ADVANCED MATHEMATICS

SENIOR FOUR

S4

Teacher's Guide

FOREWORD

Dear Teachers,

Rwanda Basic Education Board is honoured to present the teacher's guide for S4 Mathematics to be used in options where Core Mathematics is a major subject. This book serves as a guide to competence-based teaching and learning to ensure consistency and coherence in the learning of the mathematics content. The Rwandan educational philosophy is to ensure that learners achieve full potential at every level of education which will prepare them to be well integrated in society and exploit employment opportunities.

In line with efforts to improve the quality of education, the government of Rwanda emphasizes the importance of aligning teaching and learning materials with the syllabus to facilitate their learning process. Many factors influence what they learn, how well they learn and the competences they acquire. Those factors include the relevance of the specific content, the quality of teachers' pedagogical approaches, the assessment strategies and the instructional materials.

The ambition to develop a knowledge-based society and the growth of regional and global competition in the jobs market has necessitated the shift to a competence-based curriculum. This book provides active teaching and learning techniques that engage student teachers to develop competences.

In view of this, your role is to:

- Plan your lessons and prepare appropriate teaching materials.
- Organize group discussions for students considering the importance of social constructivism suggesting that learning occurs more effectively when the students works collaboratively with more knowledgeable and experienced people.
- Engage students through active learning methods such as inquiry methods, group discussions, research, investigative activities and group and individual work activities.
- Provide supervised opportunities for students to develop different competences by giving tasks which enhance critical thinking, problem solving, research, creativity and innovation, communication and cooperation.
- Support and facilitate the learning process by valuing students' contributions in the class activities.
- Guide students towards the harmonization of their findings.

- Encourage individual, peer and group evaluation of the work done in the classroom and use appropriate competence-based assessment approaches and methods.

To facilitate you in your teaching activities, the content of this book is self explanatory so that you can easily use it. It is divided in 3 parts:

The part I explain the structure of this book and give you the methodological guidance;

The part II gives a sample lesson plan;

The part III details the teaching guidance for concepts given in the student book.

Even though this teacher's guide contains the guidance on solutions for all activities given in the learner's book, you are requested to work through each question before judging student's findings.

I wish to sincerely express my appreciation to the people who contributed towards the development of this book, particularly, REB staff, Teachers from general education and experts from Local and international Organizations for their technical support.

RWANDA BASIC EDUCATION BOARD (REB)

Dr. MBARUSHIMANA Nelson

Director General, REB

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I wish to express my appreciation to the people who played a major role in the development of this teacher's guide for Senior Four Mathematics to be used in options where Core Mathematics is a major subject. It would not have been successful without active participation of different education stakeholders.

I owe gratitude to Curriculum Officers and teachers whose efforts during the editing exercise of this book were very much valuable.

Finally, my word of gratitude goes to all Rwanda Basic Education Board staffs who were involved in the whole process of in-house textbook production.

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Head of CTLR Department

Table of Contents

PART I. GENERAL INTRODUCTION

1.1 The structure of the guide

The tutor's guide of Mathematics is composed of three parts:

The Part I concerns general introduction that discusses methodological guidance on how best to teach and learn Mathematics, developing competences in teaching and learning, addressing cross-cutting issues in teaching and learning and Guidance on assessment.

Part II presents a sample lesson plan. This lesson plan serves to guide the teacher on how to prepare a lesson in Mathematics.

The Part III is about the structure of a unit and the structure of a lesson. This includes information related to the different components of the unit. This part provides information and guidelines on how to facilitate student while working on learning activities. More other, many application activities from the textbook have answers in this part.

1.2 Methodological guidance

1.2.1 Developing competences

Since 2015 Rwanda shifted from a knowledge based to a competency-based curriculum for pre-primary, primary, secondary education. This called for changing the way of learning by shifting from teacher centred to a learner centred approach. Teachers are not only responsible for knowledge transfer but also for fostering learners' learning achievement and creating safe and supportive learning environment. It implies also that learners have to demonstrate what they are able to transfer the acquired knowledge, skills, values and attitude to new situations.

The competence-based curriculum employs an approach of teaching and learning based on discrete skills rather than dwelling on only knowledge or the cognitive domain of learning. It focuses on what learner can do rather than what learner knows. Learners develop competences through subject unit with specific learning objectives broken down into knowledge, skills and attitudes through learning activities.

In addition to the competences related to Mathematics, student teachers also develop generic competences which should promote the development of the higher order thinking skills and professional skills in Mathematics teaching. Generic competences are developed throughout all units of Mathematics as follows:

Generic competences	Ways of developing generic competences
Critical thinking	All activities that require learners to calculate, convert, interpret, analyse, compare and contrast, etc have a common factor of developing critical thinking into learners
Creativity and innovation	All activities that require learners to plot a graph of a given algebraic data, to organize and interpret statistical data collected and to apply skills in solving problems of economics have a common character of developing creativity into learners
Research and problem solving	All activities that require learners to make a research and apply their knowledge to solve problems from the real-life situation have a character of developing research and problem solving into learners.
Communication	During Mathematics class, all activities that require learners to discuss either in groups or in the whole class, present findings, debate …have a common character of developing communication skills into learners.
Co-operation, interpersonal relations and life skills	All activities that require learners to work in pairs or in groups have character of developing cooperation and life skills among learners.
Lifelong learning	All activities that are connected with research have a common character of developing into learners a curiosity of applying the knowledge learnt in a range of situations. The purpose of such kind of activities is for enabling learners to become life-long learners who can adapt to the fast-changing world and the uncertain future by taking initiative to update knowledge and skills with minimum external support.

The generic competences help learners deepen their understanding of Mathematics and apply their knowledge in a range of situations. As students develop generic competences, they also acquire the set of skills that employers look for in their employees, and so the generic competences prepare students for the world of work.

1.2.2 Addressing cross cutting issues

Among the changes brought by the competence-based curriculum is the integration of cross cutting issues as an integral part of the teaching learning process-as they relate to and must be considered within all subjects to be appropriately addressed. The eight cross cutting issues identified in the national curriculum framework are: *Comprehensive Sexuality Education, Environment and Sustainability, Financial Education, Genocide studies, Gender, Inclusive Education, Peace and Values Education, and Standardization Culture*.

Some cross-cutting issues may seem specific to particular learning areas/ subjects but the teacher need to address all of them whenever an opportunity arises. In addition, learners should always be given an opportunity during the learning process to address these cross-cutting issues both within and out of the classroom.

Below are examples of how crosscutting issues can be addressed:

Cross-Cutting Issue	Ways of addressing cross-cutting issues
Comprehensive Sexuality Education: The primary goal of introducing Comprehensive Sexuality Education program in schools is to equip children, adolescents, and young people with knowledge, skills and values in an age appropriate and culturally gender sensitive manner so as to enable them to make responsible choices about their sexual and social relationships, explain and clarify feelings, values and attitudes, and promote and sustain risk reducing behaviour.	Using different charts and their interpretation, Mathematics teachers should lead students to discuss the following situation: "Alcohol abuse and unwanted pregnancies" and advise students on how they can fight those abuses. Some examples can be given when learning statistics, powers, logarithms and their properties.
Environment and Sustainability: Integration of Environment, Climate Change and Sustainability in the curriculum focuses on and advocates for the need to balance economic growth, society well-being and ecological systems. Learners need basic knowledge from the natural sciences, social sciences, and humanities to understand to interpret principles of sustainability.	Using Real life models or students' experience, Mathematics teacher should lead students to illustrate the situation of "population growth" and discuss its effects on the environment and sustainability.
Financial Education: The integration of Financial Education into the curriculum is aimed at a comprehensive Financial Education program as a precondition for achieving financial inclusion targets and improving the financial capability of Rwandans so that they can make appropriate financial decisions that best fit the circumstances of one's life.	Through different examples and calculations on interest rate problems, total revenue and total cost, Mathematics teacher can lead student to discuss how to make appropriate financial decisions.

Gender: At school, gender will be understood as family complementarities, gender roles and responsibilities, the need for gender equality and equity, gender stereotypes, gender sensitivity, etc.	Mathematics teacher should address gender as cross-cutting issue through assigning leading roles in the management of groups to both girls and boys and providing equal opportunity in the lesson participation and avoid any gender stereotype in the whole teaching and learning process.
Inclusive Education: Inclusion is based on the right of all learners to a quality and equitable education that meets their basic learning needs and understands the diversity of backgrounds and abilities as a learning opportunity.	Firstly, Mathematics teachers need to identify/recognize students with special needs. Then by using adapted teaching and learning resources while conducting a lesson and setting appropriate tasks to the level of students, they can cater for students with special education needs. They must create opportunity where student teachers can discuss how to support colleagues with special educational needs.
Peace and Values Education: Peace and Values Education (PVE) is defined as education that promotes social cohesion, positive values, including pluralism and personal responsibility, empathy, critical thinking and action in order to build a more peaceful society.	Through a given lesson, a teacher should: ▪ Set a learning objective which is addressing positive attitudes and values, ▪ Encourage students to develop the culture of tolerance during discussion and to be able to instil it in colleagues and cohabitants; ▪ Encourage students to respect ideas for others.

Standardization Culture: Standardization Culture in Rwanda will be promoted through formal education and plays a vital role in terms of health improvement, economic growth, industrialization, trade and general welfare of the people through the effective implementation of Standardization, Quality Assurance, Metrology and Testing.	With different word problems related to the effective implementation of Standardization, Quality Assurance, Metrology and Testing, students can be motivated to be aware of health improvement, economic growth, industrialization, trade and general welfare of the people.

1.2.3 Guidance on how to help students with special education needs in classroom

In the classroom, students learn in different way depending to their learning pace, needs or any other special problem they might have. However, the teacher has the responsibility to know how to adopt his/her methodologies and approaches in order to meet the learning need of each student in the classroom. Also teachers need to understand that student with special needs, have to be taught differently or need some accommodations to enhance the learning environment. This will be done depending on the subject and the nature of the lesson.

In order to create a well-rounded learning atmosphere, teachers need to:

- Remember that learners learn in different ways so they have to offer a variety of activities (e.g. role-play, music and singing, word games and quizzes, and outdoor activities);
- Maintain an organized classroom and limits distraction. This will help learners with special needs to stay on track during lesson and follow instruction easily;
- Vary the pace of teaching to meet the needs of each child. Some learners process information and learn more slowly than others;
- Break down instructions into smaller, manageable tasks. Learners with special needs often have difficulty understanding long-winded or several instructions at once. It is better to use simple, concrete sentences in order to facilitate them understand what you are asking.
- Use clear consistent language to explain the meaning (and demonstrate or show pictures) if you introduce new words or concepts;
- Make full use of facial expressions, gestures and body language;

- Pair a learner who has a disability with a friend. Let them do things together and learn from each other. Make sure the friend is not over protective and does not do everything for the one with disability. Both learners will benefit from this strategy;
- Use multi-sensory strategies. As all learners learn in different ways, it is important to make every lesson as multi-sensory as possible. Learners with learning disabilities might have difficulty in one area, while they might excel in another. For example, use both visual and auditory cues.

Below are general strategies related to each main category of disabilities and how to deal with every situation that may arise in the classroom. However, the list is not exhaustive because each child is unique with different needs and that should be handled differently.

Strategy to help learners with developmental impairment:

- Use simple words and sentences when giving instructions;
- Use real objects that learners can feel and handle. Rather than just working abstractly with pen and paper;
- Break a task down into small steps or learning objectives. The learner should start with an activity that she/he can do already before moving on to something that is more difficult;
- Gradually give the learner less help;
- Let the learner with disability work in the same group with those without disability.

Strategy to help learners with visual impairment:

- Help learners to use other senses (hearing, touch, smell and taste) and carry out activities that will promote their learning and development;
- Use simple, clear and consistent language;
- Use tactile objects to help explain a concept;
- If the learner has some sight, ask him/her what he/she can see;
- Make sure the learner has a group of friends who are helpful and who allow him/her to be as independent as possible;
- Plan activities so that learners work in pairs or groups whenever possible;

Strategy to help learners with hearing disabilities or communication difficulties

- Always get the learner's attention before you begin to speak;

- Encourage the learner to look at your face;
- Use gestures, body language and facial expressions;
- Use pictures and objects as much as possible.
- Keep background noise to a minimum.

Strategies to help learners with physical disabilities or mobility difficulties:

- Adapt activities so that learners, who use wheelchairs or other mobility aids, can participate.
- Ask parents/caregivers to assist with adapting furniture e.g. the height of a table may need to be changed to make it easier for a learner to reach it or fit their legs or wheelchair under;
- Encourage peer support when needed;
- Get advice from parents or a health professional about assistive devices if the learner has one.

Adaptation of assessment strategies:

At the end of each unit, the tutor is advised to provide additional activities to help students achieve the key unit competence. These assessment activities are for remedial, consolidation and extension designed to cater for the needs of all categories of students; slow, average and gifted students respectively. Therefore, the tutor is expected to do assessment that fits individual student.

Remedial activities	After evaluation, slow students are provided with lower order thinking activities related to the concepts learnt to facilitate them in their learning. These activities can also be given to assist deepening knowledge acquired through the learning activities for slow students.
Consolidation activities	After introduction of any concept, a range number of activities can be provided to all students to enhance/ reinforce learning.

Extended activities	After evaluation, gifted and talented students can be provided with high order thinking activities related to the concepts learnt to make them think deeply and critically. These activities can be assigned to gifted and talented students to keep them working while other students are getting up to required level of knowledge through the learning activity.

1.2.4. Guidance on assessment

Assessment is an integral part of teaching and learning process. The main purpose of assessment is for improvement of learning outcomes. Assessment for learning/ Continuous/ formative assessment intend to improve students' learning and tutor's teaching whereas assessment of learning/summative assessment intends to improve the entire school's performance and education system in general.

Continuous/ formative assessment

It is an on-going process that arises during the teaching and learning process. It includes lesson evaluation and end of sub unit assessment. This formative assessment should play a big role in teaching and learning process. The teacher should encourage individual, peer and group evaluation of the work done in the classroom and uses appropriate competence-based assessment approaches and methods.

Formative assessment is used to:

- Determine the extent to which learning objectives are being achieved and competences are being acquired and to identify which students need remedial interventions, reinforcement as well as extended activities. The application activities are developed in the learner book and they are designed to be given as remedial, reinforcement, end lesson assessment, homework or assignment
- Motivate students to learn and succeed by encouraging students to read, or learn more, revise, etc.
- Check effectiveness of teaching methods in terms of variety, appropriateness, relevance, or need for new approaches and strategies. Mathematics tutors need to consider various aspects of the instructional process including appropriate language levels, meaningful examples, suitable methods and teaching aids/ materials, etc.
- Help students to take control of their own learning.

In teaching Mathematics, formative or continuous assessment should compare performance against instructional objectives. Formative assessment should measure the student's ability with respect to a criterion or standard. For this reason, it is used to determine what students can do, rather than how much they know.

Summative assessment

The assessment can serve as summative and informative depending to its purpose. The end unit assessment will be considered summative when it is done at end of unit and want to start a new one.

It will be formative assessment, when it is done in order to give information on the progress of learners and from there decide what adjustments need to be done.

The assessment done at the end of the term, end of year, is considered as summative assessment so that the teacher, school and parents are informed of the achievement of educational objective and think of improvement strategies. There is also end of level/ cycle assessment in form of national examinations.

When carrying out assessment?

Assessment should be clearly visible in lesson, unit, term and yearly plans.

- Before learning (diagnostic): At the beginning of a new unit or a section of work; assessment can be organized to find out what students already know / can do, and to check whether the students are at the same level.
- During learning (formative/continuous): When students appear to be having difficulty with some of the work, by using on-going assessment (continuous). The assessment aims at giving students support and feedback.
- After learning (summative): At the end of a section of work or a learning unit, the Mathematics Tutor has to assess after the learning. This is also known as Assessment of Learning to establish and record overall progress of students towards full achievement. Summative assessment in Rwandan schools mainly takes the form of written tests at the end of a learning unit or end of the month, and examinations at the end of a term, school year or cycle.

Instruments used in assessment.

- **Observation:** This is where the Mathematics teacher gathers information by watching students interacting, conversing, working, playing, etc. A teacher can use observations to collect data on behaviours that are difficult to assess by other methods such as attitudes, values, and generic competences and intellectual skills. It is very important because it is used before the lesson begins and throughout the lesson since the teacher has to continue observing each and every activity.

- **Questioning**

(a) Oral questioning: a process which requires a student to respond verbally to questions

(b) Class activities/ exercises: tasks that are given during the learning/ teaching process

(c) Short and informal questions usually asked during a lesson

(d) Homework and assignments: tasks assigned to students by their teacher to be completed outside of class.

Homework assignments, portfolio, project work, interview, debate, science fair, Mathematics projects and Mathematics competitions are also the different forms/instruments of assessment.

1.2.5. Teaching methods and techniques that promote active learning

The different learning styles for students can be catered for, if the teacher uses active learning whereby learners are really engaged in the learning process.

The main teaching methods used in mathematics are the following:

- **Inductive-deductive method:** Inductive method is to move from specific examples to generalization and deductive method is to move from generalization to specific examples.
- **Analytic-synthetic method:** Analytic method proceeds from unknown to known, 'Analysis' means 'breaking up' of the problem in hand so that it ultimately gets connected with something obvious or already known. Synthetic method is the opposite of the analytic method. Here one proceeds from known to unknown.
- **Skills Laboratory method:** Laboratory method is based on the maxim "learning by doing." It is a procedure for stimulating the activities of the students and to encourage them to make discoveries through practical activities.
- **Problem solving method, Project method and Seminar Method.**

The following are some active techniques to be used in Mathematics:

- Group work
- Research
- Probing questions
- Practical activities (drawing, plotting, interpreting graphs)
- Modelling
- Brainstorming
- Quiz Technique
- Discussion Technique
- Scenario building Technique

Techniques to develop competences

The teacher can use the following techniques while teaching mathematics to support development of competences:

Techniques/ Strategies	Description
Roundtable	This is a form of cooperative learning. A question is posed by the teacher to groups of learners. Each person in the group writes one answer on a paper and passes it to the next team member. The group looks at each answer and decides which one to present to the class. Each group shares or presents their answer to the entire class. The suggestions are discussed by the class and conclusions drawn.
Questions in corners	The teacher places questions in different corners of the classroom. Groups of 3-6 learners move from corner to corner as per signal given by the teacher. They discuss and write an answer to each question taking into account answers already written by previous groups. The use of different coloured markers for each group helps to see what each group wrote for each question. Ideas for each question are discussed in plenary to come up with some conclusions at the end.

Outdoor activities Field visits	In field visits, learners go outside the classroom to observe specific organisms or phenomena, or to hear information from experts. Before the visit the teacher and learners: - agree on aims and objectives - gather relevant information prior to visit - brainstorm on key questions and share responsibilities - discuss materials needed and other logistical issues - discuss and agree on accepted behaviours during the visit - After the visit: - de-brief and discussion of what was learned and observed - evaluation of all aspects of visit - reports, presentations prepared by learners
Project work	Learners in groups or individually, are engaged in a self-directed work for an extended period of time to investigate and respond to a complex question, problem, or challenge. The work is presented to classmates and other people beyond the school. Projects are based on real-world problems that capture learners' interest. This technique develops higher order thinking as the learners acquire and apply new knowledge in a problem-solving context.

	The teacher plays the role of facilitator by: - working with learners to frame worthwhile questions - setting relevant and meaningful tasks - availing resources needed - coaching both knowledge and skills development and social skills, - assessing carefully what learners produced based on defined criteria
Group work	This is a form of peer/cooperative/ collaborative learning that values the learner-learner interaction. It is mutually beneficial and involves the sharing of knowledge, ideas and experience between learners. It offers learners opportunity to learn from each other. To be effective, teams should be heterogeneous in terms of ability levels, made of 3-4 learners in most tasks. Team members are assigned specific roles which are rotated. For elaborated work, assessment should be twofold: based on both the collective and individual work
Role play	The role play is a special kind of case study in which there is an explicit situation established with learners playing specific roles, spontaneously saying and doing what they understand their "character" would do, in that situation. The case study differs from the role play because in the case study, learners read about situations and characters; in the role play, they find themselves what to say, how to play and which material to use.
Case study	Case study as a learning technique is a story either based on real events, or from a construction of events which could reasonably take place. It involves issues or conflicts which need to be resolved. The information contained in a case study can be complex or simple. The teacher presents a problem situation and indicates how to proceed

Brainstorming	It is a technique used for creative exploration of options/ solutions in an environment free of criticism. It encourages creativity and a large number of ideas. Among ground rules there are: active participation by all members; no discussions, criticisms, compliments or other comments during the brainstorming stage. The teacher starts by reviewing the rules, sets a time limit; states and explains the question; collects and displays ideas; eliminates duplications and guides learners to draw a conclusion.
A *learning centre*/corner	It is a space set aside in the classroom that allows easy access to a variety of learning materials in an interesting and productive manner. Learners can work by themselves or with others in self-directed activities on a content related to the curriculum or not. These centres allow learners to deepen their understanding of subjects, apply their learning in a stimulating learning environment and engage in meaningful discoveries that match their individual interests. They provide learners with hands-on experiences they can pursue at their own pace and level of curiosity.
Games/play	Games are used to help learners to learn faster and better, and in enjoyable manner. Games and plays help to create a classroom experience that actively engages learners. They develop communication and other important skills such as social skills, critical thinking, problem-solving, numeracy and literacy skills in different subjects.
Research work	Each learner or group of learners is given a research topic. They have to gather information or ask experienced people and then the results are presented and discussed in class.

Practical work	Individually or in teams, learners are assigned practical tasks. To be effective, the task needs: a clear purpose with strong links and relevance to the curriculum; quality materials; learners' engagement; time for preparation and carrying out the work; support from the teacher or other experts. Such activities encourage deeper understanding of phenomena; develop skills such as observation, practical work, planning, reporting, etc.

What is Active learning?

Active learning is a pedagogical approach that engages learners in doing things and thinking about the things they are doing. Learners play the key role in the active learning process. They are not empty vessels to fill but people with ideas, capacity and skills to build on for effective learning. Thus, in active learning, learners are encouraged to bring their own experience and knowledge into the learning process.

The role of the teacher in active learning	The role of learners in active learning
- The teacher engages learners through active learning methods such as inquiry methods, group discussions, research, investigative activities, group and individual work activities. - He/she encourages individual, peer and group evaluation of the work done in the classroom and uses appropriate competence-based assessment approaches and methods. - He provides supervised opportunities for learners to develop different competences by giving tasks which enhance critical thinking, problem solving, research, creativity and innovation, communication and cooperation. - Teacher supports and facilitates the learning process by valuing learners' contributions in the class activities.	A learner engaged in active learning: - Communicates and shares relevant information with peers through presentations, discussions, group work and other learner-centred activities (role play, case studies, project work, research and investigation); - Actively participates and takes responsibility for his/her own learning; - Develops knowledge and skills in active ways; - Carries out research/ investigation by consulting print/online documents and resourceful people, and presents their findings; - Ensures the effective contribution of each group member in assigned tasks through clear explanation and arguments, critical thinking, responsibility and confidence in public speaking - Draws conclusions based on the findings from the learning activities.

Main steps for a lesson in active learning approach

All the principles and characteristics of the active learning process highlighted above are reflected in steps of a lesson as displayed below. Generally, the lesson is divided into three main parts whereby each one is divided into smaller steps to make sure that learners are involved in the learning process. Below are those main part and their small steps:

1) Introduction

Introduction is a part where the teacher makes connection between the current and previous lesson through appropriate technique. The teacher opens short discussions to encourage learners to think about the previous learning experience and connect it with the current instructional objective. The teacher reviews the prior knowledge, skills and attitudes which have a link with the new concepts to create good foundation and logical sequencings.

2) Development of the new lesson

The development of a lesson that introduces a new concept will go through the following small steps: discovery activities, presentation of learners' findings, exploitation, synthesis/summary and exercises/application activities.

❖ **Discovery activity**

Step 1

- The teacher discusses convincingly with learners to take responsibility of their learning
- He/she distributes the task/activity and gives instructions related to the tasks (working in groups, pairs, or individual to instigate collaborative learning, to discover knowledge to be learned).

Step 2

- The teacher let learners work collaboratively on the task;
- During this period the teacher refrains to intervene directly on the knowledge;
- He/she then monitors how the learners are progressing towards the knowledge to be learned and boosts those who are still behind (but without communicating to them the knowledge).

❖ **Presentation of learners' findings/productions**

- In this episode, the teacher invites representatives of groups to present their productions/findings.
- After three/four or an acceptable number of presentations, the teacher decides to engage the class into exploitation of learners' productions.

❖ **Exploitation of learner's findings/ productions**

- The teacher asks learners to evaluate the productions: which ones are correct, incomplete or false

- Then the teacher judges the logic of the learners' products, corrects those which are false, completes those which are incomplete, and confirms those which are correct.

❖ **Institutionalization or harmonization (summary/conclusion/ and examples)**

- The teacher summarizes the learned knowledge and gives examples which illustrate the learned content.

❖ **Application activities**

- Exercises of applying processes and products/objects related to learned unit/sub-unit
- Exercises in real life contexts
- Teacher guides learners to make the connection of what they learnt to real life situations. At this level, the role of teacher is to monitor the fixation of process and product/object being learned.

3) Assessment

In this step the teacher asks some questions to assess achievement of instructional objective. During assessment activity, learners work individually on the task/activity. The teacher avoids intervening directly. In fact, results from this assessment inform the teacher on next steps for the whole class and individuals. In some cases, the teacher can end with a homework/ assignment. Doing this will allow learners to relay their understanding on the concepts covered that day. Teacher leads them not to wait until the last minute for doing the homework as this often results in an incomplete homework set and/or an incomplete understanding of the concept.

PART II: SAMPLE OF LESSON

When teaching any lesson, you can follow the following steps.

Introduction

Start by reviewing previous lesson through asking some questions to learners. If there is no previous lesson, ask them prerequisites related questions for the lesson of the day.

Lesson development

In this step, activities can be more than one (exploration activity, explanation activity and elaboration activity). For each one, give an activity to learners that will be done in groups or individually. After a while, invite one or more groups for presentation of their work to other groups. If the activity is individual, ask one or more learners to present his/her work to others. After activities, capture the main points from the presentation of the learners and guide the whole class to summarize them. After this, provide application activity in their respective groups. Request learners to correct them on chalkboard where you guide every student by addressing eventual misconception.

Evaluation

Give students an activity to be done individually as an assessment. Correct every one and provide related feedback.

Conclusion

Conclude the lesson and remember to assign a home work to students. This homework may include remedial activities, consolidation activities or extended activities depending on the feedback from the assessment. Sometimes when there is no problem in the assessment, a teacher can provide a homework which will arouse the curiosity of students for the next lesson.

See example of a planned lesson here bellow.

PLANNED LESSON 1

School Name: ……… **Teacher's name: ……………………**

Term	Date	Subject	Class	Unit Nº	Lesson Nº	Duration	Class size
I	……./……./2020	Mathematics	Senior 4	2	9 of 11	40 min	...
Type of Special Educational Needs to be catered for in this lesson and number of learners in each category				3 slow learners and 2 low vision learners:			

Unit title	**Points, straight lines and circle in 2D.**
Key unit competency:	Determine algebraic representations of lines, straight lines and circles in 2D.
Title of the lesson	Distance from a point to a line
Instructional Objective	Using a T-square and a ruler, learners will be able to describe how to accurately measure the distance from a point to a line and measure practically the distance between a point on the tailed floor and the line selected there.

Plan for this Class (location: in / outside)	The lesson is held indoors, the class is organized into groups,3 slow learners are scattered in different groups ,and 2 low vision learners seat on the front desks near the blackboard in order to see and participate fully in all activities
Learning Materials (for ALL learners)	Textbooks , the classroom in which the learners are studying
References	-TTC syllabus, - S4 Mathematics textbook and Teacher's guide

<table>
<tr><td rowspan="3">Timing for each step</td><td colspan="2">Description of teaching and learning activity</td><td rowspan="3">Generic competences and cross cutting issues to be addressed + a short explanation</td></tr>
<tr><td colspan="2">-Learners work individually the questions in the introduction, and the correction is done on the chalk board by two learners, one after another under the guidance of the teacher.

-Then they discuss in groups the discovery activity, followed by the presentation by a sample group, interaction of learners and harmonization of the results under the facilitation of the teacher.

-Next, they discuss in pairs the solved example and compare their results with the answer proposed in the book

-Finally, the learners are assigned individual tasks, and the correction is done on the chalk board, and the teacher winds up the lesson by assigning homework activities to the learners.</td></tr>
<tr><td>Teacher activities</td><td>Learner activities</td></tr>
</table>

Introduction: 5 minutes	The teacher asks learners to work individually: Find the distance between: -two given points. Verify if they apply: $d(A,B)=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$ Where A (x_1,y_1) and B (x_2,y_2). - The teacher links the introduction to the lesson of the day	-Learners work individually. -Two learners, one after another, write the answers on the chalkboard.	**Communication** skills developed through the presentation and sharing ideas
Development of the lesson			

2.1 Discovery activity: 10 minutes	The teacher organizes the learners into groups Teacher gives learners activity to discuss in groups and gives instructions related to the task Teacher goes round to monitor the work of each group and provide assistance where needed	-Learners form groups -Each group analyzes and discuss the activity under the direction of the task manager of the group. If lines have the following equations: $y_1 = mx_1 + c_1$ **and** $y_{\square} = mx + c$ If they intersect, the distance is **0** at the point of intersection. If they're parallel: the distance between them will be given by the following formula : $d = \frac{\lvert c_1 - c_2 \rvert}{\sqrt{1+m^2}}$ -Learners present to the teacher their eventual problems .	**Cooperation** and **communication** skills through discussions **Peace and values education**; Cooperation , mutual respect, tolerance through discussions with people with different views and respect one's views

2.2 Presentation of learner's findings and exploitation: 15 minutes	Teacher invites the groups with different working steps to present the findings of their respective grous. The teacher encourages learners to follow attentively Teacher takes notes on key points from learners' presentation. The teacher asks learners to amend the presentation and to evaluate their work	-One students presents the work on the behalf of the group. **Expected answers** **(** Refer to solution of activity in the TG) -Learners follow the presentation. -Learners evaluate the findings of other learners -Learners evaluate their own findings.	**Cooperation** and **communication**/ attentive listening during presentations and group discussions **Critical thinking** through evaluating other's findings

2.3.Conclusion/ Summary: 5 minutes **Assessment** **5 minutes**	-Teacher facilitates the learners to elaborate the summary of the presentation -Teacher requests learners to write down the main points in their books - Teacher asks learners to individually work out the application activity -Teacher assigns the homework activities from TG (additional activity) to learners	-The learners come to the main point: -Learners take notes in their books. -Individually learners work out the application activity and finally they make a correction on the chalk board. **Expected answers** (Refer to the solution of application activity in TG) - Learners individually write down the assigned homework activities.	- **Critical thinking** and **problem solving** skills are developed through analyzing and solving real life Mathematical problem: e.g. finding the distance from the floor to the ceiling of a house - **Financial education** is addressed through good management of the money for transport by choosing the shortest distance
Observation on lesson delivery	To be completed after receiving the feed-back from the learners (what did the learners like, what challenged them...)		

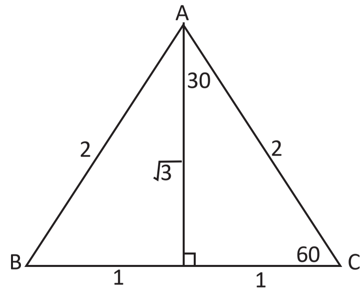
Planed lesson 2

School: .... **Teacher's name: ....**

Term	Date	Subject	Class	Unit Nº	Lesson Nº	Duration	Class size
1	8 February 2016	**Mathematics**	S4	1	4	40	45
Type of special educational needs and number of learners				Visually impairment - 1 learner			

Topic area	Trigonometry
Sub-topic area	Trigonometric circles and identities
Unit title	Fundamentals of trigonometry
Key unit competence	Use trigonometric circles and identities to determine trigonometric ratios and apply them to solve related problems
Title of the lesson	Trigonometric ratios
Instructional objective	Using geometric instruments, graph paper and pencils the learners will be able to derive the trigonometric ratios of the special angles 30° and 60° in 20 minutes, with accuracy.
Plan for this class (location: in / outside)	In the classroom
Learning materials (for all learners)	Geometric instruments (ruler, T-square, protractor), graph paper, pencils, flash cards
References	Achievers Mathematics Senior 4 Student's Book Mathematics syllabus by REB.

Timing for each step	Description of teaching and learning activity		Generic competences and cross -cutting issues to be addressed
	Teacher guides learners in recalling properties of equilateral triangles and construction to discern trigonometric ratios of the angles 30° and 60°. Learners draw an equilateral triangle, ABC , of sides 2 units in length; the draw a line AD from A perpendicular to BC. AD bisects BC giving BD=CD=1. They complete table of the trigonometric ratios (cos, sin and tan) of angles 30° and 60°.		
	Teacher activities	Learner activities	
Introduction 8 min	- Give students samples of equilateral triangles on flash cards - Ask learners the properties of equilateral triangles – size and angles. - Ensure the learner with visual impairment sits in front of the class.	- Learners will talk about the angle sizes and sides of equilateral triangles. - They will measure and discern from flash cards that the angles and sides are equal.	Generic competences: -Communication - learners discuss angle sizes and sides of equilateral triangles. - Problem - solving - learners measure angles and sides accurately Cross-cutting issues: - Peace and values education: Learners take turns when answering questions and respect the opinions of others.

Development of the lesson 27 min	1. Ensure learners are in pairs and have the required materials for construction. 2. Ask learners to follow the procedure of Activity 1.8 of the Student's Book. 3. Move round the class guiding them. 4. Let groups present their findings. 5. Write the main points from what learners have presented.	1. In pairs, learners will draw an equilateral triangle, *ABC*, of sides 2 units in length. Next they draw a line *AD* from *A* perpendicular to *BC*. AD bisects BC giving BD = DC = 1.  2. They use Pythagoras' theorem to determine the height AD = $\sqrt{3}$.	Generic competences: -Communication - learners present their findings to the rest of the class. -Cooperation - learners work together harmoniously on assigned tasks. -Problem - solving - learners draw triangles and lines accurately. They then determine the ratios of the sides.

<table>
<tr><td></td><td></td><td>3. They then determine the ratios of sides taking into account the angles.
4. Learners then present the findings in class, and fill in a table like the following.</td><td rowspan="2">Cross-cutting issues
• Standardization culture: Learners carry out accurate constructions and measurements, and adhere to the rules of trigonometric ratios.
• Peace and values education:
Learners take turns when presenting their findings and respect the opinions of others.</td></tr>
<tr><td></td><td>6. Ensure the learner with visual impairment is paired with one who has good vision.</td><td>Trigonometric ratios for the special angles 30º and 60º<table><tr><td>$\sin 60° = \frac{\sqrt{3}}{2}$</td><td>$\sin 30° = \frac{1}{2}$</td></tr><tr><td>$\cos 60° = \frac{1}{2}$</td><td>$\cos 30° = \frac{\sqrt{3}}{2}$</td></tr><tr><td>$\tan 60° = \sqrt{3}$</td><td>$\tan 30° = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$</td></tr></table></td></tr>
</table>

Conclusion 5 min	1. The teacher gives an oral summary of the ratios of the special angles 30˚ and 60˚. 2. Teacher asks learners, at random, to name the different ratios – cosine, sine and tangent of 30˚ and 60˚. 3. Gives learners homework – to write the complete table of trigonometric ratios of angles 0˚, 30˚, 60˚, 45˚, 60˚ and 90˚.	1. The learners answer the oral questions posed by the teacher; they do this in an orderly manner by raising their hands and waiting to be picked by the teacher. 2. They each draw the summary table showing the trigonometric ratios of the angles 30˚and 60˚.	Generic competences: - Communication - learners answer oral questions. - Problem-solving - learners come up with the summary table for trigonometric ratios. Cross-cutting issue: - Peace and values education - learners take turns when answering questions and respect the opinions of others.
Teacher self-evaluation			

PART III: UNIT DEVELOPMENT

UNIT

1 Fundamentals of trigonometry

Number of periods: 26

1.1. Key unit competence

Use trigonometric circle and identities to determine trigonometric ratios and apply them to solve related problems.

1.2. Learning objectives

By the end of this unit, the student must be able to do the following:

Knowledge and understanding	Skills	Attitudes and values
• Define sine, cosine, tangent, cosecant, secant and cotangent of any angle and know the special values • Convert radians to degrees and vice versa • Differentiate between complementary angles, supplementary angles and co-terminal angles.	• Represent graphically sine, cosine and tangent functions and, together with the unit circle, use them to relate values of any angle to the value of a positive acute angle. • Use trigonometry, including the sine and cosine rules, to solve problems involving triangles.	• Appreciate the relationship between the trigonometric values for different angles • Verify reasonableness of answers to exercises when solving problems

1.3. Content

1. Trigonometric concepts
    - Angle and its measurements
    - Unit circle
    - Trigonometric ratios
    - Trigonometric identities
2. Reduction to functions of positive acute angles

3. Triangles and applications:
   - Bearing
   - Air navigation
   - Inclined plane

1.4. Materials required

Geometrical instruments: rule, T-square, compass, protractors; graph papers, digital instruments such as calculators.

1.5. Generic competences

- Communication
- Problem-solving
- Research
- Cooperation
- Critical thinking

Cross-cutting issues

- **Gender studies**

  Groups to be composed of mixed gender.
- **Peace and values education**

  Learners work together harmonionsly.
- **Inclusive education**

  Learners of different abilities work together and assist each other.

1.6. Guidance on teaching and learning activities

a. Introductory activity

Organize students in groups; assign them the introductory activity 1.

Monitor how students are working and ask them some probing questions where necessary. After a while, call some groups to present their findings and then guide the whole class to discuss the findings.

During this discussion, the teacher tries to arouse the curiosity of students on the content of this unit.

(This guidance is general, it will be referred to at this point of the other units).

