SUBSIDIARY MATHEMATICS

Senior 6

Teacher's Guide

FOREWORD

Dear teacher,

Rwanda Basic Education Board is honored to present the teacher's guide for S4 Subsidiary Mathematics of the PCB combination. This book serves as a guide to competence-based teaching and learning to ensure consistency and coherence in the learning of the mathematics content.

The Rwandan educational philosophy is to ensure that learners achieve full potential at every level of education which will prepare them to be well integrated in society and exploit employment opportunities. In line with efforts to improve the quality of education, the government of Rwanda emphasizes the importance of aligning teaching and learning materials with the syllabus to facilitate their learning process. Many factors influence what they learn, how well they learn and the competences they acquire. Those factors include the relevance of the specific content, the quality of teachers' pedagogical approaches, the assessment strategies and the instructional materials.

The ambition to develop a knowledge-based society and the growth of regional and global competition in the jobs market has necessitated the shift to a competence-based curriculum. This book provides active teaching and learning techniques that engage students to develop competences.

In view of this, your role is to:

- Plan your lessons and prepare appropriate teaching materials;
- Organize group discussions for students considering the importance of social constructivism suggesting that learning occurs more effectively when the students work collaboratively with more knowledgeable and experienced people;
- Engage students through active learning methods such as inquiry methods, group discussions, research, investigative activities, group and individual work activities;
- Provide supervised opportunities for students to develop different competences by giving tasks which enhance critical thinking, problem solving, research, creativity and innovation, communication and cooperation;
- Support and facilitate the learning process by valuing students' contributions in the class activities;
- Guide students towards the harmonization of their findings;
- Encourage individual, peer and group evaluation of the work done in the classroom and use appropriate competence-based assessment approaches.

Even though this teacher's guide contains the guidance on solutions for all activities given in the learner's book, you are requested to work through each question before judging student's findings.

I wish to sincerely express my appreciation to the people who contributed towards the development of this book, particularly, REB staff, Lecturers and Teachers for their technical support.

Dr. MBARUSHIMANA Nelson

Director General, REB

ACKNOWLEDGEMENT

I wish to express my appreciation to the people who played a major role in the development of this teacher's guide for S4 Subsidiary Mathematics of the PCB combination. It would not have been successful without active participation of different education stakeholders.

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I wish to extend my sincere gratitude to lecturers and teachers whose efforts during writing exercise of this teacher's guide were very much valuable.

Finally, my word of gratitude goes to the Rwanda Basic Education Board staffs who were involved in the whole process of in-house textbook production.

Joan Murungi,

Head of CTLRD

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PART I. GENERAL INTRODUCTION

1.1 The structure of the guide

The teacher's guide of Senior 6 Subsidiary Mathematics is composed of three parts.

The Part 1 concerns general introduction that discusses methodological guidance on how best to teach and learn Mathematics, developing competences in teaching and learning Mathematics, addressing cross-cutting issues in teaching and learning subsidiary Mathematics, and Guidance on assessment.

Part 2 consists in sample lesson plan per unit. This means that since we have 4 units, there are 4 sample lesson plans developed to guide the teacher on how to prepare a lesson plan in Mathematics.

Part 3 is about the structure of a unit and the structure of a lesson. This includes information related to the different components of the unit and these components are the same for all 4 units in Senior 6 subsidiary Mathematics. This part provides information and guidelines on how to facilitate learners while working on learning activities. All application activities from the textbook have answers in this part. In part 3, the Teacher Guide (TG) provides additional information for the teacher on the unit basis and there are a variety of activities classified in 3 categories (remediation, consolidation and extended activities) to help learners enrich their concepts and content development.

1.2 Methodological guidance

5.0.1. Developing competences

Since 2015 Rwanda shifted from a knowledge based to a competency-based curriculum for pre-primary, primary and general secondary education. This called for changing the way of learning by shifting from teacher centered to a learner centered approach. Teachers are not only responsible for knowledge transfer but also for fostering learners' learning achievement and creating safe and supportive learning environment. It implies also that learners have to demonstrate what they are able to transfer the acquired knowledge, skills, values and attitude to new situations.

The competence-based curriculum employs an approach of teaching and learning based on discrete skills rather than dwelling on only knowledge or the cognitive domain of learning. It focuses on what learner can do rather than what learner knows. Learners develop competences through Subsidiary Mathematics unit with specific learning objectives broken down into knowledge, skills and attitudes through learning activities.

In addition to the competences related to Mathematics, learners also develop generic competences which should promote the development of the higher order thinking skills in Mathematics.They are seen as generic competences because they can be developed throughout the all 4 units of Senior 6 Subsidiary Mathematics as follow:

Generic competences	Ways of developing generic competences
Critical thinking	All activities that require learners to convert, interpret, analyze, compare and contrast, etc have a common factor of developing critical thinking into learners
Creativity and innovation	All activities that require learners to plot a graph given algebraic data have a common character of developing creativity into learners
Research and problem solving	All activities that require learners to make a research and apply their knowledge to the real-life situation have a character of developing research and problem solving into learners.
Communication skills	During Mathematics class, all activities that require learners to discuss either in groups or in the whole class, present findings, debate … have a common character of developing communication skills into learners.
Co-operation, interpersonal relations and life skills	All activities that require learners to work in pairs or in groups have character of developing cooperation and life skills among learners.
Lifelong learning	All activities that are connected with research have a common character of developing into learners a curiosity of applying the leant knowledge in a range of situations. The purpose of such kind of activities is for enabling learners to become life-long learners who can adapt to the fast-changing world and the uncertain future by taking initiative to update knowledge and skills with minimum external support.

The generic competences help learners deepen their understanding of Mathematics and apply their mathematical knowledge in a range of situations. As learners develop generic competences they also acquire the set of skills that employers look for in their employees, so the generic competences prepare learners for the world of work.

1.2.2 Addressing cross cutting issues

Among the changes brought by the competence-based curriculum is the integration of cross cutting issues as an integral part of the teaching learning process-as they relate to and must be considered within all subjects to be appropriately addressed. The eight cross cutting issues identified in the national curriculum framework are: *Comprehensive Sexuality Education, Environment and Sustainability, Financial Education, Genocide studies, Gender, Inclusive Education, Peace and Values Education, and Standardization Culture*. Some cross-cutting issues may seem specific to particular learning areas/ subjects but the teacher need to address all of them whenever an opportunity arises. In addition, learners should always be given an opportunity during the learning process to address these cross-cutting issues both within and out of the classroom.Below are examples of how crosscutting issues can be addressed in Subsidiary Mathematics:

Cross-Cutting Issue	Ways of addressing cross-cutting issues
Comprehensive Sexuality Education: The primary goal of introducing Comprehensive Sexuality Education program in schools is to equip children, adolescents, and young people with knowledge, skills and values in an age appropriate and culturally gender sensitive manner so as to enable them to make responsible choices about their sexual and social relationships, explain and clarify feelings, values and attitudes, and promote and sustain risk reducing behavior.	Using graphs of logarithmic and exponential functions and their interpretation, Mathematics teacher should lead learners to discuss the following situations: "Alcohol abuse and accident risk", "Alcohol abuse and unwanted pregnancies".
Environment and Sustainability: Integration of Environment, Climate Change and Sustainability in the curriculum focuses on and advocates for the need to balance economic growth, society well-being and ecological systems. Learners need basic knowledge from the natural sciences, social sciences, and humanities to understand to interpret principles of sustainability.	Using differential equations models, Mathematics teachers should lead learners to represent the situation of "**population growth**" and discuss its effects on the environment and sustainability.

Financial Education: The integration of Financial Education into the curriculum is aimed at a comprehensive Financial Education program as a precondition for achieving financial inclusion targets and improving the financial capability of Rwandans so that they can make appropriate financial decisions that best fit the circumstances of one's life.	Through different examples and calculations on interest rate problems, Mathematics teachers can lead learners to discuss on how to make appropriate financial decisions.
Gender: At school, gender will be understood as family complementarities, gender roles and responsibilities, the need for gender equality and equity, gender stereotypes, gender sensitivity, etc.	Mathematics teacher should address gender as cross-cutting issue through assigning leading roles in the management of groups to both girls and boys and providing equal opportunity in the lesson participation and avoid any gender stereotype in the whole teaching and learning process.
Inclusive Education: Inclusion is based on the right of all learners to a quality and equitable education that meets their basic learning needs and understands the diversity of backgrounds and abilities as a learning opportunity.	Firstly, mathematics teachers need to identify/recognize learners with special needs. Then by using adapted teaching and learning resources while conducting a lesson and setting appropriate tasks to the level of learners, they can cater for learners with special education needs.

Peace and Values Education: Peace and Values Education (PVE) is defined as education that promotes social cohesion, positive values, including pluralism and personal responsibility, empathy, critical thinking and action in order to build a more peaceful society.	Through a given Mathematics lesson, a teacher should: Set a learning objective which is addressing attitudes and values, Use probing questions, while teaching, in order to increase feelings in learners about the value. For instance, the teacher asks learners: 'what do you think might happen if there are no electronic materials to regulate the intensity of an AC current?' (UNIT 1) Then learners discuss this in pairs or groups and give feedback. Encourage learners to develop the culture of tolerance during discussion Encourage learners to respect others' ideas

Standardization Culture: Standardization Culture in Rwanda will be promoted through formal education and plays a vital role in terms of health improvement, economic growth, industrialization, trade and general welfare of the people through the effective implementation of Standardization, Quality Assurance, Metrology and Testing.	By providing examples where AC current is presented using complex numbers and DC current by a real number. Learners should be asked to give examples of different electronic materials (transistor, diodes, charger, etc) invented to avoid risks through regulating and transforming the intensity of complex current (AC) into direct current (DC). While solving correctly a given mathematical problem, learners are getting awareness of meeting required standards. For example, most of learners do not give importance of the arbitrary constant in integrating a function. This is an opportunity for mathematics teacher to discuss the standardization culture.

1.2.3 Guidance on how to help learners with special education needs in classroom

In the classroom, students learn in different way depending to their learning pace, needs or any other special problem they might have. However, the teacher has the responsibility to know how to adopt his/her methodologies and approaches in order to meet the learning need of each student in the classroom. Also teachers need to understand that student with special needs, need to be taught differently or need some accommodations to enhance the learning environment. This will be done depending to the subject and the nature of the lesson.

In order to create a well-rounded learning atmosphere, teachers need to:

- Remember that learners learn in different ways so they have to offer a variety of activities (e.g. role-play, music and singing, word games and quizzes, and outdoor activities);

- Maintain an organized classroom and limits distraction. This will help learners with special needs to stay on track during lesson and follow instruction easily;
- Vary the pace of teaching to meet the needs of each child. Some learners process information and learn more slowly than others;
- Break down instructions into smaller, manageable tasks. Learners with special needs often have difficulty understanding long-winded or several instructions at once. It is better to use simple, concrete sentences in order to facilitate them understand what you are asking.
- Use clear consistent language to explain the meaning (and demonstrate or show pictures) if you introduce new words or concepts;
- Make full use of facial expressions, gestures and body language;
- Pair a learner who has a disability with a friend. Let them do things together and learn from each other. Make sure the friend is not over protective and does not do everything for the one with disability. Both learners will benefit from this strategy;
- Use multi-sensory strategies. As all learners learn in different ways, it is important to make every lesson as multi-sensory as possible. Learners with learning disabilities might have difficulty in one area, while they might excel in another. For example, use both visual and auditory cues.

Below are general strategies related to each main category of disabilities and how to deal with every situation that may arise in the classroom. However, the list is not exhaustive because each child is unique with different needs and that should be handled differently.

Strategy to help learners with developmental impairment:

- Use simple words and sentences when giving instructions;
- Use real objects that learners can feel and handle. Rather than just working abstractly with pen and paper;
- Break a task down into small steps or learning objectives. The learner should start with an activity that s/he can do already before moving on to something that is more difficult;
- Gradually give the learner less help;
- Let the learner with disability work in the same group with those without disability.

Strategy to help learners with visual impairment:

- Help learners to use their other senses (hearing, touch, smell and taste) and carry out activities that will promote their learning and development;

- Use simple, clear and consistent language;
- Use tactile objects to help explain a concept;
- If the learner has some sight, ask him/her what he/she can see;
- Make sure the learner has a group of friends who are helpful and who allow him/her to be as independent as possible;
- Plan activities so that learners work in pairs or groups whenever possible;

Strategy to help learners with hearing disabilities or communication difficulties

- Always get the learner‘s attention before you begin to speak;
- Encourage the learner to look at your face;
- Use gestures, body language and facial expressions;
- Use pictures and objects as much as possible.
- Keep background noise to a minimum.

Strategies to help learners with physical disabilities or mobility difficulties:

Adapt activities so that learners who use wheelchairs or other mobility aids, can participate.

Ask parents/caregivers to assist with adapting furniture e.g. The height of a table may need to be changed to make it easier for a learner to reach it or fit their legs or wheelchair under.

- Encourage peer support wen nedded;
- Get advice from parents or a health professional about assistive devices if the learner has one.

Adaptation of assessment strategies: each unit in this teacher's guide provides additional activities to help learners achieve the key unit competence. These assessment activities are for remedial, consolidation and extension designed to cater for the needs of all categories of learners; slow, average and gifted learners respectively. Therefore, the teacher is expected to do assessment that fits individual learners.

Remedial activities	After evaluation, slow learners are provided with lower order thinking activities related to the learnt concepts to facilitate them in their learning. These activities are also given to assist deepening knowledge acquired through the learning activities for slow learners.
Consolidation activities	After introduction of any concept, a range number of activities is provided to all learners to enhance/ reinforce learning
Extended activities	After evaluation, gifted and talented learners are provided with high order thinking activities related to the learnt concepts to make them think deeply and critically. These activities are given to gifted and talented learners to keep them working while other learners are getting up to required level of knowledge through the learning activity.

1.2.4. Guidance on assessment

Assessment is an integral part of teaching and learning process. The main purpose of assessment is for improvement of learning outcomes. Assessment for learning/ Continuous/ formative assessment intends to improve learners' learning and teacher's teaching whereas assessment of learning/summative assessment intends to improve the entire school's performance and education system in general.

- **Continuous/ formative assessment** is an on-going process that arises out of interactions during teaching and learning between. It includes lesson evaluation and end of sub unit assessment. This formative assessment should play a big role in teaching and learning process. The teacher should encourage individual, peer and group evaluation of the work done in the classroom and uses appropriate competence-based assessment approaches and methods.

Formative assessment is used to:

- Determine the extent to which learning objectives are being achieved and competences are being acquired and to identify which learners need remedial interventions, reinforcement as well as extended activities. The application activities are developed in the learner book and they are designed to be given as remedial, reinforcement, end lesson assessment, homework or assignment

- Motivate learners to learn and succeed by encouraging learners to read, or learn more, revise, etc.
- Check effectiveness of teaching methods in terms of variety, appropriateness, relevance, or need for new approaches/strategies. Mathematics teachers need to consider various aspects of the instructional process including appropriate language levels, meaningful examples, suitable methods and teaching aids/ materials, etc.
- Help learners to take control of their own learning.

In teaching Mathematics, formative or continuous assessment should compare performance against instructional objectives. Formative assessment should measure the learner's ability with respect to a criterion or standard. For this reason, it is used to determine what learners can do, rather than how much they know.

Assessment should be clearly visible in lesson, unit, term and yearly plans.

- Before learning (diagnostic): At the beginning of a new unit or a section of work; assessment can be organized to find out what learners already know / can do, and to check whether the learners are at the same level.
- During learning (formative/continuous): When learners appear to be having difficulty with some of the work, by using on-going assessment (continuous). The assessment aims at giving learners support and feedback.
- After learning (summative): At the end of a section of work or a learning unit, the Mathematics teacher has to assess after the learning. This is also known as Assessment of Learning to establish and record overall progress of learners towards full achievement. Summative assessment in Rwandan schools mainly takes the form of written tests at the end of a learning unit or end of the month, and examinations at the end of a term, school year or cycle.

Instruments used in assessment.

- **Observation:** This is where the Mathematics teacher gathers information by watching learners interacting, conversing, working, playing, etc. A Mathematics teacher can use observations to collect data on behaviours that are difficult to assess by other methods such as attitudes, values, and generic competences and intellectual skills. It is very important because it is used before the lesson begins and throughout the lesson since the Mathematics teacher has to continue observing each and every activity.

- **Questioning**

  i) Oral questioning: a process which requires a learner to respond verbally to questions

  ii) Class activities/ exercise: tasks that are given during the learning/ teaching process

  iii) Short and informal questions usually asked during a lesson

  iv) Homework and assignments: tasks assigned to learners by their Mathematics teachers to be completed outside of class.

Summative assessment: The assessment can serve as summative and informative depending to its purpose. The end unit assessment will be considered summative when it is done at end of unit and want to start a new one.

It will be formative assessment, when it is done in order to give information on the progress of learners and from there decide what adjustments need to be done.

The assessment done at the end of the term, end of year, is considered as summative assessment so that the teacher, school and parents are informed of the achievement of educational objective and think of improvement strategies. There is also end of level/ cycle assessment in form of national examinations.

1.1.5 Learners' learning styles and strategies to conduct teaching and learning process

There are different teaching styles and techniques that should be catered for. The selection of teaching method should be done with the greatest care and some of the factors to be considered are: the uniqueness of subjects; the type of lessons; the particular learning objectives to be achieved; the allocated time to achieve the objective; instructional available materials; the physical/ sitting arrangement of the classroom, individual learners' needs, abilities and learning styles.

There are mainly **four different learning styles** as explained below:

a) Active and reflective learners

Active learners tend to retain and understand information best by doing something actively: discussing or applying it or explaining it to others.

Reflective learners prefer to think about information quietly then they react.

b) Sensing and intuitive learners

Sensing learners tend to like learning facts and often like solving problems by well-established methods and dislike complications and surprises.

Intuitive learners prefer discovering possibilities and relationships and like innovation and dislike repetition.

c) Visual and verbal learners

Visual learners remember best when they see pictures, diagrams, flow charts, time lines, films, demonstrations, etc.

V**erbal learners** get more out of words—written and spoken explanations.

d) Sequential and global learners

Sequential learners tend to gain understanding in linear steps, with each step following logically from the previous one.

Global learners tend to learn in large jumps, absorbing material almost randomly without seeing connections, and then suddenly "getting it."

1.2.6. Teaching methods and techniques that promote active learning

The different learning styles mentioned above can be catered for, if the teacher uses active learning whereby learners are really engaged in the learning process. The following are active techniques to be used in Mathematics:

- Group work
- Research
- Probing questions
- Practical activities (drawing, plotting, interpreting graphs)
- Modelling

What is Active learning?

Active learning is a pedagogical approach that engages learners in doing things and thinking about the things they are doing. Learners are key in the active learning process. They are not empty vessels to fill but people with ideas, capacity and skills to build on for effective learning. Thus, in active learning, learners are encouraged to bring their own experience and knowledge into the learning process.

The role of the teacher in active learning	The role of learners in active learning
The teacher engages learners through active learning methods such as inquiry methods, group discussions, research, investigative activities, group and individual work activities. He/she encourages individual, peer and group evaluation of the work done in the classroom and uses appropriate competence-based assessment approaches and methods. He provides supervised opportunities for learners to develop different competences by giving tasks which enhance critical thinking, problem solving, research, creativity and innovation, communication and cooperation. Teacher supports and facilitates the learning process by valuing learners' contributions in the class activities.	A learner engaged in active learning: Communicates and shares relevant information with peers through presentations, discussions, group work and other learner-centred activities (role play, case studies, project work, research and investigation); Actively participates and takes responsibility for his/her own learning; Develops knowledge and skills in active ways; Carries out research/investigation by consulting print/online documents and resourceful people, and presents their findings; Ensures the effective contribution of each group member in assigned tasks through clear explanation and arguments, critical thinking, responsibility and confidence in public speaking Draws conclusions based on the findings from the learning activities.

Main steps for a lesson in active learning approach

All the principles and characteristics of the active learning process highlighted above are reflected in steps of a lesson as displayed below. Generally, the lesson is divided into three main parts whereby each one is divided into smaller steps to make sure that learners are involved in the learning process. Below are those main part and their small steps:

1) Introduction

Introduction is a part where the teacher makes connection between the current and previous lesson through appropriate technique. The teacher opens short discussions to encourage learners to think about the previous learning experience and connect it with the current instructional objective. The teacher reviews the prior knowledge, skills and attitudes which have a link with the new concepts to create good foundation and logical sequencings.

2) Development of the new lesson

The development of a lesson that introduces a new concept will go through the following small steps: discovery activities, presentation of learners' findings, exploitation, synthesis/summary and exercises/application activities.

Discovery activity

Step 1

- The teacher discusses convincingly with learners to take responsibility of their learning
- He/she distributes the task/activity and gives instructions related to the tasks (working in groups, pairs, or individual to instigate collaborative learning, to discover knowledge to be learned)

Step 2

- The teacher let learners work collaboratively on the task;
- During this period the teacher refrains to intervene directly on the knowledge;
- He/she then monitors how the learners are progressing towards the knowledge to be learned and boosts those who are still behind (but without communicating to them the knowledge).

Presentation of learners' findings/productions

- In this episode, the teacher invites representatives of groups to present their productions/findings.
- After three/four or an acceptable number of presentations, the teacher decides to engage the class into exploitation of learners' productions.

Exploitation of learner's findings/ productions

- The teacher asks learners to evaluate the productions: which ones are correct,

PART II: SAMPLE LESSON PLANS

Sample Lesson for unit 1: Complex Numbers

School Name: XXXXXX **Teacher's name**: YYYYYYY

Term	Date	Subject	Class	Unit N^o	Lesson N^o	Duration	Class size
Term 1	…… /……/ 2018	Subsidiary Mathematics	S6 PCB	1	1 of 16	80 minutes	…..
Type of Special Educational Needs to be catered for in this lesson and number of learners in each category				3 slow learners and 2 low vision learners • For those with low vision provide their seats where they will be able to look on black board easily • Group slow learners with others that can help them. Remember to provide repetition when necessary and give them extra time to finish the activity.			
Unit title		**COMPLEX NUMBERS**					
Key Unit Competence:		Perform operations on complex numbers in different forms and use them to solve related problems in Physics, etc.					
Title of the lesson		Definition of complex number					
Instructional Objective		Given a complex number of the form $z = a + ib$, learners will be able to distinguish the real part from the imaginary part and provide their own examples of complex numbers z without difficulties.					
Plan for this Class (location: in / outside)		This class will be held indoors and learners will be organized in small groups; the groups containing slow learners or low vision will seat near the blackboard for facilitating them to see what is written or presentation of their classmates' findings.					
Learning Materials (for ALL learners)		• Flashcards containing the complex numbers, real numbers and pure imaginary numbers • Textbooks to facilitate research					
References		Subsidiary Mathematics S6, Advanced Mathematics for Rwanda Secondary Schools.					

	Learners'Book Senior six, etc.

<table>
<tr><th rowspan="2">Timing for each step</th><th colspan="2">Description of teaching and learning activity</th><th rowspan="3">Generic competences and cross cutting issues to be addressed + a short explanation</th></tr>
<tr><td colspan="2">Questions-answers technique is used to facilitate learners, group activities and presentation of findings is used to make learners involved in the content development and then the exploitation of the presentation follows and finally the harmonization/ institutionalisation of the group findings is made by the teacher who guides the learners to make the summary. Finally, the lesson assessment is individually and then peer assessment/ correction is done by learners after the collective correction on the board. Low vision learners are seated on front desk nearby the board. Ensure slow learners are asked level appropriate questions.</td></tr>
<tr><td></td><td>Teacher activities</td><td>Learner activities</td></tr>
<tr><td>Introduction
15 minutes</td><td>Give examples that help learners to remember how to solve equations in different sets up to $\mathbb{R}$

• Ask learners to discuss how to solve quadratic equations of the form $ax^2+bx+c=0$ in $\mathbb{R}$ (set of real numbers).
• Move around to check learners' work with focus on slow learners and provide individual</td><td>Task 1: In the whole class discussion, learners find out if:
• The equation $x-2=0$ has a solution in $\mathbb{N}$ or in $\mathbb{Z}$.
• The equation $x+4=2$ has a solution in $\mathbb{Z}$ or in $\mathbb{Q}$
• The equation $4x=5$ has a solution in $\mathbb{Z}$ or in $\mathbb{Q}$
• The equation $x^2=3$ has a solution in $\mathbb{Q}$ or in $\mathbb{R}$

Task 2: Individually, learners solve the quadratic equation: $x^2-16=0$.
The slow learners solve equations $x-4=0$ and $(x+4)(x-4)=0$.</td><td>Critical thinking is enhanced while learners are discussing on how to solve quadratic equations</td></tr>
</table>

	feedback accordingly		
Development of the lesson (Discovery actitity, Exploitation of learners' findings and harmonization, Institutionalization and summary) 50 minutes (*task 1: 20 minutes, task 2: 20 minutes, task 3:10 minutes*)	• Distribute the tasks to learners • Give instructions to learners • Ask learners to use $i^2=-1$ to solve the unsolved quadratic equations in $\mathbb{R}$. • Invites group representatives to present their findings • After presentations, harmonize the group findings by providing corrections and completing knowledge • Make a summary • Gives the application activity	**Discovery activity:** **Task 1:** In small groups, learners discuss how to solve the quadratic equation $x^2+16=0$, after replacing -1 by i^2 and then present their findings to the class. **Task 2:** In small groups, learners identify the real part and the imaginary part of a complex number $z=4+5i$ **Task 3:** Learners present their findings from tasks 1 and 2 Learners take the summary from the task1 and 2: • ***The imaginary number*** i ***is an element of a new set "set of complex numbers*** $\mathbb{C}$ • ***Given two real numbers a and b the complex number z is defined as*** $z=a+bi$ ***, a is the real part and b is the imaginary part of z*** **Mathematically the set** $\mathbb{C}$ **of complex numbers is defined as** $\mathbb{C}=\left\{a+ib;\quad where\ a,b\in\mathbb{R}\quad and\ i^2=-1\right\}$ **Application activity:** In small groups, learners read carefully a given text on application of complex numbers in physics and identify $z=a+ib$ the real number, pure imaginary number or complex number for the formula found in the text. Finally, they make a research to find out the application of complex numbers in other subjects and then share with	**Team work** and **cooperation** are enhanced through group work Communication skills will be developed through group discussions and presentation of findings **Peace and value education** is addressed through group discussions while learners freely express their ideas and respect others thoughts **Gender education** is addressed by giving equal opportunity to boys and girls in performing tasks and

		other groups during the next lessons.	activities
Assessment: 15 minutes	•Distributes the assessment task and gives instructions •Lead learners to work out the questions 2a, 2c and 3 from application activities on page 4and then monitors the learners working on the task •Evaluates the answers produced by the learners and ask them to make a collective correction and peer assessment.	Individually, learners work out the following: 1. Using the properties of the number i, find the value of $a)\ i^{25}\qquad b)i^{71}$ 2. In Electricity when dealing with direct currents (DC), we encountered Ohm's law, which states that the resistance R is the ratio between voltage and current, or $R=\frac{U}{I}$. With alternating currents (AC) both U and I are expressed by complex numbers, so the resistance is now also complex. A complex resistance we call impedance and denote it by the symbol Z. The building blocks of AC circuits are resistors (R, [Ω]), inductors (coils, L, [H=Henry]) and capacitors (C, [F=Farad]). Their respective impedances are $Z_R=R,\quad Z_L=i\omega L$ and $Z_C=\frac{1}{i\omega C}$; which of them has an imaginary part?	**Inclusive education** is addressed by providing the remediation activities & tasks to slow learners and while short sight learners are facilitated by big print flashcards
Teacher self-evaluation	To be completed by the teacher after the lesson		

Sample Lesson for unit 2: Logarithmic and exponential functions

School Name: XXXXXXXX YYYYYYYYYYYYYY **Teacher's name:**

Term	Date	Subject	Class	Unit N^o	Lesson N^o	Duration	Class size
II	.... /......./2018	Subsidiary Mathematics	S6 HEG	2	17 of17	40 min	...
Type of Special Educational Needs to be catered for in this lesson and number of learners in each category				2 slow learners and 2 low vision learners • For those with low vision provide their seats where they will be able to look on black board easily; • Group slow learners with others that can help them. Remember to provide repetition when necessary and give them extra time to finish the activity.			
Unit title	Logarithmic and exponential functions						
Key unit competency:	Extend the concepts of functions to investigate fully logarithmic and exponential functions and use them to model and solve problems about interest rates, population growth or decay, magnitude of earthquake, etc.						
Title of the lesson	Problems about alcohol and risk of car accident						
Instructional Objective	Given the equation modelling the risk of a car accident, learners will be able to accurately determine the risk corresponding to a given concentration of alcohol in the driver's blood, the concentration corresponding to a given risk and, vice versa.						
Plan for this Class (location: in / outside)	The lesson is held indoors, the class is organized into groups ,3 slow learners are scattered in different groups, and 2 low vision learners seat on the front desks near the blackboard in order to see and participate fully in all activities						

Learning Materials (for all learners)	Textbooks and charts containing graphical representations to interpret in the course of the lesson
References	Mathematics syllabus for Advanced level, Subsidiary Mathematics textbook and Teacher's guide for S6, etc.

Timing for each step	**Description of teaching and learning activity**		**Generic competences and cross cutting issues to be addressed + a short explanation**
	-Two learners chosen randomly present, one after another, their findings, from the assignment, and learners interact under the facilitation of the teacher, who, then, links the presentation to the lesson of the day. -learners are organized into groups to discuss the discovery activity and the examples, the reporter from one group, presents the findings and the learners interact. The teacher facilitates learners to capture the key concepts of the lesson through harmonization. -Finally, the learners are assigned to individual tasks, and the correction is done on the chalk board		
	Teacher activities	**Learner activities**	
Introduction: 5 minutes	The teacher asks learners to work individually and present their findings about solving the exponential equation - The teacher links the introduction to the lesson of the day	Learners solve individually the following equation: $e^{2x} = 3$ and then two learners present, one after another, their findings, and the learners interact.	Communication skills developed through the presentation and sharing ideas
Development of the lesson			
Discovery activity: **10 minutes**	- The teacher organizes the learners into	-Learners follow the instructions and in small groups they work out the activity 2.17 -Each group analyses and discuss the	• Cooperation and communication skills through discussions

	groups of 4 and assigns them the task (activity 2.17) and gives instructions related to the task (organization of the group, role of each member, duration, presentation) - Teacher goes around to monitor the work of each group and provide assistance where needed	activity 2.17 under the direction of the task manager of the group:	• Financial education through consuming limited amounts of alcohols and paying taxes • Peace and values education; Cooperation, mutual respect, tolerance through discussions with people with different views and respect one's views
Presentation of learner's findings and exploitation: **15minutes**	- Teacher invites the reporter of a sample group to present the findings of the group - The teacher encourages learners to follow attentively - Teacher takes notes on key points from learners' presentation. - The teacher asks learners to amend	-Learners present their findings: **Expected answers** **(Refer** to solution of activity 2.17, in TG) -Learners follow the presentation -Learners evaluate the findings of other learners -Learners evaluate their own findings	• Cooperation and communication/ attentive listening during presentations and group discussions • Critical thinking through evaluating other's findings

	the presentation and to evaluate their work		
Conclusion/ Summary: **5 minutes** **Assessment** **5 minutes**	-Teacher facilitates the learners to capture the main points of the presentation -Teacher requests learners to write down the main points in their books - Teacher asks learners to individually work out the application activity 2.17 -Teacher asks learners to identify other misbehaviours resulting from excess consummation of alcohol.	-The main point: The risk is modelled by an equation of the type $R(x) = R_0 e^{kx}$ -Learners take notes in their books Individually learners work out the application activity 2.17 and finally they make a correction on the chalk board. **Expected answers** **(Refer** to solution of learning activity 2.17, in TG) -Learners discuss other misbehaviours resulting from excess consummation of alcohol.	- Critical thinking and problem-solving skills are developed through analysing and solving real life Mathematical problems. - Comprehensive sexuality education developed while learners discuss about different misbehaviours related to excess consummation of alcohol
Observation on lesson delivery	To be completed after receiving the feed-back from the learners (what did the learners liked, what challenged them...)		

Sample lesson for unit 3: Integration

School Name: XXXXXXXXXX **Teacher's name**: YYYYYYYYYYYYY

Term	Date	Subject	Class	Unit N°	Lesson N°	Duration	Class size
II	……. /……./2018	Subsidiary Mathematics	S6 HEG	3	3 of 11	40 min	…

Type of Special Educational Needs to be catered for in this lesson and number of learners in each category	2 slow learners and 2 low vision learners • For those with low vision provide their seats where they will be able to look on black board easily; • Group slow learners with others that can help them. Remember to provide repetition when necessary and give them extra time to finish the activity.
Unit title	Integration
Key unit competency	Use integration as the inverse of differentiation and as the limit of a sum and then apply them to find area of a plane shapes.
Title of the lesson	Anti-derivatives
Instructional objective	Given a simple function, learners will be able to accurately determine its anti derivatives by considering the derivative in reverse order, from the answer to the question.
Plan for this Class (location: in / outside)	The lesson is held indoors, the class is organized into groups ,3 slow learners are scattered in different groups, and 2 low vision learners seat on the front desks near the blackboard in order to see and participate fully in all activities
Learning Materials (for ALL learners)	Textbooks and charts containing the table of derivatives
References	Mathematics syllabus for Advanced level, Subsidiary Mathematics textbook and Teacher's guide for S6, etc.

Timing for each step	Description of teaching and learning activity	Generic competences and cross cutting issues to be addressed + a short explanation
	-Learners work individually the introductory activity, and the correction is done on the chalk board by two learners, one after another under the guidance of the teacher.	

	-Then they discuss in groups the discovery activity, followed by the presentation by a sample group, interaction of learners and harmonization of the results under the facilitation of the teacher. -Next, they discuss in pairs the solved example and compare their results with the answer proposed in the book -Finally, the learners are assigned individual tasks, and the correction is done on the chalk board, and the teacher winds up the lesson.		
	Teacher activities	**Learner activities**	
Introduction: 5 minutes	The teacher asks learners to work individually: $f(x)=\sin x$; $f'(x)=...$ $g(x)=...$; $g'(x)=2x$ - The teacher links the introduction to the lesson of the day	-Learners work individually. -Two learners, one after another, write the answers on the chalkboard: $f'(x)=\cos x$; $g(x)=x^2$ The answer for g(x) is not unique	**Communication** skills developed through the presentation and sharing ideas
Development of the lesson			
Discovery activity: **10 minutes**	- The teacher organizes the learners into groups - Teacher gives learners activity 3.2 to discuss in groups and gives instructions related to the	-Learners form groups -Each group analyzes and discuss the activity 3.2 under the direction of the task manager of the group -Learners present to the teacher their eventual problems	• **Cooperation** and **communication** skills through discussions • **Peace and values education**; Cooperation, mutual respect, tolerance through discussions with people with different views and respect one's views

	task - Teacher goes around to monitor the work of each group and provide assistance where needed		
Presentation of learner's findings and exploitation: **15 minutes**	- Teacher invites the reporter of a sample group to present the findings of the group - The teacher encourages learners to follow attentively - Teacher takes notes on key points from learners' presentation. - The teacher asks learners to amend the presentation and to evaluate their work	-The reporter presents the work on the behalf of the group. **Expected answers** **(Refer** to solution of activity 3.2, in TG) -Learners follow the presentation -Learners evaluate the findings of other learners -Learners evaluate their own findings	• **Cooperation** and **communication**/ attentive listening during presentations and group discussions • **Critical thinking** through evaluating other's findings

Conclusion/ Summary: **5 minutes** **Assessment** **5 minutes**	-Teacher facilitates the learners to elaborate the summary of the presentation -Teacher requests learners to write down the main points in their books - Teacher asks learners to individually work out the application activity 3.2	-The learners come to the main point: **An anti-derivative of f(x) is a function F(x) such that** $[F(x)]' = f(x)$ -Learners take notes in their books -Individually learners work out the application activity 3.2. and finally, they make a correction on the chalk board. **Expected answers** **(Refer** to solution of application activity 3.2, in TG)	- **Critical thinking** and problem-solving skills are developed through analysing and solving real life Mathematical problem. - **Critical thinking** and problem-solving skills are developed through analysing and solving problems.
Observation on lesson delivery	To be completed after receiving the feed-back from the learners (what did the learners like, what challenged them, ...)		

Sample of lesson plan for unit 4: Ordinary differential equations

School Name: XXXXXXXX **Teacher's name**: YYYYYYYYYY

Term	Date	Subject	Class	Unit N°	Lesson N°	Duration	Class size
III	.... /....…/2018	Subsidiary Mathematics	S6 HEG	4	4 of15	40 min	...

Type of Special Educational Needs to be catered for in this lesson and number of learners in each category	2 slow learners and 2 low vision learners. • For those with low vision provide their seats where they will be able to look on black board easily; • Group slow learners with others that can help them. Remember to provide repetition when necessary and give them extra time to finish the activity.
Unit title	Ordinary differential equations
Key unit competency:	Use ordinary differential equations of first and second order to model and solve related problems in Physics, Economics, Chemistry, Biology, …
Title of the lesson	Differential equations and population growth
Instructional Objective	Given the equation modelling population growth, learners will be able to accurately solve the equation, plot and interpret the graph of the population growth in a given time.
Plan for this Class (location: in / outside)	The lesson is held indoors, the class is organized into groups, 3 slow learners are scattered in different groups, and the group with one of the 2 low vision learners sits on the front desks near the blackboard.
Learning Materials (for ALL learners)	Textbooks, Scientific calculators, graph drawing software such as GeoGebra
References	Mathematics syllabus for Advanced level, Subsidiary Mathematics textbook and Teacher's guide for S6, ...

Timing for each step	**Description of teaching and learning activity**	**Generic competences and cross cutting issues to be addressed**
	-Two learners chosen randomly present, one after another, their findings, from the	

	assignment, and learners interact under the facilitation of the teacher, who, then, links the presentation to the lesson of the day. -learners are organized into groups to discuss on the discovery activity and the examples, the reporter from one group presents the findings and other learners interact. The teacher facilitates learners to capture the key concepts of the lesson through harmonization. -Finally, the learners are assigned to individual tasks, and the correction is done on the chalk board.		**+ a short explanation**
	Teacher activities	**Learner activities**	
Introduction: 5 minutes	The teacher asks learners to work individually and present their findings about solving the differential equation with separable variables - The teacher links the introduction to the lesson of the day.	- Learners solve individually the following differential equation: $\sin x dx - \sin y dy = 0$ and then two learners (including one slow learner) present, one after another, their findings. - While presenting, the rest of the class follow attentively, interact with their colleagues or ask questions for better understanding.	- Communication skills developed through the presentation and sharing ideas - **Inclusive education** is addressed by providing the remediation activities & tasks to slow learners while short sight learners are facilitated by sitting on the front desks near the blackboard.
Development of the lesson			
Discovery activity: **10 minutes**	- The teacher organizes the learners into groups of 4 and assigns them the task (activity 4.4) and gives instructions related to the task (organization of the group, role of each member, duration,	- Learners follow the instructions and in small groups they work out the activity 4.4 - Each group analyzes and discuss the activity 4.4 under the direction of the task manager of the group - Learners may be given the graph showing the population growth of a city or a country and requested to give their own	- Cooperation and communication skills through discussions - Peace and values education; Cooperation, mutual respect, tolerance through discussions with people with different views and respect one's views - **Environment and Sustainability:** is addressed while discussing the effects of population growth on the environment, and on Climate Change. - Comprehensive sexuality education developed while learners discuss the issue of population growth

	presentation) - Teacher goes around to monitor the work of each group (focus on groups with slow learner) and provide assistance where needed	interpretations. They may also be requested to advise the policy makers on different majors to be taken while there is an exponential increase of the population.	through graph interpretation and adopt the family planning program as an advice.
Presentation of learner's findings and exploitation: **15minutes**	- Teacher invites the reporter of the sampled group to present the findings of the group; - The teacher encourages learners to follow attentively, - Teacher takes notes on key points from learners' presentation. - The teacher asks learners to amend the presentation and to evaluate their work.	-Learners present their findings: **Expected answers** **(Refer** to solution of activity 4.4, in TG) -Learners follow the presentation, -Learners evaluate the findings of peers and ask questions, -Learners evaluate their own findings.	- Cooperation, communication and attentive listening during presentations and group discussions; - Critical thinking through evaluating other's findings.
Conclusion/ Summary: **5 minutes** **Assessment** **5 minutes**	-Teacher facilitates the learners to capture the main points of the presentation, -Teacher requests learners to write down the main points in their notebooks,	-The main point: • The population growth is modelled by the differential equation $\frac{dP}{dt} = KP$, (P is Population, t is time and K is a constant of	- Critical thinking and problem-solving skills are developed through analysing and solving real life Mathematical problem (population growth of a city or country). - **Inclusive education** is addressed by providing the remediation activities & tasks to slow learners and while short sight learners are facilitated by letting them sit on the front desks near the blackboard.

	- Teacher asks learners to individually work out the application activity 4.4. -Teacher asks learners to identify other effects of population growth.	proportionality). -Learners write the summary in their notes books, Individually learners work out the application activity 4.4 and finally they make a correction on the chalk board. **Expected answers:** **(Refer** to solution of learning activity 4.4, in TG) -Learners discuss the impact of population growth.	
Observation on lesson delivery	To be completed after receiving the feed-back from the learners (what did the learners liked, what challenged them.)		

PART III: Unit Development

Unit 1: Complex N umbers

1

1.1 Key unit competence

Perform operations on complex numbers in different forms and use them to solve related problems in Physics, etc.

1.2 Prerequisite knowledge and skills

- Addition/ subtraction of two vectors and their geometrical representation in Cartesian plane (Senior 2: unit 7 and Senior 4: unit 7)
- Solving equations in the set of real numbers (Senior 1: unit 3; Senior 2: unit 1 and Senior 4: unit 2).
- Definition of the principle trigonometric ratios (Senior 4: Unit 1)
- Solving trigonometric equations (Senior 5: unit 4)
- Solving exponential equations (Senior 4: unit 2 and Senior 5: unit 3)

1.3 Cross-cutting issues to be addressed

- Inclusive education (promote education for all while teaching)
- Peace and value Education (respect others view and thoughts during class discussions)
- Gender (equal opportunity of boys and girls in the lesson participation)
- Standardization culture (regulation of the current intensity)

1.4 Guidance on the introductory activity

a) Learners work on the introductory activity to understand the importance of imaginary number "i" in solving equations which can't be solved in the set of real numbers.

b) Let learners read the introductory activity in the Learner's book. Through examples, let them discover the importance of the introduction of each of previous sets of numbers with regard to solving different equations.

- Equation $x-2=0$ has a solution in $\mathbb{N}$.

- Equation $x+4=0$ has no solution in $\mathbb{N}$, but it has a solution in $\mathbb{Z}$,
- Equation $4x=5$ has no solution in $\mathbb{Z}$ but in $\mathbb{Q}$, $x=\frac{5}{4}$ is the solution of this equation.
- Equation $x^2=3$ has no solution in $\mathbb{Q}$, but $x=\sqrt{3}$ or $x=-\sqrt{3}$ are solutions of the equation in $\mathbb{R}$.

c) Through question-answer, facilitate Learners to realise that $x=\pm 2$ are solutions of equation $x^2-4=0$ in the set of real numbers $\mathbb{R}$ and that the equation $x^2+4=0$ does not have solutions in $\mathbb{R}$ because the square root of a negative number is not possible in $\mathbb{R}$.

d) Through class discussions, let learners think of different ways of getting solutions of the equation $x^2+4=0$ and by introduction of imaginary number "i", let them use "$i^2=-1$" to find that are solutions of the equation $x^2+4=0$

e) Help learners to understand that the imaginary number "i" is element of a new set of numbers known as set of complex numbers "$\mathbb{C}$".

