculator

Answers

1. 2π	2. π	3. 2π

Application Activity 5.5

1. π	2. 2π
3. 2π	4. $\frac{2\sqrt{3}\pi}{3}$

5.4.2. Limits of trigonometric functions

a) Content summary

Recommended teaching periods: 9 periods

This section looks at the method used to find the limits of trigonometric functions

$$\lim_{x \to 0} \frac{\sin x}{x} = 1$$

b) Teaching guidelines

- Organise class into groups. Request each group to have a group leader who will present their findings to the class.
- Ask each group to do activity related to the lesson they are going to learn. The learners may need your assistance to do any activity. Help them to understand the activity. After group discussion, invite some or all groups for presentation of their work.

- After activity presentation, capture the main points from the presentation of the learners and summarise them. Guide the learners through given examples in Learner's book or through your own examples.
- Ask learners what they learned in day lesson to ensure that they understood what they have learned.
- Request learners to do exercises in their respective groups. Request learners to correct exercises on chalkboard and give them individual evaluation. Remember to give them homework.

c) Answers to activities

Activity 5.6

Materials

Exercise book, pens, calculator

Answers

1. a) $\lim_{x\to 0}\sin x = \sin 0 = 0$ b) $\lim_{x\to 0} x\sin x = 0\sin 0 = 0$

   c) $\lim_{x\to 0}\cos x = \cos 0 = 1$ d) $\lim_{x\to 0}\frac{1}{x} = \infty$

   e) $\lim_{x\to 0}\frac{\cos x}{x} = \lim_{x\to 0}\frac{1}{x}\lim_{x\to 0}\cos x = \infty \times 1 = \infty$

2.

x	$\frac{\sin x}{x}$
1	0.841470985
0.9	0.870363233
0.8	0.896695114
0.7	0.920310982
0.6	0.941070789
0.5	0.958851077
0.4	0.973545856
0.3	0.985067356
0.2	0.993346654

x	$\frac{\sin x}{x}$
-1	0.841470985
-0.9	0.870363233
-0.8	0.896695114
-0.7	0.920310982
-0.6	0.941070789
-0.5	0.958851077
-0.4	0.973545856
-0.3	0.985067356
-0.2	0.993346654

0.1	0.998334166
0.01	0.999983333
0.001	0.999999833
0.0001	0.999999998

-0.1	0.998334166
-0.01	0.999983333
-0.001	0.999999833
-0.0001	0.999999998

a) $\lim\limits_{x\to 0^+}\frac{\sin x}{x}=1$ b) $\lim\limits_{x\to 0^-}\frac{\sin x}{x}=1$

c) Since both limits on each side are equal to 1 then

$\lim\limits_{x\to 0}\frac{\sin x}{x}=1$

Application Activity 5.6

1. $\frac{\pi}{4}$ 2. 2 3. 0 4. 3

Activity 5.7

Materials

Exercise book, pens, calculator

Answers

1. a) Let $y=\sin^{-1}(-1)$. This equation is equivalent to $\sin y=-1, -\frac{\pi}{2}\le y\le\frac{\pi}{2}$. The only value of y satisfying these conditions is $y=-\frac{\pi}{2}$. So $\sin^{-1}(-1)=-\frac{\pi}{2}$

b) Let $y=\tan^{-1}1$. This is equivalent to $\tan y=1, -\frac{\pi}{2}<y<\frac{\pi}{2}$. The only value of y satisfying these conditions is $y=\frac{\pi}{4}$. So $\tan^{-1}(1)=\frac{\pi}{4}$

c) Let $y=\cot^{-1}(-1)$. This is equivalent to $\cot y=-1, 0<y\le\pi$. The only value of y satisfying these conditions is $y=\frac{3\pi}{4}$. So $\cot^{-1}(-1)=\frac{3\pi}{4}$

d) Let $y=\sec^{-1}(-2)$. This is equivalent to

$\sec y=-2,\ 0\le y\le\pi,\ y\ne\frac{\pi}{2}$. The only value of y satisfying these conditions is $y=\frac{2\pi}{3}$. So $\sec^{-1}(-2)=\frac{2\pi}{3}$

e) Let $y=\csc^{-1}(-2)$. This is equivalent to

$\csc y=-2,\ -\frac{\pi}{2}\le y\le\frac{\pi}{2}, y\ne 0$. The only value of y satisfying these conditions is $y=-\frac{\pi}{6}$. So $\csc^{-1}(-2)=-\frac{\pi}{6}$

2. a) $\lim_{x\to 1}\cot^{-1}\left(\frac{2x-3}{x}\right)=\cot^{-1}\left(\frac{2-3}{1}\right)=\cot^{-1}(-1)=\frac{3\pi}{4}$

b) $\lim_{x\to 1}\sin^{-1}\left(\frac{1+x}{2x}\right)=\sin^{-1}\left(\frac{1+1}{2}\right)=\sin^{-1}(1)=\frac{\pi}{2}$

c) $\lim_{x\to 0}\cos^{-1}\left(\frac{\sqrt{x+1}-1}{x}\right)=\cos^{-1}\left(\frac{0}{0}\right)$ I.C.

Remove this I.C

$\lim_{x\to 0}\frac{\sqrt{x+1}-1}{x}=\lim_{x\to 0}\frac{1}{2\sqrt{x+1}}=\frac{1}{2}$ and $\cos^{-1}\frac{1}{2}=\frac{\pi}{3}$.

Thus, $\lim_{x\to 0}\cos^{-1}\left(\frac{\sqrt{x+1}-1}{x}\right)=\frac{\pi}{3}$

d) $\lim_{x\to -1}\tan^{-1}\left(\frac{1-x^2}{2x+2}\right)=\tan^{-1}\frac{0}{0}$ I.C. Remove this I.C

$\lim_{x\to -1}\left(\frac{1-x^2}{2x+2}\right)=\lim_{x\to -1}\frac{-2x}{2}=1$ and $\tan^{-1}(1)=\frac{\pi}{4}$.

Thus, $\lim_{x\to -1}\tan^{-1}\left(\frac{1-x^2}{2x+2}\right)=\frac{\pi}{4}$

Application Activity 5.7

1) $\frac{\pi}{4}$ 2) $-\frac{\pi}{2}$ 3) 2

5.4.3. Differentiation of trigonometric functions and their inverses

a) Content summary

Recommended teaching periods: 14 periods

This section looks the derivative of trigonometric functions and their inverses.

Derivative of trigonometric functions

1. $\frac{d(\sin u)}{dx}=\cos u\frac{du}{dx}$
2. $\frac{d(\cos u)}{dx}=-\sin u\frac{du}{dx}$
3. $\frac{d(\tan u)}{dx}=\sec^2 u\frac{du}{dx}$
4. $\frac{d(\cot u)}{dx}=-\csc^2 u\frac{du}{dx}$
5. $\frac{d(\sec u)}{dx}=\sec u\tan u\frac{du}{dx}$
6. $\frac{d(\csc u)}{dx}=-\csc u\cot u\frac{du}{dx}$

Derivative of inverse trigonometric functions

1. $\frac{d(\arcsin u)}{dx}=\frac{1}{\sqrt{1-u^2}}\frac{du}{dx}$
2. $\frac{d(\arccos u)}{dx}=-\frac{1}{\sqrt{1-u^2}}\frac{du}{dx}$
3. $\frac{d(\arctan u)}{dx}=\frac{1}{1+u^2}\frac{du}{dx}$
4. $\frac{d(\operatorname{arccot} u)}{dx}=-\frac{1}{1+u^2}\frac{du}{dx}$
5. $\frac{d(\operatorname{arcsec} u)}{dx}=\frac{1}{|u|\sqrt{u^2-1}}\frac{du}{dx}$
6. $\frac{d(\operatorname{arccsc} u)}{dx}=-\frac{1}{|u|\sqrt{u^2-1}}\frac{du}{dx}$

b) Teaching guidelines

Let learners know trigonometric identities. The trigonometric identities will be used in this section

- Organise class into groups. Request each group to have a group leader who will present their findings to the class.
- Ask each group to do activity related to the lesson they are going to learn. The learners may need your assistance to do any activity. Help them to understand the activity. After group discussion, invite some or all groups for presentation of their work.
- After activity presentation, capture the main points from the presentation of the learners and summarise them. Guide the learners through given examples in Learner's book or through your own examples.

- Ask learners what they learned in day lesson to ensure that they understood what they have learned.
- Request learners to do exercises in their respective groups. Request learners to correct exercises on chalkboard and give them individual evaluation. Remember to give them homework.

c) Answers to activities

Activity 5.8

Materials

Exercise book, pens

Answers

1. $\forall x_0 \in \mathbb{R}$

$$(\sin x_0)' = \lim_{x \to x_0} \frac{\sin x - \sin x_0}{x - x_0}$$

$$= \lim_{x \to x_0} \frac{2\cos\frac{x+x_0}{2}\sin\frac{x-x_0}{2}}{x-x_0}$$

$$= \lim_{x \to x_0} \cos\frac{x+x_0}{2} \lim_{x \to x_0} \frac{2\sin\frac{x-x_0}{2}}{x-x_0}$$

$$= \lim_{x \to x_0} \cos\frac{x+x_0}{2} \lim_{x \to x_0} \frac{2\sin\frac{x-x_0}{2}}{2\frac{x-x_0}{2}}$$

$$= \lim_{x \to x_0} \cos\frac{x+x_0}{2} \lim_{x \to x_0} \frac{\sin\frac{x-x_0}{2}}{\frac{x-x_0}{2}}$$

$$= \left(\cos\frac{x_0+x_0}{2}\right) \times 1$$

$$= \cos x_0$$

Thus, $\forall x \in \mathbb{R}, \ (\sin x)' = \cos x$

2. $\cos x = \sin\left(\frac{\pi}{2} - x\right)$

$$(\cos x)' = \left[\sin\left(\frac{\pi}{2} - x\right)\right]'$$

$$= \left(\frac{\pi}{2} - x\right)' \cos\left(\frac{\pi}{2} - x\right)$$

$$= -\cos\left(\frac{\pi}{2} - x\right)$$

$$= -\sin x$$

Thus, $\forall x \in \mathbb{R},\ (\cos x)' = -\sin x$

Application Activity 5.8

1. $2x\cos(x^2+3)$
2. $6x\cos\left(x^2+4\right)\sin^2\left(x^2+4\right)$
3. $-6x\sin 3x^2$
4. $-6\cos^2 2x\sin 2x$

Activity 5.9

Materials

Exercise book, pens

Answers

1. $\tan x = \frac{\sin x}{\cos x}$

$$(\tan x)' = \left(\frac{\sin x}{\cos x}\right)'$$

$$= \frac{(\sin x)'\cos x - \sin x(\cos x)'}{(\cos x)^2}$$

$$= \frac{\cos x\cos x + \sin x\sin x}{(\cos x)^2}$$

$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x}$$

$$= \frac{1}{\cos^2 x}$$

$$\left(\tan x\right)' = \frac{1}{\cos^2 x}$$
$$= \sec^2 x$$
$$= 1 + \tan^2 x$$

2. $$\cot x = \tan\left(\frac{\pi}{2} - x\right)$$
$$\left(\cot x\right)' = \left[\tan\left(\frac{\pi}{2} - x\right)\right]'$$
$$= \frac{\left(\frac{\pi}{2} - x\right)'}{\cos^2\left(\frac{\pi}{2} - x\right)}$$
$$= \frac{-1}{\sin^2 x}$$

$$\left(\cot x\right)' = \frac{-1}{\sin^2 x}$$
$$= -\csc^2 x$$
$$= -\left(1 + \cot^2 x\right)$$

Application Activity 5.9

1. $\tan x + x\left(1 + \tan^2 x\right)$
2. $3\left[1 + \tan^2\left(3x + 2\right)\right]$
3. $-2x\left[1 + \cot^2\left(x^2 - 5\right)\right]$
4. $-4\sin x\left(1 + \cot^2 4x\right) + \cos x \cot 4x$

Activity 5.10

Materials

Exercise book, pens

Answers

1. $$\sec x = \frac{1}{\cos x}$$
$$\left(\sec x\right)' = \left(\frac{1}{\cos x}\right)'$$
$$= \frac{\sin x}{\cos^2 x}$$
$$= \frac{1}{\cos x}\frac{\sin x}{\cos x}$$
$$= \sec x \tan x$$

2. $\csc x = \dfrac{1}{\sin x}$

$$(\csc x)' = \left(\frac{1}{\sin x}\right)'$$

$$= \frac{-\cos x}{\sin^2 x}$$

$$= \frac{-1}{\sin x}\frac{\cos x}{\sin x}$$

$$= -\csc x \cot x$$

Application Activity 5.10

1. $3\sec(3x+2)\tan(3x+2)$ 2. $\theta^2 \csc 2\theta (3 - 2\theta \cot 2\theta)$
3. $12\sec^4 3x \tan 3x$

Activity 5.11

Materials

Exercise book, pens

Answers

1. $f(x) = \sin^{-1} x$ for $x \in [-1,1]$ and $x = \sin y$ for $y \in \left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]$ where $y = f(x)$.

$$(\sin^{-1} x)' = \frac{1}{(\sin y)'} = \frac{1}{\cos y} = \frac{1}{\cos(\sin^{-1} x)}$$

$$= \frac{1}{\sqrt{1-\sin^2(\sin^{-1} x)}} \quad \text{since } \cos x = \sqrt{1-\sin^2 x}$$

$$= \frac{1}{\sqrt{1-x^2}}$$

2. $f(x) = \cos^{-1} x$ for $x \in [-1,1]$ and $x = \cos y$ for $y \in [0, \pi]$ where $y = f(x)$

$$\left(\cos^{-1}x\right)' = \frac{1}{(\cos y)'} = \frac{1}{-\sin y} = \frac{-1}{\sin\left(\cos^{-1}x\right)}$$

$$= \frac{-1}{\sqrt{1-\cos^2\left(\cos^{-1}x\right)}} \quad \text{since } \sin x = \sqrt{1-\cos^2 x}$$

$$= \frac{-1}{\sqrt{1-x^2}}$$

Application Activity 5.11

1. $\dfrac{1}{|x|\sqrt{x^2-1}}$
2. $\dfrac{-2x}{\sqrt{1-x^4}}$
3. $\dfrac{-1}{\sqrt{2x-x^2}}$
4. $\dfrac{1}{\sqrt{2x}\sqrt{1-2x}}$

Activity 5.12

Materials

Exercise book, pens

Answers

1. $f(x) = \tan^{-1}x$ for $x \in \mathbb{R}$ and $x = \tan y$ for $y \in \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[$ where $y = f(x)$.

$$\left(\tan^{-1}x\right)' = \frac{1}{(\tan y)'} = \frac{1}{1+\tan^2 y} = \frac{1}{1+\tan^2\left(\tan^{-1}x\right)} = \frac{1}{1+x^2}$$

2. $f(x) = \cot^{-1}x$ for $x \in \mathbb{R}$ and $x = \cot y$ for $y \in]0, \pi[$ where $y = f(x)$

$$\left(\cot^{-1}x\right)' = \frac{1}{(\cot y)'} = \frac{-1}{1+\cot^2 y} = \frac{-1}{1+\cot^2\left(\cot^{-1}x\right)} = \frac{-1}{1+x^2}$$

Application Activity 5.12

1. $\dfrac{1}{2\sqrt{x}(1+x)}$
2. $\dfrac{1}{|x|\sqrt{x^2-1}} + \dfrac{1}{x^2+1}$
3. $\dfrac{-1}{2x\sqrt{x-1}}$

Activity 5.13

Materials

Exercise book, pens

Answers

1. $f(x) = \sec^{-1} x$ for $x \le -1$ or $x \ge 1$ and $x = \sec y$ for $y \in [0, \pi], y \ne \frac{\pi}{2}$ where $y = f(x)$

$$\left(\sec^{-1} x\right)' = \frac{1}{(\sec y)'} = \frac{1}{\sec y \tan y} = \frac{1}{x\sqrt{x^2 - 1}}$$

2. $f(x) = \csc^{-1} x$ for $x \le -1$ or $x \ge 1$ and $x = \csc y$ for $y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right], y \ne 0$ where $y = f(x)$

$$\left(\csc^{-1} x\right)' = \frac{1}{(\csc y)'} = \frac{-1}{\csc y \cot y} = \frac{-1}{x\sqrt{x^2 - 1}}$$

Application Activity 5.13

1. $\dfrac{1}{(2x+1)\sqrt{x^2 + x}}$
2. 0
3. $\dfrac{1}{x\sqrt{25x^2 - 1}}$
4. $\dfrac{-2x}{\left(x^2 + 1\right)\sqrt{x^4 + 2x^2}}$

Activity 5.14

Materials

Exercise book, pens

Answers

1. $-2\sin 2x$
2. $-4\cos 2x$
3. $8\sin 2x$
4. $16\cos 2x$
5. $-32\sin 2x$

Application Activity 5.14

1. a) $-13\sin 2x\sin 3x+12\cos 2x\cos 3x$

   b) $-\dfrac{3x+1}{4x\left(1+x^2\right)\sqrt{x}}$ c) $2\cos 2x$

   d) $\dfrac{2}{\cos^4 x}\left(1+2\sin^2 x\right)$

2. a) $\dfrac{-8x}{\left(1+x^2\right)^3}$ b) $-62\cos 2x\cos 3x+63\sin 2x\sin 3x$

   c) $\dfrac{2\sin x}{\left(4\cos^2 x-1\right)^4}\left(-64\cos^6 x+16\cos^4 x+484\cos^2 x+23\right)$

   d) $\dfrac{16\tan x}{\left(\tan^2 x-1\right)^4}\left(5\tan^6 x+19\tan^4 x+19\tan^2 x+5\right)$

3. a) $\sin\left(x+\dfrac{n\pi}{2}\right)+\cos\left(x+\dfrac{n\pi}{2}\right)$ b) $2^n\cos\left(2x+\dfrac{n\pi}{2}\right)$

5.4.4. Applications of trigonometric functions

Activity 5.15

Materials

Exercise book, pens

Answers

By reading text books or accessing internet, learners will discuss on harmonic motion and how differentiation of trigonometry functions is used to find velocity, acceleration and jerk of an object if the function representing its position is known.

If we have the function representing the position, say $S(t)$, then

- The velocity of the object is $v=\dfrac{ds}{dt}$

- The acceleration of the object is $a = \frac{d^2 s}{dt^2}$
- The jerk of the object is $j = \frac{d^3 s}{dt^3}$

Application Activity 5.15

1. a) $3m$ b) $-9\sqrt{3}\pi$ m/s c) $-27\pi\, {}^{m}\!/\!{}_{s^2}$

   d) $\frac{19\pi}{3}$ e) $\frac{3}{2}Hz$ f) $\frac{2}{3}$

2. a) amplitude x_m is $4m$

   frequency f is $0.5\,Hz$

   period T is 2

   angular frequency ω is π

   b) velocity is $\frac{dx}{dt} = -4\pi \sin\left(\pi t + \frac{\pi}{4}\right)$

   acceleration $\frac{d^2 x}{dt^2} = -4\pi^2 \cos\left(\pi t + \frac{\pi}{4}\right)$

   c) displacement at $t = 1$ is $-2\sqrt{2}$ m

   velocity at $t = 1$ is $2\pi\sqrt{2}\, {}^{m}\!/\!{}_{s}$

   acceleration at $t = 1$ is $-2\pi^2\sqrt{2}\, {}^{m}\!/\!{}_{s^2}$

   d) the maximum speed $4\pi\, {}^{m}\!/\!{}_{s}$

   maximum acceleration $4\pi^2\, {}^{m}\!/\!{}_{s^2}$

5.5. Answers for the end of unit assessment

1. a) $\frac{2\pi}{3}$ b) π c) 2π
   d) not periodic (aperiodic)
   e) 2π f) not period (aperiodic)
2. a) neither even nor odd b) even c) odd
   d) odd
3. a) 2 b) $\frac{3}{2}$ c) $\frac{1}{2}$ d) 2
   e) 7 f) $\frac{8}{5}$ g) $\frac{1}{16}$ h) 0
   i) $\frac{15}{7}$ j) $\frac{1}{2}$ k) 1 l) $\cos a$
   m) $-\sin a$ n) $\frac{1}{2}$ o) 4 p) 0
   q) 2 r) $\frac{1}{2}$ s) $\frac{3}{4}$ t) $\frac{1}{2}$
   u) $\frac{\pi^2}{2}$ v) $\pm\infty$ w) 0
4. a) $3\sec x\tan x+10\csc^2 x$ b) $-12x^{-5}-2x\tan x-x^2\sec^2 x$ c) $5\cos 2x-4\csc x\cot x$ d) $\frac{3\cos t-2}{(3-2\cos t)^2}$
   e) $-48x\sin\left(6x^2+5\right)$ f) $72x^3\sin^2\left(2x^4+1\right)\cos\left(2x^4+1\right)$
   g) $4\left(x-\cos^2 x\right)^3\left(1+\sin 2x\right)$ h) $\frac{2\sin 4x-4(2x+3)\cos 4x}{\sin^2 4x}$
   i) $\frac{-2x^2}{\sqrt{1-x^2}}$ j) $\frac{-1}{\sqrt{1-x^2}}$
   k) $\frac{-2}{x\sqrt{x^2-4}}$ l) $\frac{x^2-1}{x\sqrt{x^2-1}}$
   m) $\sin^{-1} x$

5. The amount of money in the bank account will be increasing during the following intervals: $2.1588 < t < 5.3004$, $8.4420 < t < 10$

6. $-\sqrt{2}m/\sec,\ \sqrt{2}m/\sec,\ \sqrt{2}m/\sec^2,\ \sqrt{2}m/\sec^2$

7. $0cm/\sec,\ -5\sqrt{3}cm/\sec,\ -5\sqrt{2}cm/\sec$

8. $4m/\sec, -3m/s - 4m/\sec$

9. It multiplies the velocity, acceleration, and jerk by 2, 4, and 8, respectively

10. a) $\frac{\pi}{3}$ b) $-\frac{\pi}{4}$ c) $\frac{2\pi}{3}$ d) $\frac{3\pi}{4}$

e) $\frac{6}{7}$ f) 1 g) 1 h) 1

Unit 6 Vector Space of Real Numbers

6.1. Key unit competence

Apply vectors of $\mathbb{R}^3$ to solve problems related to angles using the scalar product in $\mathbb{R}^3$ and use the vector product to solve also problems in $\mathbb{R}^3$.

6.2. Objectives

After completing this unit, the learners should be able to:

- define and apply different operations on vectors.
- show that a subset is a sub-vector space.
- define linear combination of vectors.
- find the norm of a vector.
- calculate the scalar product of two vectors.
- calculate the angle between two vectors.
- apply and transfer the skills of vectors to other area of knowledge.

6.3. Materials to be used

Exercise books, pens, instruments of geometry, calculator

6.4. Content and activities

6.4.1. Vector space $\mathbb{R}^3$

a) Content summary

Recommended teaching periods: 9 periods

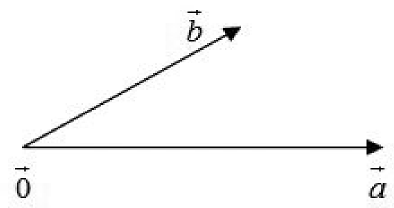
This section looks at the **operations on vectors in space**, **sub-vector spaces** and **linear combination of vectors**

The set of vectors of space with origin O is denoted by E_0 and $E_0 = \left\{\overrightarrow{0a} : a \in E\right\}$
.

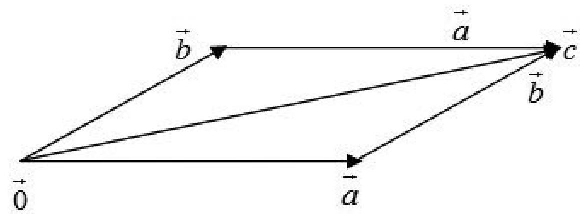
6.4.2. Operations on vectors

Sum of two vectors

Two non-parallel (or opposite) vectors of the same origin (means that their tails are together) determine one and only one plane in space.



The addition of vectors of E_0 is the application defined by $E_0 \times E_0 \to E_0$



$\vec{c}$ is then the diagonal of the parallelogram built from $\vec{a}$ and $\vec{b}$. Thus,

$\vec{a}+\vec{b}=\vec{c}$

If $\vec{a}=(a_1,a_2,a_3)$, $\vec{b}=(b_1,b_2,b_3)$, $\vec{a}+\vec{b}=(a_1+b_1,a_2+b_2,a_3+b_3)$

A vector space (also called a linear space) is a collection of objects called vectors, which may be added together and multiplied ("scaled") by numbers, called scalars in this context. Scalars are often taken to be real numbers, but there are also vector spaces with scalar multiplication by rational numbers, or generally any field.

If $(\mathbb{R},F,+)$ is a subspace of $(\mathbb{R},E,+)$, then

- $F \subset E$
- $0-vector \in F$
- $\vec{u},\vec{v}\in F,\alpha,\beta\in\mathbb{R};\quad \alpha\vec{u}+\beta\vec{v}\in F$

Sum of two sub-vector spaces

If F and G are two sub-vector spaces of E then the sum of F and G is also a sub-vector space of E. It is denoted as $F+G=\{x+y, x\in F, y\in G\}$

Theorems

- W_1 and W_2 are subspaces of V, then $W_1\cup W_2$ is a subspace $\Leftrightarrow W_1\subseteq W_2\ or\ W_2\subseteq W_1$.
- W_1 and W_2 are subspace of V, then W_1+W_2 is the smallest subspace that contains both W_1 and W_2.

