SUBSIDIARY MATHEMATICS FOR PHYSICS-CHEMISTRY-BIOLOGY (PCB) COMBINATION

SENIOR 4

Teacher's Guide

Kigali, 2022

EXPERIMENTAL VERSION

FOREWORD

Dear Teachers,

Rwanda Basic Education Board is honoured to present the teacher's guide for S4 Subsidiary Mathematics in the combination of Physics-Chemistry-Biology (PCB). This book serves as a guide to competence-based teaching and learning to ensure consistency and coherence in the learning of the Mathematics content. The Rwandan educational philosophy is to ensure that learners achieve full potential at every level of education which will prepare them to be well integrated in society and exploit employment opportunities.

In line with efforts to improve the quality of education, the government of Rwanda emphasizes the importance of aligning teaching and learning materials with the syllabus to facilitate their learning process. Many factors influence what they learn, how well they learn and the competences they acquire. Those factors include the relevance of the specific content, the quality of teachers' pedagogical approaches, the assessment strategies and the instructional materials.

The ambition to develop a knowledge-based society and the growth of regional and global competition in the jobs market has necessitated the shift to a competence-based curriculum. This book provides active teaching and learning techniques that engage student teachers to develop competences.

In view of this, your role is to:

- Plan your lessons and prepare appropriate teaching materials.
- Organize group discussions for students considering the importance of social constructivism suggesting that learning occurs more effectively when the students work collaboratively with more knowledgeable and experienced people.
- Engage students through active learning methods such as inquiry methods, group discussions, research, investigative activities and group and individual work activities.
- Provide supervised opportunities for students to develop different competences by giving tasks which enhance critical thinking, problem solving, research, creativity and innovation, communication and cooperation.

- Support and facilitate the learning process by valuing students' contributions in the class activities.
- Guide students towards the harmonization of their findings.
- Encourage individual, peer and group evaluation of the work done in the classroom and use appropriate competence-based assessment approaches and methods.

To facilitate you in your teaching activities, the content of this book is self-explanatory so that you can easily use it. It is divided in 3 parts:
The part I explain the structure of this book and give you the methodological guidance;
The part II gives a sample lesson plan;
The part III details the teaching guidance for concepts given in the student book.

Even though this teacher's guide contains the guidance on solutions for all activities given in the learner's book, you are requested to work through each question before judging student's findings.
I wish to sincerely express my appreciation to the people who contributed towards the development of this book, particularly, REB staff, Lecturers and Teachers for their technical support. A word of gratitude goes also to the Head Teachers who availed their staff for various activities.

Dr. MBARUSHIMANA Nelson
Director General, REB.

ACKNOWLEDGEMENT

I wish to express my appreciation to the people who played a major role in the development of this teacher's guide for Senior Four Subsidiary Mathematics in Physics-Chemistry-Biology (PCB) combination. It would not have been successful without active participation of different education stakeholders.
I owe gratitude to different universities and schools in Rwanda that allowed their staff to work with REB in the in-house textbooks production initiative.

I wish to extend my sincere gratitude to lecturers and teachers whose efforts during writing exercise of this teacher's guide were very much valuable.

Finally, my word of gratitude goes to the Rwanda Basic Education Board staffs who were involved in the whole process of in-house textbook writing.

MURUNGI Joan
Head of Curriculum, Teaching and Learning Resources Department/REB

CONTENTS

PART I. GENERAL INTRODUCTION

1.1 The structure of the guide

The teacher's guide of Mathematics is composed of three parts:

The Part I concerns general introduction that discusses methodological guidance on how best to teach and learn Mathematics, developing competences in teaching and learning, addressing cross-cutting issues in teaching and learning and Guidance on assessment.

Part II presents a sample lesson plan. This lesson plan serves to guide the teacher on how to prepare a lesson in Mathematics.

The Part III is about the structure of a unit and the structure of a lesson. This includes information related to the different components of the unit and these components are the same for all units. This part provides information and guidelines on how to facilitate students while working on learning activities. More other, all application activities from the textbook have answers in this part.

1.2 Methodological guidance

1.2.1 Developing competences

Since 2015 Rwanda shifted from a knowledge based to a competency-based curriculum for pre-primary, primary, secondary education. This called for changing the way of learning by shifting from teacher centred to a learner centred approach. Teachers are not only responsible for knowledge transfer but also for fostering learners' learning achievement and creating safe and supportive learning environment. It implies also that learners have to demonstrate what they are able to transfer the acquired knowledge, skills, values and attitude to new situations. The competence-based curriculum employs an approach of teaching and learning based on discrete skills rather than dwelling on only knowledge or the cognitive domain of learning. It focuses on what learner can do rather than what learner knows. Learners develop competences through subject unit with specific learning objectives broken down into knowledge, skills and attitudes through learning activities. In addition to the competences related to Mathematics, student teachers also develop generic competences which should promote the development of the higher order thinking skills and professional skills in Mathematics teaching. Generic competences are developed throughout all units of Mathematics as follows:

Generic competences	Ways of developing generic competences
Critical thinking	All activities that require students to calculate, convert, interpret, analyse, compare and contrast, etc. have a common factor of developing critical thinking into students.
Creativity and innovation	All activities that require students to plot a graph of a given algebraic data, to organize and interpret statistical data collected and to apply skills in solving problems of production/ finance/ economic have a common character of developing creativity into students
Research and problem solving	All activities that require students to make research and apply their knowledge to solve problems from the real-life situation have a character of developing research and problem solving into students.
Communication	During Mathematics class, all activities that require students to discuss either in groups or in the whole class, present findings, debate …have a common character of developing communication skills into students.
Co-operation, interpersonal relations and life skills	All activities that require students to work in pairs or in groups have character of developing cooperation and life skills among students.
Lifelong learning	All activities that are connected with research have a common character of developing into students a curiosity of applying the knowledge learnt in a range of situations. The purpose of such kind of activities is for enabling students to become life-long students who can adapt to the fast-changing world and the uncertain future by taking initiative to update knowledge and skills with minimum external support.

The generic competences help students deepen their understanding of Mathematics and apply their knowledge in a range of situations. As students develop generic competences, they also acquire the set of skills that employers look for in their employees, and so the generic competences prepare students for the world of work.

1.2.2 Addressing cross cutting issues

Among the changes brought by the competence-based curriculum is the integration of cross cutting issues as an integral part of the teaching learning process-as they relate to and must be considered within all subjects to be appropriately addressed. The eight cross cutting issues identified in the national curriculum framework are: Comprehensive Sexuality Education, Environment and Sustainability, Financial Education, Genocide studies, Gender, Inclusive Education, Peace and Values Education, and Standardization Culture.

Some cross-cutting issues may seem specific to particular learning areas/subjects but the teacher need to address all of them whenever an opportunity arises. In addition, students should always be given an opportunity during the learning process to address these cross-cutting issues both within and out of the classroom.

Below are examples of how crosscutting issues can be addressed:

Cross-Cutting Issue	**Ways of addressing cross-cutting issues**
Comprehensive Sexuality Education: The primary goal of introducing Comprehensive Sexuality Education program in schools is to equip children, adolescents, and young people with knowledge, skills and values in an age appropriate and culturally gender sensitive manner so as to enable them to make responsible choices about their sexual and social relationships, explain and clarify feelings, values and attitudes, and promote and sustain risk reducing behaviour.	Using different charts and their interpretation, Mathematics teacher should lead students to discuss the following situations: "Alcohol abuse and unwanted pregnancies" and advise students on how they can fight against them. Some examples can be given when learning statistics, powers, logarithms and the related graphical interpretation.
Environment and Sustainability: Integration of Environment, Climate Change and Sustainability in the curriculum focuses on and advocates for the need to balance economic growth, society well-being and ecological systems. Students need basic	Using Real life models or students' experience, Mathematics Teachers should lead students to illustrate the situation of "population growth" and discuss its effects on the environment and sustainability.

knowledge from the natural sciences, social sciences, and humanities to understand to interpret principles of sustainability.	
Financial Education: The integration of Financial Education into the curriculum is aimed at a comprehensive Financial Education program as a precondition for achieving financial inclusion targets and improving the financial capability of Rwandans so that they can make appropriate financial decisions that best fit the circumstances of one's life.	Through different examples and calculations on interest (simple and compound interests), interest rate problems, total revenue functions and total cost functions, supply and demand functions Mathematics Teachers can lead students to discuss how to make appropriate financial decisions.
Gender: At school, gender will be understood as family complementarities, gender roles and responsibilities, the need for gender equality and equity, gender stereotypes, gender sensitivity, etc.	Mathematics Teachers should address gender as cross-cutting issue through assigning leading roles in the management of groups to both girls and boys and providing equal opportunity in the lesson participation and avoid any gender stereotype in the whole teaching and learning process.
Inclusive Education: Inclusion is based on the right of all students to a quality and equitable education that meets their basic learning needs and understands the diversity of backgrounds and abilities as a learning opportunity.	Firstly, Mathematics Teachers need to identify/recognize students with special needs. Then by using adapted teaching and learning resources while conducting a lesson and setting appropriate tasks to the level of students, they can cater for students with special education needs. They must create opportunity where students can discuss how to cater for students with special educational needs.
Peace and Values Education: Peace and Values Education (PVE) is defined as education that promotes social cohesion, positive values, including pluralism and	Through a given lesson, a teacher should: ▪ Set a learning objective which is addressing positive attitudes and values, ▪ Encourage students to develop the

personal responsibility, empathy, critical thinking and action in order to build a more peaceful society.	culture of tolerance during discussion and to be able to instil it in colleagues and cohabitants; ▪ Encourage students to respect ideas from others.
Standardization Culture: Standardization Culture in Rwanda will be promoted through formal education and plays a vital role in terms of health improvement, economic growth, industrialization, trade and general welfare of the people through the effective implementation of Standardization, Quality Assurance, Metrology and Testing.	With different word problems related to the effective implementation of Standardization, Quality Assurance, Metrology and Testing, students can be motivated to be aware of health improvement, economic growth, industrialization, trade and general welfare of the people.

1.2.3 Guidance on how to help students with special education needs in classroom

In the classroom, students learn in different way depending to their learning pace, needs or any other special problem they might have. However, the teacher has the responsibility to know how to adopt his/her methodologies and approaches in order to meet the learning need of each student in the classroom. Also, teachers need to understand that student with special needs, need to be taught differently or need some accommodations to enhance the learning environment. This will be done depending to the subject and the nature of the lesson.

In order to create a well-rounded learning atmosphere, teachers need to:

- Remember that students learn in different ways so they have to offer a variety of activities (e.g., role-play, music and singing, word games and quizzes, and outdoor activities);
- Maintain an organized classroom and limits distraction. This will help students with special needs to stay on track during lesson and follow instruction easily;
- Vary the pace of teaching to meet the needs of each student. Some students process information and learn more slowly than others;
- Break down instructions into smaller, manageable tasks. Students with special needs often have difficulty understanding long-winded or several instructions at once. It is

better to use simple, concrete sentences in order to facilitate them understand what you are asking.

- Use clear consistent language to explain the meaning (and demonstrate or show pictures) if you introduce new words or concepts;
- Make full use of facial expressions, gestures and body language;
- Pair a student who has a disability with a friend. Let them do things together and learn from each other. Make sure the friend is not over protective and does not do everything for the one with disability. Both students will benefit from this strategy;
- Use multi-sensory strategies. As all students learn in different ways, it is important to make every lesson as multi-sensory as possible. Students with learning disabilities might have difficulty in one area, while they might excel in another. For example, use both visual and auditory cues.

Below are general strategies related to each main category of disabilities and how to deal with every situation that may arise in the classroom. However, the list is not exhaustive because each student is unique with different needs and that should be handled differently.

Strategy to help students with developmental impairment:

- Use simple words and sentences when giving instructions;
- Use real objects that students can feel and handle. Rather than just working abstractly with pen and paper;
- Break a task down into small steps or learning objectives. The student should start with an activity that she/he can do already before moving on to something that is more difficult;
- Gradually give the student less help;
- Let the student with disability work in the same group with those without disability.

Strategy to help students with visual impairment:

- Help students to use their other senses (hearing, touch, smell and taste) and carry out activities that will promote their learning and development;
- Use simple, clear and consistent language;
- Use tactile objects to help explain a concept;
- If the student has some sight, ask him/her what he/she can see;
- Make sure the student has a group of friends who are helpful and who allow him/her to be as independent as possible;

- Plan activities so that students work in pairs or groups whenever possible;

Strategy to help students with hearing disabilities or communication difficulties

- Always get the student's attention before you begin to speak;
- Encourage the student to look at your face;
- Use gestures, body language and facial expressions;
- Use pictures and objects as much as possible.
- Keep background noise to a minimum.

Strategies to help students with physical disabilities or mobility difficulties:

- Adapt activities so that students who use wheelchairs or other mobility aids, can participate.
- Ask parents/caregivers to assist with adapting furniture e.g., the height of a table may need to be changed to make it easier for a student to reach it or fit their legs or wheelchair under;
- Encourage peer support when needed;
- Get advice from parents or a health professional about assistive devices if the student has one.

Adaptation of assessment strategies:

At the end of each unit, the teacher is advised to provide additional activities to help students achieve the key unit competence. These assessment activities are for remedial, consolidation and extension designed to cater for the needs of all categories of students; slow, average and gifted students respectively. Therefore, the teacher is expected to do assessment that fits individual students.

Remedial activities	After evaluation, slow students are provided with lower order thinking activities related to the concepts learnt to facilitate them in their learning. These activities can also be given to assist deepening knowledge acquired through the learning activities for slow students.
Consolidation activities	After introduction of any concept, a range number of activities can be provided to all students to enhance/ reinforce learning.

Extended activities	After evaluation, gifted and talented students can be provided with high order thinking activities related to the concepts learnt to make them think deeply and critically. These activities can be assigned to gifted and talented students to keep them working while other students are getting up to required level of knowledge through the learning activity.

1.2.4. Guidance on assessment

Assessment is an integral part of teaching and learning process. The main purpose of assessment is for improvement of learning outcomes. Assessment for learning/ Continuous/ formative assessment intends to improve students' learning and teacher's teaching whereas assessment of learning/summative assessment intends to improve the entire school's performance and education system in general.

Continuous/ formative assessment

It is an on-going process that arises during the teaching and learning process. It includes lesson evaluation and end of sub unit assessment. This formative assessment should play a big role in teaching and learning process. The teacher should encourage individual, pair and group evaluation of the work done in the classroom and uses appropriate competence-based assessment approaches and methods.

Formative assessment is used to:

- Determine the extent to which learning objectives are being achieved and competences are being acquired and to identify which students need remedial interventions, reinforcement as well as extended activities. The application activities are developed in the student
- book and they are designed to be given as remedial, reinforcement, end lesson assessment, homework or assignment
- Motivate students to learn and succeed by encouraging students to read, or learn more, revise, etc.
- Check effectiveness of teaching methods in terms of variety, appropriateness, relevance, or need for new approaches and strategies. Mathematics teachers need to

consider various aspects of the instructional process including appropriate language levels, meaningful examples, suitable methods and teaching aids/ materials, etc.

- Help students to take control of their own learning.

In teaching Mathematics, formative or continuous assessment should compare performance against instructional objectives. Formative assessment should measure the student's ability with respect to a criterion or standard. For this reason, it is used to determine what students can do, rather than how much they know.

Summative assessment

The assessment can serve as summative and informative depending to its purpose. The end unit assessment will be considered summative when it is done at end of unit and want to start a new one.

It will be formative assessment, when it is done in order to give information on the progress of students and from there decide what adjustments need to be done.

The assessment done at the end of the term, end of year, is considered as summative assessment so that the teacher, school and parents are informed of the achievement of educational objective and think of improvement strategies. There is also end of level/ cycle assessment in form of national examinations.

When carrying out assessment?

Assessment should be clearly visible in lesson, unit, term and yearly plans.

- Before learning (diagnostic): At the beginning of a new unit or a section of work; assessment can be organized to find out what students already know / can do, and to check whether the students are at the same level.
- During learning (formative/continuous): When students appear to be having difficulty with some of the work, by using on-going assessment (continuous). The assessment aims at giving students support and feedback.
- After learning (summative): At the end of a section of work or a learning unit, the Mathematics Teacher has to assess after the learning. This is also known as Assessment of Learning to establish and record overall progress of students towards full achievement. Summative assessment in Rwandan schools mainly takes the form of written tests at the end of a learning unit or end of the month, and examinations at the end of a term, school year or cycle.

Instruments used in assessment.

- **Observation:** This is where the mathematics teacher gathers information by watching students interacting, conversing, working, playing, etc. A teacher can use observations to collect data on behaviours that are difficult to assess by other methods such as attitudes, values, and generic competences and intellectual skills. It is very important because it is used before the lesson begins and throughout the lesson since the teacher has to continue observing each and every activity.

- **Questioning**
  (a) Oral questioning: a process which requires a student to respond verbally to questions
  (b) Class activities/ exercise: tasks that are given during the learning/ teaching process
  (c) Short and informal questions usually asked during a lesson
  (d) Homework and assignments: tasks assigned to students by their teachers to be completed outside of class.

Homework assignments, portfolio, project work, interview, debate, science fair, Mathematics projects and Mathematics competitions are also the different forms/instruments of assessment.

1.2.5. Teaching methods and techniques that promote active learning

The different learning styles for students can be catered for, if the teacher uses active learning whereby students are really engaged in the learning process.

The main teaching methods used in Mathematics are the following:

- **Dogmatic method** (the teacher tells the students what to do, what to observe, how to attempt, how to conclude)
- **Inductive-deductive method:** Inductive method is to move from specific examples to generalization and deductive method is to move from generalization to specific examples.
- **Analytic-synthetic method:** Analytic method proceeds from unknown to known, 'Analysis' means 'breaking up' of the problem in hand so that it ultimately gets connected with something obvious or already known. Synthetic method is the opposite of the analytic method. Here one proceeds from known to unknown.

- **Skills lab method:** Skills lab method is based on the maxim "learning by doing." It is a procedure for stimulating the activities of the students and to encourage them to make discoveries through practical activities.
- **Problem solvingmethod, Projectmethod and Seminar Method.**

The following are some active techniques to be used in Mathematics:

- Group work
- Research
- Probing questions
- Practical activities (drawing, plotting, interpreting graphs)
- Modelling
- Brainstorming
- QuizTechnique
- DiscussionTechnique
- Scenario buildingTechnique

What is Active learning?

Active learning is a pedagogical approach that engages students in doing things and thinking about the things they are doing. Students play the key role in the active learning process. They are not empty vessels to fill but people with ideas, capacity and skills to build on for effective learning. Thus, in active learning, students are encouraged to bring their own experience and knowledge into the learning process.

The role of the teacher in active learning	**The role of students in active learning**
- The teacher engages students through active learning methods such as inquiry methods, group discussions, research, investigative activities, group and individual work activities. - He/she encourages individual, peer and group evaluation of the work done in the classroom and uses appropriate competence-based assessment	A learner engaged in active learning: - Communicates and shares relevant information with fellow students through presentations, discussions, group work and other learner-centred activities (role play, case studies, project work, research and investigation); - Actively participates and takes responsibility for his/her own learning;

approaches and methods. - He provides supervised opportunities for students to develop different competences by giving tasks which enhance critical thinking, problem solving, research, creativity and innovation, communication and cooperation. - Teacher supports and facilitates the learning process by valuing students' contributions in the class activities.	- Develops knowledge and skills in active ways; - Carries out research/investigation by consulting print/online documents and resourceful people, and presents their findings; - Ensures the effective contribution of each group member in assigned tasks through clear explanation and arguments, critical thinking, responsibility and confidence in public speaking - Draws conclusions based on the findings from the learning activities.

Main steps for a lesson in active learning approach

All the principles and characteristics of the active learning process highlighted above are reflected in steps of a lesson as displayed below. Generally, the lesson is divided into three main parts whereby each one is divided into smaller steps to make sure that students are involved in the learning process. Below are those main part and their small steps:

1) Introduction

Introduction is a part where the teacher makes connection between the current and previous lesson through appropriate technique. The teacher opens short discussions to encourage students to think about the previous learning experience and connect it with the current instructional objective. The teacher reviews the prior knowledge, skills and attitudes which have a link with the new concepts to create good foundation and logical sequencings.

2) Development of the new lesson

The development of a lesson that introduces a new concept will go through the following small steps: discovery activities, presentation of students' findings, exploitation, synthesis/summary and exercises/application activities.

- ❖ **Discovery activity**

Step 1:

- The teacher discusses convincingly with students to take responsibility of their learning
- He/she distributes the task/activity and gives instructions related to the tasks (working in groups, pairs, or individual to prompt / instigate collaborative learning, to discover knowledge to be learned)

Step 2:

- The teacher let students work collaboratively on the task;
- During this period the teacher refrains to intervene directly on the knowledge;
- He/she then monitors how the students are progressing towards the knowledge to be learned and boosts those who are still behind (but without communicating to them the knowledge).

❖ **Presentation of students' findings/productions**

- In this part, the teacher invites representatives of groups to present their productions/findings.
- After three/four or an acceptable number of presentations, the teacher decides to engage the class into exploitation of students' productions.

❖ **Exploitation of students' findings/ productions**

- The teacher asks students to evaluate the productions: which ones are correct, incomplete or false
- Then the teacher judges the logic of the students' products, corrects those which are false, completes those which are incomplete, and confirms those which are correct.

❖ **Institutionalization or harmonization (summary/conclusion/ and examples)**

- The teacher summarizes the learned knowledge and gives examples which illustrate the learned content.

❖ **Application activities**

- Exercises of applying processes and products/objects related to learned unit/sub-unit
- Exercises in real life contexts
- Teacher guides students to make the connection of what they learnt to real life situations.
- At this level, the role of teacher is to monitor the fixation of process and product/object being learned.

3) Assessment

In this step the teacher asks some questions to assess achievement of instructional objective. During assessment activity, students work individually on the task/activity. The teacher avoids intervening directly. In fact, results from this assessment inform the teacher on next steps for the whole class and individuals. In some cases, the teacher can end with a homework/ assignment. Doing this will allow students to relay their understanding on the concepts covered that day. Teacher leads them not to wait until the last minute for doing the homework as this often results in an incomplete homework set and/or an incomplete understanding of the concept.

PART II: SAMPLE LESSON

School:…… **Teacher's Names…………………………..**

Term	Date	Subject	Class	Unit N°	Lesson N°	Duration	Class size
------	----/------/-----	MATHEMATICS	S4 PCB	1	5of 7	80minutes	40 students
Type of Special Educational Needs to be catered for in this lesson and number of students in each category				2 students with hearing impairment will be seat near the teacher and the use of gestures will be improved in the lesson.			

Unit title	SET OF REAL NUMBERS
Key Unit Competence:	Think critically to understand and perform operations on the set of real numbers.
Title of the lesson	Decimal logarithms and properties
Instructional Objective	Through the given activities written on flash cards, students should be able to Use properties of decimal logarithms to solve correctly real life problems
Plan for this Class (location: in / outside)	Inside the classroom
Learning Materials (for all students)	Flash Cards, papers, Pens, Exercise Books, other supporting teaching aids such as Chalks and Chalkboard, etc...
References	S4 Student's book and Teacher's guide of Mathematics.

Steps and Timing	**Description of teaching and learning activities**	**Competences and Cross-Cutting Issues to be addressed**
	Students are organized into small groups and provided by clear instructions to discuss and work out the activity 2.7. The group members present their findings and teacher facilitates students to capture the key concepts of the lesson through harmonization. Teacher use various probing questions to guide students to explore examples and content related to properties of decimal logarithms Finally, the Students are assigned to individual tasks /	

<table>
<tr><td></td><td colspan="2">Exercise 2.7 and the collective correction is done on the chalk board.</td><td rowspan="2"></td></tr>
<tr><td></td><td>Teachers activities</td><td>Students activities</td></tr>
<tr><td>Introduction
10 min
Discovery activity (Question 1)</td><td>Decimal logarithms
- Teacher distributes flash cards to students in their small group discussions and invite them to brainstorm on the activity 2.7;
- Teacher moves around to help those who are struggling and guides them in finding definitions and properties of decimal logarithms.
- Teacher invites students to present their findings.
- Teacher harmonizes the answers from presentation.</td><td>- Students receive flashcards, discuss and brainstorm on the activity 2.7.
- They guess the definition of decimal logarithms and properties of decimal logarithms
- Group representatives present findings from groups and other students participate actively in the presentation by providing comments or by asking questions.</td><td>- Cooperation is addressed through group work where team working spirit is developed while students are working together and in small group discussions.
- Communication skills are developed through group discussions and presentation of findings.</td></tr>
<tr><td>Development of the lesson:
50 min
Discovery activity (Question 2)

- Presentation of findings</td><td>- Teacher gives instructions, invites students to brainstorm in their small groups the examples 14-18 and properties of decimal logarithm.
- Teacher moves around to each group, ask probing questions in order to help struggling students.
- Teacher invites students to present their findings.
- Teacher harmonize the students' findings and help them to summarize the learned knowledge and give examples which illustrate the properties of</td><td>- In their respective groups, Students discuss and brainstorm on examples 14-18 and discover the properties of decimal logarithm.
- Students present their findings.
- Guided by the teacher, students summarize the lesson by</td><td>- Critical thinking, problem solving skills are developed through analysing and solving Mathematical problems that involve decimal logarithm and their properties.
- Cooperation and communication skills are developed during presentations and group discussions.
- Inclusive education is addressed by providing the remediation activities and tasks to struggling students.</td></tr>
</table>

- Application activities	decimal logarithm. - Teacher asks students to individually work out the exercise 2.7 (Question 1), and then request them to do a collective correction on the chalk board	highlighting the properties of decimal logarithm, and take notes. - Individually, Students work out the exercise 2.7 (Question 1) in their textbook.	
Conclusion 20 min **- Assessment**	Teacher asks students to individually work out the exercise (questions 2-3).	Students work out independently the exercise (questions 2-3).	**Critical thinking** is developed through analysing and solving Mathematical problem that involve decimal logarithm.
Homework	Teacher gives the homework to students.	Individually, students work out the exercise 19 by giving examples where decimal logarithm is used in real life.	**Problem solving skills** is developed through giving real life examples of application of decimal logarithm. **Financial Education** is developed while students are connecting decimal logarithm with the real life problems related to production, finance and economics.
Teacher self-evaluation	To be completed after receiving the feed-back from the Students.		

PART III: UNIT DEVELOPMENT

UNIT 1: SET OF REAL NUMBERS

Note: This unit was unit 2, many teachers suggested that it may be unit 1 because it provides some prerequisites for concepts developed in the unit of trigonometry.

1.1 Key Unit Competence

Think critically to understand and perform operations on the set of real numbers.

1.2 Prerequisites

The students will perform well in this unit if they have a good background on

- Sets and operations on sets learnt in Ordinary Level (Senior 1);
- Numerical calculations;
- Sets representations (Venn diagrams).

1.3 Cross-cutting issues to be addressed

- **Inclusive education**: promote the participation of all students while teaching by providing remedial, consolidation and extended activities where needed and prepare enough and appropriate teaching and learning materials as well as providing special assistance to any case of student in need of special education needs.
- **Peace and value Education**: During group activities, teacher will encourage student to help each other and to respect opinions of colleagues.
- **Gender:** Give equal opportunities to all learners (girls and boys) to present their findings. Encourage them to participate actively in all learning and teaching activities from the beginning to the end of the lesson.

1.4 Guidance on introductory activity

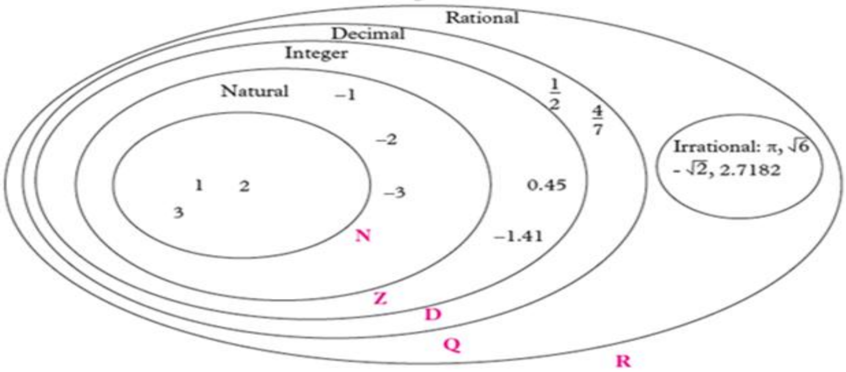
- Form small groups of students and guide them to work on the introductory **activity 2.0**
- Provide learning materials accordingly to the given activities and give clear guidance and instructions to perform the activities.
- Give time to learners to read and analyse the given activity and let them discuss about different possible solutions of the problem.
- Walk around in different groups to provide advice and facilitations where necessary and remind them to justify and support their answer / findings.
- Lead Students to recognize that the given activity should get different answers depending on the considered set of number.
- Basing on Students' experience, prior knowledge and abilities shown in answering the questions for the introductory activity, use different questions to prompt them to give their predictions and ensure that you arouse their curiosity on what is going to be leant in this unit.
- After presenting their finding, the teacher harmonizes and guides class discussions and interventions.

Answer for introductory activity 2.0

1. Lead students to know that in the question1, set of numbers they already know from senior one (S1) in secondary schools, are: ¥ , ¢ , ¤ , ¡ ,...

2. The numbers we use in counting plus zero are called Natural numbers; integers are numbers which have either negative or positive sign and includes zero. The set of integers is represented ¢ and the set of rational by ¤ ; the set of irrational numbers by numbers I form the set of real numbers. The set of real numbers is denoted by ¡

3. Some examples of numbers in each set:



4. The relationship between set of numbers is as follows: Natural numbers are part of integers, integers are part of rational numbers, rational numbers and irrational numbers are pats of real numbers. Therefore $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}$

1.5 List of lessons

Number of lesson	Lesson title	Learning objectives	Number of periods
0	Introduction on Set of real numbers.	Arise the curiosity of learners on the set of real numbers	**1**
1	Equivalent fractions, ratios, proportions, and rates	Solve problems involving equivalent fractions, ratios, proportions, and rates	**1**
2	Absolute value and its properties	Appreciate the use of absolute value in real life	**1**
3	Powers and radicals	Use properties of Powers and radicals.	**1**
4	Operation on radicals	Solve the problems in real life involving powers and radicals	**2**
5	Decimal logarithms and properties	Use properties of decimal logarithms to solve real life problems.	**2**
6	Model simple problems	Solve words problems involving	**2**

	involving decimal logarithms: population growth and decay	decimal logarithms on population growth and decay	
7	Model simple problems involving decimal logarithms: compound interest and magnitude of an earthquake	Solve words problems involving decimal logarithms on compound interest and magnitude of an earthquake	**1**
8	End unit assessment	Verify if the key unit competence was developed by students.	**1**

Lesson 1: Equivalent fractions, ratios, proportions, and rates

a) Learning objective:

Solve problems involving equivalent fractions, ratios, proportions, and rates

b) Teaching resources:

Student's book, Reference books, Ruler, Manila paper, Scientific calculators, Internet connection where applicable.

c) Prerequisites / Revision / Introduction:

Students will learn better in this lesson if they have a good understanding on concepts of shares, ratio and proportion learnt in senior 1.

d) Additional content

As a teacher, try to use this content to help students to deepen their skills and knowledge about equivalent fractions and ratios:

- **Equivalent fractions** are two or more fractions that are all equal. Therefore: $\frac{1}{2};\frac{2}{4};\frac{4}{8};\frac{5}{10}$ and $\frac{6}{12}$ are all equal and equivalent fractions.
- A **ratio** is a comparison of two quantities. The ratio of **a** to **b** can also be expressed as **a:b** or **a/b**. This relation gives us how many times one quantity is equal to the other quantity. In simple words, the ratio is the number which can be used to express one quantity as a fraction of the other ones. **Example**: The ratio of **2** to **4** is represented as **2:4 = 1:2**.
- **Proportion** is an equation which defines that the two given ratios are equivalent to each other. **Example:** The time taken by train to cover 100km per hour is equal to the time taken by it to cover the distance of 500km for 5 hours. Such as 100km/ hr = 500km/5hrs.
- **Rate:** Rates are a special type of ratio that incorporate the dimension of time into the denominator. Familiar examples include measurements of speed (kilometers per hour). If a car travels 24 kilometres in 2 hours, its average speed is a rate of 24 kilometres / 2 hours = 12 kilometres/hr.
- To understand the concept of ratio and proportion, go through the difference between ratio and proportion given here.

#	Ratio	Proportion
1	The ratio is used to compare the size of two things with the same unit	The proportion is used to express the relation of two ratios
2	It is expressed using a colon $(:)$,slash $(/)$	It is expressed using equal sign or symbol $(=)$
3	It is an expression	It is an equation
4	Keyword to identify ratio in a problem is "to every"	Keyword to identify proportion in a problem is "out of"

e) Learning activities

- Organize the students into small groups and introduce the activity to be done;
- Ask students to discuss the **activity 1. A** found below**,** and motivate them to determine how to share money respecting a given ratio. Move around to ensure all students in groups participate actively.
- Call upon groups to present their findings.
- Harmonize their findings and facilitate them to summarize the concepts of sharing quantities using ratios.
- Use different probing questions to guide students to explore examples and the content related to ratios given above.
- Guide students to perform individually the **activity 1.B** found below to assess their competences.

Activity 1.A:

1. you survey your friends about their favorite course in Physics, chemistry and Biology combination and you find that 8 out of 12 prefer chemistry and Biology.

a) Write in simplest form the fraction that represents those who do not prefer chemistry and Biology.

b) Which fraction best communicates the survey results?

c) Express to percentage, the fraction of those who prefer chemistry and Biology.

2. At a certain clinic, an hospitalized person had to pay 17,500FRW for consultation, 45,000FRW for medicine, and 30,000FRW for room in 3days. If the patient was insured by RSSB who pay 85% of the cost,

a. How much money did the RSSB pay for the patient?

b. How much money did the patient pay on his/her own?

Solution to activity 1.A:

1) a) Since $\frac{8}{12}$ prefer chemistry and Biology; $\frac{4}{12}$ do not prefer it. $\frac{4}{12}\times 100 = 33.3\%$

b) $\frac{8}{12}=\frac{2}{3}$. C) $\frac{2}{3}\times 100 = 66.7\%$

2) Total amount to be paid by the patient

$17,500\, frw + 45,000\, frw + 30,000\, frw = 92,500\, frw$

a) $\frac{85}{100}\times 92,500\, frw = 78,625\, frw$ **RSSB paid 78,625 frw**.

b) $\frac{15}{100}\times 92,500\, frw = 13,875\, frw$ **Patient paid 13875 frw**

Activity 1. B

1) In senior four physics, chemistry and biology combination, physics subject has 36 males and 48 females, whereas the Biology subject has 64 males and 80 females. You are asked to work out which cohort has the largest male to female ratio.

2) Ingabire, Mugenzi and Gahima, after investing in buying and selling of shares in the Rwanda stock exchange market, they realised a gain of 1 080 000 Frw and intend to uniquely share it in the ratio $2:3:4$ respectively. Find the share of Mugenzi.

Solution to Activity 1. B :

1) Simplifying the ratio

$36 males : 48\, females = 3 males : 4\, females (\text{for } Physics)$

$64 males : 80\, females = 4 males : 5\, females (\ for\ Bio\log y)$

Therefore, the Biology has largest ratio of males than physics.

2) Mugenzi's share

$= \frac{3}{2+3+4}$ *of* 1 080 000 FRW

$= \frac{3}{9}\times$ 1 080 000 FRW

= 360 000 FRW

Lesson 2: Absolute value and its properties

a) Learning objective:

Appreciate the use of absolute value in real life

b) Teaching resources:

Rulers, sticks, calculators, pens, pieces of chalks, etc…

c) Prerequisite:

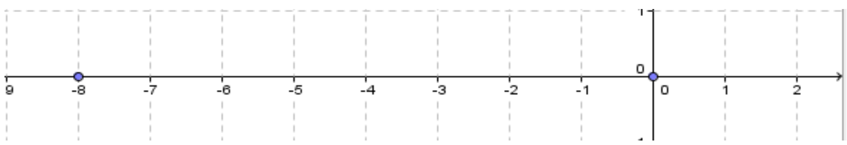
Students will perform well in this lesson if they make a revision on the absolute value of a number learnt in S2 and S3.

d) Learning activities:

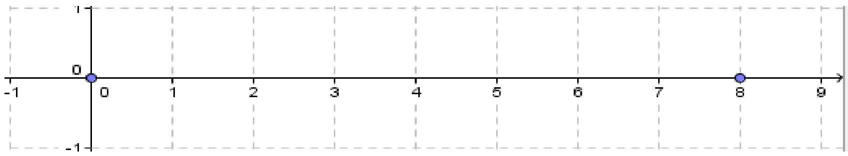
- Invite students to work in pairs the **activity 1** in student' book on absolute value.
- Ask students to share their answers with neighbouring pairs.
- Move around to different groups and verify students' works.
- Invite a member of group to present the findings while others follow for eventual comments.
- Together with students, invite them to go through 1st and the 2nd examples from content summary in student book. With clear examples, help students to discover the meaning of $|x| = \begin{cases} x, \text{ if } x \geq 0 \\ -x, \text{ if } x < 0 \end{cases}$.
- Do the same procedure above on properties of absolute value stating by the activity 2
- Insist on properties of absolute value of a number and a variable.
- Using the example of distance, guide students to brainstorm on measurements which are not expressed with negative numbers and why it can happen.
- Facilitate them to do the provided examples given in **Student's book** and work individually **Exercise 1** to check the skills they have acquired

e) Answers for activities

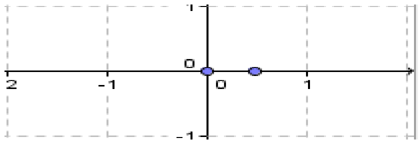
Answers for activity 1

1) 

There are 8 units between 0 and -8.

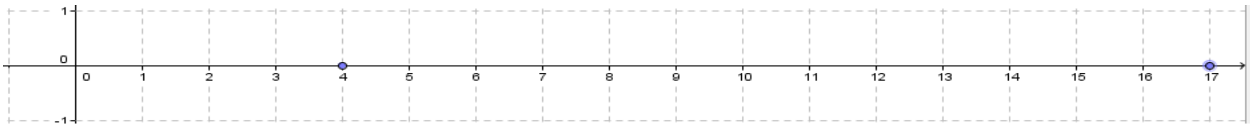
2) 

There are 8 units between 0 and 8

3) 

There are 0.5 units between 0 and $\frac{1}{2}$.

4)



There are 13 units between 4 and 17.

Answers of Exercise 1

1. -6 or 6
2. -1
3. -3 or 9
4. $\frac{-5}{2}$ or $\frac{3}{2}$
5. 1 or 5

Lesson 3: Powers and radicals

a) Learning objective

Use properties of powers and radicals.

b) Teaching resources

Rulers, sticks, Graph papers, digital materials including calculator, manila papers, etc…

c) Prerequisites

Student will perform well in this unit if they make a short revision on powers of a real number learnt in S2 and S3.

d) Learning activities

- Invite students to work in groups the **activity 3** on powers found in student' book.

- Ask students to share answers with other groups and ask support on challenging points they faced in their work.
- Move around to verify how students are working.
- Invite students to present their findings, then harmonize their answers.

- Facilitate them to do the provided examples given in **Student's book .**
- Assign students to work individually **Exercise 3** to check the skills they have acquired.
- Do the same procedure above on radical stating by the **activity 4.**
- Assign students to work individually **Exercise 4** to check the skills they have acquired.

- Provide additional activities where necessary.

e) Answers for activities

Answers for activity 3

1)

Week	**Dollars**
One	2
Two	4
Three	8
Four	16
Five	32

2)On the seventh week he will be paid 128 dollars and on the tenth week he will paid 1024 dollars.

3) Peter 's Parents Will accept the suggestion of their son because Peter 's allowance is increasing rapidly.

Answers of Exercise 3

1. x^6
2. $5x^2y^6$
3. $2y$
4. 0
5. $\frac{1}{4}$

Lesson 4: Operation on radicals

a) Learning objective:

Appreciate the importance of radicals in solving real life problems.

b) Teaching resources:

Digital materials including calculator, rulers, sticks, Graph papers, manila papers, markers, etc…

c) Prerequisites:

Student will perform well in this unit if they are enough skilled in arithmetic acquired in mathematics for senior two, unit 1.

d) Learning activities:

- In group discussions, invite student to do **activity 5** in student' book on radicals and related problems.
- Use gallery walk, student share their answers to others by rotating and ask support on challenging points they faced in their group.
- Move around to see student's progress in their respective groups.
- Invite groups with different working steps to present their answers then, harmonize the presented answers.

✓ After doing **activity 5**, use different questions and guide student to discover properties of powers and examples.

✓ Facilitate them to do the provided examples given in **Student's book** and work individually **Exercise 5**

✓ In their respective groups, invite student again to do the **activity 6** where one group will be invited to present and others will contribute with comments.

✓ Guide students to deal with operations on radicals through examples and remember to highlight different rules: simplification of radicals, rationalizing a denominator, etc.

✓ Ask student to brainstorm the use of radicals in real life: solving problems including square roots, cubic roots, etc.

✓ Facilitate them to do the provided examples given in **Student's book** and work individually **Exercise 6** to check the skills they have acquired

✓ Provide additional activities where necessary.

e) Answers for activities

Answers for activity 5

1) $\sqrt{18}+\sqrt{2}=\sqrt{9\times 2}+\sqrt{2}=3\sqrt{2}+\sqrt{2}=(3+1)\sqrt{2}=4\sqrt{2}$

2) $\sqrt{12}-3\sqrt{3}=\sqrt{4\times 3}-3\sqrt{3}=2\sqrt{3}-\sqrt{3}=(2-3)\sqrt{3}=-\sqrt{3}$

3) $\sqrt{2}\times\sqrt{3}=\sqrt{6}$

4) $\frac{\sqrt{6}}{\sqrt{2}}=\frac{\sqrt{6}\times\sqrt{2}}{\sqrt{2}\times\sqrt{2}}=\frac{\sqrt{12}}{2}=\frac{2\sqrt{3}}{2}=\sqrt{3}$

Answers of exercise 5

1. $\sqrt{20}+\sqrt{5}=3\sqrt{5}$ 2. $4\sqrt{3}-\sqrt{12}=2\sqrt{3}$ 3. $3\sqrt{7}$ 4. 12

Answers of Activity 6

1) $\frac{\sqrt{2}}{2}$ 2. $\frac{\sqrt{5}\left(2-\sqrt{3}\right)}{10}$ 3. $-\frac{2}{5}\left(1+\sqrt{6}\right)$

Answers of Exercise 6.

1. $\frac{5\sqrt{7}}{7}$

2. $1-\sqrt{2}$

3. $\frac{\sqrt{12}-\sqrt{18}-\sqrt{20}+\sqrt{50}}{4}$

4. $\frac{6\sqrt{6}-8\sqrt{2}}{23}$

5. $\frac{a\sqrt{d}-\sqrt{bd}}{d}$

6. $\frac{-3\sqrt{3}+6\sqrt{6}-2\sqrt{2}+8}{8}$

Lesson 5: Decimal logarithms and properties

a) Learning objective:

Use properties of decimal logarithms to solve real life problems.

b) Teaching resources:

Digital materials including calculator, sticks, manila papers, markers, etc…

c) Prerequisites:

Student will perform well in this lesson if they are enough skilled in properties of powers acquired in Mathematics for S2 and S3 or in previous lessons of this unit.

d) Learning activities:

- Let students work in groups and do the **activity 7** from the student' book;
- As students are working, circulate to each group and ask some questions which can lead to the objectives of this lesson;
- Ask groups to share their answers with other groups and allow them to share the challenging points they faced in their groups.
- Invite group representative to present their answers to the whole class;
- Try to harmonize students' findings;
- Ask them different questions leading them to discover the meaning of decimal logarithm of a number written in the power of 10.
- After attempting different examples, help them to formulate the decimal logarithm of a number and establish how to find it. Highlight the following properties supported with examples:

$\log x$ is the same as $\log_{10} x$ and is defined for all positive real numbers only. 10 is the base of this logarithm. In general notation we do not write this base for decimal logarithm. In the notation $y = \log x$, x is said to be the antilogarithm of y.

As a teacher, try to use this content to help students to deepen their skills and knowledge about properties of decimal logarithms.

Properties of decimal logarithms:

$\forall a, b \in \left]0, +\infty\right[$

- $\log ab = \log a + \log b$
- $\log \frac{1}{b} = -\log b$

- $\log \frac{a}{b} = \log a - \log b$
- $\log a^n = n \log a$
- $\log \sqrt{a} = \log a^{\frac{1}{2}} = \frac{1}{2} \log a$
- $\log \sqrt[m]{a^n} = \log a^{\frac{n}{m}} = \frac{n}{m} \log a$
- $\operatorname{colog} x = \log \left(\frac{1}{x} \right) = -\log x$
- Change of base formula: If u $(u > 0)$ and if a and b are positive real numbers different from 1, $\log_b u = \frac{\log_a u}{\log_a b}$ This means that if you have a logarithm in any other base, you can convert it in the decimal logarithm in the following way where $a = 10$:

$$\log_b u = \frac{\log_{10} u}{\log_{10} b} = \frac{\log u}{\log b}.$$

- There is another special logarithm called natural logarithm which has the base a number $e \approx 2.71828$. This logarithm is written as: $\log_e x = \ln x$.

- Guide students to find more examples from real life that can be solved with the intervention of logarithms.
- Invite them to do Exercise 7 to assess the competences developed.
- Provide additional activities where necessary.

e) Answers for activities

Answers for activity 7

1. The number requested is the exponent of 10 in the following expressions

$1 = 10^0$ 2. $10 = 10^1$ 3. $100 = 10^2$ 4. $1000 = 10^3$ 5. $10000 = 10^4$ 6. $100000 = 10^5$

Answers of Exercise 7

1. a) $a > b$ b) $a = b$ c) $a < b$
2. a) 2.17 b) 0.66 c) 0.30
3. a) $co\log 100 = -\log 100 = -2$

   b) $co\log 42 = -\log 42 = -1.623$

   c) $co\log 15 = -\log 15 = -1.176$

Lesson 6: Model simple problems involving decimal logarithms: population growth and decay

a) Learning objectives

Use decimal logarithms to solve population growth and decay problems .

b) Teaching resources

Student’s book and other Reference books to facilitate research, Mathematical set, calculator, manila paper, markers, pens, pencils...

c) Prerequisites/Revision/Introduction

Students will learn better in this lesson if they refer to the previous lessons of this unit and to the equations and inequalities learnt in Year 1 Unit 2 and to the lesson 1 & 2 of this Unit.

d) Additional content

As a teacher, try to use this content to help students to deepen their skills and knowledge about more application of decimal logarithms:

- **Doubling Time Model:** If D is the doubling time of a quantity (the amount of time it takes the quantity to double) and P_0 is the initial amount of the quantity then the amount of the quantity present after t units of time is $P(t) = P_0(2)^{\frac{t}{D}}$
- Half-Life Model: If H is the half-life of a quantity (the amount of time it takes the quantity be cut in half) and P_0 is the initial amount of the quantity then the amount of the quantity present after units of time is $P(t) = P_0(\frac{1}{2})^{\frac{t}{H}}$

d) Learning activities:

- Invite student to work in groups the activity 6A found below.
- Move around in the class for facilitating students where necessary and give more clarification on eventual challenges they may face during their work; Verify and identify groups with different working steps;
- Invite one member from each group with different working steps to present their work where they must explain the working steps;
- As a teacher , harmonize the findings from presentation;
- Use different probing questions and guide them to explore the content and lead them to apply decimal logarithms to solve population growth and decay problems .
- Provide additional activities where necessary.

Activity 6A:

1) Suppose that a bacteria population doubles every six hours. If the initial population is 4000 individuals, how many hours would it take the population to increase to 25,000?
2) Plutonium has a half-life of 24,000 years. Suppose that 50 pounds of it was dumped at a nuclear waste site. How long would it take for it to decay into 10 pounds?

Answer for activity 6A :

1) $P_0 = 4000 \quad and \quad D = 6$, so the doubling time model for this problem is :

$P(t) = 4000(2)^{\frac{t}{6}}$. Now, Find t when $P(t) = 25{,}000$. $25{,}000 = 4000(2)^{\frac{t}{6}}$

$6.25 = (2)^{\frac{t}{6}}$ $\log 6.25 = \log(2)^{\frac{t}{6}} \Rightarrow \log 6.25 = \left(\frac{t}{6}\right)\log 2$;

$0.7959 = \left(\frac{t}{6}\right) \times 0.3010; \frac{0.7959}{0.3010} = \frac{t}{6} \Leftrightarrow 2.644 = \frac{t}{6} \Rightarrow t = 15.9$; The population would increase to 25,000 bacteria in approximately 15.9 hours.

2) $P_0 = 50 \; and \; H = 24{,}000$, So, the half-life model for this problem is: $P(t) = 50(\frac{1}{2})^{\frac{t}{24{,}000}}$;

Now, find t when $P(t) = 10$

$$10 = 50\left(\frac{1}{2}\right)^{\frac{t}{24000}} \Rightarrow \frac{10}{50} = \left(\frac{1}{2}\right)^{\frac{t}{24000}}$$

$0.2 = \left(\frac{1}{2}\right)^{\frac{t}{24000}}$ Now, take the log of both sides of the equation.

$$\log 0.2 = \log\left(\frac{1}{2}\right)^{\frac{t}{24000}} \Rightarrow \log 0.2 = \left(\frac{t}{24000}\right)\log\left(\frac{1}{2}\right)$$

$$-0.6990 = \left(\frac{t}{24{,}000}\right) \times (-0.3010) \Rightarrow \frac{-0.6990}{-0.3010} = \frac{t}{24{,}000}$$

$$2.322 = \frac{t}{24{,}000} \Rightarrow t = 55{,}728$$

The quantity of plutonium would decrease to 10 pounds in approximately 55,728 years.