1.5 List of lessons

UNIT 1: COMPLEX NUMBERS			
SUB-UNIT 1: Algebraic form of Complex numbers and their geometric representation (16 periods)			
Introductory activity: 1 period: 40 minutes			
N°	**Lesson title**	**Learning objectives (from the syllabus including knowledge, skills and attitudes):**	**Number of periods**
1	Definition and properties of "the imaginary number i"	Identify the real part and the imaginary part of a complex number	2
2	Geometric representation of complex numbers	Represent a complex number on Argand diagram	2

3	Addition and subtraction in the set of complex numbers	Apply the properties of complex numbers to perform operations (addition, subtraction, conjugate, multiplication, powers, and division) on complex numbers in algebraic form	1
4	Conjugate of a complex number		1
5	Multiplication and powers of complex number		1
6	Division in the set of complex numbers		1
7	Modulus of a complex number	Find the modulus and the square roots of a complex number	2
8	Square roots of a complex number		2
9	Equations in the set of complex number	Solve a simple linear or quadratic equation in the set of complex numbers	2
SUB-UNIT 2: Polar form of complex numbers (7 periods)			
10	Definition and properties of a complex number z in polar form	Define a complex number in a polar form and convert a complex number from algebraic form to polar form and vice versa	2
11	Multiplication and division of complex numbers in polar form	Apply the properties of complex numbers to perform operations on complex numbers in polar form	2
12	Powers and De Moivre's formula	Apply De Moivre's theorem to calculate power of complex number	2

SUB-UNIT 3: Exponential form of complex numbers (4 periods)			
13	Definition and properties of a complex number z in exponential form	Define a complex number in exponential form and convert a complex number from algebraic or polar form to exponential form and vice versa	2
14	Euler's formula of complex numbers	Apply Euler's formula to transform trigonometric expressions	1
15	Application of complex numbers in other sciences.	Appreciate the importance of complex numbers to solve related problems such as in Physics (problem related to voltage and alternating current), …	1
16	End unit assessment		2

Notice:

For application of mathematics content to other subjects, the teacher will consider the prerequisite of learners in this domain then act accordingly; the time spent and importance given to application activities depending on the learner's level of knowledge.

Lesson 1: Definition of complex number

a) Prerequisites/Revision/Introduction:

- Through examples, let learners discuss how to solve quadratic equations in $\mathbb{R}$ (set of real numbers). For example, use the quadratic equation $x^2 - 16 = 0$ and let learners find that the solution is $x = \pm 4$ because the square root of a positive number exists.

b) Teaching resources:

- Learner's book and other Reference textbooks to facilitate research
- Flash cards containing the complex numbers, real numbers and pure imaginary numbers

c) Learning activities:

- Ask learners to use the formula for solving quadratic equation $x^2 + 16 = 0$ in the set of real numbers from activity 1.1, and by introducing the imaginary number $i = \sqrt{-1}$ or $i^2 = -1$, let them find the square root of a negative number which does not exist in $\mathbb{R}$.

- Lead learners to realize that the solution $x=\pm 4i$ is the solution of the equation $x^2+16=0$.
- Explain to the learner that the solution $x=\pm 4i$ containing the imaginary number i is an element of a new set called "set of complex numbers $\mathbb{C}$ and that $\mathbb{N}\subset\mathbb{Z}\subset\mathbb{Q}\subset\mathbb{R}\subset\mathbb{C}$.
- Through generalization, let learners discover that given two-real numbers a and b the complex number z is defined as, $z=a+ib$ where the number a is called the real part of z and the number b is called the imaginary part of z and $i^2=-1$.
- Let learners be aware that the definition of complex number is mathematically written as follows: $\mathbb{C}=\left\{a+ib;\quad where\ a,b\in\mathbb{R}\quad and\ i^2=-1\right\}$
- Through different examples, help learners to understand the importance of complex numbers by showing their application in electrical engineering, as well as in physics. For example, complex numbers are used when some quantity has a phase as well as a magnitude. Such situation occurs when one deals with sinusoidal oscillating voltage and current (other examples in physics include optics, where wave interference is important, and quantum mechanical wave functions). In other subjects, complex numbers are used to make calculations *easier.*

Application Activity 1.1

Solution for question 1

$a)\ z=4+2i,\quad \operatorname{Re}(z)=4, \operatorname{Im}(z)=2$

$b)\ z=i,\quad \operatorname{Re}(z)=0, \operatorname{Im}(z)=1$

$c)\ z=\sqrt{2}-i,\quad \operatorname{Re}(z)=\sqrt{2}, \operatorname{Im}(z)=-1$

$d)\ z=-3.5,\quad \operatorname{Re}(z)=-3.5, \operatorname{Im}(z)=0$

Solution for question 2

$a)\ 25=(6\times 4)+1,\quad i^{25}=i$

$b)\ 2310=(577\times 4)+2,\quad i^{2310}=-1$

$c) 71=(17\times 4)+3,\quad i^{71}=i^3=-i$

$d)\ 51=4\times 12+3,\quad i^{28}=i^3=-i$

$e) 28=4\times 7,\quad i^{28}=i^0=1$

Solution for question 3

The inductors and the capacitors are $Z_L = j\omega L$ and $Z_C = \frac{1}{j\omega C}$ have imaginary part.

Lesson 2: Geometric representation of complex numbers

a) Prerequisites/Revision/Introduction:

- Using the activity 1.3, guide learners in plotting points: $A(2,3)$ and $B(-3,5)$ in Cartesian plane.

b) Teaching resources:

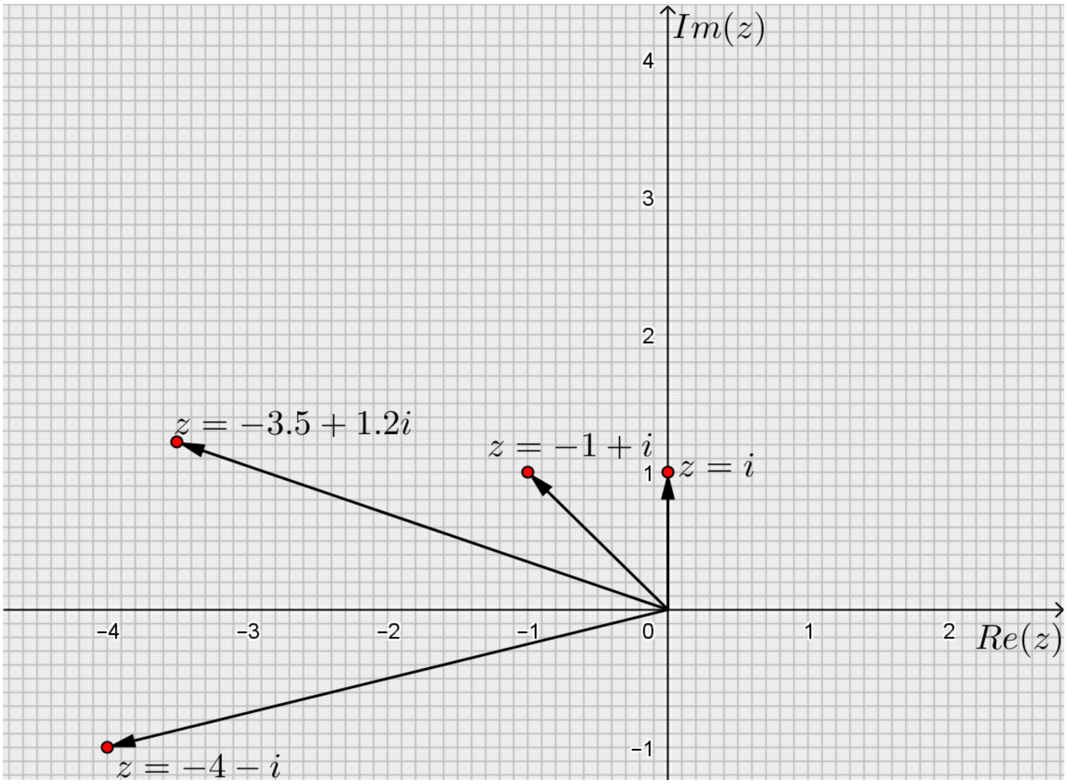
- T-square, ruler, if possible Mathematics drawing software as GeoGebra, Mathlab, graph, etc.
- Learner's book and other textbooks to facilitate research

c) Learning activities:

- Given the complex number $z=-3+5i$, ask learners to plot the point $z(-3,5)$ in a complex plane and then from activity 1.3, facilitate them to plot the point $z(a,b)$ affix of $z=a+bi$ in a complex plane.
- Through class discussions, ask learners to establish a relationship between Cartesian plane and complex plane by considering $x-axis$ as real axis and $y-axis$ axis as imaginary axis.
- Using figure 1.3, let learners deduce that the complex plane consists of two number lines that intersect in a right angle at the point $(0,0)$. The horizontal number line (or $x-axis$ in Cartesian plane) is the real axis while the vertical number line (or $y-axis$ in Cartesian plane) is the imaginary axis.

Application activity 1.2

Solution for question 1



Solution for question 2

As $X = X_L + X_C$,the net reactance vector is $jX_L + jX_C = 200j - 150j = 50j$

This is an inductive reactance, because it is positive imaginary.

Lesson 3: Addition and subtraction in the set of complex numbers

a) Prerequisites/Revision/Introduction:

- Guide learners to plot the point $A(\square\ )$ and $B(-2,4)$ then deduce the coordinate of the vector $\overrightarrow{OA}+\overrightarrow{OB}=\overrightarrow{OC}$, where C is a new point in the Cartesian plane.

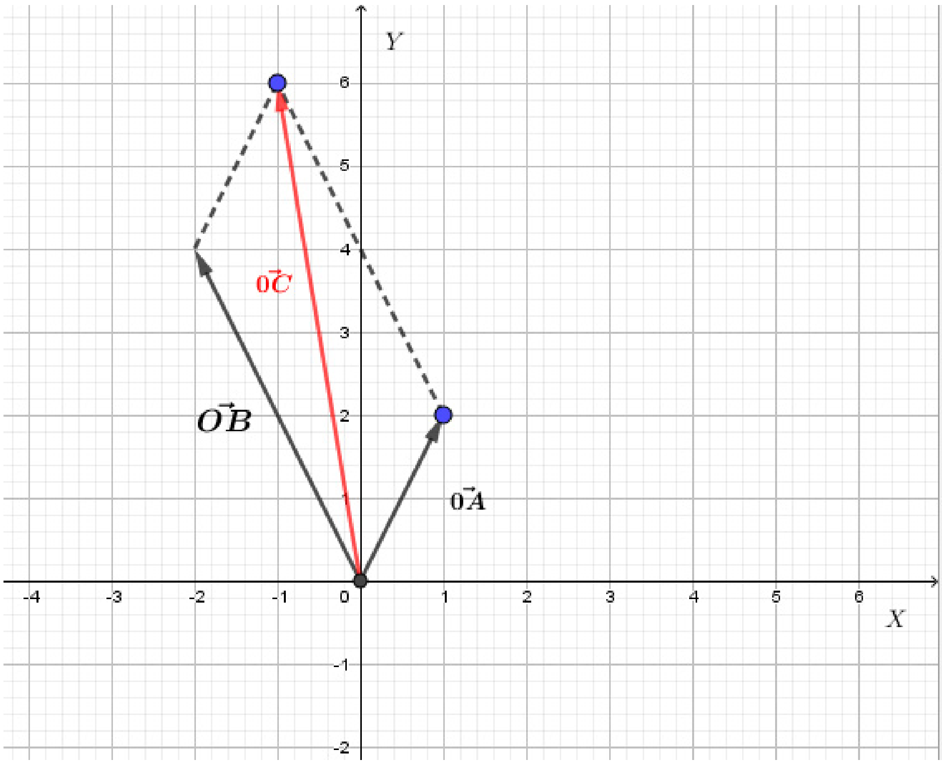


Figure 1. 1 The sum of two vectors $\vec{OA}$ *and* $\vec{OB}$

b) Teaching resources:

- T-square, ruler, if possible Math draw software as GeoGebra, math lab….
- Learner's book/ Internet or textbooks to facilitate research

c) Learning activities:

- From **Activity 1.4** and basing on the answer found in a), guide learners in plotting respectively the affix $z_1(1,2)$ and $z_2(-2,4)$ of $z_1=1+2i$ and $z_2=-2+4i$.
- Basing on the geometrical representation of $z_1(1,2)$ and $z_2(-2,4)$, facilitate learners to deduce the affix $z(-1,6)$ of the complex number z_1+z_2 .

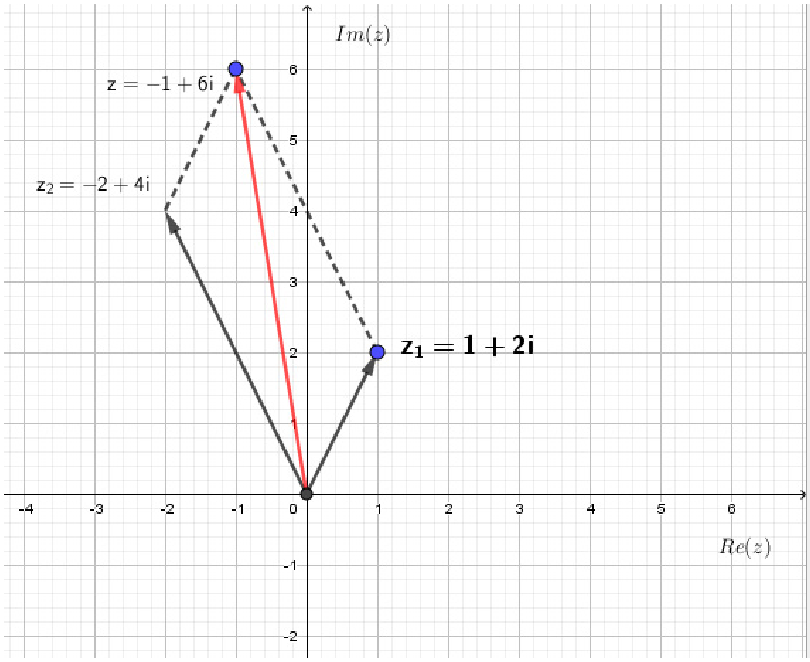


Figure 1. 2 Geometric representation of $z=-1+6i$

- Through the interpretation of the figure 1.2 by learners and parallelogram rule, let Learner to find that the sum of two complex numbers $z_1 = a+bi \text{ and } z_2 = c+di$ is obtained by adding the sum of real parts to the sum of imaginary part as follow: $\left[\text{Re}(z_1)+\text{Re}(z_2)\right]+i\left[\text{Im}(z_1)+\text{Im}(z_2)\right]$,therefore the sum of $z_1 = a+bi \text{ and } z_2 = c+di$ is obtained by $(a+bi)+(c+di)=(a+c)+(\text{b}+d)\text{i}$

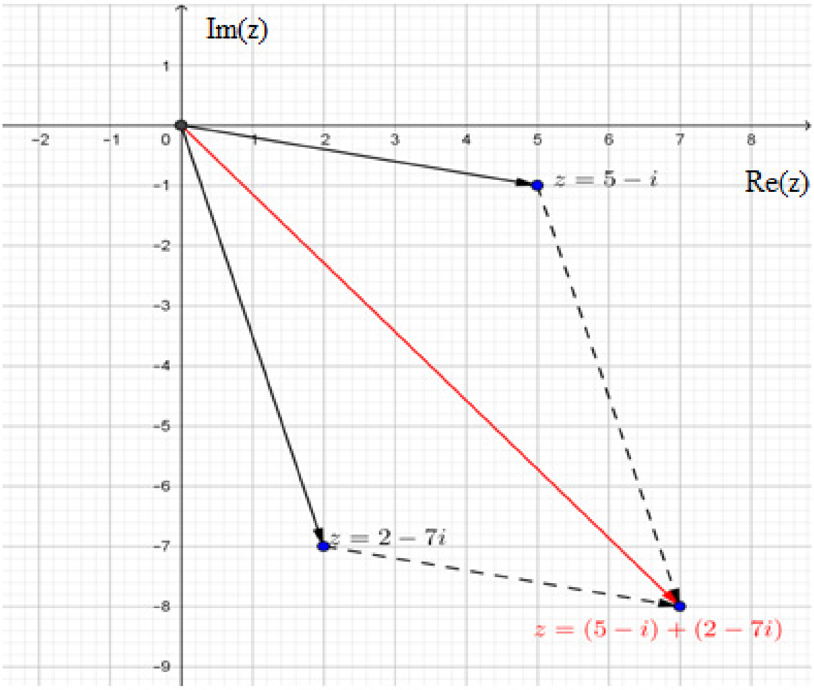
- Through a variety of examples and graphical representations of two different complex numbers in the complex plane, lead learners to calculate the difference of two complex numbers $z_1 = a+bi \text{ and } z_2 = c+di$ as follows: $(a+bi)-(c+di)=(a-c)+(\text{b}-d)\text{i}$

Application Activity 1.3

Solution for question 1

a) Facilitate learners to remember how to add vectors graphically:

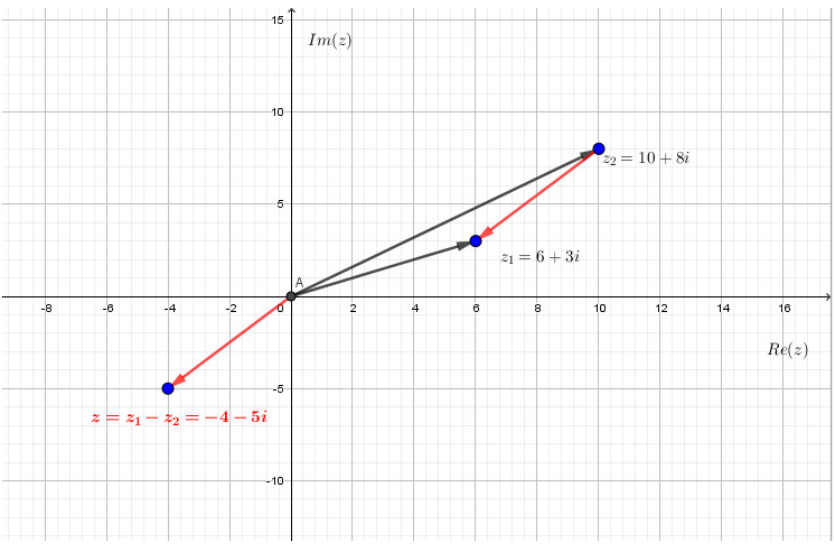
Vector addition is found geometrically by constructing a *parallelogram*, using the two vectors $z_1 = a+bi$ and $z_2 = c+di$ as two of the sides. Then, the diagonal is the resultant (or the sum vector).



From graphical representation

$(5-i)+(2-7i)=7-8i=5+2-i(1+7)$

b)

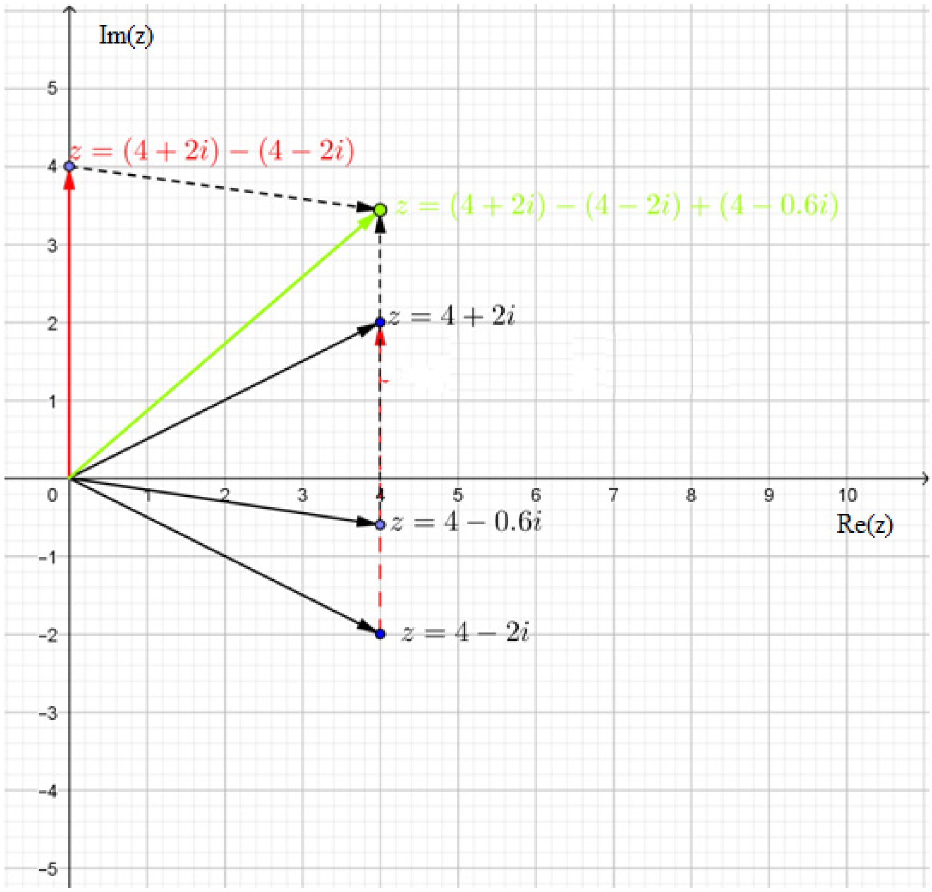


After representing graphically, $6+3i$ and $10+8i$, we deduce that

$(6+3i)-(10+8i)=-4-5i=6-10+i(3-8)$.

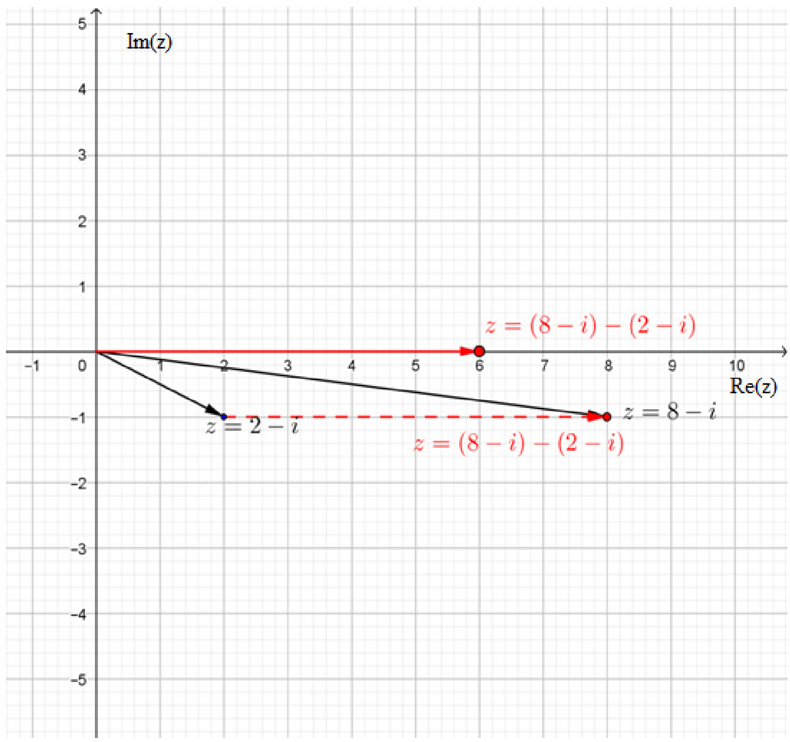
Note that $(6+3i)-(10+8i)=(6+3i)+[-(10+8i)]$

c) Graphically, we have:



From graphical representation, we get

$$(4+2i)-(4-2i)+(4-0.6i)=4i+(4-0.6i)=4+3.4i$$



From geometric presentation, we deduce that

$$(8-i)-(2-i)=6=8-2-i(1-1).$$
$$Z=R+j\left(X_L+X_C\right)=560+j\left(400-410\right)=560-10j$$

The net impedance is $Z=560-10j$.

Lesson 4: Conjugate of a complex number

a) Prerequisites/Revision/Introduction:

- Use the activity 1.5 and facilitate learners to plot the number $z=2+5i$ in the complex plane and lead them to find the image $P'(2,-5)$ of the point $P(2,5)$ affix of z by the reflection of real axis.

b) Teaching resources:

- T-square, ruler, if possible Mathematics drawing software as GeoGebra, Mathlab, etc.
- Learner's book and reference textbooks for developing learners' self-confidence through research activity.

c) Learning activities:

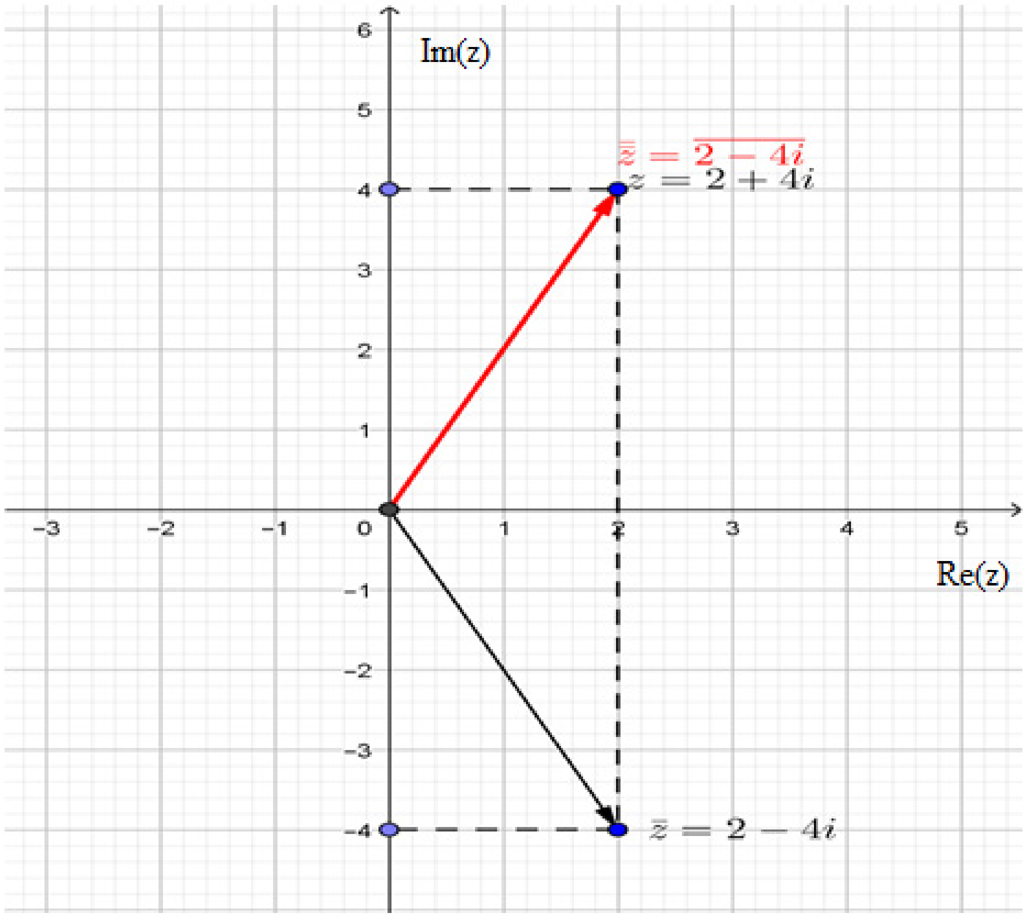
- Through the interpretation of the figure 1.8 and class discussion, let learners to discover that $z' = 2 - 5i$ is the conjugate of $z = 2 + 5i$.

- Using figure 1.8, the answer of the activity 1.5 and a variety of example 1.5, lead learners to discover that for every complex number $z = a + bi$ there is a corresponding complex number $\bar{z} = a - bi$ called conjugate of z that is the reflection of z by real axis.

Application Activity 1.4

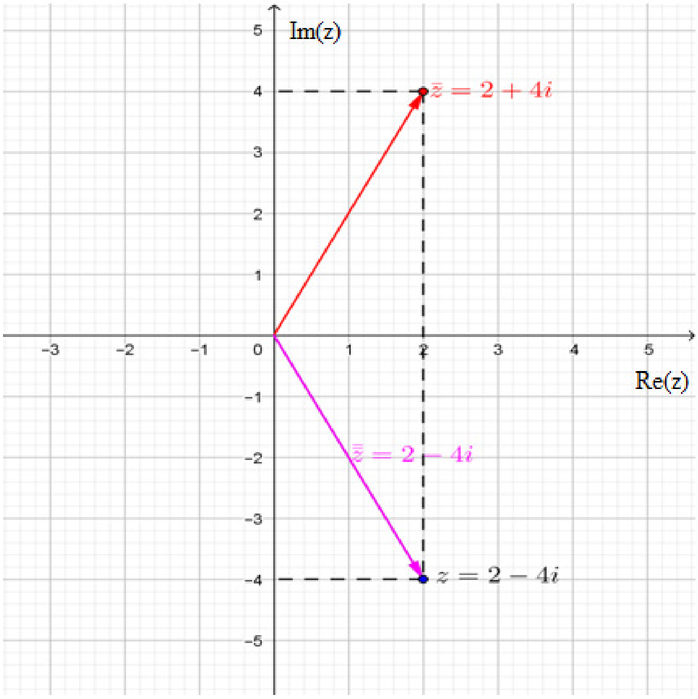
Solution for question (a)

1)



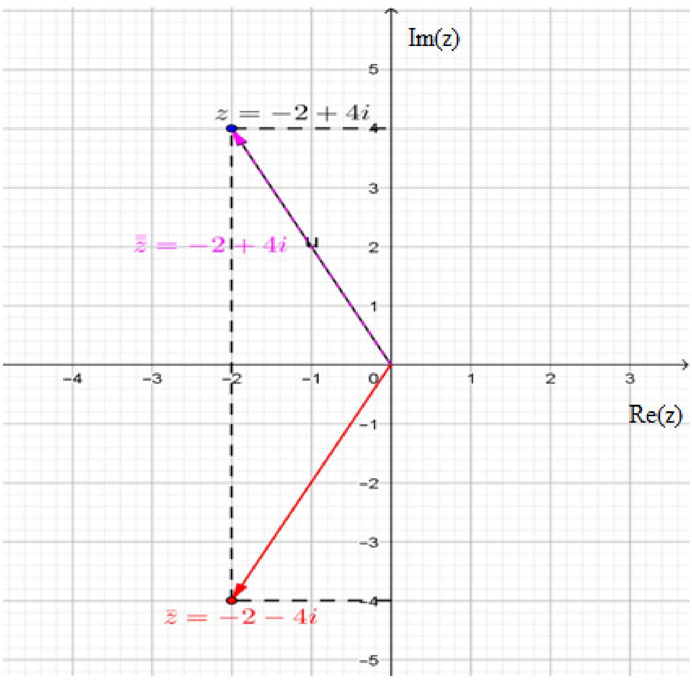
$z = 2 + 4i$, $\bar{z} = 2 - 4i$, $\bar{\bar{z}} = 2 + 4i$

2)



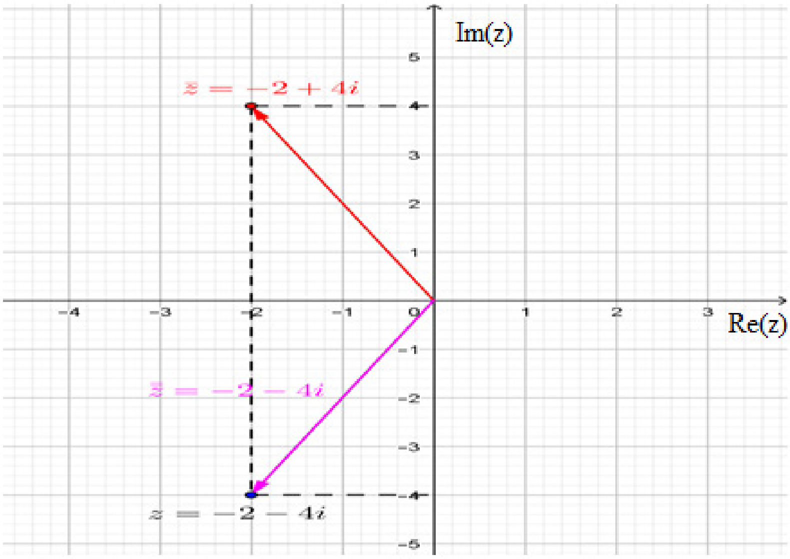
$z = 2 - 4i$, $\bar{z} = 2 + 4i$, $\bar{\bar{z}} = 2 - 4i$

3)



$z=-2+4i$, $\bar{z}=-2-4i$, $\bar{\bar{z}}=-2+4i$

4)



$z=-2-4i$, $\bar{z}=-2+4i$, $\bar{\bar{z}}=-2-4i$

Solution for question (b)

From graphical representation or numerical analysis, we find that $\bar{\bar{z}}=z$.

Lesson 5: Multiplication and powers of complex number

a) Prerequisites/Revision/Introduction:

- By means of examples, help learners to calculate the product in the set of real numbers. Let them apply the following distributive property $(a+b).(c+d)=a(c+d)+b(c+d)$

b) Teaching resources:

- Learner's book and other reference textbooks to facilitate research

c) Learning activities:

- Given two complex numbers $z_1=4-7i$ and $z_2=5+3i$, lead learners to determine the product $z_1.z_2=(4-7i).(5+3i)\Rightarrow z_1.z_2=\left[(4.5)-(-7).3\right]+i\left[(4.3)+5.(-7)\right]\Leftrightarrow z_1.z_2=41-23i$ by applying the distributive property and converting i^2 into – 1

- Using the same procedure, facilitate learners to calculate the power $z_1^2=(4-7i)^2=(4-7i)(4-7i)\Leftrightarrow z_1^2=-33-56i$.

- Let learners go through the example 1.6 and work out application activities 1.5 to emphasize their skills in calculating the product and powers of complex numbers

Application Activity 1.5

Solution

a) $z=i(3-7i)(2-i)=17-i$

b) $z=(1+i)^2-3(2-i)^3=-6+35i$

Lesson 6: Division in the set of complex numbers

a) Prerequisites/Revision/Introduction:

- Through different exercises, let learners determine the conjugate of the complex number $z=5+i$

b) Teaching resources:

- Internet and textbooks to facilitate research

c) Learning activities:

- Through the activity 1.7, let learners apply the rules of rationalizing the denominator in $\mathbb{R}$ and convert i^2 into -1, and assist them to transform the denominator of $z=\frac{2+3i}{5+i}$ into real part without changing the value of z as follow: $z=\frac{(2+3i)(5-i)}{(5+i)(5-i)}\Leftrightarrow z=\frac{(2+3i)(5-i)}{26}$
- Facilitate learners to determine the quotient of $2+3i$ and $5+i$ which is $$z=\frac{(2+i)(5-i)}{(5+i)(5-i)}=\frac{10-2i+15i+3}{25+1}=\frac{13+13i}{26}=\frac{1}{2}+\frac{1}{2}i$$

- From the activity 1.7, let learners work in small groups and make generalization on how to find $\frac{z_1}{z_2}$ given two complex numbers $z_1=a+bi$ and $z_2=c+di$.
- From the harmonization of groups work, help learners to realize that the quotient $\frac{z_1}{z_2}$ is obtained by $\frac{z_1}{z_2}=\frac{(a+bi)(c-di)}{(c+di)(c-di)}=\frac{(ac+bd)+(bc-ad)i}{c^2+d^2}$

- Let learners go through the example1.7 and work out application activities 1.6 to emphasize their skills in calculating the quotient of complex numbers

Application Activity 1.6

Solution for question 1

a) $z=\dfrac{1}{(2+i)(1-2i)}$ Or, $(2+i)(1-2i)=(2+2)+i(1-4)=4-3i$

Hence $z=\dfrac{1}{(2+i)(1-2i)}=\dfrac{1}{4-3i}=\dfrac{4+3i}{16+9}=\dfrac{1}{25}(4+3i)$.

b) $z=\left(\dfrac{\sqrt{3}-i}{\sqrt{3}+i}\right)^4\left(\dfrac{1+i}{1-i}\right)^5$. Let $z_1=\dfrac{\sqrt{3}-i}{\sqrt{3}+i}$ and $z_2=\dfrac{1+i}{1-i}$,

thus $z_1=\dfrac{(\sqrt{3}-i)(\sqrt{3}-i)}{3+1}=\dfrac{3-2\sqrt{3}i-1}{4}=\dfrac{1}{2}(1-i\sqrt{3})$ and

$z_2=\dfrac{1+i}{1-i}=\dfrac{(1+i)(1+i)}{1+1}=\dfrac{2i}{2}=i$.

Therefore,

$$z=z_1^4\cdot z_2^5=\left(\frac{\sqrt{3}-i}{\sqrt{3}+i}\right)^4\left(\frac{1+i}{1-i}\right)^5=\left[\frac{1}{2}(1-i\sqrt{3})\right]^4\cdot i^5=\frac{1}{16}(1-i\sqrt{3})^4\cdot i$$

But, $(1-i\sqrt{3})^4=1-4(i\sqrt{3})+6(i\sqrt{3})^2-4(i\sqrt{3})^3+(i\sqrt{3})^4$

$=1-4i\sqrt{3}-18+12i\sqrt{3}+9=-8+8i\sqrt{3}$.

Hence, $z=\dfrac{1}{16}(-8+8i\sqrt{3})\cdot i=\dfrac{1}{2}(-1+i\sqrt{3})\cdot i=-\dfrac{\sqrt{3}}{2}-\dfrac{1}{2}i$.

Solution for question 2

The solution of this type of problem is based on the fact that two complex numbers are equal if and only if their real parts and imaginary parts respectively are equal to each other.

a) $x+4y+xyi=12-16i\Leftrightarrow\begin{cases}x+4y=12 & (1)\\ xy=-16 & (2)\end{cases}$

From equation (1), we get $x=12-4y$. Replacing x in equation (2), we obtain:

$(12-4y)y=-16\Leftrightarrow 4y^2-12y-16=0\Leftrightarrow y^2-3y-4=0\Leftrightarrow(y-4)(y+1)=0$

Hence, $y=-1$ or $y=4$.

If $y=-1$, then $x=16$; if $y=4$, then $x=-4$.

The solutions of the given problem are $(16,-1)$ and $(-4,4)$.

b) $x-7y+8xi=6y+(6y-100)i$

$$\begin{cases} x-7y=6y \\ 8x=6y-100 \end{cases} \Leftrightarrow \begin{cases} x-13y=0 & (1') \\ 8x-6y=-100 & (2') \end{cases}$$

From (1'): $x = 13y$. Substituting the value of x in (2 ') gives

$$104y-6y=-100 \Leftrightarrow 98y=-100 \Leftrightarrow y=-\frac{100}{98}=-\frac{50}{49}.$$

Putting the value of y in (1'): $x=-\frac{\square}{49}$.

Hence, $x=-\frac{650}{49}$ and $y=-\frac{50}{49}$.

c) $\frac{1}{x+iy}+\frac{1}{1+2i}=1 \Leftrightarrow \frac{x-iy}{x^2+y^2}+\frac{1-2i}{5}=1 \quad (x\neq 0 \, and \, y\neq 0)$.

$$\Leftrightarrow \frac{5(x-iy)+(x^2+y^2)(1-2i)}{5(x^2+y^2)}=1 \Leftrightarrow \frac{5(x-iy)+(x^2+y^2)(1-2i)}{5(x^2+y^2)}=1$$

$$\Leftrightarrow \frac{(x^2+y^2+5x)-(2x^2+2y^2+5y)i}{5(x^2+y^2)}=1$$

Hence, $\begin{cases} \frac{x^2+y^2+5x}{5(x^2+y^2)}=1 & (a) \\ \frac{2x^2+2y^2+5y}{5(x^2+y^2)}=0 & (b) \end{cases}$

$$(a)\&(b) \Leftrightarrow \begin{cases} x^2+y^2+5x=5x^2+5y^2 \\ 2x^2+2y^2+5y=0 \end{cases} \Leftrightarrow \begin{cases} 4x^2+4y^2-5x=0 & (a') \\ 2x^2+2y^2+5y=0 & (b') \end{cases}$$

The linear combination $(a')-2(b')$ gives us

$-5x-10y=0 \Leftrightarrow 5(x+2y)=0 \Leftrightarrow x+2y=0$

Letting $y=\alpha$, we get $x=-2\alpha$, $\alpha\in\mathbb{R}$.

Substituting these values in (a') yields

$16\alpha^2+4\alpha^2+10\alpha=0 \Leftrightarrow 20\alpha^2+10\alpha=0$

$\Leftrightarrow 10\alpha(2\alpha+1)=0 \Leftrightarrow \alpha=0 \; or \; \alpha=-\frac{1}{2}$.

The value $\alpha = 0$ is to be eliminated since it is incompatible with the condition $x \neq 0 \, and \; y \neq 0$.

Let $\alpha = -\frac{1}{2} \Rightarrow x = 1$ and $y = -\frac{1}{2}$.These values satisfy the equations (a') and (b'), therefore the solution of the problem is and $y = -\frac{1}{2}$.

Solution for question 3

$$\frac{1+T^2}{2T} = \frac{1+\left(\frac{x-iy}{x+iy}\right)^2}{2\frac{x-iy}{x+iy}} = \frac{\frac{(x+iy)^2+(x-iy)^2}{(x+iy)^2}}{2\frac{x-iy}{x+iy}}$$

$$= \frac{\left(x^2+2xyi-y^2+x^2-2xyi-y^2\right)(x+iy)}{2(x+iy)^2(x-iy)}$$

$$= \frac{2\left(x^2-y^2\right)}{2(x+iy)(x-iy)} = \frac{x^2-y^2}{x^2+y^2} \, as \, required.$$

Lesson 7: Modulus of a complex number

a) Prerequisites/Revision/Introduction:

- Facilitate learners to plot the point $A(0,0)$, $B(3,0)$ and $C(4,3)$ in the Cartesian plane, and join those points to find a triangle and then find the distance of the side AC.

b) Teaching resources:

- T-square, ruler, if possible Math draw software as GeoGebra, Mathlab, graphcalc, etc.
- Internet and textbooks to facilitate research

c) Learning activities:

- From the activity 1.9, lead learners to calculate the modulus of $z = a + bi$ and let them discover that the modulus is always a positive real number denoted by $|z|$, such that $|z| = \sqrt{z.\bar{z}} = \sqrt{a^2+b^2}$.
- In small groups, facilitate the learners to calculate the modulus by explaining that the modulus value looks like the length of the vector with origin $(0,0)$ to the affix of the complex number z.

- By using figure 1.9, harmonize the Learners' group works and help them to get the real meaning of modulus through observation and interpretation.

- Individually, let learners go through the example1.8 and work out application activities 1.7 to emphasize their skills in calculating the modulus of complex numbers.

Application Activity 1.7

Solution for question 1

1. $z_1 = 2-3i$ then $|z_1| = \sqrt{4+9} = \sqrt{13}$,

$z_2 = 3-4i$ then $|z_2| = \sqrt{9+16} = \sqrt{25} = 5$,

$z_3 = 6+4i$ then $|z_3| = \sqrt{36+16} = \sqrt{52} = 2\sqrt{13}$,

$z_4 = 15-8i$ then $|z_4| = \sqrt{225+64} = \sqrt{289} = 17$.

$$z = \frac{(2-3i)(3+4i)}{(6+4i)(15-8i)} = \frac{z_1 \cdot z_2}{z_3 \cdot z_4} \Rightarrow |z| = \frac{|z_1|\cdot|z_2|}{|z_3|\cdot|z_4|} = \frac{5\sqrt{13}}{2\sqrt{13}\times 17} = \frac{5}{34}.$$

Solution for question 2

If $z_1 = 1 - i$, $z_2 = -2 + 4i$, $z_3 = 3 - 2i$, then

a) $|2z_2 - 3z_1| = |-7+11i| = \sqrt{170}$

b) $|z_1.\overline{z}_2 + \overline{z}_1 . z_2| = 12$

c) $$\left|\frac{z_1+z_2+1}{z_1-z_2+1}\right| = \left|\frac{(1-i)+(-2+4i)+1}{(1-i)-(-2+4i)+1}\right| = \left|\frac{3i}{4-5i}\right| = \frac{|3i|}{|4-5i|} = \frac{3}{\sqrt{41}} = \frac{3\sqrt{41}}{41}$$

d) $$|z_1^2+z_2^2|^2 + |z_3^2 - z_2^2|^2 = \left|(1-i)^2 + (-2+4i)^2\right|^2 + \left|(3-2i)^2 - (-2+4i)^2\right|^2$$

$$= |-2i + (-12-16i)|^2 + |(5-12i)-(-12-16i)|^2 = |-12-18i|^2 + |17+4i|^2$$

$$= \left(\sqrt{144+324}\right)^2 + \left(\sqrt{289+16}\right)^2 = 773$$

Solution for question 3

$$\left|\frac{z+2}{z}\right| = 2 \Leftrightarrow |z+2| = 2|z|$$

Let $z = x+iy$ where $x, y \in \mathbb{R}$

Then, $\left|\frac{z+2}{z}\right| = 2 \Leftrightarrow |x+iy+2| = 2|x+iy| \Leftrightarrow \sqrt{(x+2)^2 + y^2} = 2\sqrt{x^2+y^2}$

Squaring both sides we get

$\Leftrightarrow (x+2)^2 + y^2 = 4x^2 + 4y^2 \quad [\text{squaring both sides}]$

$\Leftrightarrow x^2 + 4x + 4 + y^2 = 4x^2 + 4y^2 \Leftrightarrow 3x^2 + 3y^2 - 4x = 4 \Leftrightarrow x^2 - \frac{4}{3}x + y^2 = \frac{4}{3}$

$\Leftrightarrow \left(x - \frac{2}{3}\right)^2 + y^2 = \frac{16}{9}$ which is the equation of a circle with centre at $\left(\frac{2}{3}, 0\right)$ and radius of $\frac{4}{3}$ *units*.