Property

If $(\mathbb{R},F,+)$ and $(\mathbb{R},G,+)$ are two sub-vector spaces of $(\mathbb{R},E,+)$ we have, $\dim(F+G)=\dim(F)+\dim(G)-\dim(F\cap G)$.

Remark

If $\dim(F\cap G)=0$, then $\dim(F+G)=\dim(F)+\dim(G)$. In this case, F and G are said to be **complementary** and the sum $F+G$ is said to be a **direct sum**; and it is denoted by $F\oplus G$.

Otherwise, F and G are said to be **supplementary**.

6.4.3. Linear combination

- The vector $\vec{u}$ is called a linear combination of the vectors $\overrightarrow{u_1},\overrightarrow{u_2},\overrightarrow{u_3}$ provided that there exists scalars c_1,c_2,c_3 such that $\vec{u}=c_1\overrightarrow{u_1}+c_2\overrightarrow{u_2}+c_3\overrightarrow{u_3}$
- Let $S=\{\overrightarrow{u_1},\overrightarrow{u_2},\overrightarrow{u_3}\}$ be a set of vectors in the vector space V. The set of all linear combinations of $\overrightarrow{u_1},\overrightarrow{u_2},\overrightarrow{u_3}$ is called the **span** of the set S, denoted by $span(S)$ or $span(\overrightarrow{u_1},\overrightarrow{u_2},\overrightarrow{u_3})$.
- The set of vectors $S=\{\overrightarrow{u_1},\overrightarrow{u_2},\overrightarrow{u_3}\}$ of a vector space V is said to be **linearly independent** provided that the equation $c_1\overrightarrow{u_1}+c_2\overrightarrow{u_2}+c_3\overrightarrow{u_3}=0$ has only the trivial solution $c_1=c_2=c_3=0$.
- A set of vectors is called **linearly dependent** if it is not linearly independent. Or if $c_1u_1+c_2u_2+c_3u_3=0$ for $c_1,c_2,c_3\neq 0$.

Theorem 1

The three vectors $\overrightarrow{u_1}, \overrightarrow{u_2}, \overrightarrow{u_3}$ in $\mathbb{R}^3$ are linearly independent if and only if the 3×3 matrix $A=\begin{bmatrix}\overrightarrow{u_1} & \overrightarrow{u_2} & \overrightarrow{u_3}\end{bmatrix}$ with the vectors as columns has non zero determinant otherwise they are linearly dependent.

b) Teaching guidelines

Let learners know how to perform operations on vectors in 2-dimension. In three dimensions, there is a third component, z

- Organise class into groups. Request each group to have a group leader who will present their findings to the class.
- Ask each group to do activity related to the lesson they are going to learn. The learners may need your assistance to do any activity. Help them to understand the activity. After group discussion, invite some or all groups for presentation of their work.
- After activity presentation, capture the main points from the presentation of the learners and summarise them. Guide the learners through given examples in Learner's book or through your own examples.
- Ask learners what they learned in day lesson to ensure that they understood what they have learned.
- Request learners to do exercises in their respective groups. Request learners to correct exercises on chalkboard and give them individual evaluation. Remember to give them homework.

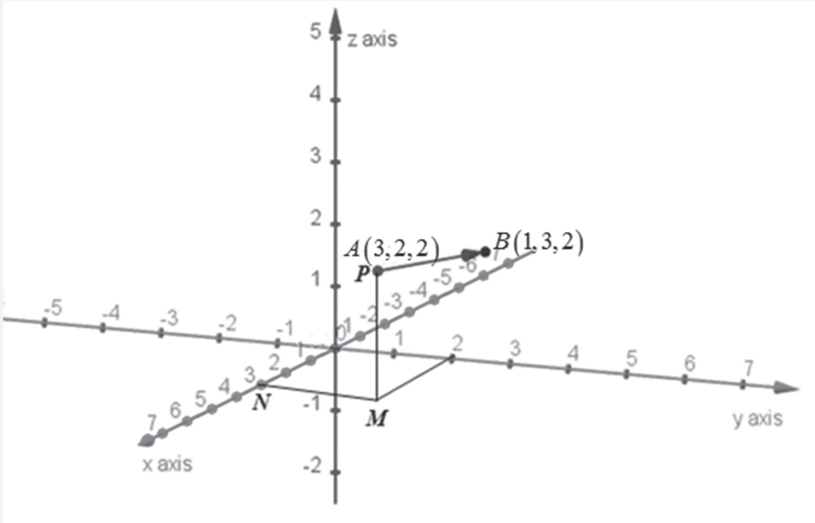
c) Answers to activities

Activity 6.1

Materials

Exercise book, pens, instruments of geometry

Answers



$B-A=(-2,1,0)$

Application Activity 6.1

1. $(-3,-3,3)$
2. $(-1,0,8)$
3. $(-14,-17,-11)$
4. $(17,21,23)$

Activity 6.2

Materials

Exercise book, pens, calculator

Answers

1. $x=0$
2. $\alpha\vec{u}+\beta\vec{v}=\alpha(0,a,3a)+\beta(0,b,3b)$

$=(0,\alpha a+\beta b,3\alpha a+3\beta b)$

$=(0,\alpha a+\beta b,3(\alpha a+\beta b))$

$=(0,c,3c)\in V,\quad for\quad \alpha a+\beta b=c$

Thus, $\alpha\vec{u}+\beta\vec{v}\in V$

Application Activity 6.2

1. $Fc\mathbb{R}^3$

   If we take $x=0, y=0$, we see that $(0,0,0)\in F$

   Consider $\vec{k}=(a,b,0)$, $\vec{t}=(c,d,0)\in F$, $\alpha,\beta\in\mathbb{R}$

   $$\alpha\vec{k}+\beta\vec{t}=\alpha(a,b,0)+\beta(c,d,0)$$
   $$=(\alpha a,\alpha b,0)+(\beta c,\beta d,0)$$
   $$=(\alpha a+\alpha c,\beta b+\beta d,0)$$

   F is a subspace of $\mathbb{R}^3$

2. $Gc\mathbb{R}^3$

   If we take $x=0, y=0$, we see that $(0,0,0)\in G$

   Consider $\vec{k}=(a,b,0)$, $\vec{t}=(c,d,0)\in G$, $\alpha,\beta\in\mathbb{R}$

   $$\alpha\vec{k}+\beta\vec{t}=\alpha(2a,2b,0)+\beta(2c,2d,0)$$
   $$=(2\alpha a,2\alpha b,0)+(2\beta c,2\beta d,0)$$
   $$=(2(\alpha a+\alpha c),2(\beta b+\beta d),0)$$

   G is a subspace of $\mathbb{R}^3$

3. $Hc\mathbb{R}^3$

   If we take $x=0, y=0$, we see that $(0,0,0)\in H$

   Consider $\vec{k}=(a,0,b)$, $\vec{t}=(c,0,d)\in H$, $\alpha,\beta\in\mathbb{R}$

   $$\alpha\vec{k}+\beta\vec{t}=\alpha(a,0,b)+\beta(c,0,d)$$
   $$=(\alpha a,0,\alpha b)+(\beta c,0,\beta d)$$
   $$=(\alpha a+\alpha c,0,\beta b+\beta d)$$

   H is a subspace of $\mathbb{R}^3$

4. $Kc\mathbb{R}^3$

   I f we take $x=0, z=0$, we see that $(0,0,0)\notin K$

   Therefore, K is not a subspace of $\mathbb{R}^3$

Activity 6.3

Materials

Exercise book, pens, calculator

Answers

$a = 3,\ b = 2$

Application Activity 6.3

1. $\vec{v} = -6\vec{e_1} + 3\vec{e_2} + 2\vec{e_3}$

2. Set $(a,b,c) = \alpha\vec{u} + \beta\vec{v} + \gamma\vec{w}$

   $\begin{cases} \alpha = a \\ \beta = b - 2a \\ \gamma = c - 2b + a \end{cases}$. Thus, $\vec{u},\vec{v}$ and $\vec{w}$ generate $\mathbb{R}^3$

3. a) $\{\vec{u},\vec{v},\vec{w}\}$ is not a basis of $\mathbb{R}^3$ b) $k = -8$

Activity 6.4

Materials

Exercise book, pens, calculator

Answers

$$\begin{cases} a = \frac{13}{8} \\ b = \frac{9}{4} \\ c = -\frac{11}{8} \end{cases}$$

Application Activity 6.4

1. $[\vec{u}]_V = (3,-2,-5)$, $[\vec{v}]_V = (3,-5,3)$
2. a) $(2,-5,7)$ b) $(c, b-c, a-b)$
3. $(3,-1,2)$
4. Suppose $\vec{v} = r\vec{e_1} + s\vec{e_2} + t\vec{e_3}$; then $[\vec{v}]_e = (r,s,t)$

From the values of $\overrightarrow{e_1}, \overrightarrow{e_2}$ and $\overrightarrow{e_3}$, we have

$$\left[\vec{v}\right] = r\left(a_1f_1 + a_2f_2 + a_3f_3\right) + s\left(b_1f_1 + b_2f_2 + b_3f_3\right) + t\left(c_1f_1 + c_2f_2 + c_3f_3\right)$$

$$\left[\vec{v}\right] = \left(ra_1f_1 + sb_1f_1 + tc_1f_1\right) + \left(ra_2f_2 + sb_2f_2 + tc_2f_2\right) + \left(ra_3f_3 + sb_3f_3 + tc_3f_3\right)$$

$$\left[\vec{v}\right] = f_1\left(ra_1 + sb_1 + tc_1\right) + f_2\left(ra_2 + sb_2 + tc_2\right) + f_3\left(ra_3 + sb_3 + tc_3\right)$$

Hence, $\left[\vec{v}\right]_f = \left(ra_1 + sb_1 + tc_1, ra_2 + sb_2 + tc_2, ra_3 + sb_3 + tc_3\right)$

On the other hand

$$\left[\vec{v}\right]_e A = \left(r, s, t\right)\begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix}$$

$$= \left(ra_1 + sb_1 + tc_1, ra_2 + sb_2 + tc_2, ra_3 + sb_3 + tc_3\right)$$

$$= \left[\vec{v}\right]_f \qquad \textit{as required}$$

6.4.4. Euclidian vector space $\mathbb{R}^3$

a) Content summary

Recommended teaching periods: 8 periods

This section talks about:

Scalar product of two vectors

The scalar product of two vectors of space is the application $E_0 \times E_0 \to \mathbb{R}$,verifying specific conditions

Algebraically, the scalar product of vectors $\vec{u} = \left(a_1, a_2, a_3\right)$ and $\vec{v} = \left(b_1, b_2, b_3\right)$ of space is defined by $\vec{u} \cdot \vec{v} = a_1b_1 + a_2b_2 + a_3b_3$.

Magnitude of a vector

The magnitude of the vector $\vec{u}$ denoted by $\left\|\vec{u}\right\|$ is defined as its length.

If $\vec{u} = \left(a, b, c\right)$ then $\left\|\vec{u}\right\| = \sqrt{a^2 + b^2 + c^2}$.

Distance between two points

The distance between two points $A(a_1,a_2,a_3)$ and $B(b_1,b_2,b_3)$ denoted, $d(A,B)$ is

$$d(A,B)=\left\|\overrightarrow{AB}\right\|=\sqrt{(b_1-a_1)^2+(b_2-a_2)^2+(b_3-a_3)^2}$$

The angle between two vectors

Angle between two vectors $\vec{u}$ and $\vec{v}$ is such that

$$\cos\left(\vec{u},\vec{v}\right)=\frac{\vec{u}\cdot\vec{v}}{\left\|\vec{u}\right\|\cdot\left\|\vec{v}\right\|}$$

Vector product of two vectors and the mixed product of three vectors.

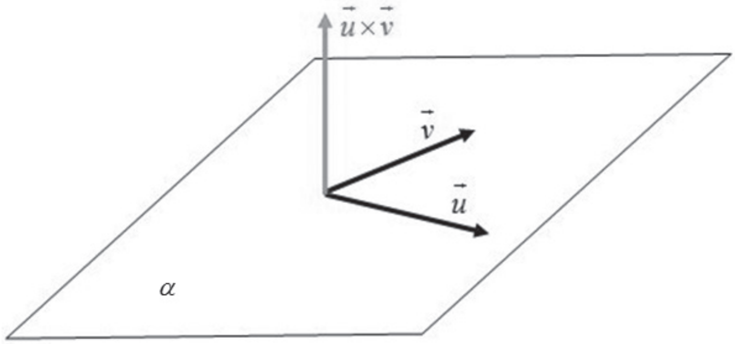
The vector product (or cross product or Gibbs vector product) of

$\vec{u}=(a_1,a_2,a_3)$ and $\vec{v}=(b_1,b_2,b_3)$ is denoted and defined by

$$\vec{u}\times\vec{v}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\end{vmatrix}=\begin{vmatrix}a_2 & a_3\\ b_2 & b_3\end{vmatrix}\vec{i}-\begin{vmatrix}a_1 & a_3\\ b_1 & b_3\end{vmatrix}\vec{j}+\begin{vmatrix}a_1 & a_2\\ b_1 & b_2\end{vmatrix}\vec{k}$$

Or

$$\vec{u}\times\vec{v}=\left(\begin{vmatrix}a_2 & a_3\\ b_2 & b_3\end{vmatrix},-\begin{vmatrix}a_1 & a_3\\ b_1 & b_3\end{vmatrix},\begin{vmatrix}a_1 & a_2\\ b_1 & b_2\end{vmatrix}\right)$$



The mixed product (also called the **scalar triple product** or **box product** or **compound product**) of three vectors is a scalar which numerically equals the vector product multiplied by a vector as the dot product.

The vector product of any two vectors is perpendicular to each of these vectors.

Then the mixed product of the vectors $\vec{u}=(a_1,a_2,a_3)$, $\vec{v}=(b_1,b_2,b_3)$ and $\vec{w}=(c_1,c_2,c_3)$ is denoted and defined by

$$\vec{u}\cdot\left(\vec{v}\times\vec{w}\right)=a_1\begin{vmatrix}b_2 & b_3\\ c_2 & c_3\end{vmatrix}-a_2\begin{vmatrix}b_1 & b_3\\ c_1 & c_3\end{vmatrix}+a_3\begin{vmatrix}b_1 & b_2\\ c_1 & c_2\end{vmatrix}.$$

b) Teaching guidelines

Let learners know how to find scalar product of two vectors, magnitude of a vector, angle between two vectors. In three dimensions, there is a third component, z

- Organise class into groups. Request each group to have a group leader who will present their findings to the class.
- Ask each group to do activity related to the lesson they are going to learn. The learners may need your assistance to do any activity. Help them to understand the activity. After group discussion, invite some or all groups for presentation of their work.
- After activity presentation, capture the main points from the presentation of the learners and summarise them. Guide the learners through given examples in Learner's book or through your own examples.
- Ask learners what they learned in day lesson to ensure that they understood what they have learned.
- Request learners to do exercises in their respective groups. Request learners to correct exercises on chalkboard and give them individual evaluation. Remember to give them homework.

c) Answers to activities

Activity 6.5

Materials

Exercise book, pens, calculator

Answers

a) 0 b) -3

Application Activity 6.5

a) 15	b) 22	c) -4	d) 3

Activity 6.6

Materials

Exercise book, pens, calculator

Answers

a) $\sqrt{29}$ b) $\sqrt{6}$

Application Activity 6.6

a) $\sqrt{461}$ unit of length b) $3\sqrt{3}$ unit of length

c) 22 unit of length d) $\sqrt{201}$ unit of length

Activity 6.7

Materials

Exercise book, pens, calculator

Answers

1. $\vec{u}\cdot\vec{v}=18$ 2. $\|\vec{u}\|\|\vec{v}\|=18$

3. $\cos^{-1}\left(\dfrac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\|\vec{v}\|}\right)=0$

Application Activity 6.7

1. a) 77.99^0 b) 85.96^0 c) 79.26^0 d) 54.74^0

2. a) $\cos\alpha=\dfrac{2}{\sqrt{29}}$, $\cos\beta=\dfrac{3}{\sqrt{29}}$, $\cos\gamma=\dfrac{4}{\sqrt{29}}$

   b) $\cos\alpha=\dfrac{4}{\sqrt{17}}$, $\cos\beta=\dfrac{-1}{\sqrt{17}}$, $\cos\gamma=0$

   c) $\cos\alpha=\dfrac{1}{\sqrt{201}}$, $\cos\beta=\dfrac{-2}{\sqrt{201}}$, $\cos\gamma=\dfrac{-14}{\sqrt{201}}$

   d) $\cos\alpha=1$, $\cos\beta=0$, $\cos\gamma=0$

Activity 6.8

Materials

Exercise book, pens, calculator

Answers

1. Let (a,b,c) be that vector. Using dot product properties

$$\begin{cases}(a,b,c)\cdot(2,-1,3)=0\\(a,b,c)\cdot(1,2,-1)=0\end{cases}\Leftrightarrow\begin{cases}2a-b+3c=0\\a+2b-c=0\end{cases}$$

$$\Leftrightarrow\begin{cases}2a-b+3c=0\\-2a-4b+2c=0\end{cases}$$

$$-5b+5c=0\Rightarrow c=b$$

$$2a-c+3c=0$$

$$\Leftrightarrow 2a+2c=0\Rightarrow a=-c$$

Then $a=-c,\ b=c$

Take $c=1$, we have $\vec{w}=(-1,1,1)$

2. $\vec{i}\begin{vmatrix}-1 & 3\\2 & -1\end{vmatrix}-\vec{j}\begin{vmatrix}2 & 3\\1 & -1\end{vmatrix}+\vec{k}\begin{vmatrix}2 & -1\\1 & 2\end{vmatrix}=-5\vec{i}+5\vec{j}+5\vec{k}$

3. Vector obtained in b is a multiple of vector obtained in a. Or we can say that the two vectors are parallel.

Application Activity 6.8

1. $(1,-2,5)$ 2. $(-2,10,-12)$ 3. $(3,3,-3)$
4. $(-66,18,83)$ 5. $(8,-18,-14)$

Activity 6.9

Materials

Exercise book, pens, calculator

Answers

1. $\left(\begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix}, -\begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix}, \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}\right)$ or

$\left(b_2c_3 - c_2b_3, -b_1c_3 + c_1b_3, b_1c_2 - c_1b_2\right)$

2. $\left(a_1\begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix}, -a_2\begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix}, a_3\begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}\right)$ or

$\left(a_1b_2c_3 - a_1c_2b_3, -a_2b_1c_3 + a_2c_1b_3, a_3b_1c_2 - a_3c_1b_2\right)$

Application Activity 6.9

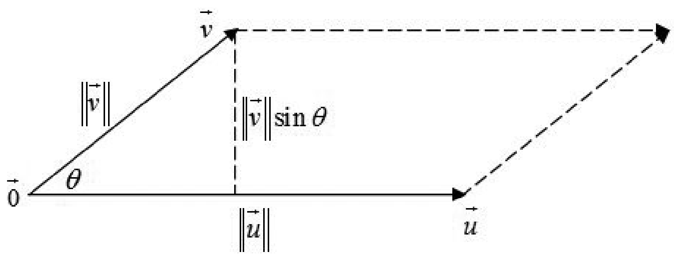
1) -4	2) 42	3) 24	4) 320	5) 40

6.4.5. Applications of vector space of real number

a) Content summary

Area of a parallelogram

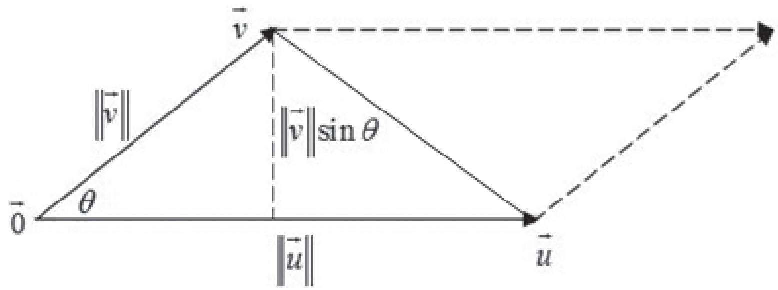
Area of a parallelogram with vectors $\vec{u}$ and $\vec{v}$ as two adjacent sides is $S_{\square} = \left\|\vec{u} \times \vec{v}\right\|$



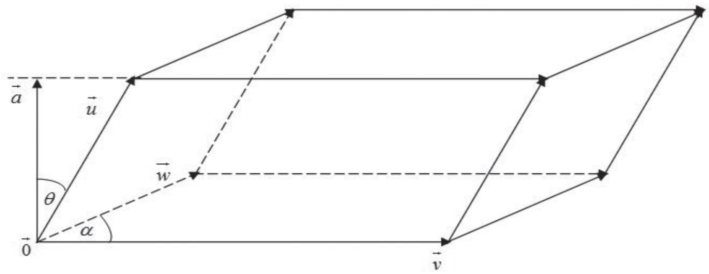
Area of a triangle

Thus, the area of triangle with vectors $\vec{u}$ and $\vec{v}$ as two sides is

$S_{\triangle} = \frac{1}{2}\left\|\vec{u} \times \vec{v}\right\|$



Volume of a parallelepiped



The volume of a parallelepiped which has vectors $\vec{u}$, $\vec{v}$ and $\vec{w}$ as three concurrent edges, where $\vec{v}$ and $\vec{w}$ define its base, is given by

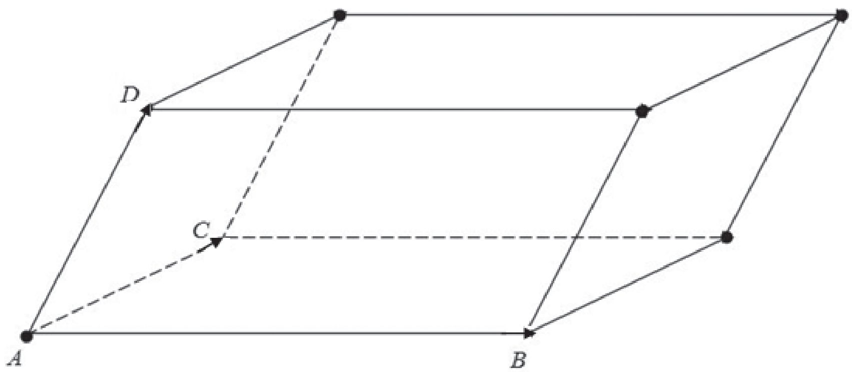
$V = \left|\vec{u}\cdot\left(\vec{v}\times\vec{w}\right)\right|$

Remember that if $\vec{u}=(a_1,a_2,a_3)$, $\vec{v}=(b_1,b_2,b_3)$ and $\vec{w}=(c_1,c_2,c_3)$, then

$$\vec{u}\cdot\left(\vec{v}\times\vec{w}\right)=\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}.= \quad a_1\begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2\begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3\begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

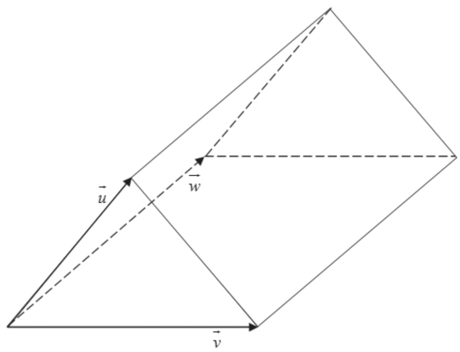
If the parallelepiped is defined by four points $A(a_1,a_2,a_3)$, $B(b_1,b_2,b_3)$, $C(c_1,c_2,c_3)$ and $D(d_1,d_2,d_3)$, its volume is

$V = \left|\overrightarrow{AD}\cdot\left(\overrightarrow{AB}\times\overrightarrow{AC}\right)\right|$



Volume of a triangular prism

The parallelepiped can be split into 2 triangular prism of equal volume. Since the volume of a parallelepiped is the magnitude of the mixed product, then the volume of a triangular prism is equal to ½ of the magnitude of the mixed product.

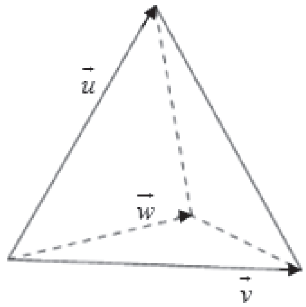


$$V = \frac{1}{2}\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \frac{1}{2}\left(\begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} a_1 - \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} a_2 + \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix} a_3\right)$$

Volume of a tetrahedron

The parallelepiped can be split into 6 tetrahedra of equal volume.

Since the volume of a parallelepiped is the magnitude of the mixed product, then the volume of a tetrahedron is equal to $\frac{1}{6}$ of the magnitude of the mixed product.



$$V=\frac{1}{6}\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}=\frac{1}{6}\left(\begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix}a_1-\begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix}a_2+\begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}a_3\right)$$

Remark

A tetrahedron is also called **triangular pyramid**.

b) Teaching guidelines

Let learners know how to find scalar product, vector product, mixed product and magnitude of a vector.