Lesson 7: Model simple problems involving decimal logarithms: compound interest and magnitude of an earthquake

a) Learning objectives

To apply decimal logarithms to solve problems related to compound interest and on magnitude of an earthquake.

b) Teaching resources

Student's book and other reference books to facilitate research, calculator, manila paper, markers, pens, pencils, etc.

c) Prerequisites/Revision/Introduction

Students will learn better in this lesson if they have a good background on the equations and inequalities learnt in Year 1 Unit 2 and on the previous lessons of this Unit.

d) Additional content

- A bank pays interest of $r\%$ per annum compounded quarterly. If P is placed in a savings account and the quarterly interest is left in the account, how much money is in the account after 1 year? The amount *A* after *t* years due to a principal *P* invested at an annual interest rate *r* compounded *n* times per year is

  $$A = P.\left(1+\frac{r}{n}\right)^{nt}$$

- Magnitudes of **earthquakes** are measured using the **Richter scale**. On this scale, the magnitude R of an earthquake is given by $R=\log\left(\frac{I}{I_0}\right)$ where I_0 is a fixed standard intensity used for comparison, and I is the intensity of earthquakes being measured.

e) Learning activities:

- Invite student to work in groups and do the following **activity 7 A** found bellow.
- Move around in the class for facilitating students where necessary and give more clarification on eventual challenges they may face during their work; Verify and identify groups with different working steps;
- Invite one member from each group with different working steps to present their work where they must explain the working steps;
- As a Teacher , harmonize the findings from presentation and guide students to determine the money saved or invested and the magnitude of an earthquake;

- Use different probing questions and guide them to explore the content and examples and lead them to apply logarithmic to solve problems related to compound interest and the magnitude of an earthquake;
- Provide additional activities where necessary.

Activity 7 A

1) A man deposits 800,000 FRW into his savings account on which interest is 15% per annum. If he makes no withdrawals, after how many years will his balance exceed 8 million FRW?
2) a)Show that if an earthquake measures $R=3$ on Richter scale, then its intensity is 1000 times the standard, that is , $I=1000I_0$.

   b)The San Francisco earthquake of 1906 registered $R=8.2$ on Richter scale. Express its intensity in terms of the standard intensity.

   c)How many times more intense is an earthquake measuring $R=8$ than on measuring $R=4$?

Answer for activity 7A

1) Here, the interest rate will be compound such that amount is $P\left(1+\dfrac{r}{100}\right)^n$, where n = *period of time*.

$$8000000=800000\left(1+\frac{15}{100}\right)^n$$

$$10=(1+0.15)^n$$

$$10=(1.15)^n$$

$$\log 10=\log(1.15)^n$$

$$1=n\log(1.15)$$

$$n=\frac{1}{\log(1.15)}$$

$$n\approx 16.5 \ \ years$$

2) a) If an earthquake measures $R=3$ on Richter scale,

then $\log\left(\dfrac{I}{I_0}\right)=3$

$$\Rightarrow \frac{I}{I_0} = 10^3$$

$$\Leftrightarrow I = 10^3 I_0$$
$$\Leftrightarrow I = 100 I_0$$

Therefore intensity is 1000 times the standard, that is , $I = 1000 I_0$.

b)The San Francisco earthquake of 1906 registered $R = 8.2$ on Richter scale. It means that $\log\left(\frac{I}{I_0}\right) = 8.2$ or $\frac{I}{I_0} = 10^{8.2} \Leftrightarrow I = 10^{8.2} I_0$ which is express its intensity in terms of the standard intensity.

c)Let E_1, E_2 be earthquakes measuring $R = 8$ and $R = 4$ respectively.

For $E_1 : R = 8 \Rightarrow \frac{I}{I_0} = 10^8 \Leftrightarrow I = 10^8 I_0$;

For $E_2 : R = 4 \Rightarrow \frac{I}{I_0} = 10^4 \Leftrightarrow I = 10^4 I_0$;

Intensity of E_1 is $I_1 = 10^8 I_0$ (1)

Intensity of E_2 is $I_2 = 10^4 I_0$ (2)

The ratio of two above equations yields

$$\frac{I_1}{I_2} = \frac{10^8 I_0}{10^4 I_0} = 10^4 \Rightarrow I_1 = 10^4 I_2 \Leftrightarrow I_2 = 10\,000 I_2;$$

An earthquake measuring $R = 8$ is $10\,000$ times more intense than one measuring $R = 4$.

1.6. Summary of the unit

1. **Equivalent fractions** are two or more fractions that are all equal.
   A **ratio** is a comparison of two quantities.
   The ratio of **a** to **b** can also be expressed as **a:b** or **a/b**. This relation gives us how many times one quantity is equal to the other quantity.
   Proportion is an equation which defines that the two given ratios are equivalent to each other.
   Rate: Rates are a special type of ratio that incorporate the dimension of time into the denominator.
   Absolute value of x is defined and denoted by : $|x| = \begin{cases} x, \text{if } x \geq 0 \\ -x, \text{if } x < 0 \end{cases}$

2. We call n^{th} power of a real number a that we note a^n, the product of n factors of a. that is

$$a^n = \underbrace{a.a.a...a}_{n\ factors} \begin{cases} \text{n is an exponent} \\ \text{a is the base} \end{cases}$$

3. The n^{th} root of a real number is $\frac{1}{n}$ power of that real number. It is noted by $\sqrt[n]{b}$,

$b \in \mathbb{R}, n \in \mathbb{N} \setminus \{1\}$. $\forall a, b \in \mathbb{R}, \sqrt[n]{b} = a \Leftrightarrow b^{\frac{1}{n}} = a \Leftrightarrow b = a^n$

$$\begin{cases} n \text{ is called the index} \\ b \text{ is called the base or radicand} \\ \sqrt[n]{\ } \text{ is called the radical sign} \end{cases}$$

4. Rationalizing is to convert a fraction with an irrational denominator to a fraction with rational denominator. To do this, if the denominator involves radicals we multiply the numerator and denominator by the conjugate of the denominator.

5. The decimal logarithm of a positive real number x is defined to be a real number y for which 10 must be raised to obtain x. We write $\forall x > 0, \quad y = \log x$

6. **Doubling Time Model:** If D is the doubling time of a quantity (the amount of time it takes the quantity to double) and P_0 is the initial amount of the quantity then the amount of the quantity present after t units of time is $P(t) = P_0(2)^{\frac{t}{D}}$

   Half-Life Model: If H is the half-life of a quantity (the amount of time it takes the quantity be cut in half) and P_0 is the initial amount of the quantity then the amount of the quantity present after units of time is $P(t) = P_0(\frac{1}{2})^{\frac{t}{H}}$

1.7 Additional Information for teachers

1. $\log_a x \pm y \neq \log_a(x \pm y)$ because $\log_a x \pm y = (\log_a x) \pm y$
2. $\log_a(x \pm y) \neq \log_a(x) \pm \log_a(y)$
3. $\log_a x^{-1} \neq \frac{1}{\log_a x}$ because $\log_a x^{-1} = \log_a \frac{1}{x}$ while $(\log_a x)^{-1} = \frac{1}{\log_a x}$
4. $(\log_a x)^n \neq \log_a x^n$ because $(\log_a x)^n = \log_a x \times \log_a x \times \log_a x \times ... \times \log_a x$ while

   $\log_a x^n = n \log_a x$ and

   $\sqrt{x \pm y} \neq \sqrt{x} \pm \sqrt{y}$

1.8 End unit assessment

Answers of End unit assessment

1)a) $\frac{xy^2z}{xy} = yz$, (b) $\left(ab^2\right)^3 + a^3b^6 = a^3b^6 + a^3b^6 = 2a^3b^6$

c) $\sqrt{2} - \sqrt{8} + \sqrt{18} = \sqrt{2} - 2\sqrt{2} + 3\sqrt{2} = 2\sqrt{2}$

1. a. $\frac{3\sqrt{35}+\sqrt{14}}{14}$ b. $2\sqrt{2}+2\sqrt{5}+\sqrt{6}+\sqrt{15}$ c. $\frac{-2-\sqrt{6}+\sqrt{10}+\sqrt{15}}{3}$

2. a. 1.08 b. -0.35 c. 0.56

1.9 Additional activities

1.9.1 Remedial activities

1. Suppose that $\$1000$ is invested at an interest rate of 9% compounded monthly. Find the new amount of principal after 5 years, after 10 years, and after 15 years, calculate the amount after those periods of time.

Solutions: we find that the amount after time t is given by $A = P\left(1+\frac{r}{4}\right)^{4t}$

After 5 years: $A = \$1000\left(1+\frac{0.09}{12}\right)^{12\times5} = \$1000(1.0075)^{60} = \$1565.68$

After 10 years, $A = \$1000\left(1+\frac{0.09}{12}\right)^{12\times10} = \$1000(1.0075)^{120} = \$2451.36$

After 15 years, $A = \$1000\left(1+\frac{0.09}{12}\right)^{12\times15} = \$1000(1.0075)^{180} = \$3838.04$

2. Simplify the following

a. $\sqrt{46656} = \sqrt{6^6} = 6^3 = 216$

b. $\sqrt[3]{ab}\times\sqrt[3]{a^2b^2} = \sqrt[3]{a^3b^3} = \sqrt[3]{(ab)^3} = ab$

c. $\sqrt{\frac{36}{81}} = \frac{\sqrt{36}}{\sqrt{81}} = \frac{6}{9} = \frac{2}{3}$

1.9.2 Consolidation activities

1) The number N of bacteria present in a culture at time t (in hours) obeys the function $N(t) = 1000e^{0.01t}$

a) Determine the number of bacteria at $t = 0\ hour$

b) What is the growth rate of the bacteria?

c) What is the population after 4 hours?

d) When will the number of bacteria reach 1700?

e) When will the number of bacteria double?

Solution :

$a)\ at\ t = 0\ ,$ N (0) = 1000 bacteria's

b) Growth rate of the bacteria is 0.01

c) $N(4) = 1000e^{0.04} = 1040.8\ bacterias$

d) $1700 = 1000e^{0.01t} \Rightarrow 1.7 = e^{0.01t}$

$\Rightarrow t = \frac{ln1.7}{0.01} = 53\ hours$

e) $2000 = 1000e^{0.01t} \Rightarrow t = \frac{ln2}{0.01} = 69.3\ hours$

2. Rationalise $\frac{\sqrt{3}+\sqrt{7}}{4\sqrt{2}}$

Solution: $\frac{\sqrt{3}+\sqrt{7}}{4\sqrt{2}} = \frac{\left(\sqrt{3}+\sqrt{7}\right)\sqrt{2}}{\left(4\sqrt{2}\right)\sqrt{2}} = \frac{\sqrt{6}+\sqrt{14}}{8}$

1.9.3 Extended activities

1) Rationalise

a. $\frac{\sqrt{2}}{\sqrt{5}-\sqrt{3}}$

$$\frac{\sqrt{2}}{\sqrt{5}-\sqrt{3}}=\frac{\sqrt{2}\left(\sqrt{5}+\sqrt{3}\right)}{\left(\sqrt{5}-\sqrt{3}\right)\left(\sqrt{5}+\sqrt{3}\right)}=\frac{\sqrt{10}+\sqrt{6}}{5-3}=\frac{\sqrt{10}+\sqrt{6}}{2}$$

UNIT 2: FUNDAMENTALS OF TRIGONOMETRY

Note: This unit was unit 1, many teachers suggested that it may be unit two because it needed some prerequisites in the concepts developed in the unit of the set of real numbers.

2.1 Key unit competence

Use the trigonometric concepts and formulas to solve related problems in Physics, Air navigation, Water navigation, bearings and Surveying.

2.2 Prerequisite

Student will easily learn this unit, if he/she has a good back ground on concept of

- vector representation learnt from S2 in Unit 7; Right-angled learnt from S3 ; Set of numbers learnt in unit 1; Algebraic fractions learnt from Senior 3, in unit 3; Isometries learnt from S2, unit 9.

2.3 Cross-cutting issues to be addressed

- Inclusive education (promote education for all while teaching)
- Peace and value Education (respect others view and thoughts during class discussions)
- Gender (equal opportunity of boys and girls in the lesson participation and making groups)

2.4 Guidance on introductory activity 1.0

- Student work on the introductory activity to understand the use of trigonometry.
- Let them read and do the introductory activity 1.0 in the Student' book.
- Make sure that all students are activating and performing well.
- Through class discussions, let students think of different ways of application of trigonometry.

- Through different examples, help students to understand the importance of trigonometry by showing their application in real life for example in construction, satellite systems and astronomy, naval and aviation industries, land surveying and in cartography (creation of maps) and so on.

Answer of introductory activity 1.0

Pythagoras theorem is not enough for finding the height of the given cathedral. By using sine rule, the required height can be determined as

$$\frac{\sin 60^0}{h}=\frac{\sin 30^0}{280} \Leftrightarrow h\sin 30^0=280\sin 60^0 \Leftrightarrow \frac{h}{2}=\frac{280\sqrt{3}}{2} \Leftrightarrow h=280\sqrt{3}\ ;\ 484.97m.$$

2.5 List of lessons

#	Lesson title	Learning objectives	Number of periods
0	Introduction activity	To arouse the curiosity of student teachers on the content of unit 8	1
1	Angles measurement	Appreciate the use of trigonometry in daily life situation	1
2	Unit circle	Appreciate the relationship between the trigonometric values for different angles	1
3	Definitions of trigonometric ratios	Define sine, cosine, and tangent (cosecant, secant and cotangent) of any angle	2
5	Trigonometric ratios of special angles $30^o, 45^o, 60^o$	Know trigonometric ratios of special angles $\left(30^0, 45^0, 60^0\right)$.	2
6	Trigonometric identities	Appreciate the relationship between the trigonometric values for different angles	2
7	Solving a triangle by cosine law	Use trigonometry, including the cosine rule to solve problems involving triangles	2
8	Solving a triangle by sine law	Use trigonometry, including the sine and cosine rules, to solve problems involving triangles	2
9	Application of trigonometry in real life: Bearing and Air plane directions	Use trigonometry, including the sine and cosine rules, to solve problems involving triangles and appreciate the use of trigonometry on Bearing and Air plane directions.	1
10	Application of trigonometry in real life: Navigation, Inclined plane.	Appreciate the use of trigonometry in daily life situation	1
	Assessment	Verify if the key unit competence	1

		was well developed by students.	

Lesson 1: Angles measurement

a) Learning objective: Appreciate the use of trigonometry in daily life situation

b) Teaching resources:

- T-square, ruler, protractor, compass, calculator, if possible, Math draw software as GeoGebra, Math lab….
- Student's book and other Reference textbooks to facilitate research

c) Prerequisites/Revision/Introduction:

Students will learn better this lesson if they have a good understanding concept of

- a vector learnt from S2 in Unit 7 ;
- Right-angled learnt from S3 in Unit 8.

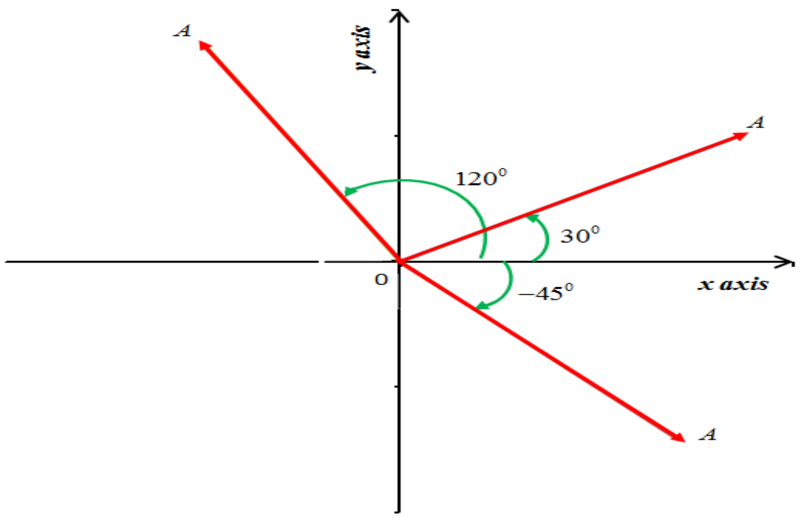
d) Learning activities

- **Guidance**

- Organize the students into groups and ask them to attempt the **Activity 1** from students book and introduce the concept of angles
- Introduce the topic by giving some examples of angles and their measurements
- Discuss with students on how a given angle is measured.
- Guide students in drawing the given angles.
- Invite one group for presentation of its work to other groups.
- Facilitate them to do the provided examples given in **Student's book** and work individually **Exercise 1** to check the skills they have acquired.
- Provide additional activities where necessary.

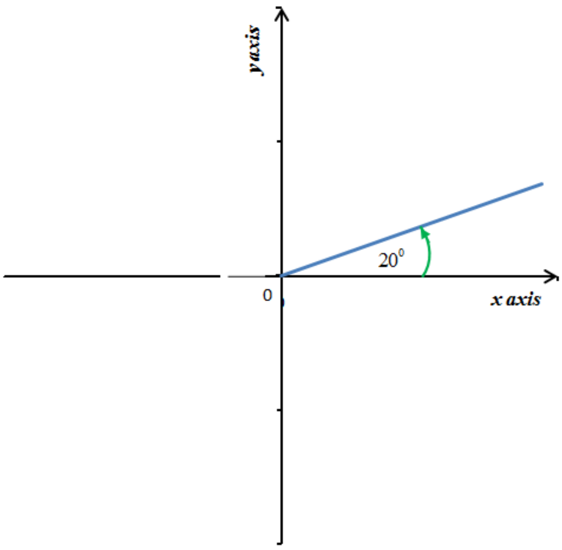
e) Answers for activities

Answers of Activity 1

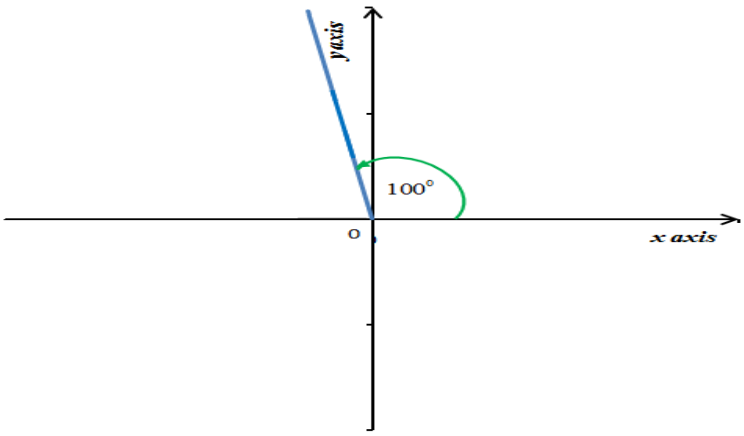


Answers of Exercise 1

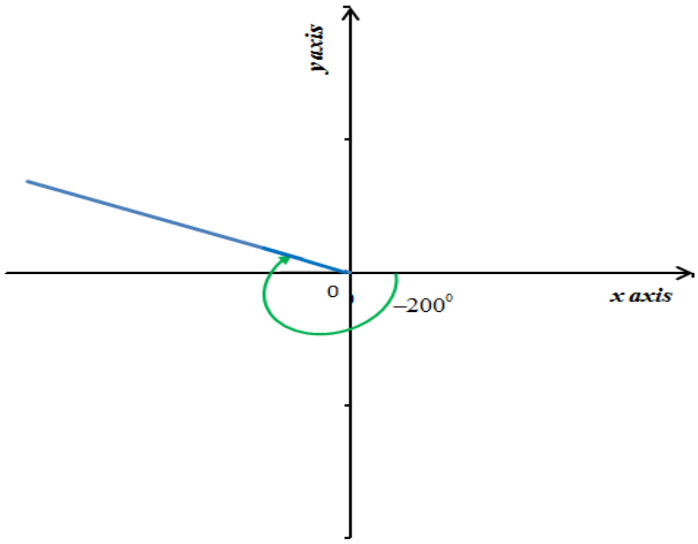
a)



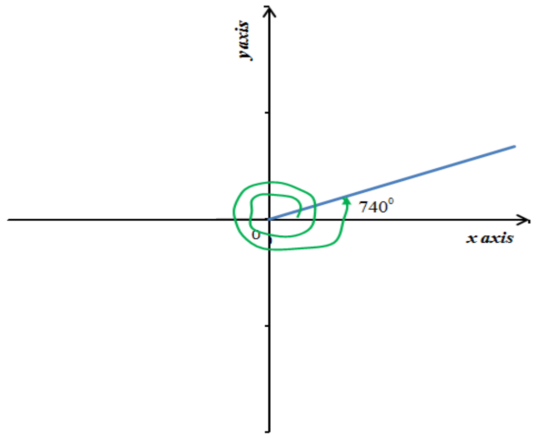
b)



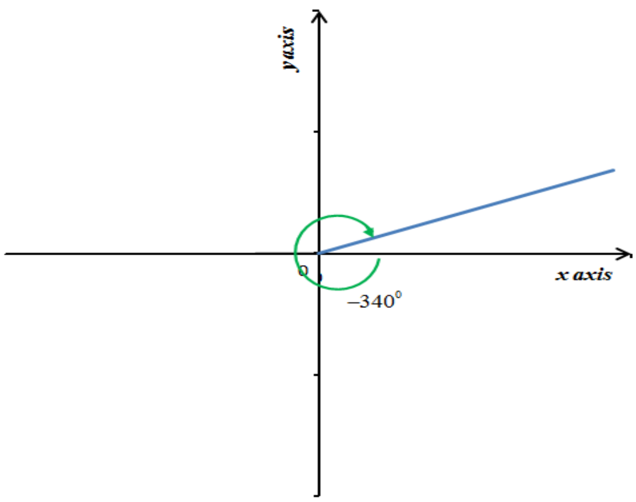
c)



d)



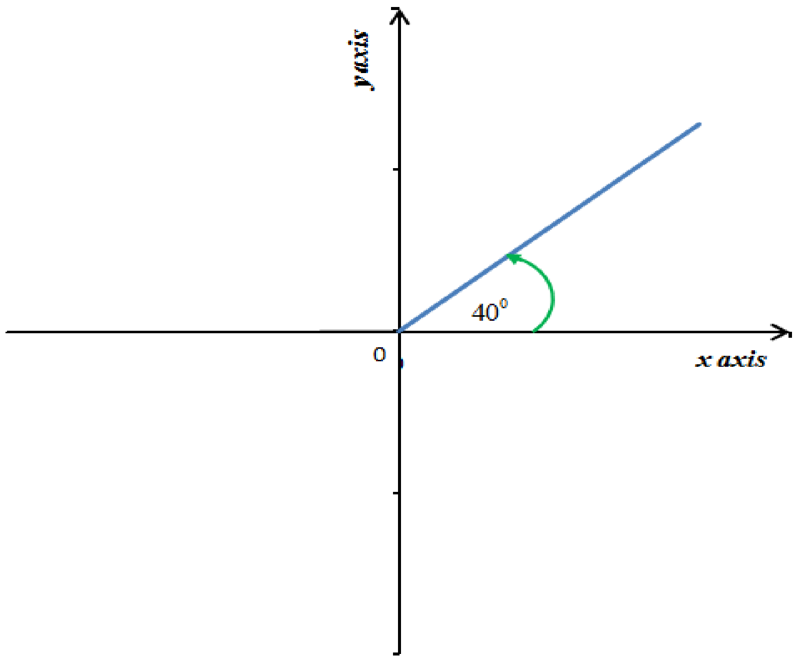
e)



740^0 and -340^0 are co terminal to 20^0

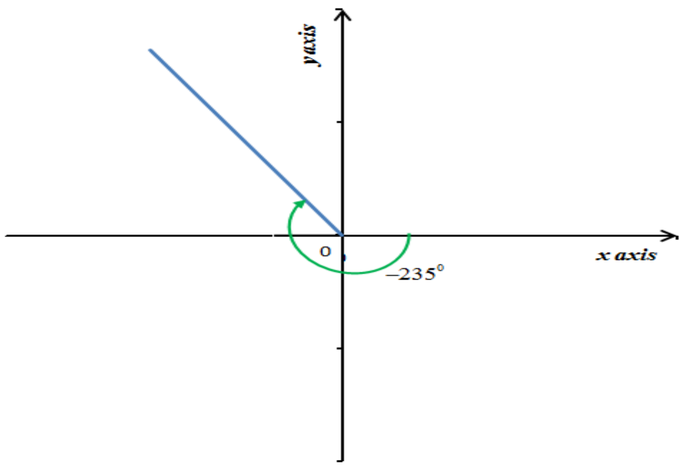
2.

a)



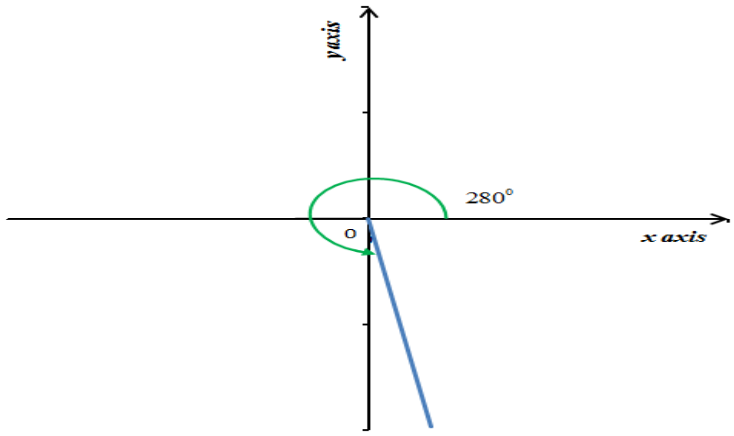
40^0 is in first quadrant

b)



-235^0 is in second quadrant

c)



280^0 is in fourth quadrant

Lesson 2: Unit circle

a) Learning objective: Appreciate the relationship between the trigonometric values for different angles

b) Teaching resources:

- Compass, T-square, ruler, protractor
- Student's book and other Reference textbooks to facilitate research

c) Prerequisites/Revision/Introduction:

Students will learn better this lesson if they have a good understanding concept of elements of a circle learnt from S3 in Unit 9.

d) Learning activities

- Organize the students into groups and ask them to attempt the **Activity 2** from students book and introduce the concept of angles Introduce the topic by giving some examples of angles and their measurements
- Discuss with students on how a given angle is measured.
- Guide students in drawing the given angles.
- Invite one group for presentation of its work to other groups.
- Facilitate them to do the provided examples given in **Student's book** and work individually **Exercise 2** to check the skills they have acquired.
- Provide additional activities where necessary.

Answers activity 2

1. $\frac{80}{3}$
2. $\frac{2480}{9}$
3. π ; $\frac{3\pi}{2}$

Answers for exercise 2

1. $\frac{11\pi}{20}$ radians , 198^0

2. $\frac{62\pi}{9}$ radians , $\frac{1240}{9}$ grades

3. 72^0

4. $5^0\ 36'\ 00''$

Lesson 3: Definitions of trigonometric ratios

a) Learning objective: Define sine, cosine, and tangent (cosecant, secant and cotangent) of any angle

b) Teaching resources:

- Ruler, T-square, Compass , protractor and calculator
- Student's book and other Reference textbooks to facilitate research

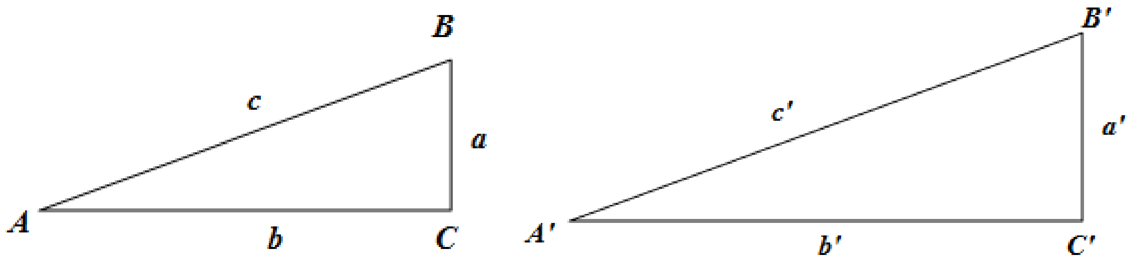
c) Prerequisites/Revision/Introduction:

Students will learn better this lesson if they have a good back ground on concept of right-angled triangle learnt from S3, in unit 8.

d) Learning activities

- **Guidance**
  - Arrange the students into groups for the discussion of **Activity 3 (in Student's book)** and make sure that everybody is engaged/ involved.
  - Facilitate students in plotting the similar figures and choices of $P(x, y)$
  - Once the group discussion is over, ask a group, chosen randomly, to present its results while other students are following attentively.
  - Facilitate them to do the provided examples given in **Student's book** and work individually **Exercise 3** to check the skills they have acquired.
  - Provide additional activities where necessary.

Answers of Activity 3



The longest side is called the hypotenuse.

Angle	$\frac{opposite\ side}{hypotenuse}$	$\frac{adjacent\ side}{hypotenuse}$	$\frac{opposite\ side}{adjacent\ side}$
A	$\frac{a}{c}$	$\frac{b}{c}$	$\frac{a}{b}$
B	$\frac{b}{c}$	$\frac{a}{c}$	$\frac{b}{a}$
C	$\frac{c}{c}=1$	$\frac{0}{c}=0$	$\frac{c}{0}$ **does not exist**

Angle	$\frac{opposite\ side}{hypotenuse}$	$\frac{adjacent\ side}{hypotenuse}$	$\frac{opposite\ side}{adjacent\ side}$
A'	$\frac{a'}{c'}$	$\frac{b'}{c'}$	$\frac{a'}{b'}$
B'	$\frac{b'}{c'}$	$\frac{a'}{c'}$	$\frac{b'}{a'}$
C'	$\frac{c'}{c'}=1$	$\frac{0}{c'}=0$	$\frac{c'}{0}$ **does not exist**

From Thales' theorem: $\frac{a}{c}=\frac{a'}{c'}, \frac{b}{c}=\frac{b'}{c'}, \frac{c}{c}=1=\frac{c'}{c'}$

For both triangle, the ratio

- Opposite side to the considered angle and hypotenuse is the same
- Adjacent side and hypotenuse is the same
- Opposite side to the considered angle and adjacent side is the same.

Answers of exercise 3

$\csc B=\frac{15}{9}=\frac{5}{3}$ $\csc A=\frac{15}{12}=\frac{5}{4}$ $\csc C=1$

$\sec B=\frac{15}{12}=\frac{5}{4}$ $\sec A=\frac{15}{9}=\frac{5}{3}$ $\sec C$ *does not exist*

$\cot B=\frac{12}{9}=\frac{4}{3}$ $\cot A=\frac{9}{12}=\frac{3}{4}$ $\cot C=0$

Lesson 4: Trigonometric ratio of special angles $30^{\circ}, 45^{\circ}, 60^{\circ}$

a) Learning objective: Know trigonometric ratios of special angles $\left(30^{0}, 45^{0}, 60^{0}\right)$.

b) Teaching resources:

- Ruler, T-square, Compass
- Student's book and other Reference textbooks to facilitate research

c) Prerequisites/Revision/Introduction:

Students will learn better this lesson if they have a good understanding concept of

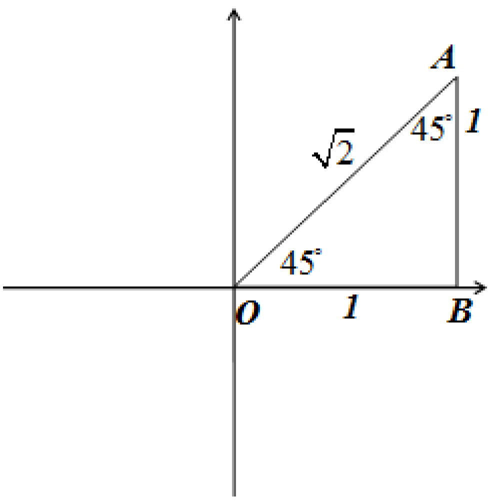
- Definition of trigonometric ratios learnt in previous lesson
- Pythagoras theorem learnt from S2, in unit 6.

d) Learning activities

- **Guidance**
  - Let Students attempt the **Activity 4 (Student's book)** and motivate them so that everybody is engaged/ involved.
  - Guide students to get trigonometric ratios of the given angle from definition.
  - Use triangles to assist students define trigonometric ratios related to 30°, 45° and 60°.
  - Invite one group for presentation of its findings to other groups.
  - Facilitate them to do the provided examples given in **Student's book** and work individually **Exercise 4** to check the skills they have acquired
  - Provide additional activities where necessary.

Answers of Activity 4

1) $\sin 45^{\circ}, \cos 45^{\circ} \textit{and} \tan 45^{\circ}$

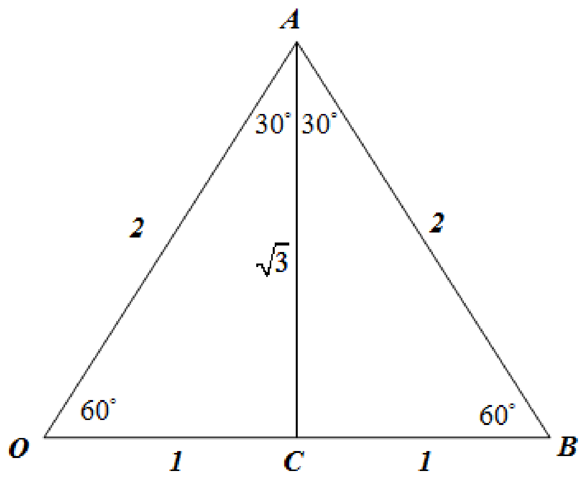


From the diagram

$\sin 45^{\circ} = \frac{1}{\sqrt{2}}$ $\quad\quad$ $\cos 45^{\circ} = \frac{1}{\sqrt{2}}$ $\quad\quad$ $\tan 45^{\circ} = \frac{1}{1} = 1$

Answer of question 2:

The six trigonometric values of 60^o *and* 30^o



From ΔOAC,

$$\sin 60^0 = \frac{\sqrt{3}}{2}, \cos 60^0 = \frac{1}{2},\ \tan 60^0 = \sqrt{3},\ \csc 60^0 = \frac{2\sqrt{3}}{3},\ \sec 60^0 = 2,\ \cot 60^0 = \frac{\sqrt{3}}{3}.$$

From ΔOAC,

$$\sin 30^0 = \frac{1}{2}, \cos 30^0 = \frac{\sqrt{3}}{2},\ \tan 30^0 = \frac{\sqrt{3}}{3},\ \csc 30^0 = 2,\ \sec 30^0 = \frac{2\sqrt{3}}{3},\ \cot 30^0 = \sqrt{3}.$$

Answers of exercise 4

1) 1   2) $\sqrt{3}$ 3) $\frac{\sqrt{3}}{3}$ 4) 0

5) does not exist   6) 0   7) does not exist   8) does not exist

9) 0   10) does not exist   11) 0

Lesson 5: Trigonometric identities

a) Learning objective: Appreciate the relationship between the trigonometric values for different angles

b) Teaching resources:

- Student's book and other Reference textbooks to facilitate research

c) Prerequisites/Revision/Introduction:

Students will learn better this lesson if they have a good understanding on

- Definition of trigonometric ratios learnt from previous lessons
- Pythagoras theorem learnt from S2, unit 6.

d) Learning activities

- **Guidance**

- Let Students attempt the **Activity 6 (Student's book)** and introduce the concept of trigonometric ratios
- Guide students in constructing right-angled triangle and labelling it
- Facilitate students in applying Pythagoras theorem
- Verify whether every student is engaged
- Facilitate them to do the provided examples given in **Student's book** and work individually **Exercise 6** to check the skills they have acquired **.**
- Provide additional activities where necessary.

Answers of Activity 6

In this triangle,

$$\sin\theta = \frac{y}{r},\ \cos\theta = \frac{x}{r},\ \sin\alpha = \frac{x}{r},\ \cos\alpha = \frac{y}{r}$$

$$\left(\frac{x}{r}\right)^2 + \left(\frac{y}{r}\right)^2 = (\sin\theta)^2 + (\cos\theta)^2 = \sin^2\theta + \cos^2\theta \text{ and then } \sin^2\theta + \cos^2\theta = 1$$

$$\sin^2\alpha + \cos^2\alpha = \left(\frac{x}{r}\right)^2 + \left(\frac{y}{r}\right)^2 = 1$$

Answer of Exercise 6

1)1

2) 0

3) $\frac{1+\tan a}{3-\tan a}$

Lesson 6: Solving a triangle by cosine law

a) Learning objective: Use trigonometry, including the cosine and cosine rules, to solve problems involving triangles

b) Teaching resources:

a) Ruler, T-square, protractor and compass

b) Student's book and other Reference textbooks to facilitate research

c) Prerequisites/Revision/Introduction:

Students will learn better this lesson if they have a good understanding on concept of

- Definition of trigonometric ratios
- Pythagoras theorem

d) Learning activities

•Guidance

- Let Students attempt the **Activity 7 (Student's book)** by introducing the concept of right-angled triangle.
- Facilitate the students while comparing the sides of the given triangle
- Make sure that everybody is engaged/ involved
- Invite one group for presentation of its work to other groups.
- Facilitate them to do the provide examples given in **Student's book and** work individually **Exercise 7** to check the skills they have acquired.
- Provide additional activities where necessary.

Answers of Activity 7

1. $\cos A = \frac{AX}{b}$

2. $b^2 = h^2 + (AX)^2 \Rightarrow h^2 = b^2 - (AX)^2$

3. $a^2 = h^2 + (XB)^2 \Rightarrow h^2 = a^2 - (XB)^2$

4. $h^2 = b^2 - (AX)^2$ and $h^2 = a^2 - (XB)^2$ gives $b^2 - (AX)^2 = a^2 - (XB)^2$

But $XB = c - AX$, then

$b^2 - (AX)^2 = a^2 - (c - AX)^2$

$\Leftrightarrow b^2 - (AX)^2 = a^2 - \left(c^2 - 2cAX + (AX)^2\right)$

$\Leftrightarrow b^2 - (AX)^2 = a^2 - c^2 + 2cAX - (AX)^2$

$\Leftrightarrow b^2 + c^2 - 2cAX = a^2$

But $\cos A = \frac{AX}{b} \Rightarrow AX = b\cos A$. Then

$\Leftrightarrow b^2 + c^2 - 2cb\cos A = a^2$

$\Leftrightarrow a^2 = b^2 + c^2 - 2bc\cos A$

Answers of exercise 7

1. $9.43\,cm$

2. $c = 21.7\,cm$

3. $A = 12.7^0,\ B = 22.3^0, c = 14.4\,cm$

Lesson 7: Solving a triangle by sine law

a) Learning objective: Use trigonometry, including the sine rule to solve problems involving triangles

b) Teaching resources:

c) Ruler, T-square, protractor

d) Student's book and other Reference textbooks to facilitate research

c) Prerequisites/Revision/Introduction:

Students will learn better this lesson if they have a good understanding on concept of

- Definition of trigonometric ratios
- Pythagoras theorem

d) Learning activities

e) Guidance

- Let Students attempt the **Activity 8 (Student's book)** by introducing the concept of right-angled triangle.
- Facilitate the students while comparing the sides of the given triangle.
- Invite one group for presentation of its work to other groups.
- Make sure that everybody is engaged/ involved.
- Facilitate them to do the provided examples given in **Student's book** and work individually **Exercise 8** to check the skills they have acquired.
- Provide additional activities where necessary.

Answers of Activity 8

1. $\sin B = \frac{h}{a}, \sin A = \frac{h}{b}$.

$h = a\sin B$ and $b\sin A = h$, then $a\sin B = b\sin A$ or $\frac{a}{\sin A} = \frac{b}{\sin B}$

2. $\sin A = \frac{k}{c}, \sin C = \frac{k}{a}$.

$k = c\sin A$ and $k = a\sin C$, then $c\sin A = a\sin C$ or $\frac{c}{\sin C} = \frac{a}{\sin A}$

3. Now, $\frac{a}{\sin A} = \frac{b}{\sin B}$ and $\frac{c}{\sin C} = \frac{a}{\sin A}$. This gives $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$

Answers of exercise 8

1. $8.45cm$

2. $A = 56.8^0$, $C = 81.2^0$ **or** $A = 123.2^0$, $C = 14.8^0$

3. $6.18cm$

Lesson 8: Application of trigonometry in real life: Bearing and Air plane directions

a) Learning objective: Use trigonometry, including the sine and cosine rules, to solve problems involving triangles and appreciate the use of trigonometry on Bearing and Air plane directions.

b) Teaching resources:

a) Ruler, T-square, protractor, calculator

b) Student's book and other Reference textbooks to facilitate research

c) Prerequisites/Revision/Introduction:

Students will learn better this lesson if they have a good understanding on concept of

- Trigonometric ratios learnt from previous lessons of this unit
- Sine law learnt in 6^{th} lesson of this unit
- Cosine law learnt in 7^{th} lesson of this unit

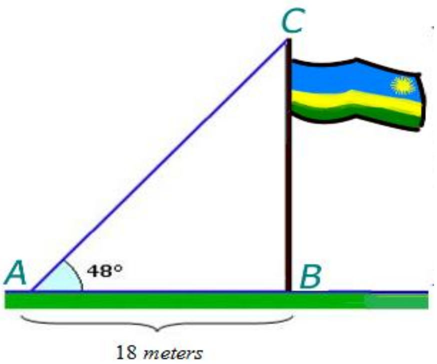
d) Learning activities

c) Guidance

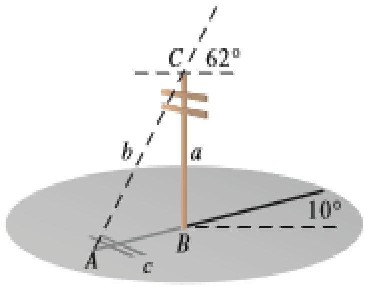
- Let Students attempt the **activity 8 A** find below and make a review on sine and cosine laws
- Guide students on how to solve triangle.
- Facilitate them while solving the given activity by checking if everybody is engaged
- Make sure that everybody is engaged/ involved.
- Help the students to use trigonometry, including the sine and cosine rules, to solve problems involving triangles.
- Ask them to state where we can apply trigonometric ratios.
- Let one group member to present their findings
- Facilitate them to do the provided examples given in **Student's book** to emphasize the skills, they have got.
- Provide additional activities where necessary.

Activity 8 A :

1. A tree is located on an incline of a hill. The tree is broken and the tip of the tree touches the hill farther down the hill and forms an angle of 30° with the hill. The broken part of the tree and the original tree form an angle of 50° at the break. The original part of the tree is 3 m tall. How tall was the tree before it broke?
2. Two ships are located 200 m and 300 m respectively from a lighthouse. If the angle formed by their paths to the lighthouse is 96°. What is the distance between the two ships?
3. From a tower of 32 m of height, a car is observed at an angle of depression of 55°. Find how far the car is from the tower.
4. A town B is 13 km south and 18 km west of a town A. Find the bearing and distance of B from A.
5. The angle of elevation of the top of a pole measures 48^o from a point on the ground 18 meters away from its base. Find the height of the flagpole.



6. A road slopes 10 above the horizontal, and a vertical telephone pole stands beside the road. The angle of elevation of the Sun is 62, and the pole casts a 14.5 metre shadow downhill along the road. Find the height of the telephone pole.



Answers of Activity 8 A

1. The tree was *8.91 m* tall.
2. The distance is *369.15 m.*
3. The distance is *45.7 m*
4. The bearing of B from A is S54°10'W and the distance of B from A is *22.2 km*.
5. *20 m*
6. *24.3m*

Lesson 9: Application of trigonometry in real life: Navigation, Inclined plane.

a) Learning objective: Use trigonometry, including the sine and cosine rules, to solve problems involving triangles and appreciate the use of trigonometry in daily life situation

b) Teaching resources:

d) Ruler, T-square, protractor, calculator

e) Student's book and other Reference textbooks to facilitate research

c) Prerequisites/Revision/Introduction:

Students will learn better this lesson if they have a good understanding on concept of

- Trigonometric ratios learnt from previous lessons of this unit
- Sine law learnt in 6^{th} lesson of this unit
- Cosine law learnt in 7^{th} lesson of this unit

d) Learning activities

f) Guidance

- Let Students discuss and attempt the **Activity 9 A** find below.
- Facilitate them while solving the given activity by checking if everybody is engaged.
- Make sure that everybody is engaged/ involved.
- Help the students to use trigonometry, including the sine and cosine rules, to solve problems involving triangles.
- Let one group member to present their findings.
- Facilitate them to do the provided examples given in **Student's book** to emphasize the skills, they have got.
- Provide additional activities where necessary.

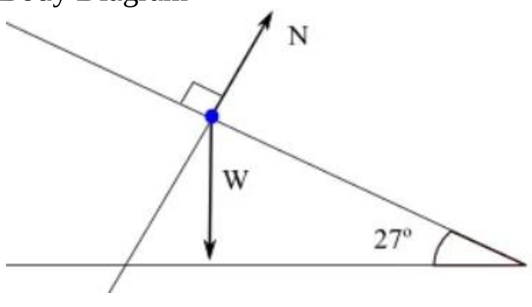
Activity 9 A

1. A 2 Kg box is put on the surface of an inclined plane at 27 ° with the horizontal. The surface of the inclined plane is assumed to be frictionless.
   a) Draw a free body diagram of the box on the inclined plane and label all forces acting on the box.
   b) Determine the acceleration a of the box down the plane.
   c) Determine the magnitude of the force exerted by the inclined plane on the box.

Solution

a)

Free Body Diagram

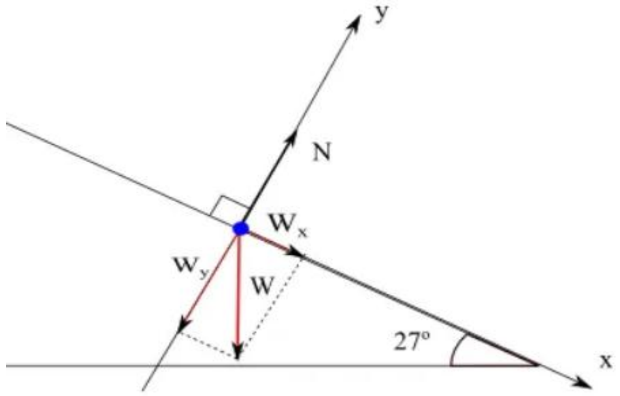


Let the small blue point be the box

Two forces act on the box: the weight W of the box and N the force normal to and exerted by the inclined plane on the box (blue point)

b)

Use system of axes x-y as shown to write all forces in their components form



vectors N, W and *a* in components form.

N = (0 , |N|)

W = (Wx , Wy) = (|W| cos (27°) , - |W| sin (27°))

a = (a_x , a_y) = (|a| , 0) , box moving down the inclined plane in the direction of positive x hence a_y = 0.

Use Newton's second law to write that the sum of all forces on the box is equal to the mass times the acceleration (vector equation)

W + N = M a , M is the mass of the box

In components form, the above equation becomes

(|W| sin (27°) , - |W| cos (27°)) + (0 , |N|) = M (|a| , 0)

For two vectors to be equal, their components must be equal. Hence

x-components are equal : |W| sin (27°) + 0 = M |a|

y-components are equal : - |W| cos (27°) + |N|= 0

M |a| = |W| sin (27°)

weight: |W| = M g ; g = 10 m/^{2}

|a| = M g sin (27°) / M = g sin (27) m/s^2 ≈ 4.5 m/s^2

c)

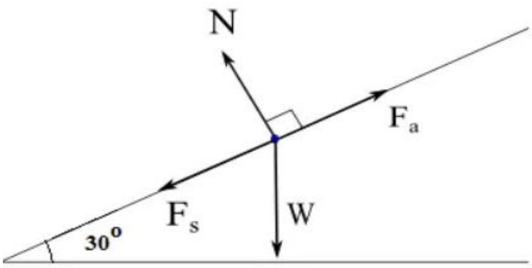
|N| = |W| cos (27°) = 2 × 10 cos (27°) ≈ 17.8 N

2. A particle of mass 5 Kg rests on a 30° inclined plane with the horizontal. A force F_a of magnitude 30 N acts on the particle in the direction parallel and up the inclined plane.
   a) Draw a Free Body Diagram including the particle, the inclined plane and all forces acting on the particle with their labels.
   b) Find the force of friction acting on the particle.
   c) Find the normal force exerted by the inclined on the particle.

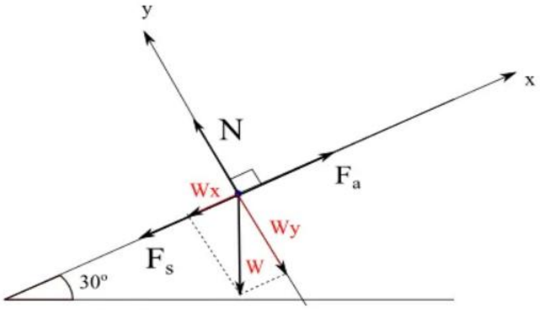
Solution

a) Free Body Diagram

The box is the small blue point. In the diagram below, W is the weight of the box, N the normal force exerted by the inclined plane on the box, F_a is the force applied to have the box in equilibrium and F_s the force of friction opposite F_a.



b)



The box is at rest, hence its acceleration is equal to 0, therefore the sum of all forces acting on the box is equal to its mass times its acceleration which is zero. (Newton's second law)

$F_a + W + N + F_s = 0$

The components form of all forces (vectors) acting on the box are:

$F_a = (30 , 0)$

$|W| = 5 \times 10 = 50$ N

$W = (W_x , W_y) = (- |W| \sin(30°) , - |W| \cos(30°)) =$

$(- 50 \sin (30°) , - 50 \cos (30°))$

$N = (0 , |N|)$

$F_s = (-|F| , 0)$

The $F_a + W + N + F_s = 0$ in components form:

$(30 , 0) + (- 50 \sin (30°) , - 50 \cos (30°)) + (0 , |N|) + (-|F| , 0) = 0$

x components: $30 - 50 \sin(30) + 0 - |F| = 0$

$|F| = - 50 \sin(30) + 30 = 5$ N

c)

y components equation:

$0 - 50 \cos (30) + |N| + 0 = 0$

$|N| = 50 \cos (30) = 25 \sqrt{3} \approx 43.3$ N

2.2. Summary of the unit

1. A **rotation angle** is formed by rotating an **initial side** through an angle, about a fixed point called **vertex**, to terminal position called **terminal side**. Angle is positive if rotated in a counter clockwise direction and negative when rotated clockwise.
2. The amount we rotate the angle is called the measure of the angle and is measured in: **degree** or **radian**.