Solution for question 4

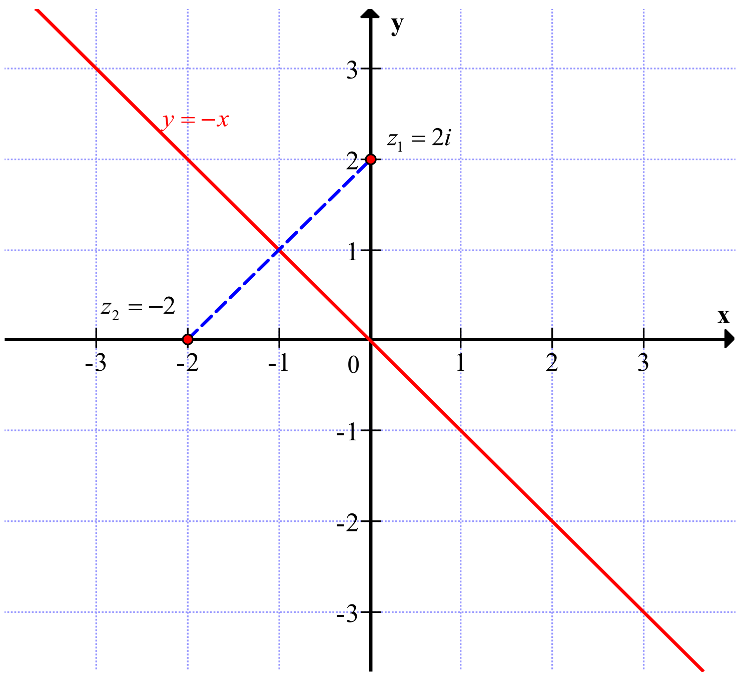
Let $z = x + yi$, we have

$|z - 2i| = |z + 2| \Leftrightarrow |x + iy - 2i| = |x + iy + 2| \Leftrightarrow \sqrt{x^2 + (y-2)^2} = \sqrt{(x+2)^2 + y^2}$

Squaring both sides and then expanding yield

$x^2 + y^2 - 4y + 4 = x^2 + 4x + 4 + y^2 \Leftrightarrow -4y = 4x \Leftrightarrow y = -x$

This is a straight line, mediator of the segment joining the points $z_1 = 2i$ and $z_2 = -2$. (See the figure bellow).



Lesson 8: Square root of a complex number

a) Prerequisites/Revision/Introduction:

- Given $z = a + bi$, let learners discuss and calculate z^2, the square of a complex number $z = a + bi$.

b) Teaching resources:

Internet and textbooks to facilitate research

c) Learning activities:

- From activity 1.9, let learners work in groups and solve $(a + ib)^2 = 6 - 4i$, where a and b are real numbers.
- Facilitate them to discuss the value of a and b and determine the square root of $z = 6 - 4i$. Help learners to solve the equation from activity 1.9 by matching similar terms of the equation both sides or identification technique and then solve the obtained similitaneous equations $\begin{cases} a^2 - b^2 = 6 \\ 2ab = -4 \end{cases}$
- While working in groups, help learners remember how to solve bi-quadratic equation of the form: $ax^4 + bx^2 + c = 0$.
- Facilitate learners to find solutions of $\begin{cases} a^2 - b^2 = 6 \\ 2ab = -4 \end{cases}$ by solving bi-quadratic equation $b^4 + 6b^2 - 4 = 0$. Let them suppose that $x = b^2$ and solve the quadratic equation $x^2 + 6b^2 - 4 = 0$
- Or alternativelly, facilitate Learners to find solutions of

  $$\begin{cases} a^2 - b^2 = 6 & (1) \\ 2ab = -4 & (2) \end{cases}$$

  by considering $|z|^2 = |z^2|$ and $a^2 + b^2 = 2\sqrt{13}$ $\quad (3)$
- Generally, harmonize the learners' group works and let them realize that when the complex number $z = x + iy$ is the square root of a complex number $a + bi$, this means that $(x + yi)^2 = (a + bi) \Rightarrow |x + iy|^2 = |a + ib| \Leftrightarrow x^2 + 2xyi - y^2 = a + ib.$

  Equality of two complex numbers gives $\begin{cases} x^2 - y^2 = a \\ 2xy = b \end{cases}$.

Therefore, $\begin{cases} x = \pm\sqrt{\frac{1}{2}\left(a+\sqrt{a^2+b^2}\right)} \\ y = \pm\sqrt{\frac{1}{2}\left(-a+\sqrt{a^2+b^2}\right)} \end{cases}$.

- Individually, let learners go through the example 1.9 and work out application activities 1.8 to emphasize their skills in determination of the square root of complex numbers

Application Activity 1.8

Solution

a) $-3+4i$, let us take $|x+iy|^2 = |-3+4i|$

$$\begin{cases} x = \pm\sqrt{\frac{1}{2}\left(a+\sqrt{a^2+b^2}\right)} \\ y = \pm\sqrt{\frac{1}{2}\left(-a+\sqrt{a^2+b^2}\right)} \end{cases}$$

or $\begin{cases} x = \pm\sqrt{\frac{1}{2}\left(-3+\sqrt{9+16}\right)} \\ y = \pm\sqrt{\frac{1}{2}\left(3+\sqrt{9+16}\right)} \end{cases} \Leftrightarrow \begin{cases} x = \pm\sqrt{1} = \pm 1 \\ y = \pm\sqrt{4} = \pm 2 \end{cases}$

Therefore, square roots of $-3+4i$ are $1+2i$ and $-1-2i$

b) $z = -2i$

It implies that $\begin{cases} a = 0 \\ b = -2 < 0 \end{cases}$ or $\begin{cases} x = \pm\sqrt{\frac{2}{2}} = \pm 1 \\ y = \pm\sqrt{\frac{2}{2}} = \pm 1 \end{cases}$

As b is less than zero, we take the different signs, and the square roots of $z = -2i$ are $1-i$ and $-1+i$

c) $z = 2-2i\sqrt{3}$

Find x and y.

$$\begin{cases} x = \pm\sqrt{\frac{1}{2}\left(2+\sqrt{4+12}\right)} = \pm\sqrt{3} \\ y = \pm\frac{1}{5}\sqrt{\frac{1}{2}\left(-2+\sqrt{4+12}\right)} = \pm 1 \end{cases}$$

As b is less than zero, we take the different signs of x and y:

Square roots of $z = 2 - 2i\sqrt{3}$ are $\sqrt{3} - i$ and $-\sqrt{3} + i$.

Lesson 9: Equations in the set of complex numbers

A. Simple equations of the form $Az + B = 0$ where A and B are two given complex numbers, $A \neq 0$

a) Prerequisites/Revision/Introduction:

- Give to learners different exercises on solving equation of the form $ax + b = 0$ in the set of real numbers.

b) Teaching resources:

Internet and textbooks to facilitate research

c) Learning activities:

- From the activity 1.10, facilitate learners to work in small groups and find the solution set of the equation $4z + 5i = 12 - i$ and let them find that:
  $4z = 12 - i - 5i \Leftrightarrow 4z = 12 - 6i \Rightarrow z = 3 - \frac{3}{2}i$
- From the activity 1.10 ,facilitate learners to have a general idea on how to find the solution set of the equation $Az + B = 0$ (where A and B are two complex numbers, A different to zero) by helping them to remember that they apply the same process as solving equation of the form $ax + b = 0$ in the set of real numbers. Let them find out that $Az + B = 0 \Rightarrow z = -\frac{B}{A}$ and z must be written in the form of $z = x + yi$
- Individually, let learners go through example1.10 and work out application activities 1.9, to emphasize their skills in solving equations of the form $Az + B = 0$.

Application Activity 1.9

Solution:

1) If $(1+3i)z = 2i + 4i$; then $z = \frac{2i + 4i}{1+3i} = \frac{9}{5} + \frac{3i}{5}$

2) $\begin{cases} 7z + (8-2i)w = 4 - 9i \\ (1+i) + (2-i)w = 2 + 7i \end{cases}$.

From the the second equation, $w=\frac{8}{5}+\frac{11i}{5}$, then replaceing w by its value in the first equation, we get the value of z: $z=\frac{23}{35}+\frac{59}{35}i$.

B. Quadratic equations of the form $Az^2+Bz+C=0$ where A, B and C are three given complex numbers

a) Prerequisites/Revision/Introduction:

- Give learners different exercises on solving equation of the form $Az^2+B=0$ in the set of real numbers.

b) Teaching resources:

Internet and textbooks to facilitate research

c) Learning activities:

- From the activity 1.11, facilitate learners to work in small groups and find the solution set of the equation $4z+5i=12-i$ and let them find that: $4z=12-i-5i \Leftrightarrow 4z=12-6i \Rightarrow z=3-\frac{3}{2}i$
- From the activity 1.11 , facilitate learners to have a general idea on how to find the solution set of simple quadratic equations in the set of complex numbers. Let them remember that it recalls the procedure of how to solve the quadratic equations in the set of real numbers considering that $\sqrt{-1}=i$. There fore, $Az^2+C=0 \Rightarrow z^2=\frac{-c}{A} \Leftrightarrow z=\sqrt{\frac{-c}{A}}$
- Through group activiy, help learners to solve the equation of the form $Az^2+Bz+C=0$, $(A\neq 0)$ and lead them to the following: $z=\frac{-B\pm\sqrt{B^2-4AC}}{2A}$

  . By considering the case where A, B, and C are the real numbers and A different to zero, facilitate sudents to find out that the equation $Az^2+Bz+C=0$ has either two real roots, one double real root or two conjugate complex roots depending on the value of $\Delta=B^2-4AC$

1) If $\Delta=0$, there is a double real root $z_1=z_2=\frac{-B}{2A}$
2) If $\Delta>0$, there are two distinct real roots $z_1=\frac{-B+\sqrt{\Delta}}{2A}$ and $z_2=\frac{-B-\sqrt{\Delta}}{2A}$
3) If $\Delta<0$, there is no real root. In this case, there are two conjugate complex roots $z_1=\frac{-B+i\sqrt{-\Delta}}{2A}$ and $z_2=\frac{-B-i\sqrt{-\Delta}}{2A}$

- Individually, let learnersgo through example1.11and work out application activities 1.10, to emphasize their skills in solving equations of the form $Az^2+Bz+C=0$

Application Activity 1.10

Solution for question 1

a) $z^2-(3+i)z+4+3i=0$

$$z=\frac{(3+i)\pm\sqrt{(3+i)^2-4.(4+3i)}}{2}=\frac{(3+i)\pm\sqrt{9+6i-1-16-12i}}{2}=\frac{(3+i)\pm\sqrt{-8-6i}}{2}$$

Let us first find $\sqrt{-8-6i}$:

$$x=\pm\sqrt{\frac{a+\sqrt{a^2+b^2}}{2}}=\pm\sqrt{\frac{-8+\sqrt{84+36}}{2}}=\pm1$$

$$y=\pm\sqrt{\frac{-a+\sqrt{a^2+b^2}}{2}}=\pm\sqrt{\frac{8+\sqrt{84+36}}{2}}=\pm3$$

$\sqrt{-8-6i}=1-3i$ or $-1+3i$.

Then $z=\dfrac{(3+i)\pm(1-3i)}{2}=2-i$ or $1+2i$.

b) $z^2+9=0$. It implies that $z=\pm3i$

Solution for question 2

$(z-4)$ is a factor of $z^3-15z-4$ if $z=4$ is a zero of $p(z)=z^3-15z-4$.

Or $p(4)=4^3-15(4)-4=64-60-4=0$;

Hence, $(z-4)$ is a factor of $z^3-15z-4$.

Since $(z-4)$ is a factor of $z^3-15z-4$, $z=4$ is one of the solutions of $z^3-15z-4=0$; other solutions could be found by factorization.

To get other factors, let us use synthetic division,

	1	0	−15	−4
4		4	16	4
	1	4	1	0

Thus, $z^3 - 15z - 4 = (z-4)(z^2 + 4z + 1)$

Solve $z^2 + 4z + 1 = 0$ by discriminant method:

$\Delta = 16 - 4 = 12$

Then,

$$z_1 = \frac{-4 + 2\sqrt{3}}{2} \text{ and } z_2 = \frac{-4 - 2\sqrt{3}}{2}, \text{thus, } z_1 = -2 + \sqrt{3} \text{ and } z_2 = -2 - \sqrt{3}$$

Therefore, the solutions of $z^3 - 15z - 4 = 0$ are $4,\ 2 + \sqrt{3}$ and $2 - \sqrt{3}$.

Lesson 10: Definition and properties of a complex number z in polar form

a) Prerequisites/Revision/Introduction:

From the activity 10.12, help learners to remember how to write a vector $\overrightarrow{M}$ using trigonometric relations. Using figure 1.10, let learners discuss and find that $a = r\cos\theta$ and $a = r\sin\theta$. Finally, facilitate learners to deduce that $M(r,\theta) = \mathrm{r}(\cos\theta + \sin\theta)$.

b) Teaching resources:

Internet and textbooks for research

c) Learning activities:

- From the activity 10.12 and the figure 1.10, let learners discuss in small groups to find out $\vec{r} = a\vec{e_1} + b\vec{e_2} = r\cos\theta\vec{e_1} + r\sin\theta\vec{e_2}$ and simillarly, facilitate them to discover that when given the complex number z with the affix $P(x,y)$ and the modulus $|z| = \sqrt{x^2 + y^2} = r$, the complex number z will be written as follow: $z = x + iy = r\cos\theta + ir\sin\theta = r(\cos\theta + i\sin\theta)$
- Help learners to understand that $z = r(\cos\theta + i\sin\theta)$ is called the polar form. Let them find out that its affix is represented by the coordinates (r,θ) where $r = \sqrt{x^2 + y^2}$ is its modulus and θ the angle between the corresponding vector and $x - axis$.

- Explain to the learners that the angle θ is called the argument of z and then given $z = x + iy$, the formula used to convert to the polar form are the following:

$$r = \sqrt{x^2 + y^2}\ ,\ x = r\cos\theta,\ y = r\sin\theta \text{ and } \theta = \arg(z) = \arctan\frac{y}{x}$$

- In small groups , let learnersgo through example 1.12 and work out application activities 1.11 to emphasize their skills in converting a complex number from algebraic form to polar form and vice versa.

Application activity 1.11

Solution for question 1

The table of trigonometric values and the scientific calculator can help the learner to find the argument of z.

Let θ be argument of the complex number z

a) If $z = -2i$ then $|z| = 2, \begin{cases} \cos\theta = 0 \\ \sin\theta = \frac{-2}{2} = -1 \end{cases}$, thus θ lies on negative $y - axis$ i.e. $\theta = -\frac{\pi}{2}[2\pi]$. The principal argument is $-\frac{\pi}{2}$.

b)If $z = 1 - i$ then $|z| = \sqrt{2}, \begin{cases} \cos\theta = \frac{1}{\sqrt{2}} \\ \sin\theta = \frac{-1}{\sqrt{2}} \end{cases}$, thus θ lies in 4th quadrant and $\theta = -\frac{\pi}{4}[2\pi]$.

The principal argument is $-\frac{\pi}{4}$.

c) If $z = 1 - i\sqrt{3}$ then $|z| = 2, \begin{cases} \cos\theta = \frac{1}{2} \\ \sin\theta = -\frac{\sqrt{3}}{2} \end{cases}$,thus θ lies in 4th quadrant and $\theta = -\frac{\pi}{3}[2\pi]$

. The principal argument is $-\frac{\pi}{3}$.

d)If $z = -1 + i\sqrt{3}$,

then $|z| = 2, \begin{cases} \cos\theta = -\frac{1}{2} \\ \sin\theta = \frac{\sqrt{3}}{2} \end{cases}$. Thus θ lies in 2nd quadrant and $\theta = \frac{2\pi}{3}[2\pi]$.

The principal argument is $\frac{2\pi}{3}$.

e)If $z=-\sqrt{3}-i$, $|z|=2,\begin{cases}\cos\theta=-\frac{\sqrt{3}}{2}\\ \sin\theta=-\frac{1}{2}\end{cases}$, thus θ lies in 3rd quadrant and $\theta=-\frac{5\pi}{6}[2\pi]$. The principal argument is $-\frac{5\pi}{6}$.

Solution for question 2

a) If $z=-1+i$ then $|z|=\sqrt{2},\begin{cases}\cos\theta=-\frac{1}{\sqrt{2}}\\ \sin\theta=\frac{1}{\sqrt{2}}\end{cases}$, thus θ lies on 2nd quadrant and $\theta=\frac{3\pi}{4}[2\pi]$. Therefore, $z=\sqrt{2}\left(\cos\frac{3\pi}{4}+i\sin\frac{3\pi}{4}\right)$.

b)If $z=-\frac{1}{2}+i\frac{\sqrt{3}}{2}$, then $|z|=1,\begin{cases}\cos\theta=-\frac{1}{2}\\ \sin\theta=\frac{\sqrt{3}}{2}\end{cases}$, thus θ lies in 2nd quadrant and $\theta=\frac{2\pi}{3}[2\pi]$. Therefore, $z=\sqrt{2}\left(\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3}\right)$.

c) $z=-2\left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)=-1-i\sqrt{3}$

$|z|=2,\begin{cases}\cos\theta=-\frac{1}{2}\\ \sin\theta=\frac{\sqrt{3}}{2}\end{cases}$, thus θ lies in 3rd quadrant and $\theta=-\frac{2\pi}{3}[2\pi]$.

Therefore, $z=2\left[\cos\left(-\frac{2\pi}{3}\right)+i\sin\left(-\frac{2\pi}{3}\right)\right]$.

If $z=2$ then $|z|=2,\begin{cases}\cos\theta=\frac{2}{2}=1\\ \sin\theta=0\end{cases}$, thus θ lies on positive $x-axis$ and $\theta=0[2\pi]$. Therefore, $z=2(\cos 0+\text{i}\sin 0)$.

e)If $z=-i$ then $|z|=1,\begin{cases}\cos\theta=0\\ \sin\theta=-1\end{cases}$, thus θ lies on negative $y-axis$ and $\theta=-\frac{\pi}{2}[2\pi]$. Therefore, $z=2\left[\cos\left(-\frac{\pi}{2}\right)+i\sin\left(-\frac{\pi}{2}\right)\right]$.

Solution for question 3

a) $z = 5cis270^0 = 5\left(\cos 270^0 + i\sin 270^0\right) = 0 + 5i$

b) $z = 4cis300^0 = 4\left(\cos 300^0 + i\sin 300^0\right) = 4\left[\cos\left(-60^0\right) + i\sin\left(-60^0\right)\right]$

$= 4\left(\frac{1}{2} - i\frac{\sqrt{3}}{2}\right) = 2 - 2i\sqrt{3}$

c) $z = \sqrt{2}cis\left(-\frac{\pi}{4}\right) = \sqrt{2}\left[\cos\left(-\frac{\pi}{4}\right) + i\sin\left(-\frac{\pi}{4}\right)\right]$

$= \sqrt{2}\left(\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}\right) = 1 - i$

d) $z = 3cis\left(\frac{\pi}{2}\right) = 3\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right) = 0 + 3i$

Lesson 11: Multiplication and division of complex numbers in polar form

a) Prerequisites/Revision/Introduction:

- Give exercises involving trigonometric identities:
  $\cos\phi\cos\theta \pm \sin\phi\sin\theta = \cos(\phi \mp \theta); \cos\phi\sin\theta \pm \cos\theta\sin\phi = \sin(\phi \pm \theta)$

- Given two complex numbers $z_1 = \sqrt{3} - i$ and $z_2 = 1 + i$, let learners work in groups or individually to find out the modulus and argument of each number.

b) Teaching resources:

Learner's book, Internet and textbooks for research

c) Learning activities:

- Write on the board the two complex numbers $z_1 = \sqrt{3} - i$ and $z_2 = 1 + i$. In small groups, ask learners to express these numbers in polar form, then compute the product $z_1.z_2$ and the quotient $\frac{z_1}{z_2}$ Ensure that learners use above trigonometric identities to come up with the following final results: $z_1 = 2(\cos\frac{\pi}{6} - i\sin\frac{\pi}{6})$ and $z_2 = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$
  Therefore $z_1.z_2 = z_1.z_2 = r_1.r_2\left[\cos\left(\theta_1 + \theta_2\right) + i\sin\left(\theta_1 + \theta_2\right)\right]$ and
  $\frac{z_1}{z_2} = \frac{r_1}{r_2}\left[\cos\left(\theta_1 - \theta_2\right) + i\sin\left(\theta_1 - \theta_2\right)\right]$. Ask learners to express these results in their own words, e.g., the modulus of the product of two complex numbers is

the product of the moduli and the argument is the sum of arguments.

- Let learners go through the example 1.13. Individually, ask them to work out application activities 1.12 to develop their skills and increase their self confidence in calculating the product and quotient of complex numbers in polar form.

Application Activity 1.12

Solution

Conversion in polar form

1) $z=1+i$. We have $|z|=\sqrt{2},\begin{cases}\cos\theta=\frac{1}{\sqrt{2}}\\ \sin\theta=\frac{1}{\sqrt{2}}\end{cases}$, thus θ lies in 1st quadrant and $\theta=\frac{\pi}{4}[2\pi]$

.Therefore, $z=\sqrt{2}\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)$

2) $w=-\sqrt{3}+i$ then $|w|=2,\begin{cases}\cos\theta=-\frac{\sqrt{3}}{2}\\ \sin\theta=\frac{1}{2}\end{cases}$, thus θ lies in the 2nd quadrant and

$\theta=\frac{5\pi}{6}[2\pi]$. Therefore, $z=2\left(\cos\frac{5\pi}{6}+i\sin\frac{5\pi}{6}\right)$.

3) $y=-3+i\sqrt{3}$, $|y|=2\sqrt{3},\begin{cases}\cos\theta=-\frac{\sqrt{3}}{2}\\ \sin\theta=\frac{1}{2}\end{cases}$, thus θ lies in 2nd quadrant and $\theta=\frac{5\pi}{6}[2\pi]$.

Therefore, $y=2\sqrt{3}\left(\cos\frac{5\pi}{6}+i\sin\frac{5\pi}{6}\right)$.

Computation for operations

1) $z\cdot w=2\sqrt{2}\left(\cos\frac{13\pi}{12}+i\sin\frac{13\pi}{12}\right)=2\sqrt{2}\left[\cos\left(-\frac{11\pi}{12}\right)+i\sin\left(-\frac{11\pi}{12}\right)\right]$

2) $z\cdot y=2\sqrt{6}\left(\cos\frac{13\pi}{12}+i\sin\frac{13\pi}{12}\right)=2\sqrt{6}\left[\cos\left(-\frac{11\pi}{12}\right)+i\sin\left(-\frac{11\pi}{12}\right)\right]$

3) $\frac{w}{y}=\frac{2}{2\sqrt{3}}(\cos 0+i\sin 0)=\frac{\sqrt{3}}{3}$.

4) $\frac{y}{z}=\frac{2\sqrt{3}}{\sqrt{2}}\left(\cos\frac{7\pi}{12}+i\sin\frac{7\pi}{12}\right)=\sqrt{6}\left(\cos\frac{7\pi}{12}+i\sin\frac{7\pi}{12}\right)$.

Lesson 12: Powers and De Moivre's formula

a) Prerequisites/Revision/Introduction:

- From activity 1.14, let learners work individually and convert $z=1-i\sqrt{3}$ in polar form.

b) Teaching resources:

Learner's textbook and other reference books for research.

c) Learning activities:

- After checking whether all learners have converted $z=1-i\sqrt{3}$ in polar form accurately, let learners work in groups and calculate z^2 .
- In the same groups, ask learners to determine $z^3=z^2.z$ in polar form
- From different products, let learners deduce **De Moivre's theorem:** $(\cos\theta+i\sin\theta)^n=\cos(n\theta)+i\sin(n\theta)$.

Let learners go through the example 1.14 and work out application activities 1.13 to develop independently their skills in calculating the power of complex numbers in polar form.

Application Activity 1.13

Solution for question 1

a) $(\cos 3\pi+i\sin 3\pi)^9=\cos 27\pi+i\sin 27\pi=\cos\pi+i\sin\pi=-1$

b) $\left(\cos\frac{5\pi}{2}+i\sin\frac{5\pi}{2}\right)^{\frac{2}{5}}=\cos\pi+i\sin\pi=-1$

c) $\left(\cos\frac{\pi}{5}+i\sin\frac{\pi}{5}\right)^5=\cos\pi+i\sin\pi=-1$

d) $(\cos 45^0+i\sin 45^0)^2=\cos 90^0+i\sin 90^0=i$

e) $(\cos 60^0+i\sin 60^0)^3=\cos 180^0+i\sin 180^0=-1$

Solution for question 2

$$a)\left(1+i\sqrt{3}\right)^6=2^6\left(\cos\frac{6\pi}{3}+i\sin\frac{6\pi}{3}\right)=2^6\left(\cos 2\pi+i\sin 2\pi\right)=2^6$$

$$b)\left(-\sqrt{3}+i\right)^{10}=2^{10}\left(\cos\frac{50\pi}{6}+i\sin\frac{50\pi}{6}\right)=2^{10}\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)=2^9\left(1+i\sqrt{3}\right)$$

$$c)\left(1-i\right)^7=\left(\sqrt{2}\right)^7\left[\cos\left(-\frac{7\pi}{4}\right)+i\sin\left(-\frac{7\pi}{4}\right)\right]=2^3\sqrt{2}\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)=8\left(1+i\right)$$

Lesson 13: Definition and properties of a complex number *z* in exponential form

a) Prerequisites/Revision/Introduction:

Using examples and activities help learners to review exponential concept and how to solve simple exponential equations learned in S5.

b) Teaching resources:

Learner's book, internet and other appropriate textbooks for research

c) Learning activities:

- Guide learners to make a research in the library or on the internet wherever possible to find out the relationship between complex number in trigonometric form with exponential function, that is, expressing $\cos\theta+i\sin\theta$ into $e^{i\theta}$. Considering that $z=\cos\theta+i\sin\theta$ is the polar form of a complex number, let learners find out the form of $z=e^{i\theta}$. Inform stundents that this notation is due to the mathematician Euler.
- Given that the voltage is expressed by $U\left(t\right)=U_0\,e^{i\omega t}$ where ω is the the angular frequency which is related to the frequency f by $\omega=2\pi f$ and t the time for which the voltage appears somewhere in the AC circuit , ask Learners to work in groups and write this voltage form as a complex number in the polar and deduce its modulus and the argument.
- From the findings of activity 1.15, harmonize the group findings by explaining that any complex number z of modulus $|z|$ and argument θ can be written as $z=|z|\left(\cos\theta+i\sin\theta\right)=|z|e^{i\theta}$.
- Explain to learners that $z=r.e^{i\theta}$ is called an exponential form of the complex number z, where r and θ are the modulus and the argument of z respectively and then let them know that all powers' properties applicable for other forms of complex numbers are also applicable for exponential form.
- Let learners go through the example 1.15 and work out application activities 1.14,

to emphasize their skills in converting a complex number in exponential form and vice versa.

- By means of examples, facilitate learners to be interested by the application of complex numbers in a range of real situations.

Application Activity 1.14

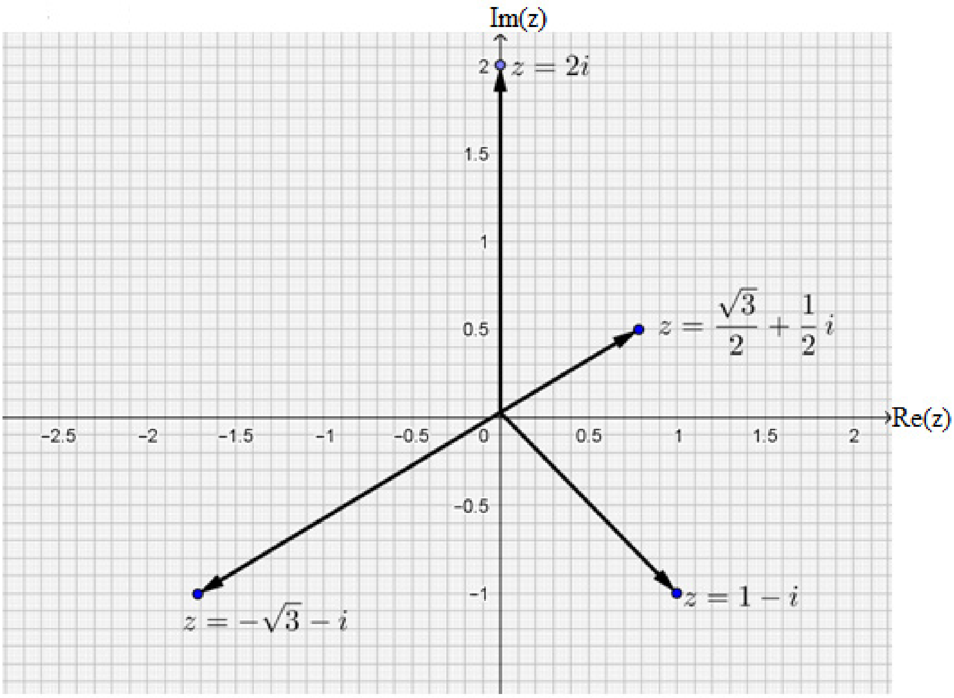
Solution for question 1

$a)1-\mathrm{i}=\sqrt{2}e^{-\frac{\pi}{4}i}$

$\mathrm{b)}\,2\mathrm{i}=2e^{\frac{\pi}{2}i}$

$\mathrm{c)}\frac{\sqrt{3}}{2}+\frac{1}{2}i=e^{\frac{\pi}{6}i}$

$\mathrm{d)}-\sqrt{3}-i=2e^{-\frac{5\pi}{6}i}$



Solution for question 2

a) $z=e^{i\frac{\pi}{3}}=\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}=\frac{1}{2}+i\frac{\sqrt{3}}{2}$

b) $z=e^{-\frac{\pi}{4}i}=\cos\left(-\frac{\pi}{4}\right)+i\sin\left(-\frac{\pi}{4}\right)=\frac{\sqrt{2}}{2}-i\frac{\sqrt{2}}{2}$

c) $z=3e^{i\frac{\pi}{6}}=3\left(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right)=\frac{\sqrt{3}}{2}+\frac{1}{2}i$

d) $z = 2e^{i\frac{2\pi}{3}} = 3\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right) = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$

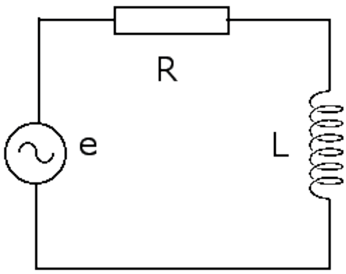
Solution for question 3

a) $z = -1 + i\sqrt{3} = 2.e^{i\frac{2\pi}{3}}$ b) $z = 3 + 4i = 5.e^{i\arctan\frac{4}{3}}$

c) $z = 2 - 2i = 2\sqrt{2}.e^{i\frac{\pi}{4}}$ d) $z = -3 + i\sqrt{3} = 2\sqrt{3}.e^{i\frac{5\pi}{6}}$

Solution for question 4

The RL series in an alternating current circuit is illustrated in the following figure.



a) The e.m.f. that is supplied to the circuit is distributed between the resistor and the inductor.

Since the elements are in series the common current is taken to have the reference phase $I = I_m.e^{j\omega t}$

Adding the potentials around the circuit:

$$E = V_R + V_L = RI + j\omega LI = (R + j\omega L)I = (R + j\omega L)I_m e^{j\omega t}$$

$$= \sqrt{R^2 + (\omega L)^2}\, e^{j\arctan\frac{\omega L}{R}} \cdot I_m \cdot e^{j\omega t} = \sqrt{R^2 + (\omega L)^2} \cdot I_m \cdot e^{j\left(\omega t + \arctan\frac{\omega L}{R}\right)}$$

$$= \sqrt{R^2 + (\omega L)^2} \cdot I_m \cdot e^{j(\omega t + \theta)} = |Z| \cdot I_m \cdot e^{j(\omega t + \theta)}$$

b) In a) we have found that $E = \sqrt{R^2 + (\omega L)^2} \cdot I_m \cdot e^{j(\omega t + \theta)} = |Z| \cdot I_m \cdot e^{j(\omega t + \theta)}$,

therefore the phase is $\theta = \arctan\frac{\omega L}{R}$.

The physical current and potentials are:

$$i = \operatorname{Im}\{I_m e^{j\omega t}\} = I_m \sin\omega t\,; V_R = \operatorname{Im}\{RI_m e^{j\omega t}\} = RI_m \sin\omega t\,;$$

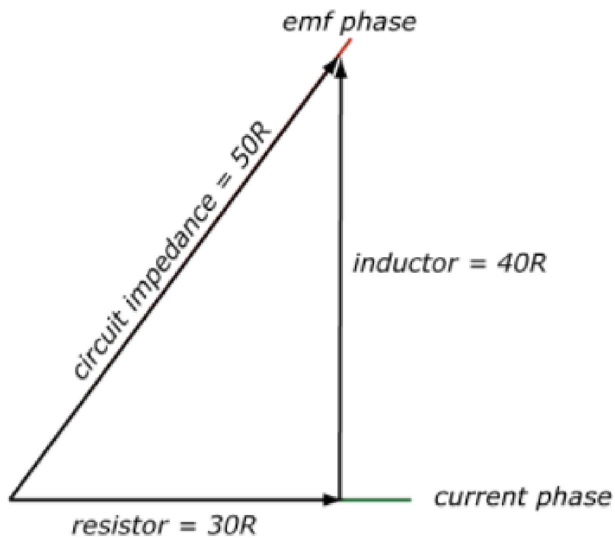
(1) $V_L = \operatorname{Im}\left\{ j\omega L I_m e^{j\omega t} \right\} = \omega L I_m \sin\left(\omega t + \frac{\pi}{2} \right)$;

$e = \operatorname{Im}\left\{ Z I_m e^{j(\omega t+\theta)} \right\} = Z I_m \sin\left(\omega t + \theta \right)$.

$\omega = 2\pi \cdot \frac{1000}{\pi} = 2000 \frac{rad}{s}$;

as $potential\ difference = 100V$ then $E_{\max} = 100\sqrt{2} = 141V \Rightarrow E = 141 e^{j2000t}$

The **complex impedance** $Z = R + j\omega L = 30 + j40$, or $Z = 50e^{0.93j}$



(2) The complex current $I = \frac{E}{Z} = \frac{141}{50} e^{j(2000t-0.93)} = 2.82 e^{j(2000t-0.93)}$,

The physical current $i = 2.82 \sin\left(2000t - 0.93 \right) A$,

The rms current or equivalent dc current is $\left| I \right| = \frac{2.82}{\sqrt{2}} = 2A$ and has no phase.

(3) Across the resistor R:

The complex potential difference is $V_R = R.I = 30 \times 2.82 e^{j(2000t-0.93)} = 84.8 e^{i(2000t-93)}$

, the physical is $V_R = 84.8 \sin\left(2000t - 0.93 \right) Volts$, the one equivalent to dc current (rms potential difference) is $V_R = \frac{84.8}{\sqrt{2}} volts = 60\ volts$.

(4) Across the inductor L:

The complex potential difference is

$$V_L = j\omega L.I = 40j \times 2.82e^{j(2000t+0.64)} = 40 \times 2.82e^{j\left(2000t-0.93+\frac{\pi}{2}\right)} = 112.8e^{j(2000t+0.64)}$$

The physical pd is $V_L = 112.8\sin(2000t + 0.64)\,Volts$, the one equivalent to dc current (rms potential difference) is $V_L = \frac{112.8}{\sqrt{2}} volts = 80\ volts$.

Lesson 14: Euler's formulae

a) Prerequisites/Revision/Introduction:

- Facilitate learners to remember De Moivre's formula or theorem and convert the polar form of a complex number into exponential form.

b) Teaching resources:

Learner's textbook and any textbooks to enable research

c) Learning activities:

- Organise learners in groups to work out the activity 1.16. Basing on the group findings and using question-answer technique, facilitate learners to realize that $\cos\theta = \frac{1}{2}\left(e^{i\theta} + e^{-i\theta}\right)$ and $sin\theta = \frac{1}{2i}\left(e^{i\theta} - e^{-i\theta}\right)$. Inform learners that these identities are called Euler's formulae.
- In groups, let learners go through the example 1.16. Individually learners work out application activities 1.15 to reinforce their learning and develop their mathematical skills. Ensure all learners are capable to achieve high level of performance through providing adequate feedback to individual learner.

Application Activity 1.15

Solution

$$a)\quad \sin^2 x\cos x = \left(\frac{e^{ix} - e^{-ix}}{2i}\right)^2 \cdot \left(\frac{e^{ix} + e^{-ix}}{2}\right) = \frac{\cos 3x}{4} - \frac{\cos x}{4}$$

$$b)\quad \cos^2 x\cos y = \frac{\cos(2x+y)}{4} + \frac{\cos y}{2} + \frac{\cos(2x-y)}{4}$$

$$c)\quad \cos^3 x = \left(\frac{e^{ix} + e^{-ix}}{2}\right)^3 = \frac{\cos 3x}{4} + \frac{3\cos x}{4}$$

Lesson 15: Application of complex numbers in other sciences

a) Prerequisites/Revision/Introduction:

- Facilitate learners to remember how to convert the algebraic form of a complex number into the polar form and then into the exponential form.

b) Teaching resources:

Internet or textbooks for research

c) Learning activities

- In groups, depending on the school facilities, let learners explore internet or textbooks to look for different applications of complex numbers in other sciences especially in Physics. This intends to support learners in developing their skills and increasing their curiosity about complex numbers and their applications in daily situations.

1.6 Unit summary

1. Definitions

- **Complex number**

Given two real numbers *a* and *b*, the complex number *z* is a number $z = a + ib$ where the number *a* is called the **real part** of *z* and the number *b* is called the **imaginary part** of *z* and $i^2 = -1$.

The set of all complex numbers is $\mathbb{C} = \left\{ z = a + ib,\ where\ a, b \in \mathbb{R}\ and\ i^2 = -1 \right\}$.

- **Conjugate of a complex number**

Conjugate of complex number $z = a + bi$ is its reflection by $x - axis$ and is denoted and defined by $\bar{z} = a - bi$.

- **Modulus of a complex number**

The modulus of $z = a + bi$ is a positive real number denoted by $|z|$, such that $|z| = \sqrt{z \cdot \bar{z}} = \sqrt{a^2 + b^2}$.

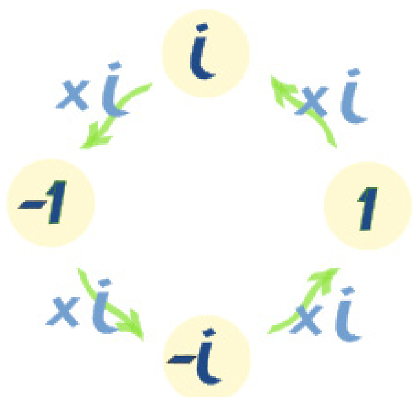
- **Argument of a complex number**

Argument of a complex number $z = a + bi$ denoted by θ is an angle between a vector of affix $z = a + bi$ with $positive\ x - axis$ and defined as foll

$$\theta = arg(z) = \begin{cases} \arctan\frac{b}{a}, if\ a > 0 \\ \pi + \arctan\frac{b}{a}, if\ a < 0\ and\ b > 0 \\ -\pi + \arctan\frac{b}{a}, if\ a < 0\ and\ b < 0 \\ \frac{\pi}{2}, if\ a = 0\ and\ b > 0 \\ -\frac{\pi}{2}, if\ a = 0\ and\ b < 0 \\ Undefined, if\ a = 0\ and\ b = 0 \end{cases}$$

2. Properties of the imaginary number "i"

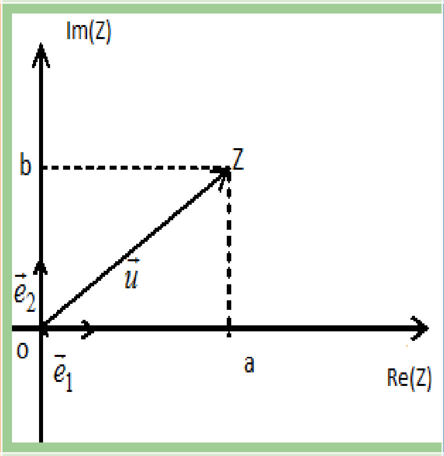
The imaginary unit, i , "cycles" through 4 different values $\{1, i, -1, -i\}$ each time we multiply.



$$i^{4n} = 1, \quad i^{4n+1} = i, i^{4n+2} = -1, i^{4n+3} = -i.$$

3. Presentation of a complex number

There are four different ways of presenting a complex number: algebraic form, geometric form, polar form and exponential form as illustrated in the following table

Form	Formula/Graphical representation	Additional information
Algebraic	$z = a + bi$	$a, b \in \mathbb{R}$ and $i^2 = -1$
Geometric		$x - axis \equiv \operatorname{Re}(z)$ and $y - axis \equiv \operatorname{Im}(z)$
Polar	$z = r(\cos\theta + i\sin\theta)$ or $z = rcis\theta$ or $r\angle\theta$	$r = \|z\|, and\ \theta = \arg(z)$
Exponential	$z = re^{i\theta}$	$r = \|z\|, and\ \theta = \arg(z)$

4. Operations in set of complex numbers

- **Addition and subtraction in the set of complex numbers**

To perform **addition** and **subtraction** of complex numbers we combine real parts together and imaginary parts separately:

Given two complex numbers $z_1 = a + bi\ and\ z_2 = c + di$, then

$z_1 + z_2 = (a + c) + (b + d)i$ and $z_1 - z_2 = (a - c) + (b - d)i.$

- **Multiplication, division and powers of complex numbers**

The **product** of two complex numbers $z_1 = a + bi\ and\ z_2 = c + di$ is given by:

$z_1 . z_2 = (a + bi)(c + di) = (ac - bd) + i(ad + cb);$

The division of two complex numbers $z_1 = a + ib$ and $z_2 = c + id$

$$\frac{z_1}{z_2}=\frac{a+bi}{c+di}=\frac{(ac+bd)+(bc-ad)i}{c^2+d^2}$$

The **power** n of $z=a+bi$ is given by: $z^n=\underbrace{z...z}_{n\,times}=(a+ib)...(a+ib)=(a+ib)^n$

Square roots of a complex number

When the complex number $z=x+yi$ is the square root of a complex number $a+bi$,

this means that $\begin{cases} x=\pm\sqrt{\frac{1}{2}\left(a+\sqrt{a^2+b^2}\right)} \\ x=\pm\sqrt{\frac{1}{2}\left(-a+\sqrt{a^2+b^2}\right)} \end{cases}$.