- Organise class into groups. Request each group to have a group leader who will present their findings to the class.
- Ask each group to do activity related to the lesson they are going to learn. The learners may need your assistance to do any activity. Help them to understand the activity. After group discussion, invite some or all groups for presentation of their work.
- After activity presentation, capture the main points from the presentation of the learners and summarise them. Guide the learners through given examples in Learner's book or through your own examples.
- Ask learners what they learned in day lesson to ensure that they understood what they have learned.
- Request learners to do exercises in their respective groups. Request learners to correct exercises on chalkboard and give them individual evaluation. Remember to give them homework.

c) Answers to activities

Activity 6.10

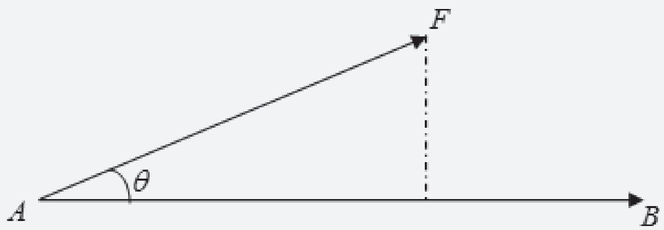
Materials

Exercise book, pens

Answers

If a constant force F acting on a particle displaces it from A to B, then,

$$
\begin{aligned}
work\ done &= (component\ of\ F)\cdot Displacement \\
&= (F\cos\theta)\cdot AB \\
&= \vec{F}\cdot\overrightarrow{AB}
\end{aligned}
$$



Application Activity 6.10

1) 16 2) 55 3) $\frac{141}{2}$

4) 20 5) 32 6) 40

Activity 6.11

Materials

Exercise book, pens

Answers

Since the base and the height of this parallelogram are $\|\vec{u}\|$ and $\|\vec{v}\|\sin\theta$ respectively, the area is $S_{\square} = \|\vec{u}\|\|\vec{v}\|\sin\theta$. But $\|\vec{u}\|\|\vec{v}\|\sin\theta = \|\vec{u}\times\vec{v}\|$. Then area is $S_{\square} = \|\vec{u}\times\vec{v}\|$.

Thus, the magnitude of the vector product of two vectors $\vec{u}$ and $\vec{v}$ represents the area of a parallelogram with vectors $\vec{u}$ and $\vec{v}$ as two adjacent sides.

Application Activity 6.11

1. a) $220\sqrt{2}$ sq. units b) $\sqrt{445}$ sq. units
2. $\frac{3\sqrt{2}}{2}$ sq. units
3. $\sqrt{563}$ sq. units
4. $6\sqrt{5}$ sq. units
5. $\sqrt{29}$ sq. units
6. $\frac{\sqrt{6}}{2}$ sq. units

Activity 6.12

Materials

Exercise book, pens

Answers

The base of this parallelepiped is defined by the vectors $\vec{v}$ and $\vec{w}$.

Then, the area of the base is $S = \|\vec{v}\times\vec{w}\| = \|\vec{v}\|\|\vec{w}\|\sin\alpha$. The height of this parallelepiped is $\|\vec{a}\|$.

Since the vector $\vec{a}$ is not known we can find the height in terms of $\|\vec{u}\|$.

We see that $\cos\theta = \frac{\|\vec{a}\|}{\|\vec{u}\|} \Leftrightarrow \|\vec{a}\| = \|\vec{u}\|\cos\theta$.

The angle θ is the angle between the vector $\vec{a}$ and vector $\vec{u}$ but it is also the angle between the vector $\vec{u}$ and the vector given by the vector product $\vec{v}\times\vec{w}$ since this cross product is perpendicular to both $\vec{v}$ and $\vec{w}$.

Now, the volume of the parallelepiped is product of the area of the base and the height.

Then,

$V = \|\vec{v}\|\|\vec{w}\|\sin\alpha\,\|\vec{u}\|\cos\theta$ $\Leftrightarrow V = \|\vec{v}\|\|\vec{w}\|\|\vec{u}\|\sin\alpha\cos\theta$

$\Leftrightarrow V = \|\vec{u}\|\left(\|\vec{v}\|\|\vec{w}\|\sin\alpha\right)\cos\theta$ $\Leftrightarrow V = \|\vec{u}\|\|\vec{v}\times\vec{w}\|\cos\theta$

$\Leftrightarrow V = \left|\vec{u}\cdot\left(\vec{v}\times\vec{w}\right)\right|$

Thus, the volume of a parallelepiped which has vectors $\vec{u}$, $\vec{v}$ and $\vec{w}$ as three concurrent edges, where $\vec{v}$ and $\vec{w}$ define its base, is given by

$V = \left|\vec{u}\cdot\left(\vec{v}\times\vec{w}\right)\right|$

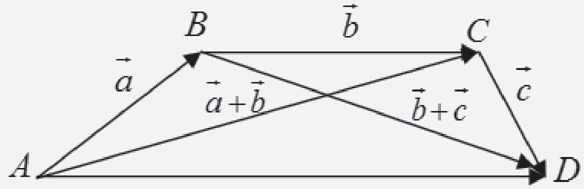
Application Activity 6.12

1. a) $\frac{3}{2}$ cubic units b) $\frac{35}{2}$ cubic units
2. a) 2 cubic units b) $\frac{10}{3}$ cubic units
3. 20 cubic units
4. $7\frac{1}{3}$ cubic units

6.5. Answers for the end of unit assessment

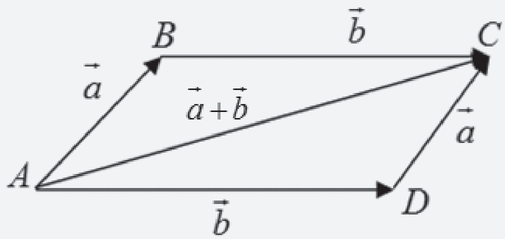
1. $\left(\vec{a}+\vec{b}\right)+\vec{c}=\overrightarrow{AC}+\overrightarrow{CD}=\overrightarrow{AD}$

   $\vec{a}+\left(\vec{b}+\vec{c}\right)=\overrightarrow{AB}+\overrightarrow{BD}=\overrightarrow{AD}$



   $\vec{a}+\vec{b}=\vec{b}+\vec{a}$

   $\overrightarrow{AC}=\overrightarrow{AB}+\overrightarrow{BC}=\overrightarrow{AD}+\overrightarrow{DC}$



2. $\vec{b}-\vec{a},\vec{b}-2\vec{a},2\left(\vec{b}-\vec{a}\right),2\vec{b}-3\vec{a},\vec{b}-\overline{2a}$
3. a) Linearly independent. b) Linearly dependent. c) Linearly independent.
4. Linearly independent
5. There are 2^n elements.
6. $\left[\vec{v}\right]_S=\left(c,b-c,a-b\right)$
7. $\left.\begin{array}{l}\vec{u}-\vec{v}=\vec{u}+\left(-\vec{v}\right)\\ k\left(-\vec{v}\right)=-k\vec{v}\end{array}\right\}k\left(\vec{u}-\vec{v}\right)=k\left(\vec{u}+\left(-\vec{v}\right)\right)=k\vec{u}+k\left(-\vec{v}\right)=k\vec{u}-k\vec{v}$
8. $s=-3p+2q+4r$

9. Set

$$a_0 + a_1t + a_2t^2 + a_3t^3 = x(1-t)^3 + y(1-t)^2 + z(1-t) + w.1,\ a_0, a_1, a_2, a_3 \in IR$$

$$\begin{cases} x = -a_3 \\ y = a_2 + 3a_3 \\ z = -a_1 - 2a_2 - 3a_3 \\ w = a_0 + a_1 + a_2 + a_3 \end{cases}$$

Hence the four polynomials generate the space of polynomials of degree ≤ 3.

10. The condition is $2a - 4b - 3c = 0$. Note, in particular, that u, v and w do not generate the whole space $\mathbb{R}^3$.

11. Set $(a,b,0) = xu + yv$ $\begin{cases} x = a \\ y = b - 2a \end{cases}$. Hence u and v generate W.

12. a) Set $(0,b,c) = x(0,1,1) + y(0,2,-1)$

$\begin{cases} x = \dfrac{b+2c}{3} \\ y = \dfrac{b-c}{3} \end{cases}$. Hence $(0,1,1)$ and $(0,2,-1)$ generate W.

b) $(0,b,c) = x(0,1,2) + y(0,2,3) + z(0,3,1)$

$\begin{cases} x = -3b + 2c + 7z \\ y = 2b - c - 5z \end{cases}$. Hence the three vectors generate W.

13. Any finite set S of polynomials contains one of maximum degree, say n. Then span of S cannot contain polynomials of degree greater than n. Accordingly, $V \neq Span(S)$, for any finite set S.

14. a) dimension is 1 b) dimension is 2

c) dimension is 2 d) dimension is 1

15. a) $\sqrt{213}$ unit of length b) $\sqrt{130}$ unit of length

c) $\sqrt{26}$ unit of length

16. $\vec{e} = \left(\dfrac{1}{3}, \dfrac{2}{3}, \dfrac{2}{3}\right)$

17. $k = 7$

18. a) 0 b) $\vec{k}$ c) $-\vec{j}$ d) $\vec{i}$ e) 0 f) 0

19. a) 7 b) 30 c) 15 d) $7\sqrt{3}$

20. a) $(-20,-67,-9)$ b) $(-78,52,-26)$
c) $(24,0,-16)$ d) $(-12,-22,-8)$
e) $(0,-56,-392)$ d) $(0,56,392)$

21. a) $\frac{\sqrt{374}}{2}$ sq. unit b) $9\sqrt{13}$ sq. unit

22. ambiguous, needs parentheses

23. a) 16 cubic unit b) 45 cubic unit

24. a) 9 cubic unit b) $\sqrt{122}$ sq. unit

25. a) $\frac{\sqrt{6}}{2}$ sq. unit b) $\frac{1}{2}\sqrt{b^2c^2 + c^2a^2 + a^2b^2}$ sq. unit

26. $\frac{1}{2}\sqrt{(y_1z_2 - y_2z_1)^2 + (z_1x_2 - z_2x_1)^2 + (x_1y_2 - x_2y_1)^2}$

27. 6 cubic unit

28. 102.12^0

29. 92

30. $\frac{41}{7}$

Unit 7 Matrices and Determinant of Order 3

7.1. Key unit competence

Apply matrix and determinant of order 3 to solve related problems.

7.2. Objectives

After completing this unit, the learners should be able to:

- Define and give example of matrix of order three.
- Perform different operations on matrices of order three.
- Find the determinant of order three.
- Find the inverse of matrix of order three.
- Solve system of three linear equations by matrix inverse method.

7.3. Materials used

Exercise books, pens, calculator

7.4. Content and activities

7.4.1. Square matrices of order 3

a) Content summary

Recommended teaching periods: 14 periods

This section looks at the definition of square matrices of order three and their operations:

- Addition and subtraction (only matrices of the same type can be subtracted)

If $A=\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$ and $B=\begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix}$, then

$$A+B=\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}+\begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix}=\begin{pmatrix} a_{11}+b_{11} & a_{12}+b_{12} & a_{13}+b_{13} \\ a_{21}+b_{21} & a_{22}+b_{22} & a_{23}+b_{23} \\ a_{31}+b_{31} & a_{32}+b_{32} & a_{33}+b_{33} \end{pmatrix}$$

$$A-B=\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}-\begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix}=\begin{pmatrix} a_{11}-b_{11} & a_{12}-b_{12} & a_{13}-b_{13} \\ a_{21}-b_{21} & a_{22}-b_{22} & a_{23}-b_{23} \\ a_{31}-b_{31} & a_{32}-b_{32} & a_{33}-b_{33} \end{pmatrix}$$

- Transpose

  If $A=\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$, then $A^t=\begin{pmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{pmatrix}$

Properties of transpose of matrices

Let A, B be matrices of order three

1. $\left(A^t\right)^t = A$
2. $\left(A + B\right)^t = A^t + B^t$
3. $\left(\alpha\times A\right)^t = \alpha\times A^t$

- **Multiplication**

  Two matrices A and B can be multiplied together if and only if the number of columns of A is equal to the number of rows of B.

  $M_{m\times n}\times M_{n\times p} = M_{m\times p}$

$$A\times B=\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}\times\begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix}$$

$$=\begin{pmatrix} a_{11}b_{11}+a_{12}b_{21}+a_{13}b_{31} & a_{11}b_{12}+a_{12}b_{22}+a_{13}b_{32} & a_{11}b_{13}+a_{12}b_{23}+a_{13}b_{33} \\ a_{21}b_{11}+a_{22}b_{21}+a_{23}b_{31} & a_{21}b_{12}+a_{22}b_{22}+a_{23}b_{32} & a_{21}b_{13}+a_{22}b_{23}+a_{23}b_{33} \\ a_{31}b_{11}+a_{32}b_{21}+a_{33}b_{31} & a_{31}b_{12}+a_{32}b_{22}+a_{33}b_{32} & a_{31}b_{13}+a_{32}b_{23}+a_{33}b_{33} \end{pmatrix}$$

Properties of multiplication of matrices

Let A, B, C be matrices of order three

1. Associative

   $A\times(B\times C)=(A\times B)\times C$

2. Multiplicative Identity

   $A\times I = A$, where *I* is the identity matrix with the same order as matrix *A*.

3. Not Commutative

   $A\times B \neq B\times A$

4. Distributive

   $A\times(B\times C)=A\times B+A\times C$

5. $(A\times B)^t = B^t \times A^t$

Notice

- If $AB=0$, it does not necessarily follow that $A=0$ or $B=0$.
- **Commuting matrices in multiplication**

  In general, the multiplication of matrices is not commutative, i.e, $AB \neq BA$, but we can have the case where two matrices A and B satisfy $AB = BA$. In this case, A and B are said to be **commuting**.

b) Teaching guidelines

Let learners know what square matrix of order two is. Square matrix of order three will have three rows and three columns.

- Organise class into groups. Request each group to have a group leader who will present their findings to the class.
- Ask each group to do activity related to the lesson they are going to learn. The learners may need your assistance to do any activity. Help them to understand the activity. After group discussion, invite some or all groups for presentation of their work.

- After activity presentation, capture the main points from the presentation of the learners and summarise them. Guide the learners through given examples in the Learner's Book or through your own examples.
- Ask learners what they learned in day lesson to ensure that they understood what they have learned.
- Request learners to do exercises in their respective groups. Request learners to correct exercises on chalkboard and give them individual evaluation. Remember to give them homework.

c) Answers to activities

Activity 7.1

Materials

Exercise book, pens

Answers

$$\begin{pmatrix} 2 & 3 & 0 \\ 1 & -1 & 2 \\ 4 & 1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

Application Activity 7.1

$$\begin{pmatrix} 2 & -3 & 11 \\ 2 & 13 & 1 \\ 0 & 11 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 23 & 24 \\ 1 & 43 & 44 \\ 1 & 1 & 2 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 12 \\ 1 & -20 & 4 \\ 0 & 18 & 6 \end{pmatrix}, \begin{pmatrix} -4 & 2 & 1 \\ -9 & 1 & 0 \\ 7 & 0 & 1 \end{pmatrix},$$

$$\begin{pmatrix} 3 & 9 & 2 \\ 5 & 6 & 4 \\ 7 & 3 & -8 \end{pmatrix}$$

There are many possible answers

Activity 7.2

Materials

Exercise book, pens, calculator

Answers

1. $\begin{pmatrix} 5 & 14 & 24 \\ 4 & 21 & 20 \\ 14 & 65 & 12 \end{pmatrix}$ 2. $\begin{pmatrix} 3 & -14 & 20 \\ 1 & -7 & -16 \\ 7 & -17 & 3 \end{pmatrix}$

3. $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ Comment: $-A$ is additive inverse of A

4. $\begin{pmatrix} 3 & 2 & 16 \\ 2 & 7 & 4 \\ 8 & 23 & 6 \end{pmatrix}$ and $\begin{pmatrix} 3 & 2 & 16 \\ 2 & 7 & 4 \\ 8 & 23 & 6 \end{pmatrix}$

The two results are equal. This implies that The two matrices are commuting for addition (Since this is true for all matrices of order 3, then the addition of matrices is commutative)

5. $\begin{pmatrix} 2 & -4 & 12 \\ 1 & 0 & -4 \\ 5 & 2 & 3 \end{pmatrix} + \begin{pmatrix} 2 & 9 & 3 \\ 1 & 9 & 12 \\ 6 & 19 & 3 \end{pmatrix} = \begin{pmatrix} 4 & 5 & 15 \\ 2 & 9 & 8 \\ 11 & 21 & 6 \end{pmatrix}$ and

$$\begin{pmatrix} 3 & 2 & 16 \\ 2 & 7 & 4 \\ 8 & 23 & 6 \end{pmatrix} + \begin{pmatrix} 1 & 3 & -1 \\ 0 & 2 & 4 \\ 3 & -2 & 0 \end{pmatrix} = \begin{pmatrix} 4 & 5 & 15 \\ 2 & 9 & 8 \\ 11 & 21 & 6 \end{pmatrix}$$

The two results are equal. This implies that The three matrices are associative for addition (Since this is true for all matrices of order 3, then the addition of matrices is associative)

6. $\begin{pmatrix} 2 & 1 & 5 \\ -4 & 0 & 2 \\ 12 & -4 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 3 \\ 6 & 7 & 21 \\ 4 & 8 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 3 \\ 3 & 2 & -2 \\ -1 & 4 & 0 \end{pmatrix}$

Application Activity 7.2

1. $\begin{pmatrix} 1 & -10 & -3 \\ -12 & -2 & -11 \\ 0 & -2 & 1 \end{pmatrix}$ 2. $\begin{pmatrix} -25 & 10 & 15 \\ -4 & 6 & -5 \\ -18 & 8 & 25 \end{pmatrix}$ 3. $\begin{pmatrix} 15 & -14 & 3 \\ 0 & 0 & -13 \\ 9 & 3 & 4 \end{pmatrix}$

Activity 7.3

Materials

Exercise book, pens, calculator

Answers

1. $\begin{pmatrix} 1 & 1 & 3 \\ 3 & -2 & -2 \\ 1 & 2 & 0 \end{pmatrix}$ and $\begin{pmatrix} 12 & 3 & -4 \\ 3 & -2 & -1 \\ -1 & 0 & 0 \end{pmatrix}$

2. $\begin{pmatrix} 13 & 4 & -1 \\ 6 & -4 & -3 \\ 0 & 2 & 0 \end{pmatrix}$ 3. $\begin{pmatrix} 13 & 6 & 0 \\ 4 & -4 & 2 \\ -1 & -3 & 0 \end{pmatrix}$

4. $\begin{pmatrix} 13 & 4 & -1 \\ 6 & -4 & -3 \\ 0 & 2 & 0 \end{pmatrix}$

5. Matrix obtained in 2 is equal to the matrix obtained in 4

6. $A = \begin{pmatrix} 1 & 3 & 1 \\ 1 & -2 & 2 \\ 3 & -2 & 0 \end{pmatrix}$

   Interchanging the rows and columns of matrix A once we get the new matrix $\begin{pmatrix} 1 & 1 & 3 \\ 3 & -2 & -2 \\ 1 & 2 & 0 \end{pmatrix}$

   Interchanging the rows and columns of matrix A twice we get the new matrix $\begin{pmatrix} 1 & 3 & 1 \\ 1 & -2 & 2 \\ 3 & -2 & 0 \end{pmatrix}$

   The last matrix is equal to matrix A

Application Activity 7.3

1. $\left(A+B\right)^t=\begin{pmatrix}1 & -3 & 9\\ 6 & 3 & 0\\ 3 & 9 & 13\end{pmatrix}$

2. $3A^t+B=\begin{pmatrix}1 & 5 & 10\\ 8 & 9 & -3\\ 12 & 20 & 29\end{pmatrix}$

3. $(-3B+4A)^t=\begin{pmatrix}-3 & 16 & -6\\ 10 & 12 & -14\\ 5 & 15 & 17\end{pmatrix}$

4. $M^t=\begin{pmatrix}1 & 2 & 4\\ 4 & 1 & 0\\ 1 & 1 & 8\end{pmatrix},\ \begin{pmatrix}1 & 2 & x^2\\ 4 & 1 & 0\\ 1 & x+3 & 8\end{pmatrix}=\begin{pmatrix}1 & 2 & 4\\ 4 & 1 & 0\\ 1 & 1 & 8\end{pmatrix}$

$\begin{cases}x^2=4\\ x+3=1\end{cases}\Rightarrow x=-2$

Activity 7.4

Materials

Exercise book, pens, calculator

Answers

$$A\times B=\begin{pmatrix}-2+3-1 & -1+2+6 & 1+3+4\\ 4+6-5 & 2+4+30 & -2+6+20\\ 0+9-4 & 0+6+24 & 0+9+15\end{pmatrix}=\begin{pmatrix}0 & 7 & 8\\ 5 & 36 & 24\\ 5 & 30 & 25\end{pmatrix}$$

Application Activity 7.4

1. $A\times B=\begin{pmatrix}-28 & 36 & 39\\ 28 & -6 & -5\\ 56 & 64 & 80\end{pmatrix}$ 2. $A\times C=\begin{pmatrix}47 & 4 & -36\\ 1 & -9 & 31\\ 112 & 8 & -28\end{pmatrix}$

3. $B\times C=\begin{pmatrix}161 & 9 & -21\\ 276 & -22 & -18\\ 123 & 7 & -17\end{pmatrix}$

Activity 7.5

Materials

Exercise book, pens, calculator

Answers

1. $A\times B=\begin{pmatrix}-1&3&-2\\-2&-1&3\\1&1&-2\end{pmatrix}$, $B\times A=\begin{pmatrix}2&-4&-1\\7&-7&-2\\-5&3&1\end{pmatrix}$

   $A\times B\neq B\times A$. Multiplication of matrices is not commutative

2. $(A\times B)^t=\begin{pmatrix}-1&-2&1\\3&-1&1\\-2&3&-2\end{pmatrix}$, $B^t\times A^t=\begin{pmatrix}-1&-2&1\\3&-1&1\\-2&3&-2\end{pmatrix}$

   $(A\times B)^t=B^t\times A^t$

3. $(B\times C)=\begin{pmatrix}3&0&1\\0&-1&1\\-2&3&2\end{pmatrix}\begin{pmatrix}2&-3&1\\4&-3&0\\-2&0&1\end{pmatrix}=\begin{pmatrix}4&-9&4\\-6&3&1\\4&-3&0\end{pmatrix}$,

   $(A\times B)\times C=\begin{pmatrix}-1&3&-2\\-2&-1&3\\1&1&-2\end{pmatrix}\begin{pmatrix}1&1&-1\\1&-2&1\\-1&1&0\end{pmatrix}=\begin{pmatrix}4&-9&4\\-6&3&1\\4&-3&0\end{pmatrix}$

   $A\times(B\times C)=(A\times B)\times C$. Multiplication of matrices is associative

4. $A\times(B+C)=\begin{pmatrix}3&0&1\\0&-1&1\\-2&3&2\end{pmatrix}\begin{pmatrix}1&2&-2\\2&-1&-1\\-2&1&1\end{pmatrix}=\begin{pmatrix}1&7&-5\\-4&2&2\\0&-5&3\end{pmatrix}$

   $A\times B+A\times C=\begin{pmatrix}-1&3&-2\\-2&-1&3\\1&1&-2\end{pmatrix}+\begin{pmatrix}2&4&-3\\-2&3&-1\\-1&-6&5\end{pmatrix}=\begin{pmatrix}1&7&-5\\-4&2&2\\0&-5&3\end{pmatrix}$

   $A\times(B+C)=A\times B+A\times C$. Multiplication of matrices is distributive over addition

Application Activity 7.5

1. a) $A\times B=\begin{pmatrix}-3&0&2\\-4&0&0\\0&0&0\end{pmatrix}$, $B\times A=\begin{pmatrix}-2&0&-2\\1&0&1\\-1&0&-1\end{pmatrix}$

b) $(A\times B)\times C=\begin{pmatrix}-3&0&2\\-4&0&0\\0&0&0\end{pmatrix}\begin{pmatrix}1&0&-1\\-2&2&0\\-1&1&0\end{pmatrix}=\begin{pmatrix}-5&2&3\\-4&0&4\\0&0&0\end{pmatrix}$

$A\times(B\times C)=\begin{pmatrix}1&0&1\\1&-1&1\\0&0&0\end{pmatrix}\begin{pmatrix}-3&1&2\\-1&2&-1\\-2&1&1\end{pmatrix}=\begin{pmatrix}-5&2&3\\-4&0&4\\0&0&0\end{pmatrix}$

c) $A\times(B+C)=\begin{pmatrix}1&0&1\\1&-1&1\\0&0&0\end{pmatrix}\begin{pmatrix}-1&0&0\\-1&2&2\\-2&1&1\end{pmatrix}=\begin{pmatrix}-3&1&1\\-2&-1&-1\\0&0&0\end{pmatrix}$,

$A\times B+A\times C=\begin{pmatrix}-3&0&2\\-4&0&0\\0&0&0\end{pmatrix}+\begin{pmatrix}0&1&-1\\2&-1&-1\\0&0&0\end{pmatrix}=\begin{pmatrix}-3&1&1\\-2&-1&-1\\0&0&0\end{pmatrix}$

d) $tr(A\times B)=tr\begin{pmatrix}-3&0&2\\-4&0&0\\0&0&0\end{pmatrix}=-3$

2. $\begin{pmatrix}\cos 2\theta&0&\sin 2\theta\\0&1&0\\-\sin 2\theta&0&\cos 2\theta\end{pmatrix}$

Activity 7.6

Materials

Exercise book, pens, calculator

Answers

1. $(1,0,2)$ 2. $(0,1,0)$

3. $(1,-1,0)$ 4. $\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 2 & 0 & 0 \end{pmatrix}$

Application Activity 7.6

1. a) $\begin{pmatrix} 3 & 2 & 0 \\ 0 & -1 & 2 \\ -1 & 0 & 1 \end{pmatrix}$ b) $\begin{pmatrix} 4 & -3 & 2 \\ -1 & 0 & 0 \\ -3 & 5 & -1 \end{pmatrix}$

2. a) $\begin{pmatrix} 2 & 1 & -6 \\ 1 & 8 & -15 \\ 11 & -20 & -22 \end{pmatrix}$ b) $\begin{pmatrix} 3 & 9 & 20 \\ -4 & -3 & 4 \\ 10 & 12 & -13 \end{pmatrix}$

c) $\begin{pmatrix} 11 & 4 & -4 \\ 9 & -19 & -10 \\ 11 & -6 & -5 \end{pmatrix}$

7.4.2. Determinant of order three

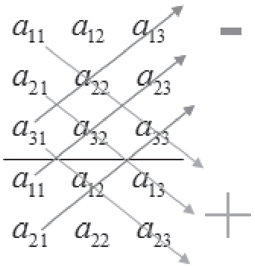
a) Content summary

Recommended teaching periods: 17 periods

This section looks at the method used to find the determinant of order three: **Rule of SARRUS**. It looks at the general method used to find determinant of order $n \geq 2$ (cofactor method). It also looks at how to find the inverse of a square matrix of order three .