3. To convert degree measure to radian measure, multiply by $\frac{\pi\, radian}{180^0}$.

4. To convert radian measure to degree measure, multiply by $\frac{180^0}{\pi\, radian}$.

5. In a triangle whose hypotenuse *r*, the adjacent side x and the opposite side y :

$$\sin\alpha = \frac{opposite\,side}{hypotenuse} = \frac{y}{r} = \frac{y}{\sqrt{x^2+y^2}} \qquad \cos\alpha = \frac{adjacent\,side}{hypotenuse} = \frac{x}{r} = \frac{x}{\sqrt{x^2+y^2}}$$

$$\tan\alpha = \frac{opposite\,side}{adjacent\,side} = \frac{y}{x} \qquad \cot\alpha = \frac{adjacent\,side}{opposite\,side} = \frac{x}{y}$$

$$\csc\alpha = \frac{hypotenuse}{opposite} = \frac{1}{\sin\alpha}, sec\alpha = \frac{hypotenuse}{adjascent} = \frac{1}{\cos\alpha}\ and\ \cot\alpha = \frac{adjacent}{opposite} = \frac{1}{\tan\alpha}$$

6. The table trigonometric number of remarkable angles

α	0^o	30^o	45^o	60^o	90^o	180^o	270^o	360^o
$\sin\alpha$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1	0	-1	0
$\cos\alpha$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	-1	0	1
$\tan\alpha$	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	**Does not exist**	0	**Does not exist**	0
$\cot\alpha$	**Does not exist**	$\sqrt{3}$	1	$\frac{\sqrt{3}}{3}$	0	**Does not exist**	0	**Does not exist**

7. Trigonometric identities

$$\cos^2\theta + \sin^2\theta = 1$$

$$1 + \tan^2\theta = \sec^2\theta$$

$$1 + \cot^2\theta = \csc^2\theta$$

8. Two angles are **equivalent** if their difference is $2k\pi,\ k \in ¢$ (or $360^o k,\ k \in ¢$). This means that the angle α and $\alpha + 2k\pi$ are equivalent angles.

$$\left.\begin{array}{l} \sin\left(\alpha+360^{\circ}k\right)=\sin\alpha \\ \cos\left(\alpha+360^{\circ}k\right)=\cos\alpha \\ \tan\left(\alpha+360^{\circ}k\right)=\tan\alpha \\ \cot\left(\alpha+360^{\circ}k\right)=\cot\alpha \end{array}\right\} k\in ¢$$

9. Angle $-\alpha$ is **opposite** of the angle α

$$\sin(-\alpha)=-\sin\alpha$$
$$\cos(-\alpha)=\cos\alpha$$
$$\tan(-\alpha)=-\tan\alpha$$
$$\cot(-\alpha)=-\cot\alpha$$

10. Two angles are said to be **complementary** if their sum is 90° (or $\frac{\pi}{2}$). Note that α and $90^{\circ}-\alpha$ are complementary.

$$\sin\left(90^{\circ}-\alpha\right)=\cos\alpha$$
$$\cos\left(90^{\circ}-\alpha\right)=\sin\alpha$$
$$\tan\left(90^{\circ}-\alpha\right)=\cot\alpha$$
$$\cot\left(90^{\circ}-\alpha\right)=\tan\alpha$$

11. Two angles are said to be **supplementary** if their sum is 180° (or π). It is easy to discover that α and $180^{\circ}-\alpha$ are supplementary.

$$\sin\left(180^{0}-\alpha\right)=\sin\alpha$$
$$\cos\left(180^{0}-\alpha\right)=-\cos\alpha$$
$$\tan\left(180^{0}-\alpha\right)=-\tan\alpha$$
$$\cot\left(180^{0}-\alpha\right)=-\cot\alpha$$

12. If a,b,c are the lengths of the sides of a triangle and A,B,C are the opposite angles respectively, then cosine law says that

$$\begin{cases} a^{2}=b^{2}+c^{2}-2bc\cos A \\ b^{2}=a^{2}+c^{2}-2ac\cos B \\ c^{2}=a^{2}+b^{2}-2ab\cos C \end{cases}$$

13. If a,b,c are the lengths of the sides of a triangle and A,B,C are the opposite angles respectively, then the sine law is

$$\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C} \quad \text{or} \quad \frac{\sin A}{a}=\frac{\sin B}{b}=\frac{\sin C}{c}$$

14. Applications

Many real situations involve right triangle. Using angles and trigonometric functions, we can solve problems involving right triangle like:

- Bearings and air navigation
- Angles of elevation and angle of depression
- Inclined plane

2.3. Additional Information for Teachers

Notice on reduction to a positive angle (reference angle)

The values of the circular functions of an angle, if they exist, are the same, up to a sign, of the corresponding circular functions of its reference angle. More specifically, if α is the reference angle for θ, then:

$\sin\theta = \pm\sin\alpha$	$\cos\theta = \pm\cos\alpha$	$\tan\theta = \pm\tan\alpha$
$\csc\theta = \pm\csc\alpha$	$\sec\theta = \pm\sec\alpha$	$\cot\theta = \pm\cot\alpha$

The choice of the (±) depends on the quadrant in which the terminal side of θ lies.

For example, 150^0 is a II quadrant angle, the reference angle is 30^0.

We find out that

$\sin 150^0 = \sin 30^0$ as the sine of any II quadrant angle is positive

$\cos 150^0 = -\cos 30^0$ as the cosine of any II quadrant angle is negative.

$\tan 150^0 = -\tan 30^0$ as the tangent of any II quadrant angle is negative.

Notice on identities

❖ **The Pythagorean Identities:**

1. $\cos^2\theta + \sin^2\theta = 1$ **Fundamental formula**

Alternative Forms:

- $1 - \sin^2\theta = \cos^2\theta$
- $1 - \cos^2\theta = \sin^2\theta$

2. $1 + \tan^2\theta = \sec^2\theta$, provided $\cos\theta \neq 0$.

Alternative Forms:

- $\sec^2\theta - \tan^2\theta = 1$
- $\sec^2\theta - 1 = \tan^2\theta$

3. $1 + \cot^2\theta = \csc^2\theta$, provided $\sin\theta \neq 0$.

Alternative Forms:

- $\csc^2\theta - \cot^2\theta = 1$
- $\csc^2\theta - \cot^2\theta = 1$

❖ **Pythagorean conjugates** are useful in proving trigonometric identities:

- $(1+\cos\theta)(1-\cos\theta) = 1 - \cos^2\theta = \sin^2\theta$
- $(1+\sin\theta)(1-\sin\theta) = 1 - \sin^2\theta = \cos^2\theta$
- $(\sec\theta + 1)(\sec\theta - 1) = \sec^2\theta - 1 = \tan^2\theta$
- $(\sec\theta + \tan\theta)(\sec\theta - \tan\theta) = \sec^2\theta - \tan^2\theta = 1$
- $(\csc\theta + 1)(\csc\theta - 1) = \csc^2\theta - 1 = \cot^2\theta$

- $(\csc\theta+\cot\theta)(\csc\theta-\cot\theta)=\csc^2\theta-\cot^2\theta=1$

2.8 Revision exercises

1. a. $\dfrac{\sin a(1+\cos a)}{(1-\cos a)(1+\cos a)}=\dfrac{\sin a(1+\cos a)}{\sin^2 a}$

$=\dfrac{1+\cos a}{\sin a}$

b. $\dfrac{1}{\cos^2 a}+\dfrac{1}{\sin^2 a}=\dfrac{\sin^2 a+\cos^2 a}{\cos^2 a\sin^2 a}$

$=\dfrac{1}{\cos^2 a}\dfrac{1}{\sin^2 a}$

$=\sec^2 a\csc^2 a$

c. $(\sec^2 a+\tan^2 a)(\sec^2 a-\tan^2 a)=(\sec^2 a+\tan^2 a)\left(\dfrac{1}{\cos^2 a}-\dfrac{\sin^2 a}{\cos^2 a}\right)$

$=(\sec^2 a+\tan^2 a)\left(\dfrac{1-\sin^2 a}{\cos^2 a}\right)$

$=(\sec^2 a+\tan^2 a)\left(\dfrac{\cos^2 a}{\cos^2 a}\right)$

$=\sec^2 a+\tan^2 a$

d. $\sqrt{\dfrac{(1-\cos a)(1-\cos a)}{(1+\cos a)(1-\cos a)}}=\sqrt{\dfrac{(1-\cos a)^2}{\sin^2 a}}=\dfrac{1-\cos a}{\sin a}$

2. $\tan\theta=3.18$
3. $\cos\theta=-0.8$; $\tan\theta=-0.75$
4. $\cos 14^0=\sqrt{1-\sin^2 14^0}$ *or* $\cos 14^0=\sin(90^0-14^0)=\sin 76^0$
5. Answers may vary. Some answers

a) 140^0 or 400^0 i.e. its **supplementary** or equivalent angle.

b) 124^0 or 416^0 i.e. its **supplementary** or equivalent angle.

c) 50^0 or 490^0 i.e. its **supplementary** or equivalent angle.

d) -120^0 or 660^0 i.e. its **supplementary** or equivalent angle.

6. $(\sin 90^0, \cos 360^0)$; $(\cos 180^0, \sin 270^0)$; $(\tan 195^0, \tan 15^0)$; $(\tan 150^0, \tan 330^0)$
7. $x\approx 37.7$
8. $p=60^0$
9. $h\approx 116.6$
10. $4.12m$

11. 7^0

12. a) $88.6\,km$

    b) $179.3\,km$

13. $954km, 133^0$

14. $76.5m/s$

15. 81.2189^0

16. $117^0\,39'\,26''$

17. $2^0\,56'$

18. 75.3^0

2.9. Additional activities

2.9.1 Remedial activities

1. In the following Exercises, convert the angle from degree measure into radian measure, giving the exact value in terms of π.

   a) 30^0 b) 240^0 c) 135^0 d) -270^0 e) -315^0 f) 150^0 g) 45^0 h) 225^0.

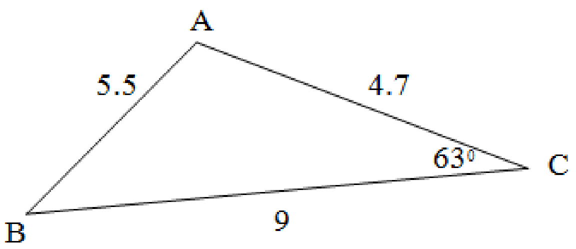
2. In the following Exercises, convert the angle from radian measure into degree measure.

   a) $\frac{2\pi}{3}$ b) $-\frac{7\pi}{6}$ c) $\frac{11\pi}{6}$ d) $\frac{\pi}{3}$ e) $-\frac{5\pi}{3}$ f) $\frac{\pi}{6}$ g) $-\frac{\pi}{2}$.

3. Find the cosine and sine of the following angles.

   a) $\theta = 270°$ b) $\theta = -\pi$ c) $\theta = 45°$ d) $\theta = \frac{\pi}{6}$ e) $\theta = 60°$

4. Using sine rule , find out the angle B



Answer:

1.

Degrees	30^0	240^0	135^0	-270^0	-315^0	150^0	45^0	225^0
radians	$\frac{\pi}{6}$	$\frac{4\pi}{3}$	$\frac{3\pi}{4}$	$-\frac{3\pi}{2}$	$-\frac{7\pi}{4}$	$\frac{5\pi}{6}$	$\frac{\pi}{4}$	$\frac{5\pi}{4}$

2.

radians	$\frac{2\pi}{3}$	$-\frac{7\pi}{6}$	$\frac{11\pi}{6}$	$\frac{\pi}{3}$	$-\frac{5\pi}{3}$	$\frac{\pi}{6}$	$-\frac{\pi}{2}$
Degrees	120	–210	330	60	–300	30	–90

3.

Angle	$\theta = 270^\circ$	$\theta = -\pi$	$\theta = 45^\circ$	$\theta = \frac{\pi}{6}$	$\theta = 60^\circ$
Cosine	0	-1	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$
Sine	-1	0	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$

4. 49.6^0

2.9.2 Consolidation activities

1. Using the given information about θ, find the indicated value

   a) If θ lies in 2^{nd} quadrant with $\sin\theta = \frac{3}{5}$, find $\cos\theta$.

   b) If $\pi < \theta < \frac{3\pi}{2}$, with $\cos\theta = -\frac{\sqrt{5}}{5}$, find $\sin\theta$.

   c)If $\sin\theta = 1$, find $\cos\theta$.

2. In order to determine the height of a tree in Nyungwe forest, two sightings from the ground, one 200m directly behind the other, are made. If the angles of inclination were 45^0 and 30^0, respectively, how tall is the tree to the nearest metre?
3. In an airport control tower A, 2 planes B and C are located at the same altitude on a radar screen. The range finder determines one plane to bear $N60^0E$ at 100km while the other bears $S50^0E$ at 150km. How far apart are the planes from each other?
4. Multiple choice

i). Which of the following trigonometric ratios could not be π?

(A) $\tan\theta$ (B) $\sec\theta$ (C) $\cot\theta$ (D) $\cos\theta$ (E) $\csc\theta$

ii). If a non horizontal line has slope $\sin\theta$, it will be perpendicular to a line with slope

(A) $\cos\theta$ (B) $-\cos\theta$ (C) $\csc\theta$ (D) $-\csc\theta$ (E) $-\sin\theta$

iii). If θ is the smallest angle in a $3-4-5$ right triangle, then $\sin\theta =$

(A) $\frac{4}{5}$ (B) $\frac{3}{5}$ (C) $\frac{5}{4}$ (D) $\frac{3}{4}$ (E) $\frac{5}{3}$

iv). Which of the following expressions does not represent a real number?

(A) sin 30° (B) tan 45° (C) cos 90° (D) csc 90° (E) sec 90°

v). A central angle in a circle of radius r has a measure of θ radians. If the same central angle were drawn in a circle of radius 2r, its radian measure would be

(A) $\frac{\theta}{2}$ (B) $\frac{\theta}{2r}$ (C) 2θ (D) θ (E) $2r\theta$

vi). If the perimeter of a sector is 4 times its radius, then the radian measure of the central angle of the sector is

(A) 2 (B) 4 (C) $\frac{4}{\pi}$ (D) $\frac{2}{\pi}$ (E) impossible to determine without knowing the radius.

vii). What is the radian measure of an angle of x degrees?

(A) $\frac{x}{180}$ (B) πx (C) $\frac{\pi x}{180}$ (D) $\frac{180}{x\pi}$ (E) $\frac{180x}{\pi}$

Answers:

1. a) From the Pythagorean identity, we get $\cos\theta = \pm\frac{4}{5}$, since θ is a Quadrant II angle, thus $\cos\theta = -\frac{4}{5}$.

a) From the Pythagorean identity, we get $\sin\theta = \pm\frac{2\sqrt{5}}{5}$, since we are given that $\pi < \theta < \frac{3\pi}{2}$, we note that θ is a Quadrant III angle, thus $\sin\theta = -\frac{2\sqrt{5}}{5}$.

b) When we substitute $\sin\theta = 1$ into $\cos^2\theta + \sin^2\theta = 1$, we get $\cos\theta = 0$.

2. The tree is approximately 273 m tall.
   i) D ii) E iii) B iv) E v) D vi) A vii) C
3. 157km

2.9.2 Extended activities

1. A 100-degree arc of a circle has a length of 7 cm. To the nearest centimeter, what is the radius of the circle?
2. When I stand 30 m away from a tree at home, the angle of elevation to the top of the tree is 50◦ and the angle of depression to the base of the tree is 10◦. What is the height of the tree? Round your answer to the nearest metre.
3. From the observation deck of the lighthouse at Rubavu Point 50 m above the surface of Lake Kivu, a lifeguard spots a boat out on the lake sailing directly toward the lighthouse. The first sighting had an angle of depression of 8.2^{o} and the second sighting had an angle of depression of 25.9^{0}. How far had the boat traveled between the sightings?
4. The broadcast tower for radio station has two enormous flashing red lights on it: one at the very top and one a few metres below the top. From a point 5000 m away from the base of the tower on level ground the angle of elevation to the top light is 7.970^{0} and to the second light is 7.1250^{0}. Find the distance between the lights to the nearest metres.
5. In a triangle ABC, determine the angle A for which $(a+b+c)(b+c-a) = 3bc$.
6. David is a cross-country skier and skis 10km in a direction N40^{0}E of the ski lodge. At this point she turns and skis S10^{0}E for 4km and arrives at a chalet. How far is David from the lodge?

Answers:

1. 4 cm
2. The tree is about 41m tall.
3. The boat has travelled about 244 m.
4. The lights are about 75 m apart.
5. $A = 60^0$
6. 8 km

UNIT 3: LINEAR, QUADRATIC EQUATIONS AND INEQUALITIES

3.1. Key unit competence

Model and solve algebraically or graphically daily life problems using linear and quadratic equations or inequalities.

3.2. Prerequisite

In this lesson, Students must be skilled in **Unit 3** of **S1,** and **Unit 6** of **S3,** i. e:

- Solve problems related to linear equations, inequalities and represent the solutions graphically
- Solve problems involving linear or quadratic functions and interpret the graphs of quadratic functions

3.3. Cross-cutting issues to be addressed

- Inclusive education (promote education for all while teaching)
- Peace and value Education (respect others view and thoughts during class discussions)
- Gender (equal opportunity of boys and girls in the lesson participation)
- Standardization culture (regulation of the current intensity)

3.4. Guidance on introductory activity 3.0

- In groups, students read and do the introductory activity 3.0 from Student 's book
- Facilitate discussions to avoid noise or other unnecessary conversation,
- Be aware of straggling Student.
- Call them to present their findings and promote gender into presentation.
- Through question-answer, facilitate Students to realize that introductory activity 3.0 stimulates them to get idea on Unit 3.
- Through class discussions, let students think on different ways of getting solutions.

Answers for Introductory activity 3.0

1) Refer to the student's book and verify answers for students.

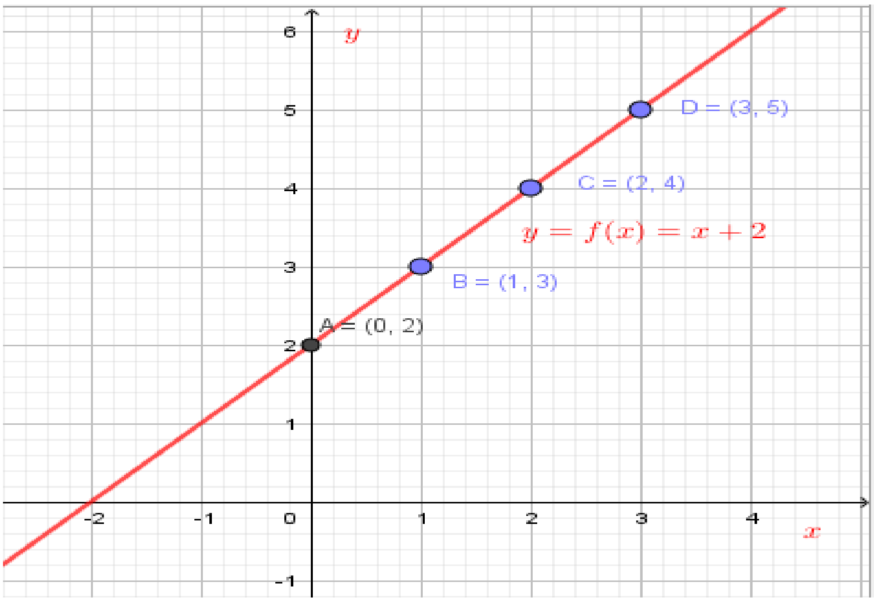
2) If x is the number of pens for a learner, the teacher decides to give him/her two more pens.

A learner with one pen will have $(1+2)\text{pens} = 3\text{pens}$

a) $y = f(x) = x+2$

X	-2	-1	0	1	2	3	4
$y = f(x) = x+2$	0	1	2	3	4	5	6
(x,y)	(-2;0)	(-1;1)	(0,2)	(1;3)	(2;4)	(3;5)	(4;6)

b) The graph obtained is the following:



c) The graph obtained is a line.

d) $y = x+2$ this is a linear equation because its graph is a line. Identically, $x+2 \geq 0$ is a linear inequality.

3) Students will give different examples. Verify whether the solution involves the linear equation.

4. Students will give different examples. Verify whether the solution involves the linear equation.

5. a) $y = -16t^2 + 1600$, for y =1000 , we have $100 = -16t^2 + 1600$ Solve this equation to find the time requested. $t = \frac{\sqrt{600}}{4} \approx 6.1$ The jumper is in free fall for about 6 seconds.

b) Table of value:

T	0	1	2	3	4	5	6
Y	1600	1616	1664	1744	1856	2000	2176

3.5. List of lessons

No	Lesson title	Learning objectives	Number of periods
0	Introductory activity	To arise the curiosity of student teachers on the content of unit 3	**1**
1	Simple linear equations in one unknown	Solve Simple linear equations in one unknown .	**1**
2	Linear equations with products or quotients	Solve Simple linear equations with products or quotients in one unknown	**1**
3	Quadratic equations	Solve quadratic equations	**1**
4	Simple Linear inequalities in one unknown	Solve Simple linear inequalities in one unknown .	**1**
5	Product and Quotient form of inequalities in one unknown	Solve product and quotient form of inequalities in one unknown .	**2**
6	Quadratic inequalities	Solve quadratic inequalities	**1**
7	Simultaneous equations in two unknowns	Solve real life problems involving simultaneous equations in two unknown	**1**
8	Application of linear equations and inequalities in real life situations	Solve problems from real life situations involving linear equations and inequalities	**2**
End unit assessment			**1**

Lesson 1: Simple linear equations in one unknown

a) Learning objective:

Solve Simple linear equations in one unknown

b) Teaching resources:

- Student's book; Reference books; Ruler, T-square, Manila paper; Scientific calculators; Internet connection
- **Note:** If possible computers with mathematical software such as Geogebra, Microsoft Excel, Math-lab or Graph-Calc and internet can be used.

c) Prerequisites / Revision / Introduction:

Students will learn better this lesson if they have a good understanding on:

- Representation and interpretation of graphs of linear functions
- Solving linear equations and represent their solutions
- Solving problems involving linear functions and interpretation of the graphs of linear functions.

d) Learning activities

- Organize the students into small groups;
- Provide clear instructions and introduce the **activity 1** by guiding students.
- Move around to ensure all students in groups participate actively.
- Call upon groups to present their findings.
- Harmonize their findings and facilitate them to summarize the concepts related to lesson.
- Guide students to perform individually **Exercise 1** to assess their knowledge and skills.
- Provide additional activities where necessary.

Answers to activity 1

The teacher guides the students to discuss about polynomials and how to the classification of polynomials is based on the number of terms. The following table identifies the types of polynomial.

1) $x+1=5$ $x+1-1=5-1$ $x=4$	2) $2x-4=0$ $2x-4+4=0+4$ $2x=4$ $\frac{2x}{2}=\frac{4}{2}$ $x=2$	3) $2x+1=-5$ $x+1-1=-5-1$ $x=-6$	4) $x-4=10$ $x-4+4=10+4$ $x=14$

e) **Answer of exercise 1**

(1) $x+5=9 \Rightarrow x+5-5=9-5 \Rightarrow x=4$

(2) $6x+5=5 \Rightarrow 6x+5-5=5-5 \Rightarrow 6x=0 \Rightarrow x=0$

(3) $x-2=3 \Rightarrow x-2+2=3+2 \Rightarrow x=5$

(4) $25=2x-5 \Rightarrow 30=2x \Rightarrow x=15$

(5) $-5=x-1 \Rightarrow -5+1=x \Rightarrow x=-4$

(6) $3x-4=2x+1 \Rightarrow x=5$

(7) $x+5=9x+1 \Rightarrow -8x=-4 \Rightarrow x=\frac{1}{2}$

(8) $-6x-5=9 \Rightarrow -6x=14 \Rightarrow x=\frac{-14}{6}=\frac{-7}{3}$

(9) $x+100=99 \Rightarrow x=99-100 \Rightarrow x=-1$

(10) $6x-51=9 \Rightarrow 6x=9+51 \Rightarrow 6x=60 \Rightarrow x=10$

Lesson 2: Linear equations with products or quotients

f) **Learning objective:**

Solve Simple linear equations with products or quotients in one unknown

g) **Teaching resources:**

- Student 's book; Reference books; Ruler, T-square, Manila paper; Scientific calculators; Internet connection
- **Note:** If possible computers with mathematical software such as Geogebra, Microsoft Excel, Math-lab or Graph-Calc and internet can be used.

h) **Prerequisites / Revision / Introduction:**

Students will learn better this lesson if they have a good understanding on:

- Representation and interpretation of graphs of linear functions
- Solving linear equations and represent their solutions
- Solving problems involving linear functions and interpretation of the graphs of linear functions.

i) Learning activities

- ➢ Organize the students into small groups;
- ➢ Provide clear instructions and introduce the **activity 2** by guiding students.
- ➢ Teacher asks students to use the student's book to discuss on **activity 2.**
- ➢ Move around to ensure all students in groups participate actively.
- ➢ Call upon groups to present their findings.
- ➢ Harmonize their findings and facilitate them to summarize the concepts related to the lesson.
- ➢ Guide students to perform individually **Exercise 2** to assess their knowledge and skills.
- ➢ Provide additional activities where necessary.

Answers for activity 2

1) $AB = 0 \Leftrightarrow A = 0 \; or \; B = 0 \; Either \; x+1=0 \; or \; x-1=0$

2) $AB = 0 \Leftrightarrow A = 0 \; or \; B = 0 \; Either \; 2x-3=0 \; or \; x=0$

3) $Cross\ product: 2(2x-3) = x, \; with \; x \neq 0$

Answers for exercise 2:

1) $S=\{-2,5\}$; 2) $S=\left\{-\frac{9}{2}\right\}$; 3) $S=\left\{-8,-\frac{1}{2}\right\}$; 1) $S=\left\{\frac{47}{11}\right\}$

Lesson 3: Quadratic equations

a) Learning objective:

Solve quadratic equations

b) Teaching resources:

- Student's book; Reference books; Ruler, T-square, Manila paper; Scientific calculators;Internet connection
- **Note:** If possible computers with mathematical software such as Geogebra, Microsoft Excel, Math-lab or Graph-Calc and internet can be used.

c) Prerequisites / Revision / Introduction:

Students will learn better this lesson if they have a good understanding on:

- Representation and interpretation of graphs of linear functions
- Solving linear equations and inequalities and checking / represent their solutions

d) Learning activities

- Organize the students into small groups;
- Provide clear instructions and introduce the **activity 12** by guiding students.
- Move around to ensure all students in groups participate actively.
- Call upon groups to present their findings.
- Harmonize their findings and facilitate them to summarize the concepts related to lesson.
- Lead them to know and use factorisation method, discriminant method and completing the square method to solve quadratic equations.
- In small group, lead them to discuss on the examples in student book.
- Guide students to perform individually **exercise 12** to assess their knowledge and skills.
- Provide additional activities where necessary

Answer for activity 12:

a) $y=-16t^2+1600$, for y =1000 , we have $100=-16t^2+1600$ Solve this equation to find the time requested. $t=\frac{\sqrt{600}}{4}\approx 6.1$ The jumper is in free fall for about 6 seconds.

b) Table of value:

T	0	1	2	3	4	5	6
Y	1600	1616	1664	1744	1856	2000	2176

Answers of exercise 3.12

1) $\{-4,2\}$; 2) $\{-1,3\}$; 3) $\{-2,-1\}$; 4) $\left\{-\frac{\sqrt{77}}{6},\frac{\sqrt{77}}{6}\right\}$

Lesson 4: Simple Linear inequalities in one unknown

a) Learning objective:

Solve Simple linear inequalities in one unknown .

b) Teaching resources:

- Student's book; Reference books; Ruler, T-square, Manila paper; Scientific calculators; Internet connection.
- **Note:** If possible computers with mathematical software such as Geogebra, Microsoft Excel, Math-lab or Graph-Calc and internet can be used.

c) Prerequisites / Revision / Introduction:

Students will learn better this lesson if they have a good understanding on taught matter in Unit 1 &3 of S1, Unit 2 of S2. This means that they should to be skilled on:

- Representation and interpretation of graphs of linear functions;
- Solving problems involving linear inequalities;
- Performing operations on inequalities.

d) Learning activities

- Organize the students into small groups;
- Provide clear instructions and introduce the activity by guiding students.
- Teacher asks students to use the student book to discuss on **activity 3**
- Move around to ensure all students in groups participate actively.
- Call upon groups to present their findings.
- Harmonize their findings and facilitate them to summarize the concepts related to lesson.
- Guide students to perform individually **exercise 3** to assess their knowledge and skills.
- Provide additional activities where necessary.

Answers to activity 3

a) $x < 5$ represents all numbers less than 5, for example 4, 3, 2, 1, 0, -1, -2, etc.

b) $x > 0$ represents all numbers greater than 0, for example 1, 2, 3, 4, 5, 6 etc

c) $-4 < x < 12$ includes for example -3,-2,-1,0,1,2,3,...,11.

Note! ***Response on this question will vary form a group to another, as a teacher, try to verify their veracity and correct them accordingly.***

Answers for exercise 3

1) $x+6<15 \Rightarrow x<9$; 2) $2x-4<16 \Rightarrow x<10$; 3) $5x \le 25 \Rightarrow x \le 5$; 4) $3x-5>21 \Rightarrow x>\frac{26}{3}$

5) $2x+8 \ge 18 \Rightarrow x \ge 5$; 6) $6+x<10 \Rightarrow x<4$; 7)$5x \le 5x+2 \Rightarrow 0x \le 2 \Rightarrow S=\mathbb{R}$;

8) Im*possible*; 9) $2x+1 \ge 12+3x \Rightarrow x \le -11$; 10) $6-x<9 \Rightarrow x>-15$

Lesson 5: Product and Quotient form of inequalities in one unknown.

a) Learning objective:

Solve product and quotient form of inequalities in one unknown .

b) Teaching resources:

Graph papers, manila papers, calculators, markers, pens, graph editors such as Geogebra (where possible).

c) Prerequisites/Revision/Introduction:

Students will perform well in this unit if they are enough skilled on linear inequalities in one unknown studied in S1, S2 and S3.

d) Learning activities

- Invite students to work in groups the **activity 4;**
- Ask students to share their answers with another group and ask them to support each other where they become more challenged.
- With the use of different questions and examples given in the student's book, guide students to establish a good method of solving different types of inequalities: involving product, quotient and absolute values. Insist on different ways of presenting the solution set of inequality.
- Help students to presents their findings and harmonizes their answers.
- Guide students to brainstorm on real life problems that involve inequalities and invite them to explore the examples given in the students' book.
- After the lesson development, invite students to do the **exercise 4** and verify if the objectives of the lesson were achieved.

- ➢ Do the same procedure above on inequalities involving absolute values stating by the **activity 5.**
- ➢ Invite students to do the **exercise 5.**
- ➢ Provide additional activities where necessary.

Answers to the Activity 4

Solve inequalities in the set of real numbers

1. Start by solving $(x+1)(x-1)=0$

$$\begin{array}{lcl} x+1=0 & \text{or} & x-1=0 \\ \Rightarrow x=-1 & & \Rightarrow x=1 \end{array}$$

In each case, first construct the sign table. The solution will be given by interval showing negative values for $<$:

x	$-\infty$		-1		1		$+\infty$
$x+1$		-	0	+		+	
$x-1$		-		-	0	+	
$(x+1)(x-1)$		+	0	-	0	+	

Since the inequality is $(x+1)(x-1)<0$; we will take the interval where the product is negative. Thus, $S=]-1,1[$

2) $\frac{2x-3}{x}<0$

x	$-\infty$		0		$\frac{3}{2}$		$+\infty$
2x-3		- - -		-	0	+ + +	
x		- - - -	0	+		+ +	
$\frac{2x-3}{x}$		+ +	0	- -	0	+ + + + +	

The table shows that $\frac{2x-3}{x}<0$ for $x\in\left]0,\frac{3}{2}\right[$

Then $S=\left]0,\frac{3}{2}\right[$.

Answers to exercise 4

1) $S =]-\infty,-3[\cup]3,+\infty[$; 2) $S = \left[\frac{3}{4},1\right]$; 3) $S =]-\infty,-5[\cup]-1,2[$; 4) $S = \left]-\frac{4}{3},-1\right[$;

5) $S =]-\infty,-2[\cup]3,+\infty[$

Answers for activity 5

1) The set of all real numbers whose number of units from zero, on a number line, are greater than 4 is $S = \{x \in \mathbb{R} : |x| > 4\}$.

2) The set of all real numbers whose number of units from zero, on a number line, are less than 6 is $S = \{x \in \mathbb{R} : |x| < 6\}$.

Answers for exercise 5

1. $]-\infty,-2[\cup]3,+\infty[$
2. $\left]-\infty,-\frac{9}{2}\right] \cup \left[\frac{3}{2},+\infty\right[$
3. $]0,3[$

Lesson 6: Quadratic inequalities

a) Learning objective:

Solve quadratic inequalities

b) Teaching resources:

- Student's book; Reference books; Ruler, T-square, Manila paper; Scientific calculators; Internet connection
- **Note:** If possible computers with mathematical software such as Geogebra, Microsoft Excel, Math-lab or Graph-Calc and internet can be used.

c) Prerequisites / Revision / Introduction:

Students will learn better this lesson if they have a good understanding on taught matter in Unit 1 &3 of S1, Unit 2 of S2. This means that they should to be skilled on:

- Representation and interpretation of graphs of linear functions

- Solving linear or quadratics equations
- Performing operations, factorizing polynomials

d) Learning activities

- Organize the students into small groups;
- Provide clear instructions and introduce the **activity 19** by guiding students.
- Teacher asks students to use the student book to discuss on **activity 19**
- Move around to ensure all students in groups participate actively.
- Call upon groups to present their findings.
- Harmonize their findings and facilitate them to summarize the concepts related to lesson.
- Guide students to perform individually **exercise 19** to assess their knowledge and skills.
- Provide additional activities where necessary.

Answers to activity 19

1. $]-\infty,-1[\cup]2,\infty[$; (2) $]1,2[$

Answers for exercise 19

1. $-3\sqrt{5}+5>x, x>3\sqrt{5}+5$

2. $\frac{1}{3}\leq x\leq\frac{1}{2}$

3. ¡

4. $\varnothing$

5. $\frac{-\sqrt{577}+17}{2}\leq x\leq\frac{\sqrt{577}+17}{2}$

Lesson 7: Simultaneous equations in two unknowns

a) Learning objective:

Solve real life problems involving simultaneous equations in two unknown.

b) Teaching resources:

Manila papers, markers, pens and calculators.

c) Prerequisites/Revision/Introduction:

Students will perform well in this unit if they have good background on solving equations as it was learnt in senior two unit 3 and senior three unit 4.

c) Learning activities

- Invite students to work in group discussions and do **activity 11** found in their Mathematics books.
- Move around in the class for facilitating various groups during their work.
- Invite group representative to present their answers to the whole class;
- Harmonize the findings highlighting how to solve simultaneous equations by equating values of two same variables;
- Guide students to explore examples given in the students' book to enhance the methods and invite them to give their own examples from real life situations.
- After doing this activity, assign students to do the **exercise 11** and verify whether the lesson's objective was achieved.

Answers for activity 11

Let $x =$ cost per cat, $y =$ cost per dog

$\begin{cases} 164x + 24y = 4240 \\ x = 2y \end{cases}$. Then, solve

Answers of exercise 11

1. The are 5 multiple choice questions. There are 15 T/F questions.
2. The small pitcher holds 2 cups of water. The large pitcher holds 4 cups of water.
3. Number of students in van is 8 and number of students in bus is 22 students.
4. 34
5. The speed of the boat in still water is 12 miles/hour and the speed of the current is 9 miles/hour.

Lesson 8: Application of linear equations and inequalities in real life situations

a) Learning objective:

- Solve problems from real life situations involving linear equations and inequalities

b) Teaching resources:

- Student 's book; Reference books; Ruler, T-square, Manila paper; Scientific calculators; Internet connection
- **Note:** If possible computers with mathematical software such as Geogebra, Microsoft Excel, Math-lab or Graph-Calc and internet can be used.

c) Prerequisites / Revision / Introduction:

Students will learn better this lesson if they have a good understanding on:

- Represent and interpret graphs of linear functions.
- Solve linear equations and inequalities, appreciate the importance of checking their solution, and represent the solution

d) Learning activities

- Organize the students into small groups;
- Provide clear instructions and introduce the activity by guiding students.
- Teacher asks students to use the student book to discuss on **activity 20.**
- Move around to ensure all students in groups participate actively.
- Call upon groups to present their findings.
- Harmonize their findings and facilitate them to summarize the concepts related to lesson.
- Guide students to perform individually **exercise 20** to assess their knowledge and skills.

Answers to activity 20

1. Linear equations can be used in daily life in many different ways like:
   - In economics, supply and demand analysis
   - Linear motion
   - Balancing equation.

2. Quadratic equations are used in daily life like:
   - Calculating areas
   - Figuring out a profit
   - In athletics Finding speeds and so on ;

There are many different answers

3.6. Summary of the unit

Linear equation

A linear equation is an equation of a straight line, it has the form of

$y = ax + b, a \neq 0 \quad and\ a, b \in ¡$.

Solve inequality

To solve inequality of the form $(ax+b)(cx+d) < 0$, we follow the following steps:

a) First we solve for $(ax+b)(cx+d) = 0$

b) We construct the table called **sign table**, find the sign of each factor and then the sign of the product or quotient if we are given a quotient.
For the quotient the value that makes the denominator to be zero is always excluded in the solution. For that value we use the symbol || in the row of quotient sign.

c) Write the interval considering the given inequality sign.

3.7. Additional Information for Teachers

- Emphasize on how to write rational fraction, we always set condition/restriction.

3.8. Answer for Revision exercise

1. $a) -19$; b)0 or -9 ; c)9 ; d)1; e) 90; f)1; g) 4; h) 3
2. $a)\ ]-\infty,-1[$; b) $]1,\infty[$; c) $]-2,0[$; d) $\left]-\infty,\frac{-5}{3}\right[$; e) $]-\infty,-1[\cup[1,\infty[$;
   f) $]-\infty,-4[\cup]-1,\infty[$; g) $]-\infty,-9[\cup]-1,2[$
3. $a)\ 7,10$; b) $\frac{-45\pm\sqrt{2569}}{8}$; c) $5\pm2\sqrt{6}$; $d)\ ]-\infty,2]\cup[5,\infty[$; e) $\left[\frac{1}{3},\frac{1}{2}\right]$; f) ¡ ;
   g) $\varnothing$; h) $\varnothing$; i) ¡ ; j) $]-\infty,-1[\cup[2,3]$; k) $]-\infty,2[\cup]3,+\infty[$
4. (a) 30; (b) $\frac{26}{3}$; (c) -2; (d) 1; (e) 3; (f) 0 or 2; (g) $-\frac{13}{8}$
5. $a)\pm\frac{\sqrt{2}}{2}$; b)25; c) $\varnothing$; d) $-\frac{1}{8}$ *or* 125

6. $a)\sqrt{2}-1$; b) $\sqrt{4}+\sqrt{3}$; c)$2\sqrt{2}-\sqrt{7}$; d)$2\left(\sqrt{2}+1\right)$; e)$\sqrt{6}-\sqrt{3}$
7. $a)18,59$
8. $a)$ \$4, \$7
9. The number is 40. The two parts are 15 and 25
10. Robert: 10 years, his father: 40 years.
11. 25 and 30
12. 39^0 *and* 51^0
13. Each chair: \$125, each table: \$165
14. The velocity is between 5m/s and 15m/s between 1 second and 3 seconds after it is thrown
15. The width must be between 1m and 7m (inclusive) and the length is 8m width
16. 12 metres by 16 metres
17. -33 and -34
18. 1.5 metres
19. 9.75 metres on a side.

3.9 . Additional activities

3.9.1 Remedial activities

Find the solution of these inequalities:

1. $(x-2)(x+5)\geq 0$

2. $(-x+1)(x-3)\leq 0$

Solutions: (1) $S=\left]-\infty,-5\right]\cup\left[2,+\infty\right[$; (2) $S=\left]-\infty,1\right]\cup\left[3,+\infty\right[$

3.9.3 Consolidation activities

A Company produced a product with 18000 frw as fixed costs. The variable cost is estimated to be 30% of the total revenue when it is sold at a rate of 20 frw per unit. Find the total revenue, total cost and profit functions.

Solution:

$R(x)=20x$ .Where x is the number of units sold.

$$C(x)=18000+\frac{30}{100}R(x)$$

$$=18000+\frac{30}{100}\times 20x$$

$$=1800+6x$$

$P(x) = R(x) - C(x)$
$= 20x - (18000 + 6x)$
$= 14x - 18000$

3 Extended activities

1) A 3 hour river cruise goes 15 km upstream and then back again. The river has a current of 2 km an hour. What is the boat's speed and how long was the upstream journey?

Solution:

There are two speeds to think about: the speed the boat makes in the water and the speed relative to the land:

Let x = the boat's speed in the water (km/h)

Let v =the speed relative to the land (km/h)

Because the river flows downstream at $2km/h$

When going up, $v = x - 2$ (its speed is reduced by $2km/h$

When going downstream, $v = x + 2$ (its speed is increased by $2km/h$

$$Time = \frac{distance}{speed}$$

Total time = time upstream + time downstream = 3 hours

$$\text{Total time} = \frac{15}{(x-2)} + \frac{15}{(x+2)} = 3h$$

$$\frac{15}{(x-2)} + \frac{15}{(x+2)} = 3h$$

$$3(x-2)(x+2) = 15(x+2) + 15(x-2)$$

$$3x^2 - 30x - 12 = 0$$

$x = -0.39$ or $x = 10.39$

Consider positive $x = 10.39$ as is more perfect then, Boat's speed is 10.39 km/h

$$\text{The upstream journey} = \frac{15}{(10.39-2)} = 1.79 hours = 1 hours 47 \min$$

$$\text{The downstream journey} = \frac{15}{(10.39+2)} = 1.21 hours = 1 hours 13 \min$$

UNIT 4: POLYNOMIAL, RATIONAL AND IRRATIONAL FUNCTIONS

4.1 Key unit competence

Use concepts and definitions of Polynomial, Rational and Irrational functions to determine the domain of Polynomial, Rational and Irrational functions and represent them graphically in simple cases…

4.2 Prerequisite

In this lesson, Students must be skilled in **Unit 1 &3** of **S1, Unit 2** of **S2** and **Unit 6** of **S3.**

4.3 Cross-cutting issues to be addressed

- Inclusive education (promote education for all while teaching)
- Peace and value Education (respect others view and thoughts during class discussions)
- Gender (equal opportunity of boys and girls in the lesson participation)
- Standardization culture (regulation of the current intensity)

4.4 Guidance on introductory activity

- In groups, facilitate students read and do the introductory **activity 4.0** from Student 's book
- Facilitate discussions to avoid noise or other unnecessary conversation,
- Be aware of straggling Students.
- Call them to present their findings and promote gender into presentation.
- Through question-answer, facilitate Students to realize that introductory activity stimulates them to get idea on Unit 4.
- Through class discussions, let students think on different ways of getting solutions.

Answers for introductory activity 4.0:

1. **i.** The use (what something is made for);
   ii. A quantity whose value depends on the value of another quantity
   iii. A quantity whose value depends on the value of another quantity
2. i. – Independent variable: x
   – Dependent variable: y
   ii. – Independent variable: r
   – Dependent variable: A
   iii. – Independent variable: A
   – Dependent variable: S

3. a) Irrational function b) Rational function c)Irrational function d)Polynomial function e) Rational function

4. i. Domain: $Dom f =]-\infty,1[\cup]1,\infty[$ [
   Range: $\operatorname{Im} f =]-\infty,0[\cup]0,\infty[$

The function $y=\frac{4x-4}{(x-1)^2}$ is equivalent to $y=\frac{4}{x+1}$; the only value that y cannot assume is 0;

ii. Domain: $Domf =]-\infty,\infty[$; [: as any expression, r can assume any value;

Range: $\operatorname{Im} f = [0,\infty[$: the expression $A=\pi r^2 \geq 0$ 2 , for any value of r

iii. As A can always take positive values or can be zero, then $Domf = [0,\infty[$ and the range is $\operatorname{Im} f = [0,\infty[$.

4.5 List of lessons

#	Lesson title	Learning objectives	Number of periods
0	Introductory activity	To arouse the curiosity of students on the content of unit 4	1
1	Generalities on numerical functions	Identify a function and recognize rules that are not function.	2
2	Types of numerical functions	Differentiate the types of functions	2
3	Domain of definition and range of numerical functions	Determine the domains of definition of different numerical functions.	2
4	Parity(even , odd) of a numerical function	Find whether a function is even, odd, or neither.	1
5	Operation on numerical functions	Perform operations on functions and use them to determine the composite functions and inverse of a function.	1
6	Graphical representation of polynomial, rational and irrational functions.	Interpret graphs of functions (linear , quadratic, polynomial and simple rational) related to practical context and make conclusions.	2
7	Solve problems related to linear , quadratic, polynomial and simple rational functions	Model and solve real life problems involving linear linear, quadratic, polynomial and simple rational functions	2
End unit assessment			1

Lesson 1: Generalities on numerical functions

a) Learning objective

Identify a function and recognize rules that are not functions

b) Teaching resources

Student's book and other Reference textbooks to facilitate research, Mathematical set, calculator, manila paper, markers, pens, pencils, etc.

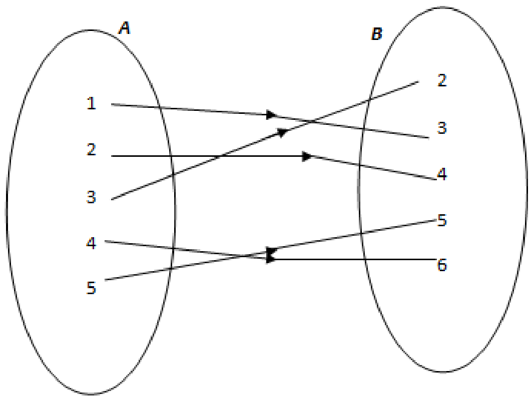
c) Prerequisites/Revision/Introduction

In this lesson, Students must be skilled in Unit 1 &3 of S1, Unit 2 of S2 and Unit 6 of S3.

d) Learning activities

- Invite students to work in group and do the activity 1 found in their Mathematics books;
- Move around in the class for facilitating where necessary and give more clarification on eventual challenges they may face during their work;
- Invite one member from each group with different working steps to present their work;
- As a teacher, harmonize the findings from presentation of students and guide them to specify the set of elements which have images, the set of all elements which have antecedents.
- Help them to check whether there exist an element which has more than one image.
- Guide them to explore the content and examples given in the student's book where they will be able to differentiate a function from relations and determine image for a point per a given function.
- After the lesson, guide students to do the exercise 1 and evaluate whether lesson objectives were achieved.
- Provide additional activities where necessary.

Answer for activity 1



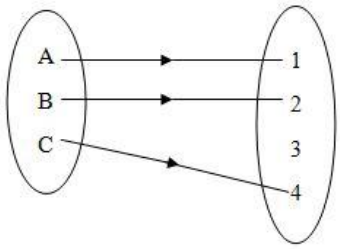
a) The set of elements of A which have images in B is $\{1;2;3;4;5\}$

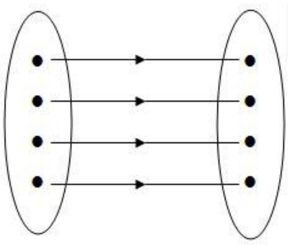
b) The set of elements in B which have antecedent in A is $\{2;3;4;5;6\}$

c) Each element of the set A has one image in the set B.

Answer of exercise 1

1. The 1st and the 2nd arrow diagrams are functions.





2. $Dom = \{a,b,c,d,e\}$ $\quad co-domain = \{1,2,3,4,5,6\}$ $\quad Range = \{1,2,3,4\}$

3.

$a)\, f(2) = 8$

$b)\, f(-2) = 0$

$c)\, f(d) = 2d + 8$

$d)\, f(a) = a \quad 2a + 8 = a$

$a = -8$

Lesson 2: Types of numerical functions

a) Learning objective

Differentiate the types of functions

b) Teaching resources

Student's book and other Reference textbooks to facilitate research, Mathematical set, calculator, Manila paper, markers, pens, pencils, etc.

c) Prerequisites/Revision/Introduction

In this lesson, Students will perform better if they revise the content on functions learnt in S2 and S3.

d) Learning activities

- Invite students to work in group and do the **activity 2** found in their Mathematics books;
- Move around in the class for facilitating where necessary and give more clarification on eventual challenges they may face during their work;
- Invite a students from each group to present their findings;
- As a teacher, harmonize the findings from presentation and guide them to explain why they take such type of function.
- Use different probing questions and guide them to explore the content and examples given in the student's book lead them to be able to differentiate different types of functions: Constant function, Identity, Monomial, Polynomial, Rational and Irrational functions.
- Guide them to classify polynomial functions either by number of terms or by degrees and guide them to establish the general form of a polynomial function as $f(x)=a_nx^n+a_{n-1}x^{n-1}+a_{n-2}x^{n-2}+...+a_2x^2+a_1x^1+a_0x^0$.

Degree 0	**Degree 1**	**Degree2**	**Degree3**	**Degree4**	**etc.**
Constant polynomial function.	Linear polynomial function/1st	quadratic polynomial function/ 2nd	Cubic polynomial function/3rd	4th degree polynomials (bi-quadratic	

	degree polynomial function.	degree polynomial function.	degree polynomial function.	polynomial functions).	

One term	**Two terms**	**Three terms**	**Four terms**	**Etc**
Monomial function	Binomial function	Trinomial function	Qua-trinomial Function	

- After this step, guide students to do the **exercise 2** and evaluate whether lesson objectives were achieved.
  - Provide additional activities where necessary.