Multiplication or division of complex numbers in polar form and exponential form

From two complex numbers in polar form $z_1=r_1cis\theta_1$ and $z_2=r_2cis\theta_2$,

The product of z_1 and z_2 is $z_1\cdot z_2=\underbrace{r_1\cdot r_2cis(\theta_1+\theta_2)}_{polar\ form}=\underbrace{r_1\cdot r_2e^{i(\theta_1+\theta_2)}}_{\exp onential\ form}$;

The quotient of z_1 and z_2 is $\frac{z_1}{z_2}=\underbrace{\frac{r_1}{r_2}cis(\theta_1-\theta_2)}_{polar\ form}=\underbrace{\frac{r_1}{r_2}e^{i(\theta_1-\theta_2)}}_{\exp onential\ form}$

De Moivre's formula and Power of a complex number using the polar/ exponential form

Given the complex number $z=r(\cos\theta+i\sin\theta)$, then, $z^n=r^n\left[\cos(n\theta)+i\sin(n\theta)\right]=r^ne^{in\theta}$

For $r=1$, we have **De Moivre's formula that** is $(\cos\theta+i\sin\theta)^n=\cos n\theta+i\sin n\theta$

Euler's formula

From polar form and exponential form of a complex number $r\angle\theta$, you get the following formulae called Euler's formulae: $\begin{cases} \cos\theta=\frac{1}{2}\left(e^{i\theta}+e^{-i\theta}\right) \\ \sin\theta=\frac{1}{2i}\left(e^{i\theta}-e^{-i\theta}\right) \end{cases}$.

Applicationof complex numbers in other sciences

Complex numbers are applied in other sciences to express certain variables or facilitate the calculation in complicated expressions. They are mostly used in Electrical

Engineering, electronics engineering, Signal analysis, Quantum mechanics, Relativity, Applied mathematics, Fluid dynamics, Electromagnetism, Civil and Mechanical Engineering.

In sciences, it is better to use j as the imaginary number instead of using i to avoid the confusion of the expression of the imaginary number and the expression of the electrical current i.

In alternating current, if the angular velocity of the wire is $\omega = 2\pi f$, respective impedances are $Z_R = R, Z_L = j\omega L$ and $\frac{1}{j\omega C}$,their modulus are the **resistance** R, the **capacitive reactance** is $|Z_C| = \frac{1}{\omega C}$ and the **inductive reactance** is given by $|Z_L| = \omega L$.

1.7. Additional information for the teacher

Application of complex numbers in other sciences

Complex numbers are applied in other subjects to express certain variables or facilitate the calculation in complicated expressions. They are mostly used in electrical engineering, electronics engineering, signal analysis, quantum mechanics, relativity, applied mathematics, fluid dynamics, electromagnetism, civil and mechanical engineering.

Application in electronics engineering

Information that expresses a single dimension, such as linear distance, is called a scalar quantity. Scalar numbers are the kind of numbers that learners use most often. In relation to science, the voltage produced by a battery, the resistance of a piece of wire (ohms), and current through a wire (amps) are scalar quantities.

When electrical engineers analysed alternating current circuits, they found that quantities of voltage, current and resistance (called impedance in AC) were not the familiar one-dimensional scalar quantities that are used when measuring DC circuits. These quantities which now alternate in direction and amplitude possess other dimensions (frequency and phase shift) that must be taken into account.

In order to analyse AC circuits, it became necessary to represent multi-dimensional quantities. In order to accomplish this task, scalar numbers were abandoned and complex numbers were used to express the two dimensions of frequency and phase shift at one time.

In mathematics, i is used to represent imaginary unit. In the study of electricity and electronics, j is used to represent imaginary unit to avoid confusion with i which represents current in electronics. It is also customary for scientists to write the complex number in the form $a + jb$.

Complex numbers play a great role in electronics. The main reason for this is that they make the whole topic of analysing and understanding alternating signals much easier.

We can now consider oscillating currents and voltages as being complex values that have a real part we can measure and an imaginary part which we can't. At first it seems pointless to create something we can't see or measure, but it turns out to be useful in a number of ways.

1. It helps us understand the behaviour of circuits which contain reactance (produced by capacitors or inductors) when we apply A.C signals.
2. It gives us a new way to think about oscillations. This is useful when we want to apply concepts like the conservation of energy to understanding the behaviour of systems which range from a simple mechanical pendulum to a quartz-crystal oscillator.

In Signal analysis

Complex numbers are used in signal analysis and other fields for a convenient description for periodically varying signals. For given real functions representing actual physical quantities, often in terms of sines and cosines, corresponding complex functions are considered of which the real parts are the original quantities. For a sine wave of a given frequency, the absolute value $|Z|$ of the corresponding z is the amplitude and the argument $\arg(z)$ the phase.

If Fourier analysis is employed to write a given real-valued signal as a sum of periodic functions, these periodic functions are often written as complex valued functions of the form $\omega f(t) = z$ where ω represents the angular frequency and the complex number Z encodes the phase and amplitude as explained above.

In Quantum mechanics

The complex number field is relevant in the mathematical formulation of quantum mechanics, where complex Hilbert spaces (infinite dimensional space over $\mathbb{C}$) provide the context for one such formulation that is convenient and perhaps most standard. The original foundation formulas of quantum mechanics - the Schrödinger equation and Heisenberg's matrix mechanics - make use of complex numbers.

The quantum theory provides a quantitative explanation for two types of phenomena that classical mechanics and classical electrodynamics cannot account for:

Some observable physical quantities, such as the total energy of a black body, take on discrete rather than continuous values. This phenomenon is called quantization, and the smallest possible intervals between the discrete values are called quanta (singular: quantum, from the Latin word for "quantity", hence the name "quantum mechanics."). The size of the quanta typically varies from system to system.

Under certain experimental conditions, microscopic objects like atoms or electrons exhibit wave-like behaviour, such as interference. Under other conditions, the same species of objects exhibit particle-like behaviour ("particle" meaning an object that can be localized to a particular region of space), such as scattering. This phenomenon is known as wave-particle duality.

In Relativity

In special and general relativity, some formulas for the metric on space time become simpler if one takes the time variable to be imaginary. (This is no longer standard in classical relativity but is used in an essential way in quantum field theory.) Complex numbers are essential to spinors, which are a generalization of the tensors used in relativity.

In Applied mathematics

In differential equations, it is common to first find all complex roots r of the characteristic equation of a linear differential equation and then attempt to solve the system in terms of base functions of the form $f(t)=e^{rt}$.

1.8.End unit assessment

This part provides the answers of end unit assessment activities designed in integrative approach to assess the key unit competence with cross reference to the textbook.

The following are standards (ST) which have been based on in setting end unit assessment questions:

ST1: Correctly apply the properties of complex numbers to perform operations on complex numbers in algebraic form, in polar form or in exponential form; reflecting analytically and logically about their learning and setting their own goals.

ST2: Accurately find and achieve the best possible solution to problems involving sets of complex numbers, having evaluated a range of alternatives.

ANSWER FOR QUESTION ONE

1. $a)\quad z_1+z_2=16+11i$; $\quad b)\quad \frac{z_1}{z_2}=\frac{(6+3i)(10-8i)}{164}=84-18i$

$c)\ z_1.z_2=36+78i$

$d)\quad \left(z_1-\overline{z_2}\right)\left(z_1+\overline{z_2}\right)=\left(6+3i-10+8i\right)\left(6+3i+10-8i\right)=-9+196i$

II.

$$z=R+j\omega L+\frac{1}{j\omega C}\Leftrightarrow z=R+j\omega L+\frac{-j\omega C}{j\omega C\left(-j\omega C\right)}$$

$$\Leftrightarrow z=R+j\omega L-\frac{j\omega C}{\omega^2C^2}\Leftrightarrow z=R+j\omega L-\frac{j}{\omega C}$$

$$\Leftrightarrow z=R+j(\omega L-\frac{1}{\omega C})$$

When $R=10, L=5, C=0.04$ and $\omega=4$, we get

$$\text{z}=10+\text{j}(4\times 5-\frac{1}{0.04\times 4})$$

$$z=10+j(20-\frac{1}{0.16})$$

$$z=10+j13.75$$

III. a) $z=3+3i=3\sqrt{2}\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)=3\sqrt{2}.e^{i\frac{\pi}{4}}$

b) $\left(3+3i\right)^5=\left[3\sqrt{2}\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)\right]^5=\left(3\sqrt{2}\right)^5\left(\cos\frac{5\pi}{4}+i\sin\frac{5\pi}{4}\right)=\left(3\sqrt{2}\right)^5 e^{i\frac{5\pi}{4}}$

c) Let $z=\left(x+yi\right)$ be square root of $Z=3+3i$ then, $\left(x+yi\right)^2=3+3i$

$x^2-y^2+2xyi=3+3i$

Equate real parts and imaginary parts to get:

$$x^2-y^2=3\ (1)$$

$$2xy=3\quad (2)$$

$$x^2+y^2=\sqrt{3^2+3^2}=\sqrt{18}=3\sqrt{2}$$

Solving (1) and (3) to get: $\begin{cases} x^2-y^2=3 \\ x^2+y^2=3\sqrt{2}\end{cases}\Leftrightarrow\begin{cases} x=\pm\sqrt{\dfrac{3+3\sqrt{2}}{2}} \\ y=\sqrt{\dfrac{-3+3\sqrt{2}}{2}}\end{cases}$

Then in algebraic form $z=\pm\left(\sqrt{\dfrac{3+3\sqrt{2}}{2}}+i\sqrt{\dfrac{-3+3\sqrt{2}}{2}}\right)$

ANSWER FOR QUESTION TWO

Let $z = r(\cos\theta + i\sin\theta)$ be complex number in polar form , and

$i = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2}$, Then,

$$r(\cos\theta + i\sin\theta)\times\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right) = r\cos\left(\theta+\frac{\pi}{2}\right) + i\sin\left(\theta+\frac{\pi}{2}\right)\pi$$

$\therefore z$ is rotated in anticlockwise with angle $\frac{\pi}{2}$.

ANSWER FOR QUESTION THREE

$$a)\quad \sin^2 x\cos x = \left(\frac{e^{ix}-e^{-ix}}{2i}\right)^2 . \left(\frac{e^{ix}+e^{-ix}}{2}\right) = \frac{e^{2ix}-2+e^{-2ix}}{-4}\cdot\frac{e^{ix}+e^{-ix}}{2}$$

$$= \frac{e^{3ix}-2e^{ix}+e^{-ix}+e^{ix}-2e^{-ix}+e^{-3ix}}{-8} = \frac{e^{3ix}-e^{ix}-e^{-ix}+e^{-3ix}}{-8}$$

$$= \frac{1}{4}\left(\frac{e^{3ix}+e^{-3ix}}{2} - \frac{e^{ix}+e^{-ix}}{2}\right) = \frac{1}{4}\cos 3x - \frac{1}{4}\cos x$$

$$\text{b)}\quad \cos^2 x\sin x = \left(\frac{e^{ix}+e^{-ix}}{2}\right)^2 . \left(\frac{e^{ix}-e^{ix}}{2i}\right) = \frac{e^{2ix}+2+e^{-2ix}}{4}\cdot\frac{e^{ix}-e^{-ix}}{2i}$$

$$= \frac{e^{3ix}+2e^{ix}+e^{-ix}-e^{ix}-2e^{-ix}-e^{-3ix}}{8i} = \frac{\left(e^{3ix}-e^{-3ix}\right)+\left(e^{ix}-e^{-ix}\right)}{8i}$$

$$= \frac{1}{4}\left(\frac{e^{3ix}-e^{-3ix}}{2i} + \frac{e^{ix}-e^{-ix}}{2i}\right) = \frac{1}{4}\left(\sin 3x + \sin x\right) = \frac{1}{4}\sin 3x + \frac{1}{4}\sin x$$

$$\text{c)}\sin^2 x\cos^2 x = \left(\frac{e^{ix}-e^{-ix}}{2i}\right)^2\left(\frac{e^{ix}+e^{-ix}}{2}\right)^2$$

$$= \left(\frac{e^{2x}-2e^{ix}e^{-ix}+e^{-2ix}}{-4}\right)\left(\frac{e^{2ix}+2e^{ix}e^{-ix}+e^{-2ix}}{4}\right)$$

$$= \left(\frac{e^{2ix}-2+e^{-2ix}}{-4}\right)\left(\frac{e^{2ix}+2+e^{-2ix}}{4}\right)$$

$$= \left(\frac{e^{4ix}+2e^{2ix}+1-2e^{2ix}-4-2e^{-2ix}+1+2e^{-2ix}+e^{-4ix}}{-16}\right)$$

$$=\left(\frac{e^{4ix}+e^{-4ix}-2}{-16}\right)=-\frac{1}{8}\left(\frac{e^{4ix}+e^{-4ix}}{2}\right)-\frac{2}{-16}=-\frac{1}{8}\cos 4x+\frac{1}{8}=\frac{1}{8}-\frac{1}{8}\cos 4x$$

d) $$\sin^3 x=\left(\frac{e^{ix}-e^{-ix}}{2i}\right)^3=\left(\frac{e^{3ix}-3e^{2ix}e^{-ix}+3e^{ix}e^{-2ix}-e^{-3ix}}{-8i}\right)$$

$$=\left(\frac{e^{3ix}-3e^{ix}+3e^{-ix}-e^{-3ix}}{-8i}\right)=\frac{\left(e^{3ix}-e^{-3ix}\right)-3\left(e^{ix}-e^{-ix}\right)}{-8i}$$

$$=-\frac{1}{4}\left(\frac{e^{3ix}-e^{-3ix}}{2i}-3\frac{e^{ix}-e^{-ix}}{2i}\right)=-\frac{1}{4}\left(\sin 3x-3\sin x\right)=\frac{3}{4}\sin x-\frac{1}{4}\sin 3x$$

ANSWER FOR QUESTION FOUR

Analytically/ numerically, let O be the starting point and A, B and C be the turning points in northeast, west of north and south of west respectively. The result of all displacements is represented by the vector $\overrightarrow{OA}+\overrightarrow{AB}+\overrightarrow{BC}=\overrightarrow{OC}$.

Or $\overrightarrow{OA}=12\left(\cos 45^0+i\sin 45^0\right)=12e^{\frac{\pi}{4}i}$, $\overrightarrow{AB}=20\left[\cos\left(30^0+90^0\right)+i\sin\left(30^0+90^0\right)\right]=20e^{\frac{2\pi}{3}i}$

and $\overrightarrow{BC}=18\left[\cos\left(60^0+180^0\right)+i\sin\left(60^0+180^0\right)\right]$

$=18\left[\cos\left(180^0-60^0\right)+i\sin\left(180^0-60^0\right)\right]=18e^{-\frac{2\pi}{3}i}$.

Therefore,

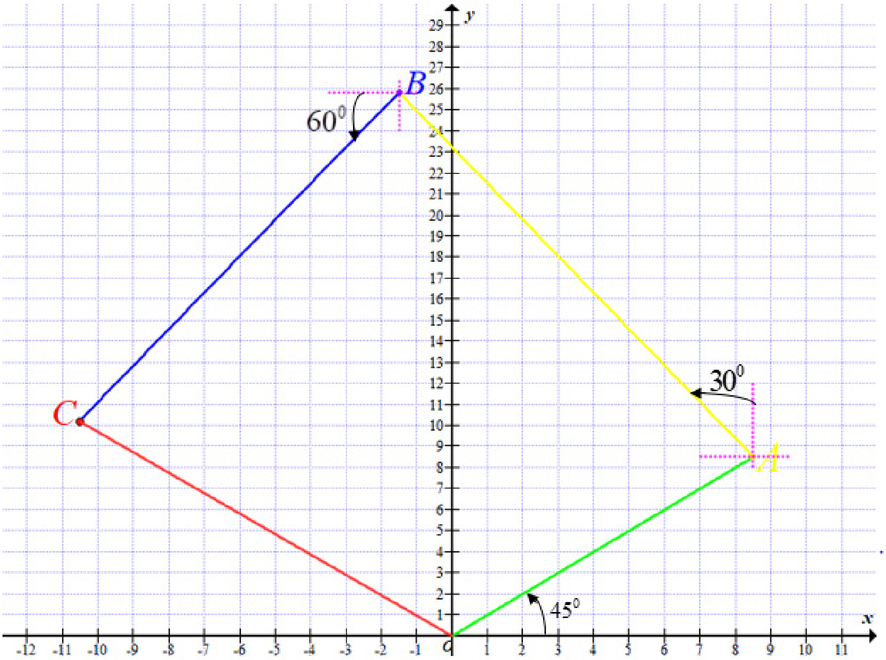
$$\overrightarrow{OC}=12e^{\frac{\pi}{4}i}+20e^{\frac{2\pi}{3}}+18e^{-\frac{2\pi}{3}i}=12\left(\frac{\sqrt{2}}{2}+i\frac{\sqrt{2}}{2}\right)+20\left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)+18\left(-\frac{1}{2}-i\frac{\sqrt{3}}{2}\right)$$

$$=\left(6\sqrt{2}-10-9\right)+i\left(6\sqrt{2}+10\sqrt{3}-9\sqrt{3}\right)=\left(6\sqrt{2}-19\right)+i\left(6\sqrt{2}+\sqrt{3}\right).$$

$$\left|\overrightarrow{OC}\right|=\sqrt{\left(6\sqrt{2}-19\right)^2+\left(6\sqrt{2}+\sqrt{3}\right)^2}\simeq 14.7;$$

$$\arg\left(\overrightarrow{OC}\right)=\pi+\arctan\frac{6\sqrt{2}+\sqrt{3}}{6\sqrt{2}-19}=2.37=135^0 49'.$$

Thus, the man is $14.7\,km$ from his starting point in a direction $135^0 49'-90^0=45^0 49'$ west of north.



Using a convenient unit of length which represent $1km$, and a protractor to measure angles, construct vectors $\overrightarrow{OA}, \overrightarrow{AB}$ and $\overrightarrow{BC}$. Then by determining the number of units in $\overrightarrow{OC}$ and the angle which $\overrightarrow{OC}$ makes with $y-axis$, we obtain the approximately $14.7\,km$ of displacement from origin (his starting point) in a direction of $45^0 49'$ west of north.

ANSWER FOR QUESTION FIVE

a) It is an open question for which the answer shows with examples that complex numbers are applicable in engineering.

b) $\omega = 2\pi.\dfrac{250}{\pi} = 500rad/s$, RMS emf =220V, R=60 ohms, $C = 50.10^{-6}F$, $L = 180.10^{-3}H$

(**i**) The complex impedance is: $Z = R + j\omega L - \dfrac{j}{\omega C}$

$= 60 + j.500.180.10^{-3} - \dfrac{j}{500.50.10^{-6}}$

$= 60 + 50j = \sqrt{60^2 - 50^2}.e^{\left(j\arctan\frac{50}{60}\right)} = 78,1.e^{0,69j}$

The impedance makes an angle $\theta = 0.69\,radians = 39.8\,\deg rees$ with the applied electromotive force.

(ii) The complex current: $I_m = \frac{E_m}{Z}$ where E_m is the complex impedance? $E = 220.\sqrt{2} = 311 volts$, which implies that $I_m = \frac{311}{78,1} e^{j(500t-0,69)} = 4.e^{j(500t-0,69)}$ because E_m is a reference phase.

The physical current is the imaginary part, $i = 4.\sin(500t - 0,69)$, it is behind *emf* at $0.69\, radians$. The root mean square current in the circuit is $i_{eq} = \frac{220}{78,1} A \quad or \quad i_{eq} = \frac{4}{\sqrt{2}} A = 2.8A$.

1.9. Consolidation, Remedial and extended activities

The teacher's guide suggests additional questions and answers to assess the key unit competence.

A. Consolidation activities: Suggestion of questions and answers for deep development of competences.

1) The table below shows examples of pure imaginary numbers in both unsimplified and simplified form.

Unsimplified form	Simplified form
$\sqrt{-9}$	$3i$
$\sqrt{-5}$	$i\sqrt{5}$
$-\sqrt{-144}$	$-12i$

a) Explain how these pure imaginary numbers are simplified

b) Explain why $\sqrt{-18}$ is an imaginary number and write it in a simplified form

Solution

a) The following property explains how the pure imaginary numbers can be simplified: For $a > 0, \sqrt{-a} = i\sqrt{a}$.

Example: the square root of -9 is an imaginary number. The square root of 9 is 3, so the square root of negative 9 is 3 imaginary units, or $3i$.

b) $\sqrt{-18}$ is an imaginary number, since it is the square root of a negative number. So, we can start by rewriting $\sqrt{-18}$ as $i\sqrt{18}$. Next we can simplify $\sqrt{18}$ using what we already know about simplifying radicals: $\sqrt{-18} = i\sqrt{9 \times 2} = 3i\sqrt{2}$.

2) It is known that $i=\sqrt{-1}$ and that $i^2=-1$, use the properties of exponents and find the value of i^3 and i^4.

Solution

We know that $i^3=i^2\cdot i$. But since $i^2=-1$, we see that $i^3=i^2\cdot i=-i$.

Similarly, $i^4=i^2\cdot i^2$. Again, using the fact that $i^2=-1$, we have the following: $i^4=i^2\cdot i^2=(-1)(-1)=1$.

3)The table below summarize the powers of i.

i^1	i^2	i^3	i^4	i^5	i^6	i^7	i^8
i	-1	$-i$	1	i	-1	$-i$	1

From the table above, it appears that the powers of i cycle through the sequence $i,-1,-i,1$.

a) Using this pattern, find i^{20}?

b) Suppose that it is possible to list the sequence $i,-1,-i,1$ up to the 138th term, but the work can take too much time. Consider that $i^4=1, i^8=1, i^{12}=1$ etc, and 136 is a multiple of 4, calculate i^{138}.

Solution

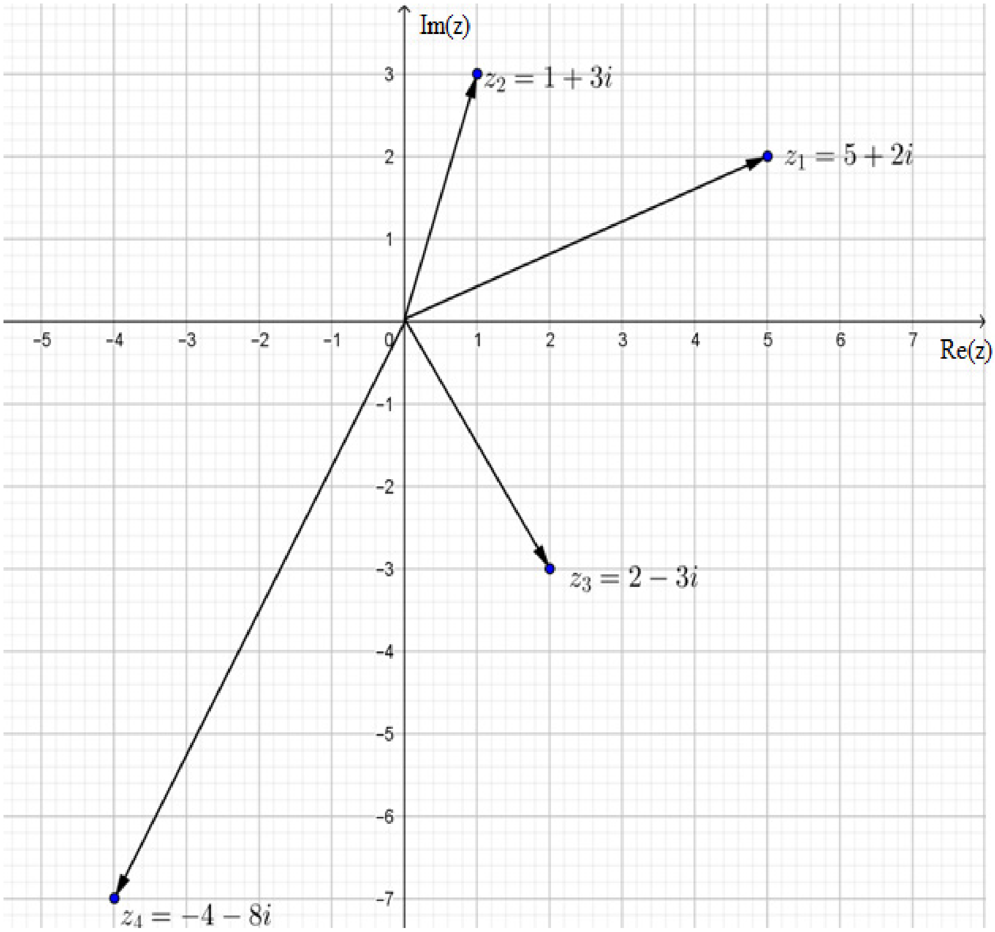
While 138 is not a multiple of 4, the number 136 is a multiple of 4. Let's use this to help us simplify i^{138}.

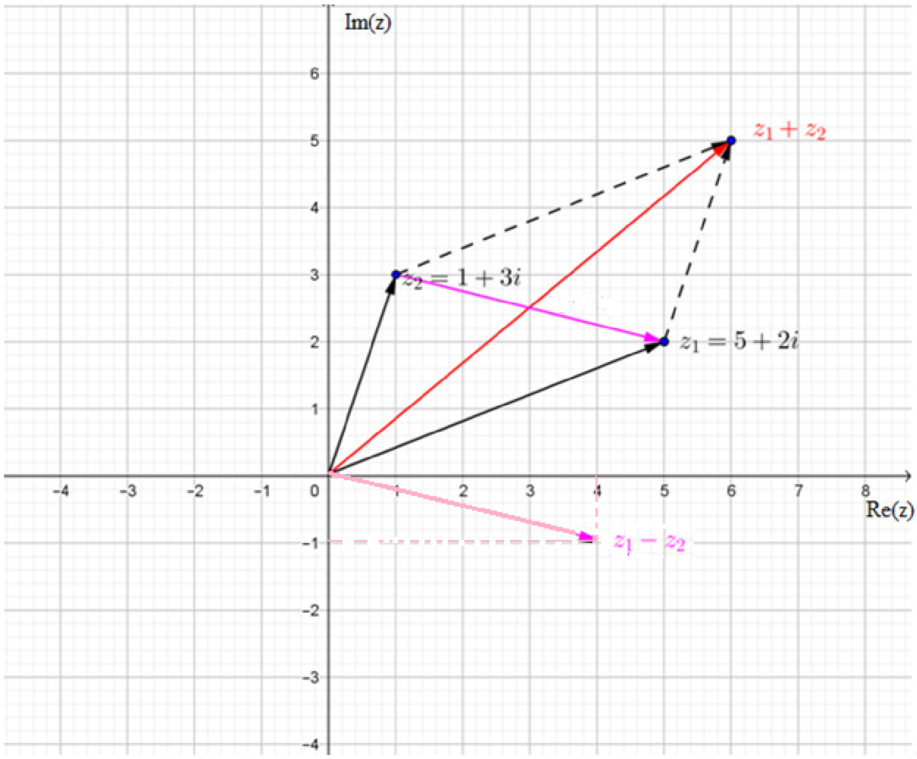
$a)i^{20}=i^{4.5}=(1)^5=1$ $\qquad$ $b)i^{138}=i^{136}.i^2=i^{4.34}.i^2=(1)^{34}.i^2=1.(-1)=-1$

4)Let $z_1=5+2i,\ z_2=1+3i,\ z_3=2-3i$ and $z_4=-4-7i$.

Plot the complex numbers z_1, z_2, z_3 and z_4 on Argand diagram or complex plane and label them. Plot the complex numbers z_1+z_2 and z_1-z_2 on the same Argand diagram. Geometrically, explain how do the positions of the numbers z_1+z_2 and z_1-z_2 relate to z_1 and z_2?

Solution





z_1+z_2 is at the endpoint of leading diagonal of parallelogram constructed from affixes of z_1 and z_2 while z_1-z_2 is at the second diagonal of the same parallelogram from affix of z_2.

5) Convert $z=-1-i$ in polar form

Solution

Let $z=a+bi$ where $a=-1$ and $b=-1$. In polar form the modulus of

$z=|z|=\sqrt{1^2+1^2}=\sqrt{2}$

$x=r\cos\theta \Leftrightarrow -1=\sqrt{2}\cos\theta \Rightarrow \cos\theta=-\frac{\sqrt{2}}{2}$ and

$y=r\sin\theta \Leftrightarrow -1=\sqrt{2}\sin\theta \Rightarrow \sin\theta=-\frac{\sqrt{2}}{2}$

$=\frac{5}{4}$, the coordinates of z in polar form are $z(r,\theta)=z\left(\sqrt{2},\frac{5\pi}{4}\right)$

5) Find the quotient of two complex numbers $\frac{20-4i}{3+2i}$

Solution

$$\frac{20-4i}{3+2i}=\frac{(20-4i)(3-2i)}{(3+2i)(3-2i)}=\frac{52}{13}-\frac{52}{13}i$$

A. Remedial Activities: Suggestion of Questions and Answers for remedial activities for slow learners.

1) Calculate the product of the following complex numbers and write the result in the form of $a+bi$

a) Multiply $-4(13+5i)$, b) Multiply $2i(3-8i)$ c) Multiply $(1+4i)(5+i)$

Solution

$a)\quad 3(2+4i)=3(2)+3(4i)=6+12i$

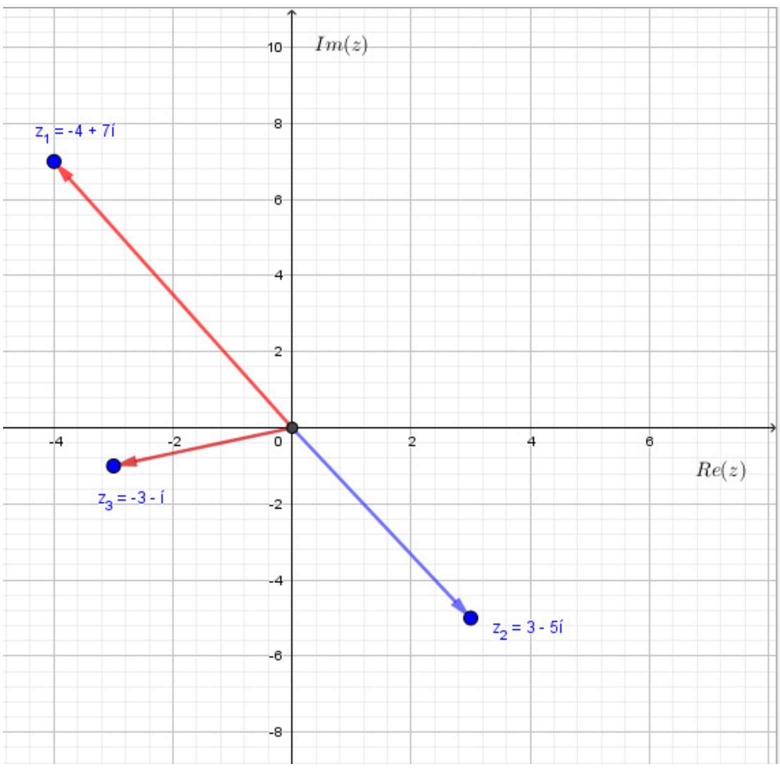
$b)\quad (5+3i)i=5i+3i^2=-3+5i$

$c)\quad (2-7i)(3+4i)=(2)(3)-(7i)(3)+(2)(4i)-(7i)(4i)=34-13i$

2) Plot the following numbers in the complex plane or on the Argand diagram

$a)\quad 3-5i;\qquad b)\quad -4+7i\qquad c)\quad -i-3$

Solution



3) Solve each of the following equations for the complex number z

$$a)\ 4+5i = z-(1-i) \qquad b)\ (1+2i)z = 2+5i$$

Solutions

$a)\ 4+5i = z-(1-i) \Leftrightarrow 4+5i+1-i = z \Rightarrow z = 5+4i$

$b)\ (1+2i)z = 2+5i \Leftrightarrow z = \frac{2+5i}{1+2i} \Rightarrow z = \frac{(2+5i)(1-2i)}{(1+2i)(1-2i)} \Leftrightarrow z = \frac{12}{5}+\frac{1}{5}i$

4) Solve the following equations for real x and y . $3+5i+x-yi = 6-2i$

Solution

$3+5i+x-yi = 6-2i \Leftrightarrow (3+x)+(5-y)i = 6-2i$

By identification of real part and imaginary part both sides, we get the following simple equations:

$3+x=6$ and $5-y=-2$. Therefore $x=3$ and $y=7$

Extended activities: Suggestion of questions and answers for gifted and talented learners.

1) Given that $z = x+iy$, do the following activities

a) Prove that there is no complex number such that $|z|-z=i$

b) Find $z\in\mathbb{C}$ such that $z^2\in\mathbb{R}$

c) Find $z\in\mathbb{C}$ such that $\mathrm{Re}[z(1+i)]+z\bar{z}=0$

Solution

a) Suppose that some $z\in\mathbb{C}$ satisfies the equation $|z|-z=i \Leftrightarrow |z| = z+i$.

Then $|z| = \mathrm{Re}(z)+i(\mathrm{Im}(z)+1)$. Since $|z|\in\mathbb{R}$, necessarily $\mathrm{Im}(z)=-1$.

Then, the given equation becomes $\sqrt{(\mathrm{Re}(z))^2+1} = \mathrm{Re}(z)$, and, squaring, we obtain $1=0$ that does not exist.

b) If $z=a+bi,\ a,b\in\mathbb{R}$, then $z^2\in\mathbb{R} \Leftrightarrow a^2-b^2+2abi\in\mathbb{R}$, that is if and only if $ab=0 \Leftrightarrow a=0$ or $b=0$. Hence $z^2\in\mathbb{R}$ if and only if $z\in\mathbb{R}$ $(b=0)$ or if z is a pure imaginary number $(a=0)$.

c) Let $z=a+bi,\ a,b\in\mathbb{R}$, thus

$\mathrm{Re}(z(1+i) = \mathrm{Re}[(a+bi)(1+i)] = \mathrm{Re}[(a-b)+i(a+b)] = a-b$.

The equation $\mathrm{Re}[z(1+i)]+z\bar{z}=0$ is then equivalent to

$a-b+a^2+b^2=0 \Leftrightarrow \left(a+\frac{1}{2}\right)^2+\left(b-\frac{1}{2}\right)^2=\frac{1}{2}$ whose solutions the points of the circle with center in $\left(-\frac{1}{2},\frac{1}{2}\right)$ and radius $\frac{\sqrt{2}}{2}$.

2) Find the least value of the positive integer for which $\left(\sqrt{3}+i\right)^n$ is

a) real b) pure imaginary

Solution

$\left(\sqrt{3}+i\right)^n = 2^n\left(\cos\frac{n\pi}{6}+i\sin\frac{n\pi}{6}\right)$ is a complex number,

a) $\left(\sqrt{3}+i\right)^n$ is real $\Leftrightarrow \sin\frac{n\pi}{6}=0 \Leftrightarrow \frac{n\pi}{6}=k\pi, k\in\mathbb{N} \Leftrightarrow n=6k$;

Therefore, $\left(\sqrt{3}+i\right)^n$ is real if n is a positive multiple of 6 and the least value of the positive integer multiple of 6 is $n=6$.

b) $\left(\sqrt{3}+i\right)^n$ is pure imaginary $\Leftrightarrow \cos\frac{n\pi}{6}=0 \Leftrightarrow \frac{n\pi}{6}=\frac{\pi}{2}+k\pi, k\in\mathbb{N} \Leftrightarrow n=3(2k+1)$;

Therefore, $\left(\sqrt{3}+i\right)^n$ is pure imaginary if n is a positive odd number that is multiple of 3 and the least value is $n=3$.

Unit 2: LOGARITHMIC AND EXPONENTIAL FUNCTIONS

2

2.1 Key unit competence

Extend the concepts of functions to investigate logarithmic and exponential functions and use them to model and solve problems about interest rates, population growth or decay, magnitude of earthquake, etc.

2.2 Prerequisite knowledge and skills

Learners will perform well in this unit if they have a good background on: properties of logarithm (Senior4: unit2), logarithmic and exponential equations (S5 unit 3), solving equations in the set of real numbers (Senior 1: unit 3; Senior 2: unit 1 and Senior 4: unit 2), domain and range of polynomial, rational and irrational functions (Senior 4: Unit 4), limits and derivatives of polynomial, rational and irrational functions (Senior 4: Unit 5).

2.3 Cross-cutting issues to be addressed

- **Inclusive education**: promote the participation of all learners while teaching.
- **Peace and value Education**: During group activities, the teacher will encourage learners to help each other and to respect opinions of colleagues. In addition, learners will be sensitized to fight alcohol abuse in the lesson on alcohol and risk of car accident).
- **Gender**: During group activities try to form heterogeneous groups (with boys and girls) or when learners start to present their findings encourage both (boys and girls) to present.
- **Environment and Sustainability:** During the lesson on population growth, guide Learners to discuss the effect of the high rate of population growth.
- **Financial education**: guide learners to discuss how to manage the mortgage loans taken from the bank.

2.4 Guidance on the introductory activity

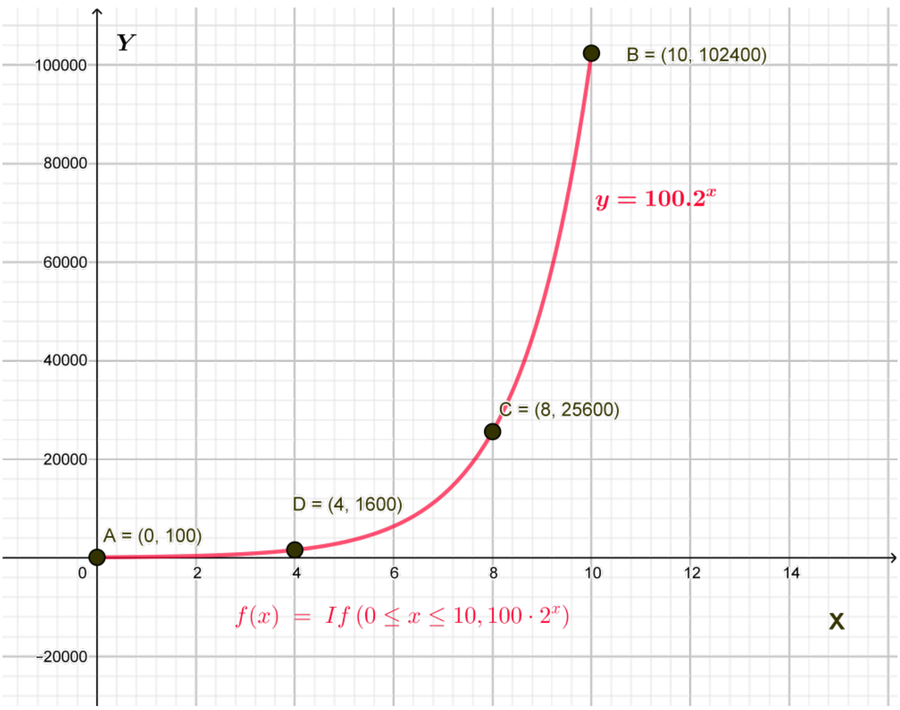
- Form groups of learners that are as heterogeneous as possible and guide them to work on the introductory activity.
- Walk around all groups to provide pieces of advice where necessary.
- After a given time invite learners to present their findings and harmonize them.

Solution:

a) Learners complete the table showing the money of the businessman from the 1st day up to 10th day.

Days	Amounts	USD
1st days	200	200
2nd day	$200x2 = 100x2x2 = 100x2^2$	400
3rd days	$100x2x2x2 = 100x2^3$	1600
4th days	$100x2x2x2x2 = 100x2^4$	3200
....		
10th day	$100x2x2x2x2... = 100x2^{10}$	102,400
Nth day	$100x2x2x2x2... = 100x2^n$	$100x2^n$

b) The graph plotted in a rectangular coordinate.



c) $f(n) = 100 \times 2^n$ USD (US dollars)

- During the presentation, let learners discover the concept of exponential function $F(t)$ starting with the property of a function with powers. $F(t) = 100 \times 2^t$

- Learners establish the function $Y(F)$ inverse of $F(t)$

$$Y(F)=F^{-1}(t)=\ln\left(\frac{t}{100}\right)=-\ln(100)+\ln t$$

$$Y(t)=-4.6+\ln(t)$$

d) The economist wants to possess the money F under the same conditions, discuss how he/she can know the number of days necessary to get such money from the beginning of the business.

The economist wants to possess the money F, using the inverse function $Y(F)=-4.6+\ln(F)$, she/he will use the equation $t=-4.6+\ln(\mathrm{F})$ to calculate the number t of days required.

Conclude that $F(t)$ and $Y(t)$ are respectively exponential function and logarithmic functions that are needed to be well studied so that they may be used without problems. This unit deals with the behaviour and properties of such essential functions and their application in real life situation.

2.5. List of lessons

UNIT TITLE: LOGARTHMICS AND EXPONENTIAL FUNCTIONS (24 periods)			
Introductory activity: 1 period: 40 minutes			
SUB-UNIT 1: Logarithmic functions (10 periods)			
No	**Lesson title**	**Learning objectives (from the syllabus including knowledge, skills and attitudes):**	**Number of periods**
1	Domain of definition of logarithmic function	• State the restrictions on the base and the variable in a logarithmic function • Find the domain and range of a logarithmic function	2
2	Limits of logarithmic functions	Calculate limit of logarithmic funCtion.	1
3	Continuity and asymptote of logarithmic functions	determine and interpret possible asymptote of logarithmic functions	2
4	Differentiation of logarithmic functions	• Determine the derivative of logarithmic function. • Relate derivative to the slope of the tangent line.	2

5.	Variation of logarithmic function	Apply derivative to investigate the minimum and maximum (extrema) of logarithmic function.	2
SUB-UNIT 2: Exponential function (5periods)			
6.	Domain of definition of exponential function	• State the restrictions on the base and the variable in exponential function • Find the domain and range of exponential function	1
7.	Limits of exponential functions	Calculate limit of exponential function	1
8.	Continuity and asymptote of exponential function	Determine and interpret possible asymptote of exponential functions.	1
9.	Differentiation of exponential functions	• Determine the derivative of exponential function • Relate derivative to the slope of the tangent line.	1
10.	Variation of exponential function	Apply derivative to investigate the minimum and maximum (extrema) of exponential functions.	1
SUB-UNIT 3: Application of logarithmic and exponential functions (7periods)			
11.	Interest rates problems	Solve related problems involving logarithmic and exponential functions.	1
12.	Mortgage problems		1
13	Population growth problems		1
14	Radioactive decay problems		1
15	Earthquake problems		1
16	Carbon dating problems		1
17	Problems about alcohol and risk of car accident		1
End unit assessment			**2**

Notice: *For application of mathematics content to other subjects, the teacher will consider the prerequisite of learners in this domain then act accordingly; the time spent and importance given to application activities depends on the learners' level of knowledge and their interest.*

Lesson 1: Domain of definition of logarithmic function

a) Prerequisites/Revision/Introduction:

Learners will perform better in this lesson if they refer to techniques for solving elementary equations involving indices and powers (S4, unit 2). For example, guide them to solve the equation such $3^x = 81$ in $\mathbb{R}$.

b) Teaching resources:

Scientific calculators to evaluate logarithms, textbooks, graph papers. If possible, the use of Mathematical software such as Geogebra to plot graph of logarithmic functions **and/or Microsoft Excel to compute values of a function and** Internet to facilitate research would be plausible.

c)Learning activities:

This lesson evaluates the set of elements for which a logarithmic function is defined. To this end,

- Form groups and ask learners to work on activity 2.1.
- Facilitate learners to use calculator or Microsoft Excel to find images of given real numbers, discuss their existence in the set of real numbers, dress a table of values for given numbers and to plot the graph of $f(x) = \log_{10}(x)$.
- Ask randomly some groups to present their findings to the whole class;
- Lead learners to give observations about images found step by step for $x \quad 1$, $x = 1$, $0 < x < 1$, and values $x < 0$,
- Facilitate learners to deduce the domain and the range for $f(x) = \log_{10}(x)$ then generalize for the function of type $y = \log_a(u(x))$ with $u(x) \geq 0$, $a \neq \square a >$ and then from their answers, write a short summary.
- Guide learners to work on example 2.1 and work individually application activities 2.1 to assess their competences.