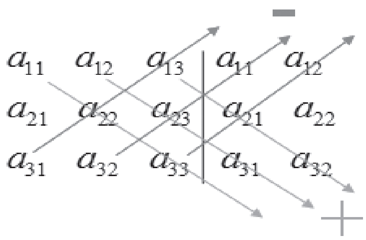
Every linear transformation $f:\mathbb{R}^3 \to \mathbb{R}^3$ can be identified with a matrix of order three, $[f]_{e_j} = (a_{ij})$, whose jth column is $f(\overrightarrow{e_j})$ where $\{\overrightarrow{e_j}\}$, $j=1,2,3$ is the standard basis of $\mathbb{R}^3$. The matrix $[f]_{e_j}$ is called matrix representation of f relative to the standard basis $\{\overrightarrow{e_j}\}$.

To calculate the 3x3 determinant, we rewrite the first two rows below the determinant (or first two columns to the right of the determinant).



$$\det = a_{11}a_{22}a_{33} + a_{21}a_{32}a_{13} + a_{31}a_{12}a_{23} - a_{31}a_{22}a_{13} - a_{11}a_{32}a_{23} - a_{21}a_{12}a_{33}$$

Or



$$\det = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{31}a_{22}a_{13} - a_{32}a_{23}a_{11} - a_{33}a_{21}a_{12}$$

General rule for $n \times n$ matrices (minor and cofactor)

General method of finding the determinant of matrix with $n \times n$ dimension $(2\times 2,\ 3\times 3,\ 4\times 4,\ 5\times 5, \ldots)$ is the use of cofactors.

Minor

The minor of an element aij, is the determinant of the matrix remains after we delete the i^th row and the j^th column

Example : consider the matrix $\begin{pmatrix} 1 & 2 & 1 \\ 2 & 5 & 4 \\ 3 & 6 & 2 \end{pmatrix}$, the minor of 5 is $\begin{vmatrix} 1 & 1 \\ 3 & 2 \end{vmatrix}$

Cofactor

The **cofactor** of the element a_{ij} is its minor prefixing:

The **+** sign if **i+j** is **even.**

The **–** sign if **i+j** is **odd.**

$\begin{vmatrix} 1 & 2 & 1 \\ [2] & 5 & 4 \\ 3 & 6 & 2 \end{vmatrix}$ the cofactor of the indicated element is $-\begin{vmatrix} 2 & 1 \\ 6 & 2 \end{vmatrix}$

The value of a determinant is equal to the sum of the products of the elements of a line (row or column) by its corresponding cofactors:

$$|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

b) Teaching guidelines

Let learners know how to find determinant of order two. For determinant of order three, we have three rows and three columns.

- Organise class into groups. Request each group to have a group leader who will present their findings to the class.
- Ask each group to do activity related to the lesson they are going to learn. The learners may need your assistance to do any activity. Help them to understand the activity. After group discussion, invite some or all groups for presentation of their work.
- After activity presentation, capture the main points from the presentation of the learners and summarise them. Guide the learners through given examples in Learner's book or through your own examples.
- Ask learners what they learned in day lesson to ensure that they understood what they have learned.
- Request learners to do exercises in their respective groups. Request learners to correct exercises on chalkboard and give them individual evaluation. Remember to give them homework.

c) Answers to activities

Activity 7.7

Materials

Exercise book, pens, calculator

Answers

1. $(1\times6\times1)+(3\times0\times2)+(5\times1\times(-4))-(2\times6\times5)-(1\times0\times1)-(1\times3\times(-4))=-62$
2. $(10\times2\times4)+((-6)\times5\times2)+(0\times3\times1)-(4\times5\times0)-(2\times3\times10)-(1\times(-6)\times2)=-70$

Application Activity 7.7

1) 82	2) 10	3) -19

Activity 7.8

Materials

Exercise book, pens, calculator

Answers

1. $|A|=0, |B|=0$
2. $|C\cdot D|=-36, |C|\cdot|D|=6\times(-6)=-36$

   $|C\cdot D|=|C|\cdot|D|$

   Determinant of product is equal to the product of determinants.
3. Product of leading diagonal elements $1\times2\times3=6$,

   $|C|=6$

   Determinant of a triangular matrix is equal to the product of leading diagonal elements

Application Activity 7.8

1. $|A|=0$, $|B|=0$, $|C|=14$, $|D|=-5$
2. $|BC|=|B|\times|C|=0$
3. $|CD|=|C|\times|D|=-70$

Activity 7.9

Materials

Exercise book, pens, calculator

Answers

1. $|A| = -1$
2. Cofactor of each element:

   $cofactor(1) = 3,\quad cofactor(1) = -5,\quad cofactor(1) = 1$

   $cofactor(2) = 1,\quad cofactor(1) = -2,\quad cofactor(-1) = 1$

   $cofactor(3) = -2,\quad cofactor(2) = 3,\quad cofactor(1) = -1$

   Cofactor matrix

   $$\begin{pmatrix} 3 & -5 & 1 \\ 1 & -2 & 1 \\ -2 & 3 & -1 \end{pmatrix}$$

3. Transpose of cofactor matrix is $\begin{pmatrix} 3 & 1 & -2 \\ -5 & -2 & 3 \\ 1 & 1 & -1 \end{pmatrix}$

4. $$\frac{1}{-1}\begin{pmatrix} 3 & 1 & -2 \\ -5 & -2 & 3 \\ 1 & 1 & -1 \end{pmatrix} = \begin{pmatrix} -3 & -1 & 2 \\ 5 & 2 & -3 \\ -1 & -1 & 1 \end{pmatrix}$$

5. $$\begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ 3 & 2 & 1 \end{pmatrix} \times \begin{pmatrix} -3 & -1 & 2 \\ 5 & 2 & -3 \\ -1 & -1 & 1 \end{pmatrix} = \begin{pmatrix} -3+5-1` & -1+2-1 & 2-3+1 \\ -6+5+1 & -2+2-1 & 4-3-1 \\ -9+10+1 & -3+4-1 & 6-6+1 \end{pmatrix}$$

   $$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I$$

   The product of these two matrices is a unity (identity) matrix I.

Application Activity 7.9

1. No inverse

2. $\begin{pmatrix} \frac{23}{268} & -\frac{29}{268} & \frac{5}{268} \\ -\frac{3}{268} & -\frac{37}{268} & \frac{11}{268} \\ -\frac{9}{268} & \frac{23}{268} & \frac{33}{268} \end{pmatrix}$

3. $\begin{pmatrix} \frac{6}{7} & -\frac{45}{14} & \frac{16}{7} \\ -\frac{5}{7} & \frac{24}{7} & -\frac{18}{7} \\ \frac{1}{7} & -\frac{11}{14} & \frac{5}{7} \end{pmatrix}$

4. $\begin{pmatrix} -\frac{2}{5} & -\frac{3}{5} & 1 \\ \frac{1}{5} & \frac{4}{5} & -1 \\ -\frac{6}{5} & -\frac{29}{5} & 8 \end{pmatrix}$

Activity 7.10

Materials

Exercise book, pens, calculator

Answers

1. $(AB)^{-1} = \begin{pmatrix} 0 & 1 & -1 \\ 0 & 1 & 0 \\ 1 & -3 & 1 \end{pmatrix}$, $B^{-1}A^{-1} = \begin{pmatrix} 1 & -1 & 0 \\ 1 & -1 & 1 \\ -2 & 3 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 & -1 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 & -1 \\ 0 & 1 & 0 \\ 1 & -3 & 1 \end{pmatrix}$

   $(AB)^{-1} = B^{-1}A^{-1}$

2. $(A^{-1})^{-1} = \begin{pmatrix} 1 & 0 & 1 \\ 1 & -1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$ $\quad (A^{-1})^{-1} = A$

3. $(4A)^{-1} = \frac{1}{4}\begin{pmatrix} 1 & 0 & -1 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $\frac{1}{4}A^{-1} = \frac{1}{4}\begin{pmatrix} 1 & 0 & -1 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

   $(kA)^{-1} = \frac{1}{k}A^{-1},\ k \neq 0$

4. $\left(A^t\right)^{-1}=\begin{pmatrix}1&1&0\\0&-1&0\\-1&0&1\end{pmatrix}$, $\left(A^{-1}\right)^t=\begin{pmatrix}1&0&-1\\1&-1&0\\0&0&1\end{pmatrix}^t=\begin{pmatrix}1&1&0\\0&-1&0\\-1&0&1\end{pmatrix}$

$\left(A^t\right)^{-1}=\left(A^{-1}\right)^t$

Application Activity 7.10

1. $A^{-1}=\frac{1}{11}\begin{pmatrix}2&5&-14\\1&-3&4\\-1&3&7\end{pmatrix}$, $B^{-1}=\begin{pmatrix}-\frac{5}{12}&\frac{1}{4}&-\frac{13}{12}\\-\frac{3}{4}&\frac{1}{4}&-\frac{3}{4}\\\frac{1}{3}&0&\frac{2}{3}\end{pmatrix}$

2. $\left(A^{-1}\right)^{-1}=\begin{pmatrix}3&7&2\\1&0&2\\0&1&1\end{pmatrix}$

3. $(10A)^{-1}=\frac{1}{110}\begin{pmatrix}2&5&-14\\1&-3&4\\-1&3&7\end{pmatrix}$

4. $\left(A^t\right)^{-1}=\frac{1}{11}\begin{pmatrix}2&1&-1\\5&-3&3\\-14&4&7\end{pmatrix}$

7.4.3. Applications

a) Content summary

Recommended teaching periods: 4 periods

This section looks at how to solve a system of three linear equations by matrix inverse and by Cramer's rule method.

Consider the following simultaneous linear equations:

$$\begin{cases} a_{11}x + a_{12}y + a_{13}z = c_1 \\ a_{21}x + a_{22}y + a_{23}z = c_2 \\ a_{31}x + a_{32}y + a_{33}z = c_3 \end{cases}$$

One of the methods of solving this, is the use Cramer's rule.

$$\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \neq 0 \qquad \Delta_x = \begin{vmatrix} c_1 & a_{12} & a_{13} \\ c_2 & a_{22} & a_{23} \\ c_3 & a_{32} & a_{33} \end{vmatrix}$$

$$\Delta_y = \begin{vmatrix} a_{11} & c_1 & a_{13} \\ a_{21} & c_2 & a_{23} \\ a_{31} & c_3 & a_{33} \end{vmatrix} \qquad \Delta_z = \begin{vmatrix} a_{11} & a_{12} & c_1 \\ a_{21} & a_{22} & c_2 \\ a_{31} & a_{32} & c_3 \end{vmatrix}$$

$x = \frac{\Delta_x}{\Delta}$, $y = \frac{\Delta_y}{\Delta}$ and $z = \frac{\Delta_z}{\Delta}$

Remember that if $\Delta = 0$, there is no solution or infinity number of solution

b) Teaching guidelines

Let learners know how to rewrite a system of linear equation in matrix form, how to find inverse of matrix and how to multiply to matrices.

- Ask each group to do activity related to the lesson they are going to learn. The learners may need your assistance to do any activity. Help them to understand the activity. After group discussion, invite some or all groups for presentation of their work.
- After activity presentation, capture the main points from the presentation of the learners and summarise them. Guide the learners through given examples in Learner's book or through your own examples.
- Ask learners what they learned in day lesson to ensure that they understood what they have learned.
- Request learners to do exercises in their respective groups. Request learners to correct exercises on chalkboard and give them individual evaluation. Remember to give them homework.

c) Answers to activities

Activity 7.11

Materials

Exercise book, pens

Answers

1. $$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}=\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$$

2. $$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}^{-1}\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}=\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}^{-1}\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix}=\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}^{-1}\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$$

Application Activity 7.11

1. $S=\{(0,0,0)\}$ 2. $S=\{\ \}$ 3. $S=\{(1,2,0)\}$

7.5. Answers for end of unit assessment

1. a) $\begin{pmatrix} -7 & -3 & 0 \\ 0 & 4 & -12 \\ 0 & -10 & -2 \end{pmatrix}$ b) $\begin{pmatrix} -9 & -23 & 6 \\ 0 & -4 & -16 \\ -4 & 2 & -8 \end{pmatrix}$

c) $\begin{pmatrix} 7 & 8 & 3 \\ 2 & 4 & -10 \\ 2 & -14 & 7 \end{pmatrix}$ d) $\begin{pmatrix} 29 & 28 & 15 \\ 10 & -34 & -21 \\ -4 & 28 & -16 \end{pmatrix}$

e) $\begin{pmatrix} 38 & 36 & 13 \\ -1 & 12 & -42 \\ 0 & 0 & -18 \end{pmatrix}$ f) $\begin{pmatrix} 118 & 120 & 37 \\ 19 & 12 & 10 \\ 14 & 0 & 76 \end{pmatrix}$

2. $\begin{pmatrix} 2 & 1 & 0 \\ 0 & 1 & -1 \\ 2 & 4 & 0 \end{pmatrix}$

3. a) $[f]_e = \begin{pmatrix} 3 & 3 & 3 \\ -6 & -6 & -2 \\ 6 & 5 & -1 \end{pmatrix}$

b) Let $\vec{v} = (a,b,c) \in \mathbb{R}^3$ $\quad [\vec{v}]_e = \begin{pmatrix} c \\ b-c \\ a-b \end{pmatrix}$

$$[f]_e [\vec{v}]_e = \begin{pmatrix} 3 & 3 & 3 \\ -6 & -6 & -2 \\ 6 & 5 & -1 \end{pmatrix} \begin{pmatrix} c \\ b-c \\ a-b \end{pmatrix} = \begin{pmatrix} 3a \\ -2a-4b \\ -a+6b+c \end{pmatrix}$$

But, $f(\vec{v}) = f(a,b,c) = (2b+c, a-4b, 3a)$,

$[f(\vec{v})]_e = \begin{pmatrix} 3a \\ -2a-4b \\ -a+6b+c \end{pmatrix}$, verified.

4. a) $\begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}$ b) no inverse c) $\begin{pmatrix} \frac{44}{207} & -\frac{8}{207} & \frac{1}{69} \\ \frac{1}{207} & -\frac{19}{207} & \frac{11}{69} \\ -\frac{13}{207} & \frac{40}{207} & -\frac{5}{69} \end{pmatrix}$

5. $X = \begin{pmatrix} 3 & -2 & -2 \\ -5 & 5 & 2 \\ 5 & -3 & 1 \end{pmatrix}$

6. a) $S = \{(0,0,0)\}$ b) $S = \{(1,1,1)\}$
   c) $S = \{(3,0,1)\}$

7. 0

8. $k \neq -\frac{3}{5}$, $A^{-1} = \frac{1}{8}\begin{pmatrix} -29 & 17 & 14 \\ -9 & 5 & 6 \\ 16 & -8 & -8 \end{pmatrix}$

9. a) $A^{-1} = \frac{1}{7}\left(4I - A^2\right)$ b) $\begin{pmatrix} 5 \\ -1 \\ 1 \end{pmatrix}$

10. $A^3 = \begin{pmatrix} 6 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 6 \end{pmatrix} = 6I$
    a) $x = -3, y = 5, z = 2$ b) $x = 2, y = 1, z = 0$

11. $-t^3 + t^2 + t - 1$

12. a) x^2 b) x^n c) $2x$ d) mx

13. $k = \frac{1}{4}$

14. a) $\lambda = 5, \mu \neq 9$ b) $\lambda \neq 5$ c) $\lambda = 5, \mu = 9$

Unit 8 Points, Straight Lines, Planes and Sphere in 3D

8.1. Key unit competence

Use algebraic representations of points, lines, spheres and planes in 3D space and solve related problems

8.2. Objectives

After completing this unit, the learners should be able to:

- plot points in three dimensions.
- find equations of straight lines in three dimensions.
- find equations of planes in three dimensions.
- find equations of sphere in three dimensions.

8.3. Materials

Exercise books, pens, instruments of geometry, calculator

8.4. Content and activities

8.4.1. Points in three dimensions

a) Content summary

Recommended teaching periods: 7 periods

This section looks at the method used to locate a point in space.

It also looks at;

- Midpoint of a segment,

  Let the points (x_1, y_1, z_1) and (x_2, y_2, z_2) be the endpoints of a line segment, then the midpoint of that segment is given by the formula:

$$\left(\frac{x_1+x_2}{2},\frac{y_1+y_2}{2},\frac{z_1+z_2}{2}\right)$$

- Centroid of a geometric figure

  The **centroid** of **geometric figure (barycentre or geometric centre)** is the arithmetic mean (average) position of all points in the shape. In physics, barycentre means the **physical centre** of mass or the **centre of gravity**.

  Let (x_1,y_1,z_1), (x_2,y_2,z_2),...., (x_n,y_n,z_n) be n points of space, their centroid is given by the formula:

$$\left(\frac{x_1+x_2+...+x_n}{n},\frac{y_1+y_2+...+y_n}{n},\frac{z_1+z_2+...+z_n}{n}\right)$$

Ratio formula:

If P is a point on the line AB such that P divides AB internally in the ratio $m:n$, then $P=\frac{mB+nA}{m+n}$ and

if P divides AB externally in the ratio $m:n$, then $P=\frac{mB-nA}{m-n}$.

b) Teaching guidelines

Let learners know how to plot points in 2-dimensions and how to find a midpoint of two points in 2-dimension.

- Organise class into groups. Request each group to have a group leader who will present their findings to the class.
- Ask each group to do activity related to the lesson they are going to learn. The learners may need your assistance to do any activity. Help them to understand the activity. After group discussion, invite some or all groups for presentation of their work.
- After activity presentation, capture the main points from the presentation of the learners and summarise them. Guide the learners through given examples in the Learner's Book or through your own examples.

- Ask learners what they learned in day lesson to ensure that they understood what they have learned.
- Request learners to do exercises in their respective groups. Request learners to correct exercises on chalkboard and give them individual evaluation. Remember to give them homework.

c) Answers to activities

Activity 8.1

Materials

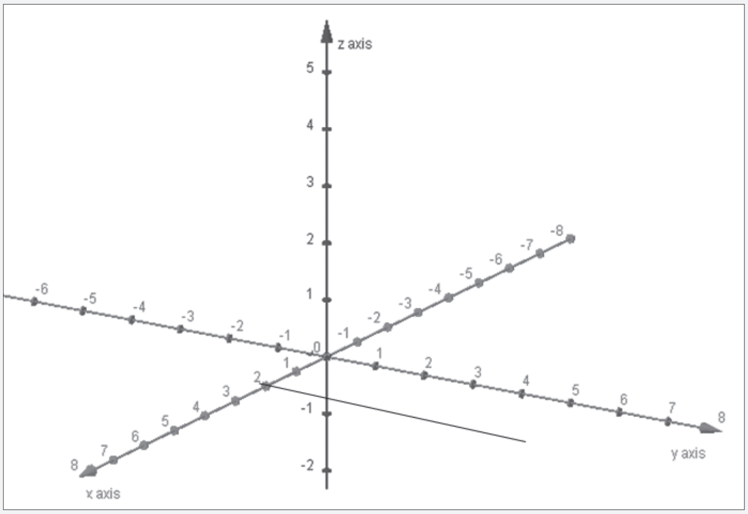
Exercise book, pens, instruments of geometry

Answers

Suppose that we need to represent the point $A(2,3,5)$ in space

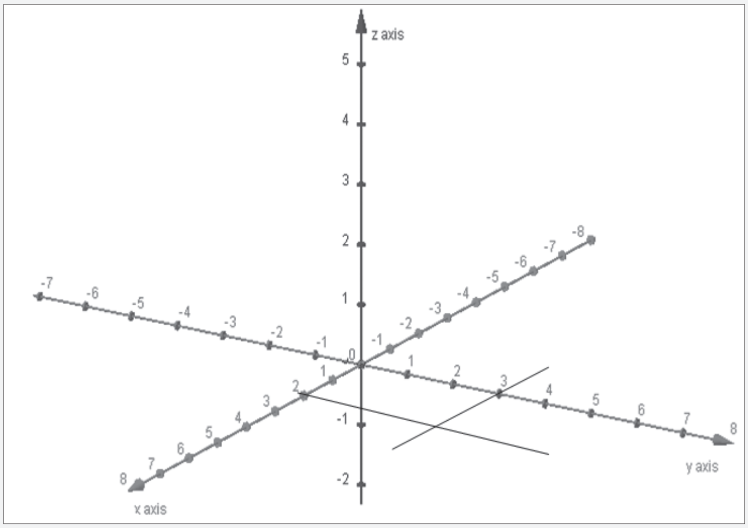
- From x-coordinate 2, draw a line parallel to y-axis

  See following figure.



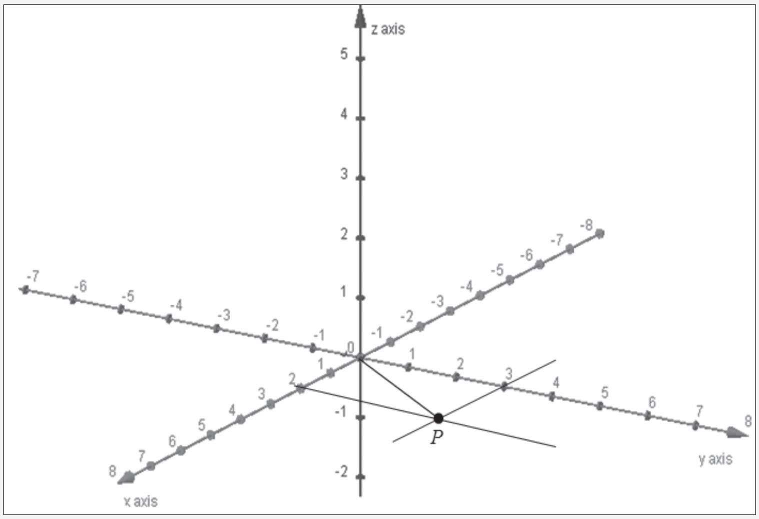
- From *y-coordinate* 3, draw another line parallel to *x-axis*

  See following figure.



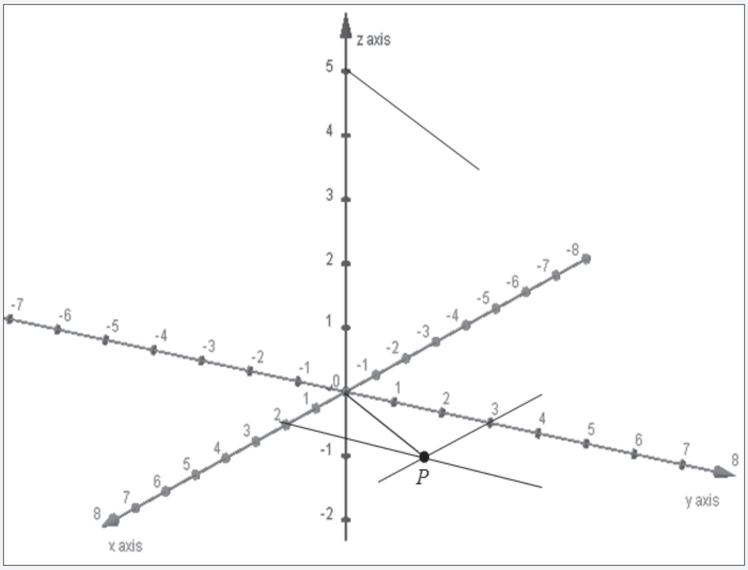
- Now you have a point of intersection of two lines, let call it *P*. From this point, draw another line parallel to *z-axis* and another joining this point and origin of coordinates which is line $0P$

  See following figure.