Answer for activity 2

Polynomial	Rational	Irrational
$f(x)=(x+1)^2$	$h(x)=\frac{x^3+2x+1}{x-4}$	$f(x)=\sqrt{x^2+x-2}$

d) Answers of exercise 2

1. $f(x)=x^3+2x^2-2$ is a polynomial function.
2. $g(x)=-2$ is a constant function.
3. $h(x)=\frac{x^3+2x^2-2}{x-5}$ is a rational function.
4. $f(x)=\frac{x^2-2}{x^2-8x+15}$ is a rational function.

Lesson 3: Domain of definition of numerical functions

a) Learning objective

Determine the domains of definition of different numerical functions.

b) Teaching resources

Student 's book and other reference textbooks to facilitate research, Mathematical set, calculator, manila paper, markers, pens, pencils, graphing software such as Geogebra, etc.

c) Prerequisites/Revision/Introduction

Students should be able to explain the difference between types of functions learnt in previous lessons of year 1.

d) Learning activities

- Invite students to work in pairs and do the **activity 3** found in their Mathematics books;
- Move around in the class for facilitating students where necessary and give more clarification on eventual challenges they may face during their work;
- Let the 2 neighbouring pairs to work together and share their works to and improve them by highlighting the set for the values obtained;
- Verify and identify groups with different working steps;
- Invite one member from each group with different working steps to present their work.
- As a teacher, harmonize the findings from presentation and guide them to explain why they took such values.
- Use different probing questions and guide them to explore the content and examples given in the student's book and lead them to determine the domain and range for specified functions: Constant, linear, quadratic, Polynomial, Rational and Irrational functions. Note that the range will be determined only for elementary functions.
- After this step, guide students to do the **exercise 3** and evaluate whether lesson objectives were achieved.
    - Provide additional activities where necessary.

Answer for activity 3

1) None. It means that for all real numbers $f(x)=x^3+2x+1$ is defined.

2) $f(x)=\dfrac{1}{x}$ is not defined for $x=0$. It means that for $x=0 \Rightarrow f(0)=\dfrac{1}{0}\notin \mathbb{R}$ (it is impossible to divide by zero in the set of real numbers).

3) $g(x)=\dfrac{x+2}{x-1}$ is not defined for $x=1$. It means that for $x=1,\ f(1)=\dfrac{3}{0}\notin \mathbb{R}$.

Answer for exercise 3

1) $f(x)=x^3+2x^2-2$

$domf=\mathbb{R}=]-\infty,+\infty[$ in this case f(x) is defined for all real numbers.

2) $g(x)=-2$

$domg=\mathbb{R}=]-\infty,+\infty[$ in this case f(x) is defined for all real numbers.

3) $h(x)=\dfrac{x^3+2x^2-2}{x-5}$

$domh=\mathbb{R}-\{5\}\ or\ \mathbb{R}\setminus\{5\}$ in this case f(x) is defined for all real numbers except 5.

4) $f(x)=\dfrac{x^2-2}{x^2-8x+15}$

$domf=\mathbb{R}\setminus\{3,5\}$ in this case f(x) is defined for all real numbers except 3 and 5.

Answer of activity 4.

1) $f(x)=\sqrt{2x+1}$, $domf=\left[-\dfrac{1}{2},+\infty\right[$

2) $f(x)=\sqrt[3]{x^2+x-2}$, $domf=\mathbb{R}=]-\infty,+\infty[$

3) $g(x)=\sqrt{\dfrac{x-2}{x+1}}$, $domg=]-\infty,-1[\cup[2,+\infty[$

Answers for exercise 4

1. $f(x)=\sqrt{4x-8}$ $domf=[2,+\infty[$

2. $g(x)=\sqrt{x^2+5x-6}$ $\quad dom g =]-\infty,-6]\cup[1,+\infty[$

3. $h(x)=\dfrac{x^3+2x^2-2}{\sqrt[3]{x+4}}$ $\quad dom h = \mathbb{R}\setminus\{-4\}$

4. $f(x)=\dfrac{x-2}{\sqrt[4]{x^2-25}}$ $\quad dom f =]-\infty,-5[\cup]5,+\infty[$

5. $f(x)=\sqrt{\dfrac{(x-1)^2}{x+4}}$ $\quad dom f = \mathbb{R} =]-4,+\infty[$

Lesson 4: Operations on the numerical functions

a) Learning objective

Perform operations on functions and use them to determine the composite functions and inverse of a function.

b) Teaching resources

Student's book and other Reference textbooks to facilitate research, Mathematical set, calculator, Manila paper, markers, pens, pencils, etc.

c) Prerequisites/Revision/Introduction

In this lesson, students must be skilled in unit 6 of S3.

c) Learning activities

- Invite students to work in groups and do the **activity 5** found in their Mathematics books;
- Move around in the class for facilitating students where necessary and ask some challenging questions to lead them to work correctly;
- Verify and identify groups with different working steps;
- Invite one member from each group with different working steps to present their work.
- As a teacher, harmonize the findings from presentation and guide them to enhance the operation of functions.

- Use different probing questions and guide them to explore the content and examples given in the student's book and lead them to explore the addition, multiplication and division of functions.
- Move to every group and verify their working steps.
- Invite all students for a whole class discussion and guide them to establish how to determine the composite of functions and the inverse of a function;
- Use different probing questions and guide them to explore the content and examples on the composite of functions and the inverse of a function as it is given in the student's book;
- After this step, guide students to do the **exercise 5** and evaluate whether lesson objectives were achieved.
  - Provide additional activities where necessary.

Answer for activity 5

1) $f(x)=\frac{x+1}{2x-3}; g(x)=x+1$

$$f(x)+g(x)=\left(\frac{x+1}{2x-3}\right)+\left(\frac{x+1}{1}\right)=\frac{(x+1)+(x+1)\cdot(2x-3)}{2x-3}$$

$$=\frac{x+1+2x^2-3x+2x-3}{2x-3}=\frac{2x^2-3}{2x-3}; x\neq\frac{3}{2}$$

2) $f(x)=\frac{x+1}{2x-3}; g(x)=x+1$

$$f(x)-g(x)=\left(\frac{x+1}{2x-3}\right)-\left(\frac{x+1}{1}\right)=\frac{(x+1)-(x+1)\cdot(2x-3)}{2x-3}$$

$$=\frac{x+1-(2x^2-x-3)}{2x-3}=\frac{-2x^2+2x+4}{2x-3}; x\neq\frac{3}{2}$$

3) $f(x)=\frac{x+1}{2x-3}; g(x)=x+1$

$$f(x)\cdot g(x)=\left(\frac{x+1}{2x-3}\right)\left(\frac{x+1}{1}\right)=\frac{(x+1)\cdot(x+1)}{2x-3}=\frac{x^2+2x+1}{2x-3}; x\neq\frac{3}{2}$$

4) $f(x)=\frac{x+1}{2x-3}; g(x)=x+1$

$$\frac{f(x)}{g(x)}=\left(\frac{\frac{x+1}{2x-3}}{\frac{x+1}{1}}\right)=2x-3$$

Answers of exercise 5

1. $2x^3+8x-5$

2. $6x^5-13x^4+28x^3-30x^2+25x-12$

3. $\left[-\frac{3}{2},+\infty\right[$

4. $]-4,+\infty[$

5. $2x^3+4x^2-9x-6$

Lesson 5: Parity of a polynomial function

a) Learning objective

Find whether a function is even, odd, or neither.

b) Teaching resources

Student's book and other Reference textbooks to facilitate research, Mathematical set, calculator, manila paper, markers, pens, pencils, …

c) Prerequisites/Revision/Introduction

Students in this lesson must be skilled in **Unit 1 &3** of **S1, Unit 2** of **S2**.

d) Learning activities

- Ask students in small groups to read and discuss on the **activity 6**
- Facilitate working, especially straggling student-teachers.
- Call students to present their findings,
- Promote gender while presenting findings, and help them to harmonize the answer.

Answer of activity 6

1.

$f(x)=x^2+2x+3$

o $f(-x)=(-x)^2+2(-x)+3=x^2-2x+3$

o $-f(x)=-(x^2+2x+3)=-x^2-2x-3$

$\therefore f(-x)\neq -f(x)$

2.

$f(x)=2x+5$

o$f(-x)=2(-x)+5=-2x+5$

o$-f(x)=-(2x+5)=-2x-5$

$\therefore f(-x)\neq -f(x)$

3.

$f(x)=x^5-3x+8$

o$f(-x)=(-x)^5-3(-x)+8=-x^5+3x+8$

o$-f(x)=-(x^5-3x+8)=-x^5+3x-8$

$\therefore f(-x)\neq -f(x)$

4.

$f(x)=x^4-3x^2$

o$f(-x)=(-x)^4-3(-x)^2=x^4-3x^2$

o$-f(x)=-(x^4-3x^2)=-x^4+3x^2$

$\therefore f(-x)\neq -f(x)$

5.

$f(x)=x^5+3x$

o$f(-x)=(-x)^5+3(-x)=-x^5-3x$

o$-f(x)=-(x^5+3x)=-x^5-3x$

$\therefore f(-x)=-f(x)$

- Help students to discover that

A function $f(x)$ is said to be **even** if the following conditions are satisfied

$\forall x\in Domf, -x\in Domf$

$f(-x)=f(x)$

A function $f(x)$ is said to be **odd** if the following conditions are satisfied

$\forall x\in Domf, -x\in Domf$

$f(-x)=-f(x)$

- In small groups, lead them to do examples
- Call students to do **Exercise 5** to master the content.

Answers of Exercise 5

1.

$f(x) = 2x^2 + 2x - 3$

o$f(-x) = 2(-x)^2 + 2(-x) - 3 = 2x^2 - 2x - 3$

o$-f(x) = -(2x^2 + 2x - 3) = -2x^2 - 2x + 3$ $\quad \therefore f(x) = 2x^2 + 2x - 3$ is neither odd nor even.

$f(-x) \neq -f(x)$ *and*

$f(-x) \neq -f(x)$

2.

$g(x) = x^3 - x$

o$g(-x) = (-x)^3 - (-x) = -x^3 + x$

o$-g(x) = -(x^3 - x) = -x^3 + x$ $\quad \therefore g(x) = x^3 - x$ is odd.

$g(-x) = -g(x)$

3.

$g(x) = x(x^2 + x) = x^3 + x^2$

o$g(-x) = (-x)^3 + (-x)^2 = -x^3 + x^2$

o$-g(x) = -(x^3 + x^2) = -x^3 - x^2$ $\quad \therefore g(x) = x(x^2 + x)$ is neither odd nor even.

$g(-x) \neq -g(x)$ *and*

$g(-x) \neq -g(x)$

Lesson 6: Plotting linear, quadratic, polynomial and simple rational functions

a) Learning objective

Interpret graphs of functions (linear and quadratic) related to practical context and make conclusions.

b) Teaching resources

Student's book and other Reference textbooks to facilitate research, Mathematical set, calculator, manila paper, markers, pens, pencils, …

c) Prerequisites/Revision/Introduction

In this lesson, Students must be skilled in **Unit 3** of **S1** and **Unit 6** of **S3.**

d) Learning activities

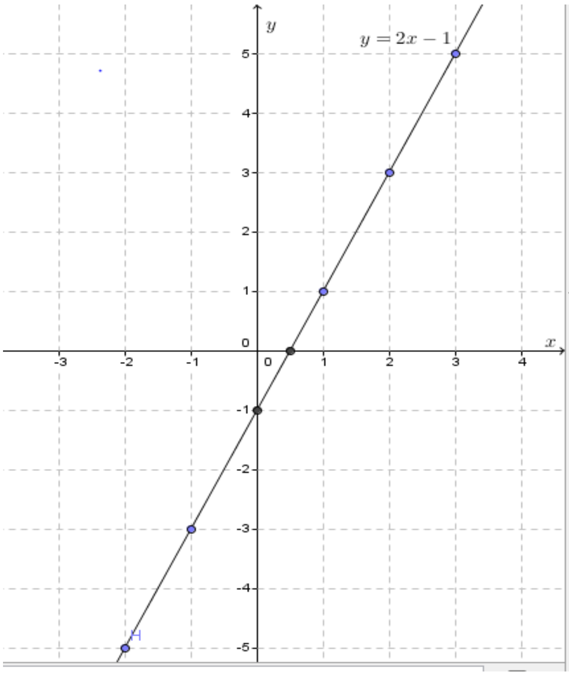
- Ask students in pairs to read and discuss on the **activity 6A** found below;
- Make sure that everybody is engaged/ involved.
- Facilitate working, especially straggling students.

- Facilitate the use of geometric materials to make accurate graphs.
- Make sure that the notebooks of students include squared papers/graph papers.
- Call students to present the findings and promote gender where possible.
- Help them to harmonize the answers.

Answer of activity 6A

1. $y = 2x - 1$ for $-3 \leq x \leq 3$

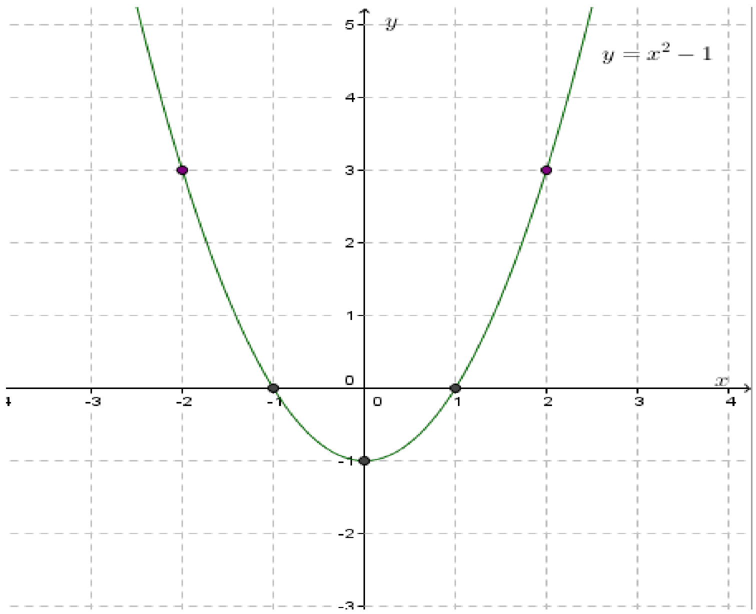
x	-3	-2	-1	0	1	2	3
$y = 2x - 1$	-7	-5	-3	-1	1	3	5



This is linear function

2. $y = x^2 - 1$ for $-3 \leq x \leq 3$

x	-3	-2	-1	0	1	2	3
$y = x^2 - 1$	8	3	0	-1	0	3	8



This is quadratic function

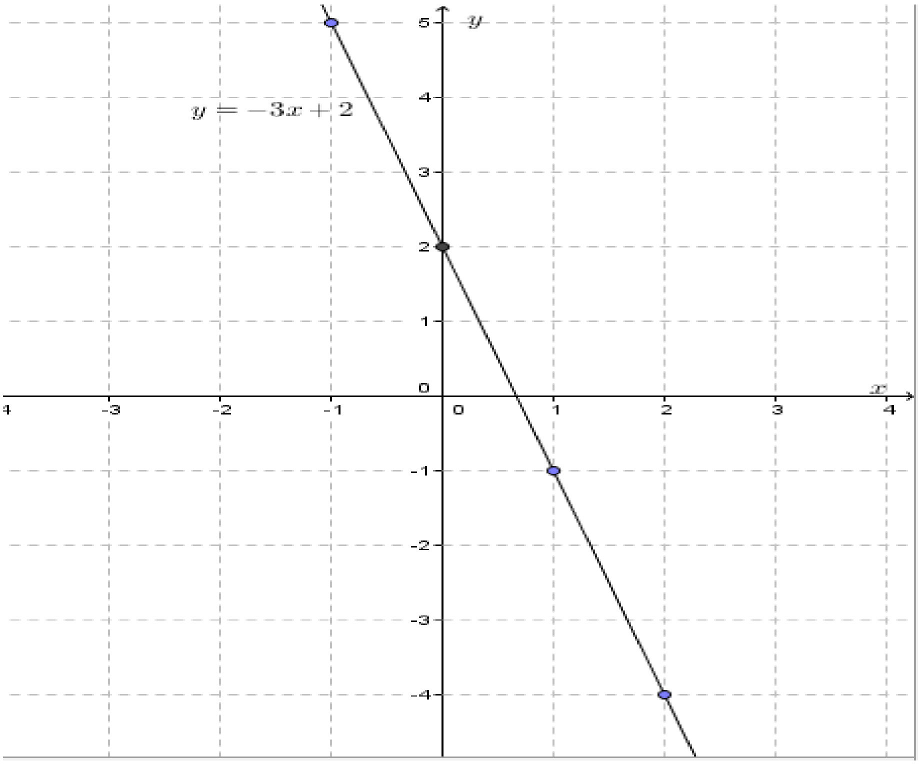
- Through examples given in student's book, guide students to know how accurately draw
  - Linear function
  - Quadratic function

- Call students to do Application activity 6A in student's book to master the content.

Answers of Application activity 6 A

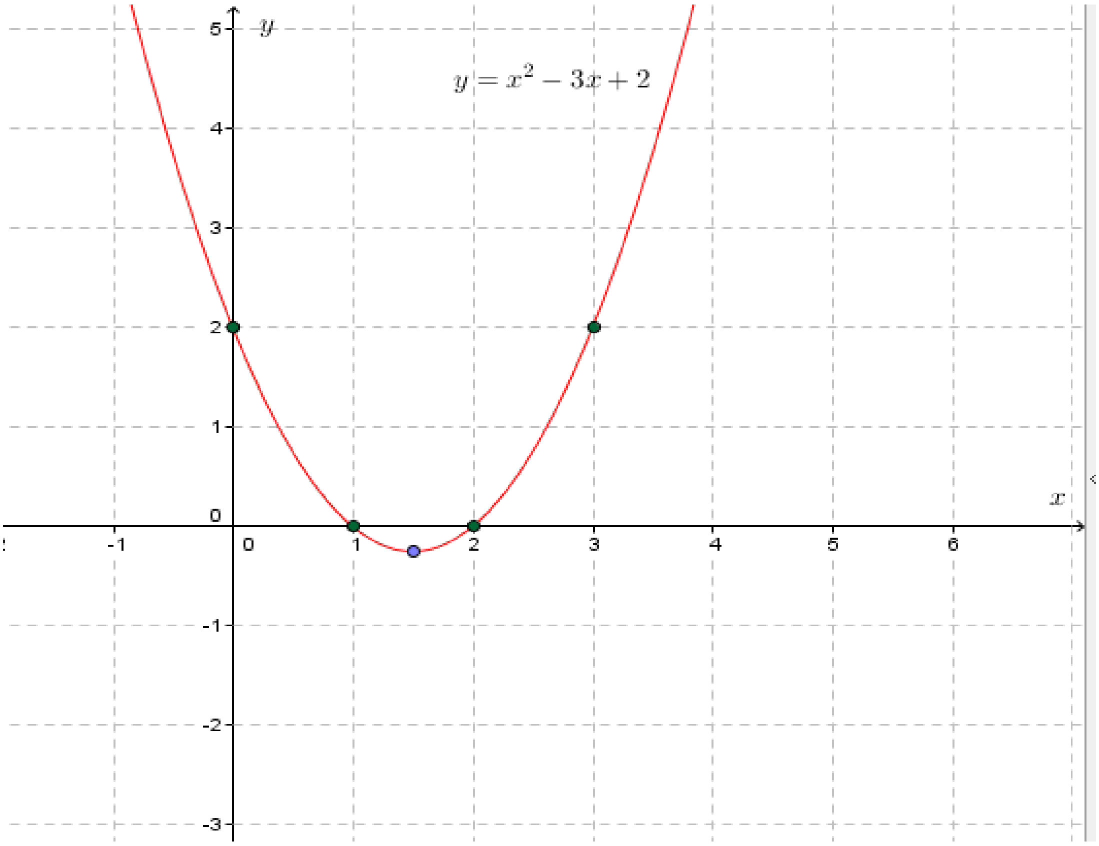
1. a) $y = -3x + 2$

x	-1	0	1	2
$y = -3x + 2$	5	-2	-1	-4



b)$y = x^2 - 3x + 2$

x	-1	0	1	2	3
$y = x^2 - 3x + 2$	6	2	0	0	2



2 (a) $y = 2x^2 + 5x - 1$

vertex $v(h,k)$ with $h = -\frac{b}{2a} = -\frac{5}{4}$ *and* $k = 2\times\left(-\frac{5}{4}\right)^2 + 5\times\left(-\frac{5}{4}\right) - 1 = -\frac{33}{8}$

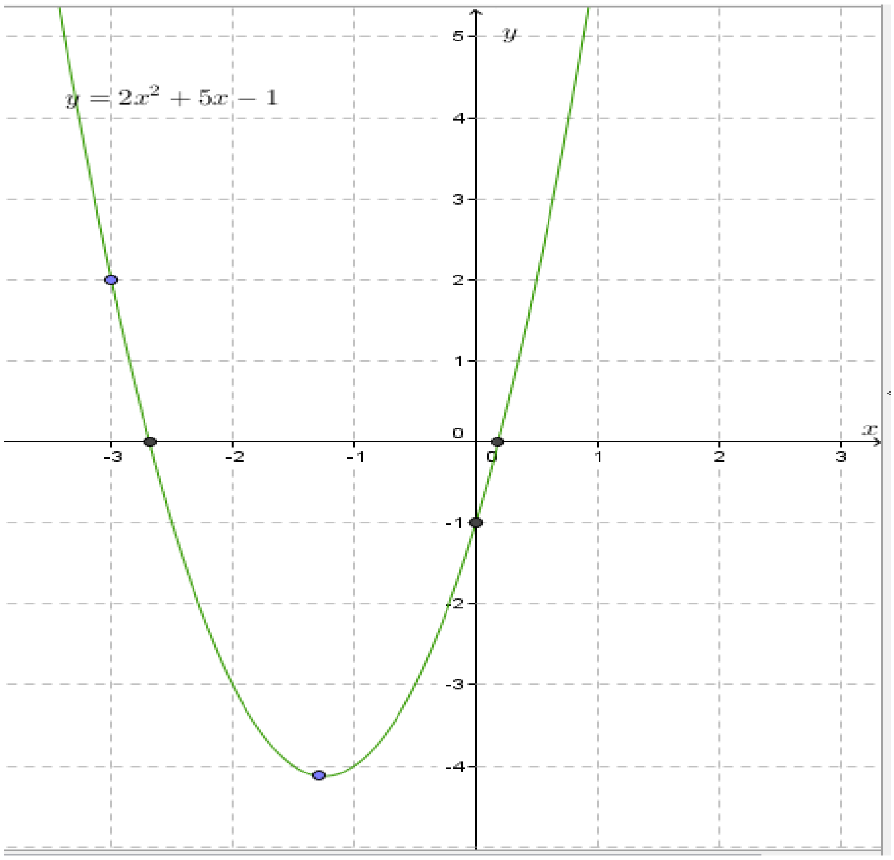
$\Rightarrow v = \left(-\frac{5}{4}, -\frac{33}{8}\right)$

axis of symmetry $x = -\frac{5}{4}$

if $x = 0 \Rightarrow y = -1$ and y-intercept is $(0,-1)$

if

$y = 0 \Rightarrow x = \frac{-5+\sqrt{33}}{4}$; 0.186 or x= $\frac{-5-\sqrt{33}}{4}$; -2.68 then x-intercept is $(0.18, 0)$ *or* $(-2.68, 0)$



(b) $y = 3x^2 + 8x - 6$

vertex $v(h,k)$ with $h = -\frac{b}{2a} = -\frac{8}{6} = -\frac{4}{3}$ *and* $k = 3\times\left(-\frac{4}{3}\right)^2 + 8\times\left(-\frac{4}{3}\right) - 6 = -\frac{34}{3}$

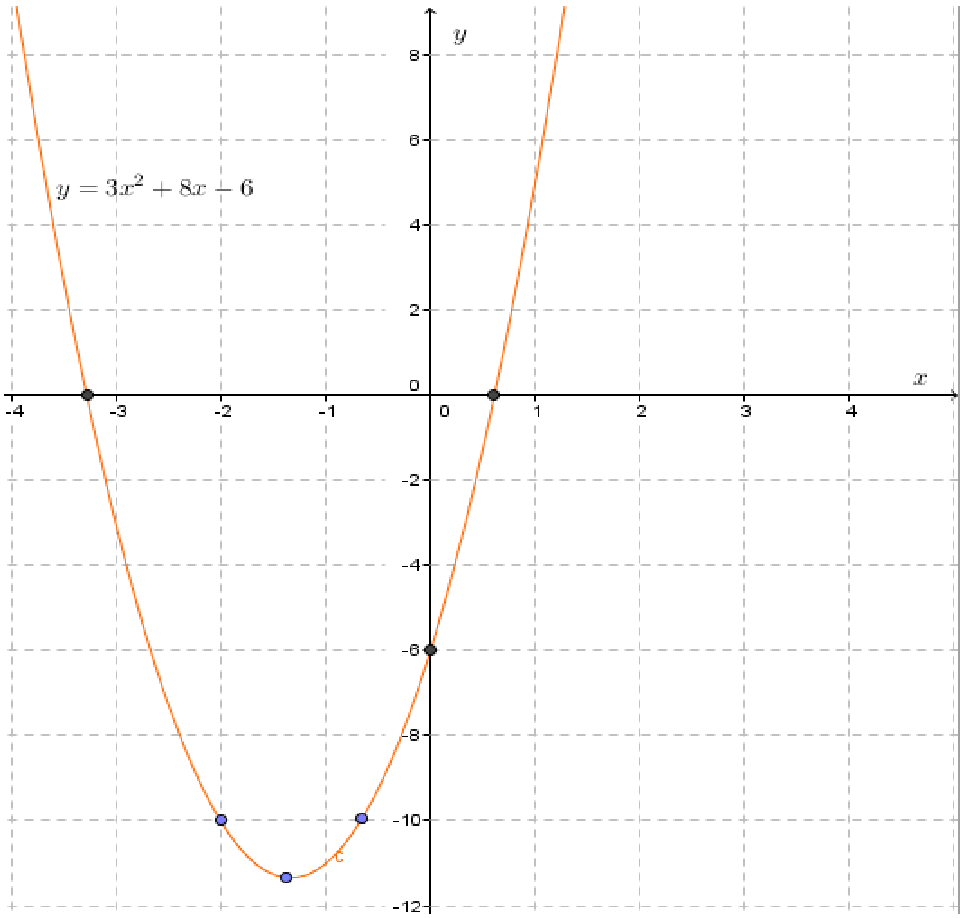
$\Rightarrow v = \left(-\frac{4}{3}, -\frac{34}{3}\right)$

axis of symmetry $x = -\frac{4}{3}$

if $x = 0 \Rightarrow y = -6$ and y-intercept is $(0,-6)$

if

$y = 0 \Rightarrow x = \frac{-8+\sqrt{136}}{6}$; 0.61 or x=$\frac{-8-\sqrt{136}}{6}$; -3.27 then x-intercept is$(0.61,0)$ *or* $(-3.27,0)$



Lesson 7: Solve problems related to linear, quadratic, polynomial and simple rational functions

a) Learning objective:

Model and solve real life problems involving linear linear, quadratic, polynomial and simple rational functions

b) Teaching resources:

Manila papers, markers, rulers, graph papers, calculators and where possible use the mathematics software for graphing such as Geogebra.

c) Prerequisites/Revision/Introduction:

Students will perform well in this unit if they learnt well the content for previous lessons in this unit.

d) Learning activities

- Invite students to work in group on the activity 8 A

- Move around in the class for facilitating where necessary and give more clarification on eventual challenges they may face during their work; Identify groups which have different working steps.
- Invite each groups with different working steps to present their answers in a whole class discussion;
- As a teacher, harmonize the findings from presentation of students and ask them to give other examples of problems from real life which involve the use of polynomial and simple rational functions especially in Health. Etc... And ask them to solve individually the **exercise 8 A.**

Activity 8 A and Answer

1) Health average systolic blood pressure is estimated by

$P(x)=0.01x^2+0.05x+107$ where x is age in years and P is pressure in millimetres of mercury (mmHg).

a) What is the healthy average systolic blood pressure of a 34 years old to the nearest tenth?

b) If a healthy average systolic blood pressure of 132.4mmHg, what is there age to the nearest year?

Answer: Health average systolic blood pressure $P(x)=0.01x^2+0.05x+107$

a) For x = 34 , then $P(34)=0.01(34)^2+0.05(34)+107$

$\therefore$ P (34)= 120.3 mmHg

b) For P (x)= 132.4 , then $132.4=0.01x^2+0.05x+107$ Using quadratic formula 2 4

$0=0.01(x)^2+0.05(x)-25.4$

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

$x\approx 48$. Therefore, the age to the nearest year is 48years old.

Exercise 8 A and Answer

1.The temperature T (in degrees Fahrenheit) of food placed in a refrigerator is modelled by

$T=10\left(\frac{4t^2+16t+75}{t^2+4t+10}\right)$; where t is the time (in hours).

a)What is the initial temperature of the food?

b)What is the temperature of the food after 3hours?

Answer for exercise 8A

$$T = 10\left(\frac{4t^2 + 16t + 75}{t^2 + 4t + 10}\right)$$

a) Initial temperature (t=0h) of the food is 75 degree Celsius

b)Temperature of the food after 3hour is 43.87 degree Celsius

4.6 Summary of the unit

A monomial is a variable, a real number, or a multiplication of one or more variables and a real number with whole-number exponents
A function that is expressible as the sum of finitely many monomials in *x* is called **polynomial in *x*.**

A function is a rule that assigns to each element in a set *A* one and only one element in set *B*. We can even define a function as any relationship which takes one element of one set and assigns to it one and only one element of second set. The second set is called a **co-domain**. The set *A* is called the **domain,** denoted by ***Domf or Df.***

Functions for which each element of the domain is associated onto a different element of the range are said to be **one-to-one.**

Relationships which are **one-to-many** can occur, but from our preceding definition, they are **not functions.**

A function $f(x)$ is said to be **even** iff $f(-x) = f(x)$ for all $x \in Domf\ \ and -x \in Domf$

A function $f(x)$ is said to be **odd** iff $f(-x) = -f(x)$ for all $x \in Domf\ \ and -x \in Domf$

A linear function is any function of the form $f(x) = mx + b$ with $m \neq 0$. The ***domain*** of this function is the set of all real numbers. The ***range*** of f is the set of all real numbers. The graph of f is a line with slope m and y intercept b.
A function $f(x) = b$, where b is a constant is called a constant function. Its graph is a horizontal line at $y = b$.

The quadratic function written as $f(x) = ax^2 + bx + c$, has: **a vertex** $v\left(-\frac{b}{2a}, f(-\frac{b}{2a})\right)$

, axis of symmetry $x = -\frac{b}{2a}$

There are two intercepts: x-intercept. (o, c) and y-intercept

$(x_1, 0)$ and $(x_2, 0)$ where x_1 and x_2 *are the roots of* $ax^2 + bx + c = 0$ with $a \neq 0$

x-intercept for any quadratic function is calculated by letting y = 0 and y- intercept is calculated by letting x = 0

Graph of a quadratic function

The graph of a quadratic function can be sketched using the table of values of x and y or by the vertex, x-intercepts and y-intercept

Some applications of polynomial functions:

- Polynomial functions can also be used to model different situations, like in the stock market to see how prices will vary over time.
- Business people also use polynomial functions to model markets, as in to see how raising the price of a good will affect its sales.
- Polynomial functions are important in calculating medicine, building structures (houses, businesses…), vehicle design, designing games, to build computers (formulas that are used to plug to computer programs), knowing how much change you should receive when making a purchase, driving (amount of gas needed for travel)

4.7 Additional Information for Teachers

- Emphasize on the graph paper/squared paper while students draw the graphs.
- Emphasize and facilitate the use of geometric materials to ameliorate the quality of graphs.
- Guide the students in scaling axes.
- Remind them to name axes (x-axis and y-axis).
- Recall them to mention/highlight the origin/intersection point of axes by 0.

4.8 Answers of **Revision Exercise**

1. a and d

2. a. (iii) b. (i) c. (iii) d. (ii)

3. a. 14 b. 50 c. 2 d. 11 e. $3t^2 + 2$

4. a. 21 b. $-\frac{5}{3}$ c. $\frac{\sqrt[3]{5}+2}{\sqrt[3]{5}}$ d. $\frac{\pi+1}{\pi-1}$ e. $\frac{a}{a-2}$

5. a. does not exist b. -8 c. $\frac{1}{3}$ d. $\frac{10}{31}$ e. does not exist

6. -3

7. 1 or $-\frac{3}{2}$

8. $a=1,\ b=-1,\ g(-4)=17$

9. a. $\{y:0\le y\le 7\}$ one to one b. $\{y:0\le y\le 9\}$ many to one

c. $\{y:0<y\le 1\}$ one to one d. $\{y\in \mathbb{R} : y\ne 0\}$ one to one

10. g is one to many since $g(2)=4\ or\ 6$ and therefore not a function.

11. a. $]-\infty,3[\cup]3,+\infty[$ b. $\left]-\infty,-\frac{7}{5}\right[\cup\left]-\frac{7}{5},+\infty\right[$ c. $]-\infty,-\sqrt{3}[\cup]\sqrt{3},+\infty[$

d. $]-\infty,-2[\cup]1,+\infty[$ e. $]-\infty,+\infty[$ f. $]-\infty,+\infty[$ g. $]-\infty,-3[$ h. $[5,+\infty[$

i. $]-\infty,+\infty[$ j. $]-\infty,-3]\cup[2,+\infty[$

12. a. $]-\infty,+\infty[$ b. $]-\infty,-5[\cup]-5,3[\cup]3,+\infty[$ c. $\left]-\infty,\frac{1}{2}\right[$ d. $[-3,4]$

e. $\left[\frac{1}{3},+\infty\right[$ f. $[-2,-1]\cup]1,+\infty]$ g. $]-2,-1[\cup]-1,1]\cup]4,+\infty[$

13. -6

14. -3

15. a. x^3+2x^2-x-2 b. x^3+4x^2-3x-2 c. $x^5+2x^4-5x^3+2x$

16. a. odd b. neither c. odd d. odd e. even f. odd g. even

4.9 Additional activities

4.9.1 Remedial activities

1. Draw the graph of a linear function and determine the properties of a function : (domain of a function, range of a function, function is/is not one-to-one function, even/odd function, coordinates of intersections with the x-axis and with the y-axis, intervals of monotonicity - increasing/decreasing function)

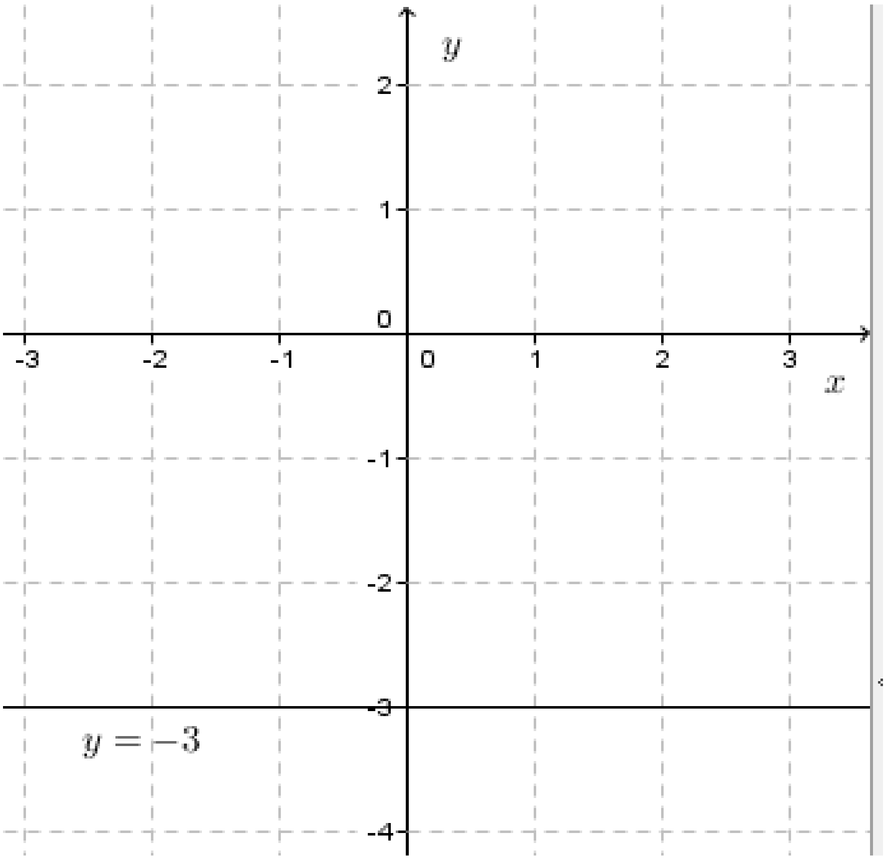
$a)\ y=-3$ $b)\ y=x$ $c)\ y=4x+2$ $d)\ 2x+4y-6=0$

Solutions

$a)\ D(f):]-\infty,+\infty[$

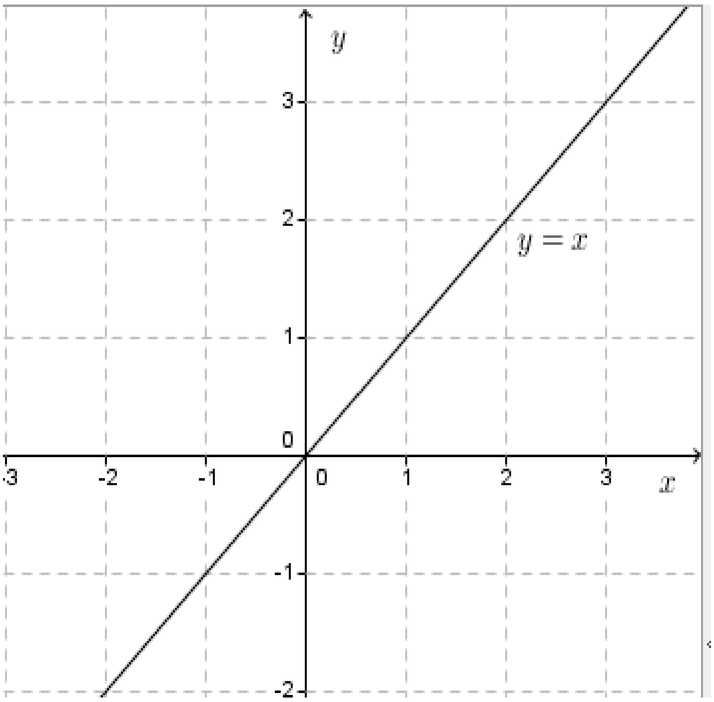
$R(f):\{-3\}$

the function $y=-3$ is not one-to-one and it is an even function, X-axis intercept: $\varnothing$, y-axis intercept: $(0,-3)$, increasing : $\varnothing$, decreasing: $\varnothing$

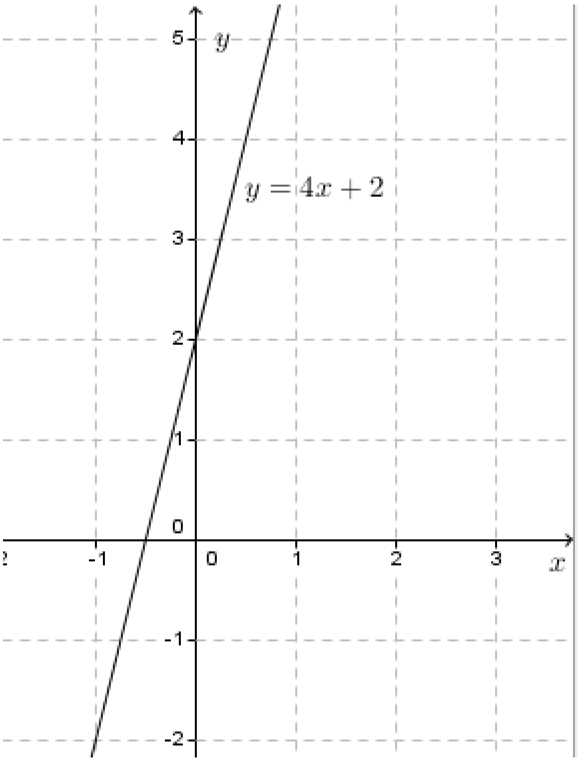


$b) D(f):]-\infty, +\infty[$

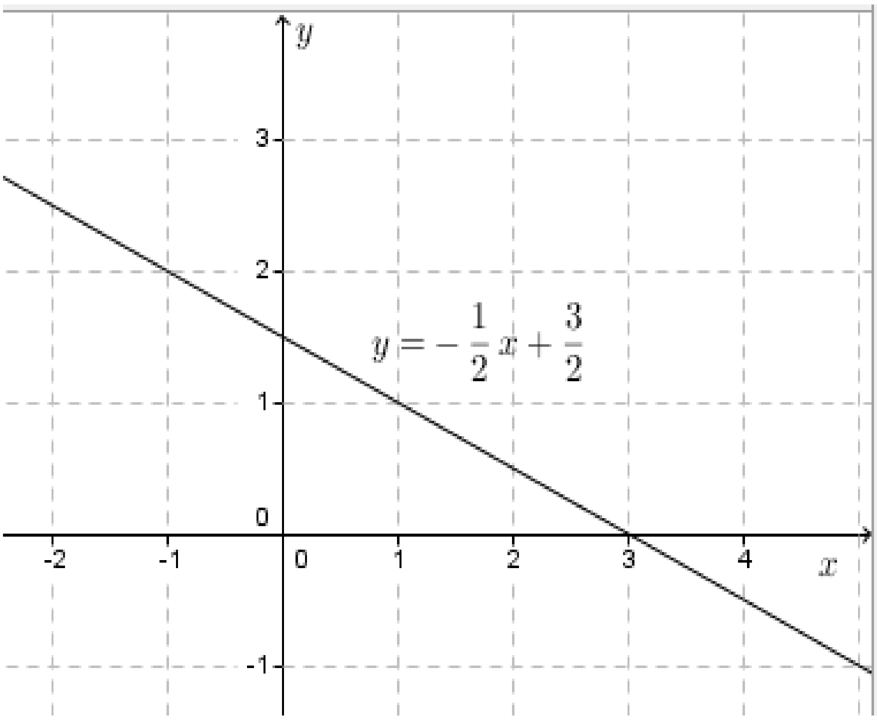
$R(f): [-\infty, +\infty[$ the function $y = x$ is one-to-one and it is an odd function , X-axis intercept: $(0,0)$, y-axis intercept: $(0,0)$, increasing : $]-\infty, +\infty[$, decreasing: $\varnothing$



$c) D(f):]-\infty,+\infty[$
$R(f): [-\infty,+\infty[$ the function $y = 4x + 2$ is one-to-one and it is neither even nor odd function, X-axis intercept: $\left(-\frac{1}{2},0\right)$, y-axis intercept: $(0,2)$, increasing : $]-\infty,+\infty[$, decreasing: $\varnothing$



$d) D(f):]-\infty,+\infty[$
$R(f): [-\infty,+\infty[$ the function $2x + 4y - 6 = 0$ is one-to-one and it is neither even nor odd function, X-axis intercept: $(3,0)$, y-axis intercept: $\left(0,\frac{3}{2}\right)$, increasing : $\varnothing$, decreasing: $]-\infty,+\infty[$



4.9.2 Consolidation activities

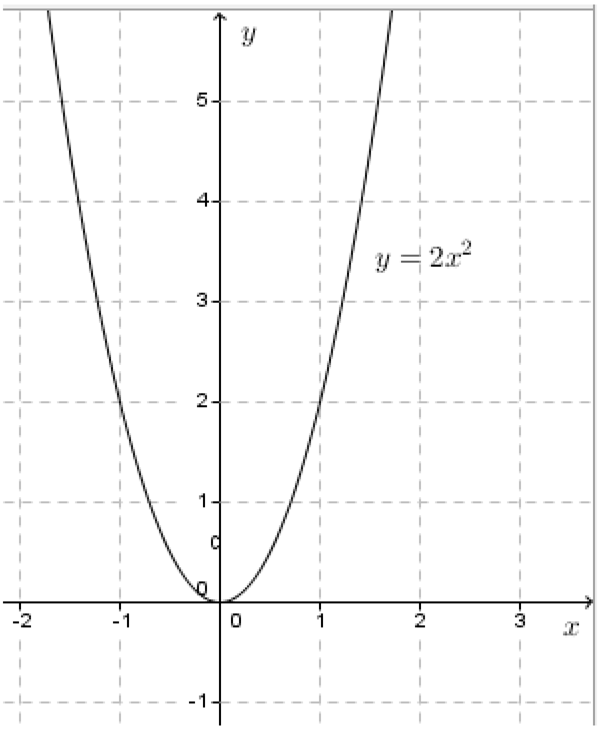
1. Draw the graph of a quadratic function and determine the properties of a function: (domain of a function, range of a function, function is/is not one-to-one function, even/odd function, vertex of a parabola, coordinates of intersections with the x-axis and with the y-axis, local extrema - local minimum & local maximum, intervals of monotonicity - increasing/decreasing function)

$$a)\, y = 2x^2 \qquad\qquad b)\, y = \frac{1}{4}x^2$$

a)$D(f):]-\infty,+\infty[$

$R(f): [0,+\infty[$

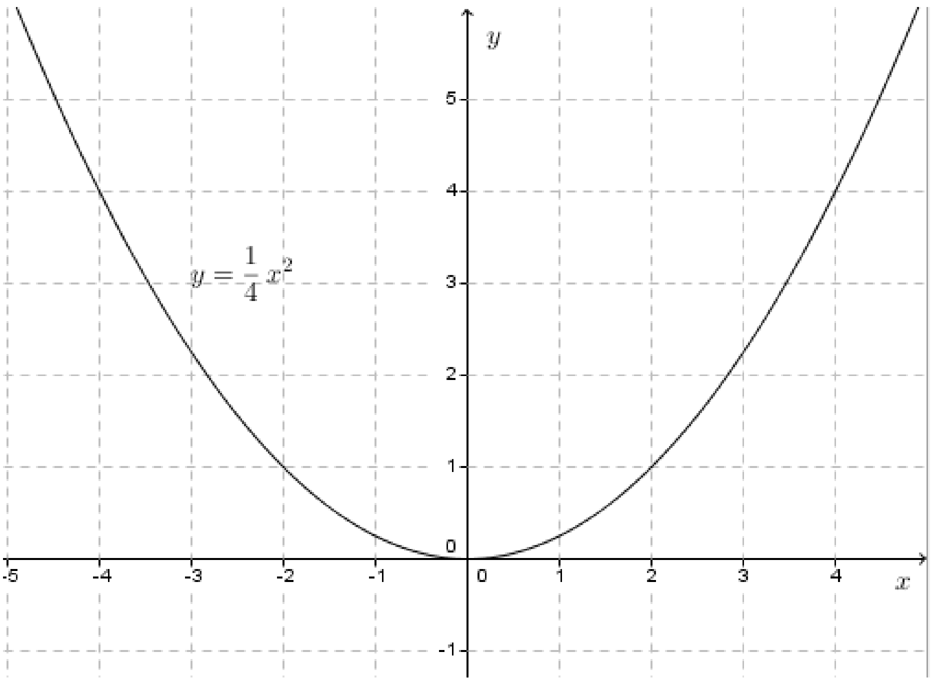
the function $y = 2x^2$ is not one-to-one and it is an even function , vertex : $(0,0)$, X-axis intercept: $(0,0)$, y-axis intercept: $(0,0)$, local minimum: $(0,0)$, local maximum : $\varnothing$, increasing : $]0,+\infty[$, decreasing: $]-\infty,0[$



b) $D(f):]-\infty, +\infty[$

$R(f): [0, +\infty[$

the function $y = \frac{1}{4}x^2$ is not one-to-one and it is an even function, vertex : $(0,0)$, X-axis intercept: $(0,0)$, y-axis intercept: $(0,0)$, local minimum: $(0,0)$, local maximum : $\varnothing$, increasing : $]0, +\infty[$, decreasing: $]-\infty, 0[$



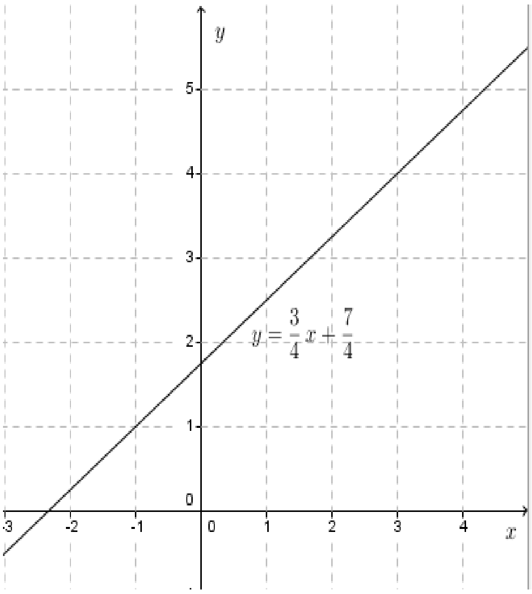
2. Find the equation of a linear function and draw the graph of a function that passes through the points:

$a)A(7,7) \quad and \quad B(-1,1)$

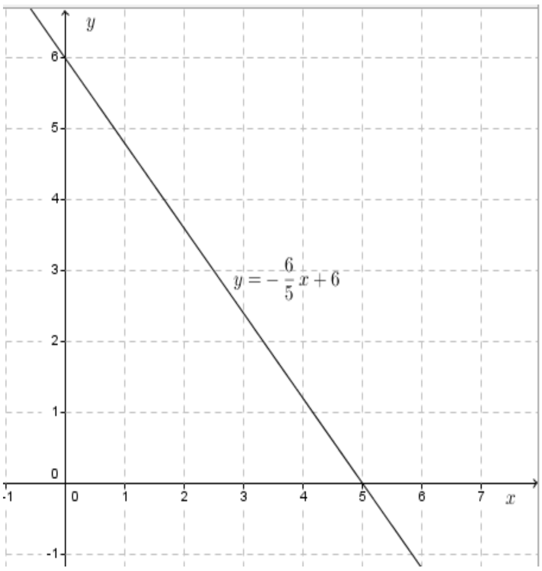
$b)C(5,0) \quad and \quad D(0,6)$

Solutions

$a)y=\frac{3}{4}x+\frac{7}{4}$



$b)y=-\frac{6}{5}x+6$



4.9.3 Extended activities

1. Draw the graph of a quadratic function and determine the properties of a function: (domain of a function, range of a function, function is/is not one-to-one function, even/odd function, vertex of a parabola, coordinates of intersections with the x-axis and with the y-axis, local extrema - local minimum & local maximum, intervals of monotonicity - increasing/decreasing function)