Solution to activity 2.1

a) Complete the table of values for $\log_{10}(x)$

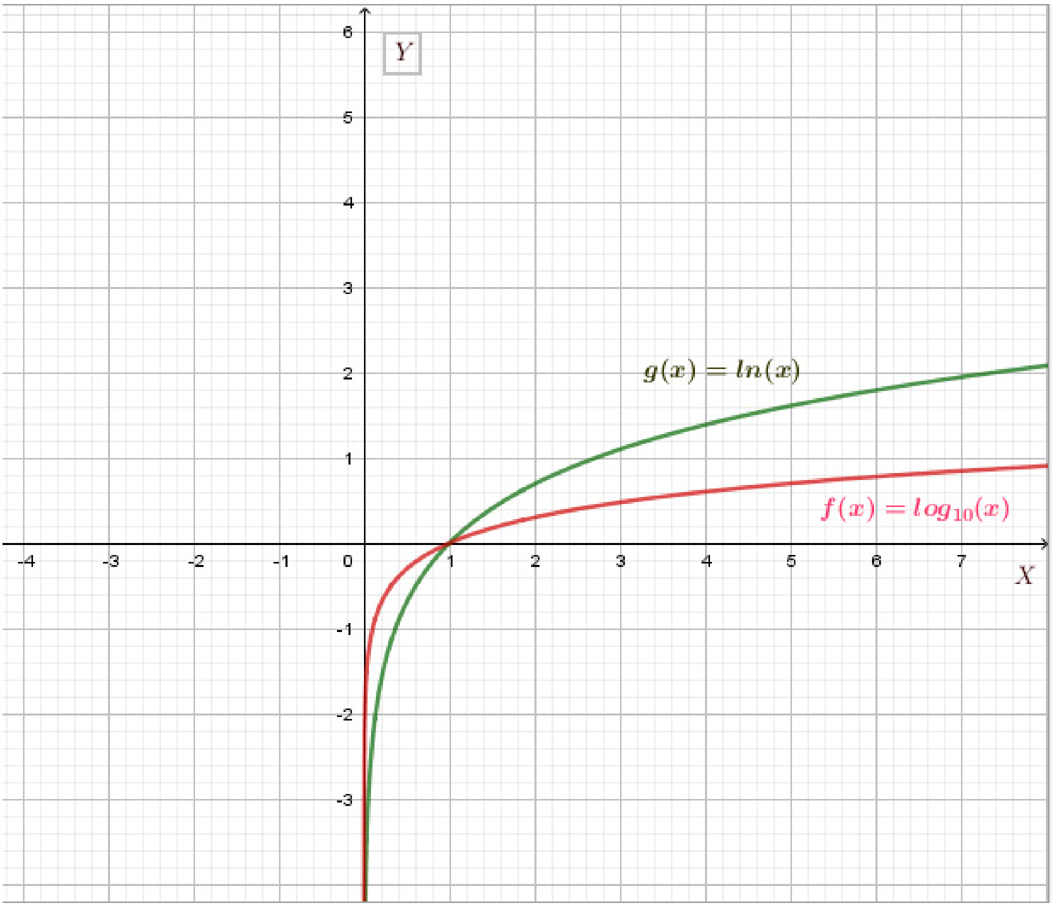
x	100	50	40	20	10	0.5	0.8	0.7	-5	-20	-30
$y=\log_{10}(x)$	2	1.69	1.6	1.30	10	-0.30	-0.09	0.15	-	-	-

b) The values of $\log_{10}(x)$ for $x<0$ do not exist in the set of real numbers.

c) Discuss the values of □ $_{10}(x)$ for $0<x<1$, $x=1$ and $x>1$.

$\log_{10}(1)=0$, $\log_{10}(x)<0\, for 0<x<1$ and $\log_{10}(x)>0\, for\ x>1$

d) The graph of $\log_{10}(x)$ for $x>0$



e) For $f:\mathbb{R}\to\mathbb{R}:x\mapsto\log_a x$,

$dom\, f=\{x\in\mathbb{R}:x>0\}=\left]0,+\infty\right[=\mathbb{R}_0^+$ and range $f=\mathbb{R}=\left]-\infty,+\infty\right[$

Solution to application activity 2.1

1) a) $y=\log_3(x-2)+4$ is defined for $x>2$. Thus $Domf=]2,+\infty[$

To find the range we proceed as following: $y=\log_3(x-2)+4 \Leftrightarrow y-4=\log_3(x-2)$

(for x in the domain) $\Leftrightarrow x-2=3^{y-4} \Leftrightarrow x=3^{y-4}+2$

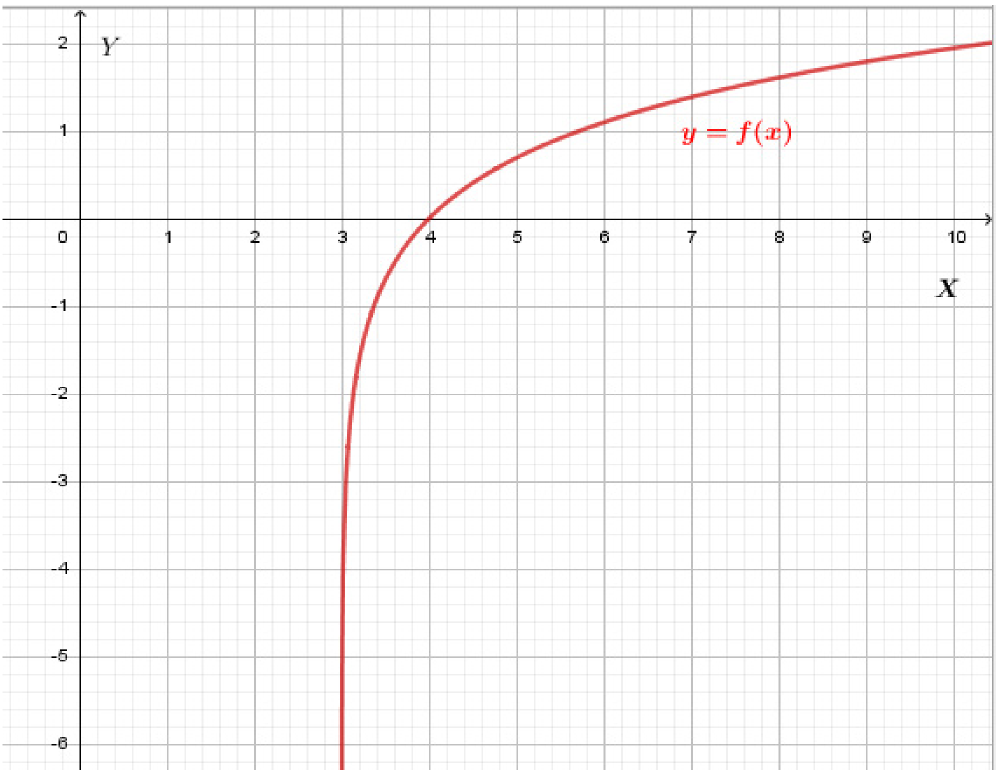
Since $3^{y-4}>0 \quad \forall y\in\mathbb{R}$, we have $x=3^{y-4}+2>2$. Thus, the range is $\mathbb{R}$

b) $y=\log_5(8-2x)$ is defined only if $8-2x>0 \Leftrightarrow -2x>-8 \Leftrightarrow x<4$ means $Domf=]-\infty,4[$

For the Range: $y=\log_5(8-2x) \Leftrightarrow 8-2x=5^y \Leftrightarrow -2x=5^y-8 \Leftrightarrow x=4-\frac{5^y}{2}$

$\frac{5^y}{2}>0$ for all values of y implies $x=4-\frac{5^y}{2}<4$. Thus, the range is $\mathbb{R}$

Observation is made to the following graph.



The domain of the function f is $Domf=]3,+\infty[$. The range is $\mathbb{R}=]-\infty,+\infty[$

Lesson 2: Limits and asymptotes of logarithmic functions

a) Prerequisites/Revision/Introduction:

Students will learn better this lesson if they have a good understanding on concepts of limits (limits of variable, one-sided limits, limits of functions at infinity) learnt in Senior 4, unit5.

b) Teaching resources:

Textbooks, Ruler, T-square, Scientific calculators, graph papers

- If possible, students may use mathematical software, such as Geogebra or Microsoft Excel to plot the graph of logarithmic functions.

c)Learning activities:

- Form groups and provide each group with activity 2.2.
- Ask them to complete the given table and discuss how to determine the required limits.
- Move around to ensure all learners in groups participate actively.
- Call upon groups to present their findings.
- Harmonize their findings by leading students to calculate $\lim_{x\to 0^+} \ln x$, the determination of the vertical asymptote and $\lim_{x\to+\infty} \ln x = +\infty$.
- Lead students to work on example 2.3 and let them work individually application activity 2.2 for assessment.

Solution to activity 2.2

x	0.5	0.001	0.0001	2	100	1001	10000
$y = \ln x$	-0.69	-6.90	-9.21	0.69	4.60	6.908	6.907

1) When the independent variable x takes values approaching 0 from the right, $y = \ln x$ takes the big negative values. We write $\lim_{x\to 0^+} \ln x = -\infty$. The graph of the function approaches the line of equation $x = 0$ considered as the vertical asymptote tothe graph.

2)When x takes greater values, $y = \ln x$ takes also greater values. Therefore, $\lim_{x\to+\infty} \ln x = +\infty$.

a) It is senseless to discuss $\lim_{x\to 0^-} \ln x$ because $y = \ln x$ is not defined for negative values of the set of real numbers.

b) The line of equation $x = 0$ is a vertical asymptote. This means that as the independent variable x takes values approaching 0, the graph of the function approaches the line of equation $x = 0$ without intercepting.

Solutions to application activity 2.2

i)Evaluate the following limits

1) $\lim_{x\to +\infty} \ln(7x^3 - x^2 + 1) = +\infty$

2) $\lim_{x\to 1^+}\left(\ln\frac{1}{x-1}\right) = \lim_{x\to 1^+}\left[\ln 1 - \ln(x-1)\right] = \lim_{x\to 1^+} \ln 1 - \lim_{x\to 1^+} \ln\left(x-1\right) = 0-(-\infty) = +\infty$

3) $\lim_{x\to 2^-} \log_5\left(x^2 - 5x + 6\right) = -\infty$

4) $\lim_{a\to 4^+} \ln\frac{a}{\sqrt{a-4}} = \ln(\lim_{a\to 4^+}\frac{a}{\sqrt{a-4}}) = +\infty$

5) $\lim_{x\to +\infty} \ln\left(x^2 - 4x + 1\right) = +\infty$

6) $\lim_{x\to +\infty} \frac{2+4\log x}{x} = 0$

ii) $\lim_{x\to +\infty}\frac{\ln x}{x} = 0$, $\lim_{x\to 0^+}\frac{\ln x}{x} = -\infty$, $\lim_{x\to}\frac{\ln x}{x} = 0$, $\lim_{x\to\frac{1}{5}}\left(\frac{\ln x}{x}\right) = \frac{\ln\frac{1}{5}}{\frac{1}{5}} = 5\left(\ln 5^{-1}\right) = -5\ln 5$

Lesson 3: Continuity and asymptote of logarithmic functions

a) Prerequisites/Revision/Introduction:

Students will learn better if they refer to the continuity of a function at a point or on interval learnt in Senior 4, unit5.

b) Teaching resources:

T-square, ruler, text book, if possible Mathematical software such as Geogebra, Microsoft Excel, Mathlab.

c) Learning activities:

- Through group discussions invite learners to do all questions of activity 2.3 and motivate them to complete table and deduce the continuity of a logarithmic function in a given point.
- Invite group representatives to present their findings, then help all learners to conclude on the continuity of a logarithmic function and how to plot its graph.
- Let learners work on example 2.3 and work individually application activities 2.3 to assess the competences.

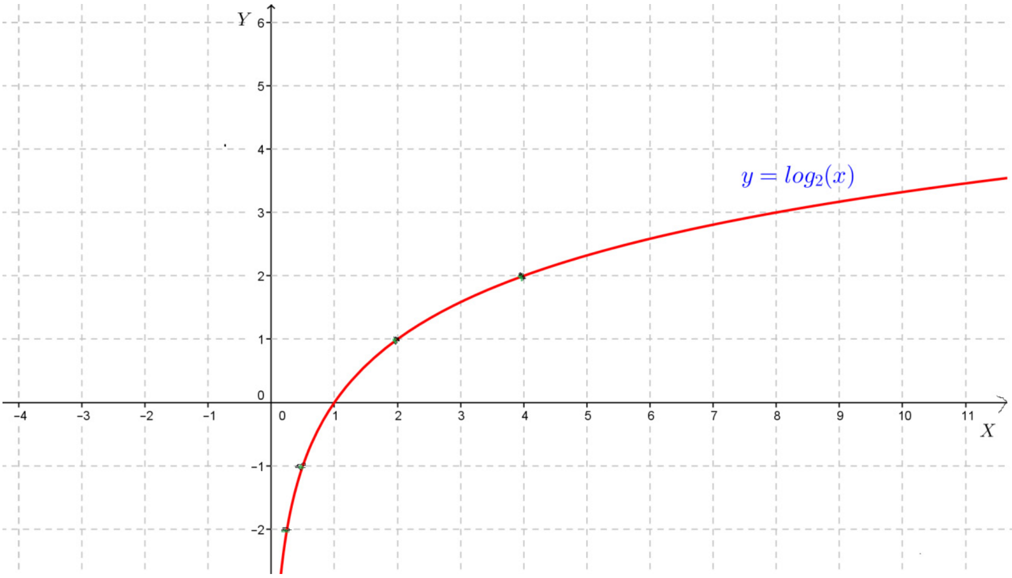
Solutions to activity 2.3:

1)Complete the table

$x = x_0$	$y = \log_2 x$	$\lim\limits_{x \to x_0} \log_2 x$
$\frac{1}{4}$	-2	-2
$\frac{1}{2}$	-1	-1
1	0	0
2	1	1
4	2	2

2)Fall all $x_0 > 0$, $\lim\limits_{x \to x_0} \log_2 x = \log_2 (x_0)$, therefore, $\log_2 x$ is continuous on $]0, +\infty[$

3)The graph of the function $y = \log_2 (x)$ can be plotted using a table of values completed using a calculator



As $x \to 0, y \to -\infty$, so the line of equation $x = 0$ (the y- axis) is an asymptote to the curve.

4)The function is continuous on an interval I if $x_0 \in I, \lim_{x \to x_0^+} f(x) = \lim_{x \to x_0^-} f(x) = f(x_0)$

Solution to application activities 2.3

1) Given the logarithmic function $y = -1 + \ln(x+1)$

i)Vertical Asymptote has equation $x = -1$

ii) $Domf =]-1, +\infty[$

$Range = \mathbb{R} =]-\infty, +\infty[$

iii) $x - \text{int}\,ercept$

The x-intercept is obtained for $y = 0$. We solve the equation $-1 + \ln(x+1) = 0$

$\ln(x+1) = 1 \Leftrightarrow x + 1 = e$ □ □

Thus, the x-intercept is *(e-1, 0).*

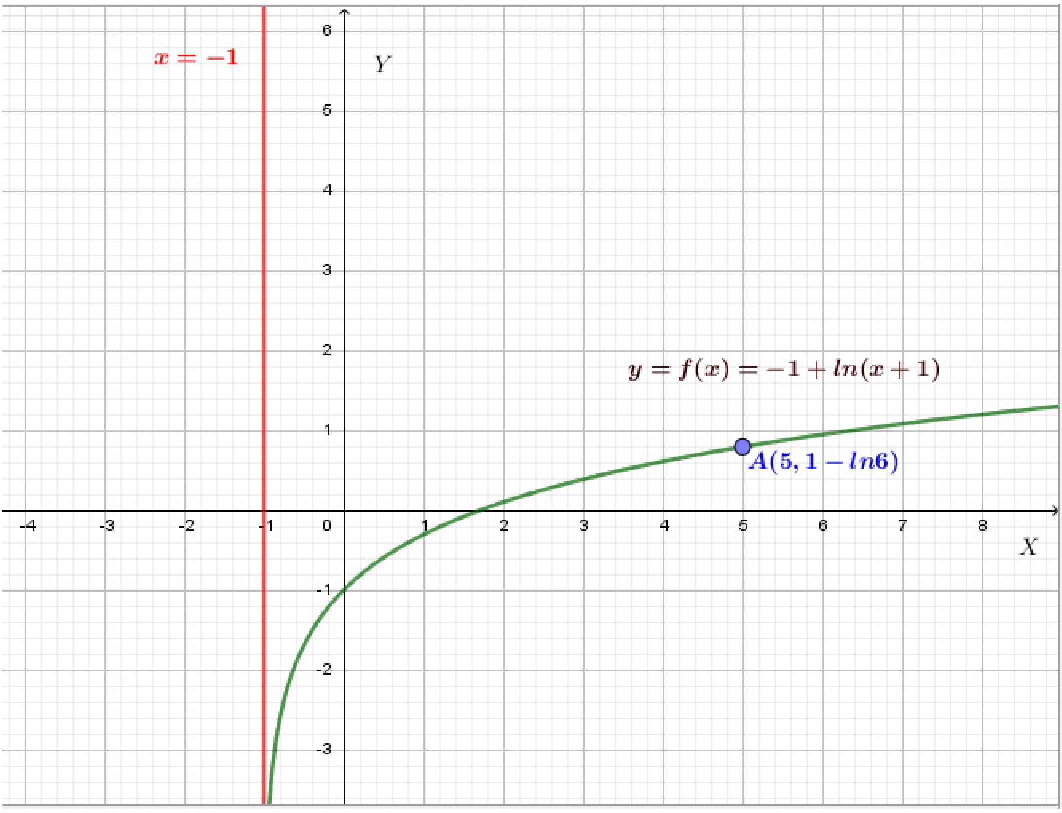
iv) $y - \text{int}\,ercept$ is obtained by Letting $x = 0$.

Thus $y = -1 + \ln(0+1) = -1$

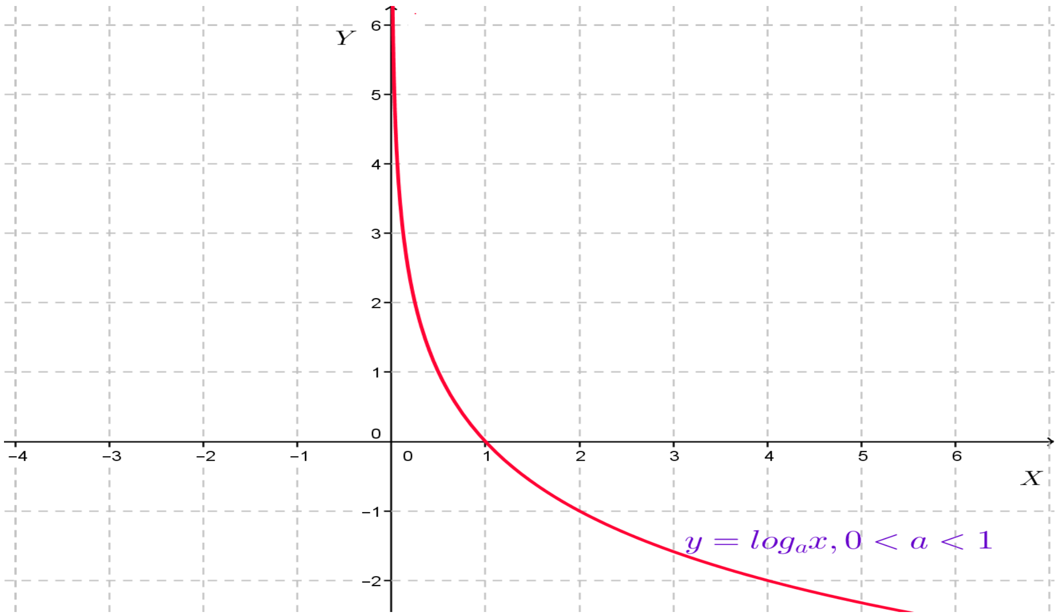
The y-intercept is (0, -1).

v)For example, when $x = 5, y = -1 + \ln(5+1) = -1 + \ln 6$ which gives the point $(5, -1 + \ln 6)$.

vi)The graph of $f(x) = y = -1 + \ln(x+1)$



The graph of the logarithmic function $f(x)=\log_a x, \quad 0<a<1$



Main characteristics of the logarithmic function $f(x)=\log_a x, \quad 0<a<1$

- The domain is $]0,+\infty[$ and $f(x)$ is continuous on this interval.
- The range is $\mathbb{R}$
- The graph intersects the $x-axis$ at $(1,0)$
- As $x \to 0, y \to +\infty$, so the line of equation $x=0$ (the y- axis) is an asymptote to the curve

Lesson 4: Differentiation of logarithmic functions

a)Prerequisites/Revision/Introduction:

Students will perform well in this lesson if they refer to the concepts of derivative and their properties learnt in Senior 4, unit 6.

b)Teaching resources:

- T-square, ruler, Learner's books, if possible Mathematical software such as geogebra, Microsoft Excel, Mathlab….

c)Learning activities:

- Form groups and invite learners to do tasks of activity 2.4,

- Walk around to different group and guide learners to determine the derivative of the function $f(x)=\ln(x)$ in a point for which $x_0=2$ by the use of the definition of derivative of a function.

- Invite some group members to present their findings.

- Harmonize the results by highlighting the derivative of the function $f(x)=\ln(x)$, $\frac{d}{dx}(\ln(u(x)))$, $\frac{d}{dx}(\log_a x)$ and $\frac{d}{dx}[\log_a u(x)]$.

- Guide learners to work through example 2.5 and work individually application activities 2.4 to assess the competences.

Solution to activity 2.4:

h	$\frac{\ln(2+h)-\ln 2}{h}$
-0.1	0.5129329
-0.001	0.5001250 $\simeq$ 1/2
-0.00001	0.5000013 $\simeq$ ½
-0.0000001	0.5000000 $\simeq$ ½
0.1	0.4879016
0.001	0.4998750 $\simeq$ ½
0.00001	0.4999988 $\simeq$ ½
0.0000001	0.50000002 $\simeq$ ½

$f'(2)=\lim_{h\to 0}\frac{\ln(2+h)-\ln 2}{h}\simeq\frac{1}{2}$, these results reflect that

$$f'(x)=\lim_{h\to 0}\frac{\ln(x+h)-\ln x}{h}=\frac{1}{x}$$

The number $f'(2)$ is the slope of the tangent line to the curve $y=f(x)=\ln x$ at the point P$(2,\ln 2)$.

Solution to application activities 2.4

1) $y=\ln\sqrt{\frac{1+x}{1-x}}$ .Here $y=\ln\sqrt{\frac{1+x}{1-x}}=\ln\sqrt{1+x}-\ln\sqrt{1-x}=\frac{1}{2}[\ln(1+x)-\ln(1-x)]$

Therefore, $\frac{dy}{dx}=\frac{1}{2}\left(\frac{1}{1+x}+\frac{1}{1-x}\right)=\frac{1}{1-x^2}$

2) To find the rate of climb (vertical velocity), we need to find the first derivative

$$\frac{d}{dt}\left[2000\ln\left(t+1\right)\right]=2000\frac{d}{dt}\ln(t+1)=\frac{2000}{t+1} .$$

At $t=3$, we have $v=\frac{2000}{3+1}=\frac{2000}{4}=500$

Therefore, the velocity is *500km/min*.

Lesson 5: Variation of logarithmic function

a) Prerequisites/Revision/Introduction:

Learners will learn better in this lesson if they refer to the following concepts: Increasing or decreasing of a function (Senior5, unit 2), Derivative of functions learnt in Senior 4, unit 6, solution of logarithmic and exponential equations learnt in Senior 5, unit 3.

b)Teaching resources:

Learner's books, ruler, T-square, scientific calculator.

c)Learning activities:

- Through group discussions invite learners to do all questions of activity 2.5.
- During group work, motivate students to verify whether the function $f(x)=\ln x$ and $g(x)=\log_{10}x$ are increasing or decreasing on a given interval, to deduce the tables of signs for $f'(x)$ and $g'(x)$ so as to establish the variation of those functions on their domain.
- Invite some group members to present their findings.
- Harmonize the results emphasizing that the function $f(x)=\log_a x$ is strictly increasing on $\mathbb{R}_0^+$ for $a>1$ and that $f(x)=\log_a x$ is strictly decreasing on $\mathbb{R}_0^+$ for $0<a<1$.
- Guide learners to work through example 2.6 and work individually application activities 2.5 to assess the competences.

Solution to activities 2.5

1) $f\left(2\right)=0.693$ and $f\left(10\right)=2.303$, $g\left(2\right)=0.301$ and $g\left(10\right)=1$

Therefore, both functions $f(x)$ *and* $g(x)$ are increasing on the closed interval $\left[2,10\right]$ because $f\left(10\right)-f(2)>0$ *and* $g(10)-g(2)>0$

2) The variation table of $f(x)$ and $f'(x)=\frac{1}{x}$ on the domain $\left]0,+\infty\right[$

x	0		e		$+\infty$
y'	+ + + + +		+ 1/e	+ + + + +	0
y	$-\infty$	↗	1	↗	$+\infty$

The variation table of $g(x)$ and $g'(x)=\frac{1}{x\ln 10}$ on the domain $\left]0,+\infty\right[$

x	0		10		$+\infty$
y'	+ + + +		1/10(ln10)	+ + + + +	0
y	$-\infty$	↗	1	↗	$+\infty$

3) The difference $f\left(10\right)-f(2)=1.61>0$ *and* $g(10)-g(2)=0.699>0$, prove that the function f is increasing faster than g on the interval $\left[2,10\right]$.

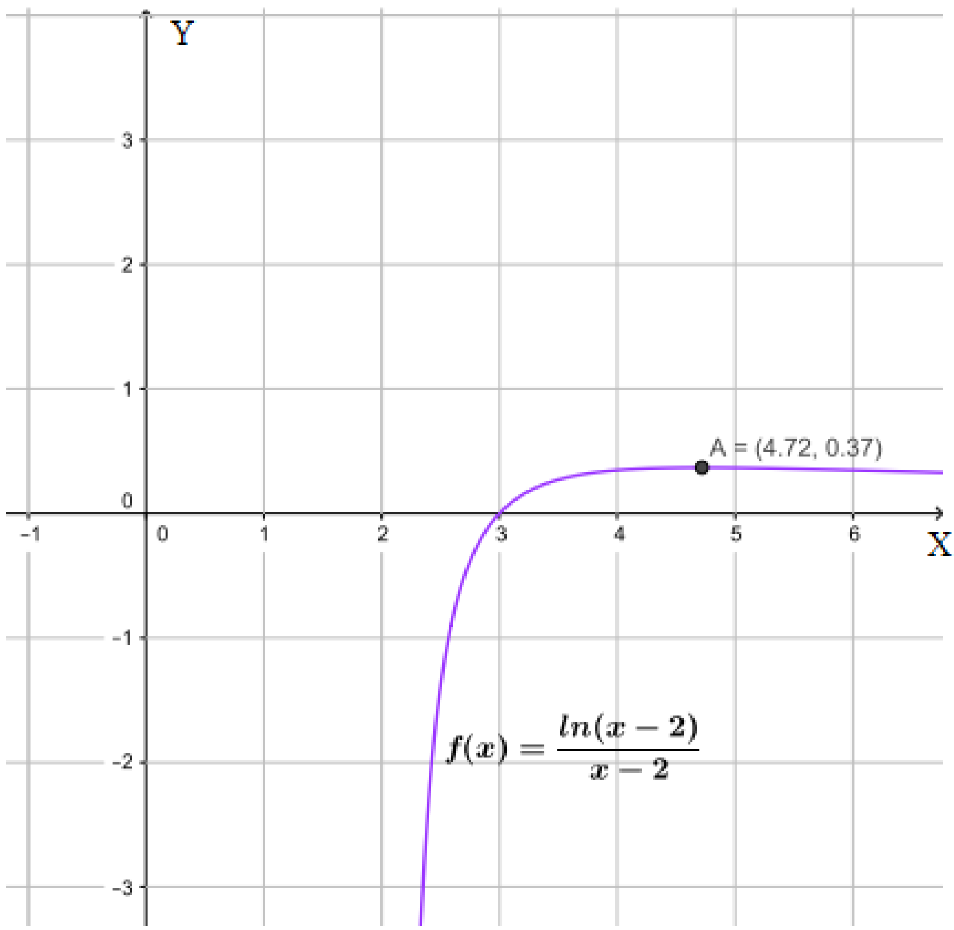
Solution to application activity 2.5

1) Variation of the function $f(x)=\frac{\ln(x-2)}{x-2}$

- $f(x)$ is defined $\Leftrightarrow x-2>0$. That if $x>2$.
- $Domf=\left]2,+\infty\right[$
- $\lim\limits_{\to 2}\frac{\ln(\quad 2)}{-2}=-\infty$, we have a vertical asymptote $x=2$
- $\lim\limits_{x\to+\infty}\frac{\ln(x-2)}{x-2}=0$, we have horizontal asymptote $y=0$
- $f'(x)=\frac{d}{dx}\left[\frac{\ln(x-2)}{x-2}\right]=\frac{\frac{1}{x-2}\times(x-2)-1\times\ln(x-2)}{(x-2)^2}=\frac{1-\ln(x-2)}{(x-2)^2}$
- $f'(x)=0\Leftrightarrow 1-\ln(x-2)=0$
- $\ln(x-2)=1\Leftrightarrow x-2=e\Leftrightarrow x=e+2$.
- $f(e+2)=\frac{\ln(e+2-2)}{e+2-2}=\frac{\ln e}{e}=\frac{1}{e}$
- Variation table of $f(x)$

x	2	$(e+2)$	$+\infty$
y'	+ + + +	0 - - - - - - -	
y	$-\infty$ ↗	$1/e$ Max[(e+2) ,1/e]	↘ 0

Graph of f(x)



1) $h(t) = 100\ln(t\ \ +1)$

a) $v(t) = \frac{d}{dt}\left[100\ln\left(t+1\right)\right] = \frac{100}{t+1}$

b) When $t = 2$ sec, $v(\mathrm{t})\ \Big|_{t=2} = \frac{100}{2+1} m/\sec = \frac{100}{3} m/\sec$

c) The velocity is increasing.

Lesson 6: Domain of definition of exponential function

a)Prerequisites/Revision/Introduction:

Learners will learn better in this lesson if they refer to the following concepts: Increasing or decreasing of a function (S5, unit 2), Derivative of functions learnt in Senior 4, unit 6 and solution of logarithmic and exponential equations learnt in Senior 5, unit 3.

b) Teaching resources:

Learner's book and other reference textbooks, ruler, T-square, scientific calculator; if possible, mathematical software and internet.

c) Learning activities:

- Ask learners to do all questions in activity 2.6 with the aim of establishing the domain of the function $g(x)=e^x$ inverse of $f(x)=\ln x$, the domain of $h(x)=3^x$ and their ranges.
- During group discussions, move around to each group and prompt them to discuss the domain and range of the function $p(x)=a^x$ in case $a>0, a\neq 1$ and $a=1$.
- Lead learners to harmonize the results by generalizing how to find the domain and the range of the function $f(x)=a^{u(x)}$ where $u(x)$ is a function of x.
- Guide learners to work through example 2.7 and work individually application activities 2.6 to assess the competences.

Solution to activity 2.6:

1) If f(x) = , let us complete the following table:

x	0	1	e	e^2	ln (3)	ln (4)
$g(x)=f^{-1}(x)$	1	e	e^e	e^{e^2}	3	4

The set of all values of $g(x)$ is composed of all positive real numbers the range of g is $\mathbb{R}^+ =]0,+\infty[$.

2) Consider the function $h(x)=3^x$ and complete the following table

x	-10	-1	0	1	10
$h(x)=3^x$	$\frac{1}{3^{10}}$	$\frac{1}{3}$	1	3	3^{10}

a) $\forall x\in\mathbb{R}, h(x)\in\mathbb{R}^+$, the domain of $h(x)$ is $\mathbb{R}=]-\infty,+\infty[$

b) All values $h(x)$ are positive, therefore, the range of $h(x)$ is $\mathbb{R}^+=]0,+\infty[$.

Solution to application activities 2.6

1) $f(x)=5e^{2x}$,

$\forall x\in\mathbb{R}, f(x)x\in\mathbb{R}^+$, we realize that $domf=]-\infty,+\infty[$ and the range is the interval $]0,+\infty[$

2) $h(x)=2^{\ln x}$

$h(x)\in\mathbb{R}$ if $x>$, therefore, $domh=]0,+\infty[$.

The range is the set of all $h(x)=2^{\ln x}, x\in\mathbb{R}^+$. That is range h= $\mathbb{R}^+=]0,+\infty[$

3) $g(x)=3^{\left(\frac{x+1}{x-2}\right)}$

Condition for the existence of $\frac{x+1}{x-2}$ in $\mathbb{R}$: $x\neq 2$.

Therefore, $Dom\ g=\mathbb{R}\setminus\{2\}=]-\infty,2[\cup]2,+\infty[$. Its range is $\mathbb{R}^+=]0,+\infty[$.

Lesson 7: Limits of exponential function

a) Prerequisites/Revision/Introduction:

Students will perform well in this lesson if they have a good understanding on Calculations on limits of polynomial, rational and irrational functions (SeniorS4, unit5), solving logarithmic and exponential equations (Senior 5, Unit3), Limits of logarithmic functions (Senior 6, unit2).

b)Teaching resources:

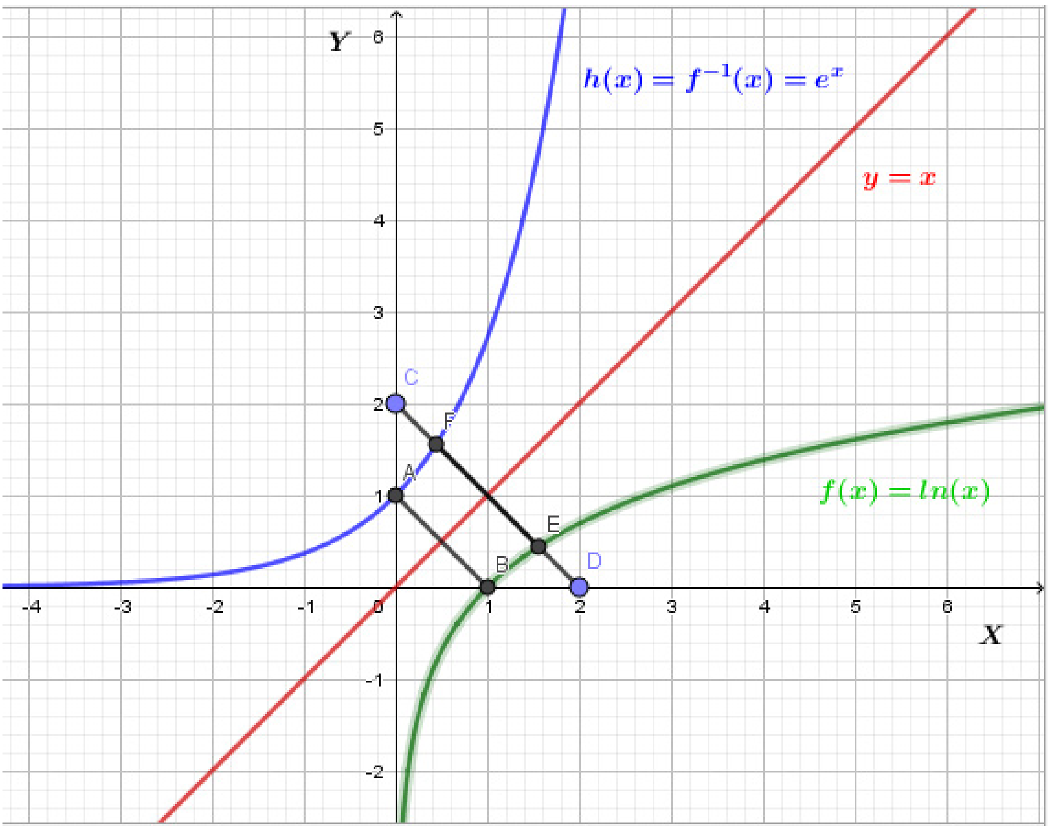
Learners' book, calculator, ruler and T-square. If possible, mathematical software such us Geogebra, Microsoft Excel, Math lab and graphcalc can be used.

c)Learning activities:

- Form groups of students and explain instructions related to the task to be done in the activity 2.8.
- Monitor how students are performing the task and provide support where necessary to guide them on how to find the graph for the inverse of a given function and how to interpret the graph of $y = e^x$ to deduce $\lim_{x\to-\infty} e^x$ and $\lim_{x\to+\infty} e^x$.
- Invite representatives of groups to present their findings.
- Decide to engage the class into exploitation of students' findings.
- Judge the logic of the students' findings, correct those which are false, complete those which are incomplete, and confirm those which are correct and guide the students to conclude about $\lim_{x\to-\infty} a^x$ and $\lim_{x\to+\infty} a^x$ for any values of a.
- Give the summary of expected feedback based on students' answers.
- Ask learners to work through example 2.7 in the learners' book and work individually application activities 2.7 to assess the competences.

Solution to activity 2.7

1) Given the graph of $f(x) = \ln x$, the graph of its inverse $y = f^{-1}(x) = e^x$ is obtained by reflecting the graph of $f(x) = \ln x$ in the axis with equation $y = x$.



2) Based on the plotted graph above, as x decreases towards $-\infty$, the graph of e^x approaches the line of equation $y=0$, therefore, $\lim\limits_{x\to-\infty} e^x = 0$ and the line $y=0$ is the horizontal asymptote. As x increases towards $+\infty$, the images increase. Therefore, $\lim\limits_{x\to+\infty} e^x = +\infty$.

3) Applying properties of limits we have:

$$\lim_{x\to-\infty}\left(\frac{1}{2}\right)^x = \frac{1}{\lim\limits_{x\to-\infty}\left(2^x\right)} = +\infty$$

$$\lim_{x\to+\infty}\left(\frac{1}{2}\right)^x = \frac{1}{\lim\limits_{x\to+\infty}\left(2^x\right)} = 0$$

4) a) **If a is less than one,** $\lim\limits_{x\to-\infty} a^x = +\infty \quad and \quad \lim\limits_{x\to+\infty} a^x = 0$

b) **If a is greater than one,** $\lim\limits_{x\to-\infty} a^x = 0 \quad and \quad \lim\limits_{x\to+\infty} a^x = +\infty$.

c) **If equals one,** the function is constant: y = 1.

Solutions to application activities 2.7

Evaluate limit of the function $f(x)$ at $+\infty$ and -∞ in each of the following case.

1)$f(x)=e^{8+2x-x^3}$ .We have $\lim\limits_{x\to+\infty} e^{8+2x-x^3} = 0$ and $\lim\limits_{x\to-\infty} e^{8+2x-x^3} = +\infty$

2) $f(x)=e^{\frac{6x^2+x}{5+3x}}$ .We have $\lim\limits_{x\to+\infty} e^{\frac{6x^2+x}{5+3x}} = e^{\lim\limits_{x\to+\infty}\frac{6x^2+x}{5+3x}} = +\infty$ and $\lim\limits_{x\to-\infty} e^{\frac{6x^2+x}{5+3x}} = e^{\lim\limits_{x\to-\infty}\frac{6x^2+x}{5+3x}} = 0$

3)$f(x)=2e^{6x}-e^{-7x}-10e^{4x}$.

$$\lim_{x\to+\infty}\left(2e^{6x}-e^{-7x}-10e^{4x}\right) = \lim_{x\to+\infty} e^{6x}\left(2-e^{-13x}-10e^{-2x}\right) = +\infty$$

4)$f(x)=3e^{-x}-8e^{-5x}-e^{10x}$

$$\lim_{x\to+\infty}\left(3e^{-x}-8e^{-5x}-e^{10x}\right) = \lim_{x\to+\infty} e^{10x}\left(3e^{-11x}-8e^{-15x}-1\right) = -\infty$$

$$\lim_{x\to-\infty}\left(3e^{-x}-8e^{-5x}-e^{10x}\right) = \lim_{x\to-\infty} e^{-5x}\left(3e^{4x}-8-e^{15x}\right) = -\infty$$

5)$f(x)=\dfrac{e^{-3x}-2e^{8x}}{9e^{8x}-7e^{-3x}}$.

We obtain $\lim_{x\to+\infty}\frac{e^{-3x}-2e^{8x}}{9e^{8x}-7e^{-3x}}=\frac{-2}{9}$ and $\lim_{x\to-\infty}\left(\frac{e^{-3x}-2e^{8x}}{9e^{8x}-7e^{-3x}}\right)=\lim_{x\to-\infty}\frac{e^{-3x}-2e^{8x}}{9e^{8x}-7e^{-3x}}$

$$=\lim_{x\to-\infty}\frac{e^{-3x}\left(1-2e^{11x}\right)}{e^{-3x}\left(-7+9e^{11x}\right)}=-\frac{1}{7}$$

Lesson 8: Continuity and asymptotes of exponential function

a) Prerequisites/Revision/Introduction:

Learners will perform better in this lesson if they have an understanding of the following concepts: calculations of limits of polynomial, rational and irrational functions (Senior 4, unit5), powers and radicals operations of real numbers (Senior 4, Unit2.), solving linear, quadratic equations and inequalities (Senior 4, Unit3), logarithmic and exponential function (Senior 5, Unit3), limits of logarithmic and exponential functions. (Senior 6, unit3,)

b) Teaching resources:

Learners' book, T-square, ruler, sheets of paper, if possible computers and Mathematics drawing software such as Geogebra, Microsoft Excel, Mathlab and Internet to facilitate research.

c) Learning activities

- Using small groups, guide learners to understand the activity 2.8, in the learners' book, and ask them to find the domain and range of $f(x)=2^{x-2}$. Help learners to find out and realize that domain of $f(x)$ is the set of all real number $Domf=\left]-\infty,+\infty\right[$ and the range of $f(x)$ is $R=\left]0,+\infty\right[$
- In the same groups, let learners find out that $\lim_{x\to-\infty}2^{(x-2)}=0$, $\lim_{x\to+\infty}f(x)=+\infty$ and $\lim_{x\to-\infty}\frac{f(x)}{x}=0$. Deduce that $y=0$ is horizontal asymptote to the graph of $f(x)$.
- Finally, invite learners to find that $\lim_{x\to0^+}f(x)=\frac{1}{4}, \lim_{x\to0^-}f(x)=\frac{1}{4}, and\ f(0)=\frac{1}{4}$ and plot the graph of $f(x)=2^{x-2}$ using the following points

  $x=4; f(4)=4;\text{and } x=-1; f(-1)=\frac{1}{8}$
- Ask groups to present their findings to the whole class and then harmonize their works to provide the lesson summary.
- Let learners work on activities in the example 2.9 and invite them to individually work out the application activity 2.8 to assess their competences on continuity

and asymptotes of exponential functions

Solution to activity 2.8

Let $f\left(x\right)=2^{x-2}$

a) $Domf=]-\infty,+\infty[$ and $\operatorname{Im} f=]0,+\infty[$

b) $\lim\limits_{x\to-\infty} 2^{x-2}=2^{-\infty-2}=\frac{1}{2^{+\infty}}=0$. The equation of Horizontal asymptote is $y=0$

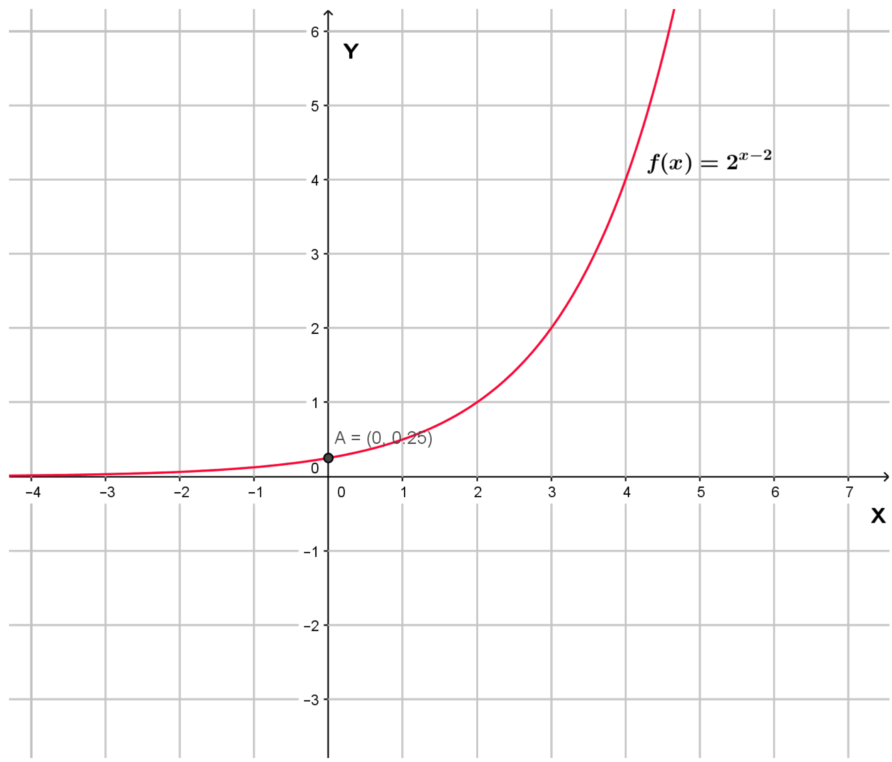
c) For $x=0, f(x)=2^{-2}=\frac{1}{4}$, therefore y-intercept of the graph is $(0,\frac{1}{4})$

d) $\lim\limits_{x\to+\infty} 2^{x-2}=2^{+\infty-2}=2^{+\infty}=+\infty$ and $\lim\limits_{x\to-\infty}\frac{2^{x-2}}{x}=\frac{2^{-\infty-2}}{-\infty}=\frac{1}{2^{\infty}(-\infty)}=\frac{1}{-\infty}=0$

e) $\lim\limits_{x\to0^+} 2^{x-2}=2^{0^+-2}=\frac{1}{2^2}=\frac{1}{4}$ and $\lim\limits_{x\to0^-} 2^{x-2}=2^{0^--2}=\frac{1}{2^2}=\frac{1}{4}$. At $x=0, f(0)=\frac{1}{4}$.