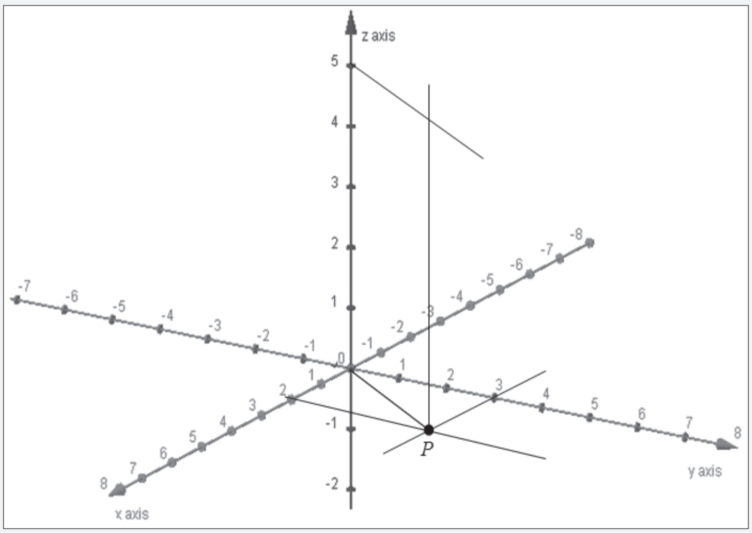
- From *z-coordinate*, 5, draw another line parallel to the line $0P$

See following figure.



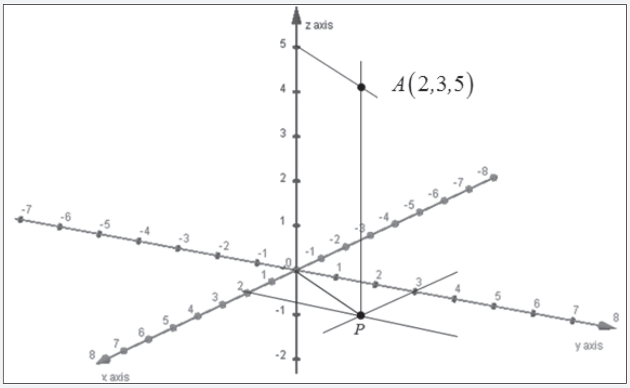
- Draw another line parallel to *z-axis* and passing through point *P*

See following figure.



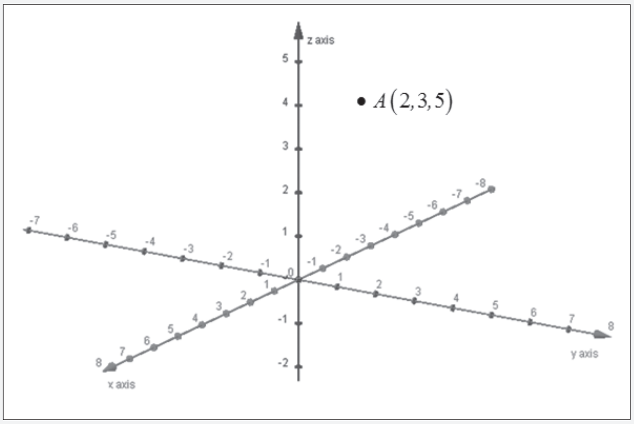
- Now, you have the intersection of the line from *z-coordinate* and the line parallel to *z-axis*. this intersection is the needed point $A(2,3,5)$

See following figure.

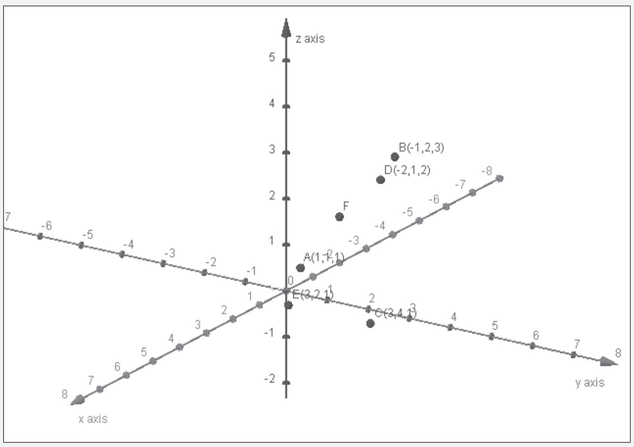


- Clean unwanted points and lines to remain with the needed point.

Finally, we have



Application Activity 8.1



Activity 8.2

Materials

Exercise book, pens, calculator

Answers

1. $\left(\frac{3}{2},\frac{5}{2},3\right)$ 2. $\left(-2,\frac{5}{3},\frac{1}{3}\right)$

Application Activity 8.2

1. a) $\left(0,\frac{7}{2},\frac{11}{2}\right)$ b) $\left(6,\frac{5}{2},-\frac{1}{2}\right)$

   c) $\left(-\frac{7}{2},\frac{11}{2},5\right)$ d) $\left(3,1,\frac{1}{2}\right)$

2. a) $\left(\frac{8}{3},2,\frac{7}{3}\right)$ b) $\left(2,-\frac{1}{3},6\right)$

   c) $\left(-\frac{1}{2},\frac{15}{4},7\right)$ d) $\left(1,0,\frac{9}{4}\right)$

Activity 8.3

Materials

Exercise book, pens

Answers

1.

$$\overrightarrow{AP}=\frac{m}{n}\overrightarrow{PB}$$

$$\Leftrightarrow P-A=\frac{m}{n}(B-P) \quad \Leftrightarrow P-A=\frac{m}{n}B-\frac{m}{n}P$$

$$\Leftrightarrow P+\frac{m}{n}P=\frac{m}{n}B+A \quad \Leftrightarrow nP+mP=mB+nA$$

$$\Leftrightarrow P(n+m)=mB+nA \Rightarrow P=\frac{mB+nA}{n+m}$$

2.

$$\overrightarrow{PA}=\frac{m}{n}\overrightarrow{PB}$$

$$\Leftrightarrow A-P=\frac{m}{n}(B-P) \quad \Leftrightarrow A-P=\frac{m}{n}B-\frac{m}{n}P$$

$$\Leftrightarrow \frac{m}{n}P-P=\frac{m}{n}B-A \quad \Leftrightarrow mP-nP=mB-nA$$

$$\Leftrightarrow P(m-n)=mB-nA \Rightarrow P=\frac{mB-nA}{m-n}$$

Or

$$\overrightarrow{AP}=\frac{m}{n}\overrightarrow{BP}$$

$$\Leftrightarrow P-A=\frac{m}{n}(P-B) \quad \Leftrightarrow P-A=\frac{m}{n}P-\frac{m}{n}B$$

$$\Leftrightarrow P-\frac{m}{n}P=A-\frac{m}{n}B \quad \Leftrightarrow nP-mP=nA-mB$$

$$\Leftrightarrow P(n-m)=nA-mB$$

$$P=\frac{nA-mB}{n-m}$$

$$=\frac{-(mB-nA)}{-(m-n)}$$

$$=\frac{mB-nA}{m-n}$$

Application Activity 8.3

1. Internally: $P=\frac{2B+3A}{5}=\frac{1}{5}(2B+3A)=\frac{1}{5}(14,9,29)$

   Externally, $P=\frac{2B-3A}{-1}=3A-2B=(-2,-3,1)$

2. $\left(\frac{21}{11},\frac{-8}{11},\frac{43}{11}\right)$

3. a) $\left(\frac{13}{5},\frac{8}{5},\frac{14}{5}\right)$ b) (17, 16, -2)

4. a) $\left(\frac{-4}{5},\frac{1}{5},\frac{3}{5}\right)$ b) (-8,17,27)

5. $\frac{1}{2}:1$ or 1:2 6. $x=9,\ z=5$

8.4.2. Straight lines in three dimensions

a) Content summary

Recommended teaching periods: 9 periods

This section looks

a) Equations of line in space.

A line parallel to the vector $\vec{v}=(a,b,c)$ and passing through the point P with position vector $\overrightarrow{0P}=(x_0,y_0,z_0)$ has:

Vector equation

$\overrightarrow{\mathrm{OQ}}=\overrightarrow{\mathrm{OP}}+\mathrm{r}\vec{\mathrm{v}}$ or $(x,y,z)=(x_0,y_0,z_0)+r(a,b,c)$

or $x\vec{i}+y\vec{j}+z\vec{k}=x_0\vec{i}+y_0\vec{j}+z_0\vec{k}+r\left(a\vec{i}+b\vec{j}+c\vec{k}\right)$, r is a parameter.

Parametric equations

$$\begin{cases} x=x_0+ra \\ y=y_0+rb \\ z=z_0+rc \end{cases}$$

Cartesian equations (or symmetric equations)

$$\frac{x-x_0}{a}=\frac{y-y_0}{b}=\frac{z-z_0}{c}$$

Remember this:

If a given line is parallel to the vector $\vec{r}=(a,b,c)$, $\vec{r}$ is called its direction vector.

For the line passing through points $P(x_0,y_0,z_0)$ and $Q(x_1,y_1,z_1)$, with $V(x,y,z)$ any point on the line , has

Vector equation

$\overrightarrow{PV} = r\overrightarrow{PQ}$, where r is a parameter.

Parametric equations:

$$\begin{cases} x = x_0 + r(x_1 - x_0) \\ y = y_0 + r(y_1 - y_0) \\ z = z_0 + r(z_1 - z_0) \end{cases}$$

Symmetric equations:

$$\frac{x - x_0}{x_1 - x_0} = \frac{y - y_0}{y_1 - y_0} = \frac{z - z_0}{z_1 - z_0}$$

Here, the direction vector is $\overrightarrow{PQ}$.

b) Condition of co-linearity of three points

The three points (a_1, a_2, a_3) ; (b_1, b_2, b_3) and (c_1, c_2, c_3) are collinear (means that they lie on the same line) if the following conditions are satisfied

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ 1 & 1 & 1 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \\ 1 & 1 & 1 \end{vmatrix} = 0$$

c) Relation between two lines

Two lines which are not parallel and do not intersect, are called skew lines.

Two lines are parallel when their direction vectors are proportional.

d) Angle between two lines

The angle between two lines is equal to the angle between their direction vectors.

Let $\vec{u}$ and $\vec{v}$ be direction vectors of two lines l_1 and l_2 respectively,

θ angle between l_1 and l_2, thus, $\theta = \arccos\left(\frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\| \cdot \|\vec{v}\|}\right)$

e) Distance from a point to a line

The distance from point $B(b_1,b_2,b_3)$ to the line passing through point $A(a_1,a_2,a_3)$ with direction vector $\vec{u}=(c_1,c_2,c_3)$ is $\frac{\|\overrightarrow{AB}\times\vec{u}\|}{\|\vec{u}\|}$.

Distance between two skew lines

Consider two skew lines $\mathrm{L}_1:\vec{\mathrm{r}}=\vec{\mathrm{a}}+\lambda\vec{\mathrm{u}}$ and $\mathrm{L}_2:\vec{\mathrm{r}}=\vec{\mathrm{b}}+\lambda\vec{\mathrm{v}}$.

The shortest distance between these lines is $d=\frac{\|\overrightarrow{ab}\cdot\vec{u}\times\vec{v}\|}{\|\vec{u}\times\vec{v}\|}$

b) Teaching guidelines

- Organise class into groups. Request each group to have a group leader who will present their findings to the class.
- Ask each group to do activity related to the lesson they are going to learn. The learners may need your assistance to do any activity. Help them to understand the activity. After group discussion, invite some or all groups for presentation of their work.
- After activity presentation, capture the main points from the presentation of the learners and summarise them. Guide the learners through given examples in Learner's book or through your own examples.
- Ask learners what they learned in day lesson to ensure that they understood what they have learned.
- Request learners to do exercises in their respective groups. Request learners to correct exercises on chalkboard and give them individual evaluation. Remember to give them homework.

c) Answers to activities

Activity 8.4

Materials

Exercise book, pens

Answers

1. $(x,y,z)=(x_0,y_0,z_0)+r(a,b,c)$

   $x\vec{i}+y\vec{j}+z\vec{k}=x_0\vec{i}+y_0\vec{j}+z_0\vec{k}+r\left(a\vec{i}+b\vec{j}+c\vec{k}\right)$

   Equate the respective components, there are three equations

   $$\begin{cases} x=x_0+ra \\ y=y_0+rb \\ z=z_0+rc \end{cases}$$

2. Eliminating parameter *r*, we have

   $$\begin{cases} r=\dfrac{x-x_0}{a} \\ r=\dfrac{y-y_0}{b} \\ r=\dfrac{z-z_0}{c} \end{cases} \quad \text{Or} \quad \frac{x-x_0}{a}=\frac{y-y_0}{b}=\frac{z-z_0}{c}$$

Application Activity 8.4

1. $x\,\vec{i}+y\,\vec{j}+z\,\vec{k}=\vec{i}+\vec{j}+\vec{k}+\lambda\left(2\vec{i}+\vec{j}+3\vec{k}\right)$

   $$\begin{cases} x=1+2\lambda \\ y=1+\lambda \\ z=1+3\lambda \end{cases} \qquad \frac{x-1}{2}=\frac{y-1}{1}=\frac{z-1}{3}$$

2. $x\,\vec{i}+y\,\vec{j}+z\,\vec{k}=-2\vec{i}+3\vec{j}+\vec{k}+\lambda\left(2\vec{i}+\vec{j}+3\vec{k}\right)$

$$\begin{cases} x = -2+2\lambda \\ y = 3+\lambda \\ z = 1+3\lambda \end{cases} \qquad \frac{x+2}{2} = \frac{y-3}{1} = \frac{z-1}{3}$$

3. $x\,\vec{i} + y\,\vec{j} + z\,\vec{k} = 9\vec{i} + 3\vec{j} + \lambda\left(\vec{i} + \vec{j} + 6\vec{k}\right)$

$$\begin{cases} x = 9+\lambda \\ y = 3+\lambda \\ z = 6\lambda \end{cases} \qquad \frac{x-9}{1} = \frac{y-3}{1} = \frac{z}{6}$$

4. $x\,\vec{i} + y\,\vec{j} + z\,\vec{k} = 4\vec{i} + 5\vec{j} + 2\vec{k} + \lambda\left(-3\vec{i} + 2\vec{j} + \vec{k}\right)$

$$\begin{cases} x = 4-3\lambda \\ y = 5+2\lambda \\ z = 2+\lambda \end{cases} \qquad \frac{x-4}{-3} = \frac{y-5}{2} = \frac{z-2}{1}$$

Activity 8.5

Materials

Exercise book, pens

Answers

1. The direction vector is

   $\overrightarrow{PQ} = (x_1, y_1, z_1) - (x_0, y_0, z_0) = (x_1 - x_0, y_1 - y_0, z_1 - z_0)$

   Now the vector equation is given by

   $\overrightarrow{PV} = r\overrightarrow{PQ}$ or $\overrightarrow{OV} = \overrightarrow{OP} + r\overrightarrow{PQ}$,

   With $V(x, y, z)$, $0(0,0,0)$ and r is a parameter.

2. Parametric equations:

$$\begin{cases} x = x_0 + r(x_1 - x_0) \\ y = y_0 + r(y_1 - y_0) \\ z = z_0 + r(z_1 - z_0) \end{cases}$$

3. Eliminating the parameter, we have the symmetric equations:

$$\frac{x - x_0}{x_1 - x_0} = \frac{y - y_0}{y_1 - y_0} = \frac{z - z_0}{z_1 - z_0}$$

Application Activity 8.5

1. $x\vec{i}+y\vec{j}+z\vec{k}=2\vec{i}+\vec{j}+4\vec{k}+\lambda\left(\vec{i}-3\vec{k}\right)$

$$\begin{cases} x=2+\lambda \\ y=1 \\ z=4-3\lambda \end{cases}$$

$$\frac{x-2}{1}=\frac{y-1}{0}=\frac{z-4}{-3} \text{ or } \frac{x-2}{1}=\frac{z-4}{-3},\ y=1$$

2. $x\vec{i}+y\vec{j}+z\vec{k}=\vec{i}+\vec{j}+3\vec{k}+\lambda\left(\vec{i}+4\vec{j}+\vec{k}\right)$

$$\begin{cases} x=1+\lambda \\ y=1+4\lambda \\ z=3+\lambda \end{cases} \qquad \frac{x-1}{1}=\frac{y-1}{4}=\frac{z-3}{1}$$

3. $x\ \vec{i}+y\ \vec{j}+z\ \vec{k}=2\vec{i}+\vec{j}+4\vec{k}+\lambda\left(4\vec{i}+2\vec{j}-2\vec{k}\right)$

$$\begin{cases} x=2+4\lambda \\ y=1+2\lambda \\ z=4-2\lambda \end{cases} \qquad \frac{x-2}{4}=\frac{y-1}{2}=\frac{z-4}{-2}$$

4. $x\vec{i}+y\vec{j}+z\vec{k}=\vec{i}+\vec{j}+\vec{k}+\lambda\left(3\vec{i}+4\vec{j}+5\vec{k}\right)$

$$\begin{cases} x=1+3\lambda \\ y=1+4\lambda \\ z=1+5\lambda \end{cases} \qquad \frac{x-1}{3}=\frac{y-1}{4}=\frac{z-1}{5}$$

Activity 8.6

Materials

Exercise book, pens, calculator

Answers

A) Equation of line passing through points $(1,2,3)$ and $(1,-4,3)$ is

$$\begin{vmatrix} x & 1 & 1 \\ y & 2 & -4 \\ 1 & 1 & 1 \end{vmatrix}=\begin{vmatrix} x & 1 & 1 \\ z & 3 & 3 \\ 1 & 1 & 1 \end{vmatrix}=0$$

If point $(-1,0,5)$ lies on this line, then

$$\begin{vmatrix} -1 & 1 & 1 \\ 0 & 2 & -4 \\ 1 & 1 & 1 \end{vmatrix} = \begin{vmatrix} -1 & 1 & 1 \\ 5 & 3 & 3 \\ 1 & 1 & 1 \end{vmatrix} = 0$$

$$\begin{vmatrix} -1 & 1 & 1 \\ 0 & 2 & -4 \\ 1 & 1 & 1 \end{vmatrix} = -2+0-4-2-4-0=-12$$

$$\begin{vmatrix} -1 & 1 & 1 \\ 5 & 3 & 3 \\ 1 & 1 & 1 \end{vmatrix} = -3+5+3+3-3-5=0$$

Since $\begin{vmatrix} -1 & 1 & 1 \\ 0 & 2 & -4 \\ 1 & 1 & 1 \end{vmatrix} = -12 \neq 0$, the given three points do not lie on the same line.

B) Equation of line passing through points $(3,4,7)$ and $(5,-2,1)$ is

$$\begin{vmatrix} x & 3 & 5 \\ y & 4 & -2 \\ 1 & 1 & 1 \end{vmatrix} = \begin{vmatrix} x & 3 & 5 \\ z & 7 & 1 \\ 1 & 1 & 1 \end{vmatrix} = 0$$

If point $(4,1,4)$ lies on this line, then

$$\begin{vmatrix} 4 & 3 & 5 \\ 1 & 4 & -2 \\ 1 & 1 & 1 \end{vmatrix} = \begin{vmatrix} 4 & 3 & 5 \\ 4 & 7 & 1 \\ 1 & 1 & 1 \end{vmatrix} = 0$$

$$\begin{vmatrix} 4 & 3 & 5 \\ 1 & 4 & -2 \\ 1 & 1 & 1 \end{vmatrix} = 16+5-6-20+8-3=0$$

$$\begin{vmatrix} 4 & 3 & 5 \\ 4 & 7 & 1 \\ 1 & 1 & 1 \end{vmatrix} = 28+20+3-35-4-12=0$$

Since $\begin{vmatrix} 4 & 3 & 5 \\ 1 & 4 & -2 \\ 1 & 1 & 1 \end{vmatrix} = \begin{vmatrix} 4 & 3 & 5 \\ 4 & 7 & 1 \\ 1 & 1 & 1 \end{vmatrix} = 0$, the given three points lie on the same line.

C) Equation of line passing through points $(1,9,3)$ and $(1,8,5)$ is

$$\begin{vmatrix} x & 1 & 1 \\ y & 9 & 8 \\ 1 & 1 & 1 \end{vmatrix} = \begin{vmatrix} x & 1 & 1 \\ z & 3 & 5 \\ 1 & 1 & 1 \end{vmatrix} = 0$$

If point $(1,10,1)$ lies on this line, then

$$\begin{vmatrix} 1 & 1 & 1 \\ 10 & 9 & 8 \\ 1 & 1 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 3 & 5 \\ 1 & 1 & 1 \end{vmatrix} = 0$$

$$\begin{vmatrix} 1 & 1 & 1 \\ 1 & 3 & 5 \\ 1 & 1 & 1 \end{vmatrix} = 3+5+1-3-5-1 = 0$$

$$\begin{vmatrix} -1 & 1 & 1 \\ 5 & 3 & 3 \\ 1 & 1 & 1 \end{vmatrix} = -3+5+3+3-3-5 = 0$$

Since $\begin{vmatrix} 1 & 1 & 1 \\ 10 & 9 & 8 \\ 1 & 1 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 3 & 5 \\ 1 & 1 & 1 \end{vmatrix} = 0$, the given three points lie on the same line.

The three points (a_1, a_2, a_3) ; (b_1, b_2, b_3) and (c_1, c_2, c_3) are lie on the same line if the following conditions are satisfied

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ 1 & 1 & 1 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \\ 1 & 1 & 1 \end{vmatrix} = 0$$

Application Activity 8.6

1. Not collinear
2. $x=1+2t,\ y=1+t,\ z=1+3t$
3. $a=7$

Activity 8.7

Materials

Exercise book, pens, calculator

Answers

1. $L_1: x=2-\lambda,\ y=2+2\lambda,\ z=1+3\lambda$,
   $L_2: x=1-\mu,\ y=1+2\mu,\ z=1+3\mu$

   L_1 and L_2 have the same direction vectors, thus L_1 and L_2 are parallel.

   Let us check if they are coincident:

   $$\begin{cases} 2-\lambda=1-\mu \\ 2+2\lambda=1+2\mu \\ 1+3\lambda=1+3\mu \end{cases}$$

   $$\Rightarrow \begin{cases} \lambda=1+\mu & (1) \\ 2+2\lambda=1+2\mu & (2) \\ 1+3\lambda=1+3\mu \Rightarrow \lambda=\mu & (3) \end{cases}$$

   From (1) and (3), we find that there is no solution.

   Therefore, L_1 and L_2 are different.

2. $L_1: x=\lambda,\ y=-2+2\lambda,\ z=5-\lambda$,
   $L_2: x=1-\mu,\ y=-3-3\mu,\ z=4+\mu$

   Direction vectors of L_1 and L_2 are not proportional, thus L_1 and L_2 are not parallel.

   Let us find their intersection:

   $$\begin{cases} \lambda=1-\mu \\ -2+2\lambda=-3-3\mu \\ 5-\lambda=4+\mu \end{cases}$$

$$\Rightarrow\begin{cases}\lambda=1-\mu\\-2+2(1-\mu)=-3-3\mu\\5-(1-\mu)=4+\mu\end{cases}\Leftrightarrow\begin{cases}\lambda=1-\mu\\-2+2-2\mu=-3-3\mu\\5-1+\mu=4+\mu\end{cases}$$

$$\Leftrightarrow\begin{cases}\lambda=1-\mu\\-2+2-2\mu=-3-3\mu\Rightarrow\mu=-3\\5-1+\mu=4+\mu\Rightarrow 0\mu=0\end{cases}$$

$$\begin{cases}\mu=-3\\\lambda=4\end{cases}$$

For $\lambda=4$, $x=4,\ y=6,\ z=1$

Intersection point is $(4,6,1)$

3. $L_1: x=5+2\lambda,\ y=4+\lambda,\ z=5+\lambda$,
$L_2: x=1+2\mu,\ y=2+\mu,\ z=3+\mu$

Direction vectors of L_1 and L_2 are proportional, thus L_1 and L_2 are parallel.

Let us check if they are coincident:

$$\begin{cases}5+2\lambda=1+2\mu\\4+\lambda=2+\mu\\5+\lambda=3+\mu\end{cases}\Rightarrow\begin{cases}\lambda=-2+\mu\\4+2(-2+\mu)=2+\mu\\5+(-2+\mu)=3+\mu\end{cases}$$

$$\Leftrightarrow\begin{cases}\lambda=2+\mu\\4-4+2\mu=2+\mu\\5-2+\mu=3+\mu\end{cases}\Leftrightarrow\begin{cases}\lambda=2+\mu\\4-4+2\mu=2+\mu\Rightarrow\mu=2\\5-2+\mu=3+\mu\Rightarrow 0\mu=0\end{cases}$$

From the values of there are many solutions. The two lines coincide.

4. $L_1:\begin{cases}x=2+8s\\y=4-3s\\z=5+s\end{cases}\quad L_2:\begin{cases}x=1+4t\\y=5-4t\\z=-1+5t\end{cases}$

Direction vectors of L_1 and L_2 are not proportional, thus L_1 and L_2 are not parallel.