$a)y=x^2-2x \qquad b)y=x^2+6x+5 \qquad c)y=-x^2-1$

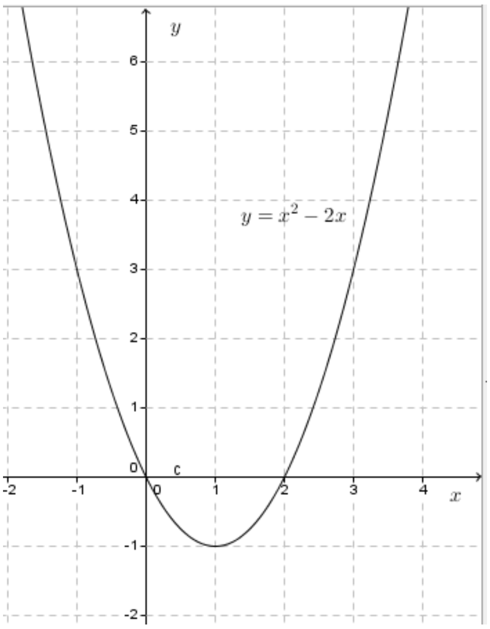
Solutions

$a)D(f):]-\infty,+\infty[$

$R(f):[-1,+\infty[$

the function $y=x^2-2x$ is not one-to-one and it is neither even nor odd function, vertex :

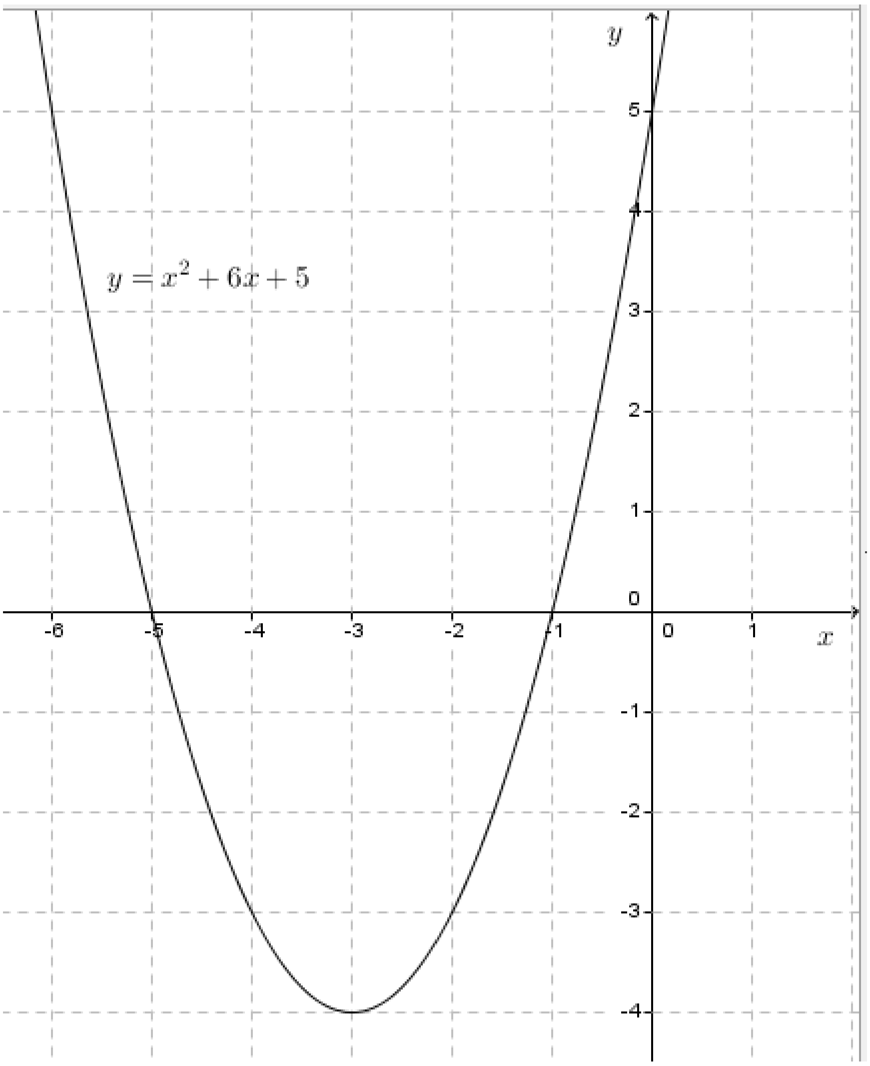
$(1,-1)$, X-axis intercept: $(0,0)$ *and* $(2,0)$, y-axis intercept: $(0,0)$, local minimum: $(1,-1)$, local maximum : $\varnothing$, increasing : $]1,+\infty[$, decreasing: $]-\infty,1[$



$b) D(f):]-\infty,+\infty[$

$R(f): [-1,+\infty[$

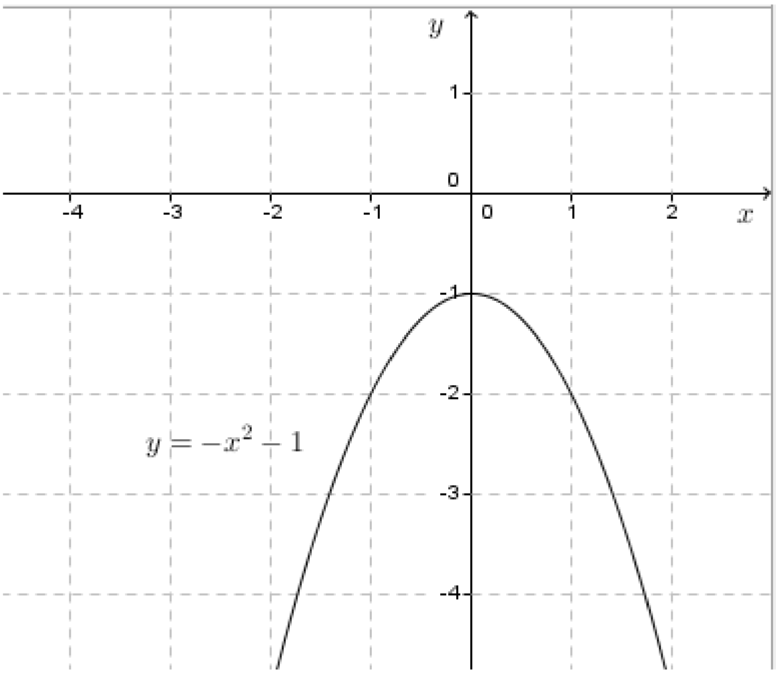
the function $y = x^2 + 6x + 5$ is not one-to-one and it is neither even nor odd function, vertex : $(-3,-4)$, X-axis intercept: $(-5,0)$ *and* $(-1,0)$, y-axis intercept: $(0,5)$, local minimum: $(-3,-4)$, local maximum : $\varnothing$, increasing : $]-3,+\infty[$, decreasing: $]-\infty,-3[$



$c) D(f):]-\infty, +\infty[$

$R(f):]-\infty, -1]$

the function $y = -x^2 - 1$ is not one-to-one and it is an even function, vertex : $(0, -1)$, X-axis intercept: $\varnothing$, y-axis intercept: $(0, -1)$, local minimum: $\varnothing$, local maximum : $(0, -1)$, increasing : $]-\infty, 0[$, decreasing: $]0, +\infty[$


y
x
1
0
-1
-2
-3
-4
-4
-3
-2
-1
0
1
2
$y = -x^2 - 1$

UNIT 5: LIMITS OF POLYNOMIAL, RATIONNAL AND IRRATIONAL FUNCTIONS

5.1. Key unit competence

Evaluate correctly limits of functions and apply them to solve related problems.

5.2 Prerequisite

Students will perform better in this unit if they:

- have a good background on numerical functions, from unit4 preceding this unit;
- are familiar with graphs and their interpretation;
- are skilled enough on intervals in the set of real numbers;
- Are able to find the Cartesian equation of lines, and their positions, as studied in ordinary level.

5.3 Cross-cutting issues to be addressed

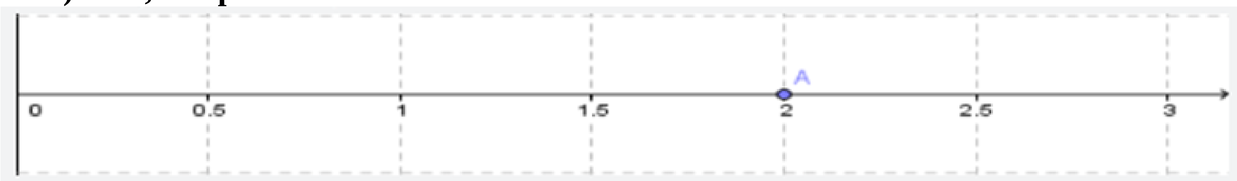
- Financial education (by providing examples reflecting economic issues)
- Inclusive education (promote education for all while teaching)
- Peace and value Education (respect others view and thoughts during class discussions)
- Gender (equal opportunity of boys and girls in the lesson participation)

5.4. Guidance on introductory activity 5.0

• Invite students to work in group and do the **activity 5.0** found in their Mathematics books;
• Move around in the class for facilitating where necessary and give more clarification on eventual challenges they may face during their work;
• As they are discussing, concentrate on slow learners for further explanation and provide assistance to groups in need
• Invite one member from each group to present their work;
• A teacher, harmonize the findings from presentation of learners and guide them to explore the content and examples given in the student's book where they will be able to differentiate the neighbourhood of a real number and the value of a function at a given point.

Answers for introductory activity 5.0

a) Yes, it is possible



b) Open intervals:]1.9;2.1[and]1.99;2.01[

c) When x approaches 2, f(x) approaches 4

$2.00001 \approx 2$ *and* $3.999985000 \approx 4$

$1.99999 \approx 2$ *and* $4.000015000 \approx 4$

The value of $f(x)$ **when** x **approaches 2 is 4**

5.5 . List of lessons

No	Lesson title	Learning objectives	Number of periods
0	Introductory unit 5	To arouse the curiosity of student teachers on the content of unit	1
1	Introduction to limits and neighborhod of a function	Define the concept of limit for a real-valued function, the neighbourhood of a real number and the value of a function at a given point of one real variable.	1
2	One-sided limits	Evaluate the condition of existence of a limit for a function at a given point.	1
3	Infinite limits and Limits at infinity	Calculate the finite and infinite limits of certain functions	1
4	The Squeeze theorem and Operations on limits	Perform operations on limits involving infinity and discuss the Squeeze theorem	2
5	Indeterminate cases $\left(\frac{\infty}{\infty}, \frac{0}{0}, \infty-\infty, 0\cdot\infty\right)$	Calculate the true values of limits by removing the indeterminate forms $\left(\frac{\infty}{\infty}, \frac{0}{0}, \infty-\infty, 0\cdot\infty\right)$	2
6	Applications of limits: Continuity of a function.	Solve problems involving continuity	2
7	Application of limits to determine asymptotes to curves of functions (Horizontal, vertical, oblique)	Extend the concept of limit to determine the asymptotes of a given function	2
8	Real life problems about limits	solve problems involving limits in real life	3
End unit assessment			1

Lesson 1: Introduction to limits and neighborhood of a function

Learning objective:

Define the concept of limit for a real-valued function, the neighbourhood of a real number and the value of a function at a given point of one real variable.

a) Teaching resources:

Student's book and other reference textbooks to facilitate research, Mathematical set, calculator, Manila paper, graph paper, ruler, markers, pens, pencils, tablets or computers for graphing etc.

b) Prerequisites/Revision/Introduction:

Students will perform well in this unit if they are skilled enough in the content of the third unit (Equations and Inequalities) of mathematics in this student's book

c) Learning activities

- Invite students to work in group and do the **activity 1** found in their Mathematics books;
- Move around in the class for facilitating where necessary and give more clarification on eventual challenges they may face during their work;
- Invite one member from each group to present their work;
- As a teacher, harmonize the findings from presentation of students and guide them to explore the content and examples given in the student's book where they will be able to differentiate the neighbourhood of a real number and the value of a function at a given point.
  - After the lesson, guide students to do the **Exercise 1** and evaluate whether lesson objectives were achieved.
  - Provide additional activities where necessary.

Answer for activity 1

Lesotho is completely surrounded by South Africa.
Swaziland is surrounded by South Africa and Mozambique.

Answer of Exercise 1

San Marino, a state surrounded by Italy

Vatican City, a state forming part of Rome, there by surrounded by Italy

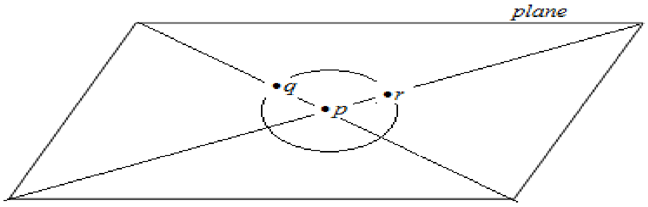
Nyarugenge surrounded by Kigali City

Nyamabuye surrounded by Muhanga.

2) There are many possible answers verify that -5 is the center of the interval. For example: $]-6;-4[$, $]-5.1;-4.9[$, $]-5.01;-4.99[$.

3) No, if the point is on the circumference, then it is impossible to center a disc at the point such that the disc is contained in the circle.

4) The following plane is a neighborhood of points *p*, *q* and *r*.



Note that! Students may give different answers, verify if they are correct. The answers will depend to the location of the city given by the students. However, the teacher need to examine if the students, to acquire well the concept of neighborhood.

Examples:

5)From a-c let student-teachers use their caluculator and approximate the limits by filling the table

They will get:

a. $\lim_{x\to 2}(3x-2)=4$

b. $\lim_{x\to 0}\frac{x}{\sqrt{x+1}\text{ -1}}=0$

c. $\lim_{x\to 2}\frac{x^2-4}{x-2}=4$

d. $\lim_{x\to 1}\left(2x^2+x-4\right)=-1$

e. $\lim_{x\to -1}\frac{x+1}{x^2-x-2}=-\frac{1}{3}$

Lesson 2: One-sided limits

a) Learning objective:

- Evaluate the condition of existence of a limit for a function at a given point.

b) Teaching resources

Student's book and other textbooks to facilitate research, Mathematical set, calculator, manila paper, markers, pens, pencils, online resources etc.

c) Prerequisites/Revision/Introduction

In this lesson, Students will perform better if they are enough skilled in:

- Interpreting graphs and functions in terms of behavior in the neighborhood of a real number ;
- Cartesian plane: plotting points, drawing lines;
- Performing operations on intervals;

d) Learning activities

Invite students to work in groups the **activity 3** in their Mathematics books;

- ➢ Move around in the class for facilitating where necessary and give more clarification on eventual challenges they may face during their work;
- ➢ Invite a student from each group to present their findings;
- ➢ As a teacher, harmonize the findings from presentation and guide them to guess the true definition of limit of a function at a given point basing on the value of the left hand limit and the right hand limit.
- ➢ Use different probing questions and guide them to explore the content and examples given in the student's book lead them to be able to determine the limit of a function at a point when applying related properties.

  - ➢ After this step, assign students to do the application **Exercise 3** and evaluate whether the lesson objectives were achieved.
- ➢ Provide additional activities where necessary.

Answers for activity 3

1) 1.8; 2) 1.9; 3)1.99; 4)2.15; 5) 2.03; 6)2.003

Answers for Exercise 3

1) 7; 2)1 ; 3) 0 ; 4) Does not exist.

Lesson 3: Infinite limits and Limits at infinity

a) Learning objective

- Calculate the finite and infinite limits of certain functions
- Perform operations on limits involving infinity and discuss the Squeeze theorem

b) Teaching resources:

Student's book and other reference textbooks to facilitate research, Mathematical set, calculator, Manila paper, markers, pens, pencils, etc.

c) Prerequisites/Revision/Introduction:

Students will perform well in this unit if they have good background in:

- Operations in the set of real numbers

d) Learning activities

- Invite students to work in group and do the **activity 4** found in their Mathematics books;
- Move around in the class for facilitating groups where necessary and ask some guiding questions on eventual challenges they may face during their work;
- Invite groups with different working steps to present their work;
- As a teacher, harmonize the findings from presentation of students and guide them to evaluate the rate of change, instantaneous velocity and instantaneous acceleration of a moving body.
- Guide them to explore the content and examples given in the student's book where they will be able to solve real problems involving limits.
- After the lesson, guide students to do the **Exercise 4** and evaluate whether lesson objectives were achieved.
- Provide additional activities where necessary.

Answers to activity 4

1.a)-65.6 b)-99 c)-199 d)201 e)101 f)67.6

2. $a)+\infty$ b) $-\infty$ c) Indetrminate
d) $-\infty$ e) $-\infty$ f) $+\infty$ g) indeterminate

Answers to Exercise 4

1. $\frac{1}{2}$

2. 0

3. ∞

4. $\lim_{x\to -4^-}\frac{x+1}{x+4}=+\infty$, $\lim_{x\to -4^+}\frac{x+1}{x+4}=-\infty$

5. $\lim_{x\to 3^-}\frac{x^2+2x+1}{x-3}=-\infty$, $\lim_{x\to 3^+}\frac{x^2+2x+1}{x-3}=+\infty$

Lesson 4: The Squeeze theorem and Operations on limits

b) Learning objective

- Perform operations on limits involving infinity and discuss the Squeeze theorem

b) Teaching resources:

Student's book and other reference textbooks to facilitate research, Mathematical set, calculator, Manila paper, markers, pens, pencils, etc.

c) Prerequisites/Revision/Introduction:

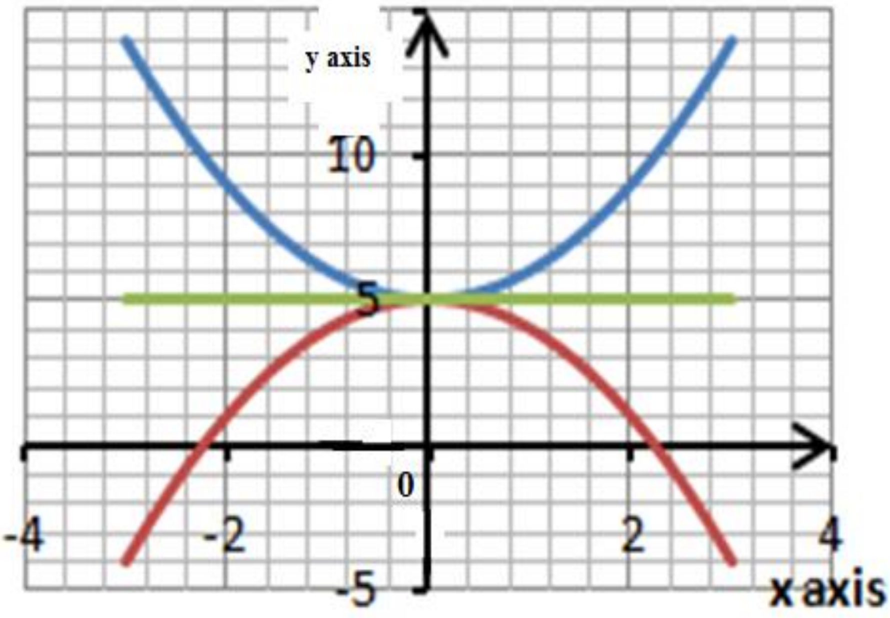
Students will perform well in this unit if they have good background in:

- Operations in the set of real numbers

d) Learning activities

- ➢ Invite students to work in group and do the **activity 6** found in their Mathematics books;
- ➢ Move around in the class for facilitating groups where necessary and ask some guiding questions on eventual challenges they may face during their work;
- ➢ Invite groups with different working steps to present their work;
- ➢ As a teacher, harmonize the findings from presentation of students and guide them to evaluate the rate of change, instantaneous velocity and instantaneous acceleration of a moving body.
- ➢ Guide them to explore the content and examples given in the student's book where they will be able to solve real problems involving limits.
  - ➢ After the lesson, guide students to do the **Exercise 6** and evaluate whether lesson objectives were achieved.
  - ➢ Provide additional activities where necessary.

Answers to activity 6



The curve of $h(x)=5$ lies between other two curves and the three curves meet at the same point $(0,5)$; $\lim_{x\to 0} f(x) = \lim_{x\to 0} g(x) = \lim_{x\to 0} h(x) = 5$

2 a. -3; -3

b. 0, -1; -1

c. -2; 5; $-\frac{2}{5}$

d. 25; 25

A constant can be moved through a limit sign,

Limit of sum is the sum of limits,

Limit of quotient is the quotient of limits, provided that the denominator is not zero,

Limit of a power is the power of limit.

Answers to Exercise 6

1. 0

2. a) 3 b) -5832

3. a) $\infty - \infty$ is indeterminate form not zero

b) $\lim_{x \to 0^+} \left(\frac{1}{x} - \frac{1}{x^2} \right) = \lim_{x \to 0^+} \frac{x-1}{x^2}$

x	$-\infty$		0		1		$+\infty$
$x-1$		-			0	+	
x^2		+	0	+			
$\frac{x-1}{x^2}$		-	||	-	0	+	

Hence, $\lim_{x \to 0^+} \left(\frac{1}{x} - \frac{1}{x^2} \right) = -\infty$

Lesson 5: Indeterminate cases

a) Learning objective:

Identify and remove indefinite cases.

b) Teaching resources

Student's book and other reference textbooks to facilitate research, Mathematical set, calculator, Manila paper, markers, pens, pencils, etc.

c) Prerequisites/Revision/Introduction

- Students will perform well in this unit if they have good background in :

  Lesson two of this unit; Radicals (surds) learnt in senior two; Polynomials factorization; Simplifying algebraic expressions.

d) Learning activities

- Invite students to work in groups and do the **activity 7** found in their Mathematics books;
- Move around in the class for facilitating students where necessary and ask some challenging questions to lead them to work correctly;
- Verify and identify groups with different working steps;
- Invite a group, randomly chosen, with to present the work of his/her.
- As a teacher, harmonize the findings from presentation and guide them to enhance the limits involving infinity and the operations on limits in general.
- Use different probing questions and guide them to explore the content and examples given in the student's book and lead them to explore the addition, multiplication and division on limits.
- Move to every group and verify their working steps.

- Invite all students for a whole class discussion and guide them to guess how to move out the indeterminate cases;
- Use different probing questions and guide them to explore the content and examples related to limits of functions at infinity and involving infinity, indeterminate cases as it is given in the student's book;
  - After this step, guide students to do the **Exercise 7** and evaluate whether lesson objectives were achieved.
  - Provide additional activities where necessary.

Answers for activity 7

1. a. $x-1$

   b. $x-2$

2. a. $\sqrt{x^2-2}-3$

   b. $\sqrt{x-2}+1$

Answers for exercise 7

1. $+\infty$

2. $-\infty$

3. 20

Lesson 6: Applications of limits: Continuity of a function.

a) Learning objective:

- Solve problems involving continuity.

b) Teaching resources:

Student's book and other Reference textbooks to facilitate research, Mathematical set, calculator, manila paper, graph papers, markers, pens, pencils, etc.

c) Prerequisites/Revision/Introduction:

Students will perform well in this unit if:

- They have good background in previous lessons of this unit;
- They are skilled in analyzing functions on the graph.

d) Learning activities

- Invite students to work in group and do the **activity 9** found in their Mathematics books;
- Move around in the class for facilitating groups where necessary and ask some guiding questions on eventual challenges they may face during their work;
- Invite few groups to present their work;
- Guide them to explore the content and examples given in the student's book where they will be able to evaluate the continuity of limits toward a point.
  - End the lesson by guiding students on the **Exercise 9** and evaluate whether lesson objectives were achieved.
  - Provide additional activities where necessary.

Answer for activity 9

1. a) 4

   b) 4

   c) $f(2)$ and $\lim_{x\to 2} f(x)$ exist and are equal, $f(2)=\lim_{x\to 2} f(x)$

Answers of Exercise 9

1. **The function is continuous on its domain**

   CE: $x^2-2x-15\neq 0$ **therefore the function is not continuous at** $x=5$ and $x=-3$.

2. Let us calculate the limit first.

   $\lim_{x\to 3}\frac{x^2-9}{x-3}=\lim_{x\to 3}\frac{(x-3)(x+3)}{x-3}=3$ but the function is continuous if $\lim_{x\to 3} f(x)=f(3)$

   $\Rightarrow k=6$ Therefore for the function to be continuous $k=6$.

3. $a=-1$ *and* $b=1$

Lesson 7. Application of limits to determine asymptotes to the curves of functions

a) Learning objective:

- Extend the concept of limit to determine the asymptotes of a given function

b) Teaching resources:

Student's book and other Reference textbooks to facilitate research, Mathematical set, calculator, manila paper, graph papers, markers, pens, pencils, etc.

c) Prerequisites/Revision/Introduction:

Students will perform well in this unit if:

- They have good background in previous lessons of this unit;
- They are skilled in analyzing functions on the graph.

d) Learning activities

- ➢ Invite students to work in group and do the **activity 12** found in their Mathematics books;
- ➢ Move around in the class for facilitating groups where necessary and ask some guiding questions on eventual challenges they may face during their work;
- ➢ Invite few groups to present their work;
- ➢ Guide them to explore the content and examples given in the student's book where they will be able to evaluate the continuity of limits toward a point.
  - ➢ End the lesson by guiding students on the **Exercise 12** and evaluate whether lesson objectives were achieved.
  - ➢ Provide additional activities where necessary.

Answers to activity 12

As x increases or decreases the curve comes closer and closer to the line B. As x approaches 3 from the right or from the left, the curve comes closer and closer to the line A

Answers to the Exercise 12

1.Horizontal asymptote: $y=1$, Vertical asymptotes: $x=-1, x=0$

2. Horizontal asymptote: $y=0$

3. Vertical asymptotes: $x=\frac{9}{4}$, Oblique asymptote: $y=\frac{1}{4}x+\frac{21}{16}$

4. No asymptote

5.Horizontal asymptote: $y=3$, Vertical asymptotes: $x=-2, x=2$

Lesson 8: Real life problems about limits

a) Learning objective

Use the properties of limits to solve real life problems involving limits.

b) Teaching resources:

Student's book and other reference textbooks to facilitate research, Mathematical set, calculator, Manila paper, markers, pens, pencils, etc.

c) Prerequisites/Revision/Introduction:

Students will perform well in this unit if they have mastered:

- Lessons from 1 to 7 of this unit
- To convert a word problem into a mathematical statement and vice versa

d) Content: Limits can be applied in different fields in real life.

1. Instantaneous rate of change of a function in physics

The **instantaneous rate of change** of $f(x)$ at a, also called the **rate of change** of $f(x)$ at a, is defined to be the limit of the average rate of change of $f(x)$ over shorter and shorter intervals around a.

Since the average rate of change is a difference quotient of the form $\frac{\Delta y}{\Delta t}$, the instantaneous rate of change is a limit of difference quotient. In practice, we often approximate a rate of change by one of these difference quotients.

2. Instantaneous velocity

Instantaneous velocity of a moving body is the limit of average velocity over an infinitesimal interval of time.

$$v = \lim_{t \to 0} \frac{\Delta s}{\Delta t}.$$

3. Instantaneous acceleration

Instantaneous acceleration for a moving body is the limit of average acceleration over an infinitesimal interval of time $a = \lim_{\Delta t \to 0} \frac{\Delta v}{\Delta t}$

(Here and elsewhere, if motion is in a straight line, vector quantities can be substituted by scalars in the equations.)

Solution (2)

Vertical asymptote is found at $p = 100$

To assess the behavior of the model, let us choose some values

Example: the cost of removing 85% of pollutants is

$$C(85\%) = \frac{80000(85)}{100-85} \approx 453.333$$

But the cost of removing 90% is

$$C(90\%) = \frac{80000(90)}{100-90} = 720.$$

We can see that the higher and higher percentage of pollutants is removed, the cost increases dramatically.

e) Learning activities

- Invite students to work in group and do the following **activity 5A** find below;
- Move around in the class for facilitating groups where necessary and ask some guiding questions on eventual challenges they may face during their work;
- Invite some representatives to present the work of their when others are following attentively;
- As a teacher, harmonize the findings from presentation of students and guide them to evaluate the rate of change, instantaneous velocity and instantaneous acceleration of a moving body.
- Guide them to explore the content and examples given in the student's book where they will be able to solve real problems involving limits.
  - After the lesson, guide students to do the following **Exercise 5A** and evaluate whether lesson objectives were achieved.
  - Provide additional activities where necessary.

Activity 5A

1. The quantity (in mg) of a drug in the blood at time t (in minutes) is given by $Q == 25(0.8)^t$. Estimate the rate of change of the quantity at t = 3 and interpret your answer.

1. The cost of removing $p\%$ in a given place is given by the model

$$\text{C(x)} = \frac{80000p}{100-p}, 0 \le p < 100$$

a)What is the vertical asymptote for this function?

b)What does this vertical asymptote mean in the context of the problem?

Solution to Activity 5A

Solution (1):

We estimate the rate of change at t = 3 by computing the average rate of change over intervals near t = 3. We can make our estimate as accurate as we like by choosing our intervals small enough.

Let's look at the average rate of change over the interval $3 \leq t \leq 3.01$:

Average rate of change $= \dfrac{\Delta Q}{\Delta t} = \dfrac{25(0.8)^{3.01} - 25(0.8)^{3}}{3.01 - 3.00} = -2.85$

A reasonable estimate for the rate of change of the quantity at $t = 3$ is -2.85 Since Q is in mg and t in minutes, the units of $\dfrac{\Delta Q}{\Delta t}$ are mg/minute. Since the rate of change is negative, the quantity of the drug is decreasing. After 3 minutes, the quantity of the drug in the body is decreasing at $2.85 mg/\text{minute}$

Solution (2):

Vertical asymptote is found at $p = 100$

To assess the behavior of the model, let us choose some values

Example: the cost of removing 85% of pollutants is

$C(85\%) = \dfrac{80000(85)}{100 - 85} \approx 453.333$

But the cost of removing 90% is

$C(90\%) = \dfrac{80000(90)}{100 - 90} = 720.$

We can see that the higher and higher percentage of pollutants is removed, the cost increases dramatically.

Exercise 5A

1. The cost C (in millions dollars) for the federal government to seize $p\%$ of a type of illegal drug as it enters the country is modeled by the

   $C = \dfrac{528p}{100 - p}, 0 \leq p < 100$

   a. Find the cost of seizing (stopping) $25\%, 50\%$ and 75%.

   b. Find the limit as $p \to 100^-$, interpret this limit in the context of the problem.

2. A business has a cost in dollars of $C = 0.5x + 500$ for producing x units.

   a. Find the average cost function $\overline{C}$.

   b. Find $\overline{C}$ when x=250 and when x=1250

   c. What is the limit of $\overline{C}$ as x approaches to infinity? Interpret the results in the context of the problem.

Answers to Exercise 5A

1.

a) $C(25\%)=\frac{528(25)}{100-25}=176$ **\$,** $C(50)=\frac{528(50)}{100-50}=352$ **\$,** $C(75)=528$

b) $\lim_{x\to 100^-}\left(\frac{528p}{100-p}\right)=+\infty$, this means that there are no more expenses of money when 100% of illegal drug is stopped. The country is safe at that time!

The tutor goes from this to inform to student-teacher that drugs destroy their brain.

2.

a. $\overline{C}=0.5+\frac{500}{x}$

b. $\overline{C}(250)=0.5+\frac{500}{250}=2.5\,\$$

c. $\overline{C}(1250)=0.5+\frac{500}{1250}=0.9\,\$$

d. $\lim_{x\to\infty}\left(0.5+\frac{500}{x}\right)=0.5\,\$$

The more quantities are sold, the average cost price decreases gradually. But there is a reserved price for the seller such that if he/she goes below that he/she can lose his/her money. So the reserved price is 0.5$.

5.6 Summary of the unit.

In this unit we dealt with limits and their applications.

We started with the concept of the “neighborhood”, in general, and the “neighborhood of a real number”, in particular. At this level, we were able to determine a δ-neighborhood of x_0, deleted or not. We were also able to find the center and the radius of an open interval.

We, then, focused on the meaning of the “limit of a variable” and the “limit of a function as the independent variable approaches x_0 (finite or infinite).We found that:

“$x\to x_0$” means that “ x is assuming values closer and closer to x_0”, from the left side and from the right side, for $x_0\in\mathbb{R}$;

“$x\to+\infty$” means that “the values of x are increasing without bound”, and

“$x\to-\infty$” means that “the values of x are decreasing without bound”

The statement "If $x \to x_0$, then $f(x) \to L$" is denoted by $\lim\limits_{x \to x_0} f(x) = L$

We set the goal of being able to determine the limit of a function algebraically, from numerical approach(table) and graphically. The calculation of the limit of a function $\lim\limits_{x \to x_0} f(x)$ brought us to three cases:

1) $\lim\limits_{x \to x_0} f(x) = \mathrm{f}(\mathrm{x}_0)$, provided $f(x_0)$ can be found by properties of elementary operations in the set of real numbers;

2) The calculation of $\lim\limits_{x \to x_0} f(x)$ involves operations in the set ¡ $\cup\{-\infty;+\infty\}$.In this set, operations, such as the following held:

$$\frac{a}{\infty} = 0; \frac{a}{0} = \infty; \frac{\infty}{b} = \infty; \infty + a = \infty; (\infty)(\infty) = \infty \text{ ; Etc}$$

3) The calculation of $\lim\limits_{x \to x_0} f(x)$ leads to one of the following:

$\frac{0}{0}; \frac{\infty}{\infty}; \infty - \infty; 0(\infty); 1^{\infty}; 0^0; \infty^0$: These were termed "**indeterminate cases**", they were hiding the true value towards which $f(x)$ gets closer and closer. To discover the true value, it was necessary to remove the indetermination. We focused on different ways of removing the indetermination, including factorization and multiplication by the conjugate.

Finally, we ended with the situations involving the use of limits: continuity and asymptotes.

5.7 Additional information for teachers.

- Ensure that your students do not confuse superscripts with signs of numbers. They need to be careful about associating the "-" superscript with negative numbers. And the "+" superscript with positive numbers. The superscripts mean either the left or right.
- The symbol "$\to$" means approaches to. Your students should avoid using it as abbreviation sign
- Use parentheses when taking the limit of an expression consisting of more than one term. For example,

$$\lim_{x \to -1}\left(3x^3 + 4x^2 - 2x + 2\right) = 3 - 1^3 + 4 - 1^2 - 2(-1) + 2$$

This is not correct

Instead,

$$\lim_{x \to -1}\left(3x^3 + 4x^2 - 2x + 2\right) = 3(-1)^3 + 4(-1)^2 - 2(-1) + 2$$

- Your students should be careful while evaluating limits of piecewise function and functions involving absolute value signs (sometimes left and right limits confuse).
- In the calculation of limits involving infinity, they should precise the sign of infinity. $\infty \neq +\infty$; For ∞, consider both cases separately: $+\infty$ and $-\infty$
- Do not omit the limit operator until the substitution phase. For example, in the calculation of

$$\lim_{x\to\infty}\sqrt{x^2+2x-1}-\sqrt{x^2-x+2}=\infty-\infty(\text{IF})$$,

We have:

$$\lim_{x\to\infty}\frac{\left(\sqrt{x^2+2x-1}-\sqrt{x^2-x+2}\right)\left(\sqrt{x^2+2x-1}+\sqrt{x^2-x+2}\right)}{\sqrt{x^2+2x-1}+\sqrt{x^2-x+2}}$$

But $(a-b)(a+b)=a^2-b^2$

$=\dfrac{(x^2+2x-1)-(x^2-x+2)}{\sqrt{x^2+2x-1}+\sqrt{x^2-x+2}}$ Wrong! (the limit sign is forgotten)

But $=\lim\limits_{x\to\infty}\dfrac{(x^2+2x-1)-(x^2-x+2)}{\sqrt{x^2+2x-1}+\sqrt{x^2-x+2}}$ correct!

$$=\lim_{x\to\infty}\frac{3x-3}{\sqrt{x^2+2x-1}+\sqrt{x^2-x+2}}$$

$$=\lim_{x\to\infty}\frac{x\left(3-\frac{3}{x}\right)}{\sqrt{x^2\left(1+\frac{2}{x}-\frac{1}{x^2}\right)}+\sqrt{x^2\left(1-\frac{1}{x}+\frac{2}{x^2}\right)}}$$

$=\lim\limits_{x\to\infty}\dfrac{x\left(3-\frac{3}{x}\right)}{2\sqrt{x^2}}$ but $\sqrt{x^2}=|x|=\begin{cases}x,\text{if } x\geq 0\\ -x,\text{if } x<0\end{cases}$

Therefore, $\lim\limits_{x\to\infty}\dfrac{x\left(3-\frac{3}{x}\right)}{2\sqrt{x^2}}=\dfrac{x\left(3-\frac{3}{x}\right)}{2x}$ wrong! the limit sign is forgotten, and at this level it is necessary to split the limit:

$\lim\limits_{x\to+\infty}\dfrac{x\left(3-\frac{3}{x}\right)}{2\sqrt{x^2}}=\lim\limits_{x\to+\infty}\dfrac{x\left(3-\frac{3}{x}\right)}{2x}=\dfrac{3}{2}$ **Correct!**

And $\lim\limits_{x\to-\infty}\dfrac{x\left(3-\frac{3}{x}\right)}{2\sqrt{x^2}}=\lim\limits_{x\to-\infty}\dfrac{x\left(3-\frac{3}{x}\right)}{-2x}=-\dfrac{3}{2}$

5.8 Answers for Revision Exercises

Revision exercises

1. $-\frac{1}{2}$

2. 2

3. 2

4. $\frac{1}{2}$

5. 8

6. 1

7. 4

8. 2

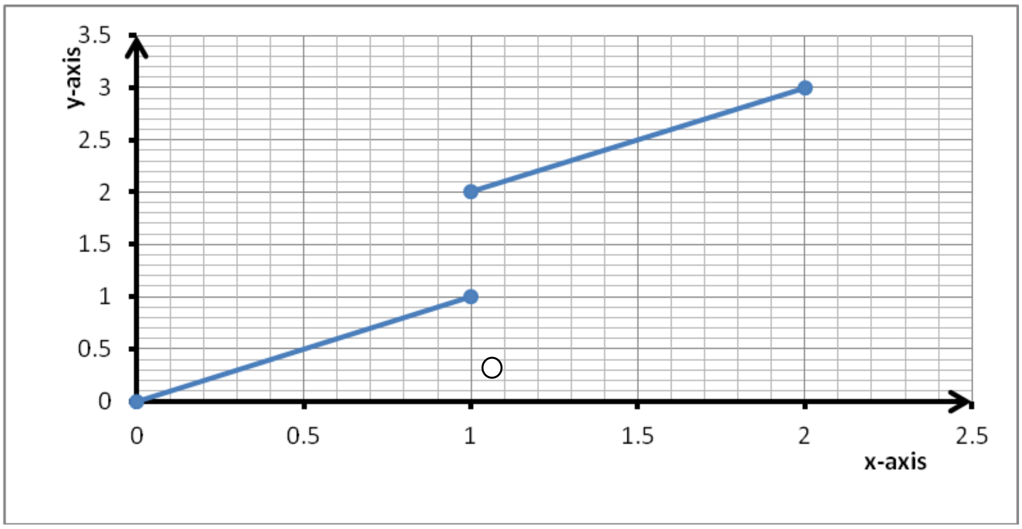
9. Does not exist

10. -6

11. $\lim_{x\to 2^-} f(x) = \lim_{x\to 2^+} f(x) = f(2)$. The given function is continuous at $x = 2$

12. 0

13. $x = 1$



14. $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x) = f(0)$. The given function is continuous at $x = 0$

$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x) = f(1)$. The given function is continuous at $x = 0$

15. Eliminable discontinuity at $x = 4$

16. Discontinuity of the second kind at $x = -2$

17. Discontinuity of the first kind at $x = 3$

18. Vertical asymptote $x = a$

19. No asymptotes

20. Vertical asymptote $x=1$, oblique asymptotes $y=-x-\frac{1}{2}$ and $y=x+\frac{1}{2}$

21. Vertical asymptote $x=0$

22. Vertical asymptote $x=2$, oblique asymptote $y=2x+2$

23. Vertical asymptote $x=-1$, oblique asymptote $y=x+1$

24. Vertical asymptote $x=2$, oblique asymptote $y=x+\frac{1}{2}$

25. Vertical asymptote $x=\frac{5}{3}$, horizontal asymptote $y=\frac{2}{3}$

26. No asymptotes

27. Vertical asymptote $x=0$, oblique asymptote $y=ax$

28. Vertical asymptotes $x=1$ and $x=-1$, horizontal asymptote $y=2$

29. Vertical asymptote $x=a$

30. Oblique asymptote $y=\frac{x}{2}-\frac{1}{2}$

31. Vertical asymptote $x=\frac{1}{2}$

32. Horizontal asymptotes: $y=-1$ and $y=1$

33. Horizontal asymptotes: $y=0$ and $y=-1$, vertical asymptote $x=0$

5.9. Additional activities

5.9.1 Remedial activities

1. For the given function $\begin{cases} x+2, x<1 \\ x^2, 1\le x\le 2 \\ 5x-6, x\ge 2 \end{cases}$

i) Identify the points where the function is discontinuity,

ii) where $f(x)$ is discontinuous or continuous

Answers

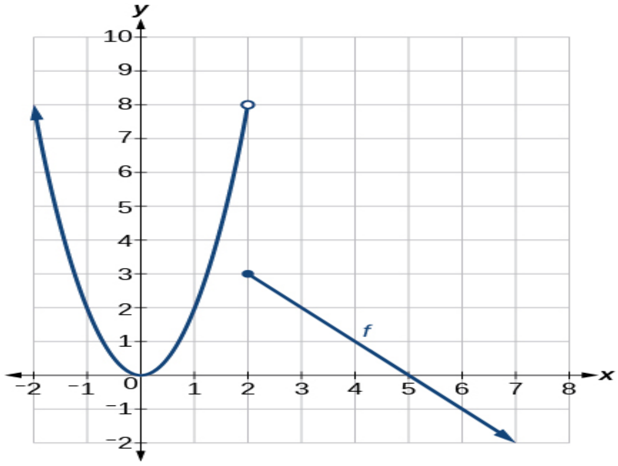
$\lim_{x\to 1^-} f(x)=3; \lim_{x\to 1^+} f(x)=1; \lim_{x\to 2^-} f(x)=4; \lim_{x\to 2^+} f(x)=4$

Conclusion

$\lim_{x\to1^-} f(x) \neq \lim_{x\to1^+} f(x)$, therefore $f(x)$ is discontinuous at $x=1$

$\lim_{x\to2^-} f(x) = \lim_{x\to2^+} f(x)$, therefore $f(x)$ is continuous at $x=2$

2. Given the graph of function *f*,



find:

a. $\lim_{x\to2^-} f(x)$ b. $\lim_{x\to2^+} f(x)$ c. $\lim_{x\to2} f(x)$ d. $f(2)$

Solution

a. $\lim_{x\to2^-} f(x)=8$ **b.** $\lim_{x\to2^+} f(x)=3$ **c.** $\lim_{x\to2} f(x)$ does not exist since $\lim_{x\to2^-} f(x) \neq \lim_{x\to2^+} f(x)$

d. $f(2)=3$

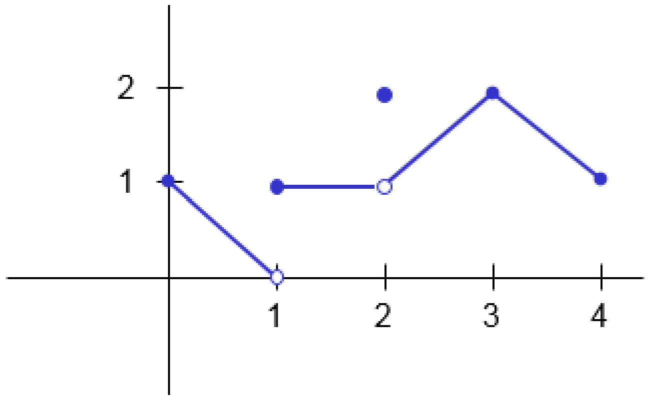
3. Discuss the continuity of the following function $f(x)=\begin{cases} x^2+1 & \text{if } x<1 \\ \dfrac{x+1}{x} & \text{if } x\geq1 \end{cases}$ at $x=1$

Solution

$\lim_{x\to1^-} f(x)=2$ and $\lim_{x\to1^+} f(x)=2$,therefore $\lim_{x\to1} f(x)=2$, hence, the function is continuous at $x=1$.

5.9.2 Consolidation activities

1. Observe the graph below, analyze it and interpret it then find out if $\lim_{x\to 3} f(x)$ exists or not.



Solution:

$\lim_{x\to 3^-} f(x)=2$ **;** $\lim_{x\to 3^+} f(x)=2$ **;** $f(3)=2$

Hence, $\lim_{x\to 3^-} f(x)=\lim_{x\to 3^+} f(x)$ then $f(x)$ exists

5.9.3 Extended activities

1. Consider the numerical function defined by $f(x)=\dfrac{x^3}{1-x^2}$

   i. Find the domain of definition of the function $f(x)$.

   ii. Find the limits at the endpoints of the domain of definition and deduce the equations of all possible asymptotes.

   iii. Discuss the symmetry of the function $f(x)$.

Solution

i. **Restrictions on x:** $1-x^2\neq 0$

$(1-x)(1+x)\neq 0 \Rightarrow x\neq 1$ and $x\neq -1$

Dom$f=\mathbb{R}\setminus\{-1,1\}$

ii. $\begin{cases}\lim_{x\to -\infty}\dfrac{x^3}{1-x^2}=+\infty\\ \lim_{x\to +\infty}\dfrac{x^3}{1-x^2}=-\infty\end{cases}$ The horizontal asymptote does not exist.

$\begin{cases}\lim_{x\to -1^-}\dfrac{x^3}{1-x^2}=+\infty\\ \lim_{x\to -1^+}\dfrac{x^3}{1-x^2}=-\infty\end{cases}$ V.A $\equiv x=-1$

$$\begin{cases} \lim\limits_{x\to 1^-} \dfrac{x^3}{1-x^2} = +\infty \\ \lim\limits_{x\to 1^+} \dfrac{x^3}{1-x^2} = -\infty \end{cases} \quad \text{V.A} \equiv x = 1$$

Let us find the oblique asymptote

$$\text{O.A} \equiv y = ax + b \text{ where } \begin{cases} a \in \mathbb{R}^* \\ b \in \mathbb{R} \end{cases}$$

$$a = \lim_{x\to\infty} \frac{f(x)}{x} \Rightarrow a = \lim_{x\to\infty} \frac{x^3}{x - x^3} = -1$$

$$b = \lim_{x\to\infty} f(x) - ax \Rightarrow b = \lim_{x\to\infty} \frac{x^3}{1-x^2} + x = 0$$

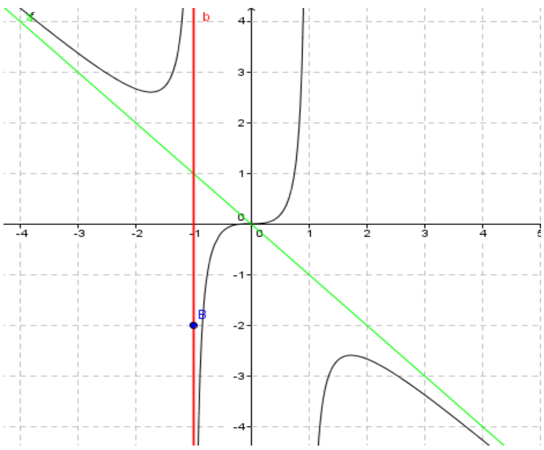
Therefore, the $\text{O.A} \equiv y = -x$

iii. The function is even if $f(x) = f(-x)$

But $f(-x) = \dfrac{(-x)^3}{1-(-x)^2} = \dfrac{-x^3}{1-x^2}$, hence the function is not even (the graph is not symmetric about the y-axis).since $f(x) \neq f(-x)$

The function is odd if and only if $f(-x) = -f(x)$

But, $-f(x) = \dfrac{-x^3}{1-x^2} = \dfrac{-x^3}{1-x^2}$, hence the function is odd since $f(-x) = -f(x)$**,** the graph of this function is symmetric about the origin. (See graph)



Note: a curve never crosses its vertical asymptote.

Unit 6: DIFFERENTIATION OF POLYNOMIALS, RATIONAL AND IRRATIONAL FUNCTIONS AND THEIR APPLICATIONS

6.1 Key unit competence:

Use the gradient of a straight line as a measure of rate of change and apply this to line tangent and normal of curves in various contexts and use these concepts of differentiation to solve and interpret related rates and optimization problems in various contexts

6.2 Prerequisite

Students will easily learn this unit, if he/she is able to

- Calculate the limits of numerical functions taught in Unit 5 ;
- Calculate the slope/gradient of graph as taught in unit5
- Determine equations of a straight line. S3(UNIT6)
- Be accurate in plotting/graphing and calculations.

6.3 Cross-cutting issues to be addressed

- Inclusive education (promote education for all while teaching)
- Peace and value Education (respect others view and thoughts during class discussions)
- Gender (equal opportunity of boys and girls in the lesson participation)
- Standardization culture (Students encouraged to adhere to rules and standards of calculations and drawing of graphs)

6.4 Guidance on introductory activity

- Students work on the introductory **activity 6.0** to be aware of this unit.
- The students have encountered gradients before. Ask them what they recall about the concept of gradients.