Through different examples, help students to understand the importance of trigonometry by showing their application in real life for example in construction, satellite systems and astronomy, naval and aviation industries, land surveying and in cartography (creation of maps) and so on.

b. Main activities

1. Introduce the topic by giving some examples of angles and their measurements.
   Guide learners in carrying out Activity 1.1 of the Student's Book. Ensure the groups are working together harmoniously and that all learners are included irrespective of gender and ability. Assist them in coming up with the explanation of what a radian is. Encourage them to appreciate each others' effort.
2. Let them attempt Activity 1.2 to introduce the concept of radians. Guide learners in converting angles from degrees to radians and vice versa. Use adequate examples. Then let them attempt Application activity 1.1 of the Student's Book.
3. Ask students to research on the trigonometric ratios. They will start with the use of the unit circles. Guide them in Activities 1.3 and 1.4 from the Student's Book. Let them discuss and give the definitions for the ratios – sine, cosine, tangent, cosecant, secant and cotangent.
4. Use triangles to assist learners define trigonometric ratios related to 30°, 45° and 60°. Let them verify them using Activities 1.5, 1.6, 1.7 and 1.8 of the Student's Book.
5. Take them through the different types of angles. Let them attempt Tasks 1.2, 1.3 and 1.4 of the Student's Book.
6. Let them work in groups to draw the trigonometric graphs of Activities 1.9, 1.10 and 1.11 of the Student's Book. Discuss with students how to read and plot the functions of sine and of cosine. Guide them in plotting graphs.
7. Help the students to use trigonometry, including the sine and cosine rules, to solve problems involving triangles. Ask them to state where we can apply trigonometric ratios. This will make them appreciate the relationship between trigonometric values of different angles. Let them attempt Tasks 1.8, 1.9 and 1.10 of the Student's Book.

c. Extra practice for learners

1. Use geometrical instruments to measure:
   - The size of a right angle
   - The size of a straight angle
   - The size of a full turn

Solution

- The measure of a right angle is $\frac{1}{4}$ turn which is 90^0;
- The measure of a straight angle is $\frac{1}{2}$ turn which is180^0;
- The measure of a full turn which is 360^0.

2. Convert $\frac{7\pi}{6}$ radians to degrees and 60^0 to radians.

Solution

$\frac{7\pi}{6}$ radians = $\frac{7\pi}{6}\left(\frac{180}{\pi}\right)$ = 210 degrees

$60° = 60°\left(\frac{\pi}{180}\right)$ radians = $\frac{\pi}{3}$ radians

Learners of varying strengths and abilities

a) Gifted and talented

Guide the learners to study units ahead of others. You can provide more advanced material to the learner. For example, let them draw extra graphs and allow them to do extra exercises. You can also make them group leaders in their groups so that they indirectly help the slower learners or the not so talented. However, ensure that they are supervised so as not to overwhelm the less gifted.

Give them individual exercises to work on as homework.

b) Slow learners

You should provide them with the non-restrictive environment that provides/maintains their degree of freedom, self-determination, dignity and integrity of mind and body. This should, however allow for participation in an inclusive setting. Give them extra tuition. Teach them to master the material skills that are commensurate with their ability levels before introducing complex components.

1.7. Additional tasks

Task 1A - For slow learners

1. Simplify the following

   a) $\cos^2 x \tan^2 x + \sin^2 x \cot^2 x$

   b) $\sin^2 x + \sin^2 x \cot^2 x$

2. Prove the following identity

$$\frac{\sin x}{1+\cos x}+\frac{1+\cos x}{\sin x}=2\csc x$$

Answers

1. (a) $\cos^2 x \tan^2 x + \sin^2 x \cot^2 x = \cos^2 x \frac{\sin^2 x}{\cos^2 x} + \sin^2 x \frac{\cos^2 x}{\sin^2 x} = \sin^2 x + \cos^2 x = 1$

(b) $\sin^2 x+\sin^2 x\cot^2 x=\sin^2 x(1+\cot^2 x) = \sin^2 x \left(1 + \frac{\cos^2 x}{\sin^2 x}\right)$

$= \sin^2 x \left(\frac{\sin^2 x + \cos^2 x}{\sin^2 x}\right) = \frac{\sin^2 x}{\sin^2 x} = 1$

2. $\frac{\sin x}{1 + \cos x} + \frac{1 + \cos x}{\sin x} = \frac{\sin^2 x + (1 + \cos x)^2}{\sin x(1 + \cos x)} = \frac{\sin^2 x + 1 + 2 \cos x + \cos^2 x}{\sin^2 x} =$

$\frac{2 + 2 \cos x}{\sin x(1 + \cos x)}$

$= \frac{2(1 + \cos x)}{\sin x(1 + \cos x)} = \frac{2}{\sin x} = 2 \left(\frac{1}{\sin x}\right) = 2 \csc x$

Task 1B - For talented learners

1. From a tower 92 m high, two rocks which are in a horizontal line through the base of the tower are observed at angles of depression of 15° and 20°. Find the distance between the rocks if they are on:

   a) the same side of the tower

   b) opposite sides of the tower.

2. Two vertical lamp posts of equal height stand on either side of a roadway which is 30 m wide. At a point in the roadway between the lamp posts, the angles of elevation for which the tops of the lamp posts are observed are 48° and 42°. Determine

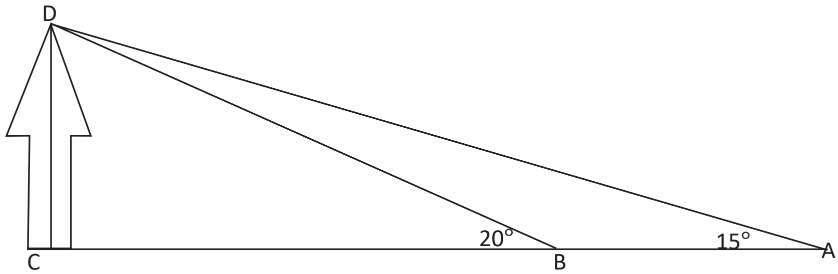
   a) the height of each lamp post.

   b) the position of the point of observation.

Answers

1.

a)



From the figure above, CD = 92m

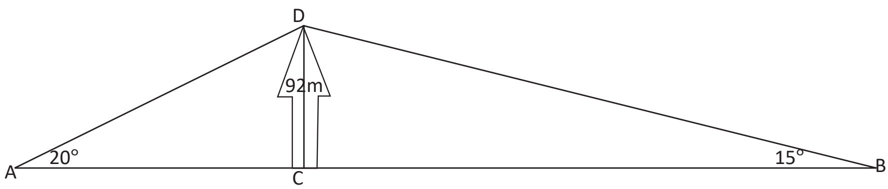
In the triangle ACD, $\tan 15° = \frac{92}{AC} \Rightarrow AC = \frac{92}{\tan 15°} = 343.35m$

In the triangle BCD, $\tan 20° = \frac{92}{AC} \Rightarrow CB = \frac{92}{\tan 20°} = 252.77m$

AB = AC – BC = 343.35 m – 252.77 m = 90.58 m.

If the rocks are on the same side of the tower, the distance between them is 90.58 m.

b)



From the figure above, CD = 92 m

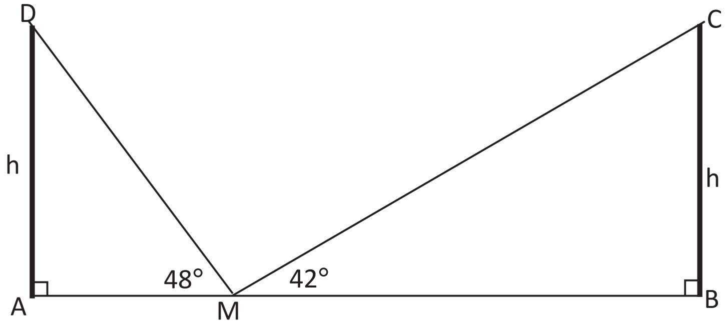
In the triangle BCD, $\tan 15° = \frac{92}{CB} \Rightarrow CB = \frac{92}{\tan 15°} = 343.35$ m

In the triangle ACD, $\tan 20° = \frac{92}{AC} \Rightarrow AC = \frac{92}{\tan 20°} = 252.77$ m

AB = AC + BC = 345.35m + 252.77 m = 596.12 m

If the rocks are on the opposite sides of the tower, the distance between them is 596.12 m.

2. ***Illustration***



From the figure above, the lamp posts are at A and B. The point of observation at M.

We have AB = 30. Let AM = x then MB = 30 – x

$$\tan 48^0 = \frac{h}{AM} = \frac{h}{x} \Rightarrow h = x\tan 48^0 \Rightarrow h = 1.11x$$

$$\tan 42^0 = \frac{h}{MB} = \frac{h}{30-x} \Rightarrow h = (30-x)\tan 42^0 \Rightarrow h = 27.01 - 0.90x$$

Let us solve for x

1.11x = 27.01 – 0.90x

2.01x = 27.01

x = 13.44

a) The point is on 13.44 m from the lamp post at A

b) h = 1.11 × 13.44 = 14.92

The height of each lamp post is 14.92 m

1.8. Additional information

Proof of trigonometric ratios formulas

1. Prove that

$1 + \tan^2 \theta = \sec^2 \theta$ ($\cos \theta \neq 0$)

Proof:

From $\sin^2 \theta + \cos^2 \theta = 1$.

Dividing both sides by $\cos^2 \theta$ gives

$$\frac{\sin^2\theta}{\cos^2\theta} + \frac{\cos^2\theta}{\cos^2\theta} = \frac{1}{\cos^2\theta}$$

Which simplifies to $1 + \tan^2 \theta = \sec^2\theta$ $(\cos \theta \neq 0)$

2. Prove that

$1 + \cot^2 \theta = \csc^2 \theta$ $(\sin \theta \neq 0)$

Proof

From $\sin^2 \theta + \cos^2 \theta = 1$.

Dividing both sides by $\sin^2 \theta$ gives

$$\frac{\sin^2 \theta}{\sin^2 \theta} + \frac{\cos^2 \theta}{\sin^2 \theta} = \frac{1}{\sin^2 \theta} \text{ and } \sin \theta \neq 0$$

Which simplifies to $1 + \cot^2 \theta = \csc^2 \theta$ $(\sin \theta \neq 0)$

3. Prove that:

$\tan \theta + \cot \theta = \sec \theta \csc \theta$ whenever both sides have meaning

Proof:

LHS = $\tan \theta + \cot \theta$

$$= \frac{\sin \theta}{\cos \theta} + \frac{\cos \theta}{\sin \theta} = \frac{\sin^2 \theta + \cos^2 \theta}{\cos \theta \sin \theta}$$

$$= \frac{1}{\cos \theta \sin \theta} = \frac{1}{\cos \theta} \frac{1}{\sin \theta}$$

$= \sec \theta \csc \theta$ = RHS

Provided $\sin \theta \neq 0$, i.e $\theta \neq \frac{k\pi}{2}$, $k \in \mathbb{Z}$.

4. Prove that:

$\text{Sin}^2 A - 4 \cos^2 A + 1 = 2 \sin^2 A - 3 \cos^2 A = 3 \sin^2 A - 2 \cos^2 A - 1$

Proof:

LSH - $\sin^2 A - 4 \cos^2 A + (\sin^2 A + \cos^2 A) = 2 \sin^2 A - 3 \cos^2 A$ = MIDDLE = $2 \sin^2 A - 2 \cos^2 A - \cos^2 A = 2 \sin^2 A - 2 \cos^2 A - (1 - \sin^2 A) = 3 \sin^2 A - 2 \cos^2 A - 1$ = RHS.

5. **Prove that**: $\dfrac{\cos^2 A}{1 + \tan^2 A} - \dfrac{\sin^2 A}{1 + \cot^2 A} = 1 - 2\sin^2 A$

whenever both sides have meaning.

Proof:

$$LSH = \frac{\cos^2 A}{\sec^2 A} - \frac{\sin^2 A}{\operatorname{cosec}^2 A} = \frac{\cos^2 A}{\frac{1}{\cos^2 A}} - \frac{\sin^2 A}{\frac{1}{\sin^2 A}} = \cos^4 A - \sin^2 A =$$

$$= \left(\cos^2 A - \sin^2 A\right)\left(\cos^2 A + \sin^2 A\right) = \cos^2 A - \sin^2 A =$$

$$= \left(1 - \sin^2 A\right) - \sin^2 A = 1 - 2\sin^2 A = RHS$$

Provided $\tan A$ and $\cot A$ both exist, i.e $\cos A \neq 0$ and $\sin A \neq 0$. Thus both sides have meaning provided

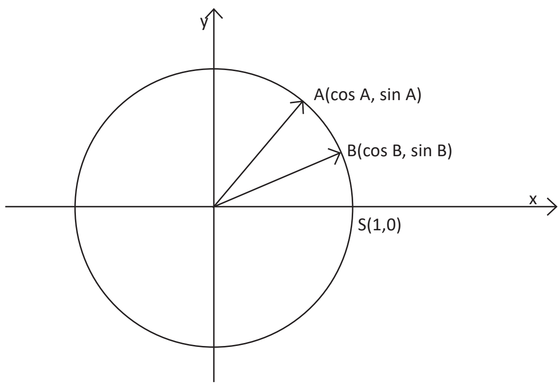
$$A \neq \frac{\pi}{2}, k \in Z$$

Additional formulae

1. cos(A – B) = cos A cos B + sin A sin B
2. cos(A + B) = cos A cos B – sin A sin B
3. sin(A + B) = sin A cos B + cos A sin B
4. sin(A – B) = sin A cos B – cos A sin B

Proof of addition formulae

1. Consider the points A(cos A, sin A) and B(cos B, sin B) on the unit circle.



AOS = A and BOS = B and AOB = A – B.

$$\cos(A - B) = \frac{\overrightarrow{OA}.\overrightarrow{OB}}{|\overrightarrow{OA}||\overrightarrow{OB}|} \begin{pmatrix}\cos A \\ \sin A\end{pmatrix}\begin{pmatrix}\cos B \\ \sin B\end{pmatrix} \text{since } |\overrightarrow{OA}| = |\overrightarrow{OB}| = 1$$

Thus cos (A – B) = cos A cos B + sin A sin B..

2. Replace B with –B in 1,

cos(A + B) = cos(A – (–B)) = cos A cos(–B) + sin A sin (–B)

cos(A + B) = cos A cos B – sin A sin B

3. Replace A with $\frac{\pi}{2}$ + A in 2.

$$\cos(\frac{\pi}{2}+A+B)=\cos(\frac{\pi}{2}+A)\cos B-\sin(\frac{\pi}{2}+A)\sin B$$

$$\sin(A+B)=\sin A\cos B+\cos A\sin B$$

$\sin(A-B) = \sin A \cos B - \cos A \sin B$

4. Replace A with –B in 3.

$\sin(A - B) = \sin A \cos(-B) + \cos A \sin(-B)$

$\sin(A - B) = \sin A \cos B - \cos A \sin B$

Additional formulae for tangent function

1. $\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$

2. $\tan(A - B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}$

Proof of Addition formulae for the tangent function

1. $\tan(A + B) = \frac{\sin(A + B)}{\cos(A + B)} = \frac{\sin A \cos B + \cos A \sin B}{\cos A \cos B - \sin A \sin B} = \frac{\frac{\sin A \cos B}{\cos A \cos B} + \frac{\cos A \sin B}{\cos A \cos B}}{\frac{\cos A \cos B}{\cos A \cos B} - \frac{\sin A \sin B}{\cos A \cos B}}$

$\frac{\tan A + \tan B}{1 - \tan A \tan B}$

2. Put –B for B in 1 then $\tan(A - B) = \frac{\tan A + \tan(-B)}{1 - \tan A \tan(-B)} = \frac{\tan A - \tan B}{1 + \tan A \tan B}$

Assessment criteria

Apply trigonometric concepts to solve problems involving triangles and angles

1.9. Answers to application activity of Unit 1 in the Student's Book

Application activity1.1

1. (a) $\frac{\pi}{2}$ (b) $\frac{\pi}{3}$ (c) $\frac{\pi}{6}$ (d) $\frac{\pi}{10}$ (e) $\frac{\pi}{20}$

(f) $\frac{3\pi}{4}$ (g) $\frac{5\pi}{4}$ (h) $\frac{3\pi}{2}$ (i) 2π (j) 4π

(l) 3π (m) $\frac{\pi}{5}$ (n) $\frac{4\pi}{9}$ (o) $\frac{23\pi}{18}$ (k) $\frac{7\pi}{4}$

2. (a) 36° (b) 108° (c) 135° (d) 10° (e) 20°

(f) 140° (g) 18° (h) 27° (i) 150° (j) 22.5°

3. (a)

Degrees	0	45	90	135	180	225	270	315	360
Radians	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π	$\frac{5\pi}{4}$	$\frac{3\pi}{2}$	$\frac{7\pi}{4}$	2π

(b)

Degrees	0	30	60	90	120	150	180	210	240	270	300	330	360
Radians	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{11\pi}{6}$	2π

Application activity 1.2

1.

	θ	sin θ	cos θ	tan θ
(a)	0°	0	1	0
(b)	90°	1	0	–
(c)	45°	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1
(d)	120°	$\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	$-\sqrt{3}$

2. (a) $\cos\theta = \pm 0.6$, $\tan\theta = \pm\frac{4}{3}$, (c) $\sin\theta = \pm\frac{2}{\sqrt{5}}$, $\cos\theta = \mp\frac{1}{\sqrt{5}}$

(b) $\sin\theta = \pm\frac{\sqrt{3}}{2}$, $\tan\theta = \pm\sqrt{3}$

3. (a) sin θ (b) cos θ (c) 2 cos θ

(d) 2 sin θ (e) 2 cos θ

4. (a) Any 4 multiples of 180°.

(b) Any 4 odd multiples of 90°.

(c) Any 4 multiples of 90°.

Application activity 1.3

1. Prove that

$1 + \tan^2\theta = \sec^2\theta$ $(\cos\theta \neq 0)$

Proof

From $\sin^2\theta + \cos^2\theta = 1$.

Dividing both sides by $\cos^2\theta$ gives

$$\frac{\sin^2\theta}{\cos^2\theta} + \frac{\cos^2\theta}{\cos^2\theta} = \frac{1}{\cos^2\theta}$$

Which simplifies to $1 + \tan^2 \theta = \sec^2 \theta$ ($\cos \theta \neq 0$)

2. Prove that

$1 + \cot^2 \theta = \csc^2 \theta$ ($\sin \theta \neq 0$)

Proof

From $\sin^2 \theta + \cos^2 \theta = 1$.

Dividing both sides by $\sin^2 \theta$ gives

$$\frac{\sin^2 \theta}{\sin^2 \theta} + \frac{\cos^2 \theta}{\sin^2 \theta} = \frac{1}{\sin^2 \theta}$$

Which simplifies to $1 + \cot^2 \theta = \csc^2 \theta$ ($\sin \theta \neq 0$)

3. Prove that:

$\tan \theta + \cot \theta = \sec \theta \csc \theta$ whenever both sides have meaning

Proof:

LHS = $\tan \theta + \cot \theta$

$$= \frac{\sin \theta}{\cos \theta} + \frac{\cos \theta}{\sin \theta} = \frac{\sin^2 \theta + \cos^2 \theta}{\cos \theta \sin \theta}$$

$$= \frac{1}{\cos \theta \sin \theta} + \frac{1}{\cos \theta} \ \frac{1}{\sin \theta}$$

Provided $\sin \theta \neq 0$ and $\cos \neq 0$, i.e. , $\theta \neq \frac{k\pi}{2}$, $k \in \mathbb{Z}$.

4. Prove that:

$\sin^2 A - 4 \cos^2 A + 1 = \sin^2 A - 3 \cos^2 A = 3 \sin^2 A - 2 \cos^2 A - 1$

Proof

LHS = $\sin^2 A - 4 \cos^2 A + (\sin^2 A + \cos^2 A) = 2 \sin^2 A - 3 \cos^2 A$ = MIDDLE = $2 \sin^2 A - 2 \cos^2 A - \cos^2 A = 2 \sin^2 A - 2 \cos^2 A - (1 - \sin^2 A) =$ $3 \sin^2 A - 2 \cos^2 A - 1$ = RHS.

5. Prove that :

$$\frac{\cos^2 A}{1+\tan^2 A} - \frac{\sin^2 A}{1+\cot^2 A} = 1 - 2\sin^2 A$$ whenever both sides have meaning.

Proof

$$LSH = \frac{\cos^2 A}{\sec^2 A} - \frac{\sin^2 A}{\mathrm{cosec}^2 A} = \frac{\cos^2 A}{\frac{1}{\cos^2 A}} - \frac{\sin^2 A}{\frac{1}{\sin^2 A}} = \cos^4 A - \sin^2 A =$$

$$= \left(\cos^2 A - \sin^2 A\right)\left(\cos^2 A + \sin^2 A\right) = \cos^2 A - \sin^2 A =$$

$$= \left(1 - \sin^2 A\right) - \sin^2 A = 1 - 2\sin^2 A = RHS$$

Provided $\tan A$ and $\cot A$ both exist, i.e $\cos A \neq 0$ and $\sin A \neq 0$. Thus both sides have meaning provided

$$A \neq \frac{k\pi}{2}, k \in Z$$

Application activity 1.4

(a)	1	(b)	$\sec^2 \frac{A}{4}$	(c)	1	(d)	$\csc^2\theta$
(e)	1	(f)	1	(g)	1	(h)	2
(i)	$\cos^2 A$	(j)	$\sin^2 2B$	(k)	$\tan^2\theta$	(l)	$-\cot^2 A$

Application activity 1.5

1. (a) $\sin 2x \cos y + \cos 2x \sin y$
   (b) $\sin 3B \cos 40° + \cos 3B \sin 40°$
   (c) $\sin 2A \cos 2B + \cos 2A \sin 2B$
   (d) $\sin x \cos 2y - \cos x \sin 2y$
   (e) $\frac{\sqrt{3}}{2} \sin\theta - \frac{1}{2} \cos\theta$
   (f) $\cos x \cos \frac{1}{2} y - \sin x \sin \frac{1}{2} y$
   (g) $\cos 3A \cos 3B - \sin 3A \sin 3B$
   (h) $\frac{\sqrt{3}}{2} \cos A + \frac{1}{2} \sin A$
   (i) $\frac{1}{2} \cos A - \frac{\sqrt{3}}{2} \sin A$
   (j) $\cos B \cos C + \sin B \sin C$
   (k) $\sin B$
   (l) $\frac{1}{2} \cos 2A - \frac{\sqrt{3}}{2} \sin 2A$
2. (a) $-\frac{24}{25}$ (b) -1 (c) 0
3. (a) $\frac{\sqrt{2}-\sqrt{6}}{4}$ (b) $\frac{\sqrt{6}-\sqrt{2}}{4}$ (c) $\frac{\sqrt{6}-\sqrt{2}}{4}$
   (d) $\frac{\sqrt{6}+\sqrt{2}}{4}$

Application activity 1.6

1. (a) 1 (b) $\cot(A - B)$ (c) $\frac{\sqrt{3}}{3}$
   (d) $\tan(\frac{\pi}{4} + A)$ (e) $\tan(\frac{\pi}{4} + A)$ (f) $\tan 2A$

Application activity 1.7

a)	0	b)	$\frac{1}{2}$	c)	$\frac{1}{4}(\sqrt{6}-\sqrt{2})$
d)	$-(2+\sqrt{3})$	e)	$\frac{1}{4}(\sqrt{6}-\sqrt{2})$	f)	$\frac{1}{4}(\sqrt{6}-\sqrt{2})$
2. a)	$\sin 3\theta$	b)	0	c)	$\tan 3A$
d)	$\tan\beta$				

3. (a) $\frac{3}{5}$ (b) $\frac{-4}{5}$ (c) $\frac{-3}{5}$

Application activity 1.8

1. (a) $x = 4.44$ cm (b) $x = 21.49$ cm (c) $x = 4$ cm
2. (a) $A = 90°$, $B = 53.13°$, $C = 36.86°$
   (b) $C = 53.13°$, $B = 90^o$
3. (a) $A = 55.77°$, $B = 41.4^o$, $C = 82.81°$
   (b) $A = 57.12°$, $B = 44.41°$, $C = 78.64°$
4. (a) $c = 4.31$ cm, $A = 55.11°$, $B = 79.87°$
   (b) $b = 13.11$ cm, $A = 49.96°$, $C = 73.18°$
   (c) $b = 5.21$ cm, $A = 35.5°$, $C = 27.5°$
   (d) $b = 13.7$ cm, $A = 49.1°$, $C = 70.9°$

Application activity 1.9

1. (a) $x = 5.45$ cm (b) $x = 5.72$ m
   (c) $x = 3.18$ cm
2. (a) $b = 17.59$ cm, $A = 58°$ (b) $a = 11.62$ cm $c = 11.62$ cm
3. (a) $b = 7.07$ cm (b) $c = 6.06$ cm (c) $B = 15.66°$
4. $C = 62.1°$m or $C = 117.9°$
5. (a) $A = 49.5°$ (b) $C = 44.3°$
   (c) $B = 72.05°$ or $107.95°$

Application activity 1.10

1. The tree was 8.91 m tall.
2. The distance is 369.15 m
3. The two cars are 60.13 km apart.
4. The distance is 45.7 m
5. The bearing of B from A is S54°10’W and the distance of B from A is 22.2 km.
6. The height of the hill is 763.94 m.

7. The length of the field is 2460.36 m.
8. Height of the tree is 34.64 m and the breadth of the river is 20 m.
9. The height of the cliff is 107.96 m.
10. The distance between the bill-boards is
    (a) 6.14 m.   (b) 49.82 m
11. 342 m
12. 384 km per hour

UNIT

2 Propositional and predicate logic

Number of periods: 14

2.1. Key unit competence

Use mathematical logic to organise scientific knowledge and as a tool of reasoning and argumentation in daily life.

2.2. Learning objectives

By the end of this topic, the student must be able to do the following:

Knowledge and understanding	Skills	Attitudes and values
• Distinguish between statements and propositions • Convert into logical formula composite propositions, and vice versa • Draw the truth table of a composite proposition • Recognise the most often used tautologies (example: De Morgan's Laws)	• Use mathematical logic to infer conclusions from given propositions • Show that a given logic statement is tautology or a contradiction	• Judge situations accurately and act with equality • Observe situations and make appropriate decisions • Appreciate and act with thoughtfulness: grasp and demonstrate carefulness. • Develop and show mutual respect • Demonstrate broadmindedness

2.3. Content

1 Introduction and fundamental definitions

2. Propositional logic:

- Truth tables
- Logical connectives
- Tautologies and contradictions

3. Predicate logic:
    - Propositional functions
    - Quantifiers
4. Applications:
    - Set theory
    - Electric circuits.

2.4. Materials required

Manila papers, markers, rulers

2.5. Generic competences

- Communication, Research , Cooperation, Problem solving, Critical thinking

2.6. Guidance on teaching and learning activities

a. Introductory activity

Guidance on introductory activity 2

See Form groups of learners that are as heterogeneous as possible and guide them to work on the introductory activity.

Answer of introductory activity 2

Lead learners to know that in the given activity, you can get different answers depending on the given conditions.

Through class discussions, let learners think of different possible solutions and justify their validity.

1) During the presentation, let learners discover the concept of logic.

Answers may vary; some of them are as follows:

i) If you give a child an orange and another an orange, they will have two oranges as .

ii) If you give a child an orange and another an orange, they will have none as they will have eaten them.

From different given and justified solutions, guide students to deduce that logic is a process of constructing arguments by careful deduction.

2. a) T; b) F; c) Neither true nor false; d) F ; e) Neither true nor false; f) Neither true nor false; g) T.

b. Reinforcement activity

Learners research on the difference between a statement and a proposition. They also research further on propositional logic and its application. Give them Activity 2.1 of the Student’s Book to work on.

Main activities

1. Introduce the unit by giving some examples of statements and propositions. Guide students to give the definitions for statement and proposition. Let them attempt, in pairs, Application activity 2.1 of the Student’s Book.
2. Let them, in groups, convert logical formulae to composite propositions and vice versa. Guide them on how they can use mathematical logic to infer conclusions from given propositions.
3. Explain to them what truth tables are. Give them Activity 2.2 and then guide them in drawing truth tables of composite propositions. Let them attempt Application activity 2.2 of the Student’s Book. Ensure they get these correct.
4. Let the students do Activity 2.3 as found of the Student’s Book. Discuss with students if a given logic statement is tautology or a contradiction. Guide them in recognition of the most commonly used tautologies, that is De Morgan’s Laws. Let them work in pairs to do Application activities 2.3 and 2.4 respectively of the Student’s Book. Let them do Activity 2.4.
5. Introduce logical quantifiers. Let them attempt Activity 2.5 of the Student's Book.
6. Introduce the students to applications of proposition and predicate logic. Let them attempt Activities 2.6 and 2.7 of the Student's Book.
7. Application activity 2.5 and 2.6 respectively should be adequate to assist them practise on Applications.
8. Guide them to develop skills and attitudes such as accurate judgement and fairness.

Learners of varying strengths and abilities

(a) Gifted and talented

Guide the learners to study units ahead of others. You can provide more advanced material to the learner. For example, let them draw extra graphs and allow them to do extra exercises. Give them individual exercises to work on as homework.

(b) Slow learners

You should provide them with the non-restrictive environment that provides/ maintains their degree of freedom, self-determination, dignity and integrity of mind and body. This should, however allow for participation in an inclusive setting. Give them extra tuition. Teach them to master the material skills that are commensurate with their ability levels before introducing complex components.

2.7. Additional tasks

Application activity 2A - For slow learners

1. Use a truth table to show that the following statement is logically equivalent:

   $\sim(p \wedge q) \equiv \sim p \vee \sim q$

2. Negate the statement $\forall x \in \mathbb{R}: x^2 \geq 0$

Answers

1. a) $\sim(p \wedge q) \equiv \sim p \vee \sim q$

P	q	~p	~q	p ∧ q	~(p ∧q)	~p V ~ q
T	T	F	F	T	F	F
T	F	F	T	F	T	T
F	T	T	F	F	T	T
F	F	T	T	F	T	T

2. $\exists x \in \mathbb{R}: x^2 < 0$

Task 2B - For talented learners

1. Find the negations of the following proposition

   a) $\forall x \in D, P(x)$

   b) $\exists x \in D, P(x)$

2. Let $P(x,y): x = y + 3$ with domain the collection of natural numbers, $\mathbb{N}$. Find the truth values of the propositions

   a) For $P(5, 2)$ and
   b) $P(6, 4)$
   c) $P(12, 9)$

Answers

1. a) $\sim[\forall x \in D, P(x)] = \exists x \in D, \sim P(x)$

b) $\sim[\exists\, x \in D, P(x)] = \forall\, x \in D, \sim P(x)$

2. By substitution in the expression of P, we find

   a) For P(5, 2), 5 = 2 + 3. Therefore P(5, 2) is true

   b) For P(6, 4), 6 ≠ 4 + 3. Therefore P(6, 4) is false

   c) For P(12, 9), 12 = 9 + 3. Therefore P(12, 9) is true

2.8. Additional information

Additional example

Deduce if the given statement is or is not a proposition

- r: 4 < 8
- s: If x = 4 then x + 3 = 7
- t: Nyanza is a chief city of Rwanda
- p: What a beautiful evening!

Solution

The statements r, s, t are logic propositions while statements p is not a logic proposition.

Venn diagrams

Venn diagrams can also be used to check the validity of arguments. An argument is the assertion that statement S follows from other statements S_1, S_2, …etc. Statements S_1, S_2, …etc. are called premises or hypothesis and the statement S is called the conclusion.

Note that the argument consists of two parts. One part of an argument consists of all hypothesis and the other part contains the conclusion derived from the hypothesis.

Proof of De Morgan's Law

1. $\sim(p \vee q) = \sim p \wedge \sim q$

p	q	~p	~q	$p \vee q$	$\sim(p \vee q)$	$\sim p \wedge \sim q$
T	T	F	F	T	**F**	**F**
T	F	F	T	T	**F**	**F**
F	T	T	F	T	**F**	**F**
F	F	T	T	F	**T**	**T**

2. $\sim(p \wedge q) = p \vee \sim q$

p	q	~ p	~q	p ∧ q	~(p ∧ q)	~p ∨ ~q
T	T	F	F	T	**F**	**F**
T	F	F	T	F	**T**	**T**
F	T	T	F	F	**T**	**T**
F	F	T	T	F	**T**	**T**

Assessment criteria

Learner is able to use mathematical logic:

- to organise scientific knowledge
- as a tool of reasoning and argumentation in daily life.

2.9. Answers for application activity of Unit 2 in the Student's Book

Application activity 2.1

(a) Proposition since it is a declarative sentence.

(b) Proposition since it is a declarative sentence.

(c) Proposition since it is a declarative sentence.

(d) Not proposition since it is not a declarative sentence.

(e) Proposition since it is a declarative sentence.

(f) Proposition since it is a declarative sentence.

Application activity 2.2

1. a)

p	~p	p∧(~p)
T	F	F
F	T	F

b)

p	~p	p∧(~p)	~[p∧(~p)]
T	F	F	T
F	T	F	T

c)

p	~ q	~ q	p ∧ ~ q
T	T	F	F
T	F	T	T
F	T	F	F
F	F	T	F

d)

p	q	~p	~p ∨ q
T	T	F	T
T	F	F	F
F	T	T	T
F	F	T	T

e)

p	q	~ q	p ∧ (~ q)	~[p ∧ (~ q)]
T	T	F	F	T
T	F	T	T	F
F	T	F	F	T
F	F	T	F	T

f)

p	q	r	q ∨ r	p ∧(q ∨ r)
T	T	T	T	T
T	T	F	T	T
T	F	T	T	T
T	F	F	F	F
F	T	T	T	F
F	T	F	T	F
F	F	T	T	F
F	F	F	F	F

g)

p	q	$p \wedge q$	$\sim p$	$(p \wedge q) \wedge (\sim p)$
T	T	T	F	F
T	F	F	F	F
F	T	F	T	F
F	F	F	T	F

h)

p	q	$\sim p$	$\sim q$	$(\sim p) \vee (\sim q)$	$\sim[(\sim p) \vee (\sim q)]$
T	T	F	F	F	T
T	F	F	T	T	F
F	T	T	F	T	F
F	F	T	T	T	F

i)

p	q	$\sim p$	$\sim q$	$\sim p \vee q$	$\sim p \wedge \sim q$	$(\sim p \vee q) \wedge (\sim p \wedge \sim q)$
T	T	F	F	T	F	F
T	F	F	T	F	F	F
F	T	T	F	T	F	F
F	F	T	T	T	T	T

2. a)

p	q	$p \wedge q$	$\sim(p \wedge q)$	$\sim p$	$\sim q \vee \sim q$	$\sim p (\sim q \wedge \sim q)$
T	T	T	F	F	F	F
T	F	F	T	F	T	T
F	T	F	T	T	F	T
F	F	F	T	T	T	T

We see that the columns of $\sim (p \wedge q)$ and $\sim p \vee \sim q$ are identical.

b)

p	q	$\sim p$	$\sim q$	$p \vee q$	$\sim p \wedge \sim q$	$\sim (\sim p \wedge \sim q)$
T	T	F	F	T	F	T
T	F	F	T	T	F	T
F	T	T	F	T	F	T
F	F	T	T	F	T	F

We see that columns of $p \vee q$ and $\sim (\sim p) \wedge (\sim q)$ are identical.

3. (a) This is a conjunction statement. So we combine by using the word "and" symbol Λ.

Now: Let p = Tuyishimire plays football
q = Tuyishimire plays netball

The statement is written as $p \wedge q$ and read p and q.

Truth table for $p \wedge q$.

$p \wedge q$ is true only when p is true and q is true and otherwise it is false.

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

(b) Let p be "I work hard" and q be "I will pass the examination".

Then the sentence is written $p \Rightarrow q$.

Truth table for $p \Rightarrow q$

p	q	$p \Rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

(c) Let p be "A number is even"

q be "it is divisible by 2"

The statement is written as $p \Leftrightarrow q$ and read as "p if and only if q", p if q.

Truth table for $p \Leftrightarrow q$.

p	q	$p \Leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

4. (a) Let p be Kalisa plays football, negation of p is the opposite of ~p denoted as ~p.

The truth table

p	~p
T	F
F	T

(b) i. $p \wedge \sim q$ = Nsengimana speaks Kinyarwanda and not French

ii. ~ (~p) = Nsengimana speaks Kinyarwanda.

iii. $\sim p \vee \sim q$ = Nsengimana does not speak Kinyarwanda or French

iv. $\sim p \wedge \sim q$ = Nsengimana does not speak neither Kinyarwanda nor French.

(c) i. Truth table for $\sim(\sim p \wedge q)$

p	q	~p	$\sim p \wedge q$	$\sim(\sim p \wedge q)$
T	T	F	F	T
T	F	F	F	T
F	T	T	T	F
F	F	T	F	T

ii. Let p be "Today is Monday"

Let q be "Nyanza Football club is playing" in symbolic form the given compound statement is written as $p \wedge \sim q$.

5. Let p: you go to the market.

q: you will need money.

r: you will be able to buy something.

In symbolic form we have, $p \Rightarrow (q \vee \sim r)$

The truth table of $P \Rightarrow ((q \vee (\sim r))$

p	q	r	~r	q v (~r)	p ⇒ ((q v (~r))
T	T	T	F	T	T
T	T	F	T	T	T
T	F	T	F	F	F
T	F	F	T	T	T
F	T	T	F	T	T
F	T	F	T	T	T
F	F	T	F	F	T
F	F	F	T	T	T

Application activity 2.3

p	q	r	p ⇒ q	q ⇒ r	(p ⇒ q) ∧ (q ⇒ r)
T	T	T	T	T	T
T	T	F	T	F	F
T	F	T	F	F	F
T	F	F	F	T	F
F	T	T	T	T	T
F	T	F	T	F	F
F	F	T	T	T	T
F	F	F	T	T	T

Application activity 2.4

1. Let p: Murerwa is not home.

   q: Iyakaremye is doing communal work

   p v r: Iyakaremye is not home or Iyakaremye is doing communal work

   We know that by De Morgan's law, ~(p ∨ q) = ~ p ∧ ~q

p	q	~p	~q	p v q	~(p v q)	(~p) ∧ (~q)
T	T	F	F	T	F	F
T	F	F	T	T	F	F
F	T	T	F	T	F	F
F	F	T	T	F	T	T

Thus the negation of p ∨ q: Murerwa is not home or Iyakaremye is doing communal work is ~(p ∨ q) = ~p ∧ ~ q

We see that ~(p v q)= (~p)∧(~q)

~(p v q)= (~p) ∧ (~q): Murerwa is home and Iyakaremye is not doing communal work.

2. Let p: Nyirarukundo buys banana

q: Muragijimana buys an orange

p∨ q: If Nyirarukundo buys banana, then Muragijimana buys an orange.

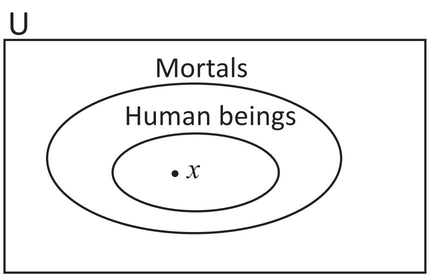
We know by De Morgan's Law that ~ p ⇒ q = (p ∧ ~q)

p	q	~p	~q	p ⇒ q	~(p ⇒ q)	p∧(~q)
T	T	F	F	T	F	F
T	F	F	T	F	T	T
F	T	T	F	T	F	F
F	F	T	T	T	F	F

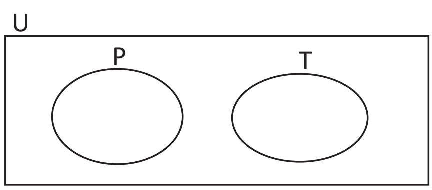
Thus, the negation of 'If Nyirarukundo buys a banana, then Muragijimana buys an orange' is 'Nyirarukundo buys an orange and Muragijimana does not buy an orange'.