Let us find their intersection:

$$\begin{cases} 2+8s=1+4t \\ 4-3s=5-4t \\ 5+s=-1+5t \end{cases} \Rightarrow \begin{cases} s=\dfrac{-1+4t}{8} \\ 4-3\dfrac{-1+4t}{8}=5-4t \\ 5+\dfrac{-1+4t}{8}=-1+5t \end{cases}$$

$$\Leftrightarrow \begin{cases} s=\dfrac{-1+4t}{8} \\ 32+3-12t=40-32t \Rightarrow 20t=5 \\ 40-1+4t=-8+40t \Rightarrow 36t=47 \end{cases}$$

There is no solution for these simultaneous equations.

Therefore L_1 and L_2 do not intersect.

Application Activity 8.7

1. Skew	2. Intersect at (2,1,-7)
3. Skew	4. Skew
5. Skew	6. Intersect at (1,1,1)

Activity 8.8

Materials

Exercise book, pens, calculator

Answers

1) 39^0	2) 39^0, 141^0

Application Activity 8.8

1) $\dfrac{\pi}{4}$	2) 1.38 radians	3) 0.82 radians	4) 79^0
5) 80.41^0	6) 48.70^0	7) 68.48^0	

Activity 8.9

Materials

Exercise book, pens

Answers

1. $\sin\theta = \dfrac{d}{\left\|\overrightarrow{AB}\right\|} \Rightarrow d = \left\|\overrightarrow{AB}\right\|\sin\theta$

2. $d = \dfrac{\left\|\overrightarrow{AB}\right\|\left\|\vec{u}\right\|\sin\theta}{\left\|\vec{u}\right\|}$ and $d = \dfrac{\left\|\overrightarrow{AB}\times\vec{u}\right\|}{\left\|\vec{u}\right\|}$

Application Activity 8.9

1) $2\sqrt{30}$ unit of length

2) 0 unit of length

3) $\dfrac{9\sqrt{42}}{7}$ unit of length

4) $\dfrac{3\sqrt{10}}{2}$ unit of length

5) $\dfrac{\sqrt{2}}{3}$ unit of length

Activity 8.10

Materials

Exercise book, pens, calculator

Answers

1. Direction vectors of the lines are $(3,4,-2)$ and $(6,-4,-1)$

   Since the two direction vectors are not proportional, the lines are not parallel.

   Check if there is a common point:

$$\begin{cases} -7+3t = 21+6s \\ -4+4t = -5+4s \\ -3-2t = 2+s \end{cases} \qquad \begin{cases} t = \dfrac{28+6s}{3} \\ -4+4\left(\dfrac{28+6s}{3}\right) = -5+4s \\ -3-2\left(\dfrac{28+6s}{3}\right) = 2+s \end{cases}$$

$$\begin{cases} t = \dfrac{28+6s}{3} \\ -12+112+24s = -15+12s \\ -9-56-12s = 6+3s \end{cases} \quad \begin{cases} 12s = -115 \\ -15s = 71 \end{cases} \text{ impossible.}$$

The lines are skew.

2. Let (a,b,c) be the perpendicular vector to both lines

$$\begin{cases} 3a+4b-2c=0 \\ 6a-4b-c=0 \Rightarrow c=6a-4b \end{cases}$$

$$3a+4b-12a+8b=0$$

$$\Leftrightarrow 12b=9a \Rightarrow b=\frac{3a}{4}$$

Let $a=1$, $b=\frac{3}{4}$ and $c=6-3=3$.

The perpendicular vector is $\left(1,\frac{3}{4},3\right)$ or $(4,3,12)$

The normalized vector of this vector is

$$\frac{1}{\sqrt{169}}(4,3,12)=\left(\frac{4}{13},\frac{3}{13},\frac{12}{13}\right)$$

3. Point on first line is $(-7,-4,-3)$, point on second line is $(21,-5,2)$. Vector joining these points $(28,-1,5)$

4. The needed scalar product is

$$\left(\frac{4}{13},\frac{3}{13},\frac{12}{13}\right)\cdot(28,-1,5)=13$$

Application Activity 8.10

1. $\frac{95\sqrt{1817}}{1817}$ unit of length 2. 0 unit of length

3. $\frac{68\sqrt{230}}{115}$ unit of length

8.4.3. Planes in three dimensions

a) Content summary

Recommended teaching periods: 9 periods

This section looks at:

1) Equations of plane in space.

Equations of plane containing point $P(x_0, y_0, z_0)$, with direction vector $\vec{u}=(x_1, y_1, z_1)$, $\vec{v}=(x_2, y_2, z_3)$ and $X(x, y, z)$ any point on this plane are the following:

Vector equation

$\overrightarrow{PX} = r\vec{u} + s\vec{v}$ where r, s are parameters.

Parametric equations

$$\begin{cases} x = x_0 + rx_1 + sx_2 \\ y = y_0 + ry_1 + sy_2 \\ z = z_0 + rz_1 + sz_2 \end{cases}$$

Cartesian equation

$$\begin{vmatrix} x-x_0 & x_1 & x_2 \\ y-y_0 & y_1 & y_2 \\ z-z_0 & z_1 & z_2 \end{vmatrix} = 0$$

We can also find the Cartesian equation by the following determinant:

$$\begin{vmatrix} x & x_0 & x_1 & x_2 \\ y & y_0 & y_1 & y_2 \\ z & z_0 & z_1 & z_2 \\ 1 & 1 & 0 & 0 \end{vmatrix} = 0$$

The Cartesian equation of plane can be written in the form $a(x-x_0)+b(y-y_0)+c(z-z_0)=0$.

If $P(x_0, y_0, z_0)$ and $Q(x_1, y_1, z_1)$ are two points of a plane whose direction vector are $\vec{v} = (x_2, y_2, z_2)$ and $X(x, y, z)$ any point on this plane , its equations are given as follows:

Vector equation

$\overrightarrow{PX} = r\overrightarrow{PQ} + s\vec{v}$ where r and s are parameters

Parametric equations

$$\begin{cases} x = x_0 + r(x_1 - x_0) + sx_2 \\ y = y_0 + r(y_1 - y_0) + sy_2 \\ z = z_0 + r(z_1 - z_0) + sz_2 \end{cases}$$

Cartesian equation

$$\begin{vmatrix} x - x_0 & x_1 - x_0 & x_2 \\ y - y_0 & y_1 - y_0 & y_2 \\ z - z_0 & z_1 - z_0 & z_2 \end{vmatrix} = 0$$

Or we can use the determinant

$$\begin{vmatrix} x & x_0 & x_1 & x_2 \\ y & y_0 & y_1 & y_2 \\ z & z_0 & z_1 & z_2 \\ 1 & 1 & 1 & 0 \end{vmatrix} = 0$$

2) Condition of co-planarity of four points

Four points

$(a_1,\ a_2,\ a_3)$; $(b_1,\ b_2,\ b_3)$;$(c_1,\ c_2,\ c_3)$ *and* $(d_1,\ d_2,\ d_3)$

are coplanar (means that they lie on the same plane) if the following condition is satisfied.

$$\begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ 1 & 1 & 1 & 1 \end{vmatrix} = 0 \text{ or } \begin{vmatrix} a_1 - d_1 & b_1 - d_1 & c_1 - d_1 \\ a_2 - d_2 & b_2 - d_2 & c_2 - d_2 \\ a_3 - d_3 & b_3 - d_3 & c_3 - d_3 \end{vmatrix} = 0$$

3) Position of a line and a plane

A line L is perpendicular to plane α if and only if each direction vector of L is perpendicular to each direction vector of α.

A line and plane are parallel if the direction vector of the line is perpendicular to the normal vector of the plane:

two possibilities occur:

a line and plane are strictly parallel or a line lies in the plane

4) Angle between lines and planes

Angle between a line and plane

The angle which line L makes with plane □ is defined to be the angle θ which is complement of angle between the direction vector of L and normal to the plane □.

Thus, $\theta = 90^0 - \arccos\left(\frac{\vec{n}.\vec{u}}{\left\|\vec{n}\right\|.\left\|\vec{u}\right\|}\right)$ or $\theta = \arcsin\left(\frac{\vec{n}\cdot\vec{u}}{\left\|\vec{n}\right\|\cdot\left\|\vec{u}\right\|}\right)$

Angle between two planes

The angle θ between planes α and β is defined to be an angle between their normal vectors $\overrightarrow{n_1}$ and $\overrightarrow{n_2}$ respectively.

Thus, $\theta = \arccos\left(\frac{\overrightarrow{n_1}\cdot\overrightarrow{n_2}}{\left|\overrightarrow{n_1}\right|\cdot\left|\overrightarrow{n_2}\right|}\right)$

5) Distance between lines and planes

Distance from a point to the plane

The distance from point $B(b_1,b_2,b_3)$ to the plane $\alpha \equiv ax+by+cz=d$ is

$$d(B,\alpha)=\frac{\left|ab_1+bb_2+cb_3-d\right|}{\sqrt{a^2+b^2+c^2}}$$

Distance between two planes

When calculating the distance between two planes, first check if the planes are parallel. If they are not, they intersect and the distance is zero. If they are parallel, find an arbitrary point in one of the planes and calculate its distance to the other plane.

Note that if two planes coincide (identical), the shortest distance is zero.

Shortest distance between a line and a plane

When calculating the distance between a line and a plane, first check if the line is parallel to the plane. If not, they intersect and the distance is zero. If they are parallel, find arbitrary point on the line and calculate its distance to the plane.

6) Projection of a line on the plane

To find the projection of the line AB on the plane □, we need a plane β containing the given line AB and perpendicular to the given plane □. The equation of the plane β and the plane □ taken together are the equations of the projection.

7) Position of planes

Position of two planes

Consider two planes

$$\alpha \equiv a_1x + b_1y + c_1z = d_1$$
$$\beta \equiv a_2x + b_2y + c_2z = d_2$$

$\alpha \parallel \beta$ if their normal vectors are proportional i.e.

$(a_1, b_1, c_1) = k(a_2, b_2, c_2), \;\; k \in \mathbb{R}_0 \;\; \Rightarrow$

The two planes coincide if

$(a_1, b_1, c_1, d_1) = k(a_2, b_2, c_2, d_2), \;\; k \in \mathbb{R}_0$

That is, $\alpha = k\beta, k \in \mathbb{R}_0$. So $S = \alpha$ or $S = \beta$

The two planes are parallel and distinct if

$(a_1, b_1, c_1, d_1) \neq k(a_2, b_2, c_2, d_2), \;\; k \in \mathbb{R}_0$

and hence no intersection.

The two planes intersect , if their normal vectors are not proportional, i.e. $(a_1,b_1,c_1)\neq k(a_2,b_2,c_2),\ \ k\in\mathbb{R}_0\ \Rightarrow \alpha \nparallel \beta$

The planes intersection is a line defined by the equations of the two planes taking together.

$$\begin{cases} a_1x+b_1y+c_1z=d_1 \\ a_2x+b_2y+c_2z=d_2 \end{cases}$$

General equation of a line

The general equation of a straight line in space is

$$\begin{cases} a_1x+b_1y+c_1z=d_1 \\ a_2x+b_2y+c_2z=d_2 \end{cases}$$

The direction vector of this line is

$$\left(\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}, -\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}, \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}\right)$$

Or to find the direction vector of the line, we can equate the right hand sides of the general equations to zero.

i.e,

$$\begin{cases} a_1x+b_1y+c_1z=0 \\ a_2x+b_2y+c_2z=0 \end{cases}$$

Position of three planes

Consider three planes

$$\alpha \equiv a_1x+b_1y+c_1z=d_1$$
$$\beta \equiv a_2x+b_2y+c_2z=d_2$$
$$\gamma \equiv a_3x+b_3y+c_3z=d_3$$

There are three possible cases:

1. These planes are parallel if and only if the left hand sides of three equations are proportional.

   That is $(a_1,b_1,c_1)=k(a_2,b_2,c_2)$ and $(a_1,b_1,c_1)=m(a_3,b_3,c_3)$

   In this case, the plane may be identical or distinct. We have two cases:

If $(a_1, b_1, c_1, d_1) = k(a_2, b_2, c_2, d_2)$, $(a_1, b_1, c_1, d_1) = m(a_3, b_3, c_3, d_3)$ and $(a_2, b_2, c_2, d_2) = n(a_3, b_3, c_3, d_3)$

The three equations are proportional and hence the three planes are coincident (identical), meaning that $\alpha \equiv \beta \equiv \gamma$

.

If $(a_1, b_1, c_1, d_1) \neq k(a_2, b_2, c_2, d_2)$ or

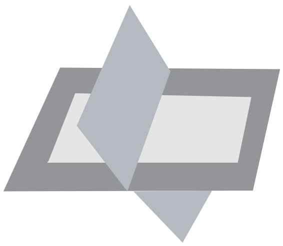
$(a_1, b_1, c_1, d_1) \neq m(a_3, b_3, c_3, d_3)$ or $(a_2, b_2, c_2, d_2) \neq n(a_3, b_3, c_3, d_3)$

There are two equations that are not proportional but with proportional left hand sides and hence two planes are parallel and distinct and the third may be coincident to one of the other two or distinct to another. Then there is no intersection.

2. Two of them are parallel and the third is secant if and only if only two equations have the left hand sides that are proportional.

   In this case, there are two planes that are parallel and the third is secant.

   If only two equations are proportional, two planes are coincident and the third is secant to them. Hence, the intersection is a straight line.



If the left hand sides of only two equations are proportional, two planes are parallel and distinct. Hence, no intersection.

3. No plane is parallel to another if and only if no left hand side of any equation is proportional to another.

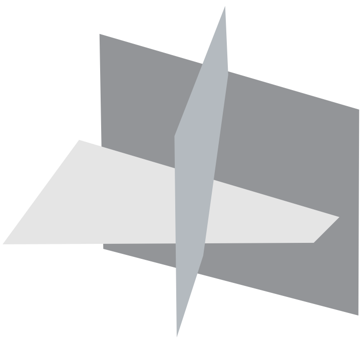
   a) There is one left hand side which is a linear combination of two others; in this case, there is a line of intersection of two planes which is parallel to the third.

      (i) If the corresponding equation is not a linear combination of two others, the line of intersection of two planes is strictly parallel to the third plane and hence there is no intersection between three planes.

      (ii) If the corresponding equation is a linear combination of two others, the line is included in the third plane and hence this line is the intersection for three planes.

To find equation of the line of intersection, we proceed in the same way as for the case of two planes by taking any two equations from the three given equations of planes.

b) No left hand side is a linear combination of others, meaning that the three equations are linearly independent; in this case, the line of intersection of two planes pierces the third plane and hence there is a point of intersection between three planes.



To find this point, we solve simultaneously the system

$$\begin{cases} a_1x+b_1y+c_1z=d_1 \\ a_2x+b_2y+c_2z=d_2 \\ a_3x+b_3y+c_3z=d_3 \end{cases}$$

b) Teaching guidelines

- Organise class into groups. Request each group to have a group leader who will present their findings to the class.
- Ask each group to do activity related to the lesson they are going to learn. The learners may need your assistance to do any activity. Help them to understand the activity. After group discussion, invite some or all groups for presentation of their work.
- After activity presentation, capture the main points from the presentation of the learners and summarise them. Guide the learners through given examples in Learner's book or through your own examples.
- Ask learners what they learned in day lesson to ensure that they understood what they have learned.
- Request learners to do exercises in their respective groups. Request learners to correct exercises on chalkboard and give them individual evaluation. Remember to give them homework.

c) Answers to activities

Activity 8.11

Materials

Exercise book, pens

Answers

1. Vector equation is

   $\overrightarrow{PV} = r\vec{u} + s\vec{v}$ or $\overrightarrow{OV} = \overrightarrow{OP} + r\vec{u} + s\vec{v}$. With $0(0,0,0)$ and r, s are parameters.

2. Parametric equations:

   From vector equation we have,

   $(x - x_0, y - y_0, z - z_0) = r(x_1, y_1, z_1) + s(x_2, y_2, z_2)$

Or

$(x,y,z)=(x_0,y_0,y_0)+r(x_1,x_1,x_1)+s(x_2,y_2,z_2)$.

Thus the parametric equations are

$$\begin{cases} x = x_0 + rx_1 + sx_2 \\ y = y_0 + ry_1 + sy_2 \\ z = z_0 + rz_1 + sz_2 \end{cases}$$

3. Cartesian equation

$$(x-x_0)\begin{vmatrix} y_1 & y_2 \\ z_1 & z_2 \end{vmatrix} - (y-y_0)\begin{vmatrix} x_1 & x_2 \\ z_1 & z_2 \end{vmatrix} + (z-z_0)\begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix} = 0$$

Let $\begin{vmatrix} y_1 & y_2 \\ z_1 & z_2 \end{vmatrix} = a$, $-\begin{vmatrix} x_1 & x_2 \\ z_1 & z_2 \end{vmatrix} = b$ and $\begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix} = c$

We have:

$$a(x-x_0)+b(y-y_0)+c(z-z_0)=0$$

Application Activity 8.11

1. Vector equation: $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \\ 1 \end{pmatrix} + r\begin{pmatrix} 1 \\ 3 \\ -1 \end{pmatrix} + s\begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$

   Parametric equations: $\begin{cases} x = 2 + r + 2s \\ y = 4 + 3r + s \\ z = 1 - r + 3s \end{cases}$

   Cartesian equation $10x - 5y - 5z = -5$

2. Vector equation: $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + r\begin{pmatrix} 4 \\ -2 \\ 1 \end{pmatrix} + s\begin{pmatrix} -2 \\ 4 \\ 3 \end{pmatrix}$

   Parametric equations: $\begin{cases} x = 1 + 4r - 2s \\ y = 1 - 2r + 4s \\ z = 1 + r + 3s \end{cases}$

   Cartesian equation $-10x - 14y + 12z = -12$

3. Vector equation: $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 6 \\ 0 \end{pmatrix} + r\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + s\begin{pmatrix} 5 \\ 1 \\ 7 \end{pmatrix}$

Parametric equations: $\begin{cases} x = 3 + r + 5s \\ y = 6 + s \\ z = r + 7s \end{cases}$

Cartesian equation $-x - 2y + z = -15$

4. Vector equation: $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 \\ 3 \\ 8 \end{pmatrix} + r\begin{pmatrix} -4 \\ 1 \\ 1 \end{pmatrix} + s\begin{pmatrix} -2 \\ 8 \\ 6 \end{pmatrix}$

Parametric equations: $\begin{cases} x = 4 - 4r - 2s \\ y = 3 + r + 8s \\ z = 8 + r + 6s \end{cases}$

Cartesian equation $-2x + 22y - 30z = -182$

Activity 8.12

Materials

Exercise book, pens

Answers

1. Vector equation is

$\overrightarrow{PX} = r\overrightarrow{PQ} + s\vec{v}$ or $\overrightarrow{OX} = \overrightarrow{OP} + r\overrightarrow{PQ} + s\vec{v}$, with $0(0,0,0)$, r and s are parameters

2. Parametric equations

$$\begin{cases} x = x_0 + r(x_1 - x_0) + sx_2 \\ y = y_0 + r(y_1 - y_0) + sy_2 \\ z = z_0 + r(z_1 - z_0) + sz_2 \end{cases}$$

3. Cartesian equation:

Eliminate the parameters in parametric equations or find the following determinant

$$\begin{vmatrix} x-x_0 & x_1-x_0 & x_2 \\ y-y_0 & y_1-y_0 & y_2 \\ z-z_0 & z_1-z_0 & z_2 \end{vmatrix} = 0$$

$$(x-x_0)\begin{vmatrix} y_1-y_0 & y_2 \\ z_1-z_0 & z_2 \end{vmatrix} - (y-y_0)\begin{vmatrix} x_1-x_0 & x_2 \\ z_1-z_0 & z_2 \end{vmatrix} + (z-z_0)\begin{vmatrix} x_1-x_0 & x_2 \\ y_1-y_0 & y_2 \end{vmatrix}$$

Application Activity 8.12

1. Vector equation: $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \\ 1 \end{pmatrix} + r\begin{pmatrix} 0 \\ -3 \\ 2 \end{pmatrix} + s\begin{pmatrix} 1 \\ 3 \\ -1 \end{pmatrix}$

   Parametric equations: $\begin{cases} x = 2+s \\ y = 4-3r+3s \\ z = 1+2r-s \end{cases}$

   Cartesian equation $-3x+2y+3z=5$

2. Vector equation: $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} + r\begin{pmatrix} 0 \\ 0 \\ 4 \end{pmatrix} + s\begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}$

   Parametric equations: $\begin{cases} x = 2-s \\ y = 1+2s \\ z = -1+4r+s \end{cases}$

   Cartesian equation $-8x-4y=-20$

3. Vector equation: $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + r\begin{pmatrix} -3 \\ 3 \\ 2 \end{pmatrix} + s\begin{pmatrix} 4 \\ -2 \\ 1 \end{pmatrix}$

   Parametric equations: $\begin{cases} x = 1-3r+4s \\ y = 1+3r-2s \\ z = 1+2r+s \end{cases}$

   Cartesian equation $7x+11y-6z=12$

4. Vector equation: $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 6 \\ 0 \end{pmatrix} + r\begin{pmatrix} 2 \\ -5 \\ 7 \end{pmatrix} + s\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$

Parametric equations: $\begin{cases} x = 3 + 2r + s \\ y = 6 - 5r \\ z = 7r + s \end{cases}$

Cartesian equation $-5x + 5y + 5z = 15$

Activity 8.13

Materials

Exercise book, pens

Answers

1. Vector equation is

$\overrightarrow{PX} = r\overrightarrow{PQ} + s\overrightarrow{PN}$ or $\overrightarrow{0X} = \overrightarrow{0P} + r\overrightarrow{PQ} + s\overrightarrow{PN}$

2. Parametric equations

$$\begin{cases} x = x_0 + r(x_1 - x_0) + s(x_2 - x_0) \\ y = y_0 + r(y_1 - y_0) + s(y_2 - y_0) \\ z = z_0 + r(z_1 - z_0) + s(z_2 - z_0) \end{cases}$$

3. Cartesian equation:

$$(x - x_0)\begin{vmatrix} y_1 - y_0 & y_2 - y_0 \\ z_1 - z_0 & z_2 - z_0 \end{vmatrix} - (y - y_0)\begin{vmatrix} x_1 - x_0 & x_2 - x_0 \\ z_1 - z_0 & z_2 - z_0 \end{vmatrix}.$$

$$+(z - z_0)\begin{vmatrix} x_1 - x_0 & x_2 - x_0 \\ y_1 - y_0 & y_2 - y_0 \end{vmatrix}$$

Application Activity 8.13

1. Vector equation: $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \\ 1 \end{pmatrix} + r\begin{pmatrix} -1 \\ -1 \\ -2 \end{pmatrix} + s\begin{pmatrix} 0 \\ -3 \\ 2 \end{pmatrix}$

Parametric equations: $\begin{cases} x = 2 - r \\ y = 4 - r - 3s \\ z = r - r + 2s \end{cases}$

Cartesian equation: $-8x + 2y + 3z = -5$

2. Vector equation: $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + r\begin{pmatrix} 3 \\ -3 \\ 0 \end{pmatrix} + s\begin{pmatrix} -3 \\ 3 \\ 2 \end{pmatrix}$

Parametric equations: $\begin{cases} x = 1 + 3r - 3s \\ y = 1 - 3r + 3s \\ z = 1 + 2s \end{cases}$

Cartesian equation: $-x - y = -2$

3. Vector equation: $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 6 \\ 0 \end{pmatrix} + r\begin{pmatrix} -2 \\ -6 \\ 1 \end{pmatrix} + s\begin{pmatrix} 2 \\ -5 \\ 7 \end{pmatrix}$

Parametric equations: $\begin{cases} x = 3 - 2r + 2s \\ y = 6 - 6r - 5s \\ z = r + 7s \end{cases}$

Cartesian equation: $-37x + 16y + 22z = -15$

4. Vector equation: $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 \\ 3 \\ 8 \end{pmatrix} + r\begin{pmatrix} -8 \\ -2 \\ -7 \end{pmatrix} + s\begin{pmatrix} -6 \\ 5 \\ -2 \end{pmatrix}$

Parametric equations: $\begin{cases} x = 4 - 8r - 6s \\ y = 3 - 2r + 5s \\ z = 8 - 7r - 2s \end{cases}$

Cartesian equation: $3x + 2y - 4z = -14$

Activity 8.14

Materials

Exercise book, pens, calculator

Answers

1. Equation of plane passing through points

$(1,2,-1),(2,3,1),(3,-1,0)$

$$\begin{vmatrix} x & 1 & 2 & 3 \\ y & 2 & 3 & -1 \\ z & -1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{vmatrix} = 0$$

The fourth point $(1,2,1)$ lies on this plane if

$$\begin{vmatrix} 1 & 1 & 2 & 3 \\ 2 & 2 & 3 & -1 \\ 1 & -1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{vmatrix} = 0$$

$$\Leftrightarrow \begin{vmatrix} 2 & 3 & -1 \\ -1 & 1 & 0 \\ 1 & 1 & 1 \end{vmatrix} - 2\begin{vmatrix} 1 & 2 & 3 \\ -1 & 1 & 0 \\ 1 & 1 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 2 & 3 \\ 2 & 3 & -1 \\ 1 & 1 & 1 \end{vmatrix} - \begin{vmatrix} 1 & 2 & 3 \\ 2 & 3 & -1 \\ -1 & 1 & 0 \end{vmatrix} = 0$$

$$\Leftrightarrow 7+6-5-18=-10$$

Thus, the four points do not lie on the same plane.