- Guide the students, to do the **activity 6.0**, in pairs, move around in order to help the slower students.
- Guide the student to present their findings

Answer of introductory activity

a)

we have $x_0 = 1$ and $h = \Delta x = 1$

$$The\ slope\ is\ given\ by\ m_p = \frac{\Delta y}{\Delta x} = \frac{f(x_0+h)-f(x_0)}{(x_0+h)-x_0} = \frac{4-2}{2-1} = 2$$

b)

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$
$$=\lim_{h\to 0}\frac{\left[(x+h)^2+1\right]-(x^2+1)}{h}$$
$$=\frac{x^2+2hx+1-x^2-1}{h}$$
$$=\frac{2hx}{h}=2x$$
$$f'(x)=2x$$
$$for\ x_0=1\Rightarrow f'(x_0)=f'(1)=2$$

The slope $m_p = f'(x_0)=2$

2) Possible answers:

The derivative of a function $f(x)$ with respect to x is denoted by $f'(x)$ or $\frac{d}{dx}f(x)$ and defined as $f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$ provided that the limit exists.

Or the **derivative of a function** $y=f(x)$ of a variable x is a measure of the rate at which the value y of the **function** changes with respect to the change of the variable x . It is called the **derivative** of $f(x)$ with respect to x.

6.5. List of lessons

No	Lesson title	Learning objectives	Number of periods
0	Introductory activity	To arouse the curiosity of student teachers on the content of unit 6	1
1	Concepts of derivative of a function : Definition and graphical interpretation.	Define the derivative of a function using the gradient and limit	1
2	Rules of differentiation of elementary functions	Explore rules and use properties of derivatives to differentiate polynomial, rational and irrational functions	2
3	Geometric interpretation of derivatives: Equation of the	Use first derivative to determine the gradient of the tangent line and normal line to a curve	2

	tangent to a curve, Equation of normal to a curve	at a point.	
4	Hospital's theorem	Apply differentiation to to determine the limit of some rational functions	2
5	Derivative and the table of variation for a function	Use derivative and the table of variation to Sketch and interpret the function's graph.	3
6	Application of differentiation: rates of change problems, optimization problems	Apply the concepts and techniques of differentiation to model, analyze and solve optimization problems in different situations.	2
7	Revision on the content of this unit	To use these periods to solve additional problems for making units more understandable.	2
End unit assessment			**1**

Lesson 1: Derivative concepts, Definition and graphical interpretation

a) Learning objective

Define the derivative of a function using the gradient and limit

b) Teaching resources

Student's book and other Reference textbooks to facilitate research, Mathematical set, calculator, manila paper, markers, pens, pencils, …

c) Prerequisites/Revision/Introduction

Students should have knowledge and skills on how to calculate the gradient (slope) and limit of numerical functions, draw axes of coordinates in Cartesian plane.

c) Learning activities

- Ask students in small groups to read and discuss on the **activity 1** and make sure that everybody is engaged/ involved.
- Facilitate working, especially straggling students.
- Call students to present their findings, and help them to harmonize the answer.
- By harmonization, lead the students to discover that the

derivative of a function $f(x)$ with respect to x is denoted by

$f'(x)$ or y' *or* $\frac{dy}{dx}$ or $\frac{df}{dx}$ or $\frac{d}{dx}f(x)$ and defined as

$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$ provided that the limit exists.

- Lead them to read through and re-do examples 1 and 2
- Guide them to do **Exercise 1** to master the content.
- Provide additional activities where necessary.

Answer of activity 1

1)

Two points are given, $P(x_{0,}x_1)$ and $Q(y_0,y_1)$

$$m_{\sec}=\frac{y_1-y_0}{x_1-x_0}$$.

$$m_{\sec}=\frac{f(x_1)-f(x_0)}{x_1-x_0}$$

2) If $x_1\to x_0$, P and Q tends to be on vertical line and the slope tends to infinity.

3)

$$m_{\tan}=\lim_{x\to x_0}m_{\sec}$$

$$m_{\tan}=\lim_{x\to x_0}\frac{f(x_1)-f(x_0)}{x_1-x_0}$$

4)Let $h=x_1-x_0$; $m_{\tan}=\lim_{h\to 0}\frac{f(x_1)-f(x_0)}{h}$

Answer of Exercise 1

1) 1; 2) 12 ; 3) $8x-1$**; 4)** $8x+3$**; 5) 0**

Lesson 2: Rules of differentiation of elementary functions

a) Learning objective

Explore rules and use properties of derivatives to differentiate polynomial, rational and irrational functions

b) Teaching resources

Student's book and other Reference textbooks to facilitate research, Mathematical set, calculator, manila paper, markers, pens, pencils, …

c) Prerequisites/Revision/Introduction

Students should have knowledge and skills on identification of polynomials, power, and composition, rational and irrational functions.

d) Learning activities

- Ask students in small groups/individual to read and discuss on the **activity 3 in student 's book** to calculate the derivatives and make sure that everybody is engaged/ involved.
- Facilitate working, especially straggling students.
- Call students to present the findings, and help them to harmonize the answer.
- Guide the students to discover the different rules of derivatives of numerical functions and using different examples.
- In small groups or individually, help the students to work out the **exercise 8**
- Provide additional activities where necessary.

Answer of activity 3

1.

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$
$$=\lim_{h\to 0}\frac{c-c}{h}$$
$$=0$$

2.

$$f'(x_0)=\lim_{x\to x_0}\frac{x^n-x_0^n}{x-x_0}=\lim_{x\to x_0}\frac{\left(x^{n-1}+x^{n-2}x_0+...+x_0^{n-1}\right)(x-x_0)}{x-x_0}$$
$$=\lim_{x\to x_0}\left(x^{n-1}+x^{n-2}x_0+...+x_0^{n-1}\right)\lim_{x\to x_0}\frac{(x-x_0)}{x-x_0}$$
$$=\left(x_0^{n-1}+x_0^{n-2}x_0+...+x_0^{n-1}\right)x'$$
$$=\left(x_0^{n-1}+x_0^{n-1}+...+x_0^{n-1}\right)x'$$
$$=nx_0^{n-1}x' \qquad (1)\quad \textit{as we have n terms}$$

Answer of exercise 8

1. $3360x^3$
2. 0

3. $\frac{-6}{(x-2)^2}$

4. $\frac{4}{x+2}-\frac{96}{(x+2)^2}+\frac{144x^2}{(x+2)^3}-\frac{96x^3}{(x+2)^4}+\frac{24(x^4-4)}{(x+2)^5}$

Lesson 3: Geometric interpretation of derivatives: Tangent line equation and Normal line equation

a) Learning objective

Use first derivative to determine the gradient of the tangent line and normal line to a curve at the point

b) Teaching resources

Student's book and other Reference textbooks to facilitate research, Mathematical set, calculator, manila paper, markers, pens, pencils, …

c) Prerequisites/Revision/Introduction

Students should have knowledge and skills on sketching the graph of function and derivative of functions **(unit 6,lesson 1&2,year2)**

d) Learning activities

- Ask students in small groups/individually to read and discuss on the **activity 9 in** student 's book
- Help the students to recall the gradient by first derivative of function **(unit 6,lesson 1&2)**
- Guide them to establish the equation tangent and normal line to the curve at the point . and make sure that everybody is engaged/ involved.
- Facilitate working, especially straggling students.
- Call students to present the findings, and help them to harmonize the answers.
- After presentation , teacher will help the students to establish the equation of the tangent and normal line to the curve at the point
- Teacher guides them to work the provided example.

  In small groups or individually, help the student to work out the following **Exercise 9** found below

Answers of activity 9

1)

$$\begin{cases} y=-x^3+3x \\ y=3x \end{cases} \Rightarrow 3x=-x^3+3x \Rightarrow x=0$$

$if\ \ x=0 \Rightarrow y=0$

So the point (0,0) is the intersection point of $y=-x^3+3x$ *and* $y=3x$

2)

$f'(x) = -3x^2 + 3$
$f'(0) = 3$
Gradient = $f'(0) = 3$

Exercise 9

1.Determine the equation of the tangent and normal line to the curve of function

a) $f(x) = \sqrt{x^2+3}$ at the point $(-1,2)$

b) $f(x) = x^3 - x + 5$ at the point $(1,5)$

c) $f(x) = x^3$ at the point where $x = 2$

2. Let $f(x) = x^2 - x$. Find the equation of tangent line with slope $m = -3$

Answer of exercise 6.9

1.

a) $T \equiv y - 2 = -\frac{1}{2}(x+1)$ **and** $N \equiv y - 2 = 2(x+1)$

b) $T \equiv y - 5 = 2(x-1)$ **and** $N \equiv y - 5 = -\frac{1}{2}(x-1)$

c) $T \equiv y - 8 = 12(x-2)$ **and** $N \equiv y - 8 = -\frac{1}{12}(x-2)$

2.

$f(x) = x^2 - x$
$f'(x) = 2x - 1$
$f'(x_0) = 2x_0 - 1 = -3 \Rightarrow x_0 = -1$
$y_0 = 2$
$So, T \equiv y - 2 = -3(x+1) \text{ or } T \equiv y = -3x + 3$

Lesson 4: Derivative and limit with indeterminate cases: Hospital's rule

a) **Learning objective**
Use first derivative to determine the limit of some rational functions

b) **Teaching resources**
Student's book and other Reference textbooks to facilitate research, calculator, manila paper, markers, pens, pencils, …

c) **Prerequisites/Revision/Introduction**

Students should have knowledge and skills on calculations of limits (**unit5)** and derivatives(**lesson 1&2 ,unit**).

d) Learning activities

- Ask students in small groups/individually to read and discuss on the **activity 10** found below.
- Help the students to recall the indeterminate forms seen in calculations of limits
- Guides them to apply the Hospital rule and make sure that everybody is engaged/ involved.
- Facilitate working, especially straggling students.
- Call students to present the findings, and help them to harmonize the answer.
- Teacher will verify if the students found the same limits by calculating the limits of derivative of numerator over derivative of denominator for above limits. And after presentation of their findings ,Teacher helps them to state the Hospital's rule.
- In small groups or individually, help the student teacher to work out the **Exercise 10 found below**

Activity 10 and its solution

Evaluate the following limits:

a) $\lim_{t\to 1}\frac{5t^4-4t^2-1}{10-t-9t^2}$

What happen if you calculate the limits of derivative of numerator over derivative of denominator for above limits?

Solutions:

a) $\lim_{t\to 1}\frac{5t^4-4t^2-1}{10-t-9t^2}=\frac{0}{0}(IF)$

$$\lim_{t\to 1}\frac{5t^4-4t^2-1}{10-t-9t^2}=\lim_{t\to 1}\frac{20t^3-8t}{1-18t}=\frac{-12}{17}$$

Exercise 10 and its solution

1. Evaluate the following limit: $\lim_{x\to -2}\frac{x+2}{x^2+3x+2}=\frac{-2+2}{(-2)^2+3(-2)+2}=\frac{0}{0}(IF)$

Apply l' Hospital's rule and reevaluate the limit:

$$\lim_{x\to -2}\frac{(x+2)'}{(x^2+3x+2)'}=\lim_{x\to -2}\frac{1}{3x+3}=\frac{1}{2(-2)+3}=\frac{1}{-1}=-1$$

2.Find $\lim_{x\to\infty}\frac{2x^2-5x+1}{3+2x+6x^2}$

Solution: plug in to evaluate the limit, if possible $\lim_{x\to\infty}\frac{2(\infty)^2-5(\infty)+1}{3+2(\infty)+6(\infty)^2}=\frac{\infty}{\infty}(IF)$

Apply l' Hospital's rule and reevaluate the limit:

$$\lim_{x\to\infty}\frac{(2x^2-5x+1)'}{(3+2x+6x^2)'}=\lim_{x\to\infty}\frac{4x-5}{2+12x}=\frac{4(\infty)-5}{2+12(\infty)}=\frac{\infty}{\infty}(\textit{still IF})$$

Apply l' Hospital's rule again, and reevaluate the limit:

$$\lim_{x\to\infty}\frac{(4x-5)'}{(2+12x)'}=\lim_{x\to\infty}\frac{4}{12}=\frac{4}{12}=\frac{1}{3}$$

Lesson 5: Derivative and the table of variation for a function

a. Learning objective

Determine and interpret the inflection points and the sense of concavity of a function's curve

b. Teaching resources

Student's book and other Reference textbooks to facilitate research, Mathematical set, calculator, manila paper, markers, pens, pencils, …

a) Prerequisites/Revision/Introduction

Student should have knowledge and skills on polynomial functions (unit 4)on sketching the graph of functions (unit 5), derivative (lesson 1, 2&3,in this unit)

b) Learning activities

- Lead the students in small groups to read and to discuss on the **activity 10A.**
- Guide them to find the answers of given **activity 10A** found below and make sure that everybody is engaged/ involved.
- Facilitate working, especially straggling students.
- Call students to present the findings, and help them to harmonize their answer.
- Teacher informs the students that the Variation table or synthetic table summarizes all information about the function: domain of definition, limits at the boundaries of domain, asymptotes to the curve, extrema points, variation of function (increasing or decreasing) , inflection points and the sense of concavity(opens up or down).
- Teacher guides the students to work out the example of studying the function completely, step by Step. In small groups or individually, help the student to work out the **exercise 10A** found below.

Answer for activity 10A

a. consider $f(x)=x^3-12x+1$

$f'(x)=3x^2-12$
$f''(x)=6x$

b.

$f'(x)=3x^2-12$
maximum point (-2,17)
$f''(x)=6x$ and $f''(x)=0 \Leftrightarrow x=0$
inflection point (0,0)

x	$-\infty$ -2 **0** **+2** $+\infty$
$f'(x)$	+ + + **0** - - - - ------------- **0** + + + +
$f''(x)$	- - - - - **0** + + + + +
$f(x)$	**(-2,17)** **0** **(2,−15)**

C) Analysis \ interpretation

$f(x)$ is defined for all $x \in ¡$

$f(x)$ is increasing on interval $]-\infty,-2]$ and $[2,+\infty[$

$f(x)$ is decreasing $[-2,2]$ and $f'(x)=0 \Leftrightarrow x=2$ or $x=-2$ it has a minimum point $(2,-15)$ and a maximum point (-2,17)

$f(x)$ is concave down $]-\infty,0]$ concave down $[0,+\infty[$ and changes the concavity at $(0,0)$

Answer of exercise 10A

1. $y=x+\frac{1}{4x}$ or $y=\frac{4x^2+1}{4x}$

Condition of existence: $4x \neq 0 \Rightarrow x \neq 0$

1) $Domf = ¡ \setminus \{0\} =]-\infty,0[\, U \,]0,+\infty[$

2) Limits at the boundaries of Domf

$$\lim_{x\to-\infty}\frac{4x^2+1}{4x}=\frac{\infty}{\infty} \qquad (IF)$$

$$\text{T.V}: \lim_{x\to-\infty}\frac{4x^2+1}{4x}=\lim_{x\to-\infty}\frac{4x^2(1+\frac{1}{4x^2})}{4x}=\lim_{x\to-\infty}x=-\infty \Rightarrow \text{There is no Horizontal asymptote}$$

$$\lim_{x\to 0}\frac{4x^2+1}{4x}=\frac{1}{0}$$

$$\lim_{x\to 0^-}\frac{4x^2+1}{4x}=\frac{1}{0^-}=-\infty \Rightarrow \textit{There is} \text{ a vertical asymptote} : x=0$$

$$\lim_{x\to 0^+}\frac{4x^2+1}{4x}=\frac{1}{0^+}=+\infty$$

$$\lim_{x\to+\infty}\frac{4x^2+1}{4x}=\frac{\infty}{\infty} \qquad (IF)$$

$$\text{T.V}: \lim_{x\to+\infty}\frac{4x^2+1}{4x}=\lim_{x\to+\infty}\frac{4x^2(1+\frac{1}{4x^2})}{4x}=\lim_{x\to+\infty}x=+\infty$$

$\Rightarrow$ There is no Horizontal asymptote

Let find the Oblique asymptote (OA)

$$OA \equiv y=ax+b \text{ with } a=\lim_{x\to\pm\infty}\frac{f(x)}{x} \text{ and } b=\lim_{x\to\pm\infty}\left[f(x)-ax\right]$$

$$a=\lim_{x\to\pm\infty}\frac{4x^2+1}{4x^2}=\lim_{x\to\pm\infty}\frac{4x^2}{4x^2}=1$$

$$b=\lim_{x\to\pm\infty}\left(\frac{4x^2+1}{4x}-x\right)=\lim_{x\to\pm\infty}\left(\frac{4x^2+1-4x^2}{4x}\right)=\lim_{x\to\pm\infty}\left(\frac{1}{4x}\right)=0$$

$a=1, b=0 \Rightarrow$ *there is no OA*

3) Party of $f(x)$

- $f(x)$ is even function iff $f(x)=f(-x)$

$$f(-x)=\frac{4(-x)^2+1}{-4x}=-\frac{4x^2+1}{4x}\neq f(x) \text{ ,so } f(x)=\frac{4x^2+1}{4x} \text{ is not even}$$

- $f(x)$ is odd function iff $-f(x)=f(-x)$

$$f(-x)=\frac{4(-x)^2+1}{-4x}=-\frac{4x^2+1}{4x}$$

- $-f(x) = -\dfrac{4x^2+1}{4x}$,so $-f(x) = \text{f}(-x)$ and $f(x)$ is odd function

4) Party of $f(x)$

The given function is not periodic (because it is not trigonometric)

5) First derivative of $f(x)$

$$f'(x) = \left(\frac{4x^2+1}{4x}\right)' = \frac{8x.4x - 4.(4x^2+1)}{16x^2} = \frac{32x^2 - 16x^2 - 4}{16x^2}$$

$$f'(x) = \frac{16x^2 - 4}{16x^2} = \frac{4x^2 - 1}{4x^2}$$

Critical points:

- Condition of existence of $f'(x)$

$4x^2 \neq 0 \Leftrightarrow x \neq 0$

Extrema points: if $x = -\dfrac{1}{2} \Rightarrow y = -1$,function has a maximum point $(-\dfrac{1}{2}, -1)$

$x = \dfrac{1}{2} \Rightarrow y = 1$, function has a minimum point $(\dfrac{1}{2}, 1)$

Sign table of $f'(x)$

x	$-\infty$		$\frac{-1}{2}$		0		$\frac{1}{2}$		$+\infty$
$f'(x)$		+ + +	0	- - - - -	‖	- - -	0	+ + + +	
$f(x)$	$-\infty$	↗	Max $\left(-\frac{1}{2}, -1\right)$	↘	$-\infty$ ‖ $+\infty$	↘	Min $\left(\frac{1}{2}, 1\right)$	↗	$+\infty$

6) Second derivative of $f(x)$

$$f''(x)=\left(\frac{4x^2-1}{4x^2}\right)'=\frac{8x.4x^2-8x.(4x^2-1)}{16x^4}=\frac{32x^3-32x^3+8x}{16x^4}=\frac{8x}{16x^4}$$

$$f''(x)=\frac{1}{2x^3}$$

Critical points: • Existence *condition* of $f''(x): x^3\neq 0 \Rightarrow x\neq 0$

• $f''(x)=0 \Leftrightarrow \frac{1}{2x^3}=0$ It doesn't exist in ¡

Sign table of $f''(x)$

x	$-\infty$	0	$+\infty$
$f''(x)$	- - - - -	||	+ + + +
$f(x)$	$-\infty$ … $-\infty$	||	$+\infty$ … $+\infty$

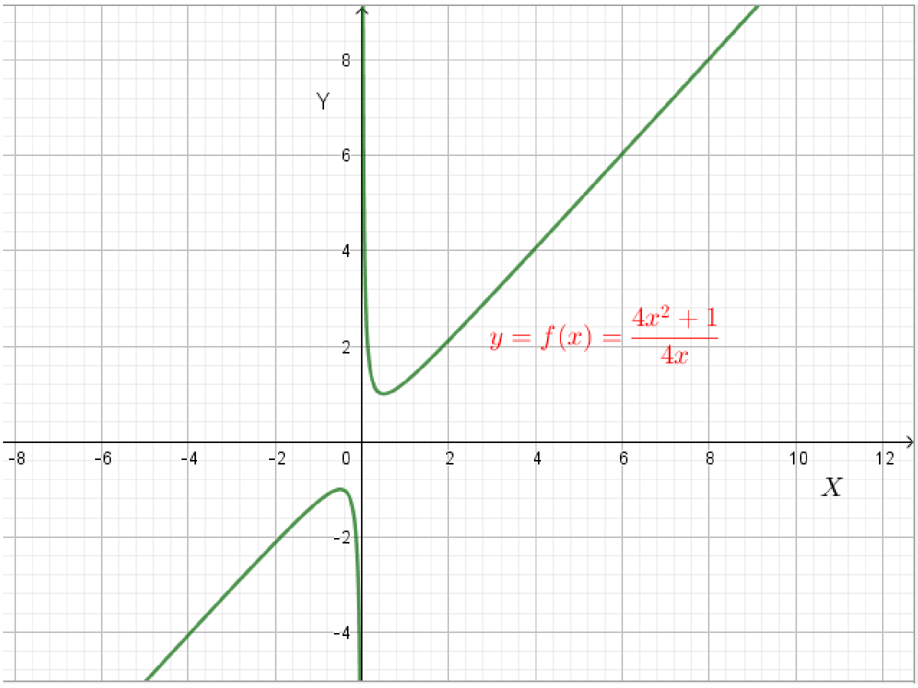
7) Variation table or synthetic table

x	$-\infty$	$\frac{-1}{2}$	0	$\frac{1}{2}$	$+\infty$
$f'(x)$	+ + +	0	- - - - - - - - -	0	+ + +
$f''(x)$	- - - - - - - -		||	+ + + + + + + + + +	
$f(x)$	$-\infty$	Max$\left(-\frac{1}{2},-1\right)$	$-\infty$ || $+\infty$	Min$\left(\frac{1}{2},1\right)$	$+\infty$

8) Supplementary points

x	2	-2	-3	3
$f(x)$	**17/8**	**-17/8**	**-37/12**	**37/12**

9) Graph of the function $f(x)=\dfrac{4x^2+1}{4x}$



2) $y=\dfrac{1}{x-2}$

Condition of existence: $x-2\neq 0$ i.e $x\neq 2$

1) $Domf=¡ \setminus\{2\} =]-\infty,2[\,U\,]2,+\infty[$

2) Limits at the boundaries of Domf

- $\lim\limits_{x\to-\infty}\dfrac{1}{x-2}=\dfrac{1}{\infty}=0$
  $\Rightarrow$ there is Horizontal asymptote $y=0$
-

  $\lim\limits_{x\to 2^+}\dfrac{1}{x-2}=\dfrac{1}{0^+}=+\infty$

  $\lim\limits_{x\to 2^-}\dfrac{1}{x-2}=\dfrac{1}{0^-}=-\infty \Rightarrow$ *There is* a vertical asymptote : $x=2$

  $\lim\limits_{x\to 2^+}\dfrac{1}{x-2}=\dfrac{1}{0^+}=+\infty$

- $\lim\limits_{x\to+\infty}\dfrac{1}{x-2}=\dfrac{1}{\infty}=0 \Rightarrow$ There is Horizontal asymptote $y=0$

Let find the Oblique asymptote (OA)

$$OA \equiv y = ax + b \text{ with } a = \lim_{x\to\pm\infty} \frac{f(x)}{x} \text{ and } b = \lim_{x\to\pm\infty}\left[f(x) - ax\right]$$

$$a = \lim_{x\to\pm\infty} \frac{1}{x^2 - 2x} = \lim_{x\to\pm\infty} \frac{1}{\infty} = 0$$

$a = 0 \Rightarrow$ *there is no OA*

3) Party of $f(x)$

- $f(x)$ is even function iff $f(x) = f(-x)$

$$f(-x) = \frac{1}{-x-2} = -\frac{-1}{x+2} \neq f(x) \text{ ,so } f(x) = \frac{1}{x-2} \text{ is not even}$$

- $f(x)$ is odd function if $-f(x) = f(-x)$

  $$f(-x) = \frac{-1}{x+2}$$

- $-f(x) = -\frac{1}{x-2} = \frac{1}{2-x}$,so $-f(x) \neq f(-x)$ and $f(x)$ is not odd function

  The given function is neither even nor odd

4) Periodicity of $f(x)$

   The given function is not periodic (because it is not trigonometric)

5) First derivative of $f(x)$

$$f'(x) = \left(\frac{1}{x-2}\right)' = \frac{-1.(1)}{(x-2)^2} = \frac{-1}{(x-2)^2}$$

$$f'(x) = \frac{-1}{(x-2)^2}$$

Critical points:

- Condition of existence of $f'(x)$ $(x-2)^2 \neq 0 \Leftrightarrow x \neq 2$

$$f'(x) = 0 \Leftrightarrow \frac{-1}{(x-2)^2} = 0 \text{ which is impossible in } \mathbb{R}$$

There are no extrema points because $f'(x) < 0$ always.

Sign table of $f'(x)$

x	$-\infty$		2		$+\infty$
$f'(x)$	- - - - - - -			- - - - - - - - - - - - -	
$f(x)$	0 ↘ $-\infty$			$+\infty$ ↘ 0	

6) **Second derivative of** $f(x)$

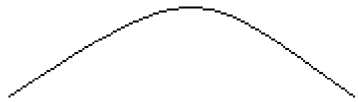
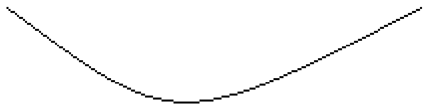
$$f''(x) = \left(\frac{-1}{(x-2)^2}\right)' = -\left[\frac{-2(x-2)}{(x-2)^4}\right] = \frac{2}{(x-2)^3}$$

Critical points:

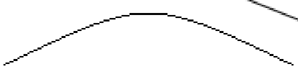
- Existence *condition* of $f''(x) : (x-2)^3 \neq 0 \Rightarrow x \neq 2$
- $f''(x) = \dfrac{2}{(x-2)^3}$ *can not be zero*

The signs of $f''(x) = \dfrac{2}{(x-2)^3}$ depends on signs of denominator where $x \neq 2$

Sign table of $f''(x)$

x	$-\infty$	2	$+\infty$
$f''(x)$	- - - - - - - -	‖	+ + + + + + + + + + + + +
$f(x)$	0  $-\infty$	‖	$+\infty$  0

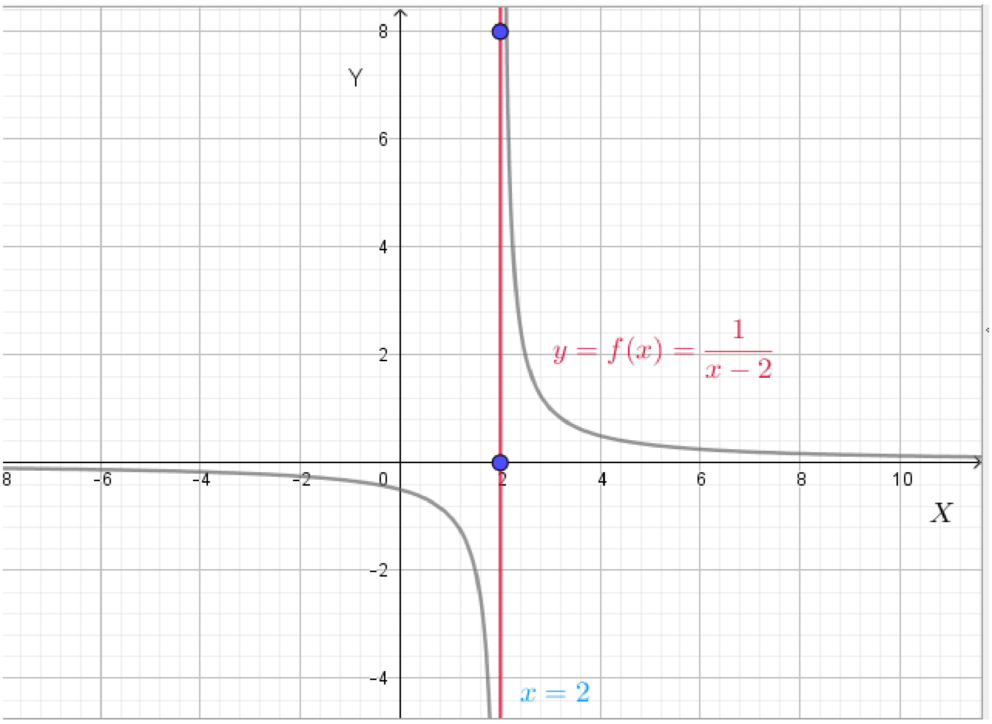
7) Variation table or synthetic table

x	$-\infty$	2	∞
$f'(x)$	– – – – –	‖	– – – — – – – –
$f''(x)$	– – – – –	‖	+ + + + + + + + +
$f(x)$	0 ↘  $-\infty$	‖	$+\infty$ ↘  0

8) Supplementary points

x	**-1**	**0**	**1**	**3**	**4**
$f(x)$	**-1/3**	**-1/2**	**-1**	**1**	**1/2**

9) Graph of function $f(x)=\dfrac{1}{x-2}$



Lesson 6: Applications of differentiation: rates of change problems, optimization problems

a) Learning objective

Apply the concepts and techniques of differentiation to model, analyse and solve optimization problems in different situations.

b) Teaching resources

Student's book and other Reference textbooks to facilitate research, calculator, manila paper, markers, pens, pencils, …

c) Prerequisites/Revision/Introduction

Students should have knowledge and skills on sketching the graph of function and derivative of functions **(unit 6,lesson 1, 2&3)**

d) Learning activities

- Ask students in small groups/individually to read and discuss on the **activity 11A** found below.
- Help the students to recall the gradient by first derivative of function **(unit 7,lesson 1&2,year2)**

- Guides them to establish the rate of change of function and make sure that everybody is engaged/ involved.
- Facilitate working, especially straggling student-teachers.
- Call students to present the findings, and help them to harmonize the answers.
- After presentation, teacher will help the students to establish the formula of rate of change and deduce the mean value , Rolle's theorem, of the function and guide them to work the **exercise 11A**.

Activity 11A and its Answers

1.You are inflating a spherical balloon at the rate of $7cm^3/\sec$. How fast is its radius increasing when the radius is $4cm$?

Solution:

Here the variables are the radius r and the volume V. We know that $\frac{dV}{dt}=7cm^3/\sec$, and we want to know $\frac{dr}{dt}$. The two variables are related by means of the equation $V=\frac{4\pi r^3}{3}$. Taking the derivative of both sides gives $\frac{dV}{dt}=4\pi r^2\dot{r}$. We now substitute the values we know at the instant in the question: $7=4\pi 4^2\dot{r}$, So $\dot{r}=\frac{7}{64\pi}cm/\sec$ or $\frac{dr}{dt}=\frac{7}{64\pi}cm/\sec$

2.A cylindrical tank standing upright (with one circular base on the ground) has radius $20cm$. How fast does the water level in the tank drop when the water is being drained at $25cm^3/\sec$?

Solution: $\dot{r}=\frac{1}{16\pi}cm/\sec$ or $\frac{dr}{dt}=\frac{1}{16\pi}cm/\sec$

Exercise 11A and its answers

1. Suppose that a company has estimated that the cost (in Rwandan francs) of producing x items is $C(x)=10000+5x+0.01x^2$. What is the marginal cost at the production level of 500 items?

Solution: Then the marginal cost function is $c'(x)=5+0.02x$

The marginal cost at the production level of 500 items is $c'(500)=5+0.02(500)=15$ Rwandan Francs per item.

2.The length l metres of a certain metal rod at temperature θ◦C is given by: $l=1+0.00005\theta+0.0000004\theta^2$. Determine the rate of change of length, in mm/◦C, when the temperature is (a) 100^0C and (b) 400^0C

Solution:

The rate of change of length means $\frac{dl}{d\theta}$

Since length $l=1+0.00005\theta+0.0000004\theta^2$, then $\frac{dl}{d\theta}=0.00005+0.0000008\theta$

(a) W

hen $\theta = 100^0 C$, $\frac{dl}{d\theta} = 0.00005 + (0.0000008)(100^{0}\text{ C}) = 0.00013 m/^0 C = 0.13 mm/^0 C$

(b) W

hen $\theta = 400^0 C$,

6.6 Unit summary

1) *The derivative of a function* with respect to *x* is denoted by or $\frac{d}{dx} f(x)$ and defined as provided that the limit exists.

2) Derivative of a constant function: $\frac{df}{dx} = \frac{d}{dx}(c) = 0$

3) Derivative of identity function: $if \quad f(x) = x, \frac{df}{dx} = \frac{dx}{dx} = 1$

4) with $c \in$ ¡

5) $\left[f^n(x) \right]' = nf^{n-1}(x) f'(x)$.

6) $\frac{d}{dx}(u+v) = \frac{du}{dx} + \frac{dv}{dx}$

7) $\frac{d}{dx}(u-v) = \frac{du}{dx} - \frac{dv}{dx}$

8) $\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$

9) $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

10) If f and g are both differentiable and F is the composite function defined by $F(x) = f(g(x))$, then F is differentiable and F' is given by the product $F'(x) = f'(g(x)) \cdot g'(x)$

11) **L**et $y = f(v) \; with \; v = \Psi(x)$ and derivative of y respect to x is given by $\frac{dy}{dx} = \frac{dy}{dv} . \frac{dv}{dx}$

12) **Theorem:**

Let f be a function differentiable on an interval $]a, b[$

a) If $f'(x) > 0$ on each point *x* of $]a, b[$, then f is increasing on $]a, b[$

b) If $f'(x) < 0$ on each point *x* of $]a, b[$, then f is decreasing on $]a, b[$

c) If $f'(x)=0$ for all $x \in]a,b[$, then f is constant on this interval, that is $f(x_1)=f(x_2)$ for all $x_1, x_2 \in]a,b[$, or equivalently, there exists a real c such that $f(x)=c$ for all $x \in]a,b[$.

13) *a. If f' changes from negative to positive at c, then f has a **local minimum** at c*

***b.** If f' changes from positive to negative at c, then f has a **localmaximum** at c*

***c.** If f' does not change sign at c (that is, f' is positive on both sides of c or negative on both sides), then f has **no** local extremum.*

*14) **Theorem**:*

Let f be a function that is defined and is twice differentiable on an open interval $]a,b[$.

1) If $f''(x)>0$ for all $x \in]a,b[$, then f is convex on $]a,b[$

2) If $f''(x)<0$ for all $x \in]a,b[$, then f is concave on $]a,b[$

15) Hospital rule states that,if the limit of $\lim_{x\to x_0}\frac{f(x)}{g(x)}$ *with* $g(x)\neq 0$ and x_0 a finite number or infinity

is indeterminate form of $\frac{0}{0}$ or $\frac{\infty}{\infty}$ then,it can be calculated by $\lim_{x\to x_0}\frac{f'(x)}{g'(x)}$ *with* $g'(x)\neq 0$**.** If this result isindeterminate form, the procedure can be repeated.

16) the equation of the tangent line on the cuve at the point is
17) equation of the Normal line on the curve at the point is of the form
18) If and (x_1, y_1) are points on the graph of , then we define to be the **average rate** at which y changes with x over the interval .
19) Mean value theorem

Suppose that $f(x)$ is a function that satisfies both of the following.

- $f(x)$ is continuous on the closed interval $[a,b]$.
- $f(x)$ is differentiable on the open interval $]a,b[$.

Then, there is a number c such that $a<c<b$ and $f'(c)=\frac{f(b)-f(a)}{b-a}$

Or, $f(b)-f(a)=f'(c)(b-a)$

Note that the Mean Value Theorem doesn't tell us what c is. It only tells us that there is at least one number c that will satisfy the conclusion of the theorem.

Also note that if $f(a)=f(b)$ we can think of Rolle's Theorem as a particular case of the Mean Value Theorem.

6.7 Answers of Revision Exercise

1. a. $4x^3$ b. $12x^2$ c. $16x$ d. $2x-4$

2. a. $2x+y=-6$ b. $y=-\frac{4}{3}$

3. a. ± 8 b. $-\frac{16}{3}$

4. a. 5, 0 b. $6x-6$, 6 c. $6x^2-10x+4$, $12x-10$ d. $3x^2+\frac{2}{x^2}$, $6x-\frac{4}{x^3}$

5. a. $(19.6-9.8t)ms^{-1}$, $-9.8ms^{-2}$ b. 2*s* c. 19.6*m* d. 0.586*s*, 3.41*s*

6. $a=-3,\ b=-12$

7. a. 3, -9 b. $\frac{32}{27}, 0$ c. $\frac{32}{27}, -\frac{49}{27}$

8. $8\frac{ft^2}{ft}$

9. $16\pi\frac{m^3}{m}$

10. a. Increases on $\left(-\infty, -\frac{2}{\sqrt{3}}\right)$ and $\left(\frac{2}{\sqrt{3}}, +\infty\right)$, decreases on $\left(-\frac{2}{\sqrt{3}}, \frac{2}{\sqrt{3}}\right)$

b. Increases on $(-2,0)$ and $(2,+\infty)$, decreases on $(-\infty,-2)$ and $(0,2)$

c. Increases on $(-\infty,3)$ and $(5,+\infty)$, decreases on $(3,5)$

11. $\frac{1-y}{2+x}$

12. $\frac{2x+y}{3y^2-x}$

13. $\frac{2-2xy^3}{3x^2y^2+1}$

14. $-\frac{3x^2+2xy}{x^2+4y}$

15. $\begin{cases} y' = -14(3-2x)^6 \\ y'' = 168(3-2x)^5 \\ y''' = -1680(3-2x)^4 \end{cases}$

16. $\begin{cases} y' = -12(x-1)^{-3} \\ y'' = 36(x-1)^{-4} \\ y''' = -144(x-1)^{-5} \end{cases}$

17. $\begin{cases} y' = \frac{1}{3}x^{-\frac{2}{3}} + \frac{1}{3}x^{-\frac{4}{3}} \\ y'' = -\frac{2}{9}x^{-\frac{5}{3}} - \frac{4}{9}x^{-\frac{7}{3}} \\ y''' = \frac{10}{27}x^{-\frac{8}{3}} + \frac{28}{27}x^{-\frac{10}{3}} \end{cases}$

$$\begin{cases} y' = \frac{5}{2}x^{\frac{3}{2}} + \frac{3}{2}x^{-\frac{1}{2}} \\ y'' = \frac{15}{4}x^{\frac{1}{2}} - \frac{3}{4}x^{-\frac{3}{2}} \\ y''' = \frac{15}{8}x^{-\frac{1}{2}} + \frac{9}{8}x^{-\frac{5}{2}} \end{cases}$$

6.8. Additional information for teachers

For the educative action of the teacher to be effective (in order to respond to all aspects of the students' needs), it is worth mentioning that the teacher needs a wide range of skills, attitudes, a rich and deep understanding of the subject matter and the pedagogical processes to develop the understanding that is required from the learner. It is, therefore, imperative for the teacher to not limit himself/herself to the only to the present book, but also to consider getting information from other relevant books, such as those mentioned in the reference.

6.9 Additional activities

6.9.1 Remedial activities

Calculate the derivative of $f(x) = (2x+1)^6$

Solution:

$f(x) = (2x+1)^6$

$f(x) = (2x+1)^6$

$f'(x) = 6(2x+1)'(2x+1)^5$

$f'(x) = 6(2)(2x+1)^5$

$f'(x) = 12(2x+1)^5$

6.9. 2 Consolidation activities

a. A firm was assumed to have the total cost function $TC = 18q$ and the total revenue function $TR = 240 + 14q$ calculate the profit-maximizing output at $q = 1$

Solution

The profit function will be $TR - TC$

$= TR - TC = 240 + 14q - 18q$

$= 240 - 4q$

The profit-maximizing output $= 240 - 4 = 236$

7.9.2 **Extended activities**

A firm faces the demand schedule $p = 200 - 2q$ and the total cost function given by the function $TC = \frac{2}{3}q^3 - 14q^2 + 222q + 50$.Derive expressions for the following functions and find out whether they have maximum or minimum points.

If they do, say what value of q this occurs at and calculate the actual value of the function at this output.

(a) Marginal cost (b) Average variable cost (c) Average fixed cost

(d) Total revenue (e) Marginal revenue (f) Profit

UNIT 7: VECTORS SPACE OF REAL NUMBERS

7.1 Key Unit Competence:

Use concepts of vectors in 2D to solve related problems such as distance, and angles.

7.2 Prerequisites

Students will easily learn this unit, if they have a good background on fundamental of trigonometry (unit 2) and on linear Equations and inequalities (unit 3).

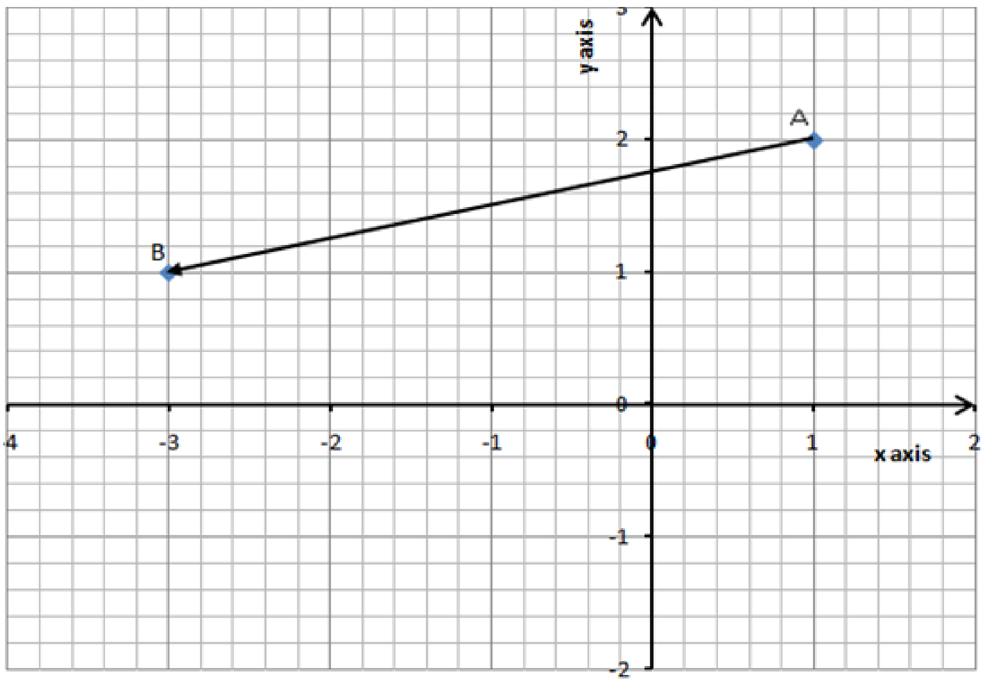
7.3 Cross-cutting issues to be addressed

- Inclusive education (promote education for all while teaching) ;
- Peace and value Education (respect others' view and thoughts during class discussions);
- Gender (provide equal opportunity to boys and girls in the lesson)

7.4 Guidance on introductory activity

- Invite students to work in groups and give them instructions on how they can perform the introductory **activity 7.0** found in student's book;
- Guide students to read and analyse the questions insisting on Using concepts of vectors in 2D to solve related problems such as distance, and angles.
- Invite some group members to present groups' findings, then try to harmonize their answers;
- Open a discussion with the students on how the vectors are drawn in the plane. This will lead to the introduction of the vector space of real number.
- Basing on students' experience, prior knowledge and abilities shown in answering the questions for this activity, use different questions to facilitate them to arouse their curiosity on what is going to be leant in this unit.

Answer of introductory activity 7.0



$B - A = (1,2) - (-3,1) = (4,1)$

7.5 List of lessons

No	Lesson title	Learning objectives	Number of periods
0	Introductory activity	To arouse the curiosity of student teachers on the content of unit 7	1
1	Euclidean vector space IR^2	Describe a vector in 2 D and determine the scalar Product and Magnitude of 2 vectors	2
2	Angle between two vectors	Calculate the angle between two vectors in a plane.	2
3	Application of vectors in Physics	Solve problems of physics involving vectors in two dimensions.	3
4	Revision on the content of this unit	To use these periods to solve additional problems for making units more understandable.	3
End Units assessment			1

Lesson 1: Euclidean vector space IR^2

a. Learning objective:

Describe a vector in 2 D and determine the vector product of 2 vectors

b. Teaching resources:

Student's book and other Reference textbooks containing the concept of vector product

c. Prerequisites/Revision/Introduction:

For students to master well the concepts in this lesson, the following prerequisites are required:

- Differentiating between a scalar quantity and a vector quantity;
- The distance between two points leant in S3;
- The right-hand rule: the axes are such that the index lies on the OX- axis, the middle finger lies on the OY-axis and the thumb lies on the OZ-axis.
- The calculation of numerical values of algebraic expressions.

d. Learning activities

- Instruct the students to form small groups;
- Ask them to discuss on the **activity 1** from student's book for a while;
- Choose at random a group to present his/her findings to the whole class while other students are following attentively;
- Through well-chosen questions, bring students to summarize the main points of the lesson, ensure that students mentioned how to calculate the dot product, and the cross product using determinants of order 2 as means;
- Let students practice individually, by solving **exercise 1**. As they are working go around different students and provide assistance where necessary;
- Make introspection about the lesson, as it took place, to write down the area to improve for next lesson.
- Provide additional activities where necessary.

Answers to activity 1

1. a. 11 b. 0

Answers to exercise 1

1. a) 5 b) $\sqrt{10}$

2. a) $\overrightarrow{AB}=(-5,-1)$ b) $(-25,-16)$ c) $\left\|\overrightarrow{AB}\right\|=\sqrt{26}$, $\left\|\vec{w}\right\|=\sqrt{881}$ d) i) -7 ii) 59

Lesson 2: Angle between two vectors

a. Learning objective:

Calculate the angle between two vectors in a plane.

b. Teaching resources:

Charts, Student's book and other Reference textbooks containing the concept Space Geometry

c. Prerequisites/Revision/Introduction:

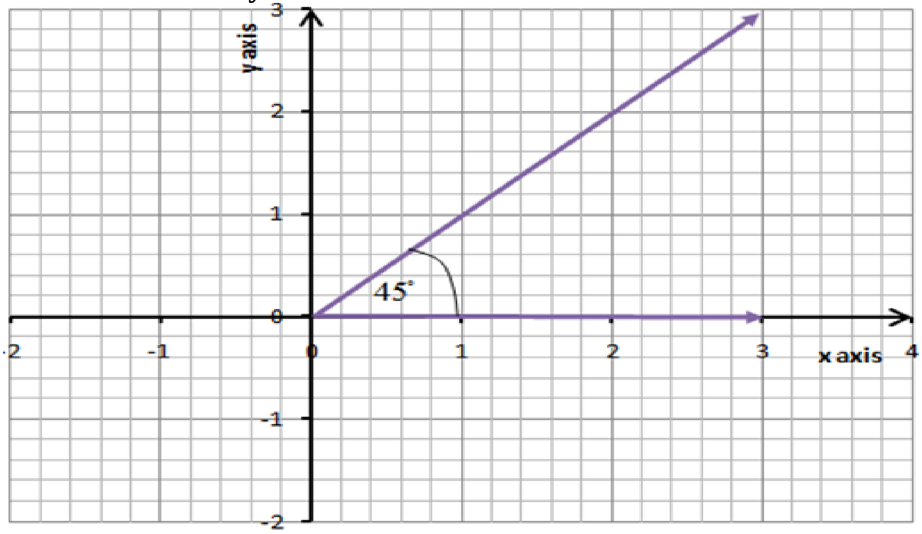
For students to master well the concepts in this lesson, the following prerequisites are required:

- Relations in a right angled triangle, from unit 2;
- The dot product of two vectors, studied in lesson 1 of this unit;
- The calculation of numerical values of algebraic expressions.

d. Learning activities

- Instruct the students to form small groups;
- Ask them to discuss the **activity 2** from student's book for a while;
- Choose at random a group to present his/her findings to the whole class while other students are following attentively;
- Request the students to try in groups the examples and to check their answers against the one proposed in the students' book;
- Through well-chosen questions, bring students to summarize the main points of the lesson, ensure that students mentioned how to calculate the angle between two vectors in a plane.
- Let students practice individually, by solving **exercise 2**. As they are working go around different students and provide assistance where necessary;
- Make introspection about the lesson, as it took place, to write down the area to improve for next lesson.
- Provide additional activities where necessary.

Answers to activity 2



The vector drawn in a. is the adjacent side with length 3 and the vector drawn in b. is the hypotenuse. The other side is the opposite side with length 3.

Then $\frac{adjacent\ side}{hypothenuse} = \frac{3}{3\sqrt{2}} = \frac{1}{\sqrt{2}} = \cos 45^{o}$

$\frac{opposite\,side}{hypothenuse} = \frac{3}{3\sqrt{2}} = \frac{1}{\sqrt{2}} = \sin 45^{0}$

Lesson 3: Application of vectors in Physics.

a) Learning objective: solve problems of physics involving vectors in two dimensions.

b) Teaching resources:

Ruler, T-square, protractor, calculator

Student's book and other Reference textbooks to facilitate research

c) Prerequisites/Revision/Introduction:

Students will learn better this lesson if they have a good understanding on concept of

- scalar product and magnitude learnt from previous lessons of this unit
- Angles between two vectors.

d)Content

Model with Vectors

Because forces can be represented by vectors, two forces "combine" the way that vectors "add." If F_1 and F_2 are two forces simultaneously acting on an object, the vector sum $F_1 + F_2$ is the resultant force. The resultant force produces the same effect on the object as that obtained when the two forces F_1 and F_2 act on the object

e) Learning activities

Guidance

- ➢ Let Students discuss and attempt the **Activity 3 A** found below
- ➢ Facilitate them while solving the given activity by checking if everybody is engaged
- ➢ Help the students to solve problems of physics involving vectors in two dimensions.
- ➢ Let one group member to present their findings
- ➢ Facilitate them to do the provided examples given in **Student's book** and to emphasize the skills, they have got.

Activity 3 A and its Solution

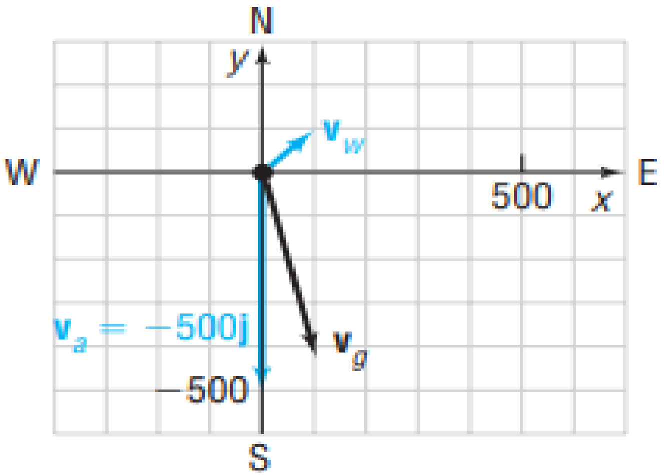
1) A Boeing 737 aircraft maintains a constant airspeed of 500 miles per hour headed due south. The jet stream is 80 miles per hour in the northeasterly direction.