Application activity 2.5

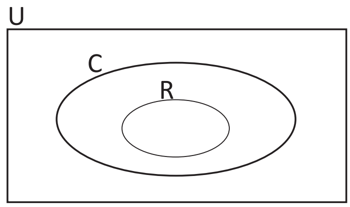
1. (a)



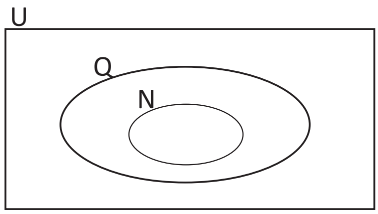
(b)



(c)

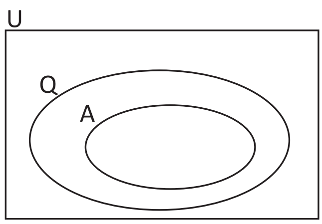


(d)



2. (a) Invalid
   (b) Invalid
   (c) Invalid
   (d) Valid

3. (a)

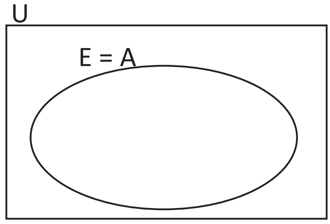


U = The set of all quadratic equations

Q = The set of all quadratic equations

A = The set of quadratic equations having two real roots

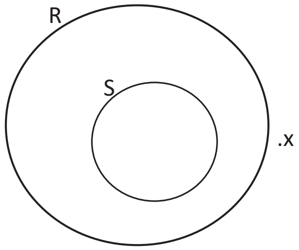
(b)



U = Set of all triangles
E = Set of all equilateral triangles

A = Set of all equiangular triangles [here sets A and E are the same].

4. (a)



The truth of the statement S_1 is represented by placing the set S entirely inside the set R, as shown in the above figure.

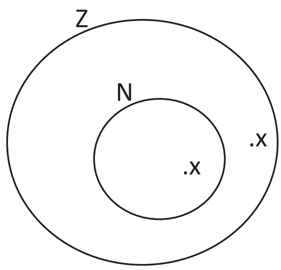
Now, the truth of the statement S_2 is represented by placing labelled 'x' outside the set R as shown in figure.

Since the dot 'x' is outside the set R of rectangles, it is necessarily outside the set S of squares.

Therefore, we conclude that 'x' is not a square.

Hence the given argument is valid.

(b) Let Z = The set of integers and N = The set of natural numbers.



The truth of the statement S_1 is represented by placing the set N entirely inside the set Z and the truth of the statement S_2 is represented by placing a do labeled 'x' inside the set Z. This dot may be inside N or outside it but inside Z.

Now S_1 and S_2 are true

The set N is entirely inside the Z and dot labeled 'x' may or may not be in N

x is not necessary a natural number

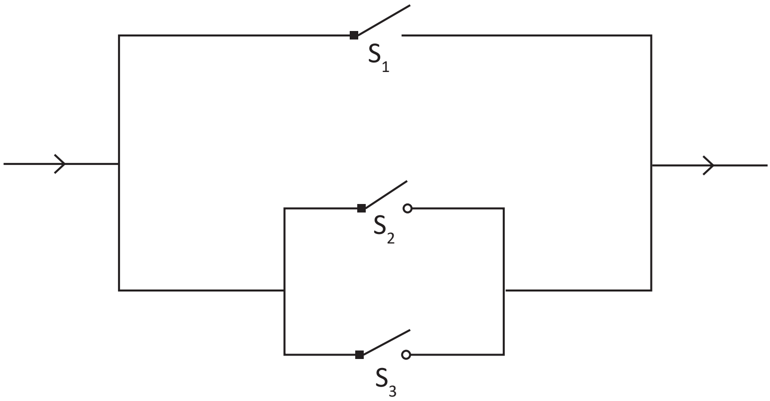
S need not be true

Hence, the given argument is valid.

Note: An argument consisting of the hypothesis is S_1, S_2, ...,S_n and conclusion S is said to be valid if S is true whenever all S_1, S_2, ..., S_n are.

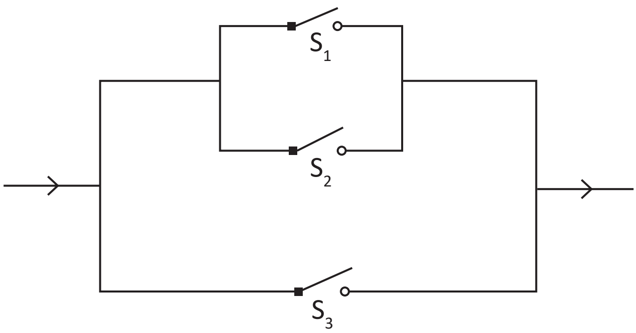
Application activity 2.6

1. The circuit is given by the diagram below.



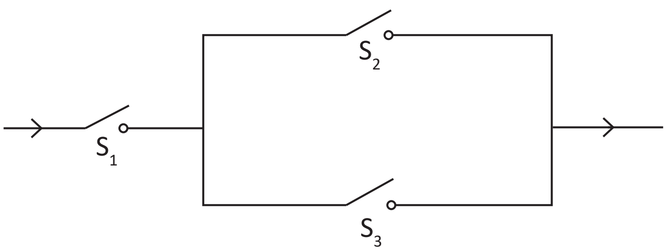
The current flows in the circuit if either S_1 is closed or both S_1 and S_3 are closed.

2. The circuit is given by the diagram



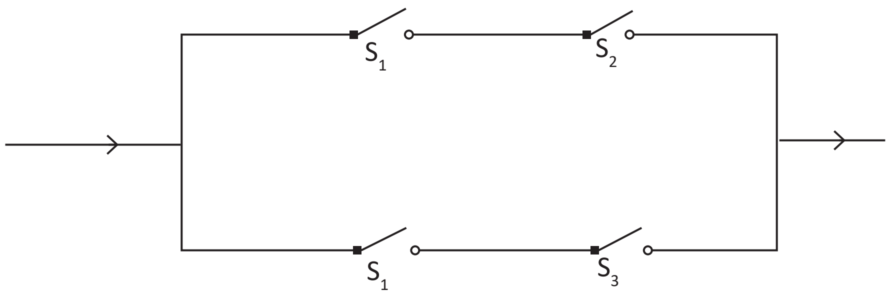
The current flows in the circuit if both S_1 and S_2 are closed or S_3 is closed.

3. The circuit is given by the diagram below



There is a flow of current in the circuit if and only if S_1 is closed and either S_2 is closed or S_3 is closed.

4. The circuit is given by the diagram below



There is a flow of current in the circuit if and only if S_1 is closed and either S_2 is closed or S_3 is closed.

In other words, the above two circuits are equivalent.

So, we have $p \wedge (q \vee r) = (p \wedge q) \vee (p \wedge r)$.

UNIT

3 Binary operations

Number of periods: 14

3.1. Key unit competence

Use mathematical logic to understand and perform operations using the properties of algebraic structures

3.2. Learning objectives

By the end of this unit, the student must be able to do the following:

Knowledge and understanding	Skills	Attitudes and values
• Define a group, a ring, an integral domain and a field • Demonstrate that a set is (or is not) a group, a ring or a field under given operations • Demonstrate that a subset of a group is (or is not) a sub group	• Determine the properties of a given binary operation • Formulate, using adequate symbols, a property of a binary operation and its negation • Construct the Cayley table of order 2, 3, 4 • Discover a mistake in an incorrect operation	• Appreciate the importance and the use of properties of binary operations • Show curiosity, patience, mutual respect and tolerance in the study of binary operations

3.3. Content

1. Definitions and properties
2. Groups and rings
3. Fields and integral domains
4. Cayley tables

3.4. Materials required

Digital instruments such as calculators, counters

3.5. Generic competences

- Problem-solving

- Cooperation
- Communication
- Lifelong learning

Cross-cutting issues

Peace and values education

Learners will be groups of various abilities and work with a respectful attitude.

Standardization culture

Learners make use of rules and standards when constructing Cayley tables.

3.6. Guidance on teaching and learning activities

a. Introductory activity

Guidance on introductory activity 3

See the guidance on introductory activity 1.

Ensure that the learners have understood what the unit will be about and they are eager to learn.

Answer of introductory activity 3:

Given the operation L defined in a set containing the element a and element b such that: $aLb = (a+b) \div b$

a) $(a+b) \div b$ is a real number only if b is different from zero.

b) The operation L is not commutative except when a=b and both are different from zero.

b. Main activities

1. Introduce the unit by giving some examples of binary operations.
2. Let them do Activity 3.2 of the Student's Book. Then let one student from each group to present their findings. Guide them in defining the term group. Let them attempt, in pairs, Application activity 3.1 of the Student's Book.
3. Guide them in defining rings. Let them attempt, in pairs, Application activity 3.2 of the Student's Book.
4. Guide them in defining fields and integral domains. Explain what Cayley tables are and demonstrate to them how to contruct tables of order 2, 3, 4.

5. Guide them on how to discover a mistake in an incorrect operation. Let them, in pairs, attempt application activity 3.3 of the Student's Book.

Learners of varying strengths and abilities

(a) Gifted and talented

Guide the learners to study units ahead of others. You can provide more advanced material to the learner. For example, let them solve additional problems on construction of Cayley tables and allow them to do extra exercises. However, ensure that they are supervised so as not to overwhelm the less gifted.

Give them individual exercises to work on as homework.

(b) Slow learners

You should provide them with the non-restrictive environment that provides and maintains their degree of freedom, self-determination, dignity and integrity of mind and body. This should, however allow for participation in an inclusive setting. Give them extra tuition. Teach them to master the material skills that are commensurate with their ability levels before introducing complex components.

3.7. Additional tasks

Task 3A - For slow learners

1. Let $*$ be a binary operation defined on the set $\mathbb{Z}$ of all integers by $x * y = x + y - 1$.

    a) Determine whether the operation is commutative

    b) Find an identity element

    c) Find a symmetric element

Answers

1. a) The operation is commutative if $x, y \in \mathbb{Z}$; $x * y = y * x$

    $x * y = x + y - 1$ and $y * x = y + x - 1 = x + y - 1$

    Thus $x * y = y * x$ and the operation is commutative

    b) There is an identity element $e \in \mathbb{Z}$ if $\forall x \in \mathbb{Z}$; $x * e = x$

    $x * e = x + e - 1 = x \Leftrightarrow e - 1 = 0 \Leftrightarrow e = 1$.

    Thus the $*$ operation admits $e = 1$ as the identity element in $\mathbb{Z}$.

    c) A symmetric (inverse) element of x in $\mathbb{Z}$ is x' if $\forall x \in \mathbb{Z}$; $x * x' = e$

$$x * x' = x + x' - 1 = e \Leftrightarrow x' = e - x + 1 = 1 - x + 1 = 2 - x$$

The $*$ operation admits $x' = 2 - x$ as a symmetric (inverse) of x in $\mathbb{Z}$.

Task 3B - For talented learners

1. Let $*$ be a binary operation defined on the set G by $\forall x, y \in G:\ a * b = (a+b) + ab$.

Show that $(G, *)$ is a commutative group.

Answers

1. The set G with the operation $*$ written $(G,*)$ is commutative group since it satisfies the following properties:

1) Closure: $\forall a, b \in G: a * b = [(a + b) + (ab)] \in G$ Verified

2) 2) Associative: $\forall a,b,c \in G: (a*b)*c = a*(b*c)$

In fact, LHS gives us

$$a*(b*c) = (a+b+ab)*c$$
$$= a+b+ab+c+(a+b+ab)c$$
$$= a+b+ab+c+ac+bc+abc$$

And RHS gives us

$$a*(b*c) = a*(b+c+bc)$$
$$= a+b+c+bc+a(b+c+bc)$$
$$= a+b+c+bc+ab+ac+abc)$$

LHS=RHS, so the associativity is verified.

3) Identity element: $\forall a \in G, \exists e \in G: a * e = e * a = a$. We find the identity element,

$a * e = a + e + ae = a \Leftrightarrow e + ae = 0 \Leftrightarrow e(1 + a) = 0 \Leftrightarrow e = 0$

The identity element is $e = 0$. Verified

4) Inverse element: $\forall a \in G, \exists a' \in G : a * a' = a' * a = 0$

$a * a' = a + a' + aa' = \Leftrightarrow a + a'(1 + a) = 0 \Leftrightarrow a'(1 + a) = -a$

$a' = \frac{-a}{1+a}$

Thus, the inverse of a is $a' = \frac{-a}{1+a}$. Verified

5) Commutative: $\forall a, b\ G: a * b = b * a$

In fact $a * b = a + b + ab$ and $b * a = b + a + ba = a + b + ab$

Thus, a + b = b ∗ a. Verified

All the 5 properties above prove that (G,∗) is a commutative group.

Assessment criteria

1. Explain why grouping, interchanging and distributing is correct or not, depending on context.
2. Carry out binary operations and determine their properties.

3.8. Answers to Application activities of Unit 3 in the Student's Book

Application activities 3.1

1. We show if the following properties hold.
   - The sum of any two real numbers is a real number, so the real numbers are closed under addition.

     $\forall x, y \in \mathbb{R} : x + y \in \mathbb{R}$
   - Addition on the real numbers is associative.

     $\forall\ x, y, z \in \mathbb{R}; (9x + y) + z = x + (y + z) \in \mathbb{R}$
   - The identity element is 0, since

     $\forall x\ ; \in \mathbb{R}; x + 0 = 0 + x = x$
   - The inverse of x is –x for all $x \in \mathbb{R}$, since

     $x + (-x) = (-x) + x = 0$
   - Addition of real numbers is commutative.

     $\forall x, y \in \mathbb{R}; x + y = y + x \in \mathbb{R}$

   Conclusion: $\mathbb{R}$ is a commutative group (Abelian Group)
2. We show if the following properties hold.
   - For any numbers a, $b \in \mathbb{Z}$; $a + b \in \mathbb{Z}$ (closure)
   - *For all* $a,b,c \in \mathbb{Z}:\ (a+b)+c = a+(b+c)$ (associative)
   - For every $a \in \mathbb{Z}$, a + 0 = 0 + a = a (0 is the identity element)
   - For every $a \in \mathbb{Z}$; a+ (–a) = –a + a = 0(–a is the inverse element of a)

   Since all the properties are satisfied, then it is a group under addition.
3. (a) Odd integers under addition is not a group

   (b) $\mathbb{Z}$ is not a group under multiplication

(c) $\mathbb{N}$ is not a group under addition

(d) $\mathbb{R}$ is a group under multiplication

(e) $\mathbb{Q}$ is a group under addition

Application activity 3.2

These are obvious as they are from the following basic properties of a commutative ring

1. If ($\mathbb{R}$,+,•) is a commutative ring, with a, b, c $\in$ $\mathbb{R}$. The following hold:

   a•0 = 0•a = 0

   a(–b) = (–a) b = –ab

   –(–a) = a

   If a + b = a + c then b = c

   If a + a = a then a = 0

   (–1) a = –a

2. Knowing that in $\mathbb{R}$, multiplication has an identity 1,

   Then $(-1)a = -a$

   If $a \in \mathbb{R}$ has a multiplicative inverse a^{-1} then $a^{-1}a = 1$

Application activity 3.3

1. (a) Commutativity
   (b) The identity element
   (c) The inverse element

2. The Cayley table for the set $S = \{1, -1, i, -i\}$ under " "

.	1	–1	i	–i
1	1	–1	i	–i
–1	–1	1	–i	i
i	i	–i	–1	1
–i	–i	i	1	–1

   (i) All the results are elements of the set S = {1, –1, i, –i }. The operation is **closure** on the set S.

(ii) There is **associativity**: For instance, i • (1• –i) = (i •1)• –i = 1

(iii) The **identity element** is 1 since For every element x of S,

x • 1 = x • 1 = x

(iv) There exists the **inverse element**: for every element x of S: For instance the inverse of 1 is –1, the inverse of i is – i.

(v) There is a **commutativity**: For instance –1 • i = i • – 1 (The entry inside the table are symmetrical about the leading diagonal.)

We conclude that (S, •) is a commutative group.

3. Cayley table for the set S = { f, g, h} under the composition operation '∘'

∘	f	g	h
f	f	g	h
g	g	f	–h
h	h	–h	f

(a) The composition is not closure since the result –h is not the element of the original set.

(b) The composition is associative since for example

$$[f(g)](h) = g(h) = -h$$

$$\text{and } f[g(h)] = f(-h) = -h$$

which shows us *that*

$$[f(g)](h) = f[g(h)]$$

(c) f is the identity element.

(d) Each function is the inverse of itself, g ∘ h = h ∘ g = –h.

(e) The composition is commutative since, for instance , f ∘ g = g ∘ f = g

UNIT

4 Set $\mathbb{R}$ of real numbers

Number of periods: 24

4.1. Key unit competence

Think critically using mathematical logic to understand and perform operations on the set of real numbers and its subsets using the properties of algebraic structures.

4.2. Learning objectives

By the end of this unit, the student must be able to do the following:

Knowledge and understanding	**Skills**	**Attitudes and values**
• Match a number and the set to which it belongs • Define a power, an exponential, a radical, a logarithm, the absolute value of a real number.	• Classify numbers into naturals, integers, rational and irrationals • Determine the restrictions on the variables in rational and irrational expressions • Illustrate each property of a power, an exponential, a radical, a logarithm and the absolute value of a real number • Use logarithm and exponentials to model simple problems about growth, decay, compound interest and magnitude of an earthquake. • Transform a logarithmic expression to equivalent power or radical form, and vice versa • Rewrite an expression containing "absolute value" using order relation	• Appreciate the importance and the use of properties of operations on real numbers • Show curiosity for the study of operations on real numbers

4.3. Content

1. Properties of real numbers
2. Absolute value and its properties
3. Powers and radicals
4. Decimal logarithms properties

4.4. Materials required

Graph papers, manila papers, digital technology instruments including calculators

4.5. Generic competences

Communication, Problem-solving, Research, Cooperation , Critical thinking

Cross-cutting issues

- Peace and values education

  Listening to others and contributing to solutions of challenges.

- Inclusive education

  For learners with impaired vision, fellow student's work with them in groups using large teaching learning visual aids. (See Activity 4.2.)

- Financial education

  Tasks and activities such application activity 4.5 that involve bank transactions and investments.

4.6. Guidance on teaching and learning activities

a. Introductory activity

Guidance on introductory activity 4

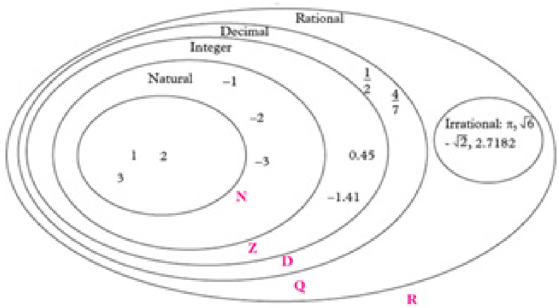
Form small groups of learners and guide them to work on the introductory activity.

After presenting their finding, the teacher harmonizes and guides class discussions and interventions.

Answer of introductory activity 4 (Answers may vary)

1. Lead learners to know that in the question1, set of numbers they already know from senior one (S1) in secondary schools, are: $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \ldots$
2. The numbers we use in counting plus zero are called Natural numbers; integers are numbers which have either negative or positive sign and includes zero. The set of integers is represented by $\mathbb{Z}$; the set of **rational numbers** $\mathbb{Q}$ and the set of **irrational numbers** I form the set of real numbers. The set of real numbers is denoted by $\mathbb{R}$

3. Some examples of numbers in each set:



4. The relationship between set of numbers is as follows: Natural numbers are part of integers, integers are part of rational numbers, rational numbers and irrational numbers are pats of real numbers. Therefore, $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}$.

Give them a mental task to find out the main facts about sets of real numbers.

b. Main activities

1. Introduce the unit by giving some examples of sets of numbers.
2. Guide them in giving the definitions of a power, an exponential, a radical, a logarithm and the absolute value of a real number. Guide them in classifying numbers.
3. Let students work on Activity 4.2 of the Student's Book.
4. Let them attempt Activity 4.3 to find out the meaning of absolute value of a real number. This should be followed by Application activity 4.1 to grasp the concept of absolute values.
5. Let them research on the meaning of the term 'power' in pairs. They will do Activity 4.4 of the Student's Book.
6. Let them do Activity 4.5 that involves researching on the meaning of radicals. You will supervise the class discussion and help them come up with a concise meaning of radicals.
7. Take them through the symbolisms and meanings of roots and surds. Guide them to illustrate each property of a power, an exponential, a radical, a logarithm and an absolute value of a real number. Explain the meaning of rationalization. Let them work on Application activity 4.2.
8. Guide them in using logarithms and exponentials to model simple problems such as on growth, decay, compound interest and magnitude of an earthquake. Give them the Activity 4.6, Application activity 4.3 and Activity 4.7 of the Student's Book to practise calculations involving logarithms. The application activity 4.4 should reinforce this.
9. The application activity 4.5 and 80 of the Student's Book should be apt for practise on applications.

Learners of varying strengths and abilities

a. Gifted and talented

You can provide more advanced material to the learners. For example, let them draw extra graphs and allow them to do extra exercises. You can also make them group leaders in their groups so that they indirectly help the slower learners or the not so talented. However, ensure that they are supervised so as not to overwhelm the less gifted.

b. Slow learners

You should provide them with the non-restrictive environment that provides/maintains their degree of freedom, self-determination, dignity and integrity of mind and body. This should, however allow for participation in an inclusive setting. Give them extra tuition. Teach them to master the material skills that are commensurate with their ability levels before introducing complex components.

c. Intellectual impairement

Try to understand the specific talents of the learner and develop them. Break the task down into small steps or learning objectives. Ensure learners start with what they can already do, then move on to a new more challenging task . Give the learner lots of practice and time. It helps to ensure the child has mastered a skill.

4.7.Additional tasks

Application activity 4A - For slow learners

1. Solve the system of equations

   $2^{x-1} + 2^{1-x} = 2$

2. Simplify the following

   i) $7\sqrt{2} - \sqrt{12} + \sqrt{48}$

   ii) $\sqrt{45} - \sqrt{20} + \sqrt{80}$

   iii) $\sqrt{50} + 3\sqrt{18} - 2\sqrt{8} - 7\sqrt{2}$

Answers

1. $2^{x-1} + 2^{1-x} = 2$

   $\frac{2^x}{2} + \frac{2}{2^x} = 2$

$$\frac{2^{2x} + 4}{2 \times 2^x} = 2$$

$2^{2x} + 4 = 4 \times 2^x$

$2^{2x} - 4 \times 2^x + 4 = 0$

Let $y = 2^x$, the equation above becomes

$y^2 - 4y + 4 = 0$

$(y - 2)^2 = 0$

$y = 2$

$2^x = 2$

$x = 1$

2. i) $7\sqrt{3} - \sqrt{12} + \sqrt{48} = 7\sqrt{3} - \sqrt{4 \times 3} + \sqrt{16 \times 3}$

$= 7\sqrt{3} - \sqrt{4} \times \sqrt{3} + \sqrt{16} \times \sqrt{3}$

$= 7\sqrt{3} - 2\sqrt{3} + 4\sqrt{3} = 9\sqrt{3}$

ii) $\sqrt{45} - \sqrt{20} - \sqrt{80} = \sqrt{9 \times 5} - \sqrt{4 \times 5} - \sqrt{16 \times 5}$

$= \sqrt{9}\sqrt{5} - \sqrt{4}\sqrt{5} - \sqrt{16}\sqrt{5}$

$= 3\sqrt{5} - 2\sqrt{5} - 4\sqrt{5} =$

iii) $\sqrt{50} + 3\sqrt{18} - 2\sqrt{8} - 7\sqrt{2} = \sqrt{25 \times 2} + 3\sqrt{9 \times 2} - 2\sqrt{4 \times 2} - 7\sqrt{2}$

$= \sqrt{25}\sqrt{2} + 3\sqrt{9}\sqrt{2} - 2\sqrt{4}\sqrt{2} - 7\sqrt{2}$

$= 5\sqrt{2} + 9\sqrt{2} - 4\sqrt{2} - 7\sqrt{2} = 3\sqrt{2}$

Task 4B - For talented learners

1. A species of lions is introduced into Akagera National Park where previously there were no lions. Six lions were introduced in 2015. It is expected that the population will increase according to $L_t = L_0 \times 2^{0.4t}$ where t is the time since the introduction.

   a) Find L_0

   b) Find the expected lion population in 2020

   c) In which year will the number of lions be 100.

Answers

1. a) $L_0 = 6$ lions

b) $L_t = L_0 \times 2^{0.4t}$

$L_5 = 6 \times 2^{0.4 \times 5}$

$L_5 = 6 \times 2^{0.4\times 5} = 6 \times 2^2 = 24$

The expected number of lions in 2020 is **24 lions**

c) $L_t = L_0 \times 2^{0.4t}$

$100 = 6 \times 2^{0.4t}$

$\frac{100}{6} = 2^{0.4t}$

$\log \frac{50}{3} = \log 2^{0.4t}$

$\log 50 - \log 3 = 0.4t \log 2$

$t = \frac{\log 50 - \log 3}{0.4 \log 2}$

$t = \frac{2.813}{0.277} = 10.15$

The population lion will be 100 in ten years ahead i.e. in 2025.

Assessment criteria

Use mathematical logic to understand and perform operations on the set of real numbers and its subsets using the properties of algebraic structures

4.8. Answers to Application activity of Unit 4 in the Student's Book

Application activity 4.1

(a) $x = \left\{\frac{5}{2}, \frac{15}{2}\right\}$

(b) $x \in [-3, 3]$

(c) $x \leq -5$ or $x \geq -2$

(d) $x = \{-1, 4\}$

(e) $x < -3$ or $x > -1$

(f) $x \in \left]-\frac{7}{2}, 4\right[$

(g) $x \in \mathbb{R}$

(h) $x \geq 4$

Application activity 4.2

1. $\frac{3}{2}\sqrt{2}$

2. $\frac{1}{7}\sqrt{7}$

3. $2\frac{\sqrt{11}}{11}$

4. $\frac{3}{5}\sqrt{10}$

5. $\frac{1}{9}\sqrt{3}$

6. $\frac{1}{2}\sqrt{2}$

7. $\sqrt{2}+1$

8. $\frac{\sqrt{3}}{23}(15-3\sqrt{2})$

9. $\frac{1}{3}(4\sqrt{3}+6)$

10. $-5(2+\sqrt{5})$

11. $\frac{1}{4}(\sqrt{7}+\sqrt{3})$

12. $4(2+\sqrt{3})$

13. $\sqrt{5}-2$

14. $\frac{1}{3}(7\sqrt{3}+2)$

15. $3+\sqrt{5}$

16. $3(\sqrt{3}+\sqrt{2})$

17. $\frac{3}{19}(10-\sqrt{3})$

18. $3+2\sqrt{2}$

19. $\frac{2}{3}(7-2\sqrt{7})$

20. $\frac{1}{2}(1+\sqrt{5})$

21. $\frac{1}{4}(\sqrt{11}+\sqrt{7})$

22. $\frac{1}{6}(9+\sqrt{3})$

23. $\frac{1}{14}(9\sqrt{2}-20)$

24. $\frac{1}{6}(3\sqrt{2}+2\sqrt{3})$

25. $\frac{1}{2}(2+\sqrt{2})$

Application activity 4.3

(a) 6 (b) 1 (c) -3 (d) 0

(e) $\frac{1}{2}$ (f) $\frac{1}{4}$ (g) $-\frac{1}{4}$ (h) $\frac{5}{2}$

(i) $\frac{2}{3}$ (j) $\frac{1}{2}$ (k) $\frac{10}{3}$ (l) $\frac{3}{2}$

(m) 2a (n) a + 3 (0) a – 1 (p) a – b

Application activity 4.4

1. (a) log 30 (b) log 8 (c) log 4
   (d) log 35 (e) log 4 (f) log 105
   (g) log 90 (h) log 2 (i) log 16
   (j) log 6,000 (k) log 4 (l) log 2
   (m) log 0.005 (n) log 20 (o) log 28
2. (a) log 32 (b) log 256 (c) 3 – log 2
   (d) 2 – log 4 (e) log 14 (f) log 2
   (g) 1 (h) 8 (i) 1

3. (a) 2 (b) $\frac{3}{2}$ (c) 4

(d) $\frac{3}{2}$ (e) $-\frac{5}{2}$ (f) $-\frac{3}{4}$

Application activity 4.5

1. (a) The growth factor for the town is 1.024.

   (b) $P(t) = 35,000(1.024)^t$.

   (c) Since the equation is designed for future, we cannot estimate the population of the town in the year 2007.

2. (a) The growth factor for the investment is 1.05.

   (b) $A(t) = 300,000(1.05)^t$ where A(t): amount of money after t years, t: period of t years.

   (c) The money will double in between 14 and 15 years.

3. (a) The decay factor for the value of the car is 0.85.

   (b) $C(t) = 25,000(0.85)^t$ where C(t): the value of the car after t years, t: period of t years.

   (c) In 10 years the car will be worth 4921.86 FRW.

4. The amount of investment at the end of 5 years is 819,308.22 FRW.

5. At the end of 96 minutes the number of bacteria would be 602,248.76.

UNIT

5 Linear equations and inequalities

Number of periods: 12

5.1. Key unit competence

Model and solve algebraically or graphically daily life problems using linear equations or inequalities.

5.2. Learning objectives

By the end of this unit, the student must be able to do the following:

Knowledge and understanding	Skills	Attitudes and values
• List and clarify the steps in modeling a problem by linear equations and inequalities	• Equations and inequalities in one unknown • Parametric equations and inequalities in one unknown. • Simultaneous equations in two unknowns. • Applications: o Economics (problems about supply and demand analysis) o Physics (linear motions, electric circuits) o Chemistry (balancing equations)	• Appreciate, value and care for situations involving two linear equations and linear inequalities in daily life situation • Show curiosity about linear equations and linear inequalities

5.3. Content

1. Equations and inequalities in one unknown
2. Parametric equations and inequalities
3. Simultaneous equations in two unknowns

4. Applications
   - Economics
   - Physics
   - Chemistry

5.4. Materials required

Geometric instruments (ruler, T-square ...), calculators.

5.5. Competences

Communication, Research , Cooperation , Problem solving , Critical thinking

Cross-cutting issues

- **Financial education**

To plan about how to use money and to do savings.

- **Inclusive education**

Those with impairements and disabilities are mixed with the rest in the groups and assisted when necessary.

5.6. Guidance on teaching and learning activities

a. Introductory activity

Guidance on introductory activity 5

Invite learners to work in groups where they read and analyse the problem in introductory activity 5.

Basing on their experience, results from their own research, prior knowledge and abilities shown in answering the questions for this activity, use different questions to facilitate learners give their predictions and ensure that you arouse their curiosity on what is going to be leant in this unit.

Answers for introductory activity 5

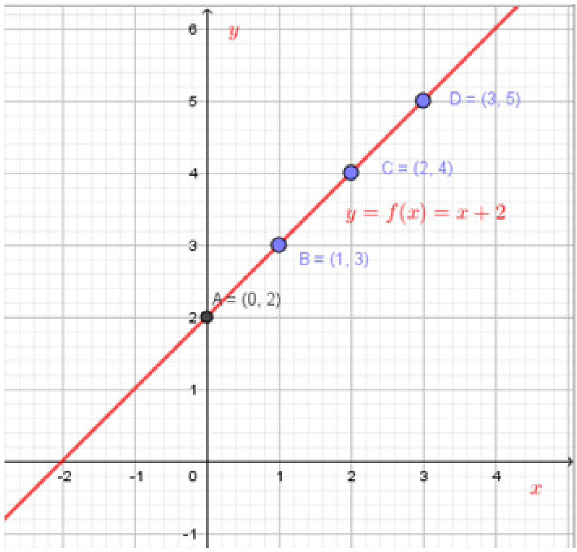
1) Refer to the student's book and verify answers for students.

2) If x is the number of pens for a learner, the teacher decides to give him/her two more pens. A learner with one pen will have(1+2)pens=3pens

a) y=f(x)=x+2

X	-2	-1	0	1	2	3	4
y=f(x)=x+2	0	1	2	3	4	5	6
(x,y)	(-2;0)	(-1;1)	(0,2)	(1;3)	(2;4)	(3;5)	(4;6)

b) The graph obtained is the following:



c) The graph obtained is a line.

d) y=x+2 this is a linear equation because its graph is a line. Identically, $x+2\geq 0$ is a linear inequality.

3) Students will give different examples. Verify whether the solution involves the linear equation.

b. Main activities

1. Let students work on Activity 5.1 of the Student's Book. Encourage them to work with cooperation when it comes to discussions.
2. Introduce the unit by listing and clarifying the steps in modelling problems of linear equations and inequalities.
3. Let them attempt Activity 5.2 and Application activity 5.1 on linear equations. They should practise until they perfect the solving of equations.
4. Let them do Activity 5.3 to introduce them to linear inequalities. Ensure they differentiate them from linear equations. They should be able to appreciate the differences.
5. Introduce them to graphs of inequalities. Let them tackle Application activity 5.2 of the Student's Book. They can do the first two numbers in pairs before working on the rest as individuals.
6. Introduce parametric equations by asking learners to research on them. Let them present their findings in class as is recommended in Activity 5.4 of the Student's Book.

7. Let them work on Activity 5.5 in groups. They should take turns in calculations and drawing of graphs. The graphs can be drawn on large manila papers to enable those with visual impairement participate fully.

8. Introduce simultaneous equations using the mental task. Give them adequate practice on finding solutions using different methods.

9. Introduce the applications of equations and inequalities. Let learners discuss and express their appreciation of these in real life, in areas of trade, linear motions, electric circuits and balancing equations. Let them attempt the Application activity 5.3, Activity 5.6 and Application activity 5.4 of the Student's Book.

Learners of varying strengths and abilities

(a) Gifted and talented

Guide the learners to study units ahead of others. You can provide more advanced material to the learner.

You can ask them to research further on application of equations and inequalities in real life and to give their presentations in class.

(b) Slow learners

You should provide them with the non-restrictive environment that provides/maintains their degree of freedom, self-determination, dignity and integrity of mind and body. This should, however allow for participation in an inclusive setting. Give them extra tuition. Teach them to master the material skills that are commensurate with their ability levels before introducing complex components.

5.7. Additional tasks

Application activity 5A - For slow learners

1. Solve and graph the solution of the following inequality

$$\frac{3(x-2)}{5} \leq \frac{5(2-x)}{3}$$

Answers

1. $\frac{3(x-2)}{5} \leq \frac{5(2-x)}{3}$

$$\frac{3(x-2)}{5} \leq \frac{10-5x}{3}$$

$$\frac{3(3x-6)}{15} \le \frac{5(10-5x)}{15}$$

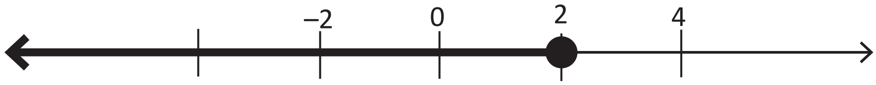
$9x - 18 < 50 - 25x$

$34x \le 68$

$x \le 2$

The solution set is

The solution can be graphed on the number line as follows:



Task 5B - For talented learners

A solution is to be kept between 77°F and 86°F. What is the range in temperature in degree Celcius (C) if the Celcius-Fahrenheit (F) conversion formula is given by

$F = \frac{9}{5}C + 32$?

Answer

We know that $F = \frac{9}{5}C + 32$?

It is given that $77 < F < 86$

$77 < \frac{9}{5}C + 32 < 86$

$77 - 32 < \frac{9}{5}C < 86 - 32$

$45 < \frac{9}{5}C < 54$

$$\frac{5}{9}\times 45 < C < \frac{9}{5}\times 54$$

$25 < C < 30$

Thus, the required range of temperature in °C is between 25°C and 30°C.

Assessment criteria

Model and solve algebraically or graphically daily life's problems using linear equations or inequalities.

5.8. Answers to Application activity of Unit 5 in the Student's Book

Application activity 5.1

(a) $(x + 2)(2x - 1) = 0$

$x + 2 = 0$ or $2x - 1 = 0$

$x = -2$ or $x = -\frac{1}{2}$

(b) $(5x - 15)(3x - 9) = 0$

$5x - 15 = 0$ or $3x - 9 = 0$

$x = \frac{15}{5}$ or $x = \frac{9}{3}$

$x = 3$ or $x = 3$

$x=3$

(c) $x^2 - 5x = 0$

$x(x - 5) = 0$

$x = 0$ or $x - 5 = 0$

$x=0$ or $x=5$

(d) $\frac{3x - 6}{x + 1} = 0$

$3x = 6$ and $x + 1 \neq 0$

$3x = 6$ and $x \neq -1$

$x = 2$ and $x \neq -1$

(e) $\frac{x - 2}{5x + 3} = 0$

$x - 2 = 0$ and $5x + 3 \neq 0$

$x = 2$ and $x \neq -\frac{3}{5}$

(f) $3(x + 7) = 0$

$3x + 21 = 0$

$x = -7$

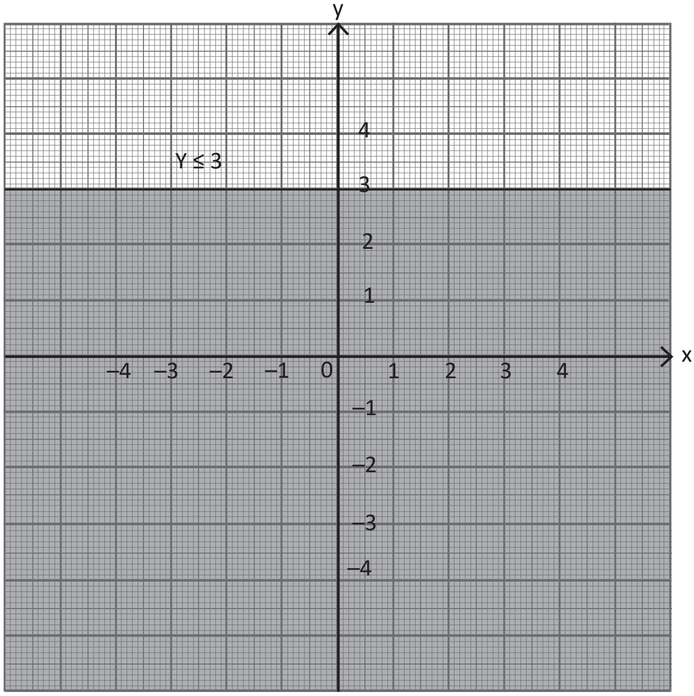
(g) $\frac{3 - x}{2x - 7} = 0$

$3 - x = 0$ and $2x - 7 \neq 0$

$x = 3$ where $x \neq \frac{7}{2}$

Application activity 5.2

1.