Since the $\lim\limits_{x\to0^+} f(x)=\lim\limits_{x\to0^-} f(x)=\frac{1}{4}$, the function has the continuity at $x=0$

f) The graph of $f(x)=2^{x-2}$



Solution to application activities 2.8

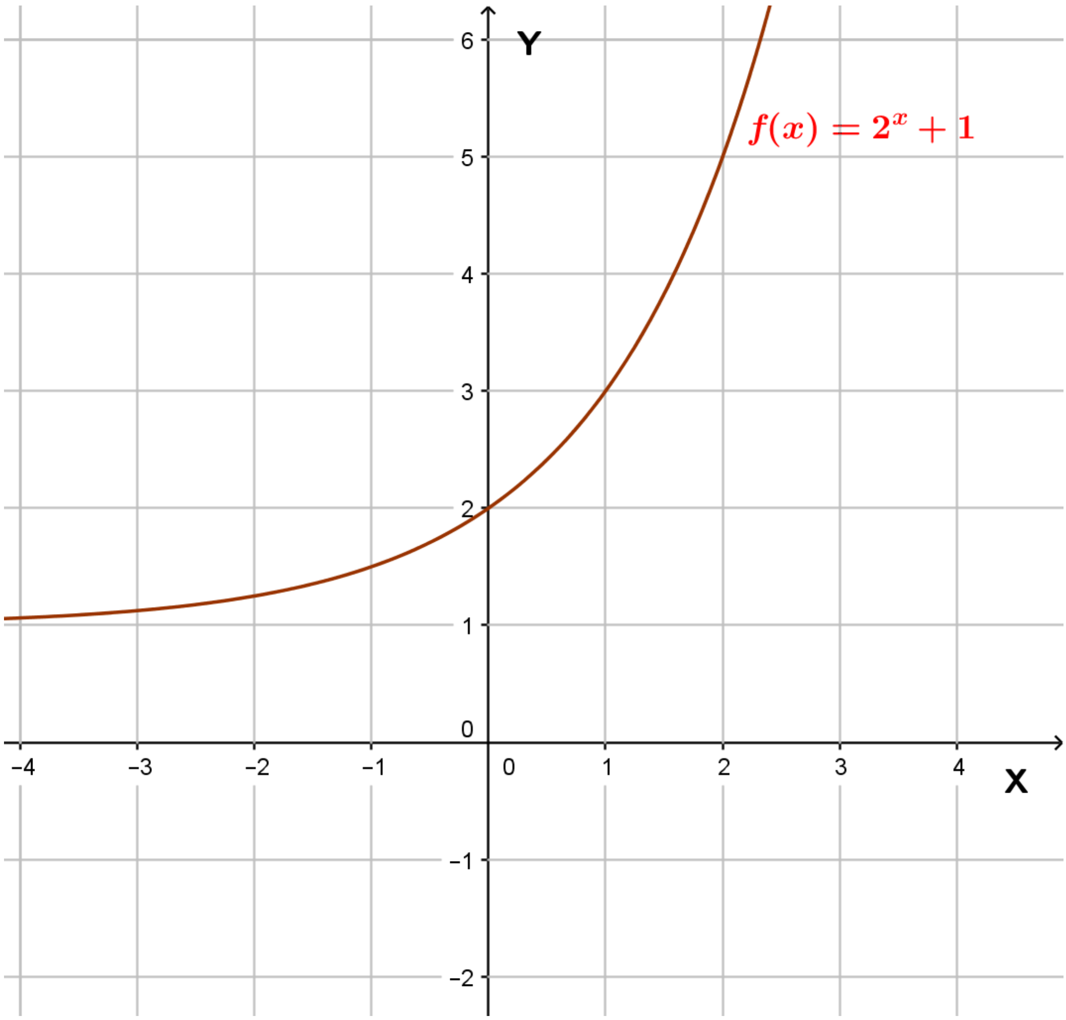
Given that f is a function given by $f(x) = 2^x + 1$

a) Dom f = $domf =]-\infty, +\infty[$ or $domf = \mathbb{R}$

b) The horizontal asymptote for the graph of $f(x)$ is the equation $y = 0$, because $\lim_{x \to 0^+} f(x) = 2, \lim_{x \to 0^-} f(x) = 2, and\ f(0) = 2$

c) The $y -$ intercept is $(0, 2)$

d) The graph of $f(x) = 2^x + 1$.



Lesson 9: Differentiation of exponential functions

a) Prerequisites/Revision/Introduction:

Learners will perform better in this lesson if they have an understanding of the concept of derivatives or differentiation, their rules and applications (Senior 4: Unit 6); differentiation of logarithmic functions (Unit3)

b) Teaching resources:

Learners' book, T-square, ruler, papers, if possible computers, Math draw software such as Geogebra, Microsoft Excel, Mathlab for graph sketching

c) Learning activities:

- Form small groups and invite learners to perform the activity 2.9 in the learners' book and determine the inverse of $f(x)$ and $g(x)$
- by using the following hint: given $f(x)=e^{u(x)}$ and $g(x)=a^{u(x)}$., we find f' *and* g' as follow: $f'=u'e^{u(x)}$ and $g'=u'(x)a^{u(x)}\ln a$
- In the same groups, ask learners to use the derivative $p'(x)=\frac{1}{x}$ of the function $p(x)=\ln x$, $k'(x)=\frac{1}{x.\ln 2}$ of the function $k(x)=\log_2 x$ and apply the following rule $\frac{1}{f'\left[f^{-1}(x)\right]}$ of differentiating inverse of logarithmic functions

  to determine that the derivative of $f'(x)=e^x$ and $g'(x)=2^x\ln 2$.
- Finally, ask learners to discuss and compare the used technique of determining the derivative of $f(x)$ and $g(x)$ to the following techniques $f'=u'e^{u(x)}$ and $g'=u'(x)a^{u(x)}\ln a$
- Ask groups to present their findings to the whole class and then lead to harmonize their works to provide the lesson summary.
- Let learners work on activities of the example 2.10 and invite them to individually work out the application activity 2.9.

Solution to activity 2.9

a) Given the functions $f(x)=e^x$ and $g(x)=2^x$, their inverse are: $f^{-1}(x)=\ln x$ and $g^{-1}(x)=\log_2 x$ respectively.

b) Given that $p(x) = \ln x$ and $k(x) = \log_2 x$, it is known that $p'(x) = \frac{1}{x}$ and $k'(x) = \frac{1}{x \ln 2}$. Then applying the rule for differentiating inverse of logarithmic functions we find $\frac{1}{p'\left[p^{-1}(x)\right]} = \frac{1}{p'\left(e^x\right)}$.

We already know that $p'(x) = \frac{1}{x}$, then $p'(e^x) = \frac{1}{e^x}$

Thus $\frac{1}{p'\left[p^{-1}(x)\right]} = \frac{1}{p'\left(e^x\right)} = \frac{1}{\frac{1}{e^x}} = e^x$ and $\frac{1}{k'\left[k^{-1}(x)\right]} = \frac{1}{k'\left(2^x\right)}$.

Since $k'(x) = \frac{1}{x \ln 2}$,

it follows that $k'(2^x) = \frac{1}{2^x \ln 2}$ and $\frac{1}{k'\left[k^{-1}(x)\right]} = \frac{1}{\frac{1}{2^x \ln 2}} = 2^x \ln 2$.

Solution to application activity 2.9

1) Given the function $f(x) = 4^x$.

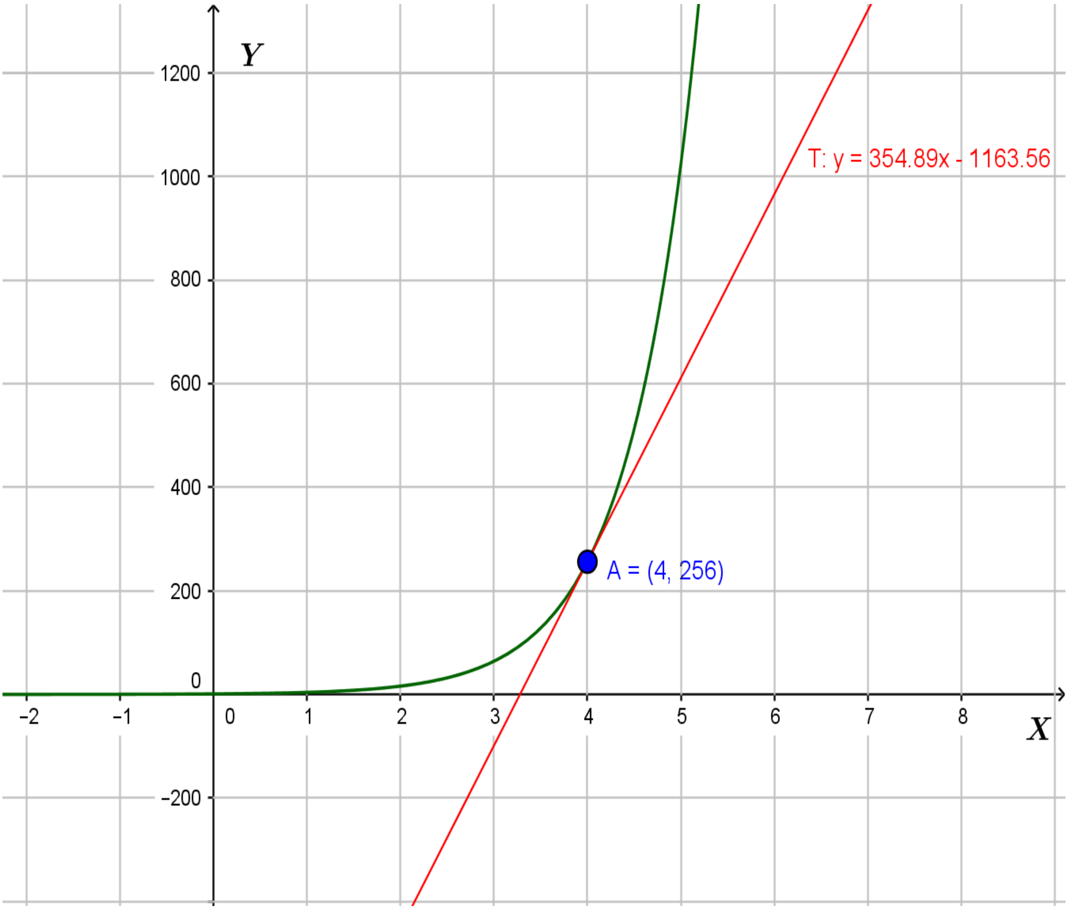
i) $f'(\ ) = 4\ \ln 4$

ii) $f(5) = 4^5 = 1024$

iii) The slope of the tangent line at $x = 5$ is $f'(5) = 4^5 \ln 4 = 1419.56$,

iv) For the function $f(x) = 4^x$, $f'(4) = 4^4 \ln 4 = 354.89$

therefore, the equation of tangent at x=4 is $y = 354.89x - 1163.56$



The tangent to the function $f(x)=4^x$ *at x=4*

2) a) $f(x)=10^{3x} \Rightarrow f'(x)=3.10^{3x}\ln 10$

b) $f(x)=xe^{x^2+1} \Rightarrow f'(x)=e^{x^2+1}(1+2x^2)$

c) $f(x)=\dfrac{3^{4x+2}}{x} \Rightarrow f'(x)=\dfrac{\left(3^{4x+2}\right)'x-3^{4x+2}(x)'}{x^2}=\dfrac{4x.\left(3^{4x+2}\right)\ln 3-3^{4x+2}}{x^2}=\dfrac{3^{4x+2}\left(4x\ln 3-1\right)}{x^2}$

Lesson 10: Variation of exponential functions

a)Prerequisites/Revision/Introduction:

Generalities on numerical functions (S4, Unit 4), Concept of derivative of functions (S4, Unit 6), Logarithmic and exponential equations (S5, unit3), Variation of logarithmic functions (S6, Unit2)

b)Teaching resources:

Learner's book and other reference books to facilitate research

c)Learning activities:

- Invite learners to work out the activity 2.10 in groups and let them find out that the functions $f(x)$ is increasing in the interval $[1,10]$, the function $g=0.5^x$ is decreasing in the interval $[1,10]$ because $a=0.5$ and $0.5<1$. From activity 2.10, lead learners to realise and conclude that the function of the form $f(x)=a^x$, with $a>1$, is always increasing and the function $g(x)=a^x$ with $0<a<1$, is always decreasing. In the same groups, let learners calculate and realize that the 1st derivative of $f(x)$ and $g(x)$ are $f'(x)=e^x$ and $g'(x)=5^x\ln 5$ respectively. Ask them to draw table of signs for $f'(x)$ and $g'(x)$ and note that the interval of variation of those function is $]-\infty,+\infty[$ and finally, let them plot the graphs of the functions: $f(x)$ and $g(x)$
- Ask groups to present their findings to the whole class and then harmonize their works to provide the lesson summary.
- Let learners read through the examples 2.11 and invite them to individually work out the application activity 2.10 to increase their knowledge and skills on variation of exponential functions

Solution to activity 2.10

Given two functions $f(x)=2^x$ and $g(x)=0.5^x$,

1) $f(1)=2$ and $f(10)=2^{10}$, $f(1)<f(10)$ and the function $f(x)$ is increasing on the interval $[1,10]$.

2) $g(1)=\frac{1}{2}$ and $g(10)=\frac{1}{2^{10}}$, $g(1)>g(10)$ and the function $g(x)$ is decreasing on the interval $[1,10]$.

3) $f'(x) = 2^x \ln 2$ and $g'(x) = (0.5)^x \ln 0.5$ or $g'(x) = \frac{1}{2^x} \ln \frac{1}{2}$

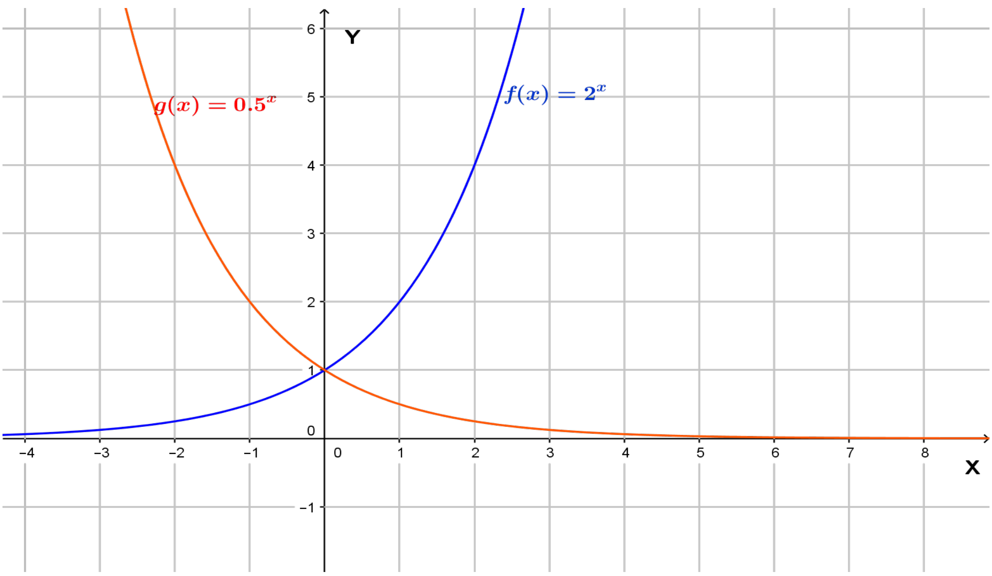
a) Table of variation of $f(x)$

x	$-\infty$		0		$+\infty$
$f'(x)$	+ + +	+	$\ln 2$	+ + + + + + +	
$f(x)$	0	↗	1	↗	$+\infty$

b) Table of variation of $g(x)$

x	$-\infty$		0		$+\infty$
$g'(x)$	- - - -	-	$\ln \frac{1}{2}$	- - - - - - - - -	
$g(x)$	$+\infty$	↘	1	↘	0

4) Plot the graphs of $f(x)$ and $g(x)$



Graph of the decresing function g(x) and increasing function f(x).

5) The exponential *function* of the form $f(x)=a^x$, with $a>1$, is always increasing and the exponential function of the form $g(x)=a^x$ with $0<a<1$, is always decreasing.

Solution to application activities 2.10

Given the function $f(x)=xe^{x^2}$

a) The derivative of $f(x)$ is $f'(x)=e^{x^2}(1+2x^2)$

b) The derivative of $f(x)$ has no zero in $\mathbb{R}$ and is always positive,

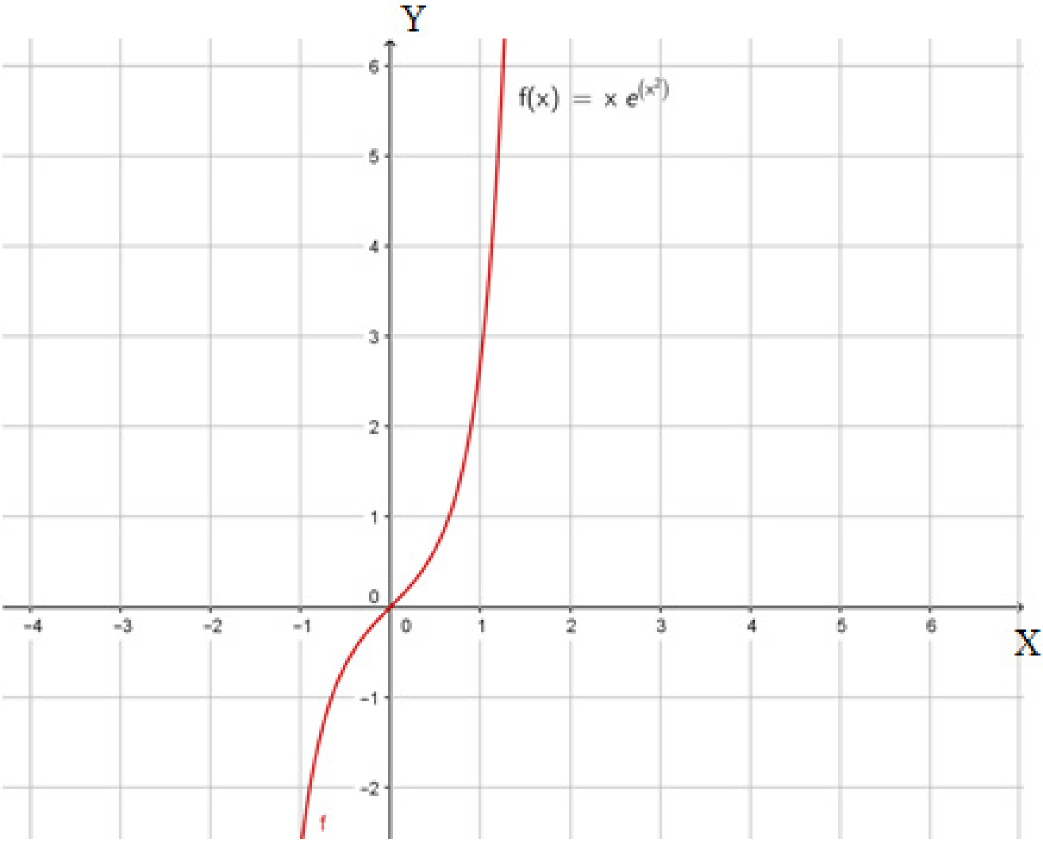
Table of variation of $f(x)$ is presented as follow:

	$-\infty$	$+\infty$
$f'(x)$	+ + + + + + + + + +	
$f(x)$	$-\infty$ ↗	$+\infty$

From the table above, the function $f(x)=xe^{x^2}$ is always increasing on the interval $]-\infty,+\infty[$

c) the function f has neither minimum nor maximum

d) Graph of the function $f(x)=xe^{x^2}$



2) Given the function $f(x)=\dfrac{e^x}{x-2}$

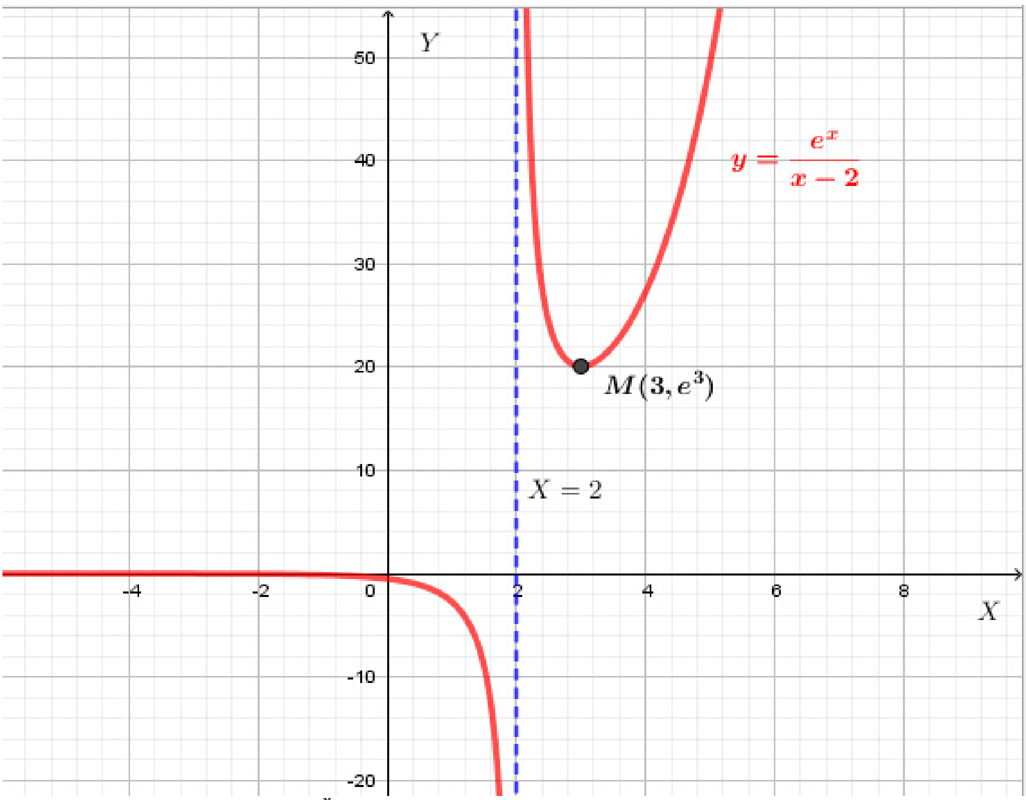
a) $Domf=\mathbb{R}-\{2\}$ and $Range\,f=\mathbb{R}$

b) $f'(x)=\dfrac{e^x(x-3)}{(x-2)^2}$, $f'(x)=0\Leftrightarrow x=3$

$\lim\limits_{x\to 2} f(x)$ is not defined because $\lim\limits_{x\to 2^+} f(x)=+\infty$ and $\lim\limits_{x\to 2^-} f(x)=-\infty$, the function $f(x)=\dfrac{e^x}{x-2}$ has a vertical asymptote of equation $x=2$

c) **Variation table of** $f(x)$

x	$-\infty$		0		2		3		$+\infty$
$f'(x)$		$-$	$-$	$-$	||	$-$	0	$+$	
$f(x)$	0	↘	$\frac{1}{-2}$	↘ $-\infty$	|| $+\infty$	↘	e^3	↗	$+\infty$



Graph of the function $f(x) = \dfrac{e^x}{x-2}$

3)The consumption of natural mineral resource M has risen from 4million tonnes at the rate of 20% per year. Assuming that growth of the consumption has been continuous following the function $M = M_0 e^{rt}$ where M is the final value, M_0 the initial consumption value, r the annual rate of growth and t the time in years.

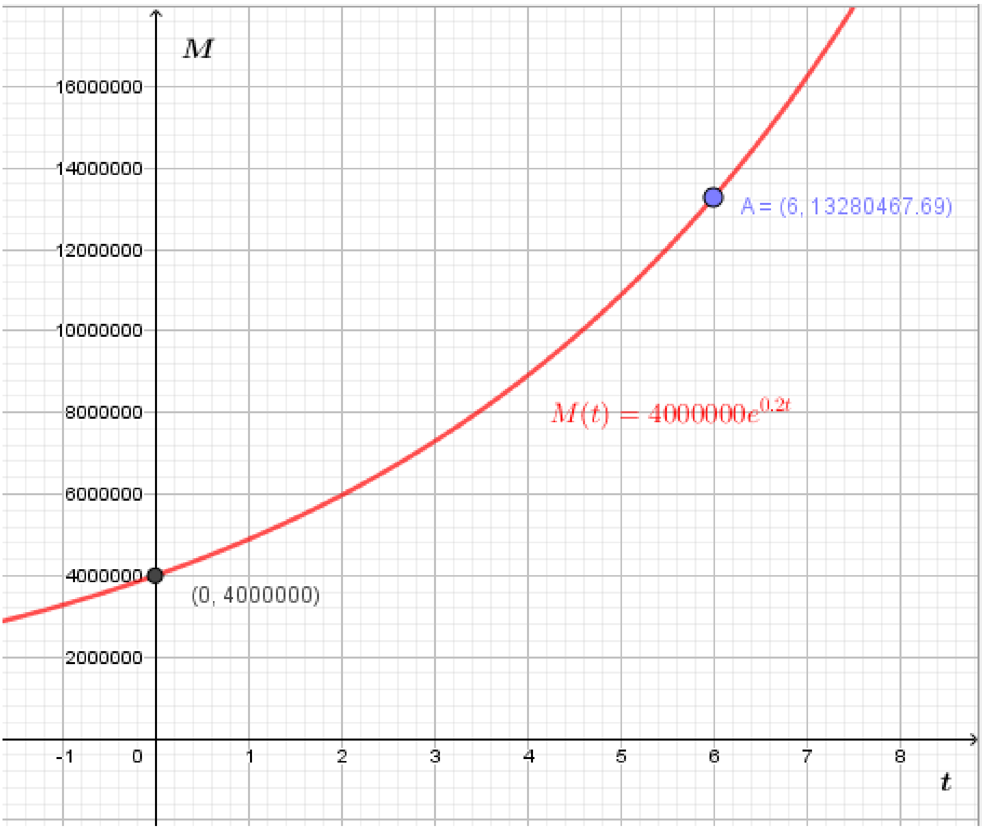
a) the consumption after 6 years if $r = 20\%$, $t = 6$ is $M = 4,000,000e^{0.2x6}$

Using calculator, we get, $M \approx 13,280,467$ T

b) Draw the graph illustrating the consumption in function of time.

Graph showing the consumption in function of time or $M(t) = 4,000,000e^{0.2t}$

Graph of $M(t) = 4,000,000e^{0.2t}$



Lesson 11: Interest rate problems

a) Prerequisites/Revision/introduction:

Learners will learn better the interest rate problems if they have a clear understanding of:

- Logarithmic and exponential functions (Senior6, unit2, previous lessons)
- The concepts: **principal**, **interest rate** and the **period for investment** (From Entrepreneurship)

b) Teaching resources:

Learner's book, scientific calculators or Microsoft Excel eventual other books where the content about rate problems can be found.

c) Learning activities:

- Organize the learners into groups and ask them to attempt activity 2.11in the learner's book
- Have them discuss the terminologies used when solving an interest rate problem and predict, from the table, the total amount at the end of t years when the interest is compounded once per year
- As they are discussing, concentrate on slow learners for further explanation and provide assistance to groups in need.
- Check how adequately the learners are using calculators and how each member of the group is contributing to the discussion.
- Once the group discussion is over, ask a group, chosen randomly, to present his results while other learners are following attentively.
- Have learners exchange their view, in mutual respect and without confrontation to establish the main points from the presentation, and to take note.
- Ask learners to work out example 2.12 under your guidance and work individually application activity 2.11 to check the skills they have acquired.

Solution to activity 2.11

At the end of	The total amount
The first year	2000+0.1(2000) =2000(1+0.1)
The second year	2000(1+0.1) +0.1[2000(1+0.1)] =2000(1+0.1)^2
The third year	2000(1+01)^3
The fifth year	2000(1+0.1)^5
The t^{th} year	2000(1+0.1)^t

Solution to application activity 2.11

With bank I, the amount at the end of the year is $A = P\left(1+\frac{r}{n}\right)^{nt}$ =

$300000\left(1+\frac{0.1}{1}\right)^{1X10}$ Frw $778,122 Frw$

With bank II, the amount at the end of the year is: $A = Pe^{nt}$ = $300000e^{0.098x10} Frw =$

799,336 frw

You should advise your aunt to invest at bank II because 999,336Frw > 778,122Frw

Lesson 12: Mortgage problems

a) Prerequisites/Revision/introduction:

Learners will learn better the mortgage problems if they have a clear understanding of:

- Logarithmic and exponential functions(previous lessons)
- The concepts: **loan, principal**, **interest rate**, **payment by instalment** and the **period for investment**(From Entrepreneurship)

b) Teaching resources:

Learner's book, scientific calculators or Microsoft Excel, eventual other books where the content about mortgage can be found.

c) Learning activities:

- Form small groups and let the learners work out activity 2.12
- Give clear instructions on the duration and how the group work is to be performed
- Let learners discuss the terminologies used when solving a mortgage problem

- Help learners to identify and to analyse the quantities involved in question2
- Have learners determine the monthly payment retained at the bank ,by proper substitution in the formula relating the different quantities and computation using a scientific calculator
- Walk around, as the learners are discussing, to provide further explanation where necessary, and to encourage each member to participate actively in the discussion
- After group discussions, have a group present his work and request other learners to follow attentively, as they are evaluating, in the light of the presentation, the work of their respective groups
- Ask learners to give their observation, in mutual respect and without confrontation, so as to come with a common understanding on the main points to be noted in the book
- Let learners proceed to example 2.13 under your guidance, and check their working against the solution proposed in the learner's book, and then request them to work individually application activity 2.12 to check the skills they have acquired

Solution to activity 2.12.

1) Collect information for the meaning of the following concepts: the periodic payment(P), annual interest rate (r), mortgage amount (M), number t of years to cover the mortgage and the number n of payments per year.

2) $P=\dfrac{\frac{rM}{n}}{1-\left(1+\frac{r}{n}\right)^{-nt}}=\dfrac{\frac{(0,06)(20000000)}{12}}{1-\left(1+\frac{0.06}{12}\right)^{(-12)(20)}}=143286.2$ The amount to pay per month should be 143286.2 FRW but in practice the bank will convert such amount into 143 287 FRW

. Generally bank offices round figure to the nearest greater integer. The last payment will be less amount than 143 287 FRW, as there will be an adjustment by considering the difference between the real amount and the amount to be paid per month.

The amount to pay per month is 143 287 FRW, the balance the brother will withdraw each month is 500,000FRW-143,287 FRW=356713 FRW

At the end of 20 years, your brother would have paid $143287\times12\times20Frw=$ 34,388,880 Frw·

The interest the bank will realize is 34,388,880 FRW-20,000,000 FRW=14,338,880FRW

Solution to application activity 2.12.

The periodic payment is P=200 000, the annual rate r=10%=0.01, the number of payments per year n=12(since the payment is monthly), the number of years to cover the mortgage is t=20.,

Solving in M $P=\dfrac{\frac{rM}{n}}{1-(1+\frac{r}{n})^{-nt}}$ yields to $M=\dfrac{P\left[1-(1+\frac{r}{n})^{-nt}\right]}{r}n$.

Replacing each quantity by its value in the formula, we have

$$M=\frac{200000\times 12\times\left[1-(1+\frac{0.1}{12})^{-12(20)}\right]}{0.1}$$

Calculations give M=20,808,156.36. So, the mortgage is 20,808,156 Frw.

Lesson13: Population growth problems

a) Prerequisites/Revision/Introduction:

Learners will learn better the population growth problems if they have a clear understanding of:

- Logarithmic and exponential functions (previous lessons)
- Graphical interpretation of functions (previous lessons)

b) Teaching resources:

Learner's book, charts containing graphical representation of exponential functions, scientific calculators, eventual other books where the content about population growth can be found, and, if possible, a computer with mathematical software such as Geogebra and Microsoft Excel.

c) Learning activities:

- Form small groupssand let learners discuss the activity 2.13
- Distribute the tasks and give clear instructions on the duration and the internal organisation of each group
- Let learners analyse the graph and answer the questions related to it, under the supervision of the task manager of the group
- When the learners are on task, provide facilitation to the groups in need

- Once discussions areover, choose a group to present their work Ask learners to give constructive remarks in order to obtain an improved information to be written by all members. Let learners proceed to example 2.14 under your guidance, and check their working against the solution proposed in the learner's book, and then ask them to work individually application activity 2.13 to check the skills they have acquired.

Solution to activity 2.13.

a)

Time t(minutes)	0	1	2	3	4			
Number of cells	1	2	4	8	16			

b) the number of cells will be $N(\text{t}) = 2^t$

- If $N(\text{t}) = N_0 e^{kt}$, then $N_0 = 1$,since it is independent of the base, and then $e^{kt} = 2^t$
- Making k the subject of the formula, by applying natural logarithm on both sides of the equation $e^{kt} = 2^t$ and then $k = \square$,such that $N(\text{t}) = e^{(\ln 2)t}$
- Using the property: If $e^{\ln u} = u$ then $n \ln u = \ln u^n$, we have $e^{5(\ln 2)} = e^{\ln 32} = 32$
- From the graph, as the time becomes elapses, the number of cells grows exponentially.

Solution to application activity 2.13

1) $N(t) = N_0 e^{tk}$. Substituting for $N_0 = 1000000$ and $N(5) = 2N_0$,we obtain

$k = \frac{1}{5} \ln 2$. The population, in 10 years would be $N(10) = 1000000 e^{10(\frac{1}{5}\ln 2)} = 4000000$

2) $\text{A}(t) = A_0 e^{tk} = 56 \times 10^9 e^{(0.025)(1.75)} = 58.504,384$

3) From $13000 == 11000 e^{6k}$, $k = \frac{1}{6} \ln \frac{13}{11}$

Lesson 14: Radioactive decay problems

a) Prerequisites/Revision/introduction:

Learners will solve easily some radioactive decay problems if they have a clear understanding of:

- Logarithmic and exponential functions (previous lessons)
- Graphical interpretation of functions (previous lessons)

b) Teaching resources:

Learner's book, charts containing graphical representation of exponential functions, scientific calculators or Microsoft Excel, eventual other books where the content about radioactive decay can be found and, if possible, a mathematical software with the application Geogebra

c) Learning activities:

- Organize learners into groups and ask them to attempt activity 2.14
- As they are discussing, concentrate on slow learners for further explanation and provide assistance to groups in need
- Check how adequately the calculators are used, and how each member of the group is contributing to the discussions
- Once the group discussion is over, ask a group, chosen randomly, to present his results while other learners are following attentively
- Have learners exchange their views in mutual respect and without confrontation to note the main points
- Ask learners to work out example 2.15 under your guidance, and work individually application activity 2.14 to check the skills they have acquired

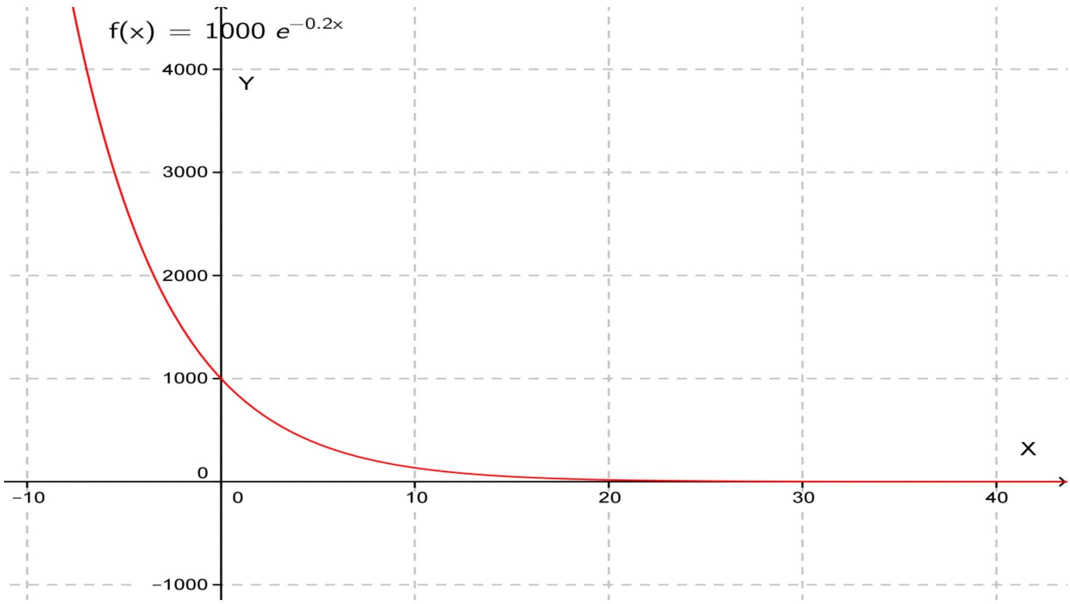
Solution to activity 2.14.

Function $N(t)=N_0e^{-0.2t}$ passes through points $(0,1000)$ and $(5,800)$,

Then $800=1000e^{-5k}$ Solving for k, we have: $k=-\frac{1}{5}\ln\frac{4}{5}=0.0446$

Using GEOGEBRA to graph the function $N(t)=1000e^{-0.2t}$,

we obtain:



Graph of the function $N(t)=1000e^{-0.2t}$

Solution to application activity 2.14

1) a) The fixed price is 5

b) As the quantity demanded becomes larger and larger, the price decreases

c) From the formula $N(\text{x})=N_0e^{kx}$, using $N(0)=5$ and $N(1)=3$,we find $N(\text{x})=5e^{-0.51x}$

2) a) 50 cm^2 b)22.46 cm^2

3) $u(t)=20+(80-20)e^{kt}=20+60e^{kt}$

$u(4)=20+60e^{k(4)}=60; k=\frac{1}{4}\ln\frac{2}{3}\approx -0.10136$

Such that $u(t)=20+60e^{-0.10136t}$; $u(\text{t})=25 \Leftrightarrow 20+60e^{-0.10136\,\text{t}}=25 \Leftrightarrow t=23.971;$

It will take about 24 minutes

Lesson15: Earthquake problems

a) Prerequisites/Revision/introduction:

Learners will learn better the earthquake problems if they have a clear understanding of:

- Logarithmic and exponential functions (, previous lessons)
- Graphical interpretation of functions (previous lessons)

b) Teaching resources:

Learner's book, scientific calculators and eventual other books where information about the characteristics and measurement of an earthquake can be found.

c) Learning activities:

- Arrange the learners into groups for the discussion of activity 2.15
- Distribute the tasks and give clear instructions on the duration and the internal organisation of each group
- Let learners discuss the terminologies used in modelling the magnitude of an earthquake and how Richter used logarithms to compare the magnitudes of earthquakes
- When learners are on task, provide any assistance to groups in need
- Once the discussions are over, choose a group to present its work when other learners are following attentively
- Ask learners to give constructive remarks, in order to obtain an improved result to be written by all members. Have the learners work in order, with mutual respect and without confrontation
- Let learners proceed to example 2.16 under your guidance, and check their working against the solution proposed in the learner's book, and then request them to work individually application activity 2.15 to check the skills they have acquired.

Solution to activity 2.15

The quantities involved in the measurement of an earthquake: the epicentre, the seismographic reading x_0 at a distance of 100 kilometres when there is no earthquake and the seismographic reading x at a distance of 100 kilometres from the epicentre when there is an earthquake.

The formula $M(x) = \log \frac{x}{x_0}$ is used to evaluate the magnitude of an earthquake.

The ratio of the seismographic readings is used to compare the intensities of two earthquakes.

Solution to application activity 2.15

a) Let x and y be the seismographic readings of the earthquakes at Ecuador and at Mexico, respectively.

Then, $\log\frac{x}{0.001}=7.8$ and $\log\frac{y}{0.001}=8.1$

This is equivalent to: $\frac{x}{0.001}=10^{7.8}$ and $\frac{x}{0.001}=10^{8.1}$

Dividing side by side, $\frac{\frac{x}{0.001}}{\frac{y}{0.001}}=\frac{10^{7.8}}{10^{8.1}}\Leftrightarrow\frac{x}{y}=10^{7.8-8.1}=10^{-0.0.3}=0.998$

This means that the earthquake at Ecuador was almost as heavy as the one that happened at Mexico.

b) Let z be the seismographic reading of the earthquake at San Francisco.

Then $\log\frac{x}{0.001}=7.8$ and $\log\frac{y}{0.001}=6.9$

This is equivalent to: $\frac{x}{0.001}=10^{7.8}$ and $\frac{x}{0.001}=10^{6.9}$

Dividing side by side, $\frac{\frac{x}{0.001}}{\frac{y}{0.001}}=\frac{10^{7.8}}{10^{6.9}}\Leftrightarrow\frac{x}{y}=10^{7.8-6.9}=10^{0.9}=7.943$

This means that the earthquake at Ecuador was is about 8 times heavier than the one that happened at San Francisco.

Lesson 16: Carbon dating problems

a) Prerequisites/Revision/introduction:

Learners will solve easily "carbon dating problems" if they have a clear understanding of:

- Logarithmic and exponential functions (previous lessons)
- Graphical interpretation of functions (previous lessons)

b) Teaching resources:

Learner's book, scientific calculators and eventual other books where information about carbon dating can be found.

c) Learning activities:

- Form small groups and let learners discuss activity 2.16
- Walk around, as learners are discussing, to provide further explanation where necessary, and to encourage each member to participate actively in the discussion
- After group discussions, ask one group present his work and request other learners to follow attentively, as they are evaluating, in the light of the presentation, the work of their respective groups
- Ask learners to give their observations, in mutual respect and without confrontation, so as to obtain the main points to be noted in the book
- Let learners proceed to example 2.17 under your guidance and check their working against the solution proposed in the learner's book, and then request them to work individually application activity 2.16 to check the skills they have acquired.

Solution to activity 2.16

$N(t) = N_0 e^{-0.693t/H}$, where H represents the half-life of Carbon-14, N_0 represents the amount of the radioactive material at time t=0 (time of the death), solving for t, in the equation $N(t) = N_0 e^{-0.693t/H}$, we obtain $t = \frac{H}{-0.693}.\ln\frac{N(t)}{N_0}$

Solution for application activity 2.16

From the formula $N(t) = N_0 e^{-0.693t/H}$,solving for t, $t = \frac{H}{-0.693}.\ln\frac{N(t)}{N_0}$, and substituting for H=5700, $\frac{N(t)}{N_0} = 0.79$,we have: $t = \frac{5700}{-0.693}.\ln(0.79) = 1938.84$

The age of the animal given that the half-life of carbon-14 is 5700 years would be about 1939 years.