(a) Express the velocity V_a of the 737 relative to the air and the velocity V_w of the jet stream in terms of **i** and **j**.

(b) Find the velocity of the 737 relative to the ground.

(c) Find the actual speed and direction of the 737 relative to the ground

Solution: (a) We set up a coordinate system in which north (N) is along the positive y-axis. See the following Figure :



The velocity of the 737 relative to the air is $V_a = -500j$. . The velocity of the jet stream V_w has magnitude 80 and direction NE (northeast), so the angle between V_w and i is 45°. We express V_w in terms of i and j as

$$V_w = 80\left(\cos 45^0 i + \sin 45^0 j\right) = 80\left(\frac{\sqrt{2}}{2} i + \frac{\sqrt{2}}{2} j\right) = 40\sqrt{2}\left(i + j\right)$$

(b) The velocity of the 737 relative to the ground V_g

is $V_g = V_a + V_w = -500j + 40\sqrt{2}\left(i + j\right) = 40\sqrt{2}i + \left(40\sqrt{2} - 500\right) j$

(c) The actual speed of the 737 is $\|V_g\| = \sqrt{\left(40\sqrt{2}\right)^2 + \left(40\sqrt{2} - 500\right)^2} \approx 447 \text{ miles per hour}$ To find the actual direction of the 737 relative to the ground, we determine the direction angle of V_g. The direction angle is found by solving $\tan\alpha = \frac{40\sqrt{2} - 500}{40\sqrt{2}}$ Then $\alpha = -82.7^0$. The 737 is traveling S 7.3° E.

7.6 Unit Summary

1. The vector joining point $A(a_1, a_2)$ and $B(b_1, b_2)$ is given by $\vec{AB} = (b_1 - a_1, b_2 - a_2)$.

2. The scalar product of vectors $\vec{u} = (a_1, a_2)$ and $\vec{v} = (b_1, b_2)$ of plane is defined by $\vec{u} \cdot \vec{v} = a_1 b_1 + a_2 b_2$

3. The magnitude of the vector $\vec{u}$ noted by $\left\|\vec{u}\right\|$ is defined as $\left\|\vec{u}\right\| = \sqrt{a^2 + b^2}$

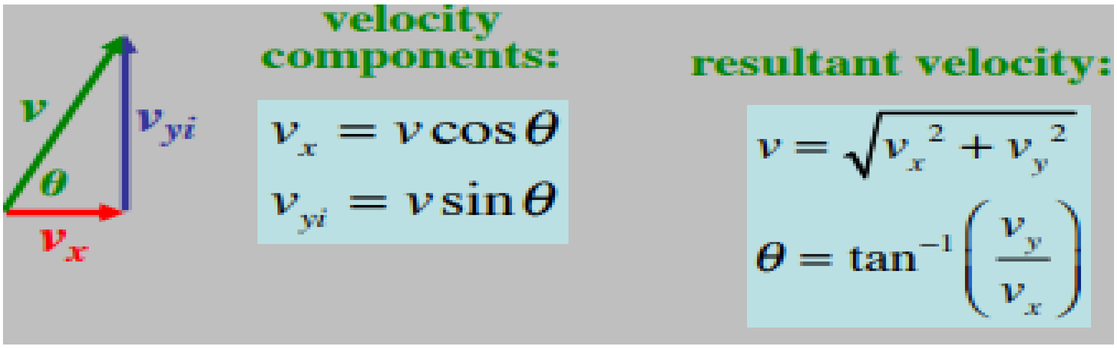
4. The angle θ between two vectors $\vec{u} = (a, b)$ and $\vec{v} = (c, d)$ is given by $\theta = \cos^{-1}\left(\frac{a.c + b.d}{\sqrt{a^2 + b^2} + \sqrt{c^2 + d^2}}\right)$. Where $\cos^{-1}$ denote the inverse function of cosine.

7.7 Additional information for Teachers

- In 2D, a point has paired coordinates (x, y) and in terms of vector, P has position vector $\overrightarrow{OP} = \binom{x}{y}$. The first number x is called the x-coordinate (or x-component), while the second number y is called the y-coordinate (or y-component. The intersection point $(0,0)$ of the two axes is called **the origin.**

- If A and B are two points we can form a vector $\overrightarrow{AB}$ and the distance between these two points denoted $d(A, B)$ is given by $\left\|\overrightarrow{AB}\right\|$. Thus, if $A(a_1, a_2)$ and $B(b_1, b_2)$ are points of plane then $d(A, B) = \left\|\overrightarrow{AB}\right\| = \sqrt{(b_1 - a_1)^2 + (b_2 - a_2)^2}$

- In 2D the Cartesian equation of the line passing through two points $A(x_A, y_A)$ *and* $B(x_B, y_B)$ and the direction vector can be defined as: $\frac{x - x_A}{x_B - x_A} = \frac{y - y_A}{y_B - y_A}$

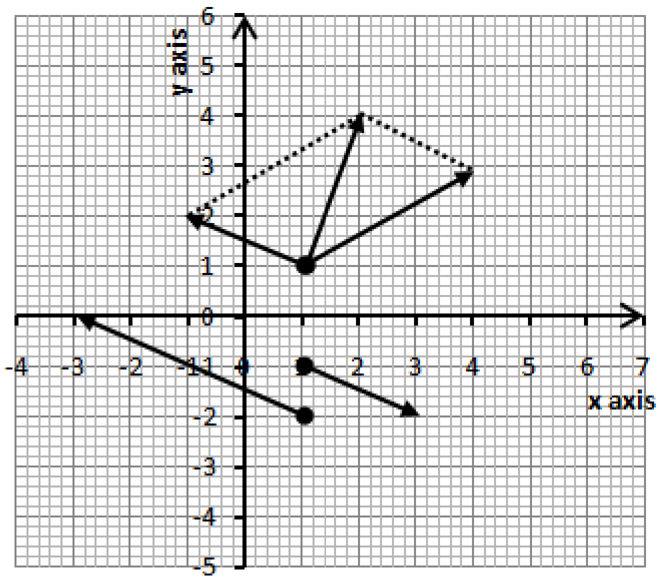


7.8 Answers to revision exercise

1. a. $(2,4),(3,-1),(5,-4),(1,-5)$

b. $(-5,-2)$ c. $(4,7)$

2.



3. $-3p+2q+4r$

4. a. subspace, $(1,0),(2,0)$ b. subspace, $(1,3),(-1,-3)$ c. not subspace, no vector $(0,0)$

5. $\frac{13\sqrt{290}}{290}$, $40.2^{\circ}, 0.70\,rad$

Additional activities

7.9.1 Remedial activities

1. Find the norm of the vector $\vec{V} = (3,4)$

   Solution: The norm is $\left\|\vec{V}\right\| = \sqrt{3^2+4^2} = 5\ \textit{length units}$

2. Find the angle between vectors $\vec{U} = (3,0)$ u = (3,0) and $\vec{V} = (5,5)$ = (5,5)

   Solution: Let α be the angle between these two vectors. Then,

   $$\alpha = \cos^{-1}\left(\frac{3.5+0.5}{\sqrt{3^2+0^2}+\sqrt{5^2+5^2}}\right) = \cos^{-1}\left(\frac{\sqrt{2}}{2}\right)$$

   $\alpha = 45^0$

3. Write the equation in point-slope form for the line through the given point that has the given slope

   a) $(0,2); m = \frac{4}{5}$

   b) $(5,-8); m = -3$

   c) $(-6,1); m = \frac{2}{3}$

   solution

   a) $y-2=\frac{4}{5}(x-0)$ b) $y+8=-3(x-5)$ c) $y-1=\frac{2}{3}(x+6)$

9.8.1. Consolidation activities

1. The points *A*, *B* and *C* have coordinates $(2,1),(7,3),\ and\ (5,k)$ respectively. If *AB* and *BC* are of equal length, find the possible values of *k*.

Solution

$k=8$ or $k=-2$

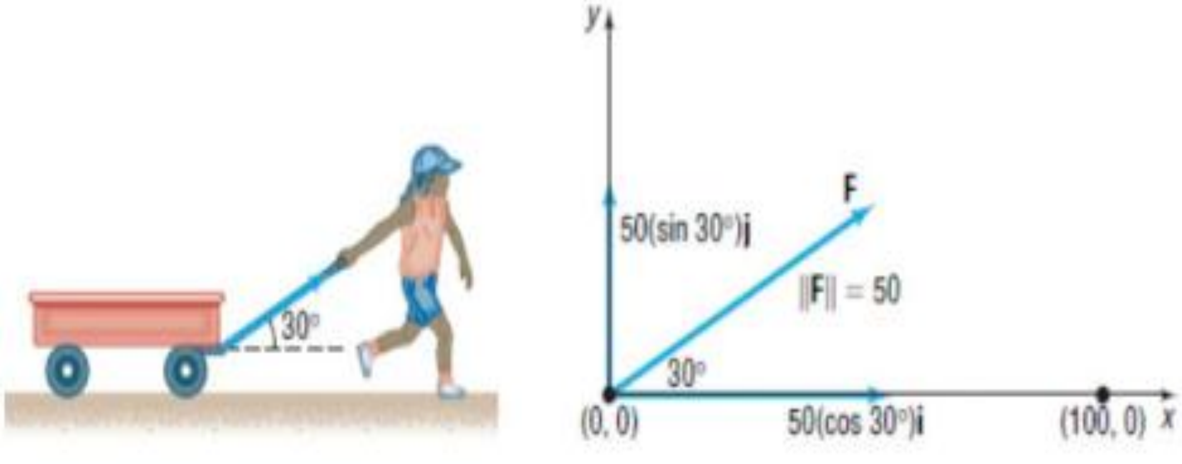
1. Find the vector equations of the lines whose Cartesian equations are given below

   a) $\frac{x-5}{1}=\frac{y+3}{-4}$

   b) $3y=2x-4$

9.8.2. Extended activities

1. The figure below shows a girl pulling a wagon with a force of 50Newtons. How much work is done in moving the wagon 100 meters if the handle makes an angle of 30 degrees with the ground?



(a) (b)

Solution:

Let us position the vector in the Cartesian plane in such a way the wagon moved from the origin O(0,0) to the point P(100,0). The motion is from O to P.

Then, $\vec{OP} = 100\vec{i}$. The force vector $\vec{F}$ is given by:

$\vec{F} = 50\left[\left(\cos 30^0\right)\vec{i} + \left(\sin 30^0\right)\vec{j}\right] = 25\left(\sqrt{3}\,\vec{i} + \vec{j}\right)$. Therefore, the work done is given by dot product: $W = \vec{F}\,\vec{OP} = 25\left(\sqrt{3}\,\vec{i}\right).\left(100\,\vec{i}\right) = 2500\sqrt{3}Joules$

UNIT 8: MATRICES AND DETERMINANTS OF ORDER 2

8.1 Key Unit Competence:

Use matrices and determinants of order 2 to solve other related problems such as organization of data in a shopping, in Cryptography, in Physics (problems about quantum or circuits), …

8.2 Prerequisites

Students will easily learn this unit, if they have a good background on Equations and inequalities (unit 3).

8.3 Cross-cutting issues to be addressed

- Inclusive education (promote education for all while teaching) ;
- Peace and value Education (respect others' view and thoughts during class discussions);
- Gender (provide equal opportunity to boys and girls in the lesson)

8.4 Guidance on introductory activity

- Invite students to work in groups and give them instructions on how they can perform the introductory **activity 8.0** found in student's book;
- Guide students to read and analyse the questions insisting on solving problems involving the system of linear equations using matrices
- Invite some group members to present groups' findings, then try to harmonize their answers;
- Basing on students' experience, prior knowledge and abilities shown in answering the questions for this activity, use different questions to facilitate them to give their predictions and ensure that you arouse their curiosity on what is going to be leant in this unit.

Answer for introductory activity 8.0

a)

Cocks	Rabbits	Prices
5	4	35,000
3	6	30,000

a) Let x be the cost of one cock and y be the cost of one rabbit, then , $\begin{cases} 5x+4y=35,000 \\ 3x+6y=30,000 \end{cases}$

$$\begin{cases} 5x+4y=35,000 & \times(3) \\ 3x+6y=30,000 & \times(-5) \end{cases} \Rightarrow \begin{cases} 15x+12y=105,000 \\ -15x-30y=-150,000 \end{cases} \Rightarrow -18y=-45,000 \Rightarrow y=2,500$$

If we replace y in the first equation we obtain

$5x+4(2500)=35,000 \Rightarrow 5x=25,000 \Rightarrow x=5,000$

Thus the cost of 1 cock is 5,000Frw and the cost of one rabbit is 2,500Frw.

8.5 List of lessons

No	Lesson title	Learning objectives	Number of periods
0	Introductory activity	To arouse the curiosity of student teachers on the content of unit 3	1
1	Definition and examples of matrices	Define and differentiate types of matrices	2
2	Addition and subtraction of matrices	Perform operations (addition, subtraction) on matrices of order 2	2
3	Multiplication of matrices	Perform operations of multiplication on matrices of order 2.	2
4	Determinant of matrices of order 2	Calculate the determinant of a matrix of order 2.	1
5	Inverse of matrix of order 2	Determine the inverse of a matrix of order 2 .	2
6	Solving Simultaneous equations with 2 unknowns	solve system of 2linear equations with 2 unknowns	2
End unit assessment			1

Lesson1: Definition and examples of matrices

a) Learning objectives

To define and differentiate types of matrices.

b) Teaching resources

Learner's book and other Reference books to facilitate research, calculator, manila paper, markers, pens, pencils...

c) Prerequisites/Revision/Introduction

Students will learn easily in this lesson, if they have a good background on linear Equations and inequalities (Unit 3).

d) Learning activities:

- Invite students to work in groups and do the **activity 1** found in their Mathematics Student books;

- Move around in the class for facilitating students where necessary and give more clarification on eventual challenges they may face during their work; Verify and identify groups with different working steps;

- Invite one member from each group with different working steps to present their work where they must explain the working steps;

- As a teacher, harmonize the findings from presentation;.

- Use different probing questions and guide them to explore the content and examples given in the student's book and lead them to discover how to define and differentiate types of matrices

- After this step, guide students to do the **Exercise 1** and evaluate whether lesson objectives were achieved.

Answer of activity 1

1) $\begin{pmatrix} 20 & 31 \\ 45 & 23 \end{pmatrix}$

2) The answers vary according to the present situation.

Answer of Exercise 1

There are many answers. Some of them: $\begin{pmatrix} 1 & 8 \\ 3 & 16 \end{pmatrix}, \begin{pmatrix} 10 & 48 \\ 33 & -6 \end{pmatrix}; \begin{pmatrix} -15 & 28 \\ 103 & 56 \end{pmatrix}$

Lesson 2: Operations on matrices, Addition and subtraction of matrices

a) Learning objectives

Perform operations (addition, subtraction) on matrices of order 2

b) Teaching resources

Student's book and other Reference books to facilitate research, calculator, manila paper, markers, pens, pencils...

c) Prerequisites/Revision/Introduction

Students will learn easily in this lesson, if they have a good background on linear and on Equations and inequalities (unit 3).

d) Learning activities:

- Invite students to work in groups and do the **activity 2** found in their Mathematics Student books;

- Move around in the class for facilitating students where necessary and give more clarification on eventual challenges they may face during their work; Verify and identify groups with different working steps;

- Invite one member from each group with different working steps to present their work where they must explain the working steps;

- As a teacher, harmonize the findings from presentation;.

- Use different probing questions and guide them to explore the content and examples given in the student's book and lead them to discover how to add and subtract matrices of order 2.

After this step, guide students to do the **Exercise 2** and evaluate whether lesson objectives were achieved.

Answer of activity 2

1) $\begin{pmatrix} 13 & 4 \\ 6 & 10 \end{pmatrix} + \begin{pmatrix} 21 & 30 \\ 9 & 12 \end{pmatrix} = \begin{pmatrix} 34 & 34 \\ 15 & 22 \end{pmatrix}$ (2) $\begin{pmatrix} 26 & 8 \\ 12 & 20 \end{pmatrix} - \begin{pmatrix} 7 & 10 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 19 & -2 \\ 9 & 16 \end{pmatrix}$

3) $\begin{pmatrix} 13 & 6 \\ 4 & 10 \end{pmatrix}, \begin{pmatrix} 7 & 3 \\ 10 & 4 \end{pmatrix}$

Answer to Exercise 2

1) **(a)** $\begin{pmatrix} 5 & 0 \\ 3 & 2 \end{pmatrix}$; **(b)** $\begin{pmatrix} -9 & -12 \\ 1 & 2 \end{pmatrix}$; **(c)** $\begin{pmatrix} 7 & 3 \\ 10 & 4 \end{pmatrix}$; **(d)** $\begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix}$; **(e)** $\begin{pmatrix} 294 & 147 \\ 98 & 49 \end{pmatrix}$

2) $x = 7;\ y = -1;\ z = \frac{-3}{2}$

a) $A = \begin{pmatrix} 8 & -3 \\ 1 & 1 \end{pmatrix}$, **b)** $A^t = \begin{pmatrix} 8 & 1 \\ -3 & 1 \end{pmatrix}$

Lesson 3: Multiplication of matrices

a) Learning objectives

To perform operations of multiplication on matrices of order 2.

b) Teaching resources

Learner's book and other Reference books to facilitate research, calculator, manila paper, markers, pens, pencils...

c) Prerequisites/Revision/Introduction

Students will well perform in this lesson, if they have a good background on Equations and inequalities (unit 3) and on lesson 1 and lesson 2 of this Unit.

d) Learning activities:

- Invite students to work in groups and do **the activity 2** found in their Mathematics Student books;
- Move around in the class for facilitating students where necessary and give more clarification on eventual challenges they may face during their work; Verify and identify groups with different working steps;
- Invite one member from each group with different working steps to present their work where they must explain the working steps;
- As a tutor, harmonize the findings from presentation: Two matrices A and B can be multiplied together if and only if the number of columns of A is equal to the number of rows of B. $A_{m\times n} \times B_{m\times p} = M_{m\times p}$
- Use different probing questions and guide them to explore the content and examples given in the student's book and lead them to discover how to perform operations of multiplication on matrices of order 2 .
- After this step, guide students to do the **Exercise 2** and evaluate whether lesson objectives were achieved.

Answer for activity 2.

1. $\begin{pmatrix} 34 & 34 \\ 15 & 22 \end{pmatrix}$

2. $\begin{pmatrix} 19 & -2 \\ 9 & 16 \end{pmatrix}$

3. $\begin{pmatrix} 13 & 6 \\ 4 & 10 \end{pmatrix}, \begin{pmatrix} 7 & 3 \\ 10 & 4 \end{pmatrix}$

Answers of Exercise 2.

1. a) $\begin{pmatrix} 5 & 0 \\ 3 & 2 \end{pmatrix}$ **b)** $\begin{pmatrix} -9 & -12 \\ 1 & 2 \end{pmatrix}$ **c)** $\begin{pmatrix} -12 & -6 \\ 8 & 4 \end{pmatrix}$ **d)** $\begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix}$ **e)** $\begin{pmatrix} 294 & 147 \\ 98 & 49 \end{pmatrix}$

2. $x = 5,\ y = -1,\ z = -1$

a. $A = \begin{pmatrix} 6 & -3 \\ 1 & 1 \end{pmatrix}$

b. $A^t = \begin{pmatrix} 6 & 1 \\ -3 & 1 \end{pmatrix}$

Lesson 4 : Determinants of matrices of order two

a) Learning objectives

To Calculate the determinant of a matrix of order 2.

b) Teaching resources

Learner's book and other Reference books to facilitate research, calculator, manila paper, markers, pens, pencils...

c) Prerequisites/Revision/Introduction

Students will learn easily in this lesson, if they well perform on operation (Addition, subtraction and multiplication)on matrices.

d) Learning activities:

- Invite students to work in groups and do the **activity 3** found in their Mathematics Student books;
- Move around in the class for facilitating students where necessary and give more clarification on eventual challenges they may face during their work; Verify and identify groups with different working steps;
- Invite one member from each group with different working steps to present their work where they must explain the working steps;
- As a teacher, harmonize the findings from presentation:
- Use different probing questions and guide them to explore the content and examples given in the student's book and lead them to discover how to calculate the determinant of a matrix of order 2 .
- After this step, guide students to do the **Exercise 3** and evaluate whether lesson objectives were achieved.
- Provide additional activities where necessary.

Answers of activity 3.

1. 1) $1\times1-2\times0=1$; 2) $-2\times6-3\times(-4)=0$

3) $3\times8-6\times1=18$; 4) $12\times9-(-2)\times3=114$

Answer of Exercise 3.

1) $\begin{vmatrix} -1 & -3 \\ 1 & 1 \end{vmatrix} = (-1\times 1)-(1\times -3) = 2$ 2) 0   3) 0   4) 2

Lesson 5. Inverse of matrices of order two

a) Learning objectives

To determine the inverse of a matrix of order 2 .

b) Teaching resources

Learner's book and other Reference books to facilitate research, calculator, manila paper, markers, pens, pencils...

c) Prerequisites/Revision/Introduction

Students will learn easily in this lesson, if they well perform on operation (Addition, subtraction and multiplication) on matrices.

d) Learning activities:

- Invite students to work in groups and do the activity 4 found in their Mathematics Student books;
- Move around in the class for facilitating students where necessary and give more clarification on eventual challenges they may face during their work; Verify and identify groups with different working steps;
- Invite one member from each group with different working steps to present their work where they must explain the working steps;
- As a teacher, harmonize the findings from presentation by helping students to realise that calculating matrix inverse of matrix *A*, is to find matrix A^{-1} such that $A\cdot A^{-1} = A^{-1}\cdot A = I$,where *I* is identity matrix.
- Use different probing questions and guide them to explore the content and examples given in the student's book and lead them to discover how to determine the inverse of a matrix of order 2.
- After this step, guide students to do the **Exercise 4** and evaluate whether lesson objectives were achieved.
- Provide additional activities where necessary.

Answers of Activity 4

$$A.A' = \begin{pmatrix} 10 & 2 \\ 6 & 3 \end{pmatrix}\begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix} = \begin{pmatrix} 10x_1+2x_3 & 10x_2+2x_4 \\ 6x_1+3x_3 & 6x_2+3x_4 \end{pmatrix};\ x_1 = \frac{3}{18}, x_2 = \frac{2}{18}, x_3 = \frac{-6}{18}, x_4 = \frac{10}{18}$$

$$\frac{1}{18}\begin{pmatrix} 3 & -2 \\ -6 & 10 \end{pmatrix}$$

Answer of exercise 4

1. $\begin{pmatrix} \frac{1}{2} & \frac{3}{2} \\ \frac{-1}{2} & \frac{-1}{2} \end{pmatrix}$ **; 2. No inverse; 3. No inverse; 4.** $\left(A^t\right)^{-1} = \begin{pmatrix} \frac{1}{2} & \frac{-1}{2} \\ \frac{3}{2} & \frac{-1}{2} \end{pmatrix}$ **; 5. No inverse.**

Lesson 6: Solving Simultaneous equations with 2 unknowns

a) Learning objectives

To solve system of 2 linear equations with 2 unknowns

b) Teaching resources

Learner's book and other Reference books to facilitate research, calculator, manila paper, markers, pens, pencils...

c) Prerequisites/Revision/Introduction

Students will learn easily in this lesson, if they well perform on operation (Addition, subtraction and multiplication)on matrices.

d) Learning activities:

- Invite students to work in groups and do the **activity 5** found in their Mathematics Student books;
- Move around in the class for facilitating students where necessary and give more clarification on eventual challenges they may face during their work; Verify and identify groups with different working steps;
- Invite one member from each group with different working steps to present their work where they must explain the working steps;
- As a teacher, harmonize the findings from presentation;
- Use different probing questions and guide them to explore the content and examples given in the student's book and lead them to discover how to solve system of 2 linear 2 linear equations with 2 unknowns .
- After this step, guide students to do the **Exercise 5** and evaluate whether lesson objectives were achieved.
- Provide additional activities where necessary.

Answer for activity 5.

1) $A=\begin{pmatrix}3 & 1\\1 & -1\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}9\\0\end{pmatrix}$ 2) $\begin{pmatrix}1 & -1\\1 & 1\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}19\\10\end{pmatrix}$

Answer for exercise 5.

1) $x=3; \quad y=1$; 2) Infinity Solution; 3) $x=1; \quad y=1$; 4) $x=2; \quad y=0$

8.6 Unit Summary

1. Square matrix of order two has the form

$$\begin{pmatrix}a_{11} & a_{12}\\a_{21} & a_{22}\end{pmatrix}$$

2. In an upper triangular matrix, the elements located below the leading diagonal are zeros
3. In a lower triangular matrix, the elements above the leading diagonal are zeros.
4. In a diagonal matrix, all the elements above and below the leading diagonal are zeros.
5. A scalar matrix is a diagonal matrix in which the leading diagonal elements are equal.
6. An identity matrix (noted by **I**) is a diagonal matrix in which the leading diagonal elements are equal to 1.
7. The determinant of matrix of order two is $\begin{vmatrix}a_{11} & a_{12}\\a_{21} & a_{22}\end{vmatrix}=a_{11}a_{22}-a_{21}a_{12}$
8. The inverse of matrix of order two $\begin{pmatrix}a & c\\b & d\end{pmatrix}$ is $A^{-1}=\dfrac{1}{\det A}\begin{pmatrix}d & -c\\-b & a\end{pmatrix}$

8.7. Additional information for teachers

This unit of matrix of order 2. It is very necessary to help students to learn it and finish all given activities in the given time. Teacher may prepare many activities to students to be performed in groups at home and then present them in written form to be marked after. This strategy will help teacher to cover all required topics and concepts in this unit. The teacher will use one example and one application task while teaching and let students do the remaining tasks themselves in groups and after class.

8.8 Answers to revision exercises

1. $a)x=-4 \quad ; y=11 \ ; \ a)x=2 \quad ; y=-1; \ \ a)x=5 \quad ; y=19$

2. 4,-11   3. $\frac{-1}{2}$ or 3

$a)x=3\quad ;y=-2\quad \text{b)}x=1\quad ;y=\frac{-2}{3}\ ;\ \text{c)}x=\frac{3}{2}\quad ;y=\frac{1}{2}$

8.9. Additional activities

8.9.1 remedial activities

1) Determine det N given that $N=\begin{pmatrix}12 & 6\\ 5 & 4\end{pmatrix}\Rightarrow |N|=\begin{vmatrix}12 & 6\\ 5 & 4\end{vmatrix}=12\times4-5\times6=48-30=18$

8.9.2 consolidation activities

1)Find the inverse of $A=\begin{pmatrix}1 & 0\\ 2 & 1\end{pmatrix}$

Solution

$\det A=1-0=1$

$A^{-1}=\frac{1}{1}\begin{pmatrix}1 & 0\\ -2 & 1\end{pmatrix}=\begin{pmatrix}1 & 0\\ -2 & 1\end{pmatrix}$

2)Find the inverse of $A=\begin{pmatrix}1 & 3\\ 2 & 6\end{pmatrix}$

Solution

$\det A=6-6=0$

Since the determinant is zero, the given matrix has no inverse.

8.9.3 Extended activity

1) A Farmer Kalisa bought in Ruhango Market 5 Cocks and 4 Rabbits and he paid 35,000Frw, on the following day, he bought in the same Market 3 Cocks and 6 Rabbits and he paid 30,000Frw.

a) Considering x as the cost for one cock and y the cost of one Rabbit, formulate equations that illustrate the activity of Kalisa;

b) Make a matrix A indicating the number of cocks and rabbits

c) If C is a matrix column made by the money paid by Kalisa, ie $C=\begin{pmatrix}35,000\\ 30,000\end{pmatrix}$,

write the equation $A\begin{pmatrix}x\\ y\end{pmatrix}=C$

d) Discuss and explain in your own words how you can determine $\begin{pmatrix} x \\ y \end{pmatrix}$ the cost of 1 Cocks and 1 Rabbit.

Solution:

a) Let consider x as the cost for one cock and y the cost of one Rabbit, the equations that illustrate the activity of Kalisa is $\begin{cases} 5x+4y=35,000 \\ 3x+6y=30,000 \end{cases}$

b) The matrix A indicating the number of cocks and rabbits is $A=\begin{pmatrix} 5 & 4 \\ 3 & 6 \end{pmatrix}$

c) In matrix form the activity of Kalisa is written as $\begin{pmatrix} 5 & 4 \\ 3 & 6 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} 35,000 \\ 30,000 \end{pmatrix}$

d) After discussion student-teacher discover that the cost of one cock is 5000Frw and the cost of one rabbit is 2500Frw.

UNIT 9: MEASURES OF DISPERSION

9.1 Key Unit Competence:

Extend understanding, analysis and interpretation of data arising from problems and questions in daily life to include the standard deviation.

9.2 Prerequisites

Students will easily learn this unit, if they have a good background on descriptive statistics leant in O-level (S3 unit 13; S2 Unit 10 & S1 unit 8)

9.3 Cross-cutting issues to be addressed

- Inclusive education (promote education for all while teaching) ;
- Peace and value Education (respect others' view and thoughts during class discussions);
- Gender (provide equal opportunity to boys and girls in the lesson)

9.4 Guidance on introductory activity

- Invite students to work in groups and give them instructions on how they can perform the introductory **activity 9.0** found in student's book;
- Guide students to read and analyse the questions insisting on analysis and interpretation of data arising from problems and questions in daily life to include the standard deviation.
- Invite some group members to present groups' findings, then try to harmonize their answers;
- Basing on students' experience, prior knowledge and abilities shown in answering the questions for this activity, use different questions to facilitate them to give their predictions and ensure that you arouse their curiosity on what is going to be leant in this unit.

Answers for the introductory activity:

The table below shows the types and the number of sold fruits in one week.

Type of fruit	A (Banana)	B(Orange)	C(Pineapple)	D(Avocado)	E(Mango)	F(apple)
Number of fruits sold	1100	962	1080	1200	884	900

a) The highest number of fruits sold is 1200 (Avocadoes)
b) The least number of fruits sold is 884 (mangoes)
c) The total number of fruits sold during the week is 6126 fruits
d) The average number of fruits sold per day is $\frac{6126}{6}=1021$

2.

a) The mean mark of the class is $\frac{3+5+6+3+8+7+8+4+8+6}{10}=\frac{58}{10}=5.8$.

b) The mark that was obtained by many student is 8

c) Comparing the mean mark of the class and the mark for every student, one can find that 4 student have the marks (3, 4 and 5) below the mean, 2 student scored the mark near the mean while 4 students have scored higher marks than the mean. Mathematics tutor should prepare remedial activities for student whom their marks are below and near the mean.

9.5 List of lessons

NO	Lesson title	Learning objectives	Number of periods
0	Introductory activity	To arouse the curiosity of students on the content of unit 9	1
1	Recall on calculation of Central tendencies (mean, median, mode) of statistical data.	Calculation and interpretation of central tendencies (mean, mode,median,quartiles)	2
2	Variance	Define and Interpret the Variance of statistical data.	3
3	Standard deviation (including combined set of data) and The Coefficient of variation	Interpretation of Standard deviation and the Coefficient of variation of statistical data.	2
4	Application of measures of dispersion in daily life	Use of measures of dispersion in real situation life	3
End unit assessment			1

Lesson 1: Recall on Central tendencies : mean, median, mode

a) Learning objective

Determine the measures of central tendencies of a given statistical series.

b) Teaching resources

Student's book and other Reference textbooks to facilitate research, Mathematical set, calculator, manila paper, markers, pens, pencils, …

c) Prerequisites/Revision/Introduction

Students should have knowledge and skills on how to

-Present data on a frequency distribution.

-Determine the mode, mean, and median of statistical data.

-Define grouped data and represent grouped data on a frequency distribution.

-Identify mode, middle class, modal class, and median of given grouped statistical data.

-Read diagram of grouped statistical data.

d) Learning activities

- Ask students in small groups to read and discuss on the **activity 9 A**
- Facilitate working, especially straggling students.
- Call students to present their findings, and help them to harmonize their answer.

Activity 9A and its solutions

1.Observe the following marks of 10 students in mathematics test: 5, 4, 10, 3, 3, 4, 7, 4, 6, 5 and complete the table below.

Marks (x)	Frequency (f)	f x
3		
4		
5		
6		
7		
10		
Total		

Calculate the mean, median, mode and range of this set

2. In three classes of S4 PCB, during the quiz of mathematics out of 5 marks, 100 students obtain marks as shown in the table below:

Marks (x)	0	1	2	3	4
Frequency (fi)	4	19	25	29	23

- What is the marks obtained by most of the students?
- You are asked to calculate the mean mark for the class. Explain how you should find it.

Marks (x)	Frequency (f)	f x
3	2	6
4	3	12
5	2	10
6	1	6
7	1	7
10	1	10

Total	10	41

i) $Mean = \frac{\sum xf}{\sum f} = \frac{41}{10} = 4.1$

ii) $Median = \frac{x_{\frac{n}{2}} + x_{\frac{n}{2}+1}}{2} = \frac{x_5 + x_6}{2} = \frac{4+5}{2} = 4.5$

iii) $Mode = 4$

iv) $Range = 10 - 3 = 7$

1. a) $3 marks$ obtained by 29 student-teachers**.**

b) Mean marks is given by the sum of product of xf divided by sum of frequencies.

$$Mean = \frac{\sum xf}{\sum f}$$

- Lead students to realize that the solution of an activity help them to reach easily to the content.
- Help students discover that for **ungrouped data**;
  - **The mean** is given by $\bar{x} = \frac{1}{n}\sum xfi$,
  - **The mode** is the number that appears the most often from the set of data
  - **The median:** The middle number in arranged data from the smallest to the largest.

    When $\sum fi = n$ is odd the median is given by

    $$Me \to \left(\frac{n+1}{2}\right)^{th} \text{ or } Me = x_{\frac{n+1}{2}}$$

    When n is even, the median is given by

    $$Me = \frac{1}{2}\left[\left(\frac{n}{2}\right)^{th} + \left(\frac{n}{2}+1\right)^{th}\right] \text{ or } Median = \frac{x_{\frac{n}{2}} + x_{\frac{n}{2}+1}}{2}$$

- Help students discover that for **grouped data**:
  - The mean is given by $\bar{x} = \frac{1}{n}\sum xfi$,
  - The mode is given by

    $$Mode = L_m + c\left(\frac{\Delta_1}{\Delta_1 + \Delta_2}\right)$$

- **The median is given by**

$$Median = L_m + c\left(\frac{\frac{n}{2} - CF_b}{f_m}\right)$$

- Facilitate them to determine the mean, mode, and median in both ungrouped and grouped data trough given examples in student book.

- Ask students to do individually the **exercise 9.A** to master the content.

Exercise 9.A and its answers.

1. A group of student-teachers from SME were asked how many books they had read in previous year; the results are shown in the frequency table below. Calculate the mean, median and mode number of books read.

Number of books	0	1	2	3	4	5	6	7	8
frequency	5	5	6	9	11	7	4	2	1

2. During oral presentation of internship report for year three student-teachers the first 10 student-teachers scored the following marks out of 10:

8, 7, 9, 10, 8, 9, 8, 6, 7 and 10

a. Calculate the mean of the group

b. Calculate the median and Mode

Solutions:

1. $Mean = 3.48$ $Mode = 4$ $Median = 3.5$

2. $Mean = 8.2$ $Mode = 10$ $Median = 4$

Lesson 2: Variance

a) Learning objective

-Define and Interpret the Variance of statistical data.

b) Teaching resources

Student's book and other Reference textbooks to facilitate research, Mathematical set, calculator, manila paper, markers, pens, pencils, …

c) Prerequisites/Revision/Introduction

Students should have knowledge and skills on how to

-Represent statistical information using frequency distribution tables,

-Determine the mode, mean, and median of statistical data.

-Define ungrouped data and represent ungrouped data on a frequency distribution.

d) Learning activities

- Ask students in small groups to read and discuss on the **activity 1** given in Student's book and make sure that everybody is engaged/ involved.
- Facilitate working, especially straggling students.
- Ask students to present the findings, and
- Help them to harmonize their answers.

- After this step, guide students to do the **Exercise 1** and evaluate whether lesson objectives were achieved.

Answer of activity 1

For $\bar{x} = 16.875$

x	f	$x-\bar{x}$	$(x-\bar{x})^2$	$f(x-\bar{x})^2$
12	4	-4.875	23.765625	95.0625
13	2	-3.875	15.015625	30.03125
15	1	-1.875	3.515625	3.515625
19	4	2.125	4.515625	18.0625
21	5	4.125	17.015625	85.078125
	$\sum f = 16$			$\sum f(x-\bar{x})^2 = 231.75$

- The variance is denoted and defined by

$$\sigma^2 = \frac{\sum_{i=1}^{n}\left(x_i - \bar{x}\right)^2}{n}$$

Developing this formula, we have

$$\sigma^2 = \frac{\sum_{i=1}^{n}\left(x_i^2 - 2x_i\bar{x} + \left(\bar{x}\right)^2\right)}{n}$$

$$= \frac{1}{n}\sum_{i=1}^{n} x_i^2 - \frac{1}{n}2\bar{x}\sum_{i=1}^{n} x_i + \frac{1}{n}\left(\bar{x}\right)^2\sum_{i=1}^{n}1$$

$$= \frac{1}{n}\sum_{i=1}^{n} x_i^2 - 2\bar{x}\bar{x} + \left(\bar{x}\right)^2$$

$$= \frac{1}{n}\sum_{i=1}^{n} x_i^2 - \left(\bar{x}\right)^2$$

Thus, the variance is also defined by

$$\sigma^2 = \frac{1}{n}\sum_{i=1}^{n} x_i^2 - \left(\bar{x}\right)^2$$

Answer of Exercise 1

1. $\frac{81}{25}$ **;** **2. 6 ;** **3.** $\frac{308}{81}$ **;** **4.** $\frac{11}{25}$ **;** **5.** $\frac{38}{9}$

Lesson 3: Standard Deviation and coefficient of variation

a) Learning objective

-Define the standard deviation and the coefficient of variation.

- Express the coefficient of variation as a measure of the spread of a set of data as a proportion of its mean.

b) Teaching resources

Student's book and other Reference textbooks to facilitate research, Mathematical set, calculator, manila paper, markers, pens, pencils, …

c) Prerequisites/Revision/Introduction

Students should have knowledge and skills on how to

-Represent statistical information using frequency distribution tables,

-Determine the mode, mean, and median of statistical data.

-Define ungrouped data and represent ungrouped data on a frequency distribution.

d) Learning activities

- Ask students in small groups to read and discuss on the **activity 2 given in** Student's book and make sure that everybody is engaged/ involved.
- Facilitate working, especially straggling students.
- Ask students to present the findings, and
- Help them to harmonise their answers.
- Lead them to do the provided examples in student's book.
- Ask students to do **Exercise 2** to master the content.

Answer of activity 2

x	f	x^2	xf	$x^2 f$
3	**2**	**9**	**6**	**18**
4	**3**	**16**	**12**	**48**
5	**5**	**25**	**25**	**125**
7	**1**	**49**	**7**	**49**
9	**6**	**81**	**54**	**486**
	$\sum f = 17$		$\sum xf = 104$	$\sum x^2 f = 726$

- The standard deviation is denoted and defined by

$$\sigma = \sqrt{\frac{\sum_{i=1}^{n}\left(x_i - \bar{x}\right)^2}{n}} \text{ or } \sigma = \sqrt{\frac{1}{n}\sum_{i=1}^{n} x_i^2 - \left(\bar{x}\right)^2}$$

Coefficient of variation

The coefficient of variation is a calculation built on other calculations: the standard deviation and the mean as follows:

$$Cv = \frac{\sigma}{\bar{x}} \times 100$$

Where:

- σ is the standard deviation
- $\bar{x}$ is the mean.

Answer of exercise 2

(1) $\frac{3\sqrt{2}}{2}$; (2) $\frac{2\sqrt{1224499}}{7}$; (3) $\frac{4\sqrt{10}}{7}$; (4) $\sqrt{2}$; (5) $\frac{2\sqrt{34}}{7}$

Answer of exercise 3

1)54.51%; (2) 20.79%; (3) 55.12%; (4) 22.22%; (5) 43.92%

Lesson 6: Application of measures of dispersion in daily life

a) **Learning objective**

Apply and explain the standard deviation as the more convenient measure of the variability in the interpretation of data.

b) Teaching resources

Learner's book and other Reference textbooks to facilitate research, Mathematical set, calculator, manila paper, markers, pens, pencils, …

c) Prerequisites/Revision/Introduction

Students should have knowledge and skills on how to

- Apply data collection to carry out a certain research.
- Represent statistical information using frequency distribution tables, bar charts, histograms, polygons, pie charts, or pictogram.
- Determine the mode, mean, and median of statistical data.
- Interpret correctly graphs involving statistical data
- Develop research and creativity.
- Read diagram of grouped statistical data

d) Learning activities

- Ask students in small groups to read and discuss on the **activity 4** to plot points and make sure that everybody is engaged/ involved.
- Facilitate working, especially straggling students.
- Call students to present the findings, and help them to harmonize the answer.

Answer of activity 4

- Let students make their own research and discover that there are different applications of statistics in real life.
- Let students do activity 9.4 to find out the usefulness of statistical terms like frequency distribution, Bar graph, Histogram, Frequency polygon, Mean.

- Let students do exercise 9.4 to find out the usefulness of statistical terms like frequency distribution, Pie chart, cumulative frequency polygon or ogive , Mode and median, Standard deviation .

1. Importance of Mean, Mode, Median, Variance , standard deviation

A measure of central tendency is a single value that attempts to describe a set of data by identifying the central position within that set of data. The mean, median and mode are all valid measures of central tendency, but under different conditions, some measures of central tendency become more appropriate to use than others. The mean is essentially a model of your data set. It is the value that is most common. You will notice, however, that the mean is not often one of the actual values that you have observed in your data set. However, one of its important properties is that it minimises error in the prediction of any one value in your data set. That is, it is the value that produces the lowest amount of error from all other values in the data set. An important property of the mean is that it includes every value in your data set as part of the calculation. In addition, the mean is the only measure of central tendency where the sum of the deviations of each value from the mean is always zero

Standard Deviation is a statistical term used to measure the amount of variability or dispersion around an average. Technically it is a measure of volatility. Dispersion is the difference between the actual and the average value. The larger this dispersion or variability is, the higher is the standard deviation.
Finance and banking is all about measuring and managing risk and standard deviation measures risk (Volatility). Standard deviation is used by all portfolio managers to measure and track risk.

The standard deviation tells those interpreting the data, how reliable the data is or how much difference there is between the pieces of data by showing how close to the average all of the data is.

- A low standard deviation means that the data is very closely related to the average, thus very reliable.
- A high standard deviation means that there is a large variance between the data and the statistical average, thus not as reliable.
- Standard deviation and variance may be basic mathematical concepts, but they play important roles throughout the financial sector, including the areas of accounting, economics, and investing. In the latter, for example, a firm grasp of the calculation and interpretation of these two measurements is crucial for the creation of an effective trading strategy.
- Standard deviation and variance are both determined by using the mean of the group of numbers in question. The mean is the average of a group of numbers, and the variance measures the average degree to which each number is different from the mean. The extent of the variance correlates to the size of the overall range of numbers—meaning the variance is greater when there is a wider range of numbers in the group, and the variance is lesser when there is a narrower range of numbers.

Some examples of situations in which standard deviation might help to understand the value of the data:

- A class of students took a math test. Their teacher found that the mean score on the test was an 85%. She then calculated the standard deviation of the other test scores and found a very small standard deviation which suggested that most students scored very close to 85%.

- A class of students took a test in Language Arts. The teacher determines that the mean grade on the exam is a 65%. She is concerned that this is very low, so she determines the standard deviation to see if it seems that most students scored close to the mean, or not. The teacher finds that the standard deviation is high. After closely examining all of the tests, the teacher is able to determine that several students with very low scores were the outliers that pulled down the mean of the entire class's scores.
- An employer wants to determine if the salaries in one department seem fair for all employees, or if there is a great disparity. He finds the average of the salaries in that department and then calculates the variance, and then the standard deviation. The employer finds that the standard deviation is slightly higher than he expected, so he examines the data further and finds that while most employees fall within a similar pay bracket, three loyal employees who have been in the department for 20 years or more, far longer than the others, are making far more due to their longevity with the company. Doing the analysis helped the employer to understand the range of salaries of the people in the department.

2. Importance of statistical graphs

One goal of statistics is to present data in a meaningful way. That is where graphs can be invaluable, allowing statisticians to provide a visual interpretation of complex numerical stories.

Good graphs convey information quickly and easily to the user. Graphs highlight the salient features of the data. They can show relationships that are not obvious from studying a list of numbers. They can also provide a convenient way to compare different sets of data.

Different situations call for different types of graphs, and it helps to have a good knowledge of what types are available. The type of data often determines what graph is appropriate to use. Qualitative data, quantitative data, and paired data each use different types of graphs.

Some examples of graphs are the following:

- Bar graph is a way to visually represent qualitative data. Data is displayed either horizontally or vertically and allows viewers to compare items, such as amounts, characteristics, times, and frequency.
- Pie chart is helpful when graphing qualitative data, where the information describes a trait or attribute and is not numerical. Each slice of pie represents a different category, and each trait corresponds to a different slice of the pie; some slices usually noticeably larger than others. By looking at all of the pie pieces, you can compare how much of the data fits in each category, or slice.
- A histogram in another kind of graph that uses bars in its display. This type of graph is used with quantitative data. Ranges of values, called classes, are listed at the bottom, and the classes with greater frequencies have taller bars.
- A stem and leaf plot breaks each value of a quantitative data set into two pieces: a stem, typically for the highest place value, and a leaf for the other place values. It provides a way to list all data values in a compact form.

Answer of application activity 9.4

1. Answers may vary with the group, as a tutor, try to harmonize them and provide feedback.

   Examples: To analyse the scores of athletes in a given competition.
   These scores may be represented using the frequency distribution table, the mean, median and standard deviation of the scores can be calculated respectively. One can explain the role of these measures

2. Answers will vary from group to another. Try to organize a session where every group will have time to present its findings and others will ask questions and provide constructive feedbacks for learning purpose.

9.6 Summary of the unit

- **Statistics** is a blanch of mathematics concerned with scientific method for collecting and presenting, organizing and summarizing and analyzing data as well as deriving valid conclusions and making reasonable decisions on the basis of this analysis.
- A sequence of observations, made on a set of objects included in the sample drawn from population, is known as **statistical data.** There are qualitative and quantitative or numerical data
- Numerical data can be further broken into discrete or continuous data.

- **Raw data:** Data which have been arranged in a systematic order are called raw data or ungrouped data.

- **Frequency distribution: A** frequency distribution is a table showing how often each value (or set of values) of the collected data occurs in a data set. A frequency table is used to summarize categorical or numerical data. Data presented in the form of a frequency distribution are called **grouped data**.

- **Cumulative frequency:** Cumulative frequency can also defined as the sum of all previous frequencies up to the current point.
- **Stem and leaf displays:** Is a plot where each data value is split into a leaf usually the last digit and a stem the other digit. The stem values are listed down, and the leaf values are listed next to them.
- In case, the range of data is large, the data must be grouped into groups **or classes.**
- **Mean, mode, median** are the measures of central tendency
- **The median:** middle data/item of arranged or ordered data. If total observation ($\sum fi = n$) then $Me = \left(\frac{n+1}{2}\right)^{th}$ *term* or $Me = x_{\frac{n+1}{2}}$ and read the number which located on this position. On the other side when n is even, $Me = \frac{1}{2}\left[\left(\frac{n}{2}\right)^{th} + \left(\frac{n}{2}+1\right)^{th}\right]$ *term* or $Median = \frac{x_{\frac{n}{2}} + x_{\frac{n}{2}+1}}{2}$ then the median is a half of the sum of number located on those two positions.

- **The mean formula of ungrouped data is the following:** $\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i f_i$
- **The mode**: The mode is the number that appears the most often from the set of data
- **The range:** In the case of ungrouped data, the range of a set of observations is the difference in values between the largest and the smallest observations in the set.
- **Formula for Mean, mode, median and range of grouped data are as follow:**

3. **Mean** is given by $\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i f_i$, where $n = \sum_{i=1}^{n} f_i$: the sum of frequencies of data.
4. $Mode = L_m + c\left(\frac{\Delta_1}{\Delta_1 + \Delta_2}\right)$, with L_m is lower boundary of modal class, C is class width: the difference between upper and lower boundary of modal class $(C = U_m - L_m)$, f_m : Frequency of modal class, $\Delta_1 = f_m - f_b$ $\Delta_2 = f_m - f_a$ f_b is frequency followed by f_m and f_a frequency follows f_m.