2. Equation of plane passing through points

$(-2,1,1),(0,2,3),(1,0,-1)$

$$\begin{vmatrix} x & -2 & 0 & 1 \\ y & 1 & 2 & 0 \\ z & 1 & 3 & -1 \\ 1 & 1 & 1 & 1 \end{vmatrix} = 0$$

The fourth point $(2,1,-1)$ lies on this plane if

$$\begin{vmatrix} 2 & -2 & 0 & 1 \\ 1 & 1 & 2 & 0 \\ -1 & 1 & 3 & -1 \\ 1 & 1 & 1 & 1 \end{vmatrix} = 0$$

$$\Leftrightarrow 2\begin{vmatrix} 1 & 2 & 0 \\ 1 & 3 & -1 \\ 1 & 1 & 1 \end{vmatrix} - \begin{vmatrix} -2 & 0 & 1 \\ 1 & 3 & -1 \\ 1 & 1 & 1 \end{vmatrix} - \begin{vmatrix} -2 & 0 & 1 \\ 1 & 2 & 0 \\ 1 & 1 & 1 \end{vmatrix} - \begin{vmatrix} -2 & 0 & 1 \\ 1 & 2 & 0 \\ 1 & 3 & -1 \end{vmatrix} = 0$$

$$\Leftrightarrow 0+10+5-5=10$$

Thus, the four points do not lie on the same plane.

3. Equation of plane passing through points $(1,0,-1),(0,2,3),(-2,1,1)$

$$\begin{vmatrix} x & 1 & 0 & -2 \\ y & 0 & 2 & 1 \\ z & -1 & 3 & 1 \\ 1 & 1 & 1 & 1 \end{vmatrix} = 0$$

The fourth point $(4,2,3)$ lies on this plane if

$$\begin{vmatrix} 4 & 1 & 0 & -2 \\ 2 & 0 & 2 & 1 \\ 3 & -1 & 3 & 1 \\ 1 & 1 & 1 & 1 \end{vmatrix} = 0$$

$$\Leftrightarrow 4\begin{vmatrix} 0 & 2 & 1 \\ -1 & 3 & 1 \\ 1 & 1 & 1 \end{vmatrix} - 2\begin{vmatrix} 1 & 0 & -2 \\ -1 & 3 & 1 \\ 1 & 1 & 1 \end{vmatrix} + 3\begin{vmatrix} 1 & 0 & -2 \\ 0 & 2 & 1 \\ 1 & 1 & 1 \end{vmatrix}$$

$$-\begin{vmatrix} 1 & 0 & -2 \\ 0 & 2 & 1 \\ -1 & 3 & 1 \end{vmatrix} = 0-20+15+5=0$$

Thus, the four points do lie on the same plane.

The four points

(a_1, a_2, a_3) ; (b_1, b_2, b_3) ;(c_1, c_2, c_3) *and* (d_1, d_2, d_3) lie on the same plane if the following condition is satisfied.

$$\begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ 1 & 1 & 1 & 1 \end{vmatrix} = 0$$

Application Activity 8.14

1. Not coplanar
2. $x = 4$
3. $a + b + c = 2$
4. $a = -1,\ x - 4y + 3z - 2 = 0$

Activity 8.15

Materials

Exercise book, pens

Answers

1. a) $\overrightarrow{AX} = (x - a_1, y - a_2, z - a_3)$

   b) $a(x - a_1) + b(y - a_2) + c(z - a_3) = 0$

   Expanding: $ax - aa_1 + by - ba_2 + cz - ca_3 = 0$ or

   $ax + by + cz = aa_1 + ba_2 + ca_3$.

   This is the equation of plane.

2. a) The line pierces the plane

   b) The line is parallel to the plane and lies in the plane

Application Activity 8.15

1. $3x - 2y - z = -3$
2. $x + 3y + 4z = 34$
3. Not parallel
4. $\begin{cases} x = 5 - 2t \\ y = 5t \\ z = -2 + 11t \end{cases}$
5. $(-1, 0, 0)$

6.	The given line lies in the given plane	
7.	$\left(\frac{3}{2},-\frac{3}{2},\frac{1}{2}\right)$	8. $(1,1,0)$
9.	The given line lies in the given plane	

Activity 8.16

Materials

Exercise book, pens, calculator

Answers

1) 77.3^0	2) 35.6^0

Application Activity 8.16

1) $\frac{\pi}{4}$	2) 1.38 radians	3) 0.82 radians
4) 35^0	5) 79^0	6) 45^0
7) 67.09^0	8) 30^0	

Activity 8.17

Materials

Exercise book, pens

Answers

Then the normalized normal vector $\vec{e}$ is

$$\vec{e}=\left(\frac{a}{\sqrt{a^2+b^2+c^2}},\frac{b}{\sqrt{a^2+b^2+c^2}},\frac{c}{\sqrt{a^2+b^2+c^2}}\right)$$

$$\overrightarrow{AB}=\left(b_1-a_1,b_2-a_2,b_3-a_3\right)$$

$$\left|\vec{e}\cdot\overrightarrow{AB}\right|=\left|\frac{a\left(b_1-a_1\right)}{\sqrt{a^2+b^2+c^2}}+\frac{b\left(b_2-a_2\right)}{\sqrt{a^2+b^2+c^2}}+\frac{c\left(b_3-a_3\right)}{\sqrt{a^2+b^2+c^2}}\right|$$

$$\Leftrightarrow\left|\vec{e}\cdot\overrightarrow{AB}\right|=\frac{\left|a\left(b_1-a_1\right)+b\left(b_2-a_2\right)+c\left(b_3-a_3\right)\right|}{\sqrt{a^2+b^2+c^2}}$$

$$\Leftrightarrow\left|\vec{e}\cdot\overrightarrow{AB}\right|=\frac{\left|ab_1-aa_1+bb_2-ba_2+cb_3-ca_3\right|}{\sqrt{a^2+b^2+c^2}}$$

$$\Leftrightarrow\left|\vec{e}\cdot\overrightarrow{AB}\right|=\frac{\left|ab_1+bb_2+cb_3-\left(aa_1+ba_2+ca_3\right)\right|}{\sqrt{a^2+b^2+c^2}}$$

Since $A\in\alpha$, $aa_1+ba_2+ca_3=d$, we have

$$\left|\vec{e}\cdot\overrightarrow{AB}\right|=\frac{\left|ab_1+bb_2+cb_3-d\right|}{\sqrt{a^2+b^2+c^2}}$$

Application Activity 8.17

1. 3 unit of length
2. $\frac{19}{5}$ unit of length
3. $\frac{5}{3}$ unit of length
4. $\frac{9\sqrt{41}}{41}$ unit of length
5. $\frac{5\sqrt{6}}{18}$ unit of length

Activity 8.18

Materials

Exercise book, pens

Answers

Point on the line: $(2,3,1)$

Direction vector of the line: $(1,-2,1)$

Normal vector of the given plane $(2,3,-2)$

Normal vector of the needed plane, say (a,b,c), is perpendicular to the direction vector of the line and also perpendicular to the normal vector of the given plane.

The needed plane has the form: $a(x-2)+b(y-3)+c(z-1)=0$

Where

$$\begin{cases} a-2b+c=0 \\ 2a+3b-2c=0 \end{cases}$$

Solving, we get

$$\begin{cases} a=\dfrac{1}{7} \\ b=\dfrac{4}{7} \\ c=1 \end{cases}$$

And the needed plane is $x+4y+7z-21=0$

Application Activity 8.18

1. $\dfrac{x}{2}=\dfrac{y-3}{-6}=\dfrac{z-6}{-5}$

2. $\begin{cases} 2x-3y+z-30=0 \\ 5x+4y+2z-15=0 \end{cases}$

Activity 8.19

Materials

Exercise book, pens

Answers

$$\frac{x-1}{1}=\frac{y-2}{-4}=\frac{z-1}{-2}$$

Intersection: $(2,-2,1)$

Application Activity 8.19

1. $(-1,-2,1)$

2. $(2,-2,-1)$

Activity 8.20

Materials

Exercise book, pens

Answers

1. $\vec{u}=(3,-2,1)$ 2. Point: $(-3,4,1)$ 3. $\begin{cases} x=-3+3t \\ y=4-2t \\ z=1+t \end{cases}$

Application Activity 8.20

1. a) $\frac{x-1}{2}=\frac{y-2}{3}=\frac{z}{4}$ b) $\frac{3x+1}{9}=\frac{3y+2}{-6}=\frac{z}{1}$

   c) $\frac{x+2}{1}=\frac{y+3}{2}=\frac{z}{1}$

2. a) $(2,-5,3)$ b) $(a,1,c)$

3. $\overrightarrow{n_1}=\vec{i}-\vec{j}$, $\overrightarrow{n_2}=\vec{i}+\vec{j}+\vec{k}$

   The planes intersect in a line

   $\vec{u}=-\vec{i}-\vec{j}+2\vec{k}$

4. $3x-3y+4z+2=0$ 5. $x-5y-3z=-7$

6. $x+6y-5z=17$ 7. $4x-2y+7z=0$

Activity 8.21

Materials

Exercise book, pens

Answers

The equation of plane α and plane γ are proportional. Then plane α and plane γ coincide.

The equation of plane β is not proportional to any other equation. Also the left hand side of the equation of plane β is not proportional to any other left hand sides of other equations, then plane β is secant to other planes.

Application Activity 8.21

1. The given planes coincide 2. No intersection
3. Point: $(1,2,3)$ 4. Point: $(-4,-3,0)$
5. No intersection

8.4.4. Sphere in three dimensions

Recommended teaching periods: 9 periods

This section looks at:

a) Equations of sphere in space

The equation of a sphere of centre (k,l,m) and radius r is given by

$S \equiv (x-k)^2 + (y-l)^2 + (z-m)^2 = r^2$

The general equation of a sphere is:

$x^2 + y^2 + z^2 + ax + by + cz + d = 0$

In this equation:

The centre is $\Omega = \left(-\frac{a}{2}, -\frac{b}{2}, -\frac{c}{2}\right)$ and the radius is given by $r = \frac{1}{2}\sqrt{a^2+b^2+c^2-4d}$, provided that $a^2+b^2+c^2-4d > 0$

b) Position of a point and sphere

Consider a sphere S with radius r and centre $\Omega(a,b,c)$ and any point $P(a_1, a_2, a_3)$.

- If $d(\Omega, P) < r$, the point lies inside the sphere S.
- If $d(\Omega, P) = r$, the point lies on the sphere S.
- If $d(\Omega, P) > r$, the point lies outside the sphere S.

In all cases, $d(\Omega, P)$ is the distance between point P and centre Ω of sphere S.

c) Position of a sphere and a line

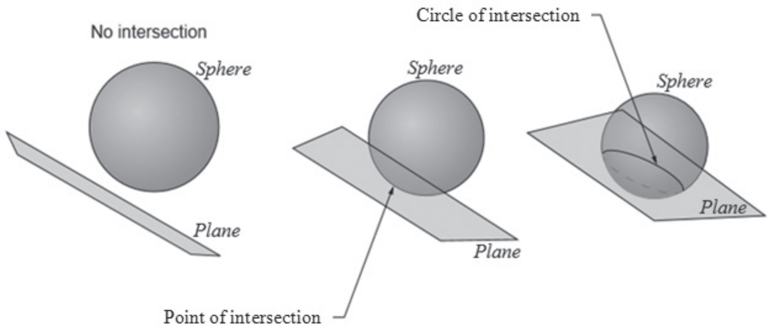
Consider a sphere S with radius r and centre $\Omega(a,b,c)$ and line L.

- If $d(\Omega, L) < r$, there are two points of intersection.
- If $d(\Omega, L) = r$, there is a single point of intersection.
- If $d(\Omega, L) > r$, there is no intersection.

In all cases, $d(\Omega, L)$ is the shortest distance between line L and centre Ω of sphere S.

d) Position of sphere and a plane

Consider a sphere $S \equiv x^2 + y^2 + z^2 + ax + by + cz + d = 0$ with centre $\Omega = (k, l, m)$ and radius r and plane $\alpha \equiv hx + ny + pz = q$, their position appears in three cases.



1. If $d(\Omega, \alpha) < r$, the plane cuts the sphere and the intersection is a circle whose centre is on the plane (on the perpendicular line of the plane passing through the centre of the sphere). When the plane cuts the sphere, we call it plane section of a sphere.

   Then, $d(P, Q) = \sqrt{[d(\Omega, Q)]^2 - [d(\Omega, P)]^2}$.

2. If $d(\Omega, \alpha) > r$, there is no intersection.

3. If $d(\Omega, \alpha) = r$, the plane is tangent to the sphere and the intersection is the point which lies on the perpendicular line of the plane passing through the centre of the sphere and it is the intersection between this perpendicular line and the plane.

In all cases, $d(\Omega, \alpha)$ is the distance between the centre $\Omega = (k, l, m)$ of the sphere $S \equiv x^2 + y^2 + z^2 + ax + by + cz + d = 0$ with radius r and plane $\alpha \equiv hx + ny + pz = q$. It is given by

$$d(\Omega, \alpha) = \frac{|hk + nl + pm - q|}{\sqrt{h^2 + n^2 + p^2}}.$$

The tangent plane at $T(x_1, y_1, z_1)$ on the sphere

$S \equiv x^2 + y^2 + z^2 + ax + by + cz + d = 0$ is

$xx_1 + yy_1 + zz_1 + a(x + x_1) + (y + y_1) + c(z + z_1) + d = 0$.

Hint. In writing equation of tangent plane to a sphere at a given point $T(x_1, y_1, z_1)$, in the sphere equation, change x^2 to xx_1, y^2 to yy_1, z^2 to zz_1, x to $\frac{1}{2}(x+x_1)$, y to $\frac{1}{2}(y+y_1)$, and z to $\frac{1}{2}(z+z_1)$

and then expand.

e) Position of two spheres

Consider two spheres with centers Ω_1 and Ω_2; radii r_1 and r_2. The position of these two spheres depends on the distance between theirs centers, $d(\Omega_1, \Omega_2)$

- If $d > r_1 + r_2$. Two spheres are exterior and hence no intersection.
- If $d < r_1 + r_2$. Two spheres are interior and hence no intersection.
- If $d = r_1 + r_2$. Two spheres are tangent exterior and hence there is a point of intersection.
- If $d = |r_1 - r_2|$. Two spheres are tangent interior and hence there is a point of intersection.
- If $|r_1 - r_2| < d < r_1 + r_2$. One sphere cuts another. The intersection is a circle.

b) Teaching guidelines

Let learners know what is a circle in 2-dimensions. When we rotate a half circle about $x-axis$ we obtain a sphere.

- Organise class into groups. Request each group to have a group leader who will present their findings to the class.
- Ask each group to do activity related to the lesson they are going to learn. The learners may need your assistance to do any activity. Help them to understand the activity. After group discussion, invite some or all groups for presentation of their work.

- After activity presentation, capture the main points from the presentation of the learners and summarise them. Guide the learners through given examples in Learner's book or through your own examples.
- Ask learners what they learned in day lesson to ensure that they understood what they have learned.
- Request learners to do exercises in their respective groups. Request learners to correct exercises on chalkboard and give them individual evaluation. Remember to give them homework.

c) Answers to activities

Activity 8.22

Materials

Exercise book, pens

Answers

1. $(x-k)^2+(y-l)^2+(z-m)^2=r^2$

   $\Leftrightarrow x^2-2kx+k^2+y^2-2ly+l^2+z^2-2mz+m^2=r^2$

   $\Leftrightarrow x^2+y^2+z^2-2kx-2ly-2mz+k^2+l^2+m^2-r^2=0$

2. Letting $-2k=a,\ -2l=b,\ -2m=c,\ k^2+l^2+m^2-r^2=d$

   Gives $k=-\dfrac{a}{2},\ l=-\dfrac{b}{2},\ m=-\dfrac{c}{2}$ and $-r^2=-k^2-l^2-m^2+d$

   Or

   $r^2=k^2+l^2+m^2-d$ $\quad r^2=\left(-\dfrac{a}{2}\right)^2+\left(-\dfrac{b}{2}\right)^2+\left(-\dfrac{c}{2}\right)^2-d$

   $r^2=\dfrac{a^2}{4}+\dfrac{b^2}{4}+\dfrac{c^2}{4}-d$ $\quad r^2=\dfrac{a^2+b^2+c^2-4d}{4}$

   $r=\dfrac{1}{2}\sqrt{a^2+b^2+c^2-4d}$

Application Activity 8.22

1. a) $x^2+y^2+z^2-2x-4y-6z-2=0$
   b) $x^2+y^2+z^2-6x+2y-2z+8=0$
   c) $x^2+y^2+z^2-8x+2z-32=0$
2. a) $(11,3,0)$; 8 b) $(-4,8,7)$; 6 c) $(0,9,3)$; 14
3. a) The interior of the sphere $x^2+y^2+z^2=4$.
   b) The solid ball bounded by the sphere
   $(x+2)^2+(y-3)^2+(z+4)^2=4$. Alternatively, the sphere
   $(x+2)^2+(y-3)^2+(z+4)^2=4$ together with its interior.
   c) The exterior of the sphere $(x-1)^2+(y+3)^2+z^2=8$.

Activity 8.23

Materials

Exercise book, pens, calculator

Answers

1. $\sqrt{38}$, outside the sphere 2. $\sqrt{6}$, on the sphere
3. $\sqrt{5}$, inside the sphere

Application Activity 8.23

1. Outside the sphere 2. Inside the sphere
3. On the sphere 4. Outside the sphere

Activity 8.24

Materials

Exercise book, pens, calculator

Answers

1. $\frac{3\sqrt{2}}{2}$, the line pierces the sphere
2. $2\sqrt{42}$, the line does not touch the sphere
3. $\frac{\sqrt{3}}{2}$, the line is tangent to the sphere

Application Activity 8.24

1. $x^2+y^2+z^2-6\sqrt{2}(x+y+z)-3\sqrt{2}=0$
2. $(1,-1,3),\ (5,2,-2)$

Activity 8.25

Materials

Exercise book, pens, calculator

Answers

1. $\frac{\sqrt{14}}{2}$, the plane cuts the sphere
2. $\frac{8\sqrt{14}}{7}$, the plane does not touch the sphere
3. $\sqrt{14}$, the plane is tangent to the sphere

Application Activity 8.25

1. $2x+2y-z+10=0,\ 2x+2y-z-8=0$
2. $2x+y-2z=9,\ x+2y+2z=9$
3. $x^2+y^2+z^2-6x-4y-2z+5=0$,
   $x^2+y^2+z^2-6x-\frac{11}{4}y-2z+5=0$
4. $x^2+y^2+z^2-2x+2y-4z+2=0$,
   $x^2+y^2+z^2-6x-4y+10z+22=0$

Activity 8.26

Materials

Exercise book, pens, calculator

Answers

Let centers of two spheres be Ω_1 and Ω_2 respectively and their radii be r_1 and r_2 respectively

1. $d(\Omega_1,\Omega_2)=\sqrt{11}$ and $r_1+r_2=5$. One sphere is inside of another
2. $d(\Omega_1,\Omega_2)=9$ and $r_1+r_2=9$. One sphere touches another
3. $d(\Omega_1,\Omega_2)=\sqrt{46}$ and $r_1+r_2=\sqrt{3}+\sqrt{4}$. One sphere is outside of another

Application Activity 8.26

1. $x^2+y^2+z^2+3y+5z-7=0$
2. $x^2+y^2+z^2-2z-8=0$
3. $13(x^2+y^2+z^2)-35x-21y+43z+176=0$

8.5. Answers for the end of unit assessment

1. a) $\frac{1}{3}\vec{A}+\frac{2}{3}\vec{B}$ b) $\frac{3}{7}\vec{A}+\frac{4}{7}\vec{B}$ c) $\frac{3}{5}\vec{A}+\frac{2}{5}\vec{B}$

   d) $\frac{3}{2}\vec{A}-\frac{1}{2}\vec{B}$ e) $-\frac{2}{3}\vec{A}+\frac{5}{3}\vec{B}$ f) $4\vec{A}-3\vec{B}$

2. $\overrightarrow{OM}=\frac{1}{2}(\vec{b}+\vec{c})$ $\overrightarrow{ON}=\frac{1}{3}(\vec{a}+\vec{b}+\vec{c})$
3. $x\vec{i}+y\vec{j}+z\vec{k}=\vec{i}+\vec{j}+\vec{k}+\lambda(2\vec{i}+3\vec{j}-\vec{k})$
4. $\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}-1\\2\\1\end{pmatrix}+\lambda\begin{pmatrix}1\\2\\3\end{pmatrix}$

5. $4\vec{i}-\vec{j}+12\vec{k}=2\vec{i}+3\vec{j}+4\vec{k}+r\left(\vec{i}-2\vec{j}+4\vec{k}\right)$

$2\vec{i}-4\vec{j}+8\vec{k}=r\left(\vec{i}-2\vec{j}+4\vec{k}\right)$

$$\begin{cases} 2=r & \Rightarrow r=2 \\ -4=-2r & \Rightarrow r=2 \\ 8=4r & \Rightarrow r=2 \end{cases}$$

Thus the given point lie on the given line.

6. $a=6,\ b=8$

7. a) $(3,3,-3)$ b) $(x,y,z)=(2,-1,1)+\lambda(3,3,-3)$

8. a) $\frac{x-2}{2}=\frac{y-3}{3}=\frac{z+1}{1}$ b) $\frac{x-3}{3}=\frac{y+1}{2}=\frac{z-2}{-4}$

c) $\frac{x-2}{2}=\frac{y-1}{-1}=\frac{z-1}{-1}$

9. $(x,y,z)=(2,5,4)+\lambda(3,-2,-1)$

10. a) $(x,y,z)=(2,2,-1)+\lambda(3,2,4)$

b) $(x,y,z)=(3,-2,3)+\lambda(1,4,-1)$

11. $(2,5,3)$ or its multiples

12. Vector equation: $(x,y,z)=(1,0,-3)+\lambda(2,-1,3)$

Parametric equations: $\begin{cases} x=1+2\lambda \\ y=-\lambda \\ z=-3+2\lambda \end{cases}$

Symmetric equations: $\frac{x-1}{2}=-y=\frac{z+3}{3}$

13. a) $(x,y,z)=(2,3,4)+\lambda(2,-3,2)+\mu(0,1,2)$

b) $(x,y,z)=(0,0,-2)+\lambda(3,3,-1)+\mu(1,-1,1)$

c) $(x,y,z)=(-2,-1,-3)+\lambda(1,0,1)+\mu(2,1,1)$

d) $(x,y,z)=(5,1,-4)+\lambda(1,-1,1)+\mu(3,-1,-1)$

14. a) $2x+3y-4z=29$ b) $2x+y-4z=12$

c) $x+y+z=0$ d) $3x+4y-5z=12$

e) $2x-y=4$ f) $5y+2z=-11$ g) $x=3$

15. $x+5y-4z=20$

16. $x-y=0$

17. $5x-6y+7z=20$

18. $x+2y-3z=0$

19. $\begin{pmatrix}7\\-5\\-4\end{pmatrix}=\begin{pmatrix}4\\3\\2\end{pmatrix}+r\begin{pmatrix}1\\-1\\-1\end{pmatrix}+s\begin{pmatrix}2\\3\\1\end{pmatrix}\Rightarrow\begin{cases}r=5\\s=-1\end{cases}$. Thus, the given point lie on the given plane.