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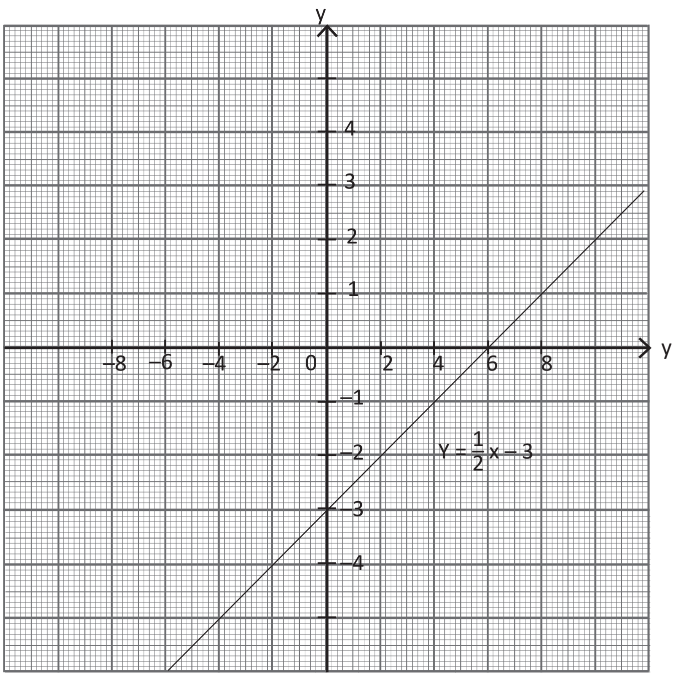

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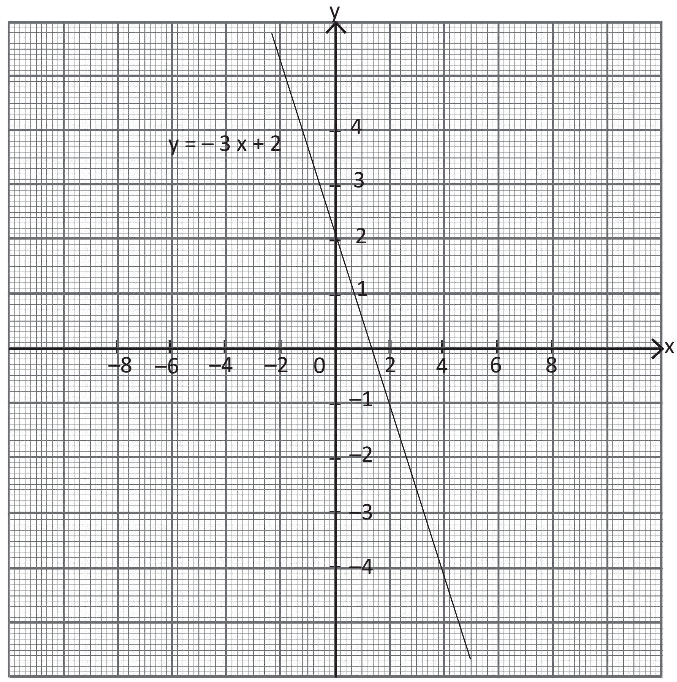

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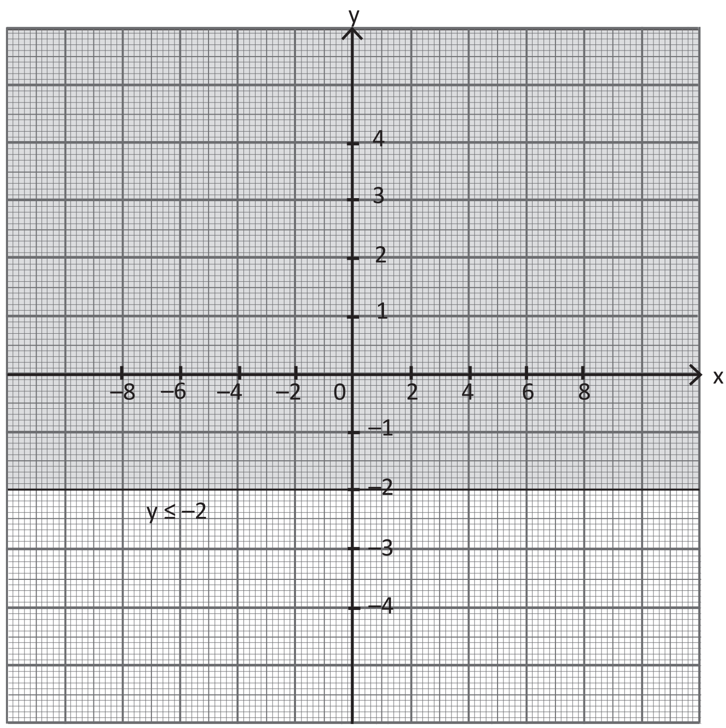

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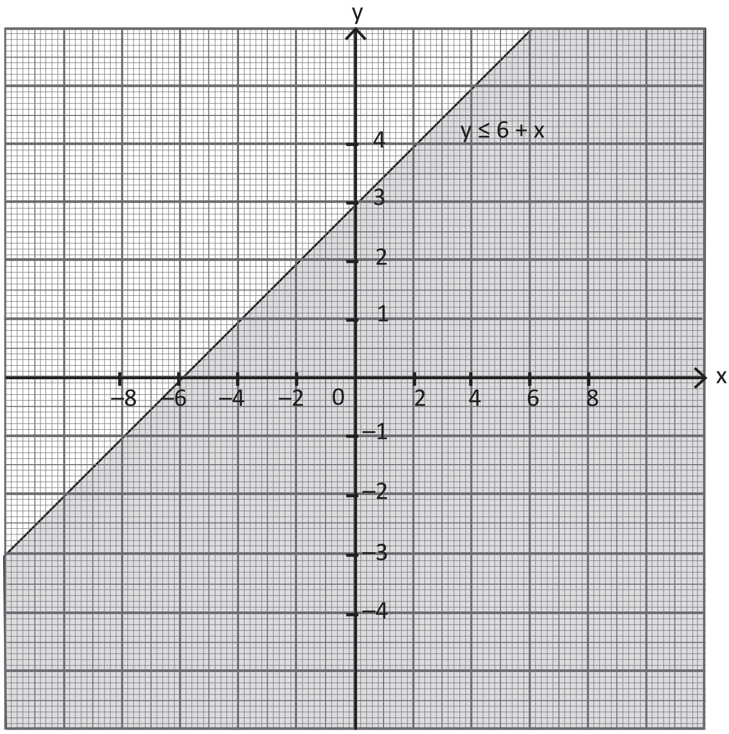

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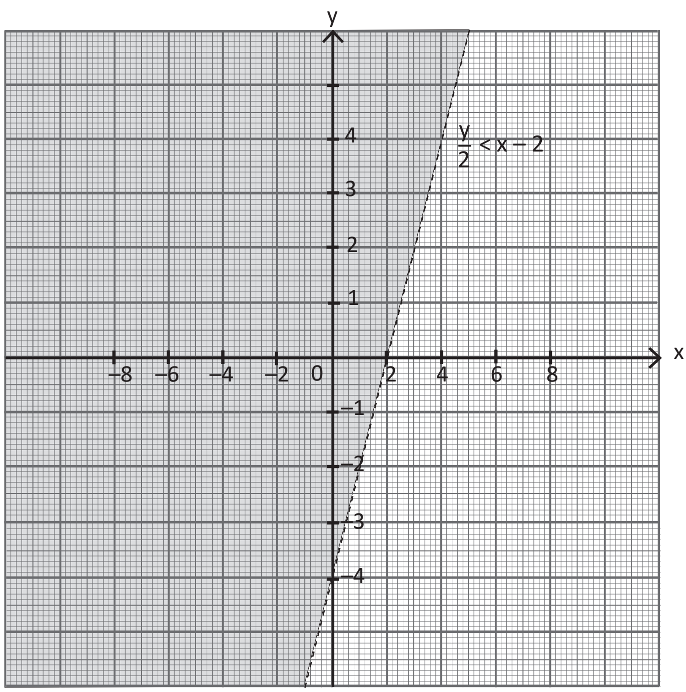


Application activity 5.3

1. Fish fillet cost x FRW

   Egg roll cost y FRW

   We form the system $\begin{cases} x = y + 500 \\ x + y = 2{,}000 \end{cases}$

   We solve for x and y

   $\begin{cases} x = y + 500 \\ x + y + 500 = 2000 \end{cases}$

   $\begin{cases} x = y + 500 \\ y + y = 1{,}500 \end{cases}$

   $\begin{cases} x = y + 500 \\ 2y = 1{,}500 \end{cases}$

   $\begin{cases} x = y + 500 \\ y = 750 \end{cases}$

   $\begin{cases} x = 1{,}250 \\ y = 750 \end{cases}$

   (a) The unit price of fish fillet is 1,250 FRW.

   (b) The unit price of egg roll is 750 FRW.

2. If x crates of Fanta and y crates of Coke are distributed

   We form a system of equations

   $\begin{cases} x + 100 = y \\ x + y = 400 \end{cases}$

   We solve for x and y

   $\begin{cases} x - y = -100 \\ x + y = 400 \end{cases}$

   $\begin{cases} x - y = -100 \\ 2x = 300 \end{cases}$

   $\begin{cases} x - y + -100 \\ x = 150 \end{cases}$

   $\begin{cases} y = 150 + 100 \\ x = 150 \end{cases}$

   $\begin{cases} y = 250 \\ x = 150 \end{cases}$

   150 crates of Fanta and 250 crates of Coke are distributed

3. A box file cost x FRW and a file folder cost y FRW.

We form the system of equations as:

$$\begin{cases} 3x = 2y + 10{,}000 \\ x + 2y = 5{,}000 \end{cases}$$

We solve for x and y

$$\begin{cases} 3x = 2y = 10{,}000 \\ -x - 2y = -5{,}000 \end{cases}$$

$$\begin{cases} 2x = 5{,}000 \\ 2y = 5{,}000 - x \end{cases}$$

$$\begin{cases} x = 2{,}500 \\ y = 1{,}250 \end{cases}$$

Rutaya spends

(i) 2,500 FRW on a box file and

(ii) 1,250 FRW on a file folder.

4. If x_1 is the number of trips on transporting sand to site 1

y_1 is the number of trips on transporting gravel to site 1

If x_2 is the number of trips on transporting sand to site 2

y_2 is the number of trips on transporting gravel to site 2

We form the system as below:

$$\begin{cases} 6{,}500x_1 + 10{,}000y_1 = 63{,}000 \\ 10{,}000x_2 + 13{,}000y_2 = 90{,}000 \end{cases}$$

Since we have the system of two equations with 4 unkowns, we cannot solve the system and thus there is no solution for the problem.

5. If x is the number of hard-covered books and y is the number of paperback

Time available for binding is 500 hours = 30,000 min

We form the system of equation and solve:

$$\begin{cases} 3x + 2y = 30{,}000 \\ x + y = 12{,}000 \end{cases}$$

$$\begin{cases} 3x + 2y = 30{,}000 \\ -2x + 2y = -24{,}000 \end{cases}$$

$$\begin{cases} x = 6000 \\ y = 12{,}000 - 6000 \end{cases}$$

$$\begin{cases} x = 6{,}000 \\ y = 6{,}000 \end{cases}$$

The number of hard-covered books is 6,000 and the number of paperbacks is 6,000.

Application activity 5.4

1. 68,260.87
2. 2h 56min 28sec
3. $7{,}500d + 3{,}000c \geq 25{,}500$

UNIT

6 Quadratic equation and inequalities

Number of periods: 18

6.1. Key unit competence

Model and solve algebraically or graphically daily life problems using quadratic equations or inequalities.

6.2. Learning objectives

By the end of this unit, the student must be able to do the following:

Knowledge and understanding	Skills	Attitudes and values
• Define a quadratic equation • Be able to solve problems related to quadratic equations	• Apply critical thinking by solving any situation related to quadratic equations (economics problems, finance problems.) • Determine the sign, sum and product of a quadratic equation • Discuss the parameter of a quadratic equation and a quadratic inequality • Explain how to reduce bi-quadratic equations to a quadratic equation and other equations of degrees greater than 2	• Appreciate value and care for situations involving to quadratic equations and quadratic inequalities in daily life situation. • Show curiosity about quadratic equations and quadratic inequalities.

6.3. Content

1. Equations in one unknown
2. Inequalities in one unknown
3. Simultaneous equations in two unknowns

4. Applications: Physics

Masonry

6.4. Materials required

Geometrical instruments (ruler, T-square), calculators.

6.5. Generic competences

Problem-solving, Cooperation, Communication , Lifelong learning

Cross-cutting issues

- **Inclusive education**

Inclusion of learners with disabilities in group work.

- **Standardisation culture**

Accuracy in calculations and drawing of graphs.

- **Financial education**

Examples on buying and selling, profit and loss.

Teaching and learning activities

a. Introductory activity

Guidance on introductory activity 6

Invite learners to work in group discussions and do activity 6 found in their Mathematics books.

Invite one member from groups with different working steps to present their answers to the whole class;

◊ Harmonize the findings and arouse the curiosity of students on the use of quadratic equations and then the necessity of being able to solve algebraically such equations.

Answers for introductory activity 6

a)The highest exponent of the variable t is 2, therefore,

$y = f(t)= -16t^2+1600$ is not a linear function.

b)Table of value:

t	0	1	2	3	4	5	6
y	1600	1616	1664	1744	1856	2000	2176

b. Main activities

1. Introduce the unit by giving some examples of quadratic equations.
2. Give them concise the definition of a quadratic equation after listening to their research fundings.
3. Help the students to solve problems related to quadratic equations. Let them attempt Application activity 6.1 of the Student's Book.
4. Guide them to be able to solve equations by looking for the sign, sum and product of a quadratic equation. Application activity 6.2 involves solving equations by use of sum and product. Let them work in pairs to tackle Activity 6.2.
5. Introduce to them solving of quadratic equations by factorisation. Let them attempt Application activity 6.3 of the Student's Book.
6. Introduce inequalities in one unknown and the use of sign diagrams. Let them attempt Application activity 6.4, Application activity 6.5 and Application activity 6.6.
7. Encourage them to define what parametric equations are. Guide them in solving parametric equations. Let them attempt Application activity 6.7 and Application activity 6.8 of the Student's Book.
8. Guide learners in researching on and discussing the applications of quadratic equations in real life. This is brought out in Activity 6.3. Assist them to appreciate this importance. Let them attempt Application activity 6.9 of the Student's Book.

Learners of varying strengths and abilities

(a) Gifted and talented

You can provide more advanced material to the learner. You can also make them group leaders in their groups so that they indirectly help the slower learners or the not so talented. However, ensure that they are supervised so as not to overwhelm the less gifted.

Give them individual exercises to work on as homework.

(b) Physical impairment

- Use cooperative learning for instance through group work and discussions. Those having difficulty with manipulative tasks should be assisted by the others in the group.
- Mix students with special needs with the rest so they can be assisted if necessary.
- Provide these students with frequent progress checks.

6.6. Additional tasks

Task 6A - For slow learners

1. Find the point(s) of intersection, if any, of the line and the parabola given by $2y + 3x - 9 = 0$ and $y = x^2 + 2$.

Answer

We form the following system of equation and solve:

$$\begin{cases} y = \dfrac{9-3x}{2} \\ y = x^2 + 2 \end{cases}$$

$$\begin{cases} y = \dfrac{9-3x}{2} \\ \dfrac{9-3x}{2} = x^2 + 2 \end{cases}$$

$$\begin{cases} y = \dfrac{9-3x}{2} \\ 2x^2 + 3x - 5 = 0 \end{cases}$$

$$\begin{cases} x = \dfrac{-3 \pm \sqrt{9+40}}{4} \\ y = \dfrac{9-3x}{2} \end{cases}$$

$$\begin{cases} y = \dfrac{9-3x}{2} \\ x = \dfrac{-3 \pm 7}{4} \end{cases}$$

$$\begin{cases} y = \dfrac{9-3(-5/2)}{2} \text{ or } y = \dfrac{9-3(1)}{2} \\ x = \dfrac{5}{2} \text{ or } x = 1 \end{cases}$$

$$\begin{cases} x = 1 \text{ or } x = \dfrac{5}{2} \\ y = 3 \text{ or } y = \dfrac{33}{4} \end{cases}$$

Thus, the solution set is $(1, 3)$ and $(\frac{-5}{2}, \frac{33}{4})$

Task 6B - For talented learners

1. All the students of a certain class were given a task, one sixth of all students were cleaning the dishes, thrice the square-root of all students were peeling potatoes, 12 remaining were cleaning the classroom. Find the number of all students of the class.

2. A swimming pool is fitted with three pipes with uniform flow. The first two pipes operating simultaneously fill the pool in the same time during which the pool is filled by the third pipe alone. The second pipe fills the pool eight hours faster than the first pipe and one hour slower than the third pipe. Find

   a) the time required by each pipe to fill the pool individually.

   b) The time required by all three pipes operating simultaneously to fill the pool

Answers

1. Let x be the number of all students.

   The number of students cleaning the dishes is $\frac{x}{6}$.

   The number of students peeling potatoes is $3\sqrt{x}$.

   The number of students cleaning the classroom is 12.

   We have the equation:

   $x = \frac{x}{6} + 3\sqrt{x} + 12$

   $x + 18\sqrt{x} - 72 = 0$

   $6x = x + 18\sqrt{x} + 72$

   Let $\sqrt{x} = y \Leftrightarrow x = y^2$

   $5y^2 - 18y - 72 = 0$

   $y = \frac{18 \pm \sqrt{324 + 1440}}{10} = \frac{18 \pm 42}{10}$

   $y = 6$ or $y = -\frac{24}{10}$ or $y = \sqrt{x}$ is positive

   Thus, the number of students in the class is 36.

2. Let V be the volume of the pool. Let x be the number of hours required for the second pipe (alone) to fill the pool. The speeds (rates) at which the pool is filled by the first, second and third pipes in one hour are

   $\frac{V}{x+8}, \frac{V}{x}$ and $\frac{V}{x-1}$ respectively.

Using the given information, we have

$$\frac{v}{x+8} + \frac{v}{x} = \frac{v}{x-1}$$

$$\frac{1}{x+8}, \frac{1}{x} \text{ and } \frac{1}{x-1}$$

$$\frac{x+x+8}{x(x+8)} = \frac{1}{x-1}$$

$$\frac{2x+8}{x(x+8)} = \frac{1}{x-1}$$

$2x^2 - 2x + 8x - 8 = x^2 + 8x$

$x^2 - 2x - 8 = 0$

$(x - 4)(x + 2) = 0$

$x = 4$ or

$x = -2$ (this value must be rejected since we cannot have negative hours)

Thus, $x = 4$

a) Thus, the time required by each of the three pipes to fill the pool is 12, 4, 3 hours respectively.

b) The speed of the three pipes all together to fill pool in one hour is

$$\frac{V}{12} + \frac{V}{4} + \frac{V}{3} = \frac{V+3V+4V}{12} = \frac{8V}{12} = \frac{2V}{3} = \frac{2}{3}\,V/h$$

The time required is $\frac{8V}{\frac{3}{2}V/h} = \frac{V}{\frac{2V}{3h}} = \frac{V}{1} \times \frac{3h}{2V} = \frac{3}{2}h = 1h30min$

Thus, the time required by all three pipes to fill the pool is 1hour and 30 minutes.

Assessment criteria

Model and solve algebraically or graphically daily life problems using linear equations or inequalities.

6.7. Answers to Application activity of Unit 6 in the Student's Book

Application activity 6.1

1. $x = \frac{1}{4}$
2. No real roots
3. $x = 1$ or $x = 5$
4. $x = -\frac{1}{2}$ or $x = 2$
5. $x = \frac{3-\sqrt{17}}{4}$ or $x = \frac{3+\sqrt{17}}{4}$
6. No real roots
7. $x = -2$

Application activity 6.2

1. $x = -5 + 4\sqrt{2}$ or $x = -5 - 4\sqrt{2}$
2. $x = 3$ or $x = -5$
3. $x = 4$ or $x = -1$
4. $x = 3$ or $x = 4$
5. $x = \frac{1}{3}$ or $x = -1$
6. $x = -1$ or $x = -6$
7. $x = 0$ or $x = 2$
8. $x = -1$ or $x = -\frac{1}{4}$
9. $x = \frac{2}{3}$ or $x = -1$
10. $x = 0$ or $x = -\frac{1}{2}$
11. $x = 0$ or $x = -6$
12. $x = 0$ or $x = 10$
13. $x = 0$ or $x = \frac{1}{2}$
14. $x = 5$ or $x = -4$
15. $x = 2$ or $x = -\frac{4}{3}$
16. $x = -1$ or $x = 0$
17. $x = 0$ or $x = 3$
18. $x = 0$ or $x = 2$
19. $x = 3$ or $x = -1$
20. $x = 1$

Application activity 6.3

(a) $x = -2$ or $x = -5$

(b) $x = -2$ or $x = -4$

(c) $x = -10$ or $x = -1$

(d) $x = 2$ or $x= 6$

(e) $x = 1$ or $x = 4$

(f) $x = 3$ 0r $x = 8$

(g) $x = -2$ or $x = -5$

(h) $x = 3$ or $x = -5$

(i) $x = 6$

(j) $x = 3$or $x = 4$

(k) $x = 7$ or $x = -6$

(l) $x = -2$ or $x = -6$

(m) $x = -1$ or $x = 15$

(n) $x = 1$ or $x = 2$

(o) $x = 7$ or $x = -4$

(p) $x = -5$ or $x = 3$

(r) $x = -5$ or $x = 12$

(q) $x \square\ 5$ *or* $x\ \ 3$

Application activity 6.4

1. $x \in \left]-\frac{1}{2}, 1\right[$
2. $x \in \left]-1, \frac{3}{2}\right[$
3. $x \in]-\infty, \infty+[$
4. $x \in \left]-\infty, \frac{-3-\sqrt{5}}{2}\right[\cup \left] \frac{-3+\sqrt{5}}{2}, +\infty\right[$
5. The answer is $x \notin \mathbb{R}$

Application activity 6.5

1. $x \in]\frac{9}{2}\ -, -2[$
2. $x \in]-\infty, -2[$
3. $x\ [\frac{6}{7}, 2[$ (here the quotient $\frac{7x-6}{2-x}$ is undefined when $x = 2$)

Application activity 6.6

1. $x \in]-\infty, -2[\cup [1, +\infty[\cup \{-1\}$ or $x = \{-1\}$
2. $x \in]-\frac{1}{3}, 3[$
3. $x < 11$
4. $x \in]-1, 0[$
5. $x < 1$ and $x > \frac{3}{2}$
6. $x \geq \frac{5}{7}$
7. $x \in]-\infty, -1[\cup]1, 2[\cup]3, +\infty[$
8. $x \in]0,1[$ and $x > 2$
9. $x < -2$ and $x \in]-1, 3[$
10. $x \in \left]-4, -\frac{2}{3}\right[$
11. $-1 < x < 2$
12. $x \in]-3, -1[$ and $x > 2$

Application activity 6.7

1. (a) $m < 7$ (b) $m > 7$ (c) $m = 7$
2. (a) $k > -\frac{1}{2}$ (b) $k < -\frac{1}{2}$ (c) $k = -\frac{1}{2}$
3. (a) $m < \frac{17}{24}$ (b) $m > \frac{17}{24}$ (c) $m = \frac{17}{24}$
4. $\left]0, \frac{4}{9}\right[$

Application activity 6.8

1. (a) $x = 2, y = 4$ or $x = -1, y = 1$ (b) $x = 3, y = 9$ or $x = -2, y = 4$
   (c) $x = 4, y = 16$ or $x = -3, y = 9$ (d) $x = y = 1$
   (e) $x = 6, y = 36$ or $x = 8, y = 64$ (f) $x = -4, y = 16$ or $x = 3, y = 9$
   (g) $x = 1, y = 7$ or $x = 7, y = 1$ (h) $x = 1, y = -5$ or $x = 5, y = -1$
   (i) $x = y = 2$ (j) $x= 3, y = -2$ or $x = -2, y = 3$
2. 4 and 16 o –2 and 4

Task 6.9 on page 111

1. The building is at h = 64 metres

   The ball rises at the height equal to 100 metres, from the top of the building, before it starts to drop downward.

   The ball hits the ground after 4 seconds, from the time of thrown.
2. George's loss is $P(0) = -100$

   The greatest possible profit is 1,006.17
3. $t \in]\frac{10-\sqrt{76}}{8}, \frac{10+\sqrt{76}}{8}[$ or $t \in]0.16, 2.34[$

UNIT

7 Polynomial, rational and irrational functions

Number of periods: 14

7.1.Key unit competence

Use concepts and definitions of functions to determine the domain of rational functions and represent them graphically in simple cases and solve related problems.

7.2. Learning objectives

By the end of this topic, the student must be able to do the following:

Knowledge and understanding	Skills	Attitudes and values
• Identify a function as a rule and recognize rules that are not functions • Determine the domain and range of a function • Construct composition of functions • Find whether a function is even , odd , or neither • Demonstrate an understanding of operations on polynomials, rational and irrational functions, and find the composite of two functions.	• Perform operations on functions • Apply different properties of functions to model and solve related problems in various practical contexts • Analyse, model and solve problems involving linear or quadratic functions and interpret the results	• Increase self-confidence and determination to appreciate and explain the importance of functions • Show concern on patience, mutual respect and tolerance • When solving problems about polynomial, rational and irrational functions

7.3. Content

1. Polynomials
2. Numerical functions
3. Domain and range of a function
4. Applications of rational and irrational functions

7.4. Materials required

Pair of compasses, graph paper, rule, digital technology equipment such as calculators

7.5. Generic competences

Problem-solving, Cooperation , Communication, Lifelong learning

Cross-cutting issues

- **Peace and values education**

Students work harmoniously in groups.

- **Inclusive education**

Groups consist of learners of various abilities.

7.6. Guidance on teaching and learning activities

a. Introductory activity

Guidance on introductory activity7

- Give clear instructions for learners to form small groups and to work on the introductory activity 7;
- As they are discussing, circulate around to note the relevancy of the discussion and to provide guidance where necessary;
- Facilitate working, especially the straggling learners
- Ensure that the learners have understood what the unit will be about and they are eager to learn; you can observe this through a clear and concise presentation of a group chosen randomly and the degree of attention other students are paying to the presentation.

b. Main activities

1. Let learners research on the definition of a polynomial and discuss this with the rest of the class. This is Activity 7.1 of the Student’s Book.
2. Explain to them what polynomials are using examples. Give them Application activity 7.1 of the Student’s Book to work in pairs.
3. Let them do Activity 7.2 of the Student's Book. Take them through the types of factorization by use of examples that include those. Give them Application activity 7.2 to attempt using the different types of factorization learnt. Allow them to explain why they choose one type over the others when solving the different problems.
4. Let them discuss and say what they know about roots of polynomials. Clarify and explain to them the concept. Help them define numerical

functions by use of examples given in the Student's Book, among others.

5. Let learners explain what they understand about the domain and range of a function. Use appropriate examples to clarify the meanings and definitions. Let them attempt Application activity 7.3 of the Student's Book.
6. Let students research on and discuss their findings on composition of functions. Give them Application activity 7.4 of the Student's Book to tackle to ensure the concept is understood well.
7. Demonstrate what is meant by parity of functions – even functions and odd functions. Give students Application activity 7.5 to attempt. Gauge their understanding of even and odd functions.
8. Let students research on applications of rational and irrational functions. This is Activity 7.3 of the Student's Book. Let them attempt Application activity 7.6.

Learners of varying strengths and abilities

(a) Gifted and talented

Guide the learners to study units ahead of others. You can provide more advanced material to the learner. For example, let them draw extra graphs and allow them to do extra exercises. You can also make them group leaders in their groups so that they indirectly help the slower learners or the not so talented. However, ensure that they are supervised so as not to overwhelm the less gifted.

Give them individual exercises to work on as homework.

(b) Intellectual impairement

Try to understand the specific talents of the learner and develop them. Break the task down into small steps or learning objectives. Ensure learners start with what they can already do, then move on to a new more challenging task. Give the learner lots of practice and time. It helps to ensure the child has mastered a skill.

7.7. Additional tasks

Task 7A - For slow learners

1. For each of the following functions f, find its domain.

   i) $f(x) = x^2 - 5$

   ii) $f(x) = \frac{2}{5x + 6}$

2. Let $p(x) = x^3 + kx^2 + x - 6$. Suppose that $(x + 2)$ is a factor of $p(x)$.

(a) Find the value of k.

(b) With the value of k found in (a), factorize $p(x)$.

Answers

1. (i) Since $f(x) = x^2 - 5$ is a polynomial, then its domain is Dom $f = \mathbb{R}$

(ii) $f(x) = \frac{2}{5x + 6}$. Condition: $5x + 6 \neq 0$.

Dom $f = \mathbb{R}\backslash\{-\frac{6}{5}\}$

2. $p(x) = x^3 + kx^2 + x - 6$

(a) Since $(x - (-2))$ is a factor of $p(x)$, it follows from the Factor Theorem that $p(-2) = 0$, that is $(-2)^3 + k(-2)^2 + (-2) - 6 = 0$.

Solving, we get $k = 4$.

(b) Using long division, we get $x^3 + 4x^2 + x - 6 = (x + 2)(x^2 + 2x - 3)$.

By inspection, we have $p(x) = (x + 2)(x + 3)(x - 1)$.

Task 7B - For talented learners

1. If α, β and γ are three roots of the cubic equation $ax^3 + bx^2 + cx + d = 0$, prove that

i) $\alpha + \beta + \gamma = -\frac{b}{a}$

ii) $\alpha\beta + \beta\gamma + \alpha\gamma = \frac{c}{a}$

iii) $\alpha\beta\gamma = -\frac{d}{a}$

Answers

1. $ax^3 + bx^2 + cx + d = 0$

$a(x^3 + \frac{b}{a}x^2 + \frac{c}{a}x + \frac{d}{a}) = 0$

If α, β and γ are the roots of $a(x^3 + \frac{b}{a}x^2 + \frac{c}{a}x + \frac{d}{a}) = 0$ (1)

then

$a(x - \alpha)(x - \beta)(x - \gamma) = 0$

$a[x^2 - \beta x - \alpha x + \alpha\beta)(x - \gamma)] = 0$

$a[x^3 - \gamma x^2 - \beta x^2 + \beta\gamma x - \alpha x^2 + \alpha\gamma x + \alpha\beta x - \alpha\beta\gamma] = 0$

$a[x^3 + (-\alpha - \beta - \gamma)x^2 + (\alpha\beta + \beta\gamma - \alpha\gamma)x - \alpha\beta\gamma] = 0$ (2)

By equating the corresponding coefficient in (1) and (2) we get

$$\begin{cases} \frac{b}{a} = -\alpha - \beta - \gamma \\ \frac{c}{a} = \alpha\beta + \beta\gamma + \alpha\gamma \\ \frac{d}{a} = -\alpha\beta\gamma \end{cases} \qquad \begin{cases} \alpha + \beta + \gamma = -\frac{b}{a} \\ \alpha\beta + \beta\gamma + \alpha\gamma = \frac{c}{a} \\ \alpha\beta\gamma = -\frac{d}{a} \end{cases}$$ as required.

Assessment criteria

Use concepts and definitions of functions to determine the domain of rational functions and represent them graphically.

7.8. Answers to Application activity of Unit 7 in the Student's Book

Application activity 7.1

a) Polynomial

b) Polynomial

c) Polynomial

Application activity 7.2

1. $(a + b)(x + y)$
2. $6x^2(2 + 3x)$
3. $(a + b)(a + c)$
4. $(px + qy)(pq + xy)$
5. $(3p + 4q)^2 = (3p + 4q)(3p + 4q)$
6. $(2x + 3)^2 = (2x + 3)(2x + 3)$
7. $(x + 9)(x + 4)$
8. $-[(2x - 3 - 3\sqrt{3})(2x - 3 + 3\sqrt{2})]$
9. $-(2x - 1)(3x + 1)$
10. $3(3y + 4z)(3y - 4z)$

Application activity 7.3

1. (a) Dom (f) = $[-7, -5] \cup [3, \infty)$

   (b) Dom (f) = $\mathbb{R}$

   (c) Dom (f) = $\mathbb{R} \setminus \{-6\}$

   (d) Dom (f) = $[2, +\infty[$

   (e) Dom (f) = $\mathbb{R} \setminus \{-1, 2\}$

   (f) Dom (f) = $\mathbb{R}_+ \setminus \{\frac{7}{4}\} = [0, \frac{7}{4}[\cup]\frac{7}{4}, +\infty[$

   (g) Dom (f) = $\{x \in \mathbb{R}: x > 7\} =]7, +\infty[$

   (h) Dom (f) = $]-\infty, -1] \cup]1, +\infty[$

(i) Dom (f) = $]-\infty, 2] \cup [3, +\infty[= \mathbb{R} \setminus]2, 3[$

(j) Dom (f) = $\{x \in \mathbb{R}: (x \geq -\frac{2}{7}) \wedge (x \neq 7) \wedge (x \neq 1)\} = [-\frac{2}{7}, 1[\cup]1,7[\cup]7, +\infty[$

(k) Dom (f) = $\{x \in \mathbb{R}: x \neq \frac{2}{3}\} =]-\infty, \frac{7}{3}, +\infty[$

2. $\left[\frac{1-\sqrt{5}}{2}, \frac{1+\sqrt{5}}{2}\right]$

3. (a) $y \geq -3$, (b) $y \geq -5$ (c) $f(x) \geq 0$,

d) $0 \leq f(x) \leq \frac{1}{2}$

4. (a) 5 (b) 4 (c) 2 (d) 0

Application activity 7.4

1. (a) $f(g(x)) = (2x + 7)^2$ (b) $f(g(x)) = 2x^2 + 7$

(c) $f(g(x)) = \sqrt{3 - 4x}$ (d) $f(g(x) = 3 - 4\sqrt{x}$

(e) $f(g(x)) = \frac{2}{x^2 + 3}$ (f) $f(g(x)) = \frac{4}{x^2} + 3$

2. (a) $f(x) = x^3, g(x) = 3x + 10$ (b) $f(x) = \frac{1}{x}, g(x) = 2x + 4$

(c) $f(x) = \sqrt{x}, g(x) = x^2 - 3x$ (d) $f(x) = \frac{10}{x^3}, g(x) = 3x - x^2$

Application activity 7.5

1. f is even 2. g is odd

3. h is neither even nor odd.

Application activity 7.6

1. 10^{th} week

2. 2 hours

3. 0.4918 FRW

UNIT

8 Limits of polynomial, rational and irrational functions

Number of periods: 14

8.1. Key unit competence

Evaluate correctly limits of functions and apply them to solve related problems.

8.2. Learning objectives

By the end of this unit, the student must be able to do the following:

Knowledge and understanding	**Skills**	**Attitudes and values**
• Define the concept of limit for real-valued functions of one real variable • Evaluate the limit of a function and extend this concept to determine the asymptotes of the given function.	• Calculate limits of certain elementary functions • Develop introductory calculus reasoning. • Solve problems involving continuity. • Apply informal methods to explore the concept of a limit including one sided limits. • Use the concepts of limits to determine the asymptotes to the rational and polynomial functions	• Show concern on the importance, the use and determination of limit of functions • Appreciate the use of intermediate-value theorem

8.3. Content

1. Concept of limits:
   - Neighbourhood of a real number
   - Limit of a variable
   - One-sided limits
2. Theorems on limits:
   - Squeeze theorem

3. Indeterminate forms
4. Applications:
    - Continuity over an interval
    - Asymptotes

8.4. Materials required

Manila papers, graph papers, ruler, markers, calculators

8.5. Generic competences

Communication, Research, Cooperation, Problem solving and Critical thinking

Cross-cutting issues

- **Peace and values education**

  Learners will be encouraged to avoid conflicts with their neighbours and classmates.

- **Standardization culture**

  Learners encouraged to adhere to rules and standards of calculations and drawing of graphs.

- **Gender education**

  Groups consist of mixed gender and all are encouraged to participate.

8.6. Guidance on teaching and learning activities

a. Introductory activity

Guidance on introductory activity 8

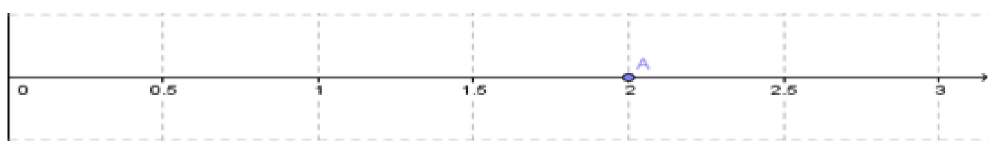
- Invite learners to work in group and do the activity 8 found in their Mathematics books;

- Invite one member from each group to present their work;

- A teacher, harmonize the findings from presentation of learners and guide them to explore the content and examples given in the student's book where they will be able to differentiate the neighbourhood of a real number and the value of a function at a given point.

Answer for activity 8

a) Yes, it is possible to put the values of x on a number line.



b) Open intervals of the number line such that their centre is x=2 are for example:]1.9;2.1[and]1.99;2.01[

c) When x approaches 2, we see that the value of f(x) approaches 4.

b. Main activities

1. Introduce the unit by assisting learners in defining the concept of limit for real-valued functions of one real variable. Explain to learners the meaning of the limit of a variable.
2. Guide students to calculate limits of certain elementary functions. Let them attempt Application activity 8.1 of the Student's Book.
3. Let them define the limit of a function and to give its graphical interpretation.
4. Guide them to discuss in groups how to evaluate the limit of a function at a point algebraically; let them extend this understanding to determine the asymptotes. Clearly explain to them the one-sided limits, the squeeze theorem, limits of functions to infinity and the various operations on limits. Use question and answer techniques to ensure full participation by learners. Let them work in pairs to tackle Application activity 8.2 of the Student's Book.
5. Guide them in understanding of indeterminate cases. Take them through the various methods such as substitution and rationalisation. Let them work in pairs to tackle Activity 8.3 and Application activity 8.3 of the Student's Book.
6. Let them research on the applications of limits. Guide them in a discussion and presentation of their findings in class. Ask them to do Activity 8.4 of the Student's Book. Place emphasis on the different types of asymptotes. Let them do Activity 8.5.
7. Let them attempt Application activity 8.4 of the Student's Book.

Learners of varying strengths and abilities

(a) Gifted and talented

Guide the learners to study units ahead of others. You can provide more advanced material to the learner. For example, let them draw extra graphs and allow them to do extra exercises. You can also make them group leaders in their groups so that they indirectly help the slower learners or the not so talented. However, ensure that they are supervised so as not to overwhelm the less gifted.

Give them individual exercises to work on as homework.

(b) Slow learners

Give them extra tuition. Teach them to master the material skills that are commensurate with their ability levels before introducing complex components.

(c) Physical impairment

- Use cooperative learning for instance through group work and discussions. Those having difficulty with manipulative tasks should be assisted by the others in the group.
- Mix students with special needs with the rest so as to be helped
- Provide these students with frequent progress checks.