5. $Median = L_m + C\left(\frac{\frac{n}{2} - CF_b}{f_m}\right)$, $n = \sum_{i=1}^{n} f_i$: the sum of frequencies of data, CF_b: cumulative frequency preceded by cumulative frequency of modal class (cumulative frequency before modal class)

- The most commonly used graphs are: Bar graph, Pie chart, Histogram, Frequency polygon, Cumulative frequency graph or Ogive.
- **Some measures of dispersion** are Quartiles, variance, Range, standard deviation, coefficient of variation

$Q_1 = \frac{1}{4}(n+1)^{th}\ term\ \ or\ \ Q_1 = x_{\frac{n+1}{4}}$. $Q_2 = \frac{1}{2}(n+1)^{th}\ term\ \ or\ Q_2 = x_{\frac{n+1}{2}} = Me$

$Q_3 = \frac{3}{4}(n+1)^{th}\ term\ \ or\ Q_3 = x_{\frac{3(n+1)}{4}}$

- **The inter-quartile range**: is given by the difference between third quartile and the first quartile.
- Variance: Variance measures how far a set of numbers is spread out. A variance of zero indicates that all the values are identical. Variance is always non-negative: a small variance indicates that the data points tend to be very close to the mean and hence to each other, while a high variance indicates that the data points are very spread out around the

mean and from each other. The variance is given by : $\sigma^2 = \frac{\sum_{i=1}^{n}\left(x_i - \bar{x}\right)^2 f_i}{n}$ or $\sigma^2 = \frac{1}{n}\sum_{i=1}^{n} x_i^2 f_i - \left(\bar{x}\right)^2$

- The standard deviation has the same dimension as the data, and hence is comparable to deviations from the mean. We define the **standard deviation** to be the square root of the variance. Thus, the standard deviation is denoted and defined by

$$\sigma = \sqrt{\frac{\sum_{i=1}^{n}\left(x_i - \bar{x}\right)^2 f_i}{n}} \text{ or } \sigma = \sqrt{\frac{1}{n}\sum_{i=1}^{n} x_i^2 f_i - \left(\bar{x}\right)^2}$$

- The coefficient of variation measures variability in relation to the mean (or average) and is used to compare the relative dispersion in one type of data with the relative dispersion in another type of data. It allows us to compare the dispersions of two different distributions if their means are positive. The greater dispersion corresponds to the value of the coefficient of greater variation. The coefficient of variation is given by: $Cv = \frac{\sigma}{\bar{x}} \times 100$ where: σ is the standard deviation and $\bar{x}$ s the mean.

9.7 Additional Information for Teachers

- Emphasize on the following results follow directly from the definitions of mean and standard deviation:

  When all the data values are multiplied by a constant *a*, the new mean and new standard deviation are equal to *a* times the original mean and standard deviation. That is, the mean of $ax_1, ax_2, ax_3, ..., ax_n$ is $a\left(\bar{x}\right)$ and the standard deviation is $a\sigma$.

  When a constant value, *b*,is added to all data values, then new mean is increased by *b*. However standard deviation does not change. That is, the mean of $ax_1, ax_2, ax_3, ..., ax_n$ is $\bar{x} + b$ and the standard deviation is σ .

- Emphasize on calculating median of ungrouped data. We need to clarify formula for $n - odd$ and for $n - even$ as highlighted in the student-teachers book and paying attention on notation.

When n is odd, the median is given by

$$Me \rightarrow \left(\frac{n+1}{2}\right)^{th} term \text{ or } Me = x_{\frac{n+1}{2}} \text{ we don't write } Me = \left(\frac{n+1}{2}\right)^{th}$$

When n is even, the median is given by

$$Me = \frac{1}{2}\left[\left(\frac{n}{2}\right)^{th} + \left(\frac{n}{2}+1\right)^{th}\right] term \text{ or } Median = \frac{x_{\frac{n}{2}} + x_{\frac{n}{2}+1}}{2}$$

- Emphasize on calculating mode and median of grouped data.

$$Mode = L_m + c\left(\frac{\Delta_1}{\Delta_1 + \Delta_2}\right)$$

$$Median = L_m + c\left(\frac{\frac{n}{2} - CF_b}{f_m}\right)$$

- Emphasize on formulae of quartiles for $n - odd$ and for $n - even$

$Q_1 \to \frac{1}{4}(n+1)^{th}$ *term* or $Q_1 = x_{\frac{n+1}{4}}$. We don't write $Q_1 = \frac{1}{4}(n+1)^{th}$

$Q_2 \to \frac{1}{2}(n+1)^{th}$ *term* or $Q_2 = x_{\frac{n+1}{2}} = Me$ we don't write $Q_2 = \frac{1}{2}(n+1)^{th}$

$Q_3 \to \frac{3}{4}(n+1)^{th}$ *term* or $Q_3 = x_{\frac{3(n+1)}{4}}$ we don't write $Q_3 = \frac{3}{4}(n+1)^{th}$

9.8 Answer for Revision Exercise

1. 53.6
2. $Mean = 32, S.D = 12.36, Median = 32.61$
3. a. 8, 9, 6 b. 194, 194, 195
4. 24, 1.19
5. a. 775, 120250 b. 157cm, 5.89cm
6. 21
7. 8
8. 7
9. $2 - 3\bar{x}$, $9\sigma^2$
10. 22.9^0, 3.27^0
11. 12; 4, 3
12. 5.10
13. 5
14. 3.74
15. 14.4, 1.68
16. 11, 14

17. $\bar{x} = 6.4, \sigma = 2.42$

a. $\bar{x} = 6.4 + 3 = 9.4, \sigma = 2.42$ b. $\bar{x} = 3 \times 6.4 = 19.2, \sigma = 3 \times 2.42 = 7.25$

c. $\bar{x} = 0.9 \times 6.4 = 5.76, \sigma = 0.9 \times 2.42 = 2.17$

$\bar{x} = 1.05 \times 6.4 = 6.72, \sigma = 1.05 \times 2.42 = 2.54$

18. 19

19. 6, 4

9.9 Additional activities

9.9.1 Remedial activities

In test of chemistry , 17 students got the following marks out of 17:

6,7,6,7,8,9,5,4, 8, 5, 7, 6, 6, 9, 4, 8,8.

b. Calculate the mean, mode, median, range and quartiles and interquartile range

c. Calculate the variance and standard deviation

d. Calculate the coefficient of variation.

9.9.2 Consolidation activities

The six runners in a 200 meters race clocked times (in seconds) of 24.2, 23.7, 25.0, 23.7, 24.0,4.6

a) Find the mean and standard deviation of these times.

b) These readings were found to be 10% too low due to faulty timekeeping. Write down the new mean and standard deviation.

c) Draw bar graph of the above information

Solution

a) $\bar{x} = \dfrac{24.2 + 23.7 + 25.0 + 23.7 + 24.0 + 24.6}{6} = 24.2 \text{ seconds}$

$$\sigma = \sqrt{\frac{(24.2-24.2)^2 + (23.7-24.2)^2 + (25.0-24.2)^2 + (23.7-24.2)^2 + (24.0-24.2)^2 + (24.6-24.2)^2}{6}}$$

$= 0.473 \text{ seconds}$

b) We must divide each term 0.9 to find the correct time. The new mean is $\bar{x} = \dfrac{24.2}{0.9} = 26.9 \text{ sec}$.

The new standard deviation is $\sigma = \dfrac{0.4726}{0.9} = 0.525 \text{ sec}$

9.9.3 Extended activities

Hospital records indicated that maternity patients stayed in the hospital for the number of days shown in the distribution:

Number of days stayed	**Frequency**
3	15
4	32
5	56
6	19
7	5
Total	**127**

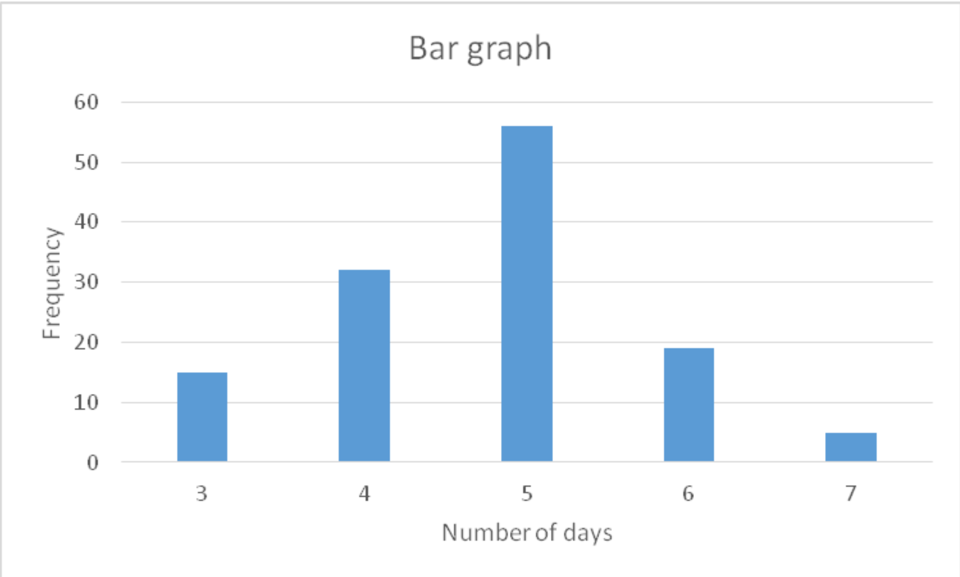
Construct

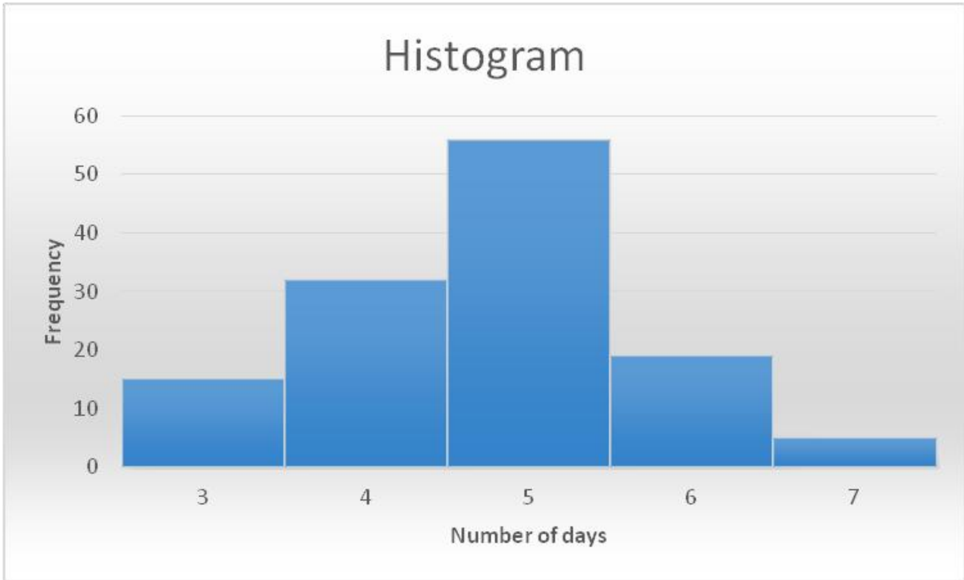
i) Bar graph

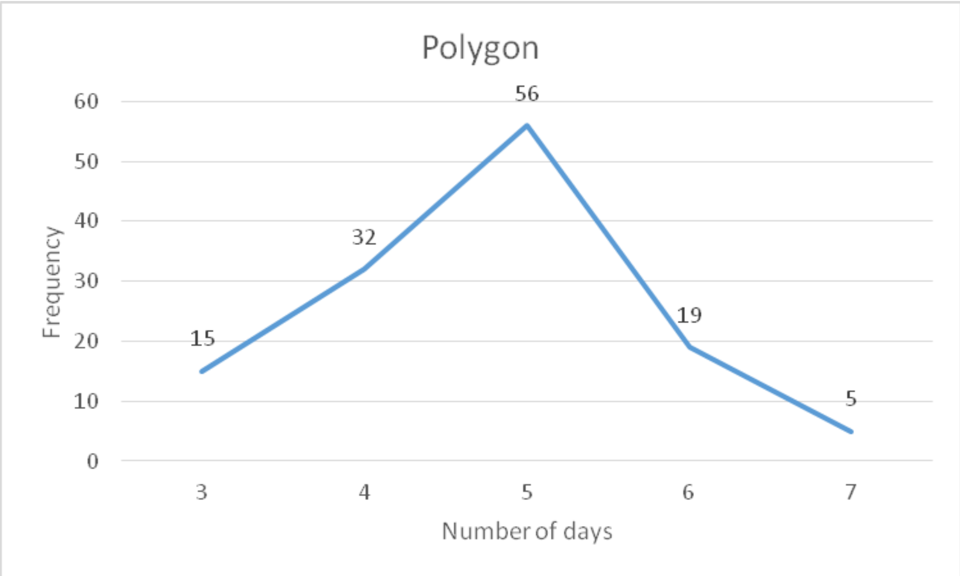
ii) Histogram

iii) Polygon

Solutions




Histogram
Frequency
60
50
40
30
20
10
0
3
4
5
6
7
Number of days


Polygon
Frequency
60
50
40
30
20
10
0
56
32
15
19
5
3
4
5
6
7
Number of days

UNIT 10: ELEMENTARY PROBABILITY

10.1 Key unit competence

Use combinations and permutations to determine probabilities of occurrence of an event

10.2 Prerequisite

Students will perform well in this unit if they make a short revision on the elementary probability learnt in S2 unit 11.

10.3 Cross-cutting issues to be addressed

- Inclusive education (promote education for all while teaching)
- Peace and value Education (respect others view and thoughts during class discussions)
- Gender (equal opportunity of boys and girls in the lesson participation and making groups)

10.4 Guidance on introductory activity 10.0

- Form groups of students and invite students to work on questions for introductory activity 10.0 found in student's book unit 10;
- Guide students to read and analyse the problem related different cases of the gender that 3 children can have: they have to write all those cases on a sheet of paper;
- Guide students to find out many possibilities as they can;
- Invite students with different working steps to present their findings to the whole class discussion;
- Basing on their experience, prior knowledge and abilities shown in answering the questions for this activity , use different questions to guide students to give their predictions and ensure that you arouse their curiosity on what is going to be leant in this unit.

Answer for introductory activity 10.0

1. Many answers may be found by students

2. To find all possible roads, students can use allows to join points or a try and fail method.

$\Omega = \{AB_1C_1, AB_1C_2, AB_1C_3, AB_2C_1, AB_2C_2, AB_2C_3,\}$; There are 6 possibilities.

3. a) There are 25 black cards in an ordinary deck of 52 cards.

b) $P(A) = \frac{n}{number\ of\ all\ cards} = \frac{26}{52} = 0.5$

c) $P(\text{A}) = \dfrac{number\ of\ outcomes\ in\ E}{Total\ number\ of\ outcomes\ in\ the\ Sample\ Space} = \dfrac{n(E)}{n(\Omega)}$

10.5. List of lessons

No	Lesson title	Learning objectives	Number of periods
0	Introductory activity	Arouse the curiosity of stents on the content of unit 10.	1
1.	Simple counting techniques	Use different counting techniques (Venn diagram, tree diagram and tables) to determine the number of possibilities or occurrence outcomes for an event.	2
2.	Arrangements	Define factorial notation and determine the number of different Arrangements of *n* different objects (unlike objects) in a row.	2
3.	Permutations	Determine the number of different permutations of n indistinguishable objects with some alike objects.	3
4	Combinations	Determine the number of different groups of *r* items that could be formed from a set of *n* distinct objects where the order of selections is being ignored.	2
5.	Binomial expansion and Pascal's triangles	Apply Pascal's triangle to complete a binomial expansion	3
6.	Concepts of probability: Event, Random experiment, Sample space	Determine the probability of an events in real life as a measure of chance	3
7	Properties and formulas	Use Properties and formulas of	1

	of probability.	probability to determine the probability of an event in real life as a measure of chance.	
8	Examples of Events in real life and determination of related probability	Appreciate the importance of probability in real life.	3
End assessment			1

Lesson 1: Simple counting techniques

a) Learning objective

Use different counting techniques (Venn diagram, tree diagram and tables) to determine the number of possibilities or occurrence outcomes for an event.

b) Teaching resources:

Graph papers, manila papers, calculators, coins, dice.

c) Prerequisites/Revision/Introduction:

Student will perform well in this lesson if they make a good revision on the introduction to probability learnt in S2 and the previous lesson.

d) Learning activities

- Form small groups of students and give them instructions on how to work on the activity 1
- Walk around to each group and ask probing questions leading them to determine the total number of roads from A to C via B;
- Ask each group to share their answers with neighbouring group and ask them to support each other for improvement;
- Invite groups with different working steps to present their findings to the whole class for discussion;
- As a teacher, harmonize their answers highlighting that there is a technique of finding the total number of outcomes for a given random experiment;
- Use different probing questions and guide them to explore examples given in the student's book and lead them to determine total number of outcomes for a given random experiment using: Venn diagram, tree diagram or a table.

- Guide them to discover that if a sequence of n events in which the first one has n_1 possibilities, the second with n_2 possibilities the third with n_3 possibilities, and so forth until n_k, the total number of possibilities of the sequence will be $= n_1.n_2.n_3...n_k$
- After this step, guide students to do the **Exercise 1** and evaluate whether lesson objectives were achieved.

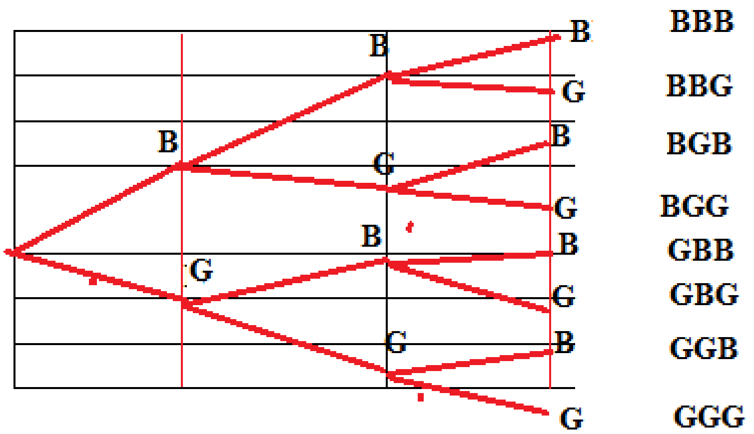
Answers for activity 1

To find all possible roads, students can use allows to join points or a try and fail method.

$\Omega = \{AB_1C_1, AB_1C_2, AB_1C_3, AB_2C_1, AB_2C_2, AB_2C_3\}$ so they are 6.

e) Answers for Exercise 1

1) Using the tree diagram, one can find:



$S = \{BBB, BBG, BGB, BGG, GBB, GBG, GGB, GGG\}$

2) The coin can land either head up or tails up.

	1	2	3	4	5	6
H	H1	H2	H3	H4	H5	H6
T	T1	T2	T4	T4	T5	T6

There are 2.6 = 12 possibilities.

Lesson 2: Arrangements of n unlike objects

a) Learning objective:

Define factorial notation and determine the number of different arrangements of *n* different objects (unlike objects) in a row.

b) Teaching resources:

Manila papers, cards with letters, calculators, coins, dice, a bench, etc.

c) Prerequisites/Revision/Introduction:

Students will perform well in this lesson if they refer to the content of the previous lesson and the properties for multiplication learnt in unit 1 for this level.

d) Learning activities

- Form small groups of students and give them instructions on how to work on the **activity2:** give each group the letter cards to be used and ask them to make all possible arrangements and permutations of those letters (for example: letter R, E and B);
- Walk around to each group and ask probing questions leading them to determine the total number of ways starting by the number of ways to chose the first letter, the second letter and the third letter;
- Ask each group to share their answers with neighbouring group and ask them to support each other for improvement;
- Invite groups with different working steps to present their findings to the whole class for discussion;
- As a teacher, harmonize their answers highlighting that the arrangement of letters is the same as ways of sitting different people on the same bench and that a permutation is an arrangement of n objects in a specific order.
- Use different probing questions and guide them to explore examples given in the student's book and lead them to discover the formula which gives the number of different permutations of *n* different objects (unlike objects) in a row.
- Guide them to discover that this number corresponds to $n!$(read *n* factorial) and explore the related properties.
- After this step, guide students to do the **Exercise 2** and evaluate whether lesson objectives were achieved.

Answers for activity 2

Possible arrangements for three letters R, E and B are $\{REB, \text{R}\,BE, ERB, EBR, BER, BRE\}$;

The possible arrangement for these three letters is 6. This can be found by: $3! = 3.2.1 = 6$

Answers for Exercise 2

1) a) $\frac{5!}{2!} = \frac{5\times4\times3\times2\times1}{2\times1} = 60$ **; b)** $\frac{10!}{6!7!} = \frac{10\times9\times8\times7!}{6\times5\times4\times3\times2\times1\times7!} = \frac{5\times2\times3\times3\times4\times2}{3\times2\times5\times4\times3\times2\times1} = 1$

2) Four different English books, five different Biology books and ten other books are to be arranged on a bookshelf:

a) (4+5+10)!= 19!= 1.216451004×10^{17};

b) Since the 3 biology books have to be together, consider these bound together as one book, there are now $(16+1)! = 17!$ books to be arranged and these can be calculated using a calculator and find 2.134124569×10^{15}.

Lesson 3: Permutation of distinguishable objects

a) Learning objective:

Determine the number of different permutations (ways) of *r* unlike objects selected from *n* different objects.

b) Teaching resources:

Manila papers, calculators, coins, dice.

c) Prerequisites/Revision/Introduction:

Students will perform well if they learnt well the previous lessons of this unit.

d) Learning activities

- Form groups of students and give them instructions on how to work on the **activity 3**: give each group the letter cards to be used and ask them to make all possible ways of selecting 3 letters from 7 given letters of the word *PRODUCT*.
- Walk around to each group and ask probing questions leading them to determine the total number of ways of selecting the first letter, the second letter and the third letter;
- Invite groups with different working steps to present their findings to the whole class for discussion;
- As a teacher, harmonize their answers highlighting that the number of ways for selecting 3 letters from 7 is the number of permutations of 3 objects taken from 7 written as ${}^{7}P_{3}$.

Guide them to enhance that the first letter can be selected in 7 ways, the second can be selected in 6 ways and the third can be selected in 5 ways;

- Ask all students to guess how they can write the product $7.6.5$ using the factorial notation which lead them to guess $7.6.5=\frac{7.6..5.4.3.2.1}{4.3.2.1}=\frac{7!}{(7-3)!}$;
- Use different probing questions and guide students to explore examples given in the student's book and lead them to discover the formula which gives the number of different permutations (ways) of *r* unlike objects selected from *n* different objects given by ${}^{n}P_{r}=\frac{n!}{(n-r)!}$ which can also be written as $P(n,r)=\frac{n!}{(n-r)!}$;
- After this step, guide students to do the **exercise 3** and evaluate whether lesson objectives were achieved.

Answers for activity 3

Students will work in different ways; as a tutor, verify whether all possible ways are given:

Selection: PRO: permutations: PRO,POR, OPR, ORP, RPO, ROP

Selection: ROD: permutations: ROD, RDO, ORD, ODR, DRO, DOR

Selection: ODU: permutations: ..., ..., ..., ..., ..., ...

Selection: DUC: Permutations: ..., ..., ..., ..., ..., ...

Selection: UCT: Permutations: ..., ..., ..., ..., ..., ...

.... : .....: ....

There are 35 lines with 6 permutations which gives the number $7.6.5=210$ permutations.

e) Answer for the Exercise 3

1) Number of permutations with 4 letters chosen from letters of the word ENGLISH:

${}^{7}P_{4}=840$

2) Number of permutations with 2 letters chosen from letters of the word PACIFIC: 13

3) Number of permutations with 5 letters chosen from letters A, B, C, D, E, F, and G is ${}^{7}P_{5}$.

4) Number of permutations with 10 letters chosen from English alphabet is ${}^{26}P_{10}$.

Lesson 4: Combinations

a) Learning objective:

Determine the number of different groups of *r* items that could be formed from a set of *n* distinct objects where the order of selections is being ignored.

b) Teaching resources:

Manila papers, calculators, coins, dice.

c) Prerequisites/Revision/Introduction:

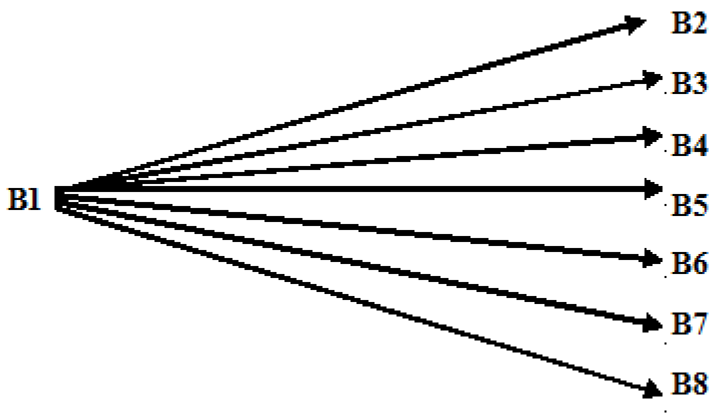
Students will perform well if they learnt well the content for the previous lessons for this level.

d) Learning activities

- Form groups of students and give them instructions on how to work on the **activity 7**
- Walk around to each group and ask probing questions leading them to determine the total number of groups each containing 2 mathematics books from 8 mathematics books;
- Invite groups with different working steps to present their findings to the whole class for discussion;
- As a teacher, harmonize their answers highlighting that in this case, the order in which books are placed is not important (the group of B1B2 is the same as the group B2B1) which is contrary to the permutation of *r* unlike objects selected from *n* different objects where the order in which those objects are placed is important.
- Lead students to see that in this case, we must divide by the 2! (or generally by the arrangement r!) as the order is not important; we get $\frac{7.8}{2}=\frac{8!}{(8-2)!2!}$.
- Use different probing questions and guide students to explore examples given in the student's book and lead them to discover the formula which gives the number of different groups of *r* items that could be formed from a set of *n* distinct objects with the order of selections being ignored is ${}^{n}C_{r}=\frac{n!}{(n-r)!r!}$
- After this step, guide students to do the **exercise 7** and evaluate whether lesson objectives were achieved.

Answers for activity 7

The book B1 can be participating in 7 different groups as follows:



Idem, every book Bi can participate in 7 different groups. This means that we have $8.7 = 56$ groups. However, as for example the group B2B3 and the group B3B2 make a same group, we have to divide by 2.

Which gives $\frac{7.8}{2}$ groups.

By the use of factorial notation we have: $\frac{7.8}{2} = \frac{8!}{(8-2)!2!} = 28$

Answers for exercise 7

1) Four men can be selected from 10 men, i.e $^{10}C_4 = \frac{10!}{(10-4)!4!}$ ways

Two women can be selected from 12women, i.e $^{12}C_2 = \frac{12!}{(12-2)!2!}$ ways

By the basic product principle of counting, there are $\left(^{10}C_4\right)\left(^{12}C_2\right)$ ways of selecting the committee.

2) In the same ways, groups containing 4 Mathematics books and 5 Physics books is formed from 9 Mathematics books and 10 Physics books in $^{9}C_4 \times {}^{10}C_5$ ways.

Lesson 5: Binomial expansion and Pascal's triangles

a) Learning objective:

Apply Pascal's triangle to complete a binomial expansion in mathematics expressions**.**

b) Teaching resources:

Manila papers, calculators, notebooks and pens.

c) Prerequisites/Revision/Introduction:

Students will perform well if they learnt well the binomial expansion and properties of powers learnt in S2.

d) Learning activities

- In small group discussions, invite students to answer the questions in **activity 8**
- Ask students to share their answers with another group and ask them to support each other where they became more challenged in solving that activity.
- Request students to present their findings in a whole class discussion.
- As a teacher, harmonize answers presented by students and guide theme to determine the coefficients of powers in a binomial expansion.

Use different probing questions and guide students to explore examples given in the student's book and lead them to discover the coefficient of $a^{n-r}b^r$ in the expansion of $(a+b)^n$ is given by ${}^nC_r = \frac{n!}{(n-r)!r!}$.

Answers for activity 8

Use these expansions and complete the table

$(a+b)^2 = a^2 + 2ab + b^2$

$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$

$(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$

$(a+b)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$

Power	Coefficient of powers of a and b					Binomial expression
0	1					$(a+b)^0$
1	1	1				$(a+b)^1$
2	1	2	1			$(a+b)^2$
3	1	3	3	1		$(a+b)^3$
4	1	4	6	4	1	$(a+b)^4$

It is clear that the coefficients of $a^{n-r}b^r$ in the expansion of $(a+b)^n$ are given by $^nC_r = \frac{n!}{(n-r)!r!}$.

Answers for Exercise 8:

1) The coefficient of x^2 in the expansion of $(4x+1)^6$ is **240**

2) The coefficient of x^3 in the expansion of $\left(x+\frac{1}{x}\right)^4$ is **0**

3) The coefficient of x^6 in the expansion of $(9x-3)^{10}$ is **9039811410**

Lesson 6: Determination of Probability of an event

a) Learning objective:

Determine the probability for outcomes of an event in real life as a measure of chance.

b) Teaching resources:

Manila papers, calculators, coins, dice, etc.

c) Prerequisites/Revision/Introduction:

Students will perform well if they learnt well all previous lessons for this unity and the introduction to probability learnt in S1 and S2.

d) Learning activities

- Let students work in groups and discuss to the **activity 9**;

- Go around to each group and ask probing questions to guide students to work towards the correct answer;
- Ask neighbouring groups to share their answers and ask them to support each other where they become more challenged in solving that activity.
- Request students to present their findings in a whole class discussion;
- As a teacher, harmonize answers for students and highlight how to determine the probability of an event using the classical probability.
- Use different probing questions and guide students to explore examples given in the student's book and lead them to establish and use properties of probability, determine probability of different events: certain event, impossible event, probability of complementary event, mutually exclusive or incompatible events.
- After this step, guide students to do the **Exercise 9** and evaluate whether lesson objectives were achieved.

Answers for activity 9

a) There are 25 black cards in an ordinary deck of 52 cards.

b) $P(\text{A}) = \dfrac{n}{number\ of\ allcards} = \dfrac{26}{52} = 0.5$

c) $P(A) = \dfrac{Number\ of\ outcomes\ \text{in} \text{E}}{Total\ \text{nu}mber\ of\ outcomes\ \text{in}\ the\ sample\ space} = \dfrac{n(\text{E})}{n(\Omega)}$

Answers for Exercise 9:

1) $P(A \cup B) = \dfrac{1}{5} + \dfrac{1}{3} = \dfrac{8}{15}$

2) $P(A \cup B) = \dfrac{1}{3} + x = \dfrac{7}{10} \Rightarrow x = \dfrac{11}{30}$

3) a) $\dfrac{3}{8}$; **b)** $\dfrac{5}{8}$; **c)** $\dfrac{1}{32}$

Lesson 7: Properties and formulas of probability

a) Learning objective:

Use Properties and formulas of probability to determine the probability of an events in real life as a measure of chance.

b) Teaching resources:

Manila papers, calculators, coins, dice, etc.

c) Prerequisites/Revision/Introduction:

Students will perform well if they learnt well all previous lessons for this unity and the introduction to probability learnt in S1 and S2.

d) Learning activities

- Let students work in groups and discuss on the **activity 10**
- Go around to each group and ask probing questions to guide students to work towards the correct answer;
- Ask neighbouring groups to share their answers and ask them to support each other where they become more challenged in solving these activities.
- Request students to present their findings ;
- As a teacher, harmonize answers for students and highlight how to determine the probability of an event using the properties and formulas of probability.
- Use different probing questions and guide students to explore examples given in the student's book and lead them to establish and use properties of probability: Sum law, mutually exclusive and exhaustive events.
- After this step, guide students to do the **Exercise 10,** then evaluate whether lesson objectives were achieved.
- Do the same procedure above on sum law of probability stating by the **activity 11,** then guide students to do the **Exercise 11**
- Again do the same procedure above on mutually exclusive events stating by the **activity 12,** then guide students to do the **Exercise 12**
- Again do the same procedure above on exhaustive events stating by the **activity 13,** then guide students to do the **Exercise 13**

Answers for activity 10

a. 11

b. 4, $\frac{4}{11}$

c. 7, $\frac{7}{11}$

d. i. Empty set

ii. $\{O, A, I, I, P, R, B, B, L, T, Y\}$

iii. $\{P, R, B, B, L, T, Y\}$

iv. $\{O, A, I, I\}$

Answers of exercise 10

1. a) $\frac{2}{11}$ b) $\frac{2}{11}$

2. a) $\frac{13}{19}$ b) $\frac{3}{19}$ c) $\frac{3}{19}$

Answers of activity 11

a. $\frac{2}{3}$

b. $\frac{2}{3}$

The two results are the same.

Answers of exercise 11.

1. $\frac{5}{6}$

2. $\frac{1}{2}$

Answers of activity 12

a. $\frac{1}{3}$

b. $\frac{2}{3}$

c. 0

d. 1

e. 1

Answers of exercise 12

1. $\frac{8}{15}$

2. $\frac{11}{30}$

Answers of activity 13

a. 1

b. 1

The two results are the same.

Answers of exercise 13

a) $\frac{3}{8}$

b) $\frac{5}{8}$

c) $\frac{1}{32}$

Lesson 8: Examples of Events in real life and determination of related probability

a) Learning objective:

Appreciate the importance of probability in social sciences

b) Teaching resources:

Manila papers, calculators, coins, dice, etc.

c) Prerequisites/Revision/Introduction:

Students will perform well in this lessons if they try to relate the content for all previous lessons for this units to real life problems involving events of chance.

d) Learning activities

- Invite students to work in small groups, discuss the betting explained in the **activity 8A** and answer to related questions;
- Ask neighbouring groups to share their answers and support each other by exchanging arguments on that activity;
- Invite group representatives to resent the findings in a whole class discussions;

- ➢ Teacher harmonizes answers for students on **activity 8A** and guide students to brainstorm their worries about the betting without a good prediction of probability for winning.
- ➢ Guide students to discuss other application of probability in real life and take decisions on eventual risks in betting and other probability related games.
- ➢ Use different probing questions and guide students to explore examples given in the student's book and lead them to realize that probability is applicable in the real life.
- ➢ After this step, guide students to do the **exercise 8A** and evaluate whether lesson objectives were achieved.

Activity 8A and its solutions

1.Two football teams in Rwanda " Rayon Sport" and "APR FC" had to play 3 matches. Two boys Matayo and Manasseh made a betting in the following ways in which the winner should be given 400,000Frw when his event succeeds.

Matayo said that APR will gain the first match only and Rayon Sport will gain the second and the third. Manasseh said that APR will gain at least two matches.

a) Between Matayo and Manasseh, discuss and determine the boy who has more chances of winning that money.

b) Is there any risk in betting? Referring to the results obtained in a) what are the points of advice you can give to the youth who spend their money in betting?

2. Carry out a research in the library or on internet to find other applications of probability in real life and present them in the classroom discussion.

Solutions for activity 8A:

1) (a) Let A stands for APR and R stands for RAYON SPORTS; $\Omega = \{RRR, RRA, RAR, RAA, ARR, ARA, AAR, AAA\}$.

Matayo said that APR will gain the first Match only and Rayon Sport will gain the second and the third, it means that $= E = \{ARR\}$ *and* $P(E) = \frac{1}{8}$

Manasseh said that APR will gain at least two matches, it means that: $F = \{RAA, ARA, AAR, AAA\}$ *and* $P(F) = \frac{4}{8} = \frac{1}{2}$

From these results, we see that Manasseh has more chances of winning that money than Matayo.

b) Normally betting is a game of chance, it is not good to bet much money without a good and clear prediction of the probability for winning. When you bet without such clear prediction, you are wasting your money. We can advise the young people not to spend much money in such games which do not have clear rules which can help the player to predict the probability of winning.

2) Students may come up with different applications of probability in real life, analyse them and organize a session for feedback in which they can discuss their strengths and weaknesses.

Exercise 8A and its solutions:

1. In one state of America, the probability that a student owns a car is 0,65, and the probability that a student owns a computer is 0.82. If the probability that a student owns both is 0.55, what is the probability that a given student owns neither a car nor a computer?

2. At a particular school with 200 male students, 58 play football, 40 play basketball, and 8 play both. What is the probability that a randomly selected male student plays neither sport?

Solutions for exercise 8A:

1) Let E: a student own a car, $P(\text{E}) = 0.65$, F: a student owns a computer; $P(\text{F}) = 0.82$

We have $P(\text{E} \cap \text{F}) = 0.55$

Question: what is the probability that a given student owns neither a car nor a computer?

i.e $1 - P(\text{E} \cup \text{F}) = ?$

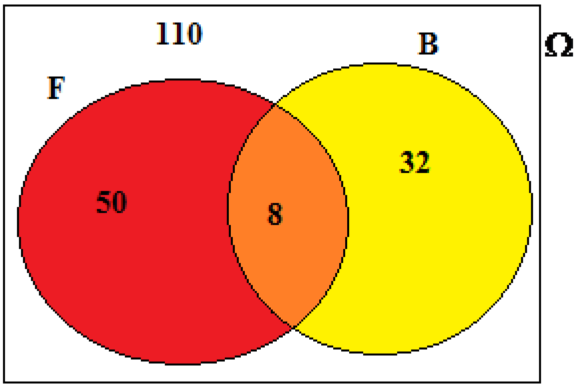
We have:

$$\begin{aligned} P(\text{E} \cup \text{F}) &= P(E) + P(E) - P(E \cup F) \\ &= 0.65 + 0.82 - 0.55 \\ &= 0.92 \end{aligned}$$

Therefore, $1 - P(\text{E} \cup \text{F}) = 1 - 0.92 = 0.08$

2) Using a Venn diagram, one can represent the number of students:

Let Ω be the sample space formed by all students, F the set of students who play Football and B the set of students who play Basket Ball. The number of students can be given in the following sets.



$n(\Omega)=200$, $n(\mathrm{F})=58$, $n(\mathrm{B})=40$, and $n(\mathrm{F}\cap\mathrm{B})=8$.

The number of students who plays Foot Ball or basket Ball is $n(\mathrm{F}\cup\mathrm{B})=90$

The number of student who play neither sport is

$$n(\mathrm{F}\cup\mathrm{B})'=n(\Omega)-n\left(F\cup B\right)$$
$$=200-90=110$$

10.6. Summary of the unit

Sample space

The totality of all possible outcome (or sample points) of a random experiment constitutes the sample space which is denoted by Ω.

Complementary events

If E is an event, then E' is the event which occurs when E does not occur. Events E and E' are said to be **complementary events**

Mutually exclusive Events

When $A\cap B=\varnothing$, the two events A and B are said to be **mutually exclusive**. This means that they cannot occur at the same time, they do not have outcomes in common.

Counting techniques

- Use of Venn diagram,

- Use of tree diagrams,

- Use of a table,

- The number of different permutations of *n* different objects (unlike objects) in a row is

$n\times(n-1)\times(n-2)\times(n-3)\times...\times 2\times 1$

- *The number of different permutations of n i**ndistinguishable** objects with* n_1 *alike,* n_2 *alike, ..., is* $\frac{n!}{n_1!n_2!...}$.

- *The number of different permutations (ways) of r unlike objects selected from n different objects is*

$^nP_r=\frac{n!}{(n-r)!}$ *or we can use the denotation* $P_r^n=\frac{n!}{(n-r)!}$ *or* $P(n,r)=\frac{n!}{(n-r)!}$

- Circular arrangements

The number of arrangements of n unlike things in a circle will therefore be $(n-1)!$. *In those cases where clockwise and anticlockwise arrangements are not considered to be different, this reduces to* $\frac{1}{2}(n-1)!$.

- Mutually exclusive situations Mutually exclusive situations

"*If Experiment* 1 *has* **m** *possible outcomes*
and if experiment 2 *has* **n** *possible outcomes,*
then an experiment which might be experiment 1 *or experiment* 2,
called **experiment 1 or 2**, *has* $(m+n)$ *possible outcomes.*"

- Combination

The number of different groups of r items that could be formed from a set of n distinct objects with the order of selections being ignored is

$$^nC_r=\frac{n!}{(n-r)!r!}.$$

Probability of an event

- The probability of an event $A \subset \Omega$, is a real number obtained by applying to A the function P defined by

$$P(A) = \frac{Number\ of\ outcomes\ in\ A}{Total\ number\ of\ outcomes\ in\ the\ sample\ space} = \frac{n(A)}{n(\Omega)}$$

- When E and E' are complementary events, $P(E) = 1 - P(E')$.

- *When two events A and B are not mutually exclusive,* $A \cap B = \phi$ *the probability that A or B occurs is given by:*

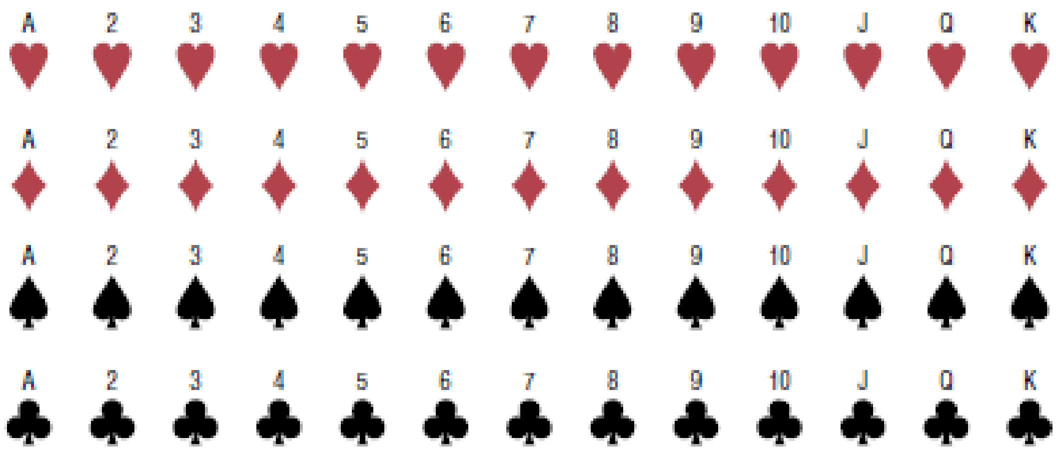
$$P(A \cup B) = \mathrm{P(A)} + \mathrm{P(B)} - \mathrm{P}(A \cap B)$$

Note:

We have to think big before taking decision regarding our engagement in the games of chance. Such games are for example: betting on card games, slot machine (ikiryabarezi), lotteries, and weather forecasting. In such games, predictions are based on probability and hypotheses are tested by using probability.

10.7 Additional Information for teachers

10.7.1 Components of an ordinary deck of cards:



10.7.2 Law of Large Numbers

When a coin is tossed one time, it is common knowledge that the probability of getting a head is $\frac{1}{2}$. But what happens when the coin is tossed 50 times? Will it come up heads 25 times? Not all the time. You should expect about 25 heads if the coin is fair. But due to chance variation, 25 heads will not occur most of the time.

If the empirical probability of getting a head is computed by using a small number of trials, it is usually not exactly $\frac{1}{2}$. However, as the number of trials increases, the empirical probability of getting a head will approach the theoretical probability of $\frac{1}{2}$, if in fact the coin is fair (i.e., balanced). This phenomenon is an example of the law of large numbers.

10.7.3 Independent events

Two events A and B are **independent events** if the fact that A occurs does not affect the probability of B occurring.

When two events are independent, the probability of both occurring is

$\text{P(A and B)} = P(A).P(B)$

Example:

A coin is flipped and a die is rolled. Find the probability of getting a head on the coin and a 4 on the die.

Solution:

P(Head and 4)= P(head).P(4)= $\frac{1}{2}.\frac{1}{4} = \frac{1}{8}$

10.8 Answers of Revision Exercise

1. a. 144 b. 72

2. 216, 72, 108

3. 18

4. 10!

5. 8!

6. 24

7. a. 210 b. $\frac{1}{110}$ c. 600 d.81

8. a. $n-1$ b. $(n+2)(n+1)$ c. $n(n^2+3n+1)$

9. 5040

10. 168

11. a. 240 b. 600

12. a. 81 b. 256

13. 36

14. 2522520

15. 151 a. 73 b. 78 c. 13 d. 138

16. a. 462 b. 56 c. 20

17. 1260

18. a. 5 b. 85 c. 365

19. a. 126 b. 280

20. 480, 172800, 462, 425

21. a. $27+27x+9x^2+x^3$ b. $125+150x+60x^2+8x^3$ c. $16+32x+24x^2+8x^3+x^4$

d. $16-32x+24x^2-8x^3+x^4$ e. $32y^5+80y^4x+80y^3x^2+40y^2x^3+10yx^4+x^5$

f. $32x^5-240x^4y+720x^3y^2-1080x^2y^3+810xy^4-243y^5$

g. $x^4-4x^2+6-\frac{4}{x^2}+\frac{1}{x^4}$ h. $x^5-10x^3+40x-\frac{80}{x}+\frac{80}{x^3}-\frac{32}{x^5}$

22. $1+12x+66x^2+220x^3$

23. $64x^5+160x^{-1}+20x^{-7}$

24. 0, 1 (trivial) and 6

25. 2

26. 30.43168

27. $a^{10}-30a^9x+405a^8x^2-3240a^7x^3$

28. $1+10x+\frac{95}{2}x^2+\frac{285}{2}x^3+\frac{4845}{16}x^4$

29. 16, $\frac{1}{8}$

30. a. ${}^{10}C_5\times 3^5$ b. ${}^{12}C_8\times 4^{10}$ c. ${}^{6}C_4\times 3^2\times 2^4$ d. $2^4\left(2\times {}^{10}C_5+{}^{10}C_6\right)$

31. a. 1.0937 b. 0.9860837 c.0.9044 d. 973.9

32. a. 0.97980 b. 10.1980 c. 2.0199 d. 1.01943 e. 2.05828

33. $1-\frac{3}{2}x+\frac{15}{8}x^2-\frac{51}{16}x^3, |x|<\frac{1}{2}$

34. 2, -9, 29, -82

35. a. ${}^8C_3\times5^5\times3^3$ b. $-{}^7C_3\times7^4\times2^3$

36. a. 560 b. -590625 c. -720 d. -448 e. 1966080 f. $-\frac{7}{144}$

37. $\pm\frac{2}{3}$

38. ${}^{12}C_8\times4^4\times x^8$

39. -20

$\frac{3}{2}, \frac{1}{2}, \frac{1970}{1393}$

10.9 Additional activities

10.9.1 Remedial activity

1) A card is drawn at random from an ordinary pack of 52 playing cards. Find the probability that the card drawn is a club or a diamond.

Solution

There are 13 clubs, then $P(\text{club})=\frac{13}{52}$

There are 13 diamonds, then $P(\text{diamond})=\frac{13}{52}$

Since a card cannot be both a club and a diamond, $P(\text{club}\cap\text{diamond})=0$

Therefore, P(a club or a diamond) = P(club) + P(diamond)

$=\frac{13}{52}+\frac{13}{52}$ $=\frac{13}{52}+\frac{13}{52}$ $=\frac{26}{52}$ $=\frac{1}{2}$

10.9.2 Consolidation activity

1) In a group of 20 adults, 4 out of the 7 women and 2 out of the 13 men wear glasses. What is the probability that a person chosen at random from the group is a woman or someone who wears glasses?

Solution

Let A be the event: "the person chosen is a woman".

B be the event: "the person chosen wears glasses".

Now, there are 7 women, then $P(A)=\frac{7}{20}$

There are 6 persons who wear glasses, then $P(B)=\frac{6}{20}$

There are 4 women who wear glasses, then $P(A\cap B)=\frac{4}{20}$

The probability that a person chosen at random from the group is a woman or someone who wears glasses is given by $P(A\, or\, B)$ which is

$$\begin{aligned}P(A\cup B)&=P(A)+P(B)-P(A\cap B)\\&=\frac{7}{20}+\frac{6}{20}-\frac{4}{20}\\&=\frac{9}{20}\end{aligned}$$

On the other hand:

There are 7 women and 6 persons who wear glasses but also 4 women who wear glasses. Then

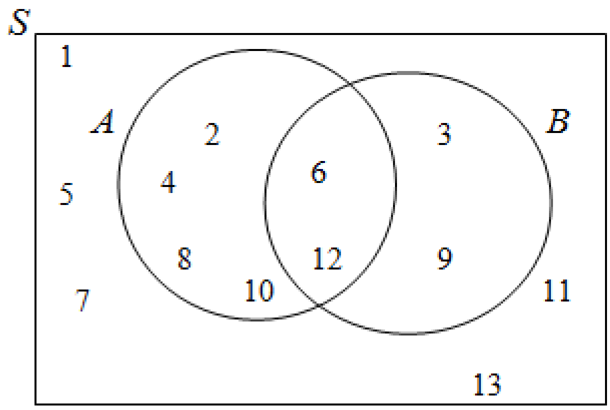
$A\cup B=9$ and $P(A\cup B)=\frac{9}{20}$.

10.9.3 Extended activity

1) An integer is chosen at random from the set $S=\{x:x\in ¢^{+},x<14\}$. Let A be the event of choosing a multiple of 2 and let B be the event of choosing a multiple of 3.

Find $P(A\cup B)$, $P(A\cap B)$ and $P(A-B)$.

Solution



From the diagram, $\#S = 13$

$A \cup B = \{2,3,4,6,8,9,10,12\} \Rightarrow \#(A \cup B) = 8$, thus $P(A \cup B) = \frac{8}{13}$

$A \cap B = \{6,12\} \Rightarrow \#(A \cap B) = 2$, thus $P(A \cap B) = \frac{2}{13}$

$A - B = \{2,4,8,10\} \Rightarrow \#(A - B) = 4$, thus $P(A - B) = \frac{4}{13}$

2) Suppose, for example, that a researcher in RAB asked 50 staff members how they go home.

The results can be categorized in a frequency distribution as shown in the table below

Method	**Frequency**
drive	20
Fly	6
Bus	24

Determine:

a) The probability of selecting a person who goes home by driving;

b) Probability of selecting a person who goes home in an air plane;

c) The probability of selecting a person who goes home in a bus.

d) The sum of the probability.

3) In a sample of 50 people, 21 had type O blood, 22 had type A blood, 5 had type B blood, and 2 had type AB blood. Set up a frequency distribution and find the following probabilities.

a) A person has type O blood.

b) A person has type A or type B blood.

c) A person has neither type A nor type O blood.

d) A person does not have type AB blood.

Solution

Type	**Frequency**
A	22
B	5
AB	2
O	21
Total	50

They are mutually exclusive.

a) $P(O)=\frac{f}{n}=\frac{21}{50}$

b) $P(\text{A or B})=\frac{22}{50}+\frac{5}{50}=\frac{27}{50}$

c) Neither A nor O means that a person has either type B or type AB blood.)

$$P(\text{neither A nor O})=\frac{5}{50}+\frac{2}{50}=\frac{7}{50}.$$

d) Find the probability of not AB by subtracting the probability of type AB from 1.

$$P(\text{not AB})=1-P(AB)=1-\frac{2}{50}=\frac{48}{50}=\frac{24}{25}$$

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