20. $2x-3y-6z=6$, $6x+3y-2z=18$

21. $(x,y,z)=(1,0,-2)+\lambda(1,1,0)+\mu(0,0,1)$

22. a) $\sqrt{2}$ units of length b) 3 units of length

    c) $\sqrt{10}$ units of length d) $\frac{1}{2}\sqrt{138}$ units of length

23. a) 7 units of length; $(1,2,3)$

    b) $2\sqrt{6}$ units of length; $(2,-1,-1)$

    c) 6 units of length; $(4,0,0)$

    d) $\frac{1}{3}\sqrt{42}$ units of length; $\left(\frac{5}{3},\frac{4}{3},\frac{11}{3}\right)$

24. 14 units of length

25. 38.31 degrees

26. 25.7 degrees

27. 80 degrees

28. 45.6 degrees

29. 40.2 degrees

30. 14.66^0 degrees

31. a) $(3,-4,5)$; 7 b) $(1,2,3)$; 3

    c) $\left(\frac{1}{2},-1,-\frac{1}{2}\right)$, 1

32. a) $x^2+y^2+z^2-2x-4y-6z+10=0$

    b) $x^2+y^2+z^2-4x-4z+4=0$

    c) $x^2+y^2+z^2-4x-6y-23=0$

33. $d = -3$

34. $7\left(x^2 + y^2 + z^2\right) - 15x - 25y - 11z = 0$

35. $x^2 + y^2 + z^2 - 4x + 6y - 2z + 5 = 0$

36. $3\left(x^2 + y^2 + z^2\right) - 2x - 2y - 2z - 1 = 0$

37. $x^2 + y^2 + z^2 - 6x - 4y + 10z + 12 = 0$

38. $x^2 + y^2 + z^2 - x - 6y - 2z + 5 = 0$

39. $5\left(x^2 + y^2 + z^2\right) - 2x - 2y - 2z - 9 = 0$, $\left(\frac{1}{5}, \frac{1}{5}, \frac{1}{5}\right)$, $\frac{4\sqrt{3}}{2}$

40. $x^2 + y^2 + z^2 + 7y - 8z + 24 = 0$

41. $x^2 + y^2 + z^2 - 4x + 6y - 8z + 4 = 0$

42. $x^2 + y^2 + z^2 + 2x + 2y + 2z + 2 = 0$

43. $x^2 + y^2 + z^2 \pm 6z - 4 = 0$

44. $4x + 15y + 26z - 75 = 0$

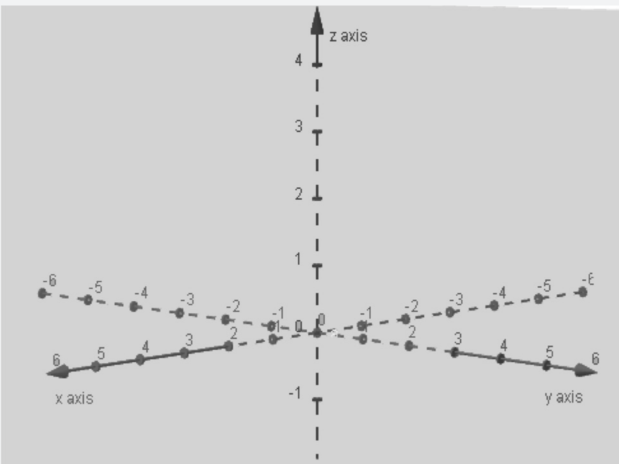
45. -4, 14

46. $(4, -2, 2), (0, 0, -2)$

47. $9\left(x^2 + y^2 + z^2\right) = 5$

48. $\sqrt{61}$

Sketch



49. $\frac{1}{2}\sqrt{b^2c^2 + c^2a^2 + a^2b^2}$

Unit 9 Bivariate Statistics

9.1. Key unit competence

Extend understanding, analysis and interpretation of bivariate data to correlation coefficients and regression lines

9.2. Objectives

After completing this unit, the learners should be able to:

- find the covariance of two quantitative variables.
- determine the linear regression line of a given series.
- calculate a linear correlation coefficient of a given double series and interpret it.

9.3. Materials used in this unit

Exercise books, pens, calculator

9.4. Content and activities

9.4.1. Covariance

a) Content summary

Recommended teaching periods: 2 periods

This section looks at formula used to find the covariance of two variable *x* and *y*.

$$\operatorname{cov}(x,y)=\frac{1}{n}\sum_{i=1}^{k} f_i x_i y_i - \bar{x}\,\bar{y}$$

b) Teaching guidelines

Let learners know how to find variance and standard deviation of a series. In bivariate statistics, we use two series.

- Organise class into groups. Request each group to have a group leader who will present their findings to the class.

- Ask each group to do activity related to the lesson they are going to learn. The learners may need your assistance to do any activity. Help them to understand the activity. After group discussion, invite some or all groups for presentation of their work.
- After activity presentation, capture the main points from the presentation of the learners and summarise them. Guide the learners through given examples in Learner's book or through your own examples.
- Ask learners what they learned in day lesson to ensure that they understood what they have learned.
- Request learners to do exercises in their respective groups. Request learners to correct exercises on chalkboard and give them individual evaluation. Remember to give them homework.

c) Answers to activities

Activity 9.1

Materials

Exercise book, pens, calculator

Answers

x	y	$x-\bar{x}$	$y-\bar{y}$	$(x-\bar{x})(y-\bar{y})$
3	6	-1.3	-2.7	3.51
5	9	0.7	0.3	0.21
7	12	2.7	3.3	8.91
3	10	-1.3	1.3	-1.6
2	7	-2.3	-1.7	3.91
6	8	1.7	-0.7	-1.19
$\sum_{1}^{6} x = 26$	$\sum_{1}^{6} y = 52$			$\sum_{1}^{6}(x-\bar{x})(y-\bar{y}) = 13.75$
$\bar{x} = 4.3$	$\bar{y} = 8.7$			

1. If you divide by total frequency you get **variance**
2. If you divide by total frequency you get **covariance**

Application Activity 9.1

1. $\text{cov}(x,y)=\frac{71}{12}$ 2. $\text{cov}(x,y)=98.75$

9.4.2. Regression lines

a) Content summary

Recommended teaching periods: 5 periods

This section looks at the adjustment of algebraic expression of two regression lines.

The regression line *y* on *x* is written as

$$L_{y/x} \equiv y-\bar{y}=\frac{\text{cov}(x,y)}{\sigma_x^2}(x-\bar{x})$$

The regression line x on y is written as

$$L_{x/y} \equiv x-\bar{x}=\frac{\text{cov}(x,y)}{\sigma_y^2}(y-\bar{y})$$

b) Teaching guidelines

Let learners know how to find mean, standard deviation and covariance.

- Organise class into groups. Request each group to have a group leader who will present their findings to the class.
- Ask each group to do activity related to the lesson they are going to learn. The learners may need your assistance to do any activity. Help them to understand the activity. After group discussion, invite some or all groups for presentation of their work.
- After activity presentation, capture the main points from the presentation of the learners and summarise them. Guide the learners through given examples in Learner's book or through your own examples.
- Ask learners what they learned in day lesson to ensure that they understood what they have learned.
- Request learners to do exercises in their respective groups. Request learners to correct exercises on chalkboard and give them individual evaluation. Remember to give them homework.

c) Answers to activities

Activity 9.2

Materials

Exercise book, pens, calculator

Answers

1. $D_b^{/}=2\sum_{i=1}^{k}(y_i-ax_i-b)(-1)$ or $D_b^{/}=-2\sum_{i=1}^{k}(y_i-ax_i-b)$

2. $\sum_{i=1}^{k}(y_i-ax_i-b)=0$ or $\sum_{i=1}^{k}y_i-\sum_{i=1}^{k}ax_i-\sum_{i=1}^{k}b=0$

   or $\sum_{i=1}^{k}b=\sum_{i=1}^{k}y_i-\sum_{i=1}^{k}ax_i$

   Dividing both sides by n gives

   $\frac{1}{n}\sum_{i=1}^{k}b=\frac{1}{n}\sum_{i=1}^{k}y_i-\frac{1}{n}\sum_{i=1}^{k}ax_i$ or $\frac{b}{n}\sum_{i=1}^{k}1=\frac{1}{n}\sum_{i=1}^{k}y_i-\frac{a}{n}\sum_{i=1}^{k}x_i$

   or $b=\overline{y}-a\overline{x}$

3. $\sum_{i=1}^{k}(y_i-ax_i-b)^2=\sum_{i=1}^{k}\left(y_i-ax_i-\overline{y}+a\overline{x}\right)^2$

   Or

   $\sum_{i=1}^{k}(y_i-ax_i-b)^2=\sum_{i=1}^{k}\left[\left(y_i-\overline{y}\right)-a\left(x_i-\overline{x}\right)\right]^2$

   Differentiation with respect to a and equating to zero:

   $$\sum_{i=1}^{k}2\left[\left(y_i-\overline{y}\right)-a\left(x_i-\overline{x}\right)\right]\left[-\left(x_i-\overline{x}\right)\right]=0$$

   $$-2\sum_{i=1}^{k}\left[\left(y_i-\overline{y}\right)-a\left(x_i-\overline{x}\right)\right]\left(x_i-\overline{x}\right)=0$$

   $$\Leftrightarrow\sum_{i=1}^{k}\left[\left(y_i-\overline{y}\right)-a\left(x_i-\overline{x}\right)\right]\left(x_i-\overline{x}\right)=0$$

   $$\Leftrightarrow\sum_{i=1}^{k}\left[\left(x_i-\overline{x}\right)\left(y_i-\overline{y}\right)-a\left(x_i-\overline{x}\right)^2\right]=0$$

$$\Leftrightarrow \sum_{i=1}^{k}\left(x_i-\overline{x}\right)\left(y_i-\overline{y}\right)-\sum_{i=1}^{k}a\left(x_i-\overline{x}\right)^2=0$$

$$\Leftrightarrow \sum_{i=1}^{k}a\left(x_i-\overline{x}\right)^2=\sum_{i=1}^{k}\left(x_i-\overline{x}\right)\left(y_i-\overline{y}\right)$$

$$\Leftrightarrow a\sum_{i=1}^{k}\left(x_i-\overline{x}\right)^2=\sum_{i=1}^{k}\left(x_i-\overline{x}\right)\left(y_i-\overline{y}\right)$$

Dividing both sides by n gives

$$\Leftrightarrow \frac{a}{n}\sum_{i=1}^{k}\left(x_i-\overline{x}\right)^2=\frac{1}{n}\sum_{i=1}^{k}\left(x_i-\overline{x}\right)\left(y_i-\overline{y}\right)$$

$$\Rightarrow a=\frac{\frac{1}{n}\sum_{i=1}^{k}\left(x_i-\overline{x}\right)\left(y_i-\overline{y}\right)}{\frac{1}{n}\sum_{i=1}^{k}\left(x_i-\overline{x}\right)^2}$$

4. The variance for variable x is $\sigma_x^2=\frac{1}{n}\sum_{i=1}^{k}\left(x_i-\overline{x}\right)^2$

and the variance for variable y is $\sigma_y^2=\frac{1}{n}\sum_{i=1}^{k}\left(y_i-\overline{y}\right)^2$

and the covariance of these two variables is

$$\operatorname{cov}(x,y)=\frac{1}{n}\sum_{i=1}^{k}\left(x_i-\overline{x}\right)\left(y_i-\overline{y}\right)$$

Then $a=\frac{\operatorname{cov}(x,y)}{\sigma_x^2}$

5. Now, we have that the regression line y on x is $y=ax+b$, where

$$\begin{cases} a=\dfrac{\operatorname{cov}(x,y)}{\sigma_x^2} \\ b=\overline{y}-a\overline{x} \end{cases}$$

Or

$$y=\frac{\operatorname{cov}(x,y)}{\sigma_x^2}x+\left(\overline{y}-\frac{\operatorname{cov}(x,y)}{\sigma_x^2}\overline{x}\right)$$

Application Activity 9.2

1. a) $y = 0.19x - 8.098$ b) $y = 4.06$
2. $x = -5.6y + 163.3, \quad y = -0.06x + 21.8$

9.4.3. Coefficient of correlation

a) Content summary

Recommended teaching periods: 3 periods

This section looks at the correlation coefficient and its properties. It also looks at the Spearman's coefficient of rank correlation.

- **The Pearson's correlation coefficient**,

  Pearson's correlation coefficient denoted by r, is a measure of the strength of linear relationship between two variables.

  The correlation coefficient between two variables x and y is given by $r = \dfrac{\operatorname{cov}(x, y)}{\sigma_x \sigma_y}$.

- **The Spearman's coefficient of rank correlation**

  The Spearman's coefficient of rank correlation is given by

  $$\rho = 1 - \frac{6\sum_{i=1}^{k} d_i^2}{n(n^2 - 1)}$$

  Where, d refers to the difference of ranks between paired items in two series.

b) Teaching guidelines

- Organise class into groups. Request each group to have a group leader who will present their findings to the class.
- Ask each group to do activity related to the lesson they are going to learn. The learners may need your assistance to do any activity. Help them to understand the activity. After group discussion, invite some or all groups for presentation of their work.

- After activity presentation, capture the main points from the presentation of the learners and summarise them. Guide the learners through given examples in Learner's book or through your own examples.
- Ask learners what they learned in day lesson to ensure that they understood what they have learned.
- Request learners to do exercises in their respective groups. Request learners to correct exercises on chalkboard and give them individual evaluation. Remember to give them homework.

c) Answers to activities

Activity 9.3

Materials

Exercise book, pens, calculator

Answers

1. $\sigma_x = 1.8,\ \sigma_y = 1.97$
2. $\text{cov}(x, y) = \frac{41}{18}$
3. $\frac{\text{cov}(x, y)}{\sigma_x \sigma_y} = 0.64$

Application Activity 9.3

1. $r = 0.94$. As the correlation coefficient is very close to 1, the correlation is very strong.
2. $r = -0.26$. As the correlation coefficient is very close to zero, the correlation is very weak.
3. $\sigma = 0.14$. There is a weak positive correlation between the English and Mathematics rankings.

9.4.4. Applications of bivariate statistics

Activity 9.4

Materials

Exercise book, pens, calculator

Answers

By reading textbooks or accessing internet, learners will discuss how bivariate statistics is used in daily life. Bivariate statistics can help in prediction of a value for one variable if we know the value of the other by using regression lines.

9.5. Answers for the end of unit assessment

1. Data set 1

   a) $y = 4.50 + 0.64x$ b) $x = 4.42 + 0.75y$

   Data set 2

   a) $y = 90.31 - 1.78x$ b) $x = 37.80 - 0.39y$

2. $y = -2.59 + 0.65x; 36.5$
3. $r = 0.918, y - 65.45 = 0.981(x - 65.18), x - 65.18 = 0.859(y - 65.45)$
4. $y = 0.611x + 10.5, x = 1.478y - 1.143, y = 28.83$
5. $y = 0.94x + 92.26, Blood\ pressure = 134.56$
6. $y = 3.8 + 1.6x, x = -2.06 + 0.59y$
7. $y = -8 + 1.2x$
8. $c = 15, d = -5$
9. 0.60, $w = 0.89h - 76$
10. $\bar{x} = -\frac{3}{29}, \bar{y} = \frac{15}{29}, r = \frac{3}{4}$
11. $r = 0.4$
12. 0.82
13. 0.77

14. -0.415

15. a) 0.954 b) $\bar{x} = 2$, $\bar{y} = 3$

16. $\bar{x} = 13, \bar{y} = 17, y = 0.8x + 6.6, x = 0.45y - 5.35, r = 0.6, \sigma_y = 4$

17. $\bar{x} = 13, \bar{y} = 17, \sigma_y = 4$

18. 0.26

19. a) 0.43

b) Some agreement between average attendances ranking a position in league, high position in league correlating with high attendance.

20. a) (i) -0.976 (ii) -0.292 (or 0.292)

b) The transport manager's order is more profitable for the seller, saleswomen is unlikely to try to dissuade.

c) (i) No, maximum value is 1

(ii) Yes, higher performing cars generally do less mileage to the gallon.

(iii) No, the higher the engine capacity, the dearer the car.

d) When only two rankings are known; when relationship is non-linear.

21. a) There is a strong positive correlation

b) 54.5

Unit 10 Conditional Probability and Bayes Theorem

10.1. Key unit competence

Solve problems using Bayes theorem and use data to make decisions about likelihood and risk.

10.2. Objectives

After completing this unit, the learners should be able to:

- use tree diagram to find probability of events.
- find probability of independent events.
- find probability of one event given that the other event has occurred.
- use and apply Bayes theorem.

10.3. Materials used in this unit

Exercise books, pens, ruler, calculator

10.4. Content and activities

10.4.1. Tree diagram

a) Content summary

Recommended teaching periods: 5 periods

This section shows the method used to find probability of events by constructing tree diagram.

A **tree diagram** is a means which can be used to show the probabilities of certain **outcomes** occurring when two or more **trials** take place in succession.

The **outcome** is written at the end of the branch and the fraction on the branch gives the probability of the outcome occurring.

For each **trial,** the number of branches is equal to the number of possible outcomes of that trial. In the diagram, there are two possible outcomes, *A* and *B*, of each trial.

b) Teaching guidelines

- Organise class into groups. Request each group to have a group leader who will present their findings to the class.
- Ask each group to do activity related to the lesson they are going to learn. The learners may need your assistance to do any activity. Help them to understand the activity. After group discussion, invite some or all groups for presentation of their work.
- After activity presentation, capture the main points from the presentation of the learners and summarise them. Guide the learners through given examples in Learner's book or through your own examples.
- Ask learners what they learned in day lesson to ensure that they understood what they have learned.
- Request learners to do exercises in their respective groups. Request learners to correct exercises on chalkboard and give them individual evaluation. Remember to give them homework.

c) Answers to activities

Activity 10.1

Materials

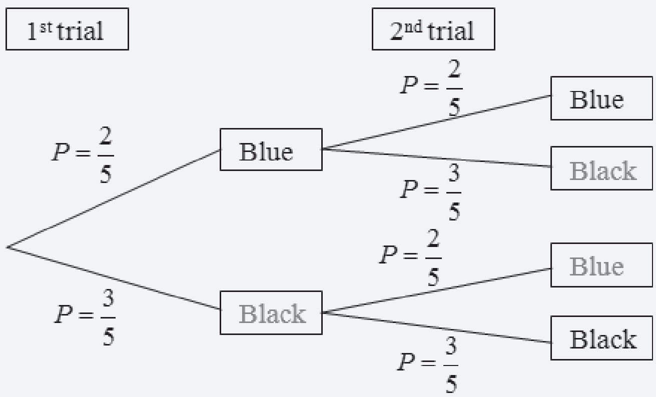
Exercise book, pens, calculator

Methodology

Facilitate learners in Group work, then questioning.

Answers

1. Probability of choosing a blue pen is $\frac{4}{10}=\frac{2}{5}$ and probability of choosing a black pen is $\frac{6}{10}=\frac{3}{5}$.
2. Probabilities on the second trial are equal to the probabilities on the first trial since after the 1st trial the pen is replaced in the box.
3. Complete figure



Application Activity 10.1

1. $\frac{1}{2}\times\frac{1}{2}\times\frac{1}{2}=\frac{1}{8}$
2. a) $P(3\,\text{boys})=\frac{10}{16}\times\frac{9}{15}\times\frac{8}{14}=0.214$

   b) $P(2\text{ boys and 1 girl})=\frac{10}{16}\times\frac{9}{15}\times\frac{6}{14}+\frac{10}{16}\times\frac{6}{15}\times\frac{9}{14}+\frac{6}{16}\times\frac{10}{15}\times\frac{9}{14}=0.482$

   c) $P(2\text{ girls and 1 boy})=\frac{10}{16}\times\frac{6}{15}\times\frac{5}{14}+\frac{6}{16}\times\frac{10}{15}\times\frac{5}{14}+\frac{6}{16}\times\frac{5}{15}\times\frac{10}{14}=0.268$

   d) $P(3\,\text{girls})=\frac{6}{16}\times\frac{5}{15}\times\frac{4}{14}=0.0357$
3. a) $\frac{1}{21}$ b) $\frac{10}{21}$ c) $\frac{11}{21}$
4. a) $\frac{1}{816}$ b) $\frac{7}{102}$ c) $\frac{7}{34}$

10.4.2. Independent events

a) Content summary

Recommended teaching periods: 5 periods

This section shows the formula used to find probability of independent events.

If probability of event B is not affected by the occurrence of event A, events A and B are said to be independent and $P(A \cap B) = P(A) \times P(B)$

This rule is the simplest form of the multiplication law of probability.

b) Teaching guidelines

- Organise class into groups. Request each group to have a group leader who will present their findings to the class.
- Ask each group to do activity related to the lesson they are going to learn. The learners may need your assistance to do any activity. Help them to understand the activity. After group discussion, invite some or all groups for presentation of their work.
- After activity presentation, capture the main points from the presentation of the learners and summarise them. Guide the learners through given examples in Learner's book or through your own examples.
- Ask learners what they learned in day lesson to ensure that they understood what they have learned.
- Request learners to do exercises in their respective groups. Request learners to correct exercises on chalkboard and give them individual evaluation. Remember to give them homework

c) Answers to activities

Activity 10.2

Materials

Exercise book, pens

Answers

The occurrence of event B does not affected by occurrence of event A because after the first trial, the pen is replaced in the box. It means that the sample space does not change.

Application Activity 10.2

1. $P(\text{red and red})=\frac{1}{5}\times\frac{1}{5}=\frac{1}{25}$
2. $P(\text{head and 3})=\frac{1}{2}\times\frac{1}{6}=\frac{1}{12}$
3. a) $\frac{1}{35}$ b) $\frac{2}{7}$

10.4.3. Conditional probability

a) Content summary

Recommended teaching periods: 5 periods

This section shows the formula used to find conditional probability.

The probability of an event B given that event A has occurred is called the conditional probability of B given A and is written $P(B|A)$.

In this case, $P(B|A)$ is the probability that B occurs considering A as the sample space, and since the subset of A in which B occurs is $A\cap B$, then

$$P(B|A)=\frac{P(B\cap A)}{P(A)}.$$

From this result, we have a general statement of the multiplication law:

$$P(A\cap B)=P(A)\times P(B|A)$$

b) Teaching guidelines

- Organise class into groups. Request each group to have a group leader who will present their findings to the class.
- Ask each group to do activity related to the lesson they are going to learn. The learners may need your assistance to do any activity. Help them to understand the activity. After group discussion, invite some or all groups for presentation of their work.
- After activity presentation, capture the main points from the presentation of the learners and summarise them. Guide the learners through given examples in Learner's book or through your own examples.
- Ask learners what they learned in day lesson to ensure that they understood what they have learned.
- Request learners to do exercises in their respective groups. Request learners to correct exercises on chalkboard and give them individual evaluation. Remember to give them homework.

c) Answers to activities

Activity 10.3

Materials

Exercise book, pens

Answers

The occurrence of event B is affected by occurrence of event A because after the first trial, the pen is not replaced in the box. It means that the sample space will be changed for the second trial.

Application Activity 10.3

1. $P(6\,|\,even)=\dfrac{\frac{1}{6}}{\frac{3}{6}}=\dfrac{1}{3}$

2. $P(White \mid Black) = \dfrac{P(Black\ and\ White)}{P(Black)} = \dfrac{0.34}{0.47} = 0.72$

3. $\dfrac{3}{13}$

10.4.4. Bayes theorem and applications

a) Content summary

Recommended teaching periods: 5 periods

This section shows how to find probability of events using Bayes theorem and its applications.

Let $B_1, B_2, B_3, ..., B_n$ be incompatible and exhaustive events and let A be an arbitrary event.

We have:

$$P(B_i \mid A) = \frac{P(B_i \cap A)}{P(A)} = \frac{P(A \mid B_i)P(B_i)}{\sum_{i=1}^{n} P(A \mid B_i)P(B_i)}$$

This formula is called Bayes' formula.

Remark

We also have (Bayes' rule)

$$P(B \mid A) = \frac{P(A \mid B)P(B)}{P(A)}$$

Teaching guidelines

- Organise class into groups. Request each group to have a group leader who will present their findings to the class.
- Ask each group to do activity related to the lesson they are going to learn. The learners may need your assistance to do any activity. Help them to understand the activity. After group discussion, invite some or all groups for presentation of their work.

- After activity presentation, capture the main points from the presentation of the learners and summarise them. Guide the learners through given examples in Learner's book or through your own examples.
- Ask learners what they learned in day lesson to ensure that they understood what they have learned.
- Request learners to do exercises in their respective groups. Request learners to correct exercises on chalkboard and give them individual evaluation. Remember to give them homework.

c) Answers to activities

Activity 10.4

Materials

Exercise book, pens

Answers

1. $P(A)=P(A|B_1)P(B_1)+P(A|B_2)P(B_2)+P(A|B_3)P(B_3)$

2. $$P(B_1|A)=\frac{P(B_1\cap A)}{P(A)}=\frac{P(A|B_1)P(B_1)}{P(A|B_1)P(B_1)+P(A|B_2)P(B_2)+P(A|B_3)P(B_3)}$$

$$P(B_2|A)=\frac{P(B_2\cap A)}{P(A)}=\frac{P(A|B_2)P(B_2)}{P(A|B_1)P(B_1)+P(A|B_2)P(B_2)+P(A|B_3)P(B_3)}$$

$$P(B_3|A)=\frac{P(B_3\cap A)}{P(A)}=\frac{P(A|B_3)P(B_3)}{P(A|B_1)P(B_1)+P(A|B_2)P(B_2)+P(A|B_3)P(B_3)}$$

Generally,

$$P(B_i|A)=\frac{P(B_i\cap A)}{P(A)}=\frac{P(A|B_i)P(B_i)}{\sum_{i=1}^{3}P(A|B_i)P(B_i)}$$

Application Activity 10.4

1. $P(engineer \mid managerial) = \dfrac{0.2\times 0.75}{0.2\times 0.75 + 0.2\times 0.5 + 0.6\times 0.2} = 0.405$

2. $P(No\ accident \mid Triggered\ alarm) = \dfrac{0.9\times 0.02}{0.1\times 0.97 + 0.9\times 0.02} = 0.157$

10.5. Answers for the end of unit assessment

1.	0.15	2. 0.13	
3.	0.56	4. $\dfrac{1}{169}$	
5.	$\dfrac{15}{128}$	6. $\dfrac{729}{1000}$	
7.	0.37	8. $\dfrac{10}{21}$	
9.	a) 0.34	b) 0.714	c) 0.0833
10.	a) 0.43	b) 0.1166	c) 0.8966
11.	a) 0.0001	b) 0.0081	
12.	a) 0.384	b) 0.512	
13.	0.1083		
14.	a) 0.5514	b) 0.2941	
15.	$\dfrac{3}{13}$		

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