(d) Hearing impairment

- Tape-record portions of textbooks, trade books, and other printed materials so students can listen (with earphones) to an oral presentation of necessary material.
- Providing written or pictorial directions to those with hearing problems
- Using of concrete objects such as models, diagrams, samples, and the like to those with impairments so as to demonstrate what you are saying by using touchable items. They can also rewrite content such as charts on large manila paper in their group work
- Facing the learner while you speak might help learners with a hearing impairment
- Use large writing on the blackboard and on visual aids

8.7. Additional tasks

Task 8A - For slow learners

1. Evaluate the following limits

(a) $\lim\limits_{x \to 2} \frac{x - \sqrt{x+2}}{\sqrt{4x+1} - 3}$

(b) $\lim\limits_{x \to \infty} \left(\sqrt{x^2 + 5x - 2} - \sqrt{x^2 - 3x - 2}\right)$

Answers

1.

(a) $\lim_{x\to 2}\frac{x-\sqrt{x+2}}{\sqrt{4x+1}-3}=\frac{0}{0}$ (Indeterminate Form)

$$\lim_{x\to 2}\frac{x-\sqrt{x+2}}{\sqrt{4x+1}-3}=\lim_{x\to 2}\frac{\left(x-\sqrt{x+2}\right)\left(x+\sqrt{x+2}\right)\left(\sqrt{4x+1}+3\right)}{\left(\sqrt{4x+1}-3\right)\left(x+\sqrt{x+2}\right)\left(\sqrt{4x+1}+3\right)}$$

$$=\lim_{x\to 2}\frac{[x^2-(x+2)]](\sqrt{4x+1}+3)}{[(4x+1)-9](x+\sqrt{x+2})}$$

$$=\lim_{x\to 2}\frac{(x^2-x-2)(\sqrt{4x+1}+3)}{(4x-8)(x+\sqrt{x+2})}=\lim_{x\to 2}\frac{(x-2)(x+1)(\sqrt{4x+1}+3)}{4(x-2)(x+\sqrt{x+2})}$$

$$=\lim_{x\to 2}\frac{(x+1)(\sqrt{4x+1}+3)}{4(x+\sqrt{x+2})}=\frac{(2+1)(3+3)}{4(2+2)}=\frac{3(6)}{4(4)}=\frac{18}{16}=\frac{9}{8}$$

(b) $\lim_{x\to +\infty}(\sqrt{x^2+5x-2}-\sqrt{x^2-3x-2})=\infty-\infty$ (Indeterminate form)

$$\lim_{x\to +\infty}=\frac{(\sqrt{x^2+5x-2}-\sqrt{x^2-3x-2})(\sqrt{x^2+5x-2}+\sqrt{x^2-3x-2})}{(\sqrt{x^2+5x-2}+\sqrt{x^2-3x-2})}$$

$$\lim_{x\to +\infty}=\frac{(x^2+5x-2-x^2+3x+2)}{(\sqrt{x^2+5x-2}+\sqrt{x^2-3x-2})}$$

$$\lim_{x\to +\infty}=\frac{8x}{\sqrt{x^2(1+\frac{5}{x}-\frac{2}{x^2})}+\sqrt{x^2(1-\frac{3}{x}-\frac{2}{x^2})}}$$

$$=\lim_{x\to +\infty}=\frac{8x}{x\left(\sqrt{1+\frac{5}{x}-\frac{2}{x^2}}+\sqrt{1-\frac{3}{x}-\frac{2}{x^2}}\right)}=\lim_{x\to +\infty}\frac{8}{2}=4$$

Application activity 8B - For talented learners

1. Given the function $f(x)=\frac{ax^2}{bx^2+6x+c}$ it is given that its curve representative has three asymptotes of respective equations x – 2 = 0, x – 1 = 0, y + 4 = 0. Find the real values of a, b and c.

Answer

$$f(x)=\frac{ax^2}{bx^2+6x+c}$$

x = 2 and x = 1 are vertical asymptotes, thus 2 and 1 are the roots of the equation $bx^2 + 6x + c = 0$.

$$\begin{cases} 4b+12+c=0 \\ b+6+c=0 \end{cases}$$

$$\begin{cases} 4b+c=-12 \\ b+c=-6 \end{cases}$$

$$\begin{cases} 3b=-6 \\ b+c=-6 \end{cases}$$

$$\begin{cases} b=-2 \\ c=-6-b \end{cases}$$

$$\begin{cases} b=-2 \\ c=-4 \end{cases}$$

y = –4 is an horizontal asymptote

$$\lim_{x\to\pm\infty}\lim_{x\to\pm\infty}\frac{ax^2}{bx^2+6x+c}=\frac{a}{b}\ \text{. so }\ \frac{a}{b}=-4 \quad\Rightarrow\quad \frac{a}{-2}=-4\Rightarrow a=8$$

$$\begin{cases} a = 8 \\ b = -2 \\ c = -4 \end{cases}$$

8.8. Answers to Tasks of Unit 8 in the Student's Book

Application activity 8.1

1. $\lim_{x\to 2}\frac{x^2-4}{x-2} = 4$

2. $\lim_{x\to -3}\frac{x^2-9}{x+3} = -6$

3. $\lim_{x\to 0}\frac{\sin x}{x} = 1$

4. $\lim_{x\to 1^-} f(x) = -1$, $\lim_{x\to 1^+} f(x) = 1$ and

   $\lim_{x\to 1} f(x)$ does not exist

Application activity 8.2

1. (a) $\lim_{x\to 2} 2x + 1 = 2(2) + 1 = 5$

   (b) $\lim_{a\to 1} a^2 - 1 = 1 - 1 = 0$

   (c) $\lim_{x\to -3}\frac{x^2+x-2}{x+1} = \frac{9-3-2}{-3+1} = \frac{4}{-2} = -2$

(d) $\lim\limits_{x \to -\infty} \frac{4x^2 + 3x - 2}{x^2 + x - 1} = \lim\limits_{x \to -\infty} \frac{x^2(4 + \frac{3}{x} - \frac{2}{x^2})}{x^2(1 + \frac{1}{x} - \frac{1}{x^2})} = \frac{\lim\limits_{x \to -\infty}(4 + \frac{3}{x} - \frac{2}{x^2})}{\lim\limits_{x \to -\infty}(1 + \frac{1}{x} - \frac{1}{x^2})}$

$$= \frac{\lim\limits_{x \to -\infty} 4 + \lim\limits_{x \to -\infty} \frac{3}{x} - \lim\limits_{x \to -\infty} \frac{2}{x^2}}{\lim\limits_{x \to -\infty} 1 + \lim\limits_{x \to -\infty} \frac{1}{x} - \lim\limits_{x \to -\infty} \frac{1}{x^2}} = \frac{4 + 0 - 0}{1 + 0 - 0} = \frac{4}{1} = 4$$

2. (a) We know that $-1 \leq \cos 2x \leq 1$

$-x \leq x \cos 2x \leq x$

$\lim\limits_{x \to 0} -x = 0 = \lim\limits_{x \to 0} x$

Therefore, $\lim\limits_{x \to 0} x\cos 2x = 0$

(b) We know that $-1 \leq \sin x \leq 1$

Since $x \in [0, +\infty[$, we have

$-\frac{1}{x} \leq \frac{\sin x}{x} \leq \frac{1}{x}$

We know that $\lim\limits_{x \to +\infty} -\frac{1}{x} = 0 = \lim\limits_{x \to +\infty} \frac{1}{x}$

Therefore, $\lim\limits_{x \to +\infty} \frac{\sin x}{x} = 0$

(c) We know that if $0 \leq x \leq \frac{\pi}{2}$ then $\sin x \leq x \leq \tan x$

Dividing each member of this inequality by $\sin x$ gives

$\frac{\sin x}{\sin x} \leq \frac{x}{\sin x} \leq \frac{\tan x}{\sin x}$

$1 \leq \frac{x}{\sin x} \leq \frac{1}{\cos x}$

Inverting member by member gives

$1 \geq \frac{\sin x}{x} \geq \cos x \Leftrightarrow \cos x \leq \frac{\sin x}{x} \leq 1$

Since $\lim\limits_{x \to 0} \cos x = 1 = \lim\limits_{x \to 0} 1$

Therefore $\lim\limits_{x \to 0} \frac{\sin x}{x} = 1$

Application activity 8.3

1. 3	2. The limit does not exist	3. 4	4. $\frac{1}{2\sqrt{3}}$
5. -1	6. $\frac{1}{2}$	7. $\frac{1}{2}$	8. $-\frac{2}{3}$
9. -2	10. $-\frac{1}{2\sqrt{3}}$	11. $-\frac{5}{2}$	12. $-\frac{3}{4}$
13. $-\frac{3}{4}$	14. $\frac{1}{2\sqrt{5}}$	15. 9	16. 7
17. $-\frac{1}{3}$	18. 0	19. $\frac{1}{5}$	20. $-\infty$
21. -1	22. 1	23. -9	24. 5
25. 0	26. 7	27. 3	28. $+\infty$
29. 1	30. $-\infty$	31. 625	32. $-\frac{1}{6}$
33. 4	34. 3	35. $-\infty$	36. 0
37. $\frac{9}{8}$	38. 0	39. Does not exist	40. $-\frac{4}{3}$
41. 3	42. $\frac{13}{3}$	43. $\frac{1}{\sqrt{2}}$	44. $\frac{3}{7}$
45. -1	46. $\frac{3}{4}$	47. $\frac{1}{2}$	48. ∞
49. $\frac{1}{4}$	50. -3	51. $\frac{1}{3}$	52. $\frac{1}{6}$
53. -15	54. 0	55. $\frac{1}{6}$	56. $-\frac{3}{2}$
57. 1	58. $\frac{1}{4}$	59. $\frac{7}{4}$	60. $\frac{1}{12}$

61. (a) $\frac{1}{2}$ (b) $\frac{1}{4}$ (c) $\frac{2}{3}$ (d) $\frac{2-\sqrt{2}}{2}$
(e) 1

62. $|x| = \begin{cases} x \text{ if } x \geq 0 \\ -x < 0 \end{cases}$ therefore $\frac{|x|-x}{x} = \begin{cases} \frac{x-x}{x} = 0 \\ \frac{-x-x}{x} = -2 \end{cases}$

(a) $\lim\limits_{x \to 0^+} y = \lim\limits_{x \to 0} + \frac{|x|-x}{x} = \lim\limits_{x \to 0^+} + 0 = 0$

(b) $\lim\limits_{x \to 0^-} y = \lim\limits_{x \to 0^+} \frac{|x|-x}{x} = \lim\limits_{x \to 0^+} -2 = -2$

(c) Since $\lim\limits_{x \to 0^+} y \neq \lim\limits_{x \to 0^+} y$ then $\lim\limits_{x \to 0} y = \lim\limits_{x \to 0} \frac{|x|-x}{x}$ does not exist.

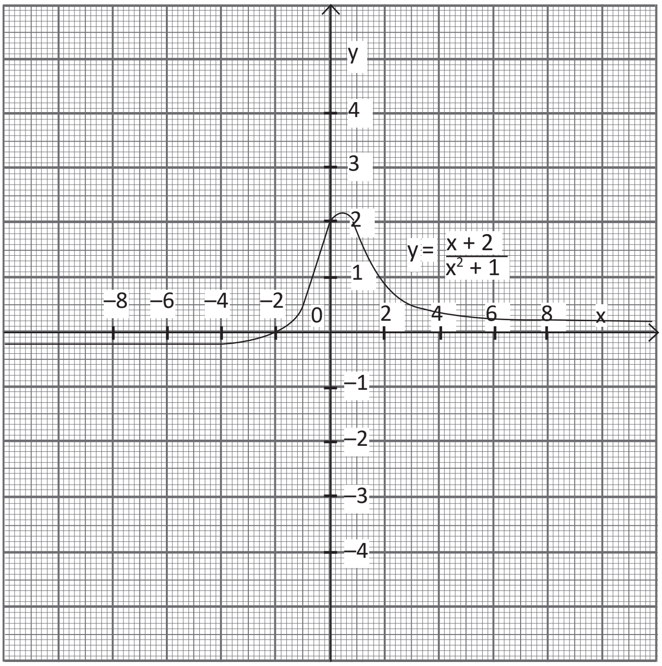
(d) $\lim\limits_{x \to 0^-} y = \lim\limits_{x \to 0^+} \frac{|x|-x}{x} = \lim\limits_{x \to 0^+} -2 = -2$

Application activity 8.4

1. Horizontal Asymptote H.A ≡ $y = 0$.

   No Vertical Asymptote. No Oblique Asymptote

   Graph

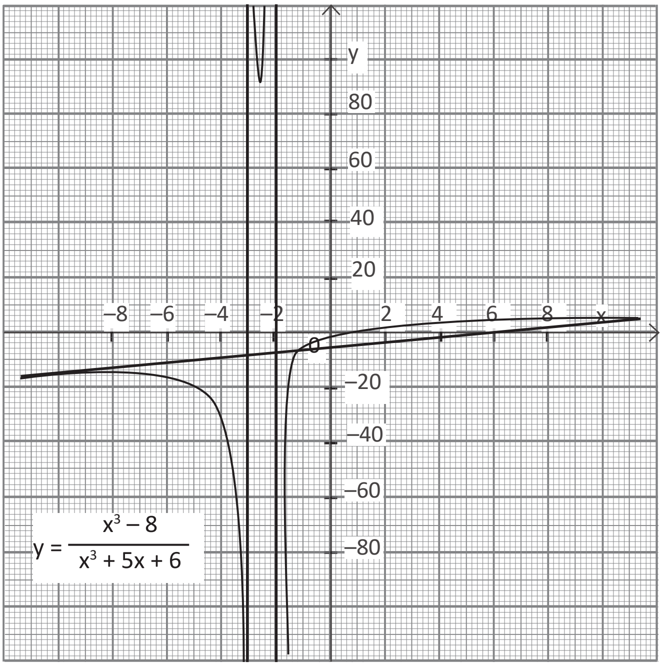


2. Vertical Asymptote V.A ≡ $x = -3$ Vertical Asymptote V.A ≡ $x = -2$

   Oblique Asymptote O.A ≡ $y = x - 5$

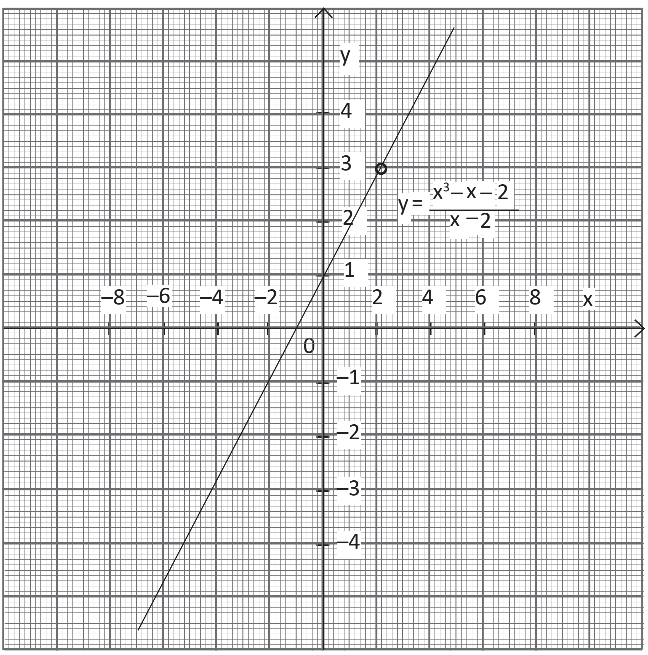
   No horizontal Asymptote

Graph



3. This can be simplified to a linear function x+1. It has a whole (not defined) at x = 2. It does not have any asymptote to it.

Graph



UNIT
9 Differentiation of polynomials, rational and irrational functions and their application

Number of periods: 21

9.1. Key unit competence

Use the gradient of a straight line as a measure of rate of change and apply this to tangent and normal of curves in various contexts and use the concepts of differentiation to solve and interpret related rates and optimisation problems in various contexts .

9.2. Learning objectives

By the end of this unit, the student must be able to do the following:

Knowledge and understanding	Skills	Attitudes and values
• Evaluate derivatives of functions using the definition of derivative • Define and evaluate from first principles the gradient at a point • Distinguish between techniques of differentiation to use in an appropriate context	• Use properties of derivatives to differentiate polynomial, rational ad irrational functions • Use first principles to determine the gradient of the tangent line to a curve at a point • Apply the concepts of and techniques of differentiation to model, analyse and solve rates or optimisation problems in different situations	• Appreciate the use of gradient as a measure of rate of change (economics) • Appreciate the importance and use of differentiation in Kinematics (velocity, acceleration) • Show concern on derivatives to help in the understanding optimization problems

9.3. Content

1. Concepts of derivatives of a function:
   - Definition
   - Differentiation from first principles
   - High order derivatives
2. Rules of differentiation

3. Application of differentiation:

- Geometric interpretation of derivatives
- Mean value theorem for derivatives
- Variations of a function
- Rates of change problems
- Optimization problems…

9.4. Materials required

Manila papers, graph papers, Digital technology including calculators, ...

9.5. Generic competences

Communication, Problem-solving, Research, Cooperation and Critical thinking

Cross-cutting issues

- **Comprehensive sexuality education**
  Learners taught to work together with opposite gender without the need to have ulterior motives.
- **Standardization culture**
  Learners encouraged to adhere to rules and standards of calculations and drawing of graphs.
- **Gender education**
  Groups consist of mixed gender and all are encouraged to participate.

9.6. Guidance on teaching and learning activities

a. Introductory activity

Guidance on introductory activity 9

Give clear instructions for learners to form small groups and to work on the introductory activity.

After correction, ensure that the learners have understood what the unit will be about and they are eager to learn; you can observe this through a clear and concise presentation of a group chosen randomly and the degree of attention other students are paying to the presentation.

Answer of introductory activity

a)

we have $x_0 = 1$ and $h = \Delta x = 1$

$$\text{The slope is given by } m_p = \frac{\Delta y}{\Delta x} = \frac{f(x_0+h)-f(x_0)}{(x_0+h)-x_0} = \frac{4-2}{2-1} = 2$$

b)

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$
$$=\lim_{h\to 0}\frac{\left[(x+h)^2+1\right]-\left(x^2+1\right)}{h}$$
$$=\frac{x^2+2hx+1-x^2-1}{h}$$
$$=\frac{2hx}{h}=2x$$
$$f'(x)=2x$$
$$for\ x_0=1\Rightarrow f'(x_0)=f'(1)=2$$

The slope $m_p=f'(x_0)=2$

2) Answers will vary, try to harmonize them.

b. Main activities

1. Introduce the topic by defining differentiation. Let students do Activity 9.1 of the Student's Book. Then guide learners to define and evaluate, from first principles, the gradient at a point. Give them Activity 9.2 of the Student's Book to try out in pairs. Let them individually tackle Application activity 9.1.
2. Demonstrate to students how to use properties of derivatives to differentiate polynomial, rational and irrational functions. Let them attempt Application activity 9.2.
3. Introduce them to higher order derivatives. Then outline the rules of differentiation. Let them tackle Application activity 9.3, 9.4 and 9.5, one at a time, after every subtopic.
4. Guide the learners on how they can distinguish between the different techniques of differentiation and to use them in an appropriate context. Let them do Application activity 9.6 and Application activity 9.7 to reinforce concepts involving theorems. Let them do Activity 9.3 and Application activity 9.8.
5. Introduce them to the concept of increasing and decreasing functions. They should be able to explain the sign of a derivative, stationary and inflection points, and concavity. Give them Application activity 9.9.
6. Help the students to use the different techniques of differentiation to model, analyse and solve rates and optimization problems. Follow up with the mental task, Application activity 9.10, Application activity 9.11 and Activity 9.4. Give them Application activity 9.12 of the Student's Book.

Learners of varying strengths and abilities

(a) Gifted and talented

Guide the learners to study units ahead of others. You can provide more advanced material to the learner. For example, let them draw extra graphs and allow them to do extra exercises. You can also make them group leaders in their groups so that they indirectly help the slower learners or the not so talented. However, ensure that they are supervised so as not to overwhelm the less gifted.

Give them individual exercises to work on as homework.

(b) Intellectual impairement

- Try to understand the specific talents of the learner and develop them.
- Break the task down into small steps or learning objectives. Ensure learners start with what they can already do, then move on to a new more challenging task
- Give the learner lots of practice and time. It helps to ensure the child has mastered a skill.

Task 9A - For slow learners

1. Find the derivative of the following function using the definition of the derivative: $g(t) = \sqrt{t}$

Answer

1. Find $g(t) = \sqrt{t}$ then $\frac{dg}{dt} = \lim_{h \to 0} \frac{g(t+h) - g(t)}{h} = \lim_{h \to 0} \frac{\sqrt{t+h} - \sqrt{t}}{h} = \frac{0}{0}$ (Indet. Form)

$$= \lim_{h \to 0} \frac{\sqrt{t+h} - \sqrt{t}}{h} = \lim_{h \to 0} \frac{(\sqrt{t+h} - \sqrt{t})(\sqrt{t+h} + \sqrt{t})}{h(\sqrt{t+h} + \sqrt{t})} = \lim_{h \to 0} \frac{t+h-t}{h(\sqrt{t+h} + \sqrt{t})}$$

$$= \lim_{h \to 0} \frac{h}{h(\sqrt{t+h} + \sqrt{t})} = \lim_{h \to 0} \frac{1}{\sqrt{t+h} + \sqrt{t}} = \frac{1}{\sqrt{t+0} + \sqrt{t}} = \frac{1}{2\sqrt{t}}$$

Task 9B - For talented learners

1. A rectangular field of 200 m of perimeter is to be enclosed. If we want a maximum of area, find the measures of its sides and the corresponding area.
2. Show that the right circular cylinder of given surface and maximum volume is such that its height is equal to the diameter of the base.

Answers

1. Let x and y be the width and the length of the field respectively.

   $2(x + y) = 200$

   $x + y = 100$

   $y = 100 – x$

   The area is $xy = x(100 – x) = 100x – x^2$

   Let $f(x) = 100x – x^2$. to find the largest value of area, is like to find the maximum of the function, f(x).

   $f'(x) = 100 – 2x$

   $f'(x) = 0 \qquad 100 – 2x = 0$

   $2x = 100$

   $x = 50$

   If x = 50 then y = 50. That gives us the area of 50 × 50 = 2500.

   The rectangular field is a square of side 50m and thus its area is $2500m^2$.

2. Let the radius of the base of the cylinder be r and height be h. Let S: surface, V: volume.

   $S = 2\pi rh + 2\pi r^2$ or $h = \frac{s - 2\pi r^2}{2\pi r}$

   $V = \pi r^2 h = \pi r^2\left(\frac{s - 2\pi r^2}{2\pi r}\right)$

   $V = \frac{1}{2}(Sr – 2\pi r^3)$

   $\frac{dV}{dr} = \frac{1}{2}(S – 6\pi r^2)$

   For maximum or minimum volume $\frac{dV}{dr} = 0$

   $\frac{1}{2}(S – 6\pi r^2) = 0 \qquad S = 6\pi r^2$

   $h = \frac{s - 2\pi r^2}{2\pi r} = \frac{6\pi r^2 - 2\pi r^2}{2\pi r} = \frac{4\pi r^2}{2\pi r} = 2r$

   Therefore, the cylinder has maximum volume if the height is equal to the diameter.

9.7. Additional information

Proof of the rules of differentiation

Each of the rules can be proved using the first first definition of derivative of a function .

The following proof are worth examining.

1. If f(x) = cu(x) where c is a constant then f′(x) = cu′(x) .

Proof:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{c\,u(x+h) - cu(x)}{h} = \lim_{h \to 0} c\left[\frac{u(x+h) - u(x)}{h}\right]$$

$$= c\lim_{h \to 0}\left[\frac{u(x+h) - u(x)}{h}\right] = cu'(x)$$

2. If f(x) = u(x) + v(x) then f′(x) = u′(x) + v′(x)

Proof:

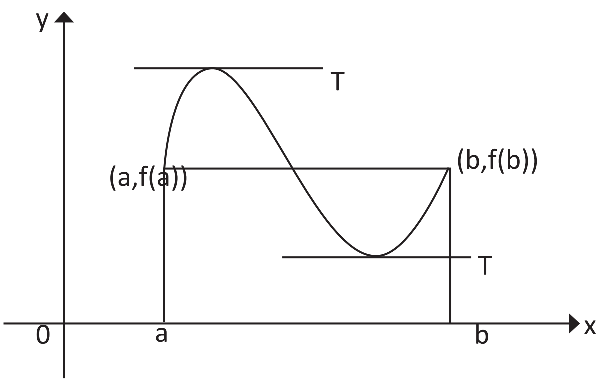
$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - u(x)}{h} = \lim_{h \to 0} \left(\frac{u(x+h) + v(x+h) - [u(x) + v(x)]}{h}\right)$$

$$= \lim_{h \to 0} \left(\frac{u(x+h) - u(x) + v(x+h) - v(x)}{h}\right)$$

$$= \lim_{h \to 0} \frac{u(x+h) - u(x)}{h} + \lim_{h \to 0} \frac{v(x+h) - v(x)}{h} = u'(x) + v'(x)$$

Geometrical significance of Rolle's Theorem

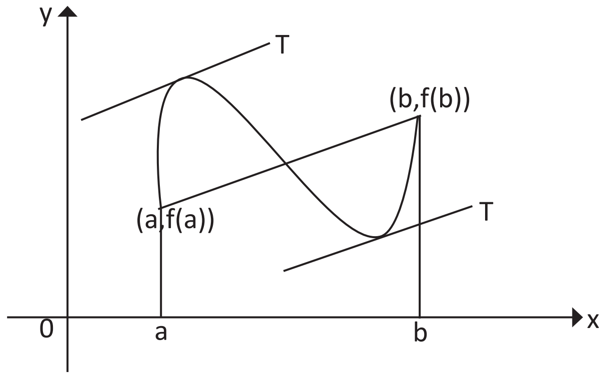
Rolle's Theorem states that the curve representing of graph of the function f must have a tangent parallel **to x-**axis, at least at one point between **a** and **b**.



Mean value theorem

The expression $\frac{f(b) - f(a)}{b - a}$ represent the slope of the chord joining the points (a, f(a)) and (b,f(b)).

Then, since f′(x)is the slope of the tangent line to the curve, the mean theorem states that there is always a number c between a and b such that the slope of the tangent at the point (c,f(c)) is the same as the slope of the chord. i.e tangent line and chord are parallel.



Assessment criteria

Use the gradient of a straight line as a measure of rate of change and apply this to line tangent or normal curves in various contexts. Use these concepts to solve and interpret related rates and optimization problems in various contexts.

9.8. Answers to Application activities of Unit 9 in the Student's Book

Application activity 9.1

1. (a) 0 (b) $1 + 3x^2$ (c) $3x^2 + 2$ (d) $4x^3 - \frac{1}{3}$
2. (a) $4 + 4x$ (b) 12 (c) -4
3. (a) $-\frac{1}{(x+2)^2}$ (b) $-\frac{2}{(2x-1)^2}$ (c) $-\frac{2}{x^3}$ (d) $-\frac{3}{x^3}$
4. (a) $\frac{1}{2\sqrt{x+2}}$ (b) $-\frac{1}{2x\sqrt{x}}$ (c) $\frac{1}{2\sqrt{2x+2}}$

Application activity 9.2

1. (a) 4 (b) -4 (c) 5 (d) 1
   (e) -12 (f) 3
2. (a) -1 (b) -1 (c) $-\frac{3}{4}$ (d) 1
   (e) $\frac{-1}{32}$ (f) -12 (g) -9 (h) $\frac{-45}{289}$
3. (a) $\frac{1}{2}$ (b) 1 (c) $\frac{-1}{27}$ (d) $\frac{1}{4}$

Application activity 9.3

(a) $f'(x) = 3x^2 + 2$ and $f''(x) = 6x$

(b) $f'(x) = -\frac{7}{(x-2)^2}$ and $f''(x) = \frac{14}{(x-2)^3}$

(c) $f'(x) = -\frac{2}{x^2}$ and $f''(x) = \frac{4}{x^3}$

(d) $f'(x) = \frac{x^2 + 4x + 1}{(x+2)^2}$ and $f''(x) = \frac{6}{(x+2)^3}$

(e) $f'(x) = 4x - \frac{1}{x^2} - \frac{1}{2\sqrt{x}}$ and $f''(x) = 4 + \frac{2}{x^3} + \frac{1}{4x\sqrt{x}}$

(f) $f'(x) = \frac{5x^2 + 2x + 7}{(x^2 + 2x - 1)^2}$ and $f''(x) = \frac{-10x^3 - 6x^2 - 42x - 30}{(x^2 + 2x - 1)^3}$

Application activity 9.4

1. (a) $4x$ (b) $2x + 3$ (c) $3x^2 + 6x + 4$

   (d) $20x^3 - 12x$ (e) $\frac{6}{x^2}$ (f) $-\frac{2}{x^2} + \frac{6}{x^3}$

   (j) $2x - \frac{5}{x^2}$ (h) $2x + \frac{3}{x^2}$ (i) $-\frac{1}{2x\sqrt{x}}$

   (j) $8x - 4$ (k) $3x^2 + 12x + 12$

2. a) $3x^2 + 6x$ (b) $10x$ (c) $-\frac{2}{5x^3}$

   d) 100 (e) 15 (f) $12x^2$

3. a) 1 (b) $\frac{3\sqrt{x}}{2}$ (c) $2x - 10$

   d) $2 - 9x^2$ (e) $4 + \frac{1}{4x^2}$ (f) -11

4. (a) 2 (b) $-\frac{16}{729}$ (c) -7

   (d) $\frac{13}{4}$ (e) $\frac{1}{8}$ (f) -11

5. (a) $2 - \frac{1}{2\sqrt{x}}$ (b) $\frac{1}{3\sqrt[3]{x^2}}$ (c) $\frac{1}{x\sqrt{x}}$

   (d) $\frac{2}{\sqrt{x}} + 1$ (e) $-\frac{2}{x\sqrt{x}}$ (f) $6x - \frac{3}{2}\sqrt{x}$

   (g) $-\frac{25}{2x^3\sqrt{x}}$ (h) $2 + \frac{9}{2x^2\sqrt{x}}$

6. (a) $\frac{dy}{dx} = 3 + \frac{2}{x^2}$, $\frac{dy}{dx}$ is the slope function of $y = 3x - \frac{2}{x}$ from which the slope at any point can be found.

   (b) $\frac{ds}{dt} = 2t + 6$ metres per second, $\frac{ds}{dt}$ is the instantaneous rate of change in position at the time t, i.e.. it is the velocity function.

   (c) $\frac{dC}{dx} = 4 + \frac{1}{200}x$ FRW per pen, $\frac{dC}{dx}$ is the instantaneous rate of change in cost as the number of pens changes.

Application activity 9.5

1. (a) $8(4x-5)$ (b) $2(5-2x)^{-2}$

(c) $\frac{1}{2}(3x-x^2)^{-1/2}(3-2x)$ (d) $-12(1-3x)^2$

(e) $-18(5-x)^2$ (f) $\frac{1}{3}(2x^3-x^2)^{-2/3}(6x^2-2x)$

(g) $-60(5x-4)^{-3}$ (h) $-4(3x-x^2)^{-2}(3-2x)$

(i) $18x^5-72x^2+\frac{72}{x}-\frac{6(x^3-2)^3}{x^4}$

2. (a) $-\frac{1}{\sqrt{3}}$ (b) -18 (c) -8

(d) -6 (e) $-\frac{3}{32}$ (f) 0

3. (a) $\frac{dy}{dx}=3x^2$,

(b) $\frac{dx}{dy}=\frac{1}{3}y^{-2/3}$ ……here we substitute $y=x^3$

Application activity 9.6

1. (a) $2x(2x-1)+2x^2$ (b) $2x(x-3)(2x-3)$

(c) $2x\sqrt{3-x}-\frac{x^2}{2\sqrt{3-x}}$ (d) $\frac{3(x-1)}{2\sqrt{x}}$

(e) $\frac{x^2(7x-6)}{2\sqrt{x-1}}$ (f) $\frac{1}{2\sqrt{x}}(x-x^2)^3+3\sqrt{x}(x-x^2)^2(1-2x)$

2. (a) -48 (b) -3

(c) $\frac{13}{3}$ (d) $\frac{11}{2}$

3. $x=3$ or $\frac{3}{5}$

4. (a) $\frac{2}{(x+2)^2}$ (b) $\frac{9}{(x+5)^2}$

(c) $\frac{(x^2-3)-2x^2}{(x^2-3)^2}$ (d) $\frac{6-8x}{(2x+3)^4}$

(e) $\frac{3x^2-6x+9}{(3x-x^2)^2}$ (f) $\frac{\sqrt{1-3x}+\frac{3x}{2\sqrt{1-3x}}}{1-3x}$

(g) $\frac{2}{(x^2+2)^{3/2}}$

5. (a) 1 (b) 1 (c) $-\frac{7}{324}$

(d) $-\frac{28}{27}$

Application activity 9.7

1. (a) $T \equiv y = 2x - 5$, $N \equiv 2y + x + 5 = 0$

(b) $T \equiv y = 4x - 2$, $N \equiv 4y + x + 8 = 0$

(c) $T \equiv y + x + 2$, $N \equiv y = x$

(d) $T \equiv y = 5$, $N \equiv x = 0$

(e) $T \equiv y + x = 3$, $N \equiv y = x - 1$

(f) $T\ y = 19x + 26$, $N \equiv 19y + x + 230 = 0$

2. (a) $y = -7x + 11$ (b) $4y = x + 8$

(c) $y = -2x - 2$ (d) $y = -2x + 6$

3. (a) $6y = -x + 57$ (b) $7y = -x + 36$

(c) $3y = x + 11$ (d) $x + 6y = 43$

4. (a) $(\frac{1}{2}, 2\sqrt{2})$ (b) $y = 21$ and $y = -6$

(c) $m = -5$ (d) $y = -3x + 1$

5. (a) $3y = x + 5$ (b) $9y = x + 4$

(c) $16y = x - 3$ (d) $y = -4$

6. (a) $y = 2x - \frac{7}{4}$ (b) $y = -27x - \frac{242}{3}$

(c) $57y = -4x + 1042$ (d) $2y = x + 1$

7. $m = 4, n = 3$

Application activity 9.8

1. 0 3. $-\frac{1}{8}$ 4. $-\frac{7}{3}$

2. $-\frac{6}{11}$ 5. -4

Application activity 9.9

a) i. $]-\infty, 0[\cup]2, +\infty[$

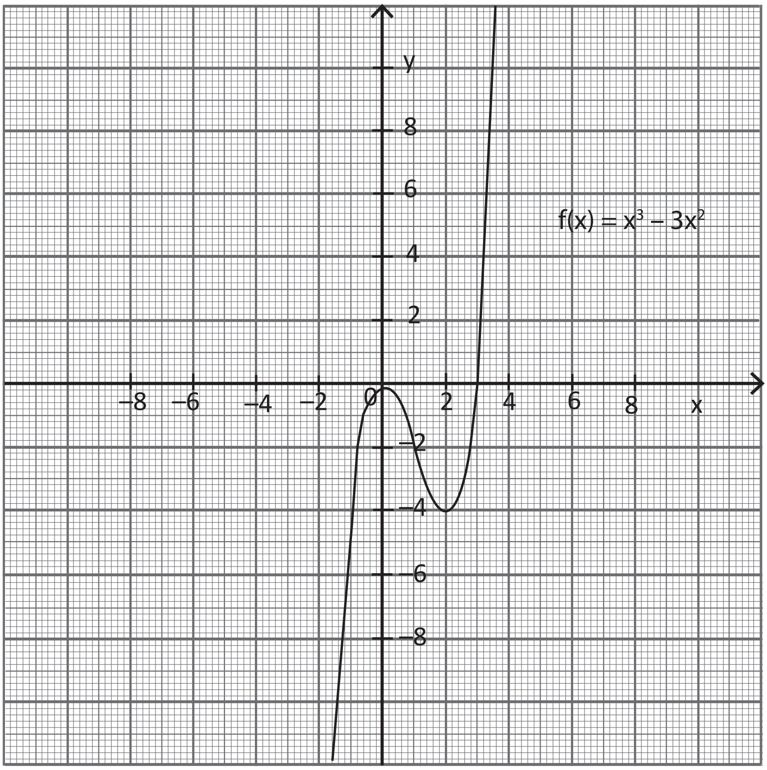
ii) $x = 0$ or $x = 2$

iii) $]1,+\infty[$

iv) $x = 1$

v) The extreme points are (0, 0) and 2, –4. The point of inflection is (1, –2)

Graph



b)

i)]–∞, 0 [∪] 3, +∞[

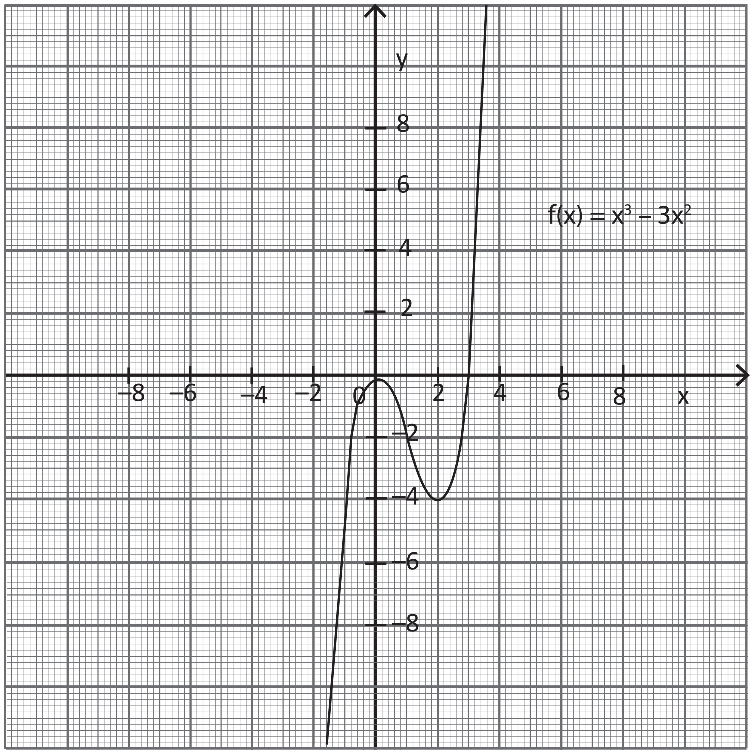
ii) x = 0 or x = 3

iii)]–∞, 0 [∪] 2, +∞[

iv) x = 0 or x = 2

v) The extreme points are (0, 0) and (3, –27). The points of inflection are (0, 0) and (2, –16).