Lesson17: Problems about alcohol and risk of car accident

a) Prerequisites/Revision/introduction:

Learners will learn better "problems about alcohol and risk of car accident" if they have a clear understanding of:

- Logarithmic and exponential functions (previous lessons)
- Graphical interpretation of functions (previous lessons)

b) Teaching resources:

Learner's book, scientific calculators, charts and eventual other books where the content can be found, and, if possible computer with the software Geogebra.

c) Learning activities:

- Organize learners into groups to discuss the activity 2.17
- Distribute the tasks and give clear instructions on the duration and the internal organisation of each group
- Let learners discuss the relationship between the alcohol concentration in the blood of a driver and the risk of car accident.
- When the learners are on task, provide support to the groups in need
- Once discussions are over, choose a group to present his work when other learners are following attentively
- Ask learners to give constructive remarks and complements, in order to obtain a conclusion to be noted by all learners.
- Let learners proceed to example 2.18 under your guidance, and check their working against the solution proposed in the book, and then request them to work individually application activity 2.17 to check the skills they have acquired

Solution to activity 2.17

a) Excess of alcohol taken by the driver can yield to car accident

b)

i)The risk when there is no alcohol in the driver's blood is 1; it is not zero because the car accident is not due only to excess of alcohol in the driver's blood.

ii) More the concentration of alcohol in the driver's blood, more the risk of accident.

c) Since the risk grows exponentially, the equation is of the type $R(x) = R_0 e^{kx}$ $R(0) = 1$

and $R(4) = 5$ give $R(x) = e^{(\frac{1}{4}\ln 5)x}$

Solution to application activity 2.17

a) For the concentration of alcohol in the blood of 0.05 and a risk of 8%, we have:

$$8 = 4e^{k(0.05)} \Leftrightarrow e^{0.05k} = 2 \Leftrightarrow 0.05k = \ln 2 \Leftrightarrow k = \frac{\ln 2}{0.05} = 13.86$$

b) Using k=13.86 and x=0.18, we have: $R = 4e^{(13.86)(0.18)} = 48.477$

For a concentration of alcohol in the blood of 0.18, the risk of accident is about 48.5%

c) $100 = 4e^{13.86x} \Leftrightarrow 13.86x = \ln 25 \Leftrightarrow x = 0.2442$: for a concentration of alcohol of 0.24, the risk of accident is 100%.

2.6 Unit summary

I. Logarithmic functions

- Definition: $\log_a x = y \Leftrightarrow a^y = x$, where $a > 0, a \neq 1, x > 0$; this definition is used to determine the domain and the range.
- Formula for changing the base, from base a to base e: $\log_a x = \frac{\ln x}{\ln a}$
- Limits of a logarithmic function:

For $f(x) = \log_a x$ the domain is $]0,+\infty[$, If $x_0 \in]0,+\infty[$, then $\lim_{x \to x_0} \log_a x = \log_a x_0$

$\lim_{x \to 0^+} \log_a x = \begin{cases} -\infty; & ;a > 1 \\ +\infty; & 0 < a < 1 \end{cases}$; in particular, $\lim_{x \to 0^+} \ln x = -\infty$.

$\lim_{x \to +\infty} \log_a x = \begin{cases} +\infty; & ;a > 1 \\ -\infty; & 0 < a < 1 \end{cases}$; in particular, $\lim_{x \to +\infty} \ln x = +\infty$

Indeterminate cases $\frac{0}{0};\frac{\infty}{\infty}$: the indeterminate can be removed by applying Hospital's rule

Indeterminate cases $\infty - \infty; 0 \times \infty; 1^{\infty}; 0^0; \infty^0$: Re-write the limit to obtain $\frac{0}{0}$ or $\frac{\infty}{\infty}$ and then apply Hospital's rule

- Derivative of a logarithmic function:

  $(\log_a \mathrm{u})' = \frac{u'}{\mathrm{u} \ln a}$; $(\ln \mathrm{u})' = \frac{u'}{\mathrm{u}}$, where u is function of variable x;

For more elaborated functions, such as product, power, quotient, etc, containing logarithms ,the rules for differentiation still apply

- Variations and graphs of logarithmic functions: either graph the function, using software, such as Geogebra, and then analyse the graph to draw the conclusion about maximum, minimum, increasing, decreasing, concavity, inflection point or, study the sign of the first derivative (eventually the second derivative) and draw conclusion about the variations, then graph the function.

II. Exponential functions

- Definition: $f(x) = a^x$, where $a > 0, a \neq 1$, ;this definition is used to determine the domain and the range of an exponential function

- Limits of an exponential function

  For $f(x) = a^x$ the domain is $]-\infty, +\infty[$

  - If $x_0 \in]-\infty, +\infty[$, then $\lim_{x \to x_0} a^x = a^{x_0}$
  - $\lim_{x \to -\infty} a^x = \begin{cases} 0 \ ; a > 1 \\ +\infty; \ 0 < a < 1 \end{cases}$; in particular, $\lim_{x \to -\infty} e^x = 0$
  - $\lim_{x \to +\infty} a^x = \begin{cases} +\infty; \ ; a > 1 \\ 0; \ 0 < a < 1 \end{cases}$; in particular, $\lim_{x \to +\infty} e^x = +\infty$

Indeterminate cases $\frac{0}{0}; \frac{\infty}{\infty}$: the indeterminate form can be removed by applying Hospital's rule.

Indeterminate cases $\infty - \infty; 0 \times \infty; 1^\infty; 0^0; \infty^0$: Re-write the limit to obtain $\frac{0}{0}$ or $\frac{\infty}{\infty}$ and then apply Hospital's rule.

- **Derivative of an exponential function:**

  $(a^u)' = u' a^u \ln a$; $(e^u)' = u' e^u$, where u is function of variable x .

  Note that for compounded functions such us power, product and quotient involving exponential functions, the differentiation rules are still applied.

- Variations and graphs of exponential functions: either graph the function, using software, such as Geogebra, and then analyse the graph to draw the conclusion about maximum, minimum, increasing, decreasing, concavity, inflection point or, study the sign of the first derivative (eventually the second derivative) and draw conclusion about the variations, then graph the function.

III. Applications of logarithmic and exponential functions

- Interest compounded n times per year, r: rate of annual interest, P: Principal

  $A = P\left(1 + \frac{r}{n}\right)^{nt}$

- Interest compounded continuously: $A = Pe^{rt}$

- Formula connecting the quantities involved in a mortgage problem: $P = \frac{\frac{rM}{n}}{1 - \left(1 + \frac{r}{n}\right)^{-nt}}$

  where P: Amount after t years, M: Mortgage, r: Annual rate interest, n: Number of payment per year, t: Number of years to cover the Mortgage

- Law of exponential growth or decay

  $P(t) = P_0 e^{kt}$ where P: Population at time t, P_0 :Initial population, $k > 0$ or $k < 0$, t: time

- Magnitude of earthquake with seismographic reading x : $M(x) = \log \frac{x}{0.001}$:
- Risk of car accident corresponding to concentration x of alcohol in the driver's blood $R(x) = R_0 e^{kx}$.

2.7. Additional information for the teacher

Given that the knowledge of the teacher must be wider than the one of the learners, the following information is useful for the teacher, though not stated in the learner's book:

- The study of logarithmic and exponential functions can follow the study of integrals. In this case, the natural logarithm of x is defined as $\ln x = \int_1^x \frac{dt}{t}$,where $x > 0$
- The concepts of logarithmic functions and exponential functions can be taught interchangeably. Exponential functions can be taught first: $f(x) = a^x$, where $a > 0$ and $a \neq 1$
- The roots of some equations involving logarithms or exponentials can be approximated using Taylor's expansion. Similarly, in the calculation of limits, some indeterminate cases can be removed by approximating the function involved in the limit, using Taylor's expansion.

  The Taylor's expansion of f(x) at x=0,or the Maclaurin's expansion of $f(x) = e^x$ and for $f(x) = \ln(1+x)$ are given below:

  $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$ for any value of x

  $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots$ for values of x in the neighbourhood of 0

Examples:

a) Approximate e^x by a quadratic function and approximate the roots of the equation

$e^x - x^2 = 0$

b) Approximate e^x and $\ln(1+x)$ by quadratic functions then calculate $\lim_{x \to 0} \frac{x^3 e^x}{xe^{-x} - \ln(x+1)}$

Solution:

a) $e^x - x^2 = 0 \Leftrightarrow (1+x+\frac{x^2}{2}) - x^2 = 0 \Leftrightarrow -\frac{1}{2}x^2 + x + 1 = 0 \Leftrightarrow x_1 = -1+\sqrt{3}; x_2 = -1-\sqrt{3}$

b) Substituting x in the limit, we obtain the indeterminate case $\frac{0}{0}$

Then $\lim_{x\to 0}\frac{x^2 e^x}{xe^{-x} - \ln(x+1)} = \lim_{x\to 0}\frac{x^2(1+x+\frac{x^2}{2})}{x(1-x+\frac{x^2}{2})-(x-\frac{x^2}{2})} = -2$

2.8. End unit assessment

This part provides the answers of end unit assessment activities designed in integrative approach to assess the key unit competence with cross reference to the textbook.

The following are standards (ST) which have been based on in setting end unit assessment questions:

ST3: Thoroughly explore and investigate logarithmic and exponential functions; reflecting analytically and logically about their learning and setting their own goals.

ST4: Correctly apply logarithmic and exponential functions in solving problems, selecting the appropriate mathematical operations and calculations.

ANSWERS FOR QUESTION ONE

a) $f(x) = \log_2(3x-2)$

$f(x) = \log_2(3x-2)$ is defined if and only if $(3x-2) > 0 \Leftrightarrow 3x > 2 \Leftrightarrow x > \frac{2}{3}$

Thus, $Domf = \left]\frac{2}{3}, +\infty\right[$. From the function, $0 < 3x-2 < +\infty$.

Then $-\infty < \log_2(3x-2) < +\infty$. therefore, the range is $\mathbb{R} =]-\infty, +\infty[$

b) $f(x) = \ln(x^2-1)$

$f(x) = \ln(x^2-1)$ is defined if and only if $x^2-1>0 \Leftrightarrow (x-1)(x+1) > 0$

Using the variation table of $\Leftrightarrow (x-1)(x+1) > 0$,

x	$-\infty$ $\quad$ -1 $\quad$ 1 $\quad$ $+\infty$
$x-1$	- 0 + + + + + + + + +
$x+1$	- - - - - - - - 0 + + + + + + + + + + + + + + + + + +
$(x-1)(x+1)$	+ + + + + + + 0- - - - - - - - - - - - - 0+ + ++ + + + + + +

We deduce the domaine requested: $Domf =]-\infty,-1[\cup]1,+\infty[$

From the function, $0 < x^2 - 1 < +\infty$ .Then $-\infty < \ln(x^2 - 1) < +\infty$ .Therefore, the range is $\mathbb{R} =]-\infty,+\infty[$

c) $f(x) = 2e^{3x+1}$

$Domf = \mathbb{R} =]-\infty,+\infty[$

From the function, $-\infty < 3x+1 < +\infty$;

Then $0 < e^{3x+1} < +\infty$; or $\quad 0 < 2e^{3x+1} < +\infty$ .Therefore, the range is $]0,+\infty[$

d) $f(t) = 4^{\sqrt{3t+1}}$

$f(t) = 4^{\sqrt{3t+1}}$ is defined if and only if $3t+1 \geq 0 \Leftrightarrow t \geq \frac{-1}{3}$

$Domf = \left[\frac{-1}{3},+\infty\right[$

From the function, $0 \leq 3t+1 < +\infty$; Then $0 \leq \sqrt{3t+1} < +\infty$;

$1 \leq 4^{\sqrt{3t+1}} < +\infty$. Therefore, the range is $[1,+\infty[$

ANSWERS FOR QUESTION TWO

a) $\lim_{x \to 0^+} \frac{\ln x}{x} = -\infty$ .We have a vertical asymptote which has equation $x = 0$

b) $\lim_{x \to +\infty} \left(3 + x^2 \ln x\right) = +\infty$ .No horizontal asymptote

ANSWERS FOR QUESTION THREE

1)a)

$$f(x) = \log_2 \sqrt{\frac{x^2-4}{x+2}} = \frac{1}{2}\left[\log_2(x-2) + \log_2(x+2) - \log_2(x+2)\right] = \frac{1}{2}\log_2(x-2)$$

$$\frac{d}{dx} f(x) = \frac{1}{(2\ln 2)(\mathrm{x}-2)}$$

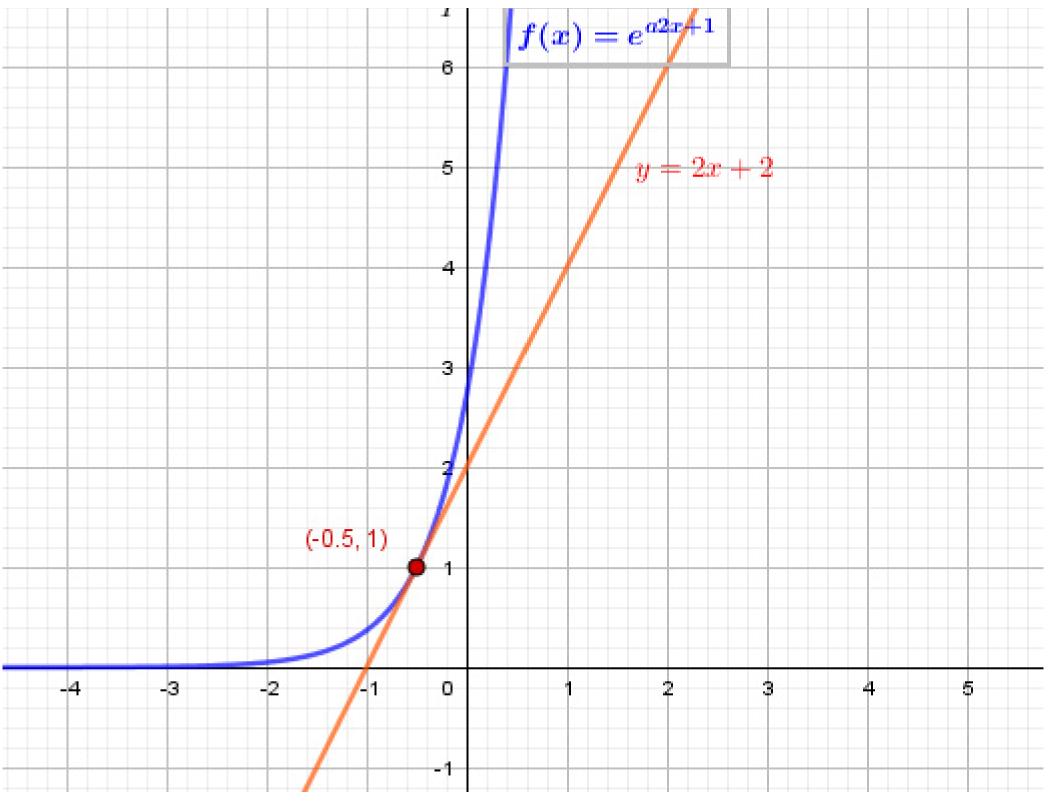
b) $\frac{d}{dx}h(x)=\frac{d}{dx}[\frac{1}{3}(4^{2x+5})]=\frac{2}{3}(4^{2x+5})\ln 4$

2) i)The value of the function $y=e^{2x+1}$ at $x=-\frac{1}{2}$ is $y=e^{2(-\frac{1}{2})+1}=1$

The intersection point is $A(-\frac{1}{2},1)$

ii) $\frac{dy}{dx}=2e^{2x+1}$. At $x=-\frac{1}{2}$, $\frac{dy}{dx}=2e^{2(-\frac{1}{2})+1}=2$

The equation of the tangent is $y-1=2(x+\frac{1}{2})$, which is equivalent to $y=2x+2$



Graph of the tangent of the curve at x= -0.5

ANSWERS FOR QUESTION FOUR

$y=xe^{-x}$;

$\frac{dy}{dx}=(e^{-x}-xe^{-x})=(1-x)e^{-x}$

$\frac{dy}{dx} = 0 \Leftrightarrow x = 1$, the extrema is $(1, \frac{1}{e})$ .From the study of the sign of $\frac{dy}{dx}$,
$\frac{dy}{dx} > 0$ for $x < 1$,and $\frac{dy}{dx} < 0$ for $x > 1$.Therefore, the stationary point, the extrema is a maximum

ANSWERS FOR QUESTION FIVE

1) Five applications of logarithmic or exponential functions:

- In Geography: the magnitude of an earthquake is found using logarithms
- In Entrepreneurship: the interest rate problems can be solved using logarithm and exponential functions
- In Chemistry: the radioactive decay problems are solved using logarithmic and exponential functions
- In History: archaeology involves carbon dating whose principles are based on the use of logarithms
- In Social studies: logarithms and exponentials are used in the determination of risk corresponding to a given concentration of alcohol

2) Assuming exponential growth, $N(t) = 5.7e^{0.02t}$

The population will reach 114 billion after t years such that $114 = 5.7e^{0.02t}$ .solving for t,
$t = \frac{1}{0.02} \ln \frac{114}{5.7} = 26.278$ years. The population will reach 114 billion in the year 2021.

2.9. Remedial, consolidation and extended activities

The teacher's guide suggests additional questions and answers to assess the key unit competence.

a) Remedial activities: Suggestion of questions and answers for remedial activities for slow learners.

1) Find the domain and the range of the function;

a) $f(x) = \log(x-1)$ b) $f(x) = 4^{\sqrt{6x}}$

2)Calculate: $\lim_{x \to +\infty} \left(2 - \frac{3}{\ln x} + e^{-x} \right)$

3) Given the logarithmic function $f(x) = \log_2(x-5)$

a) What is the equation of the asymptote line?

b) If $x=7$ find y

Answers:

1) **a) Domain:** $]1,+\infty[$ and **Range:** $\mathbb{R}=]-\infty,+\infty[$

b) Domain: $[0,+\infty[$ and **Range:** $\mathbb{R}=[1,+\infty[$

2) $\lim\limits_{x\to+\infty}\left(2-\dfrac{3}{\ln x}+e^{-x}\right)=2-\dfrac{3}{\lim\limits_{x\to+\infty}\ln x}+\lim\limits_{x\to+\infty}e^{-x}=2-\dfrac{3}{+\infty}+e^{-\infty}=2-0+0=2$

3)a) Vertical asymptote: $x=5$, since $\lim\limits_{x\to5^+}\log_2(x-5)=-\infty$

$y=f(7)=\log_2(7-5)=\log_2 2=1$

b) Consolidation activities: Suggestion of questions and answers for deep development of competences.

1)Consider the function $f(x)=6^{x-2}$

a) Determine $f'(x)$

b) Find the equation of the tangent to the graph of the function at the point where $x=3$

c) Graph the function and its tangent

2)Suppose the function $f(x)=2x-\ln x$

a) State the domain and range

b) Find the 1st derivative.

c) Solve for $f'(\mathrm{x})=0$

d) Determine a point through which the graph passes

e) Draw the variation table of $f(x)$, and find the stationary point and its nature.

f) Sketch the graph

3) Two earthquakes took place at town B, and at town K. Their magnitudes, on Richter's scale, were 5.6 and 5.2, respectively. Compare the two earthquakes by finding the ratio of their seismographic readings.

4) An amount of 1, 000 000 FRW is invested at a bank that pays an interest rate of 10% compounded annually.

a) How much will the owner have at the end of 15 years, in each of the following alternatives?The interest rate is compounded:

i) Once a year.

ii) Twice a year

b) Compare the two types of compounding, and explain which one is the best

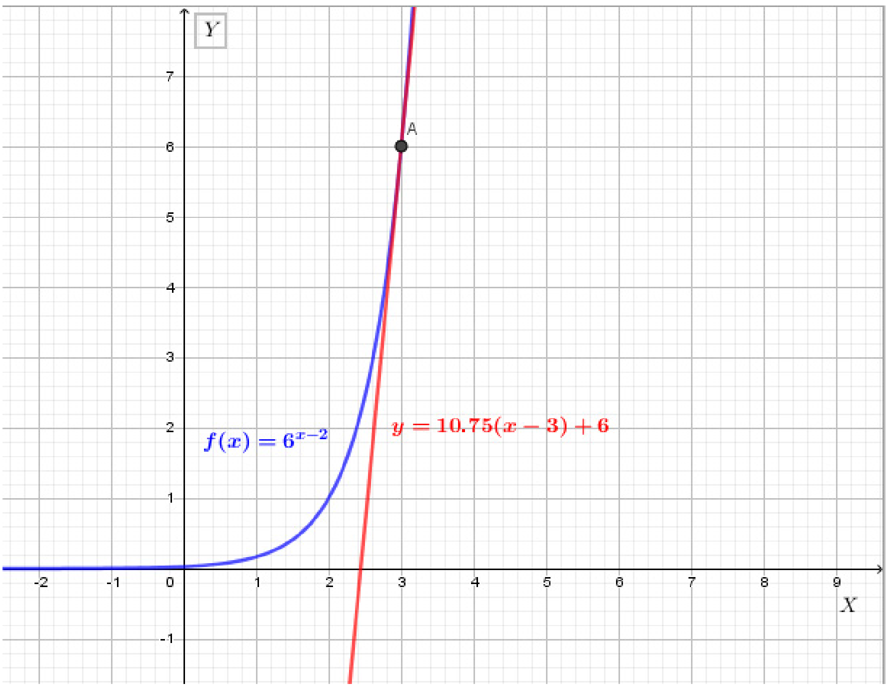
5)How long will it take money to double at 5% interest when compounded quarterly?

Solution:

1)a) $f'(x) = (6^{x-2})' = 6^{x-2}\ln 6$

b) $f'(3) == 6^{3-2}\ln 6 = 6\ln 6$ and $f(3) = 6^{3-2} = 6$

c) the equation of the tangent is then $y - 6 = (6\ln 6)(x-3)$



Graphic of the tangent to the graph of the function f(x) at the point $x = 3$

2) Function $f(x) = 2x - \ln x$

a) Domain: $]0, +\infty[$

The range is $\mathbb{R} =]-\infty, +\infty[$

b) The first derivative: $f'(x) = 2 - \frac{1}{x}$

c) Calculate $f'(\text{x}) = 0 \Leftrightarrow 2 - \frac{1}{x} = 0 \Leftrightarrow \frac{1}{x} = 2 \Leftrightarrow 2x = 1 \Leftrightarrow x = \frac{1}{2}$

d) An example of a point through which the graph of $f(x)$ passes is $\text{A}(1, 2)$

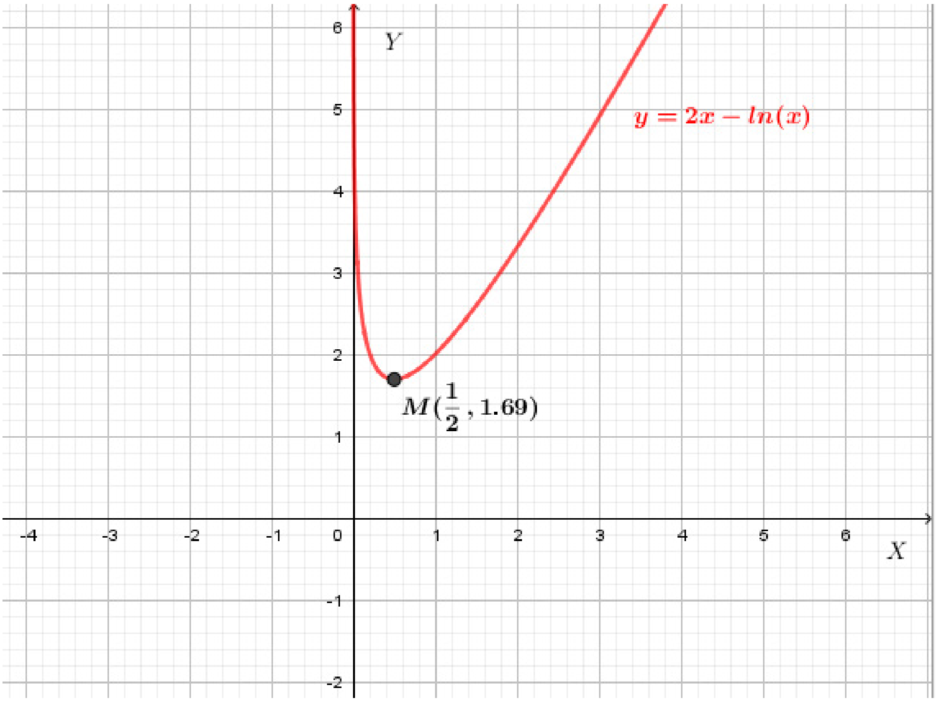
e) Variation table of $f(x)$

x	0		$\frac{1}{2}$		$+\infty$
$f'(x)$		$-------$	0	$+++$	
$f(x)$	$+\infty$	↘	$1+\ln 2$	↗	$+\infty$

Function f is decreasing on interval $]0, \frac{1}{2}[$ and increasing on $]\frac{1}{2}, +\infty[$.

The graph has a minimum at point $M\left(\frac{1}{2}, 1+\ln 2\right)$

f) The graph of f(x)



Graph of the function $f(x) = 2x - \ln x$

3)Let x and y be the seismographic readings of the earthquakes at B and at K, respectively.

Then, $\log\frac{x}{0.001} = 5.6$ and $\log\frac{y}{0.001} = 5.2$ .This is equivalent to: $\frac{x}{0.001} = 10^{5.6}$ and $\frac{y}{0.001} = 10^{5.2}$

Dividing side by side, $\frac{\frac{x}{0.001}}{\frac{y}{0.001}} = \frac{10^{5.6}}{10^{5.2}} \Leftrightarrow \frac{x}{y} = 10^{5.6-5.2} = 10^{0.4} = 2.5118$

This means that the earthquake at town B is about 2.5 times heavier than the earthquake at town K .

4) a) i) For once a year, at the end of 15 years the owner will have

$$A = P(1+r)^t = 1,000,000(1+0.10)^{15}$$

$$= 1,000,000(1.10)^{15} = 4,177,248.16\, frw$$

ii) For twice a year, at the end of 15 years, the owner will have

$$A = P(1+\frac{r}{2})^{2t} = 1,000,000(1+\frac{0.10}{2})^{2(15)}$$

$$= 1,000,000(1.05)^{30} = 4,321,942.37\, Frw$$

b)Conclusion: since $4,321,942.3 > 4,177,248.1$, compounding many times per year is better.

5) $A = P(1+\frac{0.05}{4})^{4t} = 2P \Leftrightarrow (1.0125)^{4t} = 2$. Calculating t we obtain $t = 13.95$ years

c) Extended activities: (Suggestion of Questions and Answers for gifted and talented learners).

1) The revenue R obtained by selling x units of a certain item at price p per unit is $R = xp$.

If x and p are related by $p(x) = 8.25e^{-0.02x}$ .Find the price and the number of units to sell for the revenue to be maximized

2) Organic waste is dumped into a pond. As the waste material oxidizes, the level of oxygen in the pond is given by $f(t) = \frac{t-e^{-t}}{t}$, where t is the time in weeks.

Find the level of oxygen in the pond as the time gets larger and larger (express the answer in percentage).

Solution:

1)The revenue is $R(x) = 8.25xe^{-0.02x}$ The revenue is maximum if $R'(x) = (8.25xe^{-0.02x})' = 0 \Leftrightarrow (8.25e^{-0.02x} - 0.165xe^{-0.02x})' = 0$

$\Leftrightarrow 1-0.02x = 0 \Leftrightarrow x = 50$ .Thus $p(50) = 8.25e^{-0.02(50)} = \frac{8.25}{e} = 3.035$

2) $\lim_{t\to+\infty} \frac{t-e^{-t}}{t} = 1 - \lim_{t\to+\infty} \frac{e^{-t}}{t} = 1 - 0 = 1$. As the time gets larger and larger, the level of oxygen approaches 1, that is 100%

UNIT 3: INTEGRATION

3

3.1 Key Unit Competence:

Use integration as the inverse of differentiation and apply the definite integrals to find area of a plane shapes.

3.2 Prerequisite knowledge and skills

- Solving exponential equations (Senior 4: unit 2 and Senior 5: unit 3)
- Differentiation of polynomials, rational and irrational functions and their applications (Senior 4, unit 6).
- Logarithmic and exponential functions and their applications
- (Senior5: unit3 and Senior 6: unit 2)
- Trigonometric and inverse of trigonometric functions where there is application of derivatives in solving real life problems; differentiation and linearization of trigonometric function (senior 5: unit 4 and Senior 6: unit 1)

3.3 Cross-cutting issues to be addressed

- Inclusive education (promote education for all while teaching)
- Peace and value Education (respect others view and thoughts during class discussions)
- Gender (equal opportunity of boys and girls in the lesson participation)
- Financial education (mortgage, rate of change, marginal cost, marginal utility, demand function, marginal demand....)
- Environment and sustainability (population growth,)
- Comprehensive sexuality (alcohol abuse ...)

3.4. Guidance on the introductory activity

- Invite learners to form groups and lead them to work on introductory activity to understand the concept of the anti- derivative; using integration as the inverse of differentiation and to calculate the area of a plane shape.

- Allow learners present their findings

- Harmonize their works and ensure that they got exact solution:

Solution for the introductory activity:

a) One unit stands for one meter:

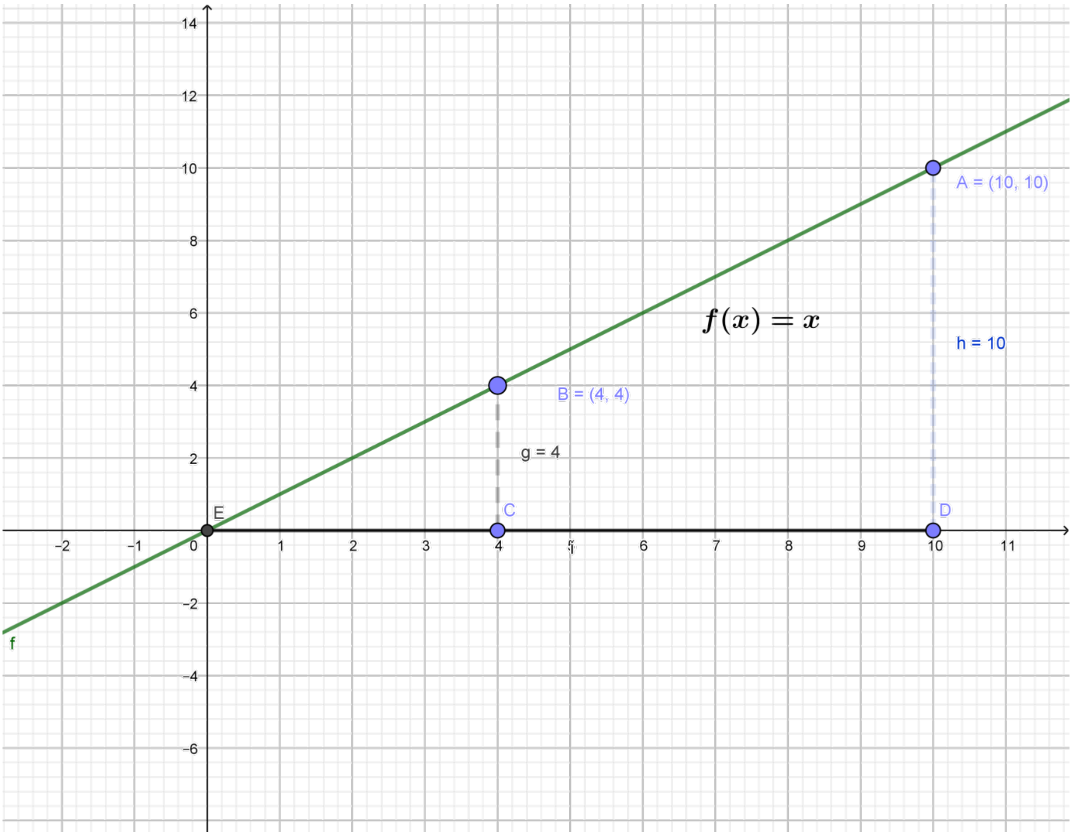


Figure: The quadrilateral field

1) The area A_1 found by the first group is $A_1 = area(\Delta EDA) - area(\Delta ECB)$

Calculate the area of $area(\Delta EDA) = \frac{10x10}{2} = 50m^2$

Calculate the area of $area(\Delta ECB) = \frac{4X4}{2} = 8m^2$

Therefore $A_1 = (50-8) = 42m^2$

2) The second group with high critical thinking skills used a function $F(x)$ that was differentiated to find $f(x) = x$ (which means $F'(x) = f(x)$) and the x-coordinate d of D and the x- coordinate c of C in the following way: $A_2 = F(d) - F(c)$.

Therefore $F(x) = \frac{x^2}{2} + c$, where c is a given constant.

$F(x)$ is said to be an integral of $f(x)$ or anti-derivative of $f(x)$, because $F'(x) = f(x) = x$ in this case.

3) The area A_2 found by the second group using $F(x)$ is

$A_2 = F(d) - F(c) = F(10) - F(4)$. Then $F(10) = \frac{10^2}{2} = 50$ and $F(4) = \frac{4^2}{2} = 8$

Therefore $A_2 = F(10) - F(4) \Leftrightarrow 50 - 8 = 42m^2$

4) We realize that A_1 and A_2 are equal (see results found in (1) and in (2).

Therefore, referring to the graph of the function $f(x)$ on the figure 3.1, you can find the area bounded by a function $f(x)$ and x-axis and the lines of equations are $x = x_1$ and $x = x_2$.

If $F'(x) = f(x)$, the area is calculated using $Area = F'(x_2) - F'(x_1)$.

3.5. List of lessons

UNIT TITLE: INTEGRATION (30 periods)			
Introductory activity: 1 period = 40 minutes			
SUB-UNIT 1: INDEFINITE INTEGRAL (5periods)			
	Lesson title	**Learning objectives (from the syllabus including knowledge, skills and attitudes):**	**Number of periods**
1	Differential of a function	Define and interpret geometrically the differential of a function Use differentials to approximate the value of a function at a point	2
2.	Anti-derivatives	State and clarify the relationship between derivative and antiderivatives of a function	1

3.	Definition and properties of indefinite integral	Use properties of integrals to simplify the calculation of indefinite integrals	2
SUB-UNIT 2: Techniques of integration: (12periods)			
4	Basic integration formulae	Calculate indefinite integrals using appropriate techniques	3
5.	Integration by change of variable		3
6.	Integration by parts		3
7	Application indefinite integrals:	Use integrals to solve problems in Physics (work,..), Economics (marginal cost, revenue, utility profit and total cost functions,etc).	3
SUB-UNIT 3:Definite integrals (9 periods)			
8.	Definition and properties of definite integrals	Differentiate between indefinite and definite integrals Use properties of integrals to simplify the calculation of definite integrals	3
9.	Techniques of integration	Calculate definite integrals using appropriate techniques	3
10.	Applications of definite integrals: Calculation of the area of a plane surface	Apply definite integrals to calculate the area of a plane surface Use integrals to solve problems in Physics (work,..), Economics (marginal , utility and total cost,etc.)	3
End of unit assessment (3periods)			

Notice: *For application of mathematics content to other subjects, the teacher will consider the prerequisite of learners in this domain then act accordingly; the time spent and importance given to application activities depends on the learner's level of knowledge as well as their major subjects.*

Lesson 1: Differentials

a) Prerequisites/Revision/introduction:

Learners will get a better understanding of the content of this lesson if they refer to derivatives of functions (Senior 4, unit 6), increment or change in a variable or in a function (Senior 4, unit 6).

For example, guide them to find the derivative of the function $f(x)=\sqrt{x}$ and the change in the function $f(x)=\sqrt{x}$ when the variable x changes from 9 to 10.

b) Teaching resources:

Learner's book, Reference books, scientific calculators. If possible computers with mathematical software such as Geogebra, Microsoft Excel, Math-lab or Graph-Calc and internet can be used.

c) Learning activities:

- Form groups and ask learners to attempt activity 3.1, in the learner's book.
- Facilitate learners to use scientific calculator to find the numerical values of the given function $y=f(x)$ at 2 and 10. Let learners discuss and illustrate the relationship among increment of x, the change in y and the derivative $f'(x)$.
- Ask one group to present his/her findings to the whole class.
- Guide learners to follow attentively and to interact about the findings and to write a short summary.
- Let learners work out example 3.1 under your guidance and work individually application activity 3.1 for assessment.

Solutions to activity 3.1:

a) $y=f(x)=4+0.5x+0.1\sqrt{x}$; The consumption at x= 2 is

$f(2)=4+0.5(2)+0.1\sqrt{2}=5+0.1\sqrt{2}$. The consumption at x=10 is

$f(10)=4+0.5(10)+0.1\sqrt{10}=9+0.1\sqrt{10}$

b) The corresponding increment of y is

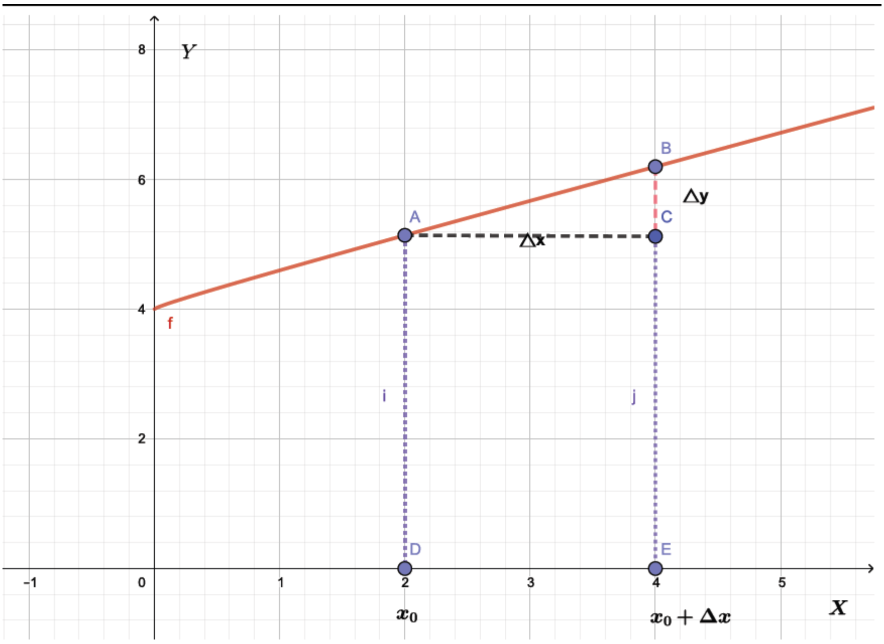
$f(10)-f(2)=(9+0.1\sqrt{10})-(5+0.1\sqrt{2})\approx 4.175$

c) If x changes from x_0 to x_1 where $(x_1>x_0)$,then $f(x)$ changes from $f(x_0)$ to $f(x_1)$

the increment of x is $\Delta x=x_1-x_0$ and the change in y is $\Delta y=f(x_1)-f(x_0)=\ldots$

d) If Δx is very small, then we have $\lim_{\Delta x\to 0}\Delta y=\lim_{\Delta x\to 0}f'(x_0)\Delta x=dy=f'(x_0)dx$

e) Graphical interpretation:



In this regard, $dy = 0.5 + \dfrac{0.1}{2\sqrt{x}} dx$.

Solutions for application activity 3.1:

1)a) $d(\sin 3x) = 3\cos 3x dx$ b) $d(x^2 e^x) = (2x + x^2)e^x dx$ c) $d(\frac{\ln x}{x}) = (\frac{1-\ln x}{x^2})dx$

2) If x is the length of the side, then the volume of the tank is given by $y = x^3$; $dy = 3x^2 dx$

For $x = 4$ and $dx = 0.02$, $dy = 3(4)^2(0.02) = 0.96$ .The volume of the tank is $x^3 = 4^3\ m^3 = 64\ m^3$ with an error of 0.96. In litres, the volume is 64000 litres with an error of 0.96.

Lesson 2: Anti-derivatives

a) Prerequisites/Revision/introduction:

Students will learn better the anti-derivatives if they have mastered the following concepts: Derivatives of functions (Senior 4: unit6; Senior 6: unit2), Differential of a function (Senior 6: unit3, lesson1)

For example, guide the learners to find the derivative $y' = f'(x)$ for the function $y = f(x)$ if i) $y = \sin x$, ii) $y = e^x$, iii) $y = \ln x$.

b) Teaching resources:

Learner's books and any other books where the content can be found.

c) Learning activities:

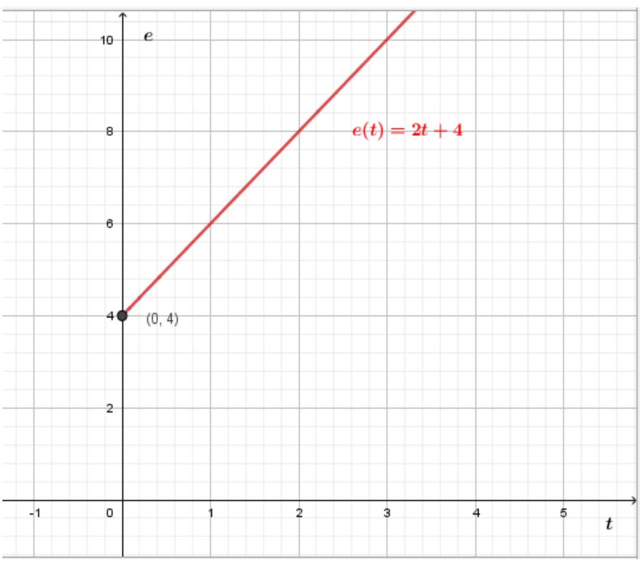
- Form groups and ask learners to attempt activity 3.2 in the learner's book.
- As they are working, walk around to ensure that they are performing the task effectively and cooperatively.
- Provide assistance where necessary by facilitating learners to determine the positions of the caterpillars, to draw the graphs and to guess a function differentiated to find a given derivative.
- Ask one group to present his/her findings to the whole class
- Guide learners to follow attentively their classmates and to interact about the findings and to write a short summary
- Guide learners to work out example 3.2 under your guidance and work individually application activity 3.2.

Solution for activity 3.2:

1) If v is the constant velocity, the position of a moving body at time t is given by $e(t) = vt + e_0$ where e_0 is the initial position.

i) $e(t) = 2t + 1$ ii) $e(t) = 2t + 2$ iii) $e(t) = 2t + 4$

2) For the third caterpillar, $e(t) = 2t + 4$. Therefore, $e'(t) = 2$ m/min which is the velocity.



$e'(t) = 4 = v(t)$

3) i) $F(x) = \sin x + k$ where k is a constant, since $F'(x) = (\sin x + k)' = \cos x$

ii) There are infinitely many possibilities for $F(x)$ because k can take different values in the set of real numbers.
iii) They all differ by a constant.

Solution to application activity 3.2:

1) a) The position of the student is $s(t) = \int (2t+1)dt = t^2 + t + C$

From initial condition we have $s(0) = 0^2 + 0 + C = 500$ .Then, C=500. Therefore, $s(t) = t^2 + t + 500$

b) The school is $\left((20)^2 + 20 + 500\right) meters = 920\ meters$ from the office (considered as origin).

In 20 minutes, the brother (or sister) would have travelled (50 m /min) (20 min) =1000 m, which is greater than 920 m. Therefore, the elder brother will meet the student before reaching the school.

Lesson 3: Definition and properties of indefinite integrals

a) Prerequisites/Revision/introduction:

Students will learn better the definition and properties of indefinite integrals if they have mastered the following concepts from previous years or previous lessons: Anti derivatives of functions (Senior 6: unit 3, lesson 2), derivative of the sum, the product or the quotient of two functions (Senior 4: unit 6).

For example, guide the learners to discuss the following:

Given the derivatives $f'(x)$ and $g'(x)$ of functions $f(x)$ and $g(x)$ respectively, and a constant k, find $(f+g)'(x)$ and $(kf)'(x)$

b) Teaching resources:

Learner's book and any other books where the content can be found.

c) Learning activities:

- Form groups and ask learners to attempt activity 3.3, in the learner's book.
- Monitor the work of different groups and facilitate the learners in need. Ensure that the learners do not confuse **derivative** and **anti-derivative**
- Choose randomly a group to present the findings to the whole class, while the

audience is following attentively,

- Facilitate learners to exchange their views about the presentation of their classmates
- Harmonize their findings and help them to conclude that the process of finding the anti-derivatives or indefinite integral of a function is called **integration**.
- Initiate learners to summarize the key points of the presentation.
- Guide learners to work out example 3.3 and example 3.4, and work individually application activity 3.3, in the learner's book to assess the skills they have acquired from the lesson.