Graph



c) (i) $]-\infty, \frac{6}{5}[\cup]2, +\infty[$

(ii) $x = 0$ or $x = 2$ or $x = \frac{6}{5}$

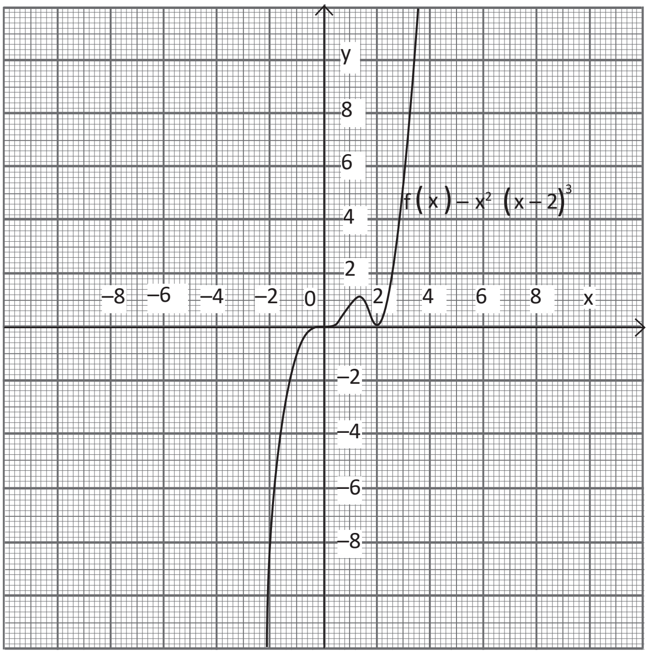
(iii) $]0, \frac{6-\sqrt{6}}{5}[\cup]\ \frac{6+\sqrt{6}}{5}, +\infty[$

(iv) $x = 0$ or $x = \frac{6-\sqrt{6}}{5}$ or $x\ \frac{6+\sqrt{6}}{5}$

(v) Extreme points are (0, 0), (2, 0) and $(\frac{6}{5}, \frac{3456}{3125}$). The inflection

points are at (0, 0), $(\frac{6-\sqrt{6}}{5}, f(\frac{6-\sqrt{6}}{5})$ and $(\frac{6+\sqrt{6}}{5}, f(\frac{6+\sqrt{6}}{5})$

Graph



Application activity 9.10

1. Let x be the side of a square sheet. If A is the area of a square sheet at time t, then we have $A = x^2$. Differentiating both sides with respect to time t, we get $\frac{dA}{dt} = 2x\frac{dx}{dt}$. Rate of increase of side x with respect to time t is 4 cm/sec. i.e.$\frac{dx}{dt} = 4$ cm/sec.

   Then, the rate of change of area A with respect to time t is $\frac{dA}{dt} = 2x \times 4 = 8x$.
   Hence, rate of increase of area A with respect to t when x = 8 is 8 × 8 = 64 cm^2/sec.

2. Let the radius of circle be r (in cm)

   Given: Rate of change of radius be $\frac{dr}{dt} = 3$ cm.

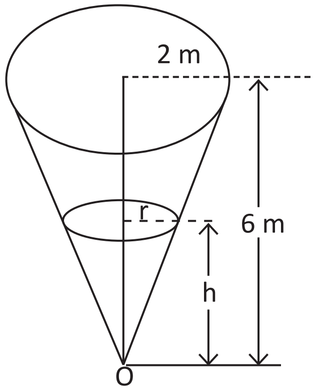
   Area of circle $A = \pi r^2$.

   Differentiating both sides with respect to time t, we get $\frac{dA}{dt} = 2\pi r\frac{dr}{dt}$

   When $\frac{dr}{dt} = 3$ therefore $\frac{dA}{dt} = 2\pi r \times 3 = 6\pi r$

   When r = 10 the rate of change of area is $\frac{dA}{dt} = 6\pi \times 10 = 60$ cm^2/sec .

3. Let r be the radius of the cone of oil where h denotes the height of the oil at a given time. Now $V = \frac{\pi r^2 H}{3}$.



From the figure, using similar triangles.

$\frac{r}{R} = \frac{h}{H} \Rightarrow \frac{r}{2} = \frac{h}{6} \Rightarrow r = \frac{h}{3}$

Then, the volume of oil at height h is given by $V = \frac{\pi\left(\frac{h}{3}\right)^2 h}{3} = \frac{\pi h^3}{23}$

Differentiating both sides with respect to time t, we get

$\frac{dv}{dt} = \frac{\pi}{9} h^2 \frac{dh}{dt}$

Given that $\frac{dv}{dt} = 0.002\ m^3/min$

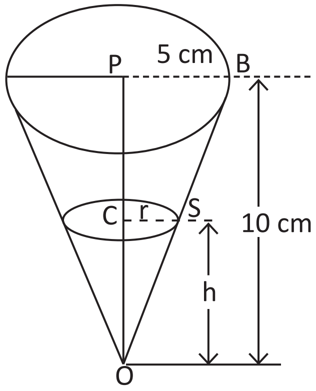
For h = 3 m, we get

$0.002 = \frac{\pi}{9} (3)^2 \frac{dh}{dt}$

$\frac{dh}{dt} = \frac{0.002}{3.14} = 6.37 \times 10^{-4}$ m/min

Hence, the rate at which the height of the oil is increasing is 6.37×10^{-4} m/min.

4. Let r (in cm) be the radius of the circular water-surface when the depth is h cm; at the end of time t minutes.



From figure, using similar triangles;

$\frac{r}{R} = \frac{h}{H} \Rightarrow \frac{r}{5} = \frac{h}{10} \Rightarrow r = \frac{h}{2}$

Let v be the volume of water at time t minutes.

Then $v = \frac{\pi\left(\frac{h}{2}\right)^2 h}{3} = \frac{\pi h^3}{12}$

Now $v = \frac{\pi h^3}{12}$

Differentiating both sides with respect to t, we get

$\frac{dv}{dt} = \frac{3}{12}\pi h^2 \frac{dh}{dt} = \frac{1}{4}\pi h^2 \frac{3}{8\pi}$

Given that $\frac{dv}{dt}$ = 1.5 cm^3/min and h = 4 cm, we get

$\frac{1}{4}\pi(4)^2 \frac{dh}{dt} = 1.5$

$$4\pi \frac{dh}{dt} = 1.5$$

$$\frac{dh}{dt} = \frac{1.5}{4\pi} = \frac{15}{40\pi} = \frac{3}{8\pi} \quad cm/\min$$

Hence, the rate at which the level of water is rising is $\frac{3}{8\pi}$ cm/min.

5. Suppose that each tin has radius r and perpendicular height h and S is the total area. We require the dimensions for the surface area S to be minimum and so we need an expression for S which is $S = 2\pi r^2 + 2\pi rh$. We cannot differentiate this expression in this form as S is given as a function of two variables r and h.

However the volume is $V = \pi r^2 h$ and this volume is to be 128π cm^3.

$128\pi = \pi r^2 h \Leftrightarrow h = \frac{128}{r^2}$ (*)

$S = 2\pi r^2 + 2\pi rh = 2\pi r^2\ 2\pi r(\frac{128}{r^2}) = 2\pi r^2 + \frac{256\pi}{r}$ and $\frac{dS}{dr} = 4\pi r - \frac{256\pi}{r^2}$

When $\frac{dS}{dr} = 0$ then $4\pi r - \frac{256\pi}{r^2} = 0 \Leftrightarrow \frac{4\pi r^3 - 256\pi}{r^2} = 0 \Leftrightarrow 4\pi(r^3 - 64) = 0 \Leftrightarrow$

$r^3 = 64 \Leftrightarrow r = 4cm$(**)

Using (**) in (*), we get h = 8 cm

The sign of $\frac{dS}{dr} = \frac{4\pi(r^3 - 64)}{r^2} = \frac{4\pi(r-4)(r^2 + 4r + 16)}{r^2}$

depends on the sign of $(r - 4)(r^2 + 4r + 16)$

Factors \ r	$-\infty$		4		$+\infty$
$r - 4$		$-$	0	$+$	
$r^2 + 4r + 16$		$+$		$+$	
$(r-4)(r^2 + 4r + 16)$		$-$	0	$+$	
S		↘		↗	

The surface is minimum at r = 4 and h = 8 cm.

Each tin should have a base radius of r = 4 cm and perpendicular height h = 8 cm.

6. Area $A = \frac{1}{2} \times 6 \times 8 \times \sin\theta = 24 \sin\theta \text{ cm}^2$

$\frac{dA}{d\theta} = 24 \cos\theta$

When $\theta = \frac{\pi}{4}$, $\cos\theta = \frac{\sqrt{2}}{2}$

$\frac{dA}{d\theta} = 24 \left(\frac{\sqrt{2}}{2}\right) = 12\sqrt{2} \text{ cm}^2$ per radian

Application activity 9.11

1. (a) 5, 0

   (b) $6x - 6, 6$

   (c) $6x^2 - 10x + 4, 12x - 10$

   (d) $3x^2 + \frac{2}{x^2}, 6x - \frac{4}{x^3}$

   (e) $\frac{3}{2x^{\frac{1}{2}}} - \frac{1}{x^{\frac{3}{2}}}, -\frac{3}{4x^{\frac{3}{2}}} + \frac{3}{2x^{\frac{5}{2}}}$

   (f) $-\frac{6}{x^2} + \frac{6}{x^3} - \frac{12}{x^4}$ and $\frac{12}{x^3} - \frac{18}{x^4} + \frac{48}{x^5}$

2. (a) Body changes direction at $t = 2$

   (b) Body does not change direction in the first 3 seconds.

3. 28 m

4. (a) $(19.6 - 9.8t)\ ms^{-1}, -9.8\ ms^{-2}$ (b) 2s

   (c) 19.6 m (d) 0.586 s, 3.41s

Application activity 9.12

1. Maximum profit is 12

2. Let the radius of the base of the cylinder be r and height h.

   Thus, $S = 2\pi rh + 2\pi r^2$ or $h = \frac{s - 2\pi r^2}{2\pi r}$

   $V = \pi r^2 h = \pi r^2 \left(\frac{s - 2\pi r^2}{2\pi r}\right)$

   Volume, $V = \frac{1}{2}(sr - 2\pi r^2)$

   $\frac{dV}{dr} = \frac{1}{2}(s - 6\pi r^2)$

For maximum or minimum,

$$\frac{dV}{dr} = 0$$

$$\frac{1}{2}(s - 6\pi r^2) = 0$$

$$s - 6\pi r^2 = 0$$

$$r = \sqrt{\frac{s}{6\pi}}$$

Note: $\frac{d^2V}{dr^2} = \frac{1}{2}(-12\pi r) = -6\pi r = -6\pi\sqrt{\frac{s}{6\pi}} = -\sqrt{6\pi s}$ (negative)

We have $h = \frac{s - 2\pi r^2}{2\pi r} = \frac{6\pi r^2 - 2\pi r^2}{2\pi r} = \frac{4\pi r^2}{2\pi r} = 2r.$

Hence, the height of the cylinder = diameter of the base

UNIT

10 Vector space of real numbers

Number of periods: 16

10.1. Key unit competence

Determine the magnitude and angle between two vectors and be able to plot these vectors and point out dot product of two vectors

10.2. Learning objectives

By the end of this unit, the student must be able to do the following:

Knowledge and understanding	Skills	Attitudes and values
• Define the scalar product of two vectors • Give examples of scalar product • Determine the magnitude of a vector and angle between two vectors	• Calculate the scalar product of two vectors • Analyse a vector in terms of size. • Calculate the angle between two vectors	• Apply and transfer the skills of dot product and magnitude to other areas of knowledge

10.3. Content

1. Vector spaces $\mathbb{R}^2$:
   - Definitions and operations on vectors
   - Properties of vectors
   - Sub-vector spaces
2. Vector spaces of plane vectors $\mathbb{R}$, V, +:
   - Linear combination of vectors
   - Basis and dimension
3. Euclidian vector space:
   - Modulus (or magnitude) of vectors
   - Dot product and properties

10.4. Materials required

Manila papers, graph papers, geometric instruments: ruler, T-square, protractors, computers.

10.5. Generic competences

Problem-solving, Communication, Cooperation and Research

Cross-cutting issues

- **Standardization culture**

  Learners encouraged to adhere to rules and standards of calculations and drawing of graphs.

- **Gender education**

  Groups consist of mixed gender and all are encouraged to participate.

10.6. Guidance on teaching and learning activities

a. Introductory activity

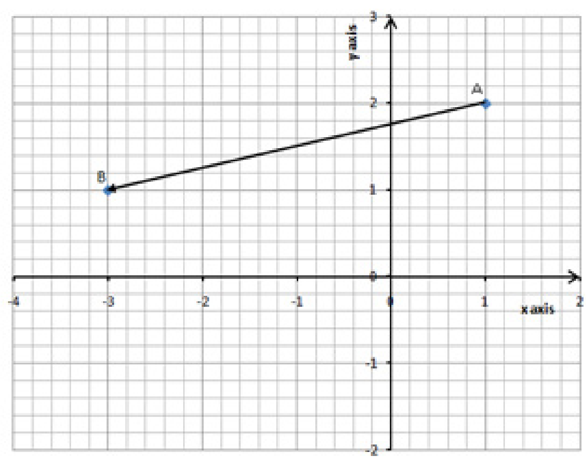
Guidance on introductory activity10

- Form groups of students and invite learners to work on the introductory activity found in student's book

- Guide the learners to present their findings and help them to harmonize their findings basing on their experience, prior knowledge and abilities shown in answering the questions for this activity.

- Open a discussion with the students on how the vectors are drawn in the plane XY . This will lead to the introduction of the vector space of real number.

Answer for introductory activity



b. Main activities

1. Introduce the unit by defining the main concepts used in vectors. This to include the operations – addition, subtraction and multiplication involving vectors.

2. Guide students to define and understand the scalar product of two vectors. Use appropriate examples.
3. Guide learners on the operations and definitions involving vector spaces. Give them Application activity 10.2 of the Student's Book to attempt in pairs.
4. Let them carry out research in advance on linear combination of vectors. Expound on spanning vectors and linear independent vectors. Let them do Application activity 10.3.
5. Use examples to expound on linear dependent vectors.
6. Let students work in groups and research on the similarities between vector spaces and Euclidian spaces. Let them do Activity 10.1 of the Student's Book.
7. Learners should be given a task to determine the magnitude and the angle between two vectors, plot these vectors and point out the dot product of two vectors. Give them Activity 10.2 and Activity 10.3 to attempt in groups or in pairs.

Reinforcement activity

Learners discuss about the scalar product of two vectors.

Learners of varying strengths and abilities

(a) Gifted and talented

Guide the learners to study units ahead of others. You can provide more advanced material to the learner. For example, let them draw extra graphs and allow them to do extra exercises. You can also make them group leaders in their groups so that they indirectly help the slower learners or the not so talented. However, ensure that they are supervised so as not to overwhelm the less gifted.

Give them individual exercises to work on as homework.

(b) Slow learners

You should provide them with the non-restrictive environment that provides/maintains their degree of freedom, self-determination, dignity and integrity of mind and body. This should, however allow for participation in an inclusive setting. Give them extra tuition. Teach them to master the material skills that are commensurate with their ability levels before introducing complex components.

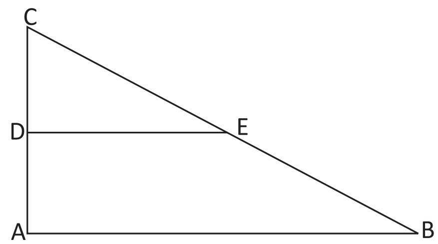
10.7. Additional tasks

Tasks 10 A - For slow learners

1. Show by using vectors that the line segment joining the mid-points of two sides of a triangle is parallel to the third side and is half of its length.
2. In a regular hexagon of vertices ABCDEF, prove that AB + AC + AD + AE + AF = 3AD

Answers

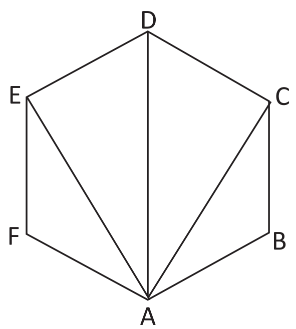
1. Let us illustrate the situation, where D and E are the mid-points of the sides AC and CB respectively.



We see that DE = DC + CE = $\frac{1}{2}$AC + $\frac{1}{2}$CB = $\frac{1}{2}$(AC + CB) = $\frac{1}{2}$AB

Therefore, DE = $\frac{1}{2}$AB

2. We make use of illustration below:



Since AB = ED and AF = CD, then

AB + AC + AD + AE + AF = ED + AC + AD + AE + CD = (AC + CD) + (AE + ED) + AD = AD + AD + AD = 3AD

Therefore, AB + AC + AD + AE + AF = 3AD

Application activity 10B - For talented learners

1. Let $\vec{e_1}$ = (1, 0) and $\vec{e_2}$ = (1, 0); Show that { $\vec{e_1}$, $\vec{e_2}$ } is a basis of vector space $\mathbb{R}^2$.

2. Let $\vec{u} = (1, -1)$ and $\vec{v} = (3, 2)$

(a) Show that $S = \{\vec{u}, \vec{v}\}$ is a basis of $\mathbb{R}^2$

(b) Find dim $\mathbb{R}^2$.

Answers

1. We have $\vec{e_1} = (1, 0)$ and $\vec{e_2} = (0, 1)$

The set $\{\vec{e}_1, \vec{e}_2\}$ is a basis if it is independent and its generating set $\{\vec{e}_1, \vec{e}_2\}$ is linearly independent.

$\{\vec{e}_1, \vec{e}_2\}$ is linear independent if….

$\alpha\vec{e_1} + \beta\vec{e_2} = 0 \Rightarrow \alpha = \beta = 0$

In fact;

$\alpha\begin{pmatrix}1\\0\end{pmatrix} + \beta\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$

$\begin{pmatrix}\alpha\\0\end{pmatrix} + \begin{pmatrix}0\\\beta\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$

$\begin{pmatrix}\alpha\\\beta\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$

$\alpha = \beta = 0$

Thus, $\{\vec{e_1}, \vec{e_2}\}$ is an independent set

$\{\vec{e_1}, \vec{e_2}\}$ is a generating set iff

$\forall x, y \in \mathbb{R}^2, \exists \alpha, \beta \in \mathbb{R} : \begin{pmatrix}x\\y\end{pmatrix} = \alpha\vec{e_1} + \beta\vec{e_2}$

$\begin{pmatrix}x\\y\end{pmatrix} = \alpha\begin{pmatrix}1\\0\end{pmatrix} + \beta\begin{pmatrix}0\\1\end{pmatrix}$

$\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}\alpha\\0\end{pmatrix} + \begin{pmatrix}0\\\beta\end{pmatrix}$

$\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}\alpha\\\beta\end{pmatrix}$

$\alpha = x$ and $\beta = y$

Thus, $\{\vec{e_1}, \vec{e_2}\}$ is a generating set.

Since $\{\vec{e_1}, \vec{e_2}\}$ is linear independent and generating set. Hence we conclude that $\{\vec{e_1}, \vec{e_2}\}$ is a basis of $\mathbb{R}^2$.

2. (a) We have to show that the vectors $\vec{u} = (1, -1)$ and $\vec{v} = (3, 2)$ are independent and generating vectors of $\mathbb{R}^2$.

We set $\alpha(1, -1) + \beta(3, 2) = (0, 0)$ and we determine α and β to see if the result will give us $\alpha = \beta = 0$

$$\begin{cases} \alpha + 3\beta = 0 \\ -\alpha + 2\beta = 0 \end{cases}$$

$$\begin{cases} 5\beta = 0 \\ \alpha = -2\beta \end{cases}$$

$$\begin{cases} \alpha = 0 \\ \beta = 0 \end{cases}$$

$\beta = 0$ and $\alpha = (0, 0)$ and . Thus the vectors $\vec{u}$ and $\vec{v}$ are independent.

We have to show that every vector in $\mathbb{R}^2$ lies in the span of S. Indeed, If $(x, y) \in \mathbb{R}^2$ we must show that there exist scalars α and β such

that $(x, y) = \alpha(1, -1) + \beta(3, 2)$

$$\begin{cases} \alpha + 3\beta = x \\ -\alpha + 2\beta = y \end{cases}$$

$$\begin{cases} 5\beta = x + y \\ \alpha = 2\beta - y \end{cases}$$

$$\begin{cases} \beta = \frac{x + y}{5} \\ \alpha = x - 3(\frac{x + y}{5}) = \frac{5x - 3x - 3y}{5} = \frac{2x - 3y}{5} \end{cases}$$

$\beta = \frac{x + y}{5}$ and $\alpha = \frac{2x - 3y}{5}$

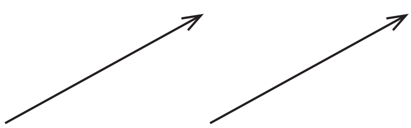
This shows that $\mathbb{R}^2$ = span S.

Hence, since $S = \{\vec{u}, \vec{v}\}$ is independent and generating set of $\mathbb{R}^2$ then it is the basis of $\mathbb{R}^2$.

(b) Since the number of vectors in the basis of $\mathbb{R}^2$ is 2 then, dim $\mathbb{R}^2 = 2$.

10.8. Answers to Tasks of Unit 10 in the Student's Book

Application activity 10.1

1. A vector is defined to be something that has magnitude and direction.
2. A quantity that is a real number, in other words, not a vector, is called a scalar. Unlike a vector, a scalar does not have direction.
3. 

Application activity 10.2

1. The proof is based on the properties of the vector space $\mathbb{R}$.

   (a) We verify if $(F(\mathbb{R}),$ is an abelian group, that is:

   (i) $(f + g)(x) = f(x) + g(x) = g(x) + f(x) = (g + f)(x)$ where we have used the fact that the addition of numerical functions is commutative.

(ii) $[(f + g) + h](x) = (f + g)(x) + h(x) = (fx) + g(x) + h(x)$

$= f(x) + (g(x) + h(x)) =$

$f(x) + (g + h)(x) = [f + (g + h)](x)$. (Associativity)

(iii) Let 0 be the zero function. Then for any $f \in F(\mathbb{R})$ we have

$(f + 0)(x) = f(x) + 0(x) = f(x)$ (Additive identity is 0)

(iv) $[f + (-f)](x) = f(x) + (-f(x)) = f(x) - f(x) = 0$.

(Additive inverse of $f(x)$ is $-f(x)$)

(b) We verify if the scalar multiplication satisfies:

(i) $[\alpha(\beta f)](x) = \alpha[\beta f(x)] = \alpha\beta[f(x)] = [(\alpha\beta)f](x)$.

(Associativity of multiplication)

(ii) $(1f)(x) = 1f(x) = f(x)$. (Multiplicative identity is 1)

(iii) $[\alpha(f + g)](x) = \alpha(f + g)(x) = \alpha f(x) + \alpha g(x) = (\alpha f + \alpha g)(x)$.

(Right distributivity)

(iv) $[\alpha(+ \beta)f](x) = (\alpha + \beta)f(x) = \alpha f(x) + \beta f(x) = (\alpha f + \beta f)(x)$. (Left distributivity)

Thus, $f(\mathbb{R})$ is a Vector Space.

2. For any $(x, y) \in V$ with $x, y > 0$ we have $-(x, y) \notin V$. The inverse element is not verified. Thus, V is not a vector space.

Application activity 10.3

1. Two vectors $\vec{u}$ and $\vec{v}$ are dependent if and only if one is a multiple of the other.

   (a) No, (b) yes, for $\vec{v} = 3\vec{u}$

2. Since $\alpha = \beta = 0$, $\vec{i}$ and $\vec{j}$ are linear independent.

3. (a) For any vector $\vec{x} = \begin{bmatrix} a \\ b \end{bmatrix}$ in $\mathbb{R}^2$, we have $c_1 = (\frac{5a - 2b}{7})$ and $c_2 = \frac{-4a + 3b}{7}$ for $\vec{x} = c_1\vec{v} + c_2\vec{w}$.

   (b) $c_1\vec{v} + c_2\vec{w} = \begin{bmatrix} a \\ b \end{bmatrix}$ implies $c_1 = c_2 = 0$.

UNIT

11 Concepts and operations on linear transformation in 2D

Number of periods: 14

11.1. Key unit competence

Determine whether a transformation of IR xIR is linear or not and perform operations on linear transformations.

11.2. Learning objectives

By the end of this unit, the student must be able to do the following:

Knowledge and understanding	Skills	Attitudes and values
• Define and distinguish between linear transformations in 2D • Define central symmetry, orthogonal projection of a vector, identical transformation • Define a rotation through an angle about the origin, reflection in the x axis, in y axis, in the line y – x • Show that a linear transformation is isomorphism in 2D or not	• Perform operations on linear transformations in 2D • Construct the composite of two linear transformations in 2D • Determine whether a linear transformation in 2D is isomorphism or not. • Determine the analytic expression of the inverse of an isomorphism in 2D	• Appreciate the importance and the use of operations on transformation in 2D • Show curiosity for the study of operations on transformations in 2D

11.3. Content

1. Linear transformations in 2D: Definition and properties
2. Geometric transformations: Definition and properties
3. Kernel and range
4. Operations of linear transformations

11.4. Materials required

Manila papers, graph papers, Geometric instruments (ruler, pair of compasses, T-square), digital technology instruments including calculators

11.5. Generic competences

Research, Cooperation, Communication and Problem-solving

Cross-cutting issues

- **Standardization culture**

  Learners encouraged to adhere to rules and standards of calculations and drawing of graphs.

- **Gender education**

  Groups consist of mixed gender and all are encouraged to participate.

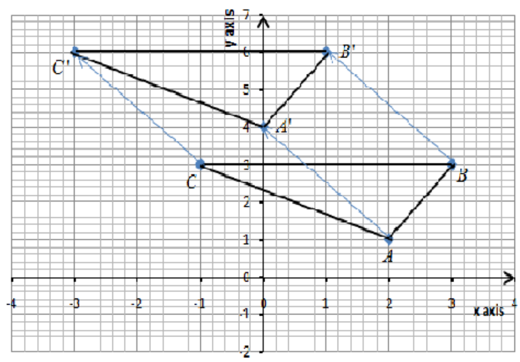
11.6. Guidance on teaching and learning activities

a. Introductory activity

Guidance on introductory activity 11

- Give clear instructions for learners to form small groups and to work on the introductory activity;
- Ensure that the learners have understood what the unit will be about and they are eager to learn; you can observe this through a clear and concise presentation of a group chosen randomly and the degree of attention other students are paying to the presentation.

Answer for introductory activity



The two triangle have the same size but different position. All vertices of triangle ABC have been moved in the same direction and same distance to form triangle A'B'C'

b. Main activities

1. Introduce the topic by defining a linear transformation in 2D. Guide them through definitions that are related to linear transformations.
2. Give them Applicatioon activity 11.1 of the Student's Book to discuss whether each operation in 2D is linear or not.

3. Ask learners to carry out research and discussions on geometric transformation in 2D. Let them do Activity 11.2 of the Student's Book. Take them through the transformations of rotation and reflection.
4. Help the students to determine whether a linear transformation in 2D is an isomorphism or not.
5. Guide them in determining the analytical expression of the inverse of an isomorphism in 2D. Let them tackle Applicatioon activity 11.2 of the Student's Book.
6. Ensure they appreciate the importance and use of operations on transformation in 2D. Let them do Activity 11.3 of the Student's Book. Give them Applicatioon activity 11.3 to work on in pairs.

Learners of varying strengths and abilities

(a) Gifted and talented

Guide the learners to study units ahead of others. You can provide more advanced material to the learner. For example, let them draw extra graphs and allow them to do extra exercises.

Give them individual exercises to work on as homework.

(b) Intellectual impairement

Try to understand the specific talents of the learner and develop them. Break the task down into small steps or learning objectives. Ensure learners start with what they can already do, then move on to a new more challenging task. Give the learner lots of practice and time. It helps to ensure the child has mastered a skill.

11.7. Additional tasks

Task 11A - For slow learners

1. A linear transformation T has matrix $\begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix}$. Find
   (a) the image of the point (2, 3) under T,
   (b) the coordinates of the point having an image of (7, 2) under T.

Answers

1. If (x', y') is the image of (x, y) then $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$
   (a) In this case $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 2(2) - 1(3) \\ 1(2) + 1(3) \end{pmatrix} = \begin{pmatrix} 4 - 3 \\ 2 + 3 \end{pmatrix} = \begin{pmatrix} 1 \\ 5 \end{pmatrix}$$

Thus (1, 5) is the image of the point (2, 3) under T.

(b) In this case (x', y') = (7, 2)

$$\begin{pmatrix}7\\2\end{pmatrix}=\begin{pmatrix}2 & -1\\1 & 1\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}$$

$$\begin{pmatrix}2 & -1\\1 & 1\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}7\\2\end{pmatrix}$$

$$\begin{cases}2x - y = 7\\x + y = 2\end{cases}$$

$x = 3$ and $y = -1$

Thus (3, –1) is the point having the image of (7, 2) under T.

Application activity 11B - For talented learners

Show that the following mapping f is homomorphism

$f: \mathbb{R}^2 \to \mathbb{R}^2$ defined by $f(x,y) = (2x, x - y)$.

Answer

1. $f(x, y) = 2x, x - y)$

We verify the two conditions of the definition.

For the first condition, we let (x_1, y_1) and (x_2, y_2) in $\mathbb{R}^2$ and compute

$$\begin{aligned}f((x_1, y_1) + (x_2, y_2)) = f(x_1, + x_2, y_1 + y_2) &= (2(x_1 + x_2), (x_1 + x_2) - (y_1 + y_2))\\ &= (2x_1 + 2x_2, x_1 + x_2 - y_1 - y_2)\\ &= (2x_1, x_1 - y_1) + (2x_2, x_2 - y_2)\\ &= f(x_1, y_1) + f(x_2, y_2)\end{aligned}$$

This proves the first condition.

For the second condition, we let $\alpha \in \mathbb{R}$ and $(x, y) \in \mathbb{R}^2$ and we compute

$f(\alpha(x, y)) = f(\alpha x, \alpha y) = (2\alpha x, \alpha x - \alpha y) = \alpha(2x, x - y) = \alpha f(x, y)$

This proves the second condition

Since the two conditions are satisfied, we conclude that is homomorphism (linear transformation)

Assessment criteria

Demonstrate that a transformation of $\mathbb{R}^2$ is linear. Perform operations on linear transformations.

11.8. Answers to Application activity of Unit 11 in the Student's Book

Application activity 11.1

1. (a) $f: \mathbb{R}^2 \to \mathbb{R}^2: f(x, y) = (y, 0)$

We verify the two conditions of the definition.

For the first condition, we let (x_1, y_1) and (x_2, y_2) in $\mathbb{R}^2$ and compute:

$\mathbf{f((x_1, y_1) + (x_2, y_2))} = f(x_1 + x_2, y_1 + y_2) = (y_1 + y_2, 0)$

$= (y_1, 0) + (y_2, 0) = \mathbf{f(x_1, y_1) + f(x_2, y_2)}$

This proves the first condition.

For the second condition, we let $\alpha \in \mathbb{R}$ and $(x, y) \in \mathbb{R}^2$ and we compute

$\mathbf{f(\alpha(x, y)} = f(\alpha x, \alpha y) = (\alpha y, 0) = \alpha(y, 0) = \mathbf{\alpha f(x, y).}$

This proves the second condition.

Since the two conditions are satisfied, f is a linear transformation.

(b) $f: \mathbb{R}^2 \to \mathbb{R}^2 : f(x,y) = (x^2,0)$

We verify the two conditions of the definition.

For the first condition, we let (x_1, y_1) and (x_2,y_2) in $\mathbb{R}^2$ and compute:

$\mathbf{f((x_1,y_1) + (x_2,y_2))} = f(x_1 + x_2, y_1 + y_2) = ((x_1 + x_2)^2, 0)$

$\neq ((x_1)^2, 0) + ((x_2)^2, 0) = f(x_1, y_1) + f(x_2, y_2)$

This does not prove the first condition. There is no need to continue for the second condition.

Since the first condition is not verified, the given transformation f is not linear.

(c) $f: \mathbb{R}^2 \to \mathbb{R}^2 : f(x,y) = (x, 2x - y)$

We verify the two conditions of the definition.

For the first condition, we let (x_1,y_1) and (x_2,y_2) in $\mathbb{R}^2$ and compute:

$(f((x_1y_1) + (x_2, y_2)) = f(x_1 + x_2, y_1 + y_2) = (x_1 + x_2, 2(x_1 + x_2) - (y_1 + y_2))$

$= (x_1 + x_2, 2x_1 + 2x_2 - y_1 - y_2)$

$= (x_1 + x_2, 2x_1 - y_1 + 2x_2 - y_2)$

$= (x_1, 2x_1 - y_1) + (x_2, 2x_2 - y_2)$

$= \mathbf{f(x_1, y_1) + f(x_2, y_2)}$

This proves the first condition.

For the second condition, we let $\alpha \in \mathbb{R}$ and $(x, y) \in \mathbb{R}^2$ and we compute $f(\alpha(x, y)) = f(\alpha x,\alpha y) = (\alpha x, 2\alpha x - \alpha y) = \alpha(x ,2x - y) = \alpha f(x,y)$.

This proves the second condition

Since the two conditions are satisfied, f is a linear transformation.

(d) $f: \mathbb{R}^2 \to \mathbb{R}^2: f(x, y) = (1, xy)$

We verify the two conditions of the definition.

For the first condition, we let (x_1,y_1) and (x_2,y_2) in $\mathbb{R}^2$ and compute:

$$f((x_1,y_1) + (x_2,y_2)) = f(x_1 + x_2, y_1 + y_2) = (1, (x_1 + x_2)(y_1 + y_2))$$
$$= (1, x_1y_1 + x_2y_2 + x_1y_2 + x_2y_1)$$
$$\neq (1, x_1y_1) + (1_1, x_2y_2) = \mathbf{f(x_1,y_1) + f(x_2,y_2)}$$

This does not prove the first condition. There is no need to proceed to the second condition.

Since one condition is not verified, the given transformation f is not linear.

2. $f(x,y) = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} (x,y) = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x + 2y \\ 3x + 4y \end{pmatrix} = (x + 2y, 3x + 4y)$

$f: \mathbb{R}^2: \to \mathbb{R}^2$ $(f(x, y) = (x + 2y, 3x + 4y)$

We verify the two conditions of the definition.

For the first condition, we let (x_1, y_1) and (x_2, y_2) in $\mathbb{R}^2$ and compute

$$\mathbf{f((x_1, y_1) + (x_2, y_2))} = f(x_1 + x_2, y_1 + y_2)$$
$$= ((x_1 + x_2) + 2(y_1 + y_2), 3(x_1 + x_2) + 4(y_1 + y_2))$$
$$= (x_1 + x_2 + 2y_1 + 2y_2, 3x_1 + 3x_2 + 4y_1 + 4y_2)$$
$$= (x_1 + 2y_1, 3x_1 + 4y_1) + (x_2 + 2y_2, 3x_2 + 4y_2)$$
$$= \mathbf{f(x_1, y_1) + f(x_2,y_2)}$$

This proves the first condition.

For the second condition, we let $\alpha \in \mathbb{R}$ and $(x,y) \in \mathbb{R}^2$ and we compute

$\mathbf{f(\alpha(x,y))} = f(\alpha x, \alpha y) = (\alpha x + 2\alpha y), 3\alpha x + 4\alpha y) =$

$\alpha(x + 2y, 3x + 4y) = \mathbf{\alpha f(x,y)}$.

This proves the second condition.

Since the two conditions are satisfied, is a linear transformation.

Application activity 11.2

$L = \mathbb{R}^2 \rightarrow \mathbb{R}^2$ defined by $L(x, y) = \begin{bmatrix} 3x + 2y \\ -x + y \end{bmatrix}$.

$\ker f = \{v \in \mathbb{R}^2 : f(v) = 0 \in \mathbb{R}^2\} = \{(x,y) \in \mathbb{R}^2 : (3x + 2y, -x + y) = (0, 0)\}$

$= \left\{(x, y) \in \mathbb{R}^2 : \begin{cases} 3x + 2y = 0 \\ -x + y = 0 \end{cases}\right\} = \left\{(x, y) \in \mathbb{R}^2 : \begin{cases} 3x + 2y = 0 \\ 2x - 2y = 0 \end{cases}\right\}$

$= \left\{(x, y) \in \mathbb{R}^2 : \begin{cases} 5x = 0 \\ x = y \end{cases}\right\} = \{(x, y) \in \mathbb{R}^2 : x = y = 0\} = \{(0, 0) \in \mathbb{R}^2\}$

ker f $= \{(0,0) \in \mathbb{R}^2\}$.

Thus, **dim ker f = 0**

Application activity 11.3

1. $f: = \mathbb{R}^2 \rightarrow \mathbb{R}^2 : f(x, y) = (3x - y, 2x + y)$

   We have to verify if f is one-to-one and onto.

   Let $u = (x_1, x_2)$ and $v = (y_1, y_2)$, then if $f(u) = f(v)$, we have

   $(3x_1 - x_2, 2x_1 + x_2) = (3y_1 - y_2, 2y_1 + y_2)$. We have the system

   $\begin{cases} 3x_1 - x_2 = 3y_1 - y_2 \\ 2x_1 + x_2 = 2x_1 + y_2 \end{cases}$ solving the system, we have $\begin{cases} 5x_1 = 5y_1 \\ 2x_1 + x_2 = 2y_1 + y_2 \end{cases} \Leftrightarrow$

   $\begin{cases} x_1 = y_1 \\ 2x_1 + x_2 = 2x_1 + y_2 \end{cases} \Leftrightarrow \begin{cases} x_1 = y_1 \\ x_2 = y_2 \end{cases}$

   Thus, f is one-to-one.

   Let (a,b) be an arbitrary element in the range space, then we have to verify if the following system has solution for all values of a and b.

   $\begin{cases} 3x - y = a \\ 2x + y = b \end{cases} \Leftrightarrow \begin{cases} 5x = a + b \\ y = b - 2x \end{cases} \Leftrightarrow \begin{cases} x = \frac{a + b}{5} \\ y = b - 2\left(\frac{a + b}{5}\right) \end{cases} \Leftrightarrow \begin{cases} x = \frac{a + b}{5} \\ y = \frac{5b - 2a - 2b}{5} \end{cases} \Leftrightarrow$

   $\begin{cases} x = \frac{a + b}{5} \\ y = \frac{3b - 2a}{5} \end{cases}$. We have solution for x and y.

   Thus, the transformation is onto.

   Hence, the transformation f is isomorphism.

2. $f: = \mathbb{R}^2 \rightarrow \mathbb{R}^2 : f(x, y) = (2x - 4y, -3y + 6y)$

   We have to verify if f is one-to-one and onto.

   Let $u = (x_1, x_2)$ and $y = (y_1, y_2)$, then if $f(u) = (f(v)$, we have

   $(2x_1 - 4x_2, -3x_1 + 6x_2) = (2y_1 - 4y_2, -3y_1 + 6y_2)$ we have the system

$$\begin{cases} 2x_1 - 4x_2 = 2y_1 - 4y_2 \\ -3x_1 + 6x_2 = -3y_1 + 6y_2 \end{cases}$$

solving the system, we have

$$\begin{cases} 6x_1 - 12x_2 = 6y_1 - 12y_2 \\ -6x_1 + 12x_2 = -6y_1 + 12y_2 \end{cases}$$

$0x_1 + 0x_2 = 0y_1 + 0y_2$.