Solution to activity 3.3:

a) An anti-derivative of $f(x)=x+2\cos x$ is $F(x)=\frac{1}{2}x^2+2\sin x$

b) No, it is not unique. Other anti-derivatives are, for example, $F_1(x)=\frac{1}{2}x^2+2\sin x-1$

$F_2(x)=\frac{1}{2}x^2+2\sin x+4$, $F_3(x)=\frac{1}{2}x^2+2\sin x+7$.

c) The part $\frac{1}{2}x^2+2\sin x$ is common to all anti derivatives;

All anti derivatives differ by an additive constant.

d) The systematic way of finding the set of all anti-derivatives of a given function is the integration of $f(x)=x+2\cos x$.

Solutions to application activity 3.3:

1) a) $\int(x^3+3\sqrt{x}-7)dx=\frac{1}{4}x^4+2\sqrt{x^3}-7x+C$

b) $\int(4x-12x^2+8x-9)dx=2x^2-4x^3+4x^2-9x+C$

c) $\int(\frac{1}{x^2}+e^{-x}-\frac{2}{x})dx=-\frac{1}{x}-e^{-x}-2\ln x+C$

2) It is not correct because the integral of a quotient is not the quotient of integrals.

$$\int\frac{x^3-2}{x^3}dx=\int(1-2x^{-3})dx=x+\frac{1}{x^2}+C$$

3) $\int\frac{x^3-5}{x^2}dx=\int(x-5x^{-2})dx=\frac{1}{2}x^2+\frac{5}{x}+C$

$F(x)=\frac{1}{2}x^2+\frac{5}{x}+C;F(1)=\frac{1}{2}+5+C=\frac{1}{2};C=-5$

Therefore, $F(x)=\frac{1}{2}x^2+\frac{5}{x}-5$

4) $f(x)=\int(1+50x-4x^2)dx=x+25x^2-\frac{4}{3}x^3+C$

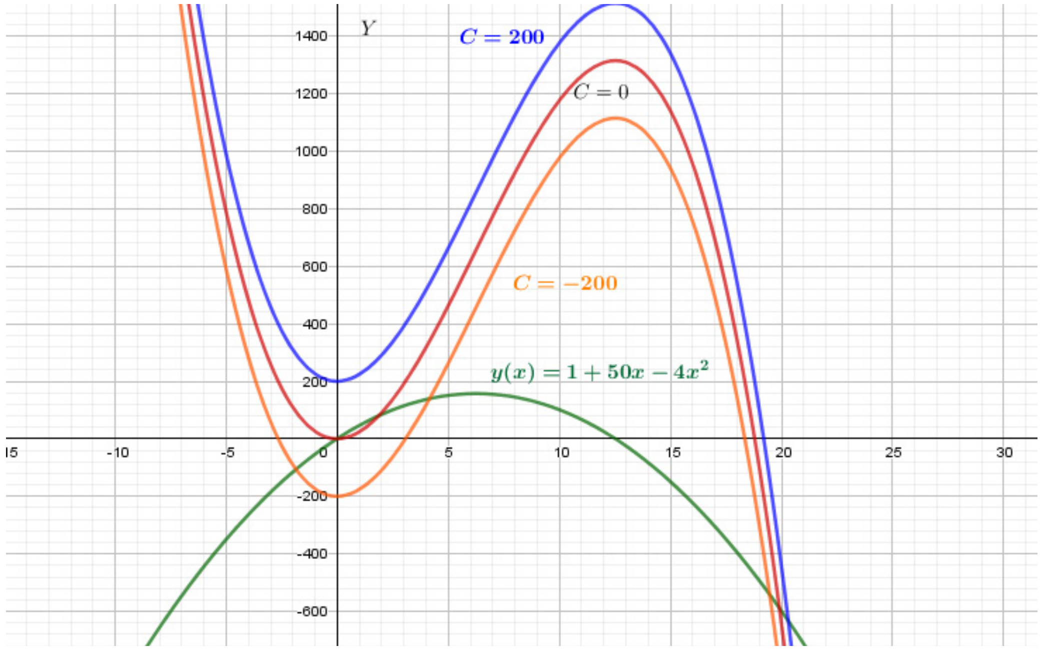


Figure: graph of the marginal cost and three of its corresponding possible total costs

Lesson 4: Basic integration formula

a) Prerequisites/Revision/Introduction:

Students will learn better in this lesson if they have mastered the following concepts, from previous years or previous lessons: derivative of functions (Senior 5: unit 4), Differential of a function (S6: unit 3, Lesson 1), Anti-derivatives (Senior 6: unit 3, lessons 2 and3).

b) Teaching resources:

Textbooks and if possible the internet to facilitate research

c) Learning activities:

- Let learners form groups and perform activity 3.4;
- Walk around to each group and facilitate them to use the properties of anti-derivatives to find integrals of given functions.
- Give each group time to present their findings.

- Through harmonization encourage learners to realize that $\int \frac{1}{1+x^2}dx = \arctan x + C$ and $\int a^x dx = \frac{a^x}{\ln a} + C$ using anti-derivatives properties from the given data.
- Help them realize that indefinite integral of a given function is a set of all anti-derivatives of that function and that any anti-derivatives F of the function f , every possible anti-derivative of f can be written in the form of $F(\text{x})+c$,where c is any constant.
- Invite learners to explore the list of basic integration formula of indefinite integral in their textbook.
- Invite learners to workout individually application activities 3.4 to improve their skills in calculating indefinite integral of functions by using definition and basic integration formula.

Solutions to activity 3.4

Given $(Arc\tan x)' = \frac{1}{1+x^2}$, it follows $\int \frac{1}{1+x^2}dx = \arctan x + C$

Since $\left(\frac{a^x}{\ln a}\right)' = a^x$ then, $\int a^x dx = \frac{a^x}{\ln a} + C$

The indefinite integral of a function is a set of all anti-derivatives of that function. Therefore, given any anti-derivative F of the function f , every possible anti-derivative of f can be written in the form of $F(\text{x})+c$,where c is any constant.

Solution to application activity 3.4:

Find the following integrals:

1) $\int e^{3x+1}dx = \frac{e^{3x+1}}{3} + C$

2) $\int \text{cosec}^2(2x+3)dx = \frac{1}{2}\cot(2\text{x}+3) + \text{C}$

3) $\int 3^x dx = \frac{3^x}{\ln 3} + C$

4) $\int (10x + \cos x)dx = \int 10x dx + \int \cos x dx = 10\int x dx + \sin x + c$

$= 10\frac{x^2}{2} + \sin x + C = 5x^2 + \sin x + C$

5) $\int \frac{4dx}{\cos^2 x} = 4\int \frac{dx}{\cos^2 x} = 4\tan x + C$

6) $\int (8 - x^5)\,\text{dx} = \int 8dx - \int x^5 dx = 8x - \frac{x^6}{6} + C$

Lesson 5: Integration by changing variables

a) Prerequisites/Revision/Introduction:

The students will perform well in this lesson if they refer to: Differential of a function,

Anti-derivatives of a function and basic integration formula for integration seen in previous lessons (Senior 6: unit 3, Lessons 1, 2 and 3).

b) Teaching resources:

T-square, ruler and textbooks. If possible Mathematical software such as geogebra, Mathlab, graphcalc and internet.

c) Learning activities:

- Instruct learners to form groups and work on activity 3.5,
- Follow up the working steps of different group to give support where necessary and motivate learners to mention any difficulty met when integrating $\int 2x(x^2+4)^5 dx$ if any.
- Stimulate learners to let $u = x^2 + 4$, find 1st derivative of $u = x^2 + 4$ and through group discussion, allow them to determine $\int 2x(x^2+4)^5 dx$ using expression of u.
- Facilitate each group to present their findings.
- Harmonize the learners 'works and help them realize that it is not easy to determine integral of the form $\int 2x(x^2+4)^5 dx$ by using basic formula but the task becomes very simple if we change the variable to obtain a new expression. It means that if we cannot integrate $\int h(x)dx$ directly, we should find a new variable u and function $f(u)$ for which $\int h(x)dx = \int f(u(x))\frac{du}{dx}dx = \int f(u)du$. The method is the integration by changing variables or integration by substitution.
- Individually, let learners go through the example 3.5 and work out application activities 3.5 to enhance their skills in calculating integral of functions by changing variables or by substitution.

Solution to activity 3.5:

1) $i) \int x^5 dx = \frac{x^{5+1}}{5+1} + C = \frac{x}{} + C$

ii) Cannot be integrated using basic formulae.

2. It is open question (teacher leads learners to discuss the answer)

3) $\int 2x(x^2+4)^5 dx \quad ii)\int 2x(x^2+4)^5 dx =$

To integrate this immediately is more difficult because you can't find the integration formula which can be applicable immediately.

Therefore, substitution method or changing variable can help as follows:

Let $u = x^2+4$ thus $\frac{du}{dx} = 2x$. By multiplying all sides by dx we get $du = 2xdx$
Replacing variables u by x^2+4 and du by $2xdx$, gives $\int 2x(x^2+4)^5 dx = \int u^5 du$

To integrate $\int u^5 du$ is easy by using basic integration formula seen previously $\int u^5 du = \frac{u^6}{6} + C$

By substituting u by x^2+4, $\frac{u^6}{6} + C = \frac{(x+4)^6}{6} + C$. Therefore,

$\int 2x(x^2+4)^5 dx = \frac{(x+4)^6}{6} + C$

Solution for application activity 3.5

Determine the following integrals

1) $\int xe^{x^2} dx$ .Let $u = x^2 \Rightarrow du = 2xdx$, thus

$\int xe^{x^2} dx = \int \frac{1}{2} e^u du = \frac{1}{2}\int e^u du$

$= \frac{1}{2}e^u + C = \frac{1}{2}e^{x^2} + C$

Therefore $\int xe^{x^2} dx = \frac{1}{2}e^{x^2} + C$

2) $\int \frac{dx}{(1-2x)^2}$

Let $u = 1-2x \Rightarrow du = -2dx \Rightarrow dx = -\frac{1}{2}du$

Then, $\int \frac{dx}{(1-2x)^2} = \frac{-1}{2}\int \frac{du}{u^2} = \frac{-1}{2}\int u^{-2} du \quad = \frac{-1}{2}(\frac{u^{-2+1}}{-2+1}) + C$

$= \frac{-1}{2}(-u^{-1}) + C = \frac{1}{2}(u^{-1}) + C$

3) $\int \frac{x+x^2}{(-3x^2+4-2x^3)^3} dx =$ Substituting u by $1-2x$, we get

$$\int \frac{dx}{(1-2x)^2} = \frac{1}{2}[(1-2\mathrm{x})^{-1}] + C$$

Let us take $u = 4 - 3x^2 - 2x^3$, $du = (-6x - 6x^2)dx \Rightarrow dx = \dfrac{du}{-6(\mathrm{x}+\mathrm{x}^2)}$

Substituting u by $4 - 3x^2 - 2x^3$ and dx by $\dfrac{du}{-6(\mathrm{x}+\mathrm{x}^2)}$

$$\int \frac{x + x^2}{(-3x^2 + 4 - 2x^3)^3} dx = \frac{-1}{6}\int \frac{du}{u^3} = \frac{1}{12u^2} + c$$

Coming back to the variable x,

$$4)\int \frac{x}{(1-2x^2)^{\frac{1}{3}}} dx =$$

Let $u = 1 - 2x^2$, $du = -4xdx \Rightarrow dx = \dfrac{du}{-4x}$

$$\Rightarrow \int \frac{x}{(1-2x^2)^{\frac{1}{3}}} dx = \int -\frac{1}{4u^{\frac{1}{3}}} du = -\frac{1}{4}\int u^{-\frac{1}{3}} du = -\frac{1}{4}\left(\frac{u^{\frac{2}{3}}}{\frac{2}{3}}\right) + \mathrm{C} \square \quad \frac{3}{8}(u^{\frac{2}{3}}) \quad C$$

Finally, we find $\displaystyle\int \frac{x}{(1-2x^2)^{\frac{1}{3}}} dx = -\frac{3}{8}(1-2x^2)^{\frac{2}{3}} + C$

$$5)\int x\sqrt{-1+x^2}\,dx =$$

Putting $u = -1 + x^2$ we have $du = 2xdx \Rightarrow dx = \dfrac{du}{2x}$

$$\int x\sqrt{-1+x^2}\,dx = \int \frac{1}{2}\sqrt{u}\,du = \frac{2}{6}u^{\frac{3}{2}} + C$$

By substitution we get $\displaystyle\int x\sqrt{-1+x^2}\,dx = = \frac{2}{6}(-1+\mathrm{x}^2)^{\frac{3}{2}} + C$

Lesson 6: Integration by parts

a)Prerequisites/Revision/Introduction:

Learners will perform well in this lesson if they refer to: Differential of a function, anti-derivatives of a function, basic formula for integration and integration by substitution seen in previous lessons (S6, unit 3, Lessons 1, 2, 3 and 4).

b)Teaching resources:

Textbooks, mathematics softwares and Internet if available.

c) Learning activities:

- Form groups and invite learners to attempt activity 3.5;
- Walk around to different groups to provide support where necessary.
- Let each group present their findings.
- Through harmonization guide learners to find that there are some integrals that can not be found using the substitution method. To overcome this situation, we use other methods such as the integration by parts where the integration of the product of two functions u and dv can be obtained by using the formula $\int udv = uv - \int vdu$ which is deduced from the derivative of a product of two functions u and v.
- Let learners go through the example 3.6, and individually work out application activities 3.6 to develop their skills in calculating integral of functions by integration by parts.

Solution to activity 3.6.

1) $\int 3x^2(x^3+1)dx =$

Let $u = x^3 + 1$,

$du = 3x^2 dx$

$\int udu = \frac{u^2}{2} + C$

$\int 3x^2(x^3+1)dx = \int \frac{1}{3}udu = \frac{1}{3}\frac{u^2}{2} + C = \frac{1}{6}u^2 + C$

$\int 3x^2(x^3+1)dx = \frac{1}{2}(x^3+1)^2 + C = \frac{1}{2}(x^6 + 2x^3 + 1) + C$

2) $\int xe^x dx =$

i) It is not easy to integrate this integral by substitution.

ii) $du = dx$, $dv = e^x dx \Rightarrow v = \int dv = \int e^x dx = e^x$

iii) $\int udv = \int xe^x dx$

iv) $\int udv = uv - \int vdu = xe^x - \int e^x dx = xe^x - e^x + c$

Therefore, $\int xe^x dx = xe^x - e^x + c$

Solution for application activities 3.6

1) $\int 3xe^{-x}dx$ $\int 3xe^{-x}dx = 3\int xe^{-x}dx$

Integrate this by integration by parts method: let $u = x$ then $du = dx$

$dv = e^{-x}dx \Rightarrow v = \int e^{-x}dx = -e^{-x}$

Thus $\int udv = u.v - \int vdu = x(-e^{-x}) - \int (-e^{-x})dx = -xe^{-x} - e^{-x} + C$

$\int 3xe^{-x}dx = 3(-xe^{-x} - e^{-x}) + C$

2) $\int x^2 \ln xdx$

Using integration by parts,

Set $\begin{cases} u = \ln x \\ dv = x^2 dx \end{cases}$, then $\begin{cases} du = \dfrac{1}{x}dx \\ v = \dfrac{x^3}{3} \end{cases}$

$\int udv = u.v - \int vdu$

$\int udv = (\frac{x^3}{3})\ln x - \int \frac{x^3}{3}\frac{dx}{x} = \frac{x^3}{3}\ln x - \frac{1}{3}\int x^2 dx = \frac{x^3}{3}\ln x - \frac{1}{9}x^3 + C$

$\int x^2 \ln xdx = \frac{x^3}{3}\ln x - \frac{1}{9}x^3 + C = \frac{x^3}{3}(\ln x - \frac{1}{3}) + C$

3) $\int \frac{x}{3}\sin 2xdx = \frac{1}{3}\int x\sin 2xdx$

$For \quad I = \int x\sin 2xdx$, let us take

$I = \int \text{x} \sin 2xdx$

$\begin{cases} u = x \\ dv = \sin 2xdx \end{cases} \Rightarrow \begin{cases} du = dx \\ v = -\dfrac{1}{2}\cos 2x \end{cases}$

$I = -\frac{x}{2}\cos 2x - \int \frac{-\cos 2x}{2}dx$ and $I = -\frac{x}{2}\cos 2x + \frac{1}{4}\sin 2x + C$. Thus

$\int \frac{x}{3}\sin 2xdx = \frac{1}{3}I = \frac{1}{3}(-\frac{x}{2}\cos 2x + \frac{1}{4}\sin 2x) + C$

$4.\int x\sqrt{x+5}dx =$

Integration by parts

$$\begin{cases} u = x \\ dv = \sqrt{x+5} \end{cases} \Rightarrow \begin{cases} du = dx \\ v = \frac{2}{3}(x+5)^{\frac{3}{2}} \end{cases}$$

$$\int x\sqrt{x+5}dx = \frac{2}{3}x(x+5)^{\frac{3}{2}} - \int \frac{2}{3}(x+5)^{\frac{3}{2}} dx$$

Therefore $\int x\sqrt{x+5}dx = \frac{2}{3}x(x+5)^{\frac{3}{2}} - \frac{4}{15}(x+5)^{\frac{5}{2}} + C$

$5.\int 8x\cos x dx = 8\int x\cos x dx$

Consider $I = \int x\cos x dx$

$$\begin{cases} u = x \\ dv = \cos x dx \end{cases} \Rightarrow \begin{cases} du = dx \\ v = \sin x \end{cases},$$

$$I = \int x\cos x dx = x\sin x - \int \sin x dx$$

$$I = x\sin x + \cos x + c$$

Therefore,

$$\int 8x\cos x dx = 8I = 8(\text{xsinx} + \text{cosx}) + \text{C}$$

Lesson 7: Applications of indefinite integrals

a) Prerequisites/Revision/Introduction:

Learners will get a better understanding of the content of this lesson if they refer to: Trigonometric and inverse of trigonometric functions where there is application of derivatives in solving real life problems (Senior 5: unit 4), Differential of a function (Senior 6: unit 3, Lesson 1), Anti-derivatives (S6: unit 3: Lesson 2), Definition and properties of indefinite integrals (Senior 6: unit 3, Lesson 3), Basic integration formula (Senior 6: unit 3, Lesson 4),Integration by changing variables (Senior 6: unit 3,Lesson 5), and Integration by parts (Senior 6: unit 3,Lesson 6).

b) Teaching resources:

Learners' book, Internet and textbooks to facilitate research.

c) Learning activities:

- Form groups and invite learners to work on activity 3.7,
- From the activity 3.7 ,ask learners to plot the graph of the weight F as function of the height x above the ground level and interpret the obtained result.
- Through group discussions, allow learners to find an expression for the work W done in lifting the gallon if $dW = F(x)dx$.
- In group, let learners find the work done when the gallon is 6 meters above the ground level when the work done by the gallon is 40 Joules for 2 meters above the ground
- Invite learners to plot the graph of the work and interpret it in their respective groups.
- Invite learners to present their findings and harmonize their works to make the lesson summary. Lead learners to realize that indefinite integrals are used to determine the work done by a force moving through an axis using the following formulas: $s = \int v(t)dt$, $v = \int a(t)dt$ and $W = \int F(x)dx$ where s is the position, v the speed, a acceleration, t time and W the work. Individually, let learners read through the example3.7 to enable them work out application activities 3.7, to reinforce their skills .

Solutions to activity 3.7

a) Complete the table below

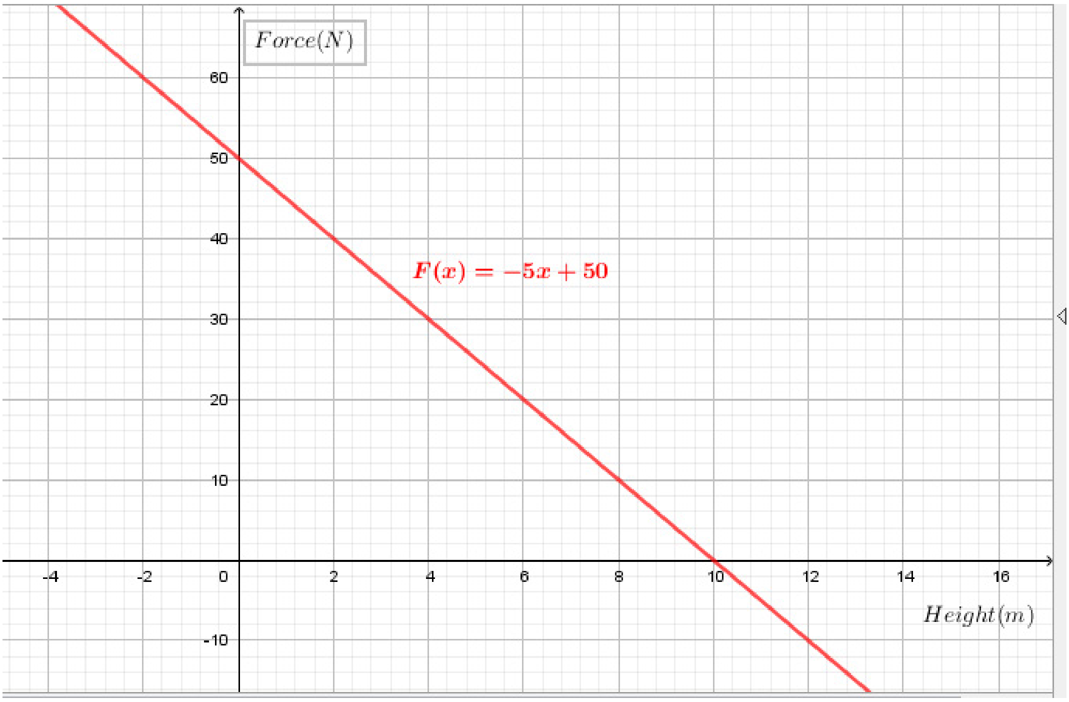
Height above the ground (m)	Weight(N)
0	50
10	0

$\text{F}(0) = a \times 0 + b$ and $a \times 0 + b = 50 \Leftrightarrow b = 50$

$\text{F}(10) = a \times 10 + 50$ and $a \times 10 + 50 = 0 \Leftrightarrow a = -5$

Therefore, $\text{F}(x) = -5x + 50$

b) The graph is shown below

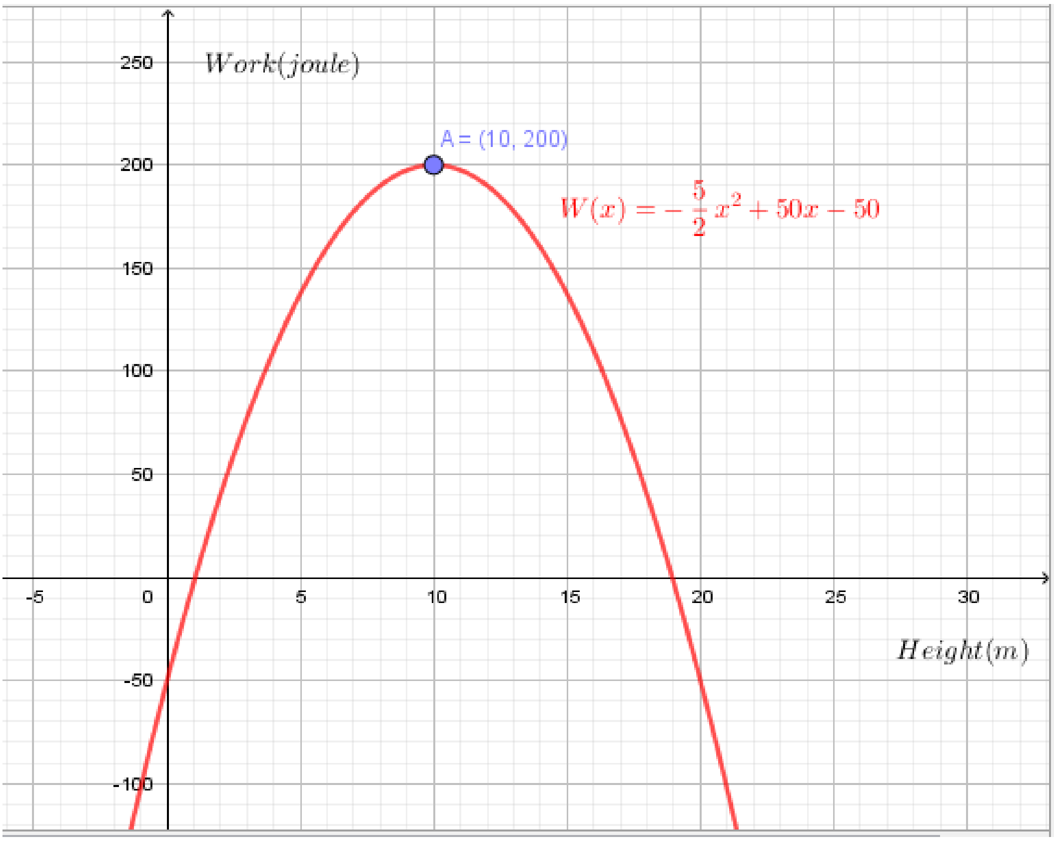


c) The work is found by integrating dW

$$W(x) = \int (-5x + 50)dx = -\frac{5}{2}x^2 + 50x + C \text{ means } W(x) = -\frac{5}{2}x^2 + 50x - 50$$

When x=6, $W(6) = -\frac{5}{2}(6)^2 + 50(6) - 50 = 160$. When the gallon is 6 meters above the ground level, the work done is 160 Joules.

d)



The work increases up to its maximum value, occurring for *x=10*, then decreases.

Solution to application activity 3.7

1)The total revenue is $R(x)=\int(30-10e^{-\frac{1}{40}x})dx = 30x+400e^{-\frac{1}{40}x}+C$, that is

$R(x)=30x+400e^{-\frac{1}{40}x}+C$ and $R(0)=30(0)+400e^{-\frac{1}{40}(0)}+C=100 \Leftrightarrow C=-300$

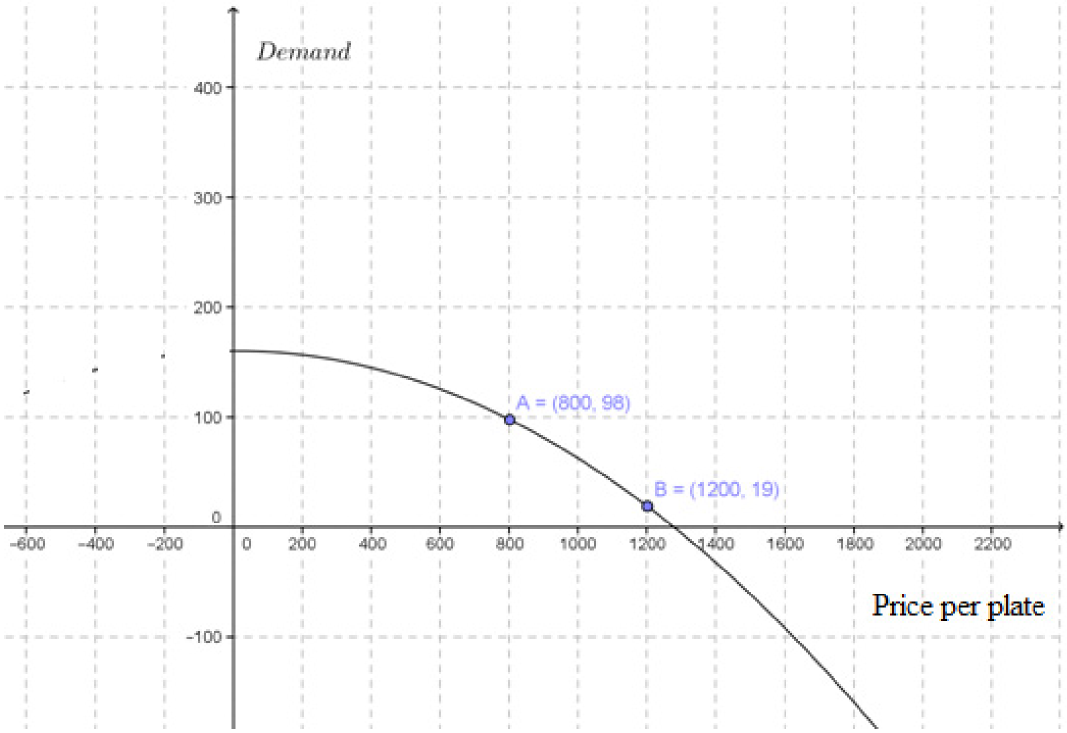
Therefore, $R(x)=30x+400e^{-\frac{1}{40}x}-300$

2) a) The demand function is $\mathrm{D}(x)=\int(-0.0002x+\frac{1}{400})dx=-0.0001x^2+\frac{1}{400}x+C$

or $\mathrm{D}(x)=-0.0001x^2+\frac{1}{400}x+C$;

$$D(800) = -0.0001(800)^2 + \frac{1}{400}(800) + C = 98 \Leftrightarrow C = 160$$

Therefore, $D(x) = -0.0001x^2 + \frac{1}{400}x + 160$



Note: D(x) is a mathematical function, keep in mind that price can not be negative

b) If the price per plate is 1200 Frw, then the demand is $D(1200) = -0.0001(1200)^2 + \frac{1}{400}(1200) + 160 = 19$, which is confirmed by the graph above.

c) As the price of food per plate increases, the demand decreases drastically/ significantly.

d) The restaurant owner should be advised not to increase the price per plate, otherwise he/she will loose all the customers.

Lesson 8: Definition and properties of define integrals

a) Prerequisites/Revision/Introduction:

Learners will perform better in this lesson if they have understanding on: Rules of differentiation of polynomials, rational and irrational functions (Senior4: Unit6), Differentiation of trigonometric functions (Senior 4: Unit 6), Differentiation of logarithmic and exponential functions (Senior 6: Unit 2). Calculation of anti-derivative of any function (Senior6: Unit 3, lesson2)

b) Teaching resources:

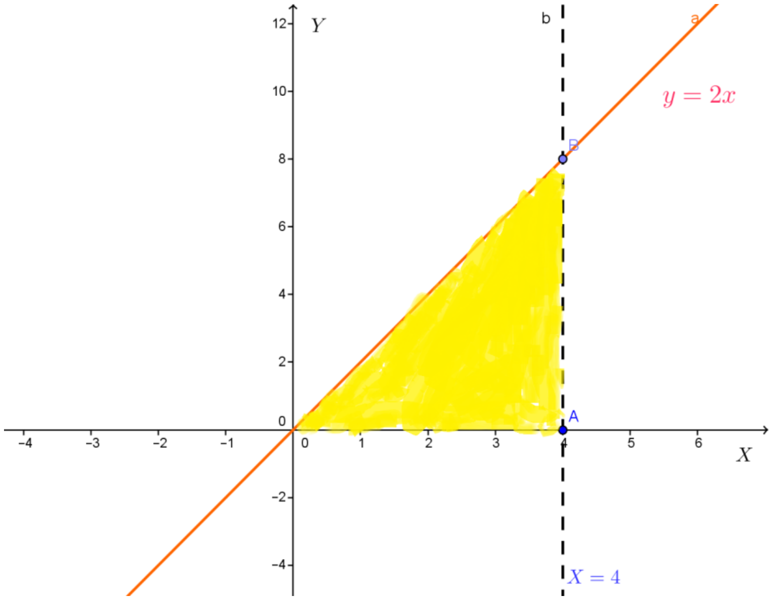
- Scientific calculators, ruler, textbooks, graph papers.
- Internet to facilitate research
- If possible, the use of Mathematical software such as Geogebra to plot graphs of functions is essential.

a) Learning activities:

- Organize learners into groups and let them attempt activity 3.8; In their respective groups, ask learners to plot in a Cartesian plane the following: $f(x)=2x$, $y=0$, $x=0$, and $x=4$ and let learners identify and name the shape made by the lines $y=2x$, $y=0$, $x=0$, and $x=4$;
- In same groups, ask learners to determine the area of the obtained triangle using the formula and then request learners to find the anti-derivative $F(x)$ of $f(x)=2x$, then calculate $F(4)-F(0)$ and then compare the area of the shape and the result of $F(4)-F(0)$.
- Invite randomly some groups to present their findings to the whole class and after presentation facilitate learners to have a lesson summary
- Individually, invite learners to read through example 3.7 and then work out the application activities 3.8 to enhance their knowledge and skills about definite integrals.

Solutions to activity 3.8

a)



The shape obtained is the Triangle that has three vertices $O(0,0),\ A(4,0),\ and\ B(4,8)$

b) Area of Triangle $=\dfrac{BH}{2}$. The base $B=4$ units of length, and the height $H=8\ units\ of\ length$.

Then, the $Area=\dfrac{4\times 8}{2}=16\ square\ units$. The area of the triangle is $16\ square\ units$.

c) The antiderivative of $f(x)=2x\ is\ F(x)=x^2+c$.

$F(4)-F(0)=\left[(4^2+c)-(0^2+c)\right]=16+c-c=16$.

Comparison shows that the findings are the same.

Thus, the area of the triangle = $F(4)-F(0)=16\ square\ units$.

Solution to application activity 3.8

1)a) $\displaystyle\int_1^2(4x^2-3x)dx=\left[\frac{4x^3}{3}-\frac{3x^2}{2}\right]_1^2=\left[\left(\frac{4\times 2^3}{3}-\frac{3\times 2^2}{2}\right)-\left(\frac{4\times 1^3}{3}-\frac{3\times 1^2}{2}\right)\right]$

$$=\left(\frac{32}{3}-6\right)-\left(\frac{-1}{6}\right)=\frac{14}{3}+\frac{1}{6}=\frac{29}{6}$$

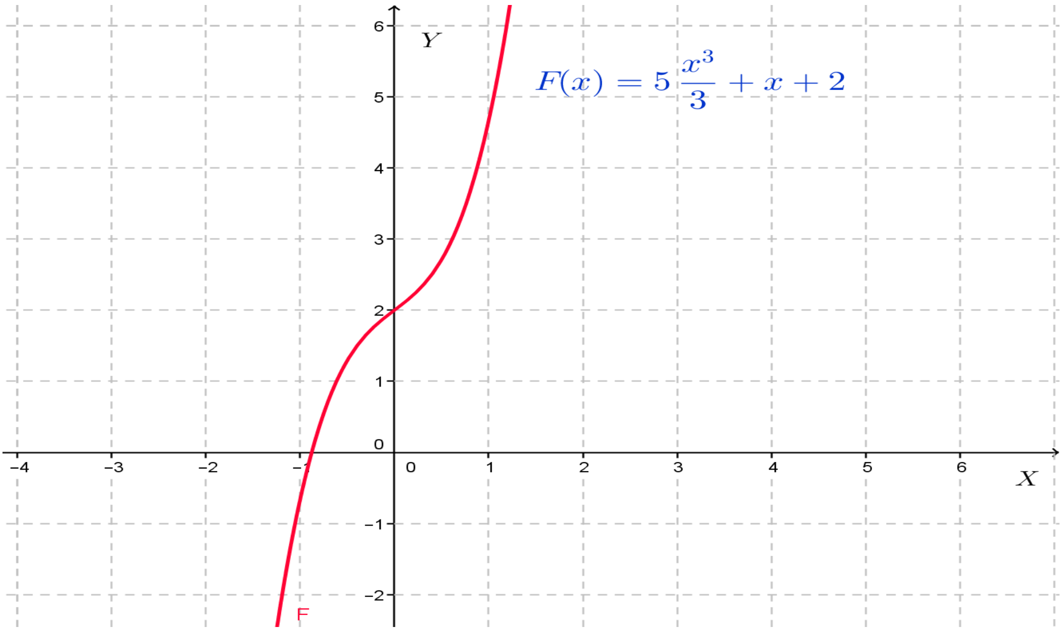
b) $\displaystyle\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}(\cos 2x-2e^x)dx=\left[\frac{\sin 2x}{2}-2e^x\right]_{-\frac{\pi}{4}}^{\frac{\pi}{4}}=\left(\frac{\sin\frac{2\times\pi}{4}}{2}-2e^{\frac{\pi}{4}}\right)-\left(\frac{\sin\frac{-2\times\pi}{4}}{2}-2e^{\frac{-\pi}{4}}\right)$

$$=\left(\frac{1}{2}-2e^{\frac{\pi}{4}}\right)-\left(\frac{-1}{2}-2e^{\frac{-\pi}{4}}\right)=1-2e^{\frac{\pi}{4}}+2e^{\frac{-\pi}{4}}=1-2\left(e^{\frac{\pi}{4}}-e^{\frac{-\pi}{4}}\right)$$

2) $F(x)=\int F'(x)dx=\int(5x^2+1)dx=\frac{5x^3}{3}+x+c$ and

$F(0)=2\Leftrightarrow\frac{5\times 0}{3}+0+c=2\Leftrightarrow c=2$ thus $F(x)=\frac{5x^3}{3}+x+2$

The graph of $F(x)=\frac{5x^3}{3}+x+2$



2) Given $f(x)=y=30-2x-x^2$

a) If $x_0=3$, $xo=3$, $y_0=30-2\times3-3^2=30-6-9=30-15=15$

The consumer's surplus $=\int_0^{x_0} f(x)dx-x_0y_0=\left[30x-2\frac{x^2}{2}-\frac{x^3}{3}\right]_0^3-(3\times15)$

$=[(30\times3)-9-9]-45$

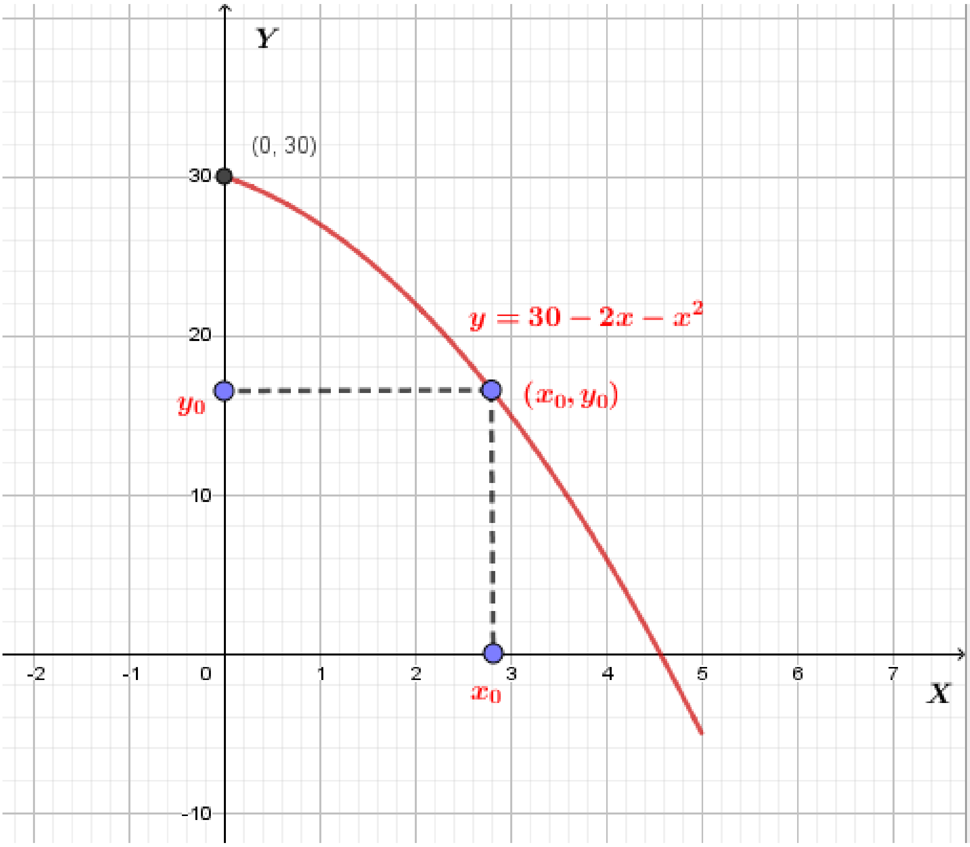
$=90-18-45=27$

$=\left[30x-2\frac{x^2}{2}-\frac{x^3}{3}\right]_0-(3\times15)$

$=[(30\times3)-9-9]-45$

$=90-18-45=27$

b) The graph of $f(x)=y=30-2x-x^2$



Lesson 9: Techniques of Integration of definite integrals

a) Prerequisites/Revision/Introduction:

Learners will learn better the techniques of integration of definite integrals if they have good understanding on the calculation of anti-derivative of various types of functions (Senior 6: Unit3, lesson2) and on the techniques of integration of indefinite integrals (Senior 6: Unit3, lesson 4,5 and 6).

b) Teaching resources:

Scientific calculators, geometrical materials, textbooks and graph papers. If possible, the use of Mathematical software such as Geogebra to plot the graph of functions and the internet to facilitate research is essential.

c) Learning activities:

- Form groups of learners and guide them to discuss activity 3.9;
- Walk around to different groups to provide support where necessary and guide them to discuss what happens to integral's boundaries when one applies the substitution method;

- Invite randomly some groups to present their findings to the whole class;
- Trough class discussions, let learners to analyse their findings and establish the general rule used, then write summary in their notebooks;
- Guide learners to work through examples 3.8 and 3.9 and work individually application activities 3.9 to assess the competences.

Solution to activity 3.9

1) $f(x)=e^{x^2}$

i) $t=x^2$, then $x=0\Rightarrow t=0$ and $x=2\Rightarrow t=2^2=4$

ii) $t=x^2\Rightarrow dt=2xdx$

iii) $\int_0^2 2xe^{x^2}dx$ $\int_0^2 2xe^{x^2}dx=\int_0^4 e^t dt=e^t|_0^4=e^4-e^0=e^4-1$

iv) It is clear that when we apply the substitution method, we also substitute boundaries to keep integral the same.

2) Evaluation of $\int_1^e \mathrm{x}^2\ln x dx$

Let evaluate this integral by parts.

Let $u=\ln x$ and $dv=x^2dx$, $du=\frac{dx}{x}$ and $v=\int x^2dx=\frac{x^3}{3}$

$$\int_1^e \mathrm{x}^2\ln x dx=\frac{x^3}{3}\ln x\Big|_0^e-\int_0^e\frac{x^3}{3}\frac{dx}{x}=\frac{x^3}{3}\ln x|_0^e-\frac{1}{9}x^3|_0^e=\frac{e^3}{3}-\frac{e^3}{9}=\frac{2e^3}{9}$$

Solutions to application activity 3.9

1) $\int_0^{\frac{\pi}{2}}\cos xe^{\sin x}dx$ using integration by substitution

Let $\begin{cases} t=\sin x \\ dt=\cos xdx \end{cases}$

When $x=0,\ t=0$

When $x=\frac{\pi}{2},\ t=1$

$$I=\int_0^{\frac{\pi}{2}}\cos xe^{\sin x}dx=\int_0^1 e^t dt=\left[e^t\right]_0^1=e^1-e^0=e-1$$

2) $\int_0^1 \ln(1+x)dx$ using integration by parts

Let $u=\ln(1+x) \Rightarrow du=\frac{1}{1+x}$ *and* $dv=dx \Rightarrow v=x+c$

$$I=\int_a^b udv = uv-\int_a^b vdu \quad becomes \ I=\int_0^1 \ln(1+x)=[x\ln(1+x)]_0^1-\int_0^1 \frac{x}{1+x}dx$$

$$=\ln 2-\int_0^1\left(1-\frac{1}{1+x}\right)dx=\ln 2-\left[x-\ln(1+x)\right]_0^1=\ln 2-1+\ln 2=-1+2\ln 2=-1+\ln 4$$

Lesson 10: Application of definite integrals

a) Prerequisites/Revision/Introduction:

Learners will perform better in this lesson if they have good understanding on:

- Application of indefinite integrals (Senior 6, unit3, lesson7)
- Calculation of definite integrals using appropriate techniques (Senior 6, unit3, lesson9)

b) Teaching resources:

Scientific calculators-square, ruler, textbooks, graph papers. If possible, the use of Mathematical software such as Geogebra to graph functions and the internet to facilitate research is essential.