Some different vectors of the domain have the same image.

Thus f is not one-to-one. There is no need to verify if the transformation is onto.

Hence, the transformation f is not isomorphism.

UNIT
12 Matrices and determinants of order 2

Number of lessons: 12

12.1. Key unit competence

Use matrices and determinants of order 2 to solve systems of linear equations and to define transformations of 2D.

12.2. Leaning objectives

By the end of this unit, the student should be able to do the following:

Knowledge and understanding	Skills	Attitudes and values
• Define the order of a matrix • Define a linear transformation in 2D by a matrix • Define operations on matrices of order 2 • Show whether a square matrix of order 2 is invertible or not	• Reorganise data into matrices • Determine the matrix of a linear transformation in 2D • Perform operations on matrices of order 2 • Construct the matrix of the composite of two linear transformations in 2D • Construct the matrix of the inverse of an isomorphism of IR2 • Determine the inverse of a matrix of order 2	• Appreciate the importance and the use of matrices in organising data • Show curiosity for the study of matrices of order 2

12.3. Content

1. Matrix of a linear transformation: Definition and operations
2. Matrix of geometric transformation
3. Operations on matrices:
    - Equality of matrices
    - Addition
    - Multiplication of matrices
    - Transpose of a matrix
    - Inverse of a matrix

4. Determinant of a matrix of order 2:
   - Definition
   - Applications of determinants

12.4. Materials required

Manila papers, markers,

12.5. Generic competences

Critical thinking, Cooperation, Communication and Research

Cross-cutting issues

- **Standardization culture**

  Learners encouraged to adhere to rules and standards of calculations and drawing of graphs.
- **Gender education**

  Groups consist of mixed gender and all are encouraged to participate.

12.6. Guidance on teaching and learning activities

a. Introductory activity

Guidance on introductory activity12

- Form groups of students and invite learners to work on the introductory activity found in student's book

- Guide the learners to present their findings and help them to harmonize their findings basing on their experience, prior knowledge and abilities shown in answering the questions for this activity.

- Open a discussion with the students on how the numbers were presented before solving the problems. This will lead to the introduction of the matrix concept.

Answer for introductory activity

a)

Cocks	Rabits	Prics
5	4	35,000
3	6	30,000

b)Let x be the cost of one cock and y be the cost of one rabbit, then ,

$$\begin{cases}5x+4y=35,000\\3x+6y=30,000\end{cases}$$

$$\begin{cases}5x+4y=35,000 & \times(3)\\3x+6y=30,000 & \times(-5)\end{cases}\Rightarrow\begin{cases}15x+12y=105,000\\-15x-30y=-150,000\end{cases}\Rightarrow -18y=-45,000\Rightarrow y=2,500$$

If we replace y in the first equation we obtain

$$5x+4(2500)=35,000\Rightarrow 5x=25,000\Rightarrow x=5,000$$

Thus the cost of 1 cock is 5,000Frw and the cost of one rabbit is 2,500Frw.

These numbers can also be presented as follows: $\begin{pmatrix}5 & 4\\3 & 6\end{pmatrix}$ and $\begin{pmatrix}35000\\30000\end{pmatrix}$ respectively.

b. Main activities

1. Guide them to define the order of a matrix. Let them also reorganise data into matrices. Let them be able to appreciate the importance and use of matrices in organising data.
2. Expound on the unit by defining a linear transformation by matrix in 2D.
3. Give them the Activity 12.2 to carry out.
4. Guide them in carrying out operations on matrices – addition, subtraction and multiplication. Let them do Activity 12.3 then Application activity 12.1 and Application activity 12.2 of the Student's Book.
5. Explain to them what a transpose matrix is and let them attempt Application activity 12.3 of the Student's Book.
6. Give them Activity 12.4 to find out what a determinant of a matrix is.
7. Help learners to determine the inverse of a matrix of order 2. Learners in groups to show whether or not a matrix of order 2 is invertible. Let them attempt Application activity 12.4 of the Student's Book.
8. Let them practise on uses of matrices by doing Application activity 12.5.

Learners of varying strengths and abilities

(a) Gifted and talented

You can provide more advanced material to the learner. Allow them to do extra exercises. You can also make them group leaders in their groups so that they indirectly help the slower learners or the not so talented. However, ensure that they are supervised so as not to overwhelm the less gifted.

Give them individual exercises to work on as homework.

(b) Slow learners

You should provide them with the non-restrictive environment that provides/maintains their degree of freedom, self-determination, dignity and integrity of mind and body. This should, however allow for participation in an inclusive setting. Give them extra tuition. Teach them to master the material skills that are commensurate with their ability levels before introducing complex components.

12.7. Additional tasks

Application activity 12A - For slow learners

Find x, y, z and t if $3\begin{pmatrix} x & y \\ z & t \end{pmatrix} = \begin{pmatrix} x & 6 \\ -1 & 2t \end{pmatrix} + \begin{pmatrix} 4 & x+y \\ z+t & 3 \end{pmatrix}$

Answers

$$3\begin{pmatrix} x & y \\ z & t \end{pmatrix} = \begin{pmatrix} x & 6 \\ -1 & 2t \end{pmatrix} + \begin{pmatrix} 4 & x+y \\ z+t & 3 \end{pmatrix}$$

$$\begin{pmatrix} 3x & 3y \\ 3z & 3t \end{pmatrix} = \begin{pmatrix} x+4 & 6+x+y \\ -1+z+1 & 2t+3 \end{pmatrix}$$

We form the system of equation below and solve

$$\begin{cases} 3x = x+4 \\ 3y = 6+x+y \\ 3z = -1+z+t \\ 3t = 2t+3 \end{cases} \qquad \begin{cases} x=2 \\ y=4 \\ 2z=-1+t \\ t=3 \end{cases} \Rightarrow \begin{cases} x=2 \\ y=4 \\ z=1 \\ t=3 \end{cases}$$

Therefore $x=2, y=4, z=1$ *and* $t=3$

Application activity 12B - For talented learners

1. Given that $A = \begin{pmatrix} 1 & 2 \\ 3 & -4 \end{pmatrix}$, $f(x) = 2x^2 - 3x + 5$ and $g(x) = x^2 + 3x - 10$. Determine:

   (a) A^2 (c) $f(A)$

   (b) A^3 (d) $g(A)$

Answers

1. (a) $A^2 = \begin{pmatrix} 1 & 2 \\ 3 & -4 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 3 & -4 \end{pmatrix} = \begin{pmatrix} 1+6 & 2-8 \\ 3-12 & 6+16 \end{pmatrix} = \begin{pmatrix} 7 & -6 \\ -9 & 22 \end{pmatrix}$

   (b) $A^3 = A^2A = \begin{pmatrix} 7 & -6 \\ -9 & 22 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 3 & -4 \end{pmatrix} = \begin{pmatrix} 7-18 & 14+24 \\ -9+66 & -18-88 \end{pmatrix}$

   $= \begin{pmatrix} -11 & 38 \\ 57 & -106 \end{pmatrix}$

   (c) $f(A) = 2A^2 - 3A + 5I = 2\begin{pmatrix} 7 & -6 \\ -9 & 22 \end{pmatrix} - 3\begin{pmatrix} 1 & 2 \\ 3 & -4 \end{pmatrix} + 5\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$$= \begin{pmatrix} 14 & -12 \\ -18 & 44 \end{pmatrix} - \begin{pmatrix} 3 & 6 \\ 9 & -12 \end{pmatrix} + \begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix} = \begin{pmatrix} 14-3+5 & -12-6+0 \\ -18-9+0 & 44+12+5 \end{pmatrix}$$

$$= \begin{pmatrix} 16 & -18 \\ -27 & 61 \end{pmatrix}$$

(d) $g(A) = A^2 + 3A - 10I = \begin{pmatrix} 7 & -6 \\ -9 & 22 \end{pmatrix} + 3\begin{pmatrix} 1 & 2 \\ 3 & -4 \end{pmatrix} - 10\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$$= \begin{pmatrix} 7 & -6 \\ -9 & 22 \end{pmatrix} + \begin{pmatrix} 3 & 6 \\ 9 & -12 \end{pmatrix} - \begin{pmatrix} 10 & 0 \\ 0 & 10 \end{pmatrix} = \begin{pmatrix} 7+3-10 & -6+6-0 \\ -9+9-0 & 22-12-10 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

we can say that a matrix is a zero (root) of the polynomial g(x) .

Assessment criteria

Use matrices and determinants of order 2 to solve systems of linear equations and to define transformations.

12.8. Answers to application activity of Unit 12 in the Student's Book

Application activity 12.1

$x = 4$, $y = -6$ and $z = 9$

Application activity 12.2

a) $\begin{bmatrix} 2 & -3 \\ -4 & 2 \end{bmatrix} + \begin{bmatrix} -1 & -5 \\ 3 & -2 \end{bmatrix} = \begin{bmatrix} 1 & -8 \\ -1 & 0 \end{bmatrix}$

b) $\begin{bmatrix} 2 & -3 \\ -4 & 2 \end{bmatrix} - \begin{bmatrix} -1 & -5 \\ 3 & -2 \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ -7 & 4 \end{bmatrix}$

c) $\begin{bmatrix} 3 & -5 & 4 \\ -1 & 4 & 6 \end{bmatrix} + \begin{bmatrix} -1 & 4 & 2 \\ -5 & -2 & 3 \end{bmatrix} = \begin{bmatrix} 2 & -1 & 6 \\ -6 & 2 & 9 \end{bmatrix}$

d) $\begin{bmatrix} 3 & -5 & 4 \\ -1 & 4 & 6 \end{bmatrix} - \begin{bmatrix} -1 & 4 & 2 \\ -5 & -2 & 3 \end{bmatrix} = \begin{bmatrix} 4 & -9 & 2 \\ 4 & 6 & 3 \end{bmatrix}$

Application activity 12.3

a) $AB = \begin{bmatrix} 3 & 1 \\ 4 & 2 \end{bmatrix} \times \begin{bmatrix} 2 & 3 \\ 1 & 5 \end{bmatrix} = \begin{bmatrix} 7 & 14 \\ 10 & 22 \end{bmatrix}$

b) $AB = \begin{bmatrix} 3 & -1 & 2 \\ -2 & 4 & 0 \end{bmatrix} \times \begin{bmatrix} 2 & 0 \\ -1 & 4 \\ -3 & 2 \end{bmatrix} = \begin{bmatrix} 13 & 0 \\ -8 & 16 \end{bmatrix}$

c) $QP = \begin{bmatrix} 4 & -2 \\ 3 & 2 \end{bmatrix} \times \begin{bmatrix} -2 & 5 \\ 1 & -3 \end{bmatrix} = \begin{bmatrix} -10 & 26 \\ -4 & 9 \end{bmatrix}$

d) $BA = \begin{bmatrix} 2 & -1 & 0 \\ 3 & -5 & 2 \\ 1 & 4 & -2 \end{bmatrix} \times \begin{bmatrix} 3 & -2 & 5 \\ 0 & -1 & 6 \\ -4 & 2 & -1 \end{bmatrix} = \begin{bmatrix} 6 & -3 & 4 \\ 1 & 3 & -17 \\ 11 & -10 & 31 \end{bmatrix}$

Application activity 12.4

1. If $A = \begin{pmatrix} 2 & -1 \\ 4 & 6 \end{pmatrix}$, then $2A = 2\begin{pmatrix} 2 & -1 \\ 4 & 6 \end{pmatrix} = \begin{pmatrix} 4 & -2 \\ 8 & 12 \end{pmatrix}$

   $-A = -\begin{pmatrix} 2 & -1 \\ 4 & 6 \end{pmatrix} = \begin{pmatrix} -2 & 1 \\ -4 & -6 \end{pmatrix}$

   $\frac{1}{2}A = \frac{1}{2}\begin{pmatrix} 2 & -1 \\ 4 & 6 \end{pmatrix} = \begin{pmatrix} 1 & -\frac{1}{2} \\ 2 & 3 \end{pmatrix}$

2. $AB = \begin{pmatrix} 2 & 3 \\ 4 & 6 \end{pmatrix}\begin{pmatrix} 3 & 6 \\ -2 & -4 \end{pmatrix} = \begin{pmatrix} 6-6 & 12-12 \\ 12-12 & 24-24 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

3. If $B = \begin{pmatrix} 1 & -2 \\ 0 & 3 \end{pmatrix}$, then $B^0 = I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

   $B^2 = \begin{pmatrix} 1 & -2 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} 1 & -2 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} 1-0 & -2-6 \\ 0+0 & 0+9 \end{pmatrix} = \begin{pmatrix} 1 & -8 \\ 0 & 9 \end{pmatrix}$

   $B^3 = B^2B = \begin{pmatrix} 1 & -8 \\ 0 & 9 \end{pmatrix}\begin{pmatrix} 1 & -2 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} 1-0 & -2-24 \\ 0+0 & 0+27 \end{pmatrix} = \begin{pmatrix} 1 & -26 \\ 0 & 27 \end{pmatrix}$.

4. det A = 2

5. det D = 0. D is a singular matrix

6. $AB = \begin{pmatrix} 4 & 5 \\ -1 & 4 \end{pmatrix} \neq \begin{pmatrix} 8 & 7 \\ -3 & 0 \end{pmatrix} = BA$

7. The matrix A does not have an inverse because det A=0.

While det B = 1 ≠ 0 thus, inverse of B is $B^{-1} = \begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix}$

Application activity 12.5

1. 30,000 FRW for Panadol tin 12,000 FRW for Fansidar tin.

2. x = 12 cm y = 6 cm

3. The costs of a bacon and a sausage are 4,700FRW and 1,100FRW respectively.

4. Airfares for first class and third class are 25,000 FRW and 14,5000 FRW respectively.

UNIT

13 Points, straight lines and circles in 2D

Number of periods: 21

13.1. Key unit comppetence

Determine algebraic representations of lines, straight lines and circles in 2D.

13.2. Learning objectives

By the end of this unit, the student must be able to d0 the following:

Knowledge and understanding	Skills	Attitudes and values
• Define the coordinate of a point in 2-D • Define a straight line knowing its: - 2 points - direction vector - gradient	• Represent a point and or a vector in 2D - calculate the distance between two points in 2D and the mid-point of a segment in 2D • Determine equations of a straight line (vector equation, parametric equation, Cartesian equation) • Apply knowledge to find the centre, radius, and diameter to find the equation of a circle • Perform operations to determine the intersection of a circle and a line	• Appreciate that a point is a fixed position in a plane • Show concern patiently and mutual respect in representations and calculations • Be accurate in representations and calculations • Manifest a team spirit and think critically in problem solving related to the positions of straight lines in 2D

13.3. Content

1. Points in 2D:
    - Cartesian coordinates of a point
    - Distance between two points
    - Mid-point of a line segment
2. Lines in 2D:
    - Equations of a line
    - Vector equation
    - Parametric equations

- Cartesian equation

3. Problems on points and straight lines in 2D
   - Position, angle, distance
4. The circle
   - Cartesian equation of a circle
   - Problems involving position of a circle and point or position of a circle and lines in 2D

13.4. Materials required

Manila paper, graph paper, geometric instruments (ruler, T-square), digital technology equipment including calculators.

13.5. Generic competences

Critical thinking, Cooperation, Communication and Research

Cross cutting issues

- **Standardization culture**

  Learners encouraged to adhere to rules and standards of calculations and drawing of graphs.

- **Gender education**

  Groups consist of mixed gender and all are encouraged to participate.

13.6. Guidance on teaching and learning activities

a. Introductory activity

Guidance on introductory activity 13

- Let learners read and do the introductory activity in the Learner's book by stating location of some places and use direction to know the place with respect to another place.
- Through question-answer, facilitate Learners to discover the importance of points and direction in daily activities.

Answers of introductory activity 13

a) At *A3* there is Change rooms.

b) The Main pool covers 3squares.

c) i) Diving area is at *F2* .

ii) Canteen is at *A1* .

d) i) Water slides is East from the Kids'pool.

ii) Change rooms is North from Canteen.

iii) Main pool is West from Diving.

b. Main activities

1. Revise with learners how to get the distance between two points in the Cartesian plane. Also guide them in locating the mid-point of a line. Then let them work in pairs or small groups to tackle Application activity 13.1 of the Student's Book.
2. Guide them to determine algebraic representations of straight lines in 2D. Let them work on Application activity 13.2 and Application activity 13.3 of the Student's Book.
3. In groups learners discuss the distance between two vectors.
4. Learners represent on graph papers some chosen points, lines and or circles and determine their parametric or Cartesian equations. Give them Application activity 13.4 on page 26 to tackle in pairs, or groups.
5. Assist them to represent the equations of a straight line in 2D.
6. Let them ponder over the Mental Application activity of the Student's Book to recall the main points and parts of a circle.
7. Ask them to research on what a unit circle is and to present their findings in a discussion in class. Allow them to explain with examples on the chalk board.
8. Expound on the concept of intersection of lines and circles then on collinear points. Give them Application activity 13.5 of the Student's Book.

Learners of varying strengths and abilities

(a) Gifted and talented

Guide the learners to study units ahead of others. You can provide more advanced material to the learner. For example, let them draw extra graphs and allow them to do extra exercises. You can also make them group leaders in their groups so that they indirectly help the slower learners or the not so talented. However, ensure that they are supervised so as not to overwhelm the less gifted.

Give them individual exercises to work on as homework.

(b) Slow learners

You should provide them with the non-restrictive environment that provides/maintains their degree of freedom, self-determination, dignity and integrity of mind and body. This should, however allow for participation in an inclusive setting. Give them extra tuition. Teach them to master the material skills that are commensurate with their ability levels before introducing complex components.

13.7. Additional tasks

Application activity 13A - For slow learners

1. Let L be the line that passes through the points A(1, 3) and B(2, -4). Find an equation for the line L.

Answer

1. We are given points A(1, 3) and (2, -4)

$L \equiv y - 3 = \frac{3+4}{1-2}(x - 1)$

$L \equiv y - 3 = \frac{7}{-1}(x - 1)$

$L \equiv y = -7x + 10$

$L \equiv y = 10 - 7x$

Application activity 13B - For talented learners

1. If one end of a diameter of the circle $x^2 + y^2 - 4x - 6y + 11 = 0$ is (3, 4), then find the other end of the diameter on the circle.

Answer

1. Equation of the circle is $x^2 + y^2 - 4x - 6y + 11 = 0$

$x^2 - 4x + y^2 - 6y = -11$

By completing squares

$(x^2 - 4x + 4) + (y^2 - 6y + 9) = -11 + 4 + 9$

$(x - 2)^2 + (y - 3)^2 = 2$

The centre of the circle is (2, 3) and the radius is $\sqrt{2}$.

That centre is the midpoint of the diameter.

Let the other end point of the diameter be (a, b)

$(2, 3) = \left(\frac{3+a}{2}, \frac{4+b}{2}\right)$

We have the system of two equations with two unknowns $\begin{cases} \frac{3+a}{2} = 2 \\ \frac{4+b}{2} = 3 \end{cases}$

$$\begin{cases} 3 + a = 4 \\ 4 + b = 6 \end{cases}$$

$$\begin{cases} a = 1 \\ b = 2 \end{cases}$$

The other end of the diameter on the circle is (1, 2).

Assessment criteria

Determine algebraic representations of lines and circles in 2D.

13.8. Answers to Tasks of Unit 13 in the Student's Book

Application activity 13.1

1. (a) (6, 5) (b) (3, 1) (c) $(\frac{7}{2}, 5)$
   (d) (1, 3) (e) (–5, –5) (f) (4, 2)
   (g) (–9, 4)
2. (a) $\begin{pmatrix} 4 \\ 6 \end{pmatrix}$ (b) $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$ (c) $\begin{pmatrix} -3 \\ 1 \end{pmatrix}$
   d() $3\vec{i}$ (e) $6\vec{i} + 2\vec{j}$ (f) $-4\vec{i} - 2\vec{j}$
3. B (2, 3)
4. P = 5 and q = –1
5. E(–9, –8)
6. (a) (3, 1) (b) (6, 5) (c) 5
7. If we have A(9, 9), B(3, 2) and C(9, 4)

   (a) The coordinates of M, the midpoint of BC is $\frac{1}{2}BC = \left(\frac{3+9}{2}, \frac{2+4}{2}\right) = (6,3)$

   (b) The length of the median from A to M is

   $|AM| = \sqrt{(6-9)^2 + (3-9)^2} = \sqrt{9 + 36} = \sqrt{45} = \sqrt{9 \times 5} = 3\sqrt{5}$
8. A(0, 1), B(2, 7) and C(4, –1)

   The midpoints are

   $M_{BC} = \left(\frac{2+4}{2}, \frac{7-1}{2}\right) = (3, 3)$

   $M_{AB} = \left(\frac{0+2}{2}, \frac{1+7}{2}\right) = (1, 4)$, and

   $M_{AC} = \left(\frac{0+4}{2}, \frac{1-1}{2}\right) = (2, 0)$

   The lengths of the medians are

$|AM_{BC}| = \sqrt{(3-0)^2 + (3-1)^2} = \sqrt{9+4} = \sqrt{13}$

$|CM_{AB}| = \sqrt{(1-4)^2 + (4+1)^2} = \sqrt{9+25} = \sqrt{34}$

$|BM_{AC}| = \sqrt{(2-2)^2 + (0-7)^2} = \sqrt{0+49} = 7$

9. We are given the points of a parallelogram P(–1, 5) , Q(8, 10) , R(7, 5) and S. We have to find the coordinates of S.

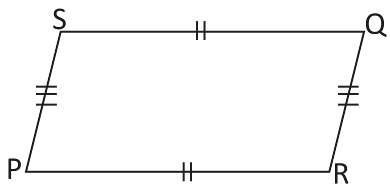
$|PQ| = \sqrt{(8+1)^2 + (10-5)^2} = \sqrt{81+25} = \sqrt{106}$

$|QR| = \sqrt{(7-8)^2 + (5-10)^2} = \sqrt{1+25} = \sqrt{26}$

$|PR| = \sqrt{(7+1)^2 + (5-5)^2} = \sqrt{64+0} = \sqrt{64} = 8$

Since |PQ| is longer, it is a diagonal.

We can illustrate this as below:



We have P(–1, 5), Q(8, 10), R(7, 5).

We find the gradient of the line RQ which is the same as the one of the line PS and is $\frac{10-5}{8-7}$

The equation of the line PS is

$y - 5 = 5(x + 1)$

$y = 5x + 10$

We find the gradient of the line PR which is the same as the one of the line QS and is $\frac{5-5}{7+1} = 0$

The equation of the line QS is

$y - 10 = 0(x - 8)$

$y = 10$

We have to find the point S which is the intersection of the lines

$y = 5x + 10$ and $y = 10$. Therefore, to find S, we have to solve the following system:

$$\begin{cases} y = 5x + 10 \\ y = 10 \end{cases}$$

$$\begin{cases} 10 = 5x + 10 \\ y = 10 \end{cases}$$

$$\begin{cases} x = 0 \\ y = 10 \end{cases}$$

The coordinates of the point S are (0, 10).

10. We are given the points of a parallelogram A(1, 1) , B(2, 7) , C(13, 7) and D.

(a) We have to find the coordinates of D.

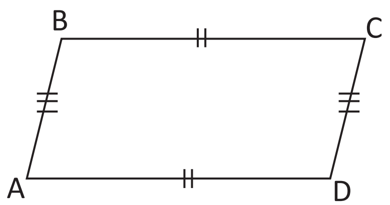
$|AB| = \sqrt{(2-1)^2 + (7-1)^2} = \sqrt{1 + 36} = \sqrt{37}$

$|AC| = \sqrt{(13-1)^2 + (7-1)^2} = \sqrt{144 + 36} = \sqrt{170}$

$|BC| = \sqrt{(13-2)^2 + (7-7)^2} = \sqrt{121 + 0} = \sqrt{121} = 11$

Since |AC| is longer, it is a diagonal.

We can illustrate this as below:



We have A(1, 1) , B(2, 7) , C(13, 7).

We find the gradient of the line AB which is the same as the one of the line CD and is $\frac{7-1}{2-1} = 6$

The equation of the line CD is

$y - 7 = 6(x - 13)$

$y = 6x - 71$

We find the gradient of the line BC which is the same as the one of the line AD and is $\frac{7-7}{13-2} = 0$

The equation of the line AD is

$y - 1 = 0(x - 1)$

$y = 1$

We have to find the point D which is the intersection of the lines $y = 6x - 71$ and $y = 1$. Therefore, to find D, we have to solve the following system:

$$\begin{cases} y = 6x - 71 \\ y = 1 \end{cases}$$

$$\begin{cases} I = 6x - 71 \\ y = 1 \end{cases}$$

$$\begin{cases} x = 12 \\ y = 1 \end{cases}$$

The coordinates of the point D are (12, 1).

(b) The coordinates of the midpoint of the diagonal DB is

$$\left(\frac{12 + 2}{2}, \frac{7 + 1}{2}\right) = (7, 4)$$

(c) The coordinates of the midpoint of the diagonal AC is

$$\left(\frac{1 + 13}{2}, \frac{1 + 7}{2}\right) = (7, 4)$$

Application activity 13.2

1. (a) (1, 2) and (0, 2)

$$y - 2 = \frac{2 - 2}{1 - 0}(x - 1)$$

$$y - 2 = 0(x - 1)$$

$$y = 2$$

(b) (1, 3) and (–2, 5)

$$y - 3 = \frac{3 - 5}{1 + 2}(x - 1)$$

$$y = -\frac{2}{3}(x - 1) + 3$$

$$y = -\frac{2}{3}x + \frac{11}{3}$$

(c) (–3, 0) and (1, 4)

$$y - 0 = \frac{4 - 0}{1 + 3}(x + 3)$$

$$y = 1(x + 3)$$

$$y = x + 3$$

(d) (8, 2) and (3, –5)

$$y - 2 = \frac{2 + 5}{8 - 3}(x - 8)$$

$$y = \frac{7}{5}(x - 8) + 2$$

$y = \frac{7}{5}x - \frac{46}{5}$

(e) (4, –2) and (6, –1)

$y + 2 = \frac{-1 + 2}{6 - 4}(x - 4)$

$y = \frac{1}{2}(x - 4) + 2$

$y = \frac{1}{2}x - 4$

2.

(a) (2, 3) and gradient is 2

$y - 3 = 2(x - 2)$

$y = 2x - 1$

(b) (1, 5) and gradient is – 1

$y - 5 = -1(x - 1)$

$y = -x + 6$

(c) (–3, 0) and gradient is 3

$y - 0 = 3(x + 3)$

$y = 3x + 9$

(d) (4, 2) and gradient is 1

$y - 2 = 1(x - 4)$

$y = x - 2$

(e) (9, –3) and gradient is –2

$y + 3 = -2(x - 9)$

$y = -2x + 15$

3. C(1, 3), D(4, 2) , E(–3, –1) and F(–1, 5)

$M_{CD} = \left(\frac{1 + 4}{2}, \frac{3 + 2}{2}\right) = (\frac{5}{2}, \frac{5}{2})$

$M_{EF} = \left(\frac{-3 - 1}{2}, \frac{-1 + 5}{2}\right) = (-2, 2)$

$y - 2 = \frac{\frac{5}{2} - 2}{\frac{5}{2} + 2}(x + 2)$

$$y - 2 = \frac{\frac{1}{2}}{\frac{9}{2}}(x + 2)$$

$$y = \frac{2}{9}(x + 2) + 2$$

$$y = \frac{2}{9}x + \frac{22}{9}$$

4. A(2, 4) and B(–1, 3)

   Gradient of AB is $\frac{4-3}{2+1} = \frac{1}{3}$

   The required line is perpendicular to the line AB and its gradient is –3.

   The required line passes through the midpoint of AB which is $\left(\frac{2-1}{2}, \frac{4+3}{2}\right) = \left(\frac{1}{2}, \frac{7}{2}\right)$.

   The equation of the line is

   $y - \frac{7}{2} = -3(x - \frac{1}{2})$

   $y = -3x + 5$

5. A(3, 1) and B(4, 8)

   (a) The mid-point of AB is $\left(\frac{3+4}{2}, \frac{1+8}{2}\right) = (\frac{7}{2}, \frac{9}{2})$

   (b) The gradient of AB is $\frac{8-1}{4-3} = 7$

   (c) The required line is perpendicular to the line AB and its gradient is $-\frac{1}{7}$

   The required line passes through the midpoint of AB which is $(\frac{7}{2}, \frac{9}{2})$.

   The equation of the line is

   $y - \frac{9}{2} = -\frac{1}{7}(x - \frac{7}{2})$

   $y = -\frac{1}{7}x + 5$

6. A(0, 7), B(9, 4) and C(1, 0)

   (a) The gradient of the line AB is $\frac{4-7}{9-0} = -\frac{1}{3}$

   The required line is perpendicular to the line AB and its gradient is 3.

   The required line passes through the point C(1, 0)

   The equation of the required line is

   $y - 0 = 3(x - 1)$

   $y = 3x - 3$

   (b) The required line passes through the point A(0, 7) and the midpoint of

BC which is $\left(\frac{9+1}{2}, \frac{4+0}{2}\right) = (5,2)$. The equation of the required line is

$y - 2 = \frac{7-2}{0-5}(x - 5)$

$y - 2 = -1(x - 5)$

$y = -x + 7$

7. A(1, 0) , B(5, 2) and C(1, 6)

The midpoints are:

$M_{BC} = \left(\frac{5+1}{2}, \frac{2+6}{2}\right) = (3, 4)$

$M_{AC} = \left(\frac{1+1}{2}, \frac{0+6}{2}\right) = (1, 3)$

$M_{AB} = \left(\frac{1+5}{2}, \frac{0+2}{2}\right) = (3, 1)$

The median from A to BC has equation

$y - 4 = \frac{4-0}{3-1}(x - 3)$

$y - 4 = 2(x - 3)$

$y = 2x - 2$

The median from B to AC has equation

$y - 3 = \frac{3-2}{1-5}(x - 1)$

$y - 3 = -\frac{1}{4}(x - 1)$

$y = -\frac{1}{4}x + \frac{13}{4}$

The median from C to AB has equation

$y - 1 = \frac{6-1}{1-3}(x - 3)$

$y = -\frac{5}{2}(x - 3) + 1$

Application activity 13.3

(a) (4, –1)

(b) (2, 1)

(c) (2, 3)

(d) (3, 5)

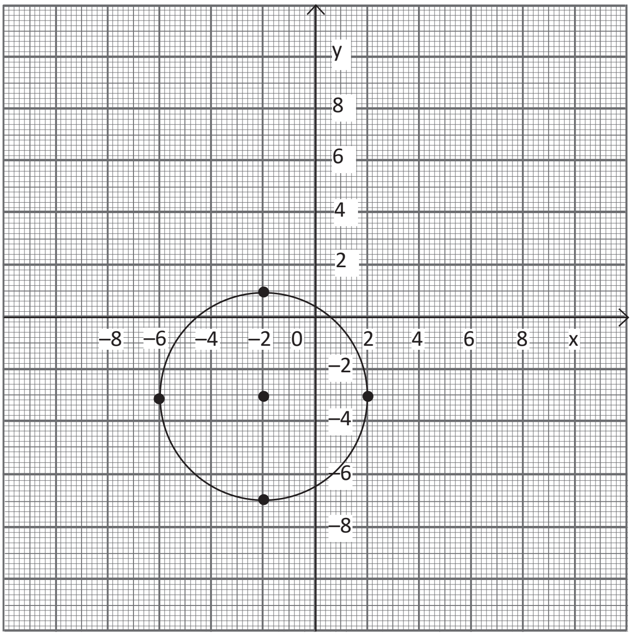
Application activity 13.4

1. (a) $\frac{1}{3}$ (b) $\frac{3}{4}$ (c) $-\frac{11}{2}$ or $-5\frac{1}{2}$

2. 26.6°, 45°, 108.4°

Application activity 13.5

1. $x^2+y^2+22x-18y+186=0$
2. $(x+7)^2+(y-11)^2=317$
3. 6
4. (-4, 6)
5. The centre is (–2, –3) and the radius is 5 units of length.



UNIT

14 Measures of dispersion

Number of periods: 7

14.1. Key unit competence

Extend understanding, analysis and interpretation of data arising from problems and questions in daily life to include the standard deviation.

14.2. Learning objectives

By the end of this unit, the student must be able to do the following:

Knowledge and understanding	**Skills**	**Attitudes and values**
• Define the variance, standard deviation and the coefficient of variation • Analyse and interpret critically data and infer conclusion.	• Determine the measures of dispersion of a given statistical series. • Apply and explain the standard deviation as the more convenient measure of the variability in the interpretation of data • Express the coefficient of variation as a measure of the spread of a set of data as a proportion of its mean.	• Appreciate the importance of measures of dispersion in the interpretation of data • Show concern on how to use the standard deviation as measure of variability of data.

14.3. Content

1. Introduction - measures of central tendency
2. Measures of dispersion
3. Coefficient of variation
    - Problems to include measure of dispersion and explain the standard deviation as the more convenient measure of variability in the interpretation of data.

- Problems to include measure of dispersion and express the coefficient of variation as a measure of the spread of a set of data as a proportion of its mean.

14.4. Materials required

Manila papers, graph papers, ruler, digital technology including calculators.

14.5. Generic competences

Critical thinking, Problem solving, Cooperation and Communication

Cross-cutting issues

- **Financial education**

  Use of examples that encourage planning for and prudent use of money.

- **Standardization culture**

  Learners encouraged to adhere to rules and standards of calculations and drawing of graphs.

- **Gender education**

  Groups consist of mixed gender and all are encouraged to participate.

14.6. Teaching and learning activities

a. Introductory activity

Guidance on introductory activity14

- In small groups or pairs, let learners read and do the introductory activity 14 found in the students' book.
- After a given time invite learners to present their findings and harmonize them.
- From presentations, try to arouse the curiosity of students on the content of this unit.

Answers for the introductory activity 14

1. The table below shows the types and the number of sold fruits in one week.

Type of fruit	A (Banana)	B (Orange)	C (Pineapple)	D (Avocado)	E (Mango)	F (apple)
Number of fruits sold	1100	962	1080	1200	884	900

a) The highest number of fruits sold is 1200 (Avocadoes)

b) The least number of fruits sold is 884 (mangoes)

c) The total number of fruits sold during the week is 6126 fruits

d) The average number of fruits sold per day is $\frac{6126}{6} = 1021$.

b. Main activities

1. In groups, learners will discuss the measures of dispersion, interpret them and represent their findings. Let them do Application activity 14.2 of the Student's Book.
2. Discuss and explain the range, to include the interquartile range and the semi-interquartile range.
3. Explain variability of data.
4. Introduce them to standard deviation and variance. Use a number of examples. Encourage learners to work out some examples on the chalkboard for all to discuss. Give them Application activity 14.1 of the Student's Book to work out.
5. Let students research on the meaning of coefficient of variation. Discuss their findings to come up with a concise definition.
6. Give them Application activity 14.3 to research on and discuss real life applications of measures of dispersion. Let them do Application 14.2 of the Student's Book.
7. Help them appreciate the importance of measures of dispersion.

Learners of varying strengths and abilities

(a) Physical impairment

- Use cooperative learning for instance through group work and discussions. Those having difficulty with manipulative tasks should be assisted by the others in the group.
- Mix students with special needs with the rest so as to be helped.
- Provide these students with frequent progress checks.

(b) Intellectual impairement

- Try to understand the specific talents of the learner and develop them.
- Break the task down into small steps or learning objectives. Ensure learners start with what they can already do, then move on to a new more challenging task.
- Give the learner lots of practice and time. It helps to ensure the child has mastered a skill.

14.7. Additional tasks

Application activity 14A - For slow learners

1. The numbers of incorrect answers on a true-false competency test for a random sample of 15 students were recorded as follows:

   2, 1, 3, 0, 1, 3, 6, 0, 3, 3, 5, 2, 1, 4, and 2.

   Find the mean, the median, the mode and the sample standard deviation.

Answers

1. The mean is

   $\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i = \frac{1}{15}(2 + 1 + 3 + 0 + 1 + 3 + 6 + 0 + 3 + 3 + 5 + 2 + 1 + 4 + 2)$

   $= \frac{1}{15}(36) = 2.4$

   The median is the middle observation in the ascending data:

   0, 0, 1, 1, 1, 2, 2, 2 , 3, 3, 3, 3, 4, 5, 6 . So, the median is 2.

   The mode is the most frequently occurring observation in a sample. So the mode is 3.

   The sample standard deviation is $\sigma = \sqrt{\frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^2}$. We need a table:

N^0	x_i	$x_i - \bar{x}$	$(x_i - \bar{x})^2$
1	0	–2.4	5.76
2	0	–2.4	5.76
3	1	–1.4	1.96
4	1	–1.4	1.96
5	1	–1.4	1.96
6	2	–0.4	0.16
7	2	–0.4	0.16
8	2	–0.4	0.16
9	3	0.6	0.36
10	3	0.6	0.36
11	3	0.6	0.36
12	3	0.6	0.36
13	4	1.6	2.56
14	5	2.6	6.76
15	6	3.6	12.96
			$\sum_{i=1}^{n}(x_i - \bar{x})^2 = 41.6$

$\sigma = \sqrt{\frac{1}{15}\Sigma_{i=1}^{n}(x_i - \bar{x})^2} = \sqrt{\frac{1}{15}(41.6)} = \sqrt{2.77} = 1.66$

Application activity 14B - For talented learners

1. If the sum of the first n natural numbers and the sum of the first square natural numbers is given by

   $\Sigma x_i = 1 + 2 + 3 + 4 + 5 + \ldots + n = \frac{n(n+1)}{2}$ and

   $\Sigma (x_i)^2 = 1^2 + 2^2 + 3^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}$, respectively;

   show that their mean and variance is $\frac{n+1}{2}$ and $\frac{n^2-1}{12}$, respectively.

Answer

1. $\bar{x} = \frac{1}{n}\Sigma x_i = \frac{1}{n}(1 + 2 + 3 + 4 + 5 + \ldots + n) = \frac{1}{n}(\frac{n(n+1)}{2}) = \frac{n+1}{2}$

   Variance $(\sigma^2) = \frac{1}{n}\Sigma x_i^2 - (\bar{x})^2 = \frac{1}{n}(\frac{n(n+1)(2n+1)}{6}) - (\frac{n+1}{2})^2$

   $= \frac{(n+1)(2n+1)}{6} - \frac{(n+1)^2}{4}$

   $= (n+1)\left[\frac{2n+1}{6} - \frac{n+1}{4}\right] = (n+1)\left[\frac{4n+2-3n-3}{12}\right] = (n+1)\left[\frac{n-1}{12}\right]$

   $= \frac{(n+1)(n-1)}{12} = \frac{n^2-1}{12}$

Assessment criteria

Extend understanding, analysis and interpretation of data arising from problems and questions in daily life to include standard deviation.

14.8. Answers to application activity of Unit 14 in the Student's Book

Application activity 14.1

1.
   (a) $\bar{x} = 5$ and $\sigma = 2$
   (b) $\bar{x} = 8.5$ and $\sigma = 1.80$
   (c) $\bar{x} = 18.8$ and $\sigma = 6.46$

2.
   (a) $\bar{x} = 10.834$ and $\sigma = 4.10$
   (b) $\bar{x} = 3.42$ and $\sigma = 1.91$
   (c) $\bar{x} = 205$ and $\sigma = 3.16$

Application activity 14.2

1. The inter-quartile range is 30
2. The standard deviation is 14.3614
3. The variance is 8. The standard deviation is $2\sqrt{2}$
4. a) $x = 11$
   b) The modes are 4, 10 and 11
   c) The median is 10
   d) The range is 16
   e) The mean deviation is $\frac{1}{n}\sum |x_1 - \bar{x}| = \frac{1}{10}(40) = 4$
   f) The standard deviation is 4.8785
   g) 9.5
   h) 75

UNIT

15 Combinatorics

Number of periods: 18

15.1. Key unit activity

Use combinations and permutations to determine the number of ways a random experiment occurs.

15.2. Learning objectives

By the end of this unit, the student must be able to do the following:

Knowledge and understanding	**Skills**	**Attitudes and values**
• Define the combinatorial analysis • Recognise whether repetition is allowed or not. and if order matters on not in performing a given experiment • Construct Pascal's triangle • Distinguish between permutations and combinations	• Determine the number of permutations and combinations of "n" items, "r" taken at a time. • Use counting techniques to solve related problems. • Use properties of combinations	• Appreciate the importance of counting techniques • Show concern on how to use the counting techniques

15.3. Content

1. Counting techniques:
    - Venn diagram
    - Tree diagrams
    - Contingency table
    - Multiplication principles
2. Arrangements and permutations:

- Arrangements with or without repetition
- Permutations with or without repetition

3. Combinations:
   - Definitions and properties
   - Pascal's triangles
   - Binomial expansion

15.4. Materials required

Manila papers, graph papers, ruler, digital components including calculators.

15.5. Generic competences

Cooperation, Communication, Problem-solving and Research

Cross-cutting issues

- **Inclusive education**

People with special needs must be included in solving the country problems.

- **Gender education**

Group work to consist of both gender in full participation. The examples given also have both gender in the groups.

15.6. Guidance on teaching and learning activities

a. Introductory activity

Guidance for the introductory activity 15

- In small groups or pairs, let learners read and do the introductory activity 15 in the students' book.
- Walk around to each group and ask probing questions leading them to determine the total number of roads from A to C via B;
- Invite groups with different working steps to present their findings to the whole class for discussion;
- As a facilitator, harmonize their answers highlighting that there is a technique of finding the total number of outcomes for a given random experiment;

Answers for introductory activity 15

To find all possible roads, students can use allows to join points or a try and fail method.

$\Omega = \left\{ AB_1C_1, AB_1C_2, AB_1C_3, AB_2C_1, AB_2C_2, AB_2C_3 \right\}$ so they are 6.

b. Main activities

1. Discuss with learners what tree diagrams are and how they are used in representing relationships.
2. Discuss with the students and explain to them the use of a contingency table.
3. Give them the Mental Application activity of the Student's Book to introduce them to the generalities of permutations and combinations.
4. Explain the factorial notation.
5. Give them Activity 15.1 of the Student's Book to introduce them to the real concept of permutations. After that, guide them to be able to distinguish between those with repetition and those without.
6. Discuss with them the arrangement in a circle formation as brought out of the Student's Book.
7. Explain and give examples and practice questions on conditional arrangements. Let them do Application activity 15.2 of the Student's Book.
8. Give them Activity 15.2 of the Student's Book to assist them grasp the concept of combinations. Expound on the definitions and explain using examples the different possible scenarios on combinations.
9. Let them do Application activity 15.3 of the Student's Book.
10. Explain the concept of Pascal's triangles and guide students in linking it to the binomial expansion. Let them do Application activity 15.4 and Application activity 15.5 in groups or pairs.

Learners of varying strengths and abilities

(a) Physical impairment

- Use cooperative learning, for instance through group work and discussions. Those having difficulty with manipulative tasks should be assisted by the others in the group.
- Mix students with special needs with the rest so as to be helped
- Provide these students with frequent progress checks.

(b) Hearing impairment

- Record portions of textbooks, trade books, and other printed materials in audio so students can listen (with earphones) to an oral presentation of necessary material.

- Providing written or pictorial directions to those with hearing problems
- Using of concrete objects such as models, diagrams, samples, and the like to those with so as to demonstrate what you are saying by using touchable items. They can also rewrite content such as charts on large manila paper in their group work
- Facing the learner while you speak might help learners with a hearing impairment
- Use large writing on the blackboard and on visual aids

(c) Intellectual impairement

- Try to understand the specific talents of the learner and develop them;
- Break the task down into small steps or learning objectives. Ensure learners start with what they can already do, then move on to a new more challenging task
- Give the learner lots of practice and time. It helps to ensure the child has mastered a skill.

15.7. Additional tasks

Application activity 15A - For slow learners

1. If $(n + 2)! = 20n!$, find n.
2. (a) If license plates of cars in Rwanda are made of the letter R followed by any two letters of the alohabet and any three digits followed by any one letter, how many possible different license plates can be made?

   (b) If any MTN Rwanda line must contain 10 digits where the first three of which are 078, find the number of all possible different MTN lines.

Answers

1. $(n + 2)(n + 1)n! = 20n!$

   $(n + 2)(n + 1) = 20$

   $n^2 + 3n + 2 - 20 = 0$

   $n^2 + 3n - 18 = 0$

   $(n - 3)(n + 6) = 0$

   $n = 3$ and $n = -6$ must be rejected

   Thus, $n = 3$ is the only solution

2. (a) We have 26 alphabet letters and 10 digits to choose from where the first letter must be R and there is one possible choice of R.

The number of possible different license plates is

$1 \cdot 26 \cdot 26 \cdot 10 \cdot 10 \cdot 10 \cdot 26 = 17{,}576{,}000$

(b) We have 10 digits to choose from where the first three digits must be 078

The number of all possible different MTN lines is

$1 \times 10 \times 10 \times 10 \times 10 \times 10 \times 10 \times 10 = 10{,}000{,}000$

Application activity 15B - For talented learners

1. Calculate the number of different ways in which six people can form

(a) A queue (that is, a single file) of six people

(b) A queue of two people and another queue of four people

(c) a group of two people and another group of four people

(d) two groups of three people each

(e) three pairs

(f) first, second and third pairs

2. How many odd numbers between 2,000 and 3,000 can be formed from the digits 1, 2, 3, 4, 5 and 6 using each of them only once in each number?

Answers

1. (a) The number of different ways in which six people can form a queue is 6! = 720 ways

(b) This is a permutation of 6 people in 6 places. The number of ways is 6! = 720

(c) Here we find the ways by forming one group of 2 people from 6 and the other group will be formed automatically. The number of ways

is $\binom{6}{2} = \frac{6!}{2!4!} = \frac{6 \times 5 \times 4!}{2 \times 4!} = 15$

(d) Here we find the ways by forming one group of 3 people from 6 and the other group will be formed automatically. The number of ways

is $\binom{6}{3} = \frac{6!}{3!3!} = \frac{6 \times 5 \times 4 \times 3!}{3! \times 3!} = \frac{6 \times 5 \times 4}{6} = 20$

(e) This is to form the group of two people from six.

The number of ways is $\binom{6}{2} = \frac{6!}{2!4!} = \frac{6 \times 5 \times 4!}{2 \times 4!} = 15$

(f) The number of ways we can form any of the three groups is

$\binom{6}{2} = \frac{6!}{2!4!} = \frac{6 \times 5 \times 4!}{2 \times 4!} = 15.$

The number of different permutations is $3! = 6$

Therefore, the number of different ways of having the first, second and third pairs is $15 \cdot 6 = 90$.

2. Because the number is between 2000 and 3000, it has only four digits and the first must only be 2. The number is odd. So, the last digits must be taken from 1, 3 and 5. Once the first and the last digits are placed there are four digits available for second place and three for the third place.

	1st	2nd	3rd	4th (Last)
Number of ways of selecting digit	1	4	3	3

There are $1 \cdot 4 \cdot 3 \cdot 3 = 36$ numbers.

Assessment criteria

Calculate accurately combinations or permutations of 'n' items, taking 'r' at a time.

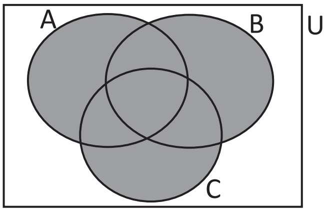
15.8. Answers to Application activity of Unit 15 in the Student's Book

Application activity 15.1

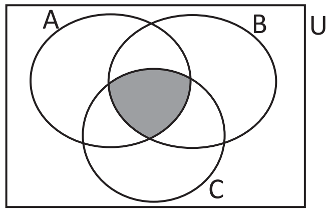
1. (a) {1, 3, 4, 5, 6, 7} (b) {3, 5, }
   (c) {4, 6} (d) {1, 2, 3, 5, 7}
   (e) {1} (f) {2}

2. (a) {3, 4, 5, 6} (b) {1, 4, 9, 16, 25}
   (c) {2, 3, 4, 5, 6} (d) {1, 11, 13, 143}
   (e) {1} (f) {1}

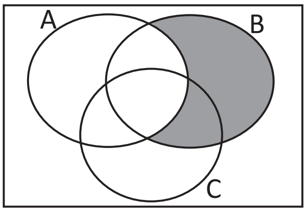
3. (a)



(b)



(c)



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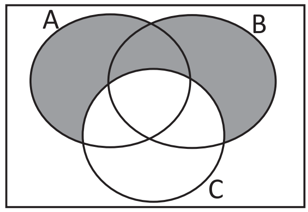

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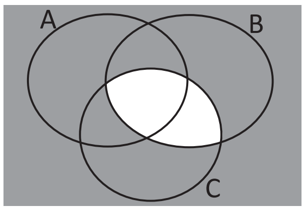
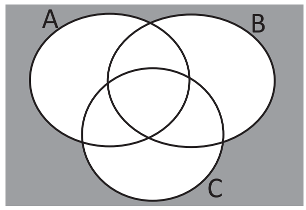


(f)



Application activity 15.2

1. (a) 24 (b) 120
2. 720
3. 3,628,800
4. (a) 125 (b) 60
5. 120
6. 5,040
7. 168
8. (a) 240 (b) 600
9. (a) 1,000 (b) 720 (c) 990
10. (a) 103,680 (b) 34,560

Application activity 15.3

1. (a) 6 (b) 84 (c) 1820
2. 1140
3. 525
4. (a) 462 (b) 56 (c) 20
5. 1,260
6. (a) 924 (b) 34,650
7. (a) 729 (b) 28
8. 10

9. 61 (including the given straight line)

10. 56

11. (a) 286 (b) 84

Application activity 15.4

1. $x^3 + 9x^2 + 27x + 27$

2. $x^4 - 8x^3 + 24x^2 - 32x + 16$

3. $x^4 + 4x^3 + 6x^2 + 4x + 1$

4. $8x^3 + 12x^2 + 6x + 1$

5. $x^5 - 15x^4 + 90x^3 - 270x^2 + 405x - 243$

6. $p^4 - 4p^3q + 6p^2q^2 - 4pq^3 + q^4$

7. $8x^3 + 36x^2 + 54x + 27$

8. $x^5 - 20x^4 + 160x^3 - 640x^2 + 1280x - 1{,}024$

9. $81x^4 - 108x^3 + 54x^2 - 12x + 1$

10. $1 + 20a + 150a^2 + 500a^3 + 625a^4$

Application activity 15.5

1. (a) $x^4 + 4x^3y + 6x^2y^2 + 4xy^2 + y^4$

   (b) $a^7 - 7a^6b + 21a^5b^2 - 35a^4b^3 + 35a^3b^4 - 21a^2b^5 + 7ab^6 - b^7$

   (c) $64 + 192p^2 + 240p^4 + 160p^6 + 60p^8 + 12p^{10} + p^{12}$

   (d) $32h^5 - 80h^4k + 80h^3k^2 - 40h^2k^3 + 10k^4 - k^5$

   (e) $x^3 + 3x + 3x^{-1} + x^{-3}$

   (f) $z^8 - 4z^6 + 7z^4 - 7z^2 + \frac{35}{8} - \frac{7}{4}z^{-2} + \frac{7}{16}z^{-4} - \frac{1}{16}z^{-6} + \frac{1}{256}z^{-8}$

2. $64x^5 + 160x^{-1} + 20x^{-7}$

3. 0, 1 (trivial) and 6.

4. 2

5. 30.43168

6. (a) 560 (b) –590,625 (c) –720 (d) –448

   (e) 1,966,080

UNIT

16 Elementary probability

Number of periods: 7

16.1. Key unit competence

Use counting techniques and concepts of probability to determine the probability of possible outcomes of events occurring under equally likely assumptions.

16.2. Learning objectives

By the end of this unit, students must be able to do the following:

Knowledge and understanding	Skills	Attitudes and values
• Define probability and explain probability as a measure of chance • Distinguish between mutually exclusive and non-exclusive events and compute their probabilities	• Use and apply properties of probability to calculate the number possible outcomes of occurring event under equally likely assumptions • Determine and explain expectations from an experiment with possible outcomes	• Appreciate the use of probability as a measure of chance • Show concern on patience, mutual respect, tolerance and curiosity in the determination of the number of possible outcomes of a random experiment

16.3. Content

1. Concepts of probability:
   - Random experiment
   - Sample space
2. Finite probability spaces
3. Sum and product laws
4. Conditional probability

16.4. Materials required

Manila paper, graph paper, ruler, digital components including calculators.

16.5. Generic competences

Problem solving, Cooperation, Communication and Research

Cross-cutting issues

- **Standardization culture**

  Learners encouraged to adhere to rules and standards of calculations and drawing of graphs.

- **Gender education**

  Groups consist of mixed gender and all are encouraged to participate. Examples given in the Student's Book and Teacher's Guide are inclusive of all gender.

16.6. Guidance on teaching and learning activities

a. Introductory activity

Guidance on introductory activity16

- In small groups or pairs, let learners read and do the introductory activity in the students' book.
- Request learners to present their findings in a whole class discussion;

As a facilitator, harmonize answers for students and arouse their curiosity on the content of this unit.

Answers for the introductory activity 16

a) There are 25 black cards in an ordinary deck of 52 cards.

b) $P(\text{A}) = \dfrac{n}{number\ of\ allcards} = \dfrac{26}{52} = 0.5$

c) If $P(A)$ is the probability of selecting a black card, probability for any this event is the quotient of the number of outcomes of this event by the total number of outcomes in the sample space.

b. Main activities

1. Let learners discuss problems associated with gambling and report their results to the group. This is the requirement of Activity 16.1 of the Student's Book.
2. Assist them to define probability and explain probability as a measure of chance and explain expectations from an experiment with possible outcomes.
3. Guide students to distinguish between mutually exclusive and non-

exclusive events and compute their probabilities.

4. Help them on how to use and apply properties of probability to calculate the number of possible outcomes occurring under equally likely assumptions
5. Introduce permutations and combination in probability theory by giving them Activity 16.2 of the Student's Book. Learners are given a task of sitting 3 men and 4 women at random in a row. In groups, they discuss about the probability that all the men are seated together then they give feedback.
6. Explain to them the concept, to include probability spaces. Let them tackle Application activity 16.1 found of the Student's Book.
7. Explain sum and product laws and let them tackle, in pairs, application activity 16.2 of the Student's Book.
8. Take them through the various types of conditional probability and let them work on application activity 16.3 of the Student's Book.

Learners of varying strengths and abilities

(a) Gifted and talented

Guide the learners to study units ahead of others. You can provide more advanced material to the learner.

Give them individual exercises to work on as homework.

(b) Slow learners

Give them extra tuition. Teach them to master the material skills that are commensurate with their ability levels before introducing complex components.

(c) Intellectual impairement

- Try to understand the specific talents of the learner and develop them;
- Break the task down into small steps or learning objectives. Ensure learners start with what they can already do, then move on to a new more challenging task
- Give the learner lots of practice and time. It helps to ensure the child has mastered a skill.

16.7. Additional tasks

Application activity 16A - For slow learners

1. A fair (unbiased) coin is tossed 5 times. Find the probability of obtaining
   (a) 5 tails
   (b) Exactly 4 tails
   (c) At least one head

Answer

1. We are given an unbiased coin which is tossed 5 times. $P(H) = P(T) = \frac{1}{2}$.
   (a) P(obtaining 5 tails) = P(T, T, T, T, T)

   $= P(\text{obtaining no head}) = \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{32}$

   (b) P(exactly 4 tails) =

   P(H, T, T, T, T) + P(T, H, T, T, T) + P(T, T, H, T, T) + P(T, T, T, H, T)

   $+ P(T, T, T, T, H) = 5(\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2}) = \frac{5}{32}$

   (c) $P(\text{at least one head}) = 1 - P(\text{obtaining no head}) = 1 - \frac{1}{32} = \frac{32-1}{32} = \frac{31}{32}$

Application activity 16B - For talented learners

1. Show that sets A and B are independent if and only if $P(A \cap B) = P(A) \times P(B)$
2. Each seat of Gacaca Judiciary court of a cell is made up of 9 persons of integrity.
   (a) How many ways of choosing a committee of coordination of judiciary if each member of Gacaca seat is a candidate to one post and selections for the posts are made in 3 successive phases:

   Phase 1: one president

   Phase 2: first and second vice presidents

   Phase 3: two secretaries

   (b) If the seat of Gacaca Court of the cell is composed of 4 ladies and 5 men:
   i) what is the probability of choosing a lady as the president of the judiciary court?
   ii) what is the probability for the two posts of first and second vice presidents to be occupied by ladies?
   iii) what is the probability of choosing two secretaries of different

sexes if the president and one of the vice presidents are men?

Answers

1. We know that $P(A|B) = \frac{P(A\cap B)}{P(B)} = P(A)$ this gives us $P(A\cap B) = P(A) \times P(B)$

   and $P(B|A) = \frac{P(B\cap A)}{P(A)} = \frac{P(A\cap B)}{P(A)} = P(B)$ this gives us $P(A\cap B) = P(A) \times P(B)$

2.

   (a) We select 1 out of 9, then 2 out of 8 remaining (those can interchange positions), then 2 out of 6 remaining: The number of ways is

   $\binom{9}{1} \times \binom{8}{2} P_2^2 \times \binom{6}{2} = \frac{9!}{1! \times 8!} \times 2 \times \frac{8!}{2! \times 6!} \times \frac{6!}{2! \times 4!} = 9 \times 2 \times 28 \times 15 = 7560$ or

   $\binom{9}{1} \times P_2^8 \times \binom{6}{2} = \frac{9!}{1! \times 8!} \times \frac{8!}{6!} \times \frac{6!}{2! \times 4!} = 9 \times 56 \times 15 = 7560$

   (b) i) The probability of choosing a woman is $\frac{4}{9}$.

   ii) Here is a case where we have 4 women if the elected president is a man and the case where we have 3 women if the elected president is a woman.

   The probability is $\frac{P_2^4 + P_2^3}{P_2^8} = \frac{\frac{4!}{2!} + \frac{3!}{1!}}{\frac{8!}{6!}} = \frac{12 + 6}{56} = \frac{18}{56} = \frac{9}{28}$

   Or since we are interested in that the two vice presidents are women, we can ignore the order and we find the probability is $\frac{\binom{4}{2} + \binom{3}{2}}{\binom{8}{2}} = \frac{6 + 3}{28} = \frac{9}{28}$

   iii) Here we remain by 3 men and 3 women and one on each gender is selected.

   The probability is $\frac{\binom{3}{1} \times \binom{3}{1}}{\binom{6}{2}} = \frac{3 \times 3}{15} = \frac{9}{15} = \frac{3}{5}$

Assessment criteria

Use counting techniques and concept of probability to determine the probability of possible outcomes occurring under equally likely assumptions.

16.8. Answers to Application activity of Unit 16 in the Student's Book

Application activity 16.1

1. (a) $\frac{1}{2}$ (b) $\frac{1}{2}$ (c) $\frac{1}{2}$

2. (a) $\frac{1}{4}$ (b) $\frac{3}{4}$ (c) $\frac{7}{24}$

(d) $\frac{11}{12}$

3. (a) $\frac{1}{17}$ (b) $\frac{15}{34}$

4. (a) $\frac{1}{6}$ (b) $\frac{5}{126}$

Application activity 16.2

1. $P(A) = \frac{3}{8}$, $P(B) = \frac{5}{12}$ and $P(A \cap B) = \frac{1}{4}$

   $P(A \cup B) = P(A) + P(B) - P(A \cap B) = \frac{3}{8} + \frac{5}{12} - \frac{1}{4} = \frac{9 + 10 - 6}{24} = \frac{13}{24}$

2. $P(A - B) = 0.3$, $P(B - A) = 0.4$, $P(A' \cap B') = 0.1$

   (a) We know that $P(A' \cap B') = P(A \cup B)' = 0.1$

   $P(A \cap B) = 1 - [P(A - B) + P(B - A) + P(A \cup B)']$

   $= 1 - [0.3 + 0.4 + 0.1] = 1 - 0.8 = 0.2$

   (b) $P(A) = P(A - B) + P(A \cap B) = 0.3 + 0.2 = 0.5$

   (c) $P(B) = P(B - A) + P(A \cap B) = 0.4 + 0.2 = 0.6$

3. A and B are independent events such that $P(A) = \frac{3}{4}$, $P(B) = \frac{5}{6}$.

   (a) $P(A \cap B) = P(A) \times P(B) = \frac{3}{4} \times \frac{5}{6} = \frac{15}{24} = \frac{5}{8}$

   (b) $P(A \cup B) = P(A) + P(B) - P(A \cap B) = \frac{3}{4} + \frac{5}{6} - \frac{5}{8} = \frac{18 + 20 - 15}{24} = \frac{23}{24}$

   (c) $P(A \cap B') = P(A) \times P(B') = \frac{3}{4} \times \frac{1}{6} = \frac{3}{24} = \frac{1}{8}$

   (d) $P(A' \cup B') = P(A \cap B)' = 1 - P(A \cap B) = 1 - \frac{5}{8} = \frac{8 - 5}{8} = \frac{3}{8}$

4. We are given $P(6) = \frac{1}{3}$

   (a) $P(\text{2 sixes}) = P(6,6) = \frac{1}{3} \times \frac{1}{3} = \frac{1}{9}$

   (b) $P(\text{at least one six}) = P(6,6) + P(6,\bar{6}) + P(\bar{6},6) = \frac{1}{3} \times \frac{1}{3} + \frac{1}{3} \times \frac{2}{3} + \frac{2}{3} \times \frac{1}{3}$

   $= \frac{1}{9} + \frac{2}{9} + \frac{2}{9} = \frac{5}{9}$

5. An unbiased die is thrown three times

   (a) $P(\text{obtaining three sixes}) = P(6, 6, 6) = \frac{1}{6} \times \frac{1}{6} \times \frac{1}{6} = \frac{1}{216}$

   (b) $P(\text{obtaining exactly two sixes}) = P(6, 6, \bar{6}) + P(6, \bar{6}, 6) + P(\bar{6}, 6, 6)$

   $= \frac{1}{6} \times \frac{1}{6} \times \frac{5}{6} + \frac{1}{6} \times \frac{5}{6} \times \frac{1}{6} + \frac{5}{6} \times \frac{1}{6} \times \frac{1}{6}$

   $= \frac{5}{216} + \frac{5}{216} + \frac{5}{216} = \frac{15}{216} = \frac{5}{72}$

   (c) $P(\text{obtaining at least one six})$

   $= P(6, 6, 6) + P(6, 6, \bar{6}) + P(6, \bar{6}, 6) + P(\bar{6}, 6, 6) + P(\bar{6}, \bar{6}, 6) + P(\bar{6}, 6, \bar{6})$

$+ P(6, \bar{6}, \bar{6})$

$$= \frac{1}{216} + 3\left(\frac{15}{216}\right) + 3\left(\frac{25}{216}\right) = \frac{1 + 45 + 75}{216} = \frac{121}{216}$$

Application activity 16.3

1. The sample space is $\Omega = \{(3,3), (4,2), (2,4), (5,1), (1,5)\}$

   P(one is 2) = $\frac{2}{5}$

2. (a) P(3 red marbles are followed by 1 blue marble, marble is replaced)

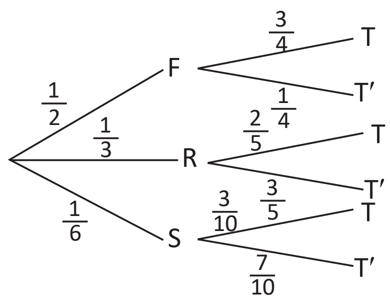
   $= \frac{5}{8} \text{ X } \frac{5}{8} \times \frac{5}{8} \times \frac{3}{8} = \frac{375}{4096}$

   (b) P(3 red marbles are followed by 1 blue marble, marble is not replaced.

   $= \frac{5}{8} \times \frac{4}{7} \times \frac{3}{6} \times \frac{3}{5} = \frac{180}{1680} = \frac{3}{28}$

3. We are given P(fine day) = $\frac{1}{2}$, P(raining day) = $\frac{1}{3}$, P(snowing day) = $\frac{1}{6}$.

   We make use of a tree diagram below.



P(Student is on time|fine day) = P(T|F) = $\frac{3}{4}$

P(student is on time|raining day) = P(T|R) = $\frac{2}{5}$

P(Student is on time|snowing day) = P(T|S) = $\frac{3}{10}$

(a) P(Student is on time on any given day)

$= P(T) = P(T|F) + P(T|R) + P(T|S)$

$= \frac{1}{2} \times \frac{3}{4} + \frac{1}{3} \times \frac{2}{5} + \frac{1}{6} \times \frac{3}{10} = \frac{3}{8} + \frac{2}{15} + \frac{3}{60}$

$= \frac{45 + 16 + 6}{120} = \frac{67}{120}$

(b) $P(R \setminus T') = \dfrac{P(R \cap T')}{P(T')}$

$P(T') = 1 - P(T) = 1 - \dfrac{67}{120} = \dfrac{53}{120}$

$P(R \setminus T') = \dfrac{\frac{1}{3} \times \frac{3}{5}}{\frac{53}{120}} = \dfrac{\frac{3}{15}}{\frac{53}{120}}$

$= \dfrac{3}{15} \times \dfrac{120}{53} = \dfrac{24}{53}$

4. (a) We are given a tin with 4 red and 6 blue marbles. Three mar bles are withdrawn without replacement

   (a) P(first two are red and the third is blue) = $\frac{4}{10} \times \frac{3}{9} \times \frac{6}{8} = \frac{72}{720} = \frac{1}{10}$

   (b) P(two are red marbles and one is blue)

   $= \frac{4}{10} \times \frac{3}{9} \times \frac{6}{8} + \frac{4}{10} \times \frac{6}{9} \times \frac{3}{8} + \frac{6}{10} \times \frac{4}{9} \times \frac{3}{8}$

   $= 3\left(\frac{72}{720}\right) = \frac{3}{10}$

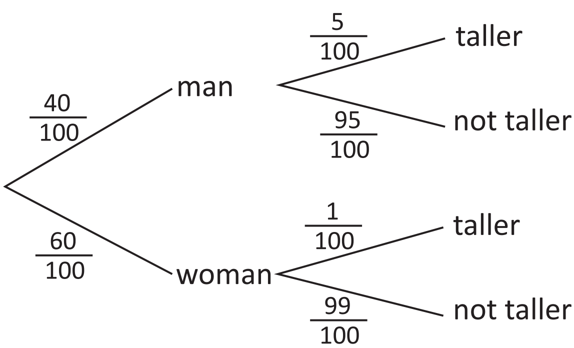
5. The probability that the fifth card dealt to him is the fourth ace is $\frac{1}{48}$.

6. We are given that P(man is taller than 180 cm) = $\frac{5}{100}$

   P(woman is taller than 180 cm) = $\frac{1}{100}$

   P(Student is a man) = $\frac{40}{100}$ and P(Student is a woman) = $\frac{60}{100}$

   We make use of the tree diagram below:



   P(Student is taller than 180 cm) = $\frac{40}{100} \times \frac{5}{100} + \frac{60}{100} \times \frac{1}{100} = \frac{10}{500} + \frac{3}{500} = \frac{13}{500}$

   P(Student is a woman|is taller than 180 cm)

$= \frac{P(\text{Student is a woman and is taller than 180 cm})}{P(\text{Student is taller than 180cm})}$

$$= \frac{\frac{60}{100} \times \frac{1}{100}}{\frac{13}{500}} = \frac{\frac{60}{10000}}{\frac{13}{500}} = \frac{3}{500} \times \frac{500}{13} = \frac{3}{13}$$

If a student is selected at random and found to be taller than 180 cm, then the probability that this student is a woman is $\frac{3}{13}$.

16.9. Answers to Practice Tasks

Practice application activity 1

1.

p	q	~p	~q	p ∨q	(p ∨q)	(~p)∧(~q)
T	T	F	F	T	F	F
T	F	F	T	T	F	F
F	T	T	F	T	F	F
F	F	T	T	F	T	T

2. (a) $\exists x \in \mathbb{R} : x^2 \leq 0$

   (b) $\exists x \in \mathbb{R}; \forall n \in \mathbb{N}: x \geq n$

3. (a) 1, 105° (b) 1,30° (c) 1,300°

   (d) 1,330°

4. (a) $x = -\frac{3}{7}$ (b) $a = 4$ (c) $a = \pm 64$

5. (a) $p \geq 9$ and $p \leq 1$ (b) $p \geq 5$ and $p \leq 1$

6. (a) 5 (b) $-\frac{1}{4}$ (c) $\frac{1}{3}$

   (d) $\frac{\sqrt[3]{294}}{7}$ (e) $-\frac{1}{4}$

7. Vertical asymptotes V.A $\equiv x = 2$

   Horizontal asymptote H.A. $\equiv y = 0$

8. $\left[\frac{1}{12}, \frac{1}{4}\right]$

9. Set of solutions is $\{(1,1); (\frac{1}{4}, 4)\}$

10. (a) $\frac{3}{5}$ (b) $-\frac{4}{5}$ (c) $-\frac{3}{5}$

11. $\frac{dg}{dt} = \frac{1}{2\sqrt{t}}$

12. $0 < a < \frac{4}{9}$

13. 1,760 ways.

14. $P(x) = x^2 - x + 3$

15. Number of permutations is $\frac{8!}{3!2!} = \frac{40,320}{6 \times 2} = 3,360$

16. (a) Dom $f = (-\infty, 3) \cup (3, +\infty)$

(b) x-intercept is (–1, 0). y-intercept is $(0, \frac{1}{9})$

(c) The vertical asymptote is VA $\equiv x = 3$. The horizontal asymptote is HA $\equiv y = 0$

(d) $f'(x) = \frac{(x+5)}{(x-3)^3}$ and $f''(x) = \frac{2(x+9)}{(x-3)^4}$

(e) The graph of f has a local minimum point at $(-5, -\frac{1}{16})$

(f) The graph of f is increasing on (–5, 3), f is decreasing on (–∞,5) and (3, + ∞).

(g) The graph of f has inflection point at $(-9, -\frac{1}{18})$.

(h) The graph of f is concave up on (–9, 3) and (3, +∞) , and concave down on (–∞, –9).

17. (a) $f(x) \geq -3$ (b) $f(x) \geq -5$ (c) $f(x) \geq 0$

(d) $0 \leq f(x) \leq \frac{1}{2}$

18. (a) P(2 survive) = 0.384 (b) P(3 survie) = 0.512

19. (a) The mean is 2.4 (b) The median is 2

(c) The mode is 3 (d) The sample standard deviation is 1.66

20. (a) $\tan 2\theta = \frac{24}{7}$ (b) The height of the flagpole is 10m

Practice Application activity 2

1. (a) The centre is at (–4, 1) and radius is 4 units of length

(b) The centre is at $\left(-\frac{1}{2}, -\frac{3}{2}\right)$ and radius is $\frac{3\sqrt{2}}{2}$ units of length

(c) The centre is at (–3,0) and radius is $\sqrt{14}$ units of length

2. (a) The probability that dictionary is selected is $\frac{1}{3}$

(b) The probability that 2 novels and 1 book of poems are selected is $\frac{5}{14}$

3. $A^{-1} = 5I + A$

4. (a) $-4 < x < -\frac{2}{3}$ (b) $2 < x < 3$

(c) $x < -2$

5. The * operation admits $e = -3$ as the identity element in $\mathbb{Z}$.

The * operation admits $x' = -x - 6$as a symmetric (inverse) of x in $\mathbb{Z}$.

6. (a) 0 (b) $\frac{1}{2}$ (c) $\frac{1}{4}(\sqrt{6} - \sqrt{2})$

(d) $-(2 + \sqrt{3})$ (e) $\frac{1}{4}(\sqrt{6} - \sqrt{2})$ (f) $\frac{1}{4}(\sqrt{6} + \sqrt{2})$

7. (a) $\sin 3\theta$ (b) 0

8. (a) $A^{-1} = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} \\ \frac{2}{3} & -\frac{1}{3} \end{pmatrix}$ (b) $B^{-1} = \begin{pmatrix} \frac{3}{2} & \frac{1}{2} \\ 1 & 0 \end{pmatrix}$

(c) $(AB)^{-1} = \begin{pmatrix} \frac{5}{6} & \frac{1}{3} \\ \frac{1}{3} & -\frac{1}{3} \end{pmatrix}$ (d) $(BA)^{-1} = \begin{pmatrix} \frac{5}{6} & \frac{1}{6} \\ \frac{2}{3} & \frac{1}{3} \end{pmatrix}$

(e) $A^{-1}B^{-1} = \begin{pmatrix} \frac{5}{6} & \frac{1}{6} \\ \frac{2}{3} & \frac{1}{3} \end{pmatrix}$ (f) $B^{-1}A^{-1} = \begin{pmatrix} \frac{5}{6} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} \end{pmatrix}$

9. (a) $\frac{\log 3}{2 \log 3 - 3 \log 2} = 9.33$

(b) The answer is $-\frac{1}{3} < x < 3 \Rightarrow x \in]-\frac{1}{3}, 3[$

10. (a) The amount of material in the original sample is 80 g.

(b) The half-life is 100 years.

(c) It will take 632 years for the material to decay to 1.

11. The vertex is $(-\frac{3}{4}, -\frac{49}{8})$, and the axis is $X = -\frac{3}{4}$

12. (a) For two distinct real roots $\Delta > 0$ and so $k < -3$ or $k > 1$.

(b) For no real roots $\Delta < 0$ and so $-3 < k < 1$.

13. (a) $(g \circ f)(x) = g[f(x)] = g(x + 2) = 2(x + 2) + 3 = 2x + 4 + 3 = 2x + 7$

(b) $(f \circ g)(x) = f[g(x)] = f(2x + 3) = (2x + 3) + 2 = 2x + 3 + 2 = 2x + 5$

(c) $(f \circ f)(x) = f[f(x)] = f(x + 2)= (x + 2) + 2 = x + 4$

(d) $(g \circ g)(x) = g[g(x)] = g(2x + 3) = 2(2x + 3) + 3 = 4x + 6 + 3 = 4x + 9$

14. The three vectors are linearly dependent.

15. (a) For any vector $\vec{u} = \binom{x}{y}$ in $\mathbb{R}^2$, we have $c_1 \frac{5x - 2y}{7}$ and $c_2 = \frac{3y - 4x}{7}$ for $\vec{u} = c_1\vec{v} + c_2\vec{w}$.

(b) $c_1\vec{v} + c_w\vec{w} = \binom{0}{0}$ implies $c_1 = c_2 = 0$.

16. (a) (i) $\frac{1}{52}$ (ii) $\frac{7}{26}$ (iii) $\frac{10}{13}$

(b) $\frac{13}{51}$

17. The perimeter of the triangle is 156 cm.

18. (a) $(f \circ g)(x) = (2 - x)^2$ and Dom $(f \circ g) = \mathbb{R}$ and range $(f \circ g) = [0, +\infty)$

(b) $(g \circ f)(x) = 2 - x^2$ and Dom $(g \circ f) = \mathbb{R}$ and range $(g \circ f) = [-\infty, 2)$

19. (a) $a = 1$ and $b = -5$

(b) $f(x) = (x - 1)(x - 1)(x + 3)$

20. (a) 5,040 b) 720

Practice Test 3

1. The distance is $\frac{3}{5}$ unit of length.

2. (a) $\frac{7}{20}$ (b) $\frac{11}{20}$ (c) $\frac{3}{20}$

(d) $\frac{3}{4}$

3. $T \equiv y = 4x - 20$ or $T \equiv y = -4x + 4$ and the tangents meet at $(3, -8)$

4. The number of arrangements in the word **MIYOVE** starting with a consonant is $3(5!) = 3 \times 5 \times 4 \times 3 \times 2 \times 1 = 360$

5. (a) The centre of the circle is $(-2, -3)$ and the radius is 4.

(b) The shortest distance between the centre of the circle and the line L is 4.

(c) Since the shortest distance between the centre of the circle and line L is exactly equal to the radius of the circle, then the line L is the tangent to the circle.

(d) The point of tangency is $(-\frac{22}{5}, \frac{1}{5})$

6. They can be formed into 36 numbers.

7. (a) 1 (b) $\frac{7}{8}$ (c) 2 (d) -2

(e) $-\frac{1}{2}$

8. (a) 120 permutations (b) 240 permutations

9. (a) $6561 - 34992x + 81648x^2 - ...$ (b) $1 - 5x + \frac{45}{4}x^2 + ...$

10. (a) The slope at x = 2 is 72

(b) Tangent has equation $T \equiv y = -1$

(c) The points are (0, 2), (–1, –1) , and (1, –1)

11. The graph of the function f is increasing on the interval $[3, \infty)$.

The graph of the function f is decreasing on the interval $(-\infty, 3]$.

12. (a) The mean is 6. (b) The median is 5

(c) The mode does not exist. (d) The range is 9

(e) The variance is 10.

(f) The standard deviation is $\sqrt{10}$.

13. (a) The inflection numbers of f are 0 and 2 and the inflection points are (0,5) and (2, –11).

(b) The graph of is bending up on the interval $(-\infty, 0)$ and $(2, \infty)$ and bending down on the interval (0, 2).

14. $k = -\frac{1}{6}$

15. (a) The slope is 0 at the points where $x = -1 + \sqrt{2}$ and $x = -1 - \sqrt{2}$.

(b) The slope is 2 at the points where x = -3 and x = 1.

(c) The slope is -1 at the point where x = 0 and x = –2.

16. The height of the tree is 50 m.

17. The largest possible value for xy is 100.

18. The angle of elevation of the sun is 40°.

19. (a) $\frac{1}{9}$ (b) $\frac{1}{35}$

20. (a) $A^{-1} = \begin{pmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{pmatrix}$

(b) x = 1, y = 2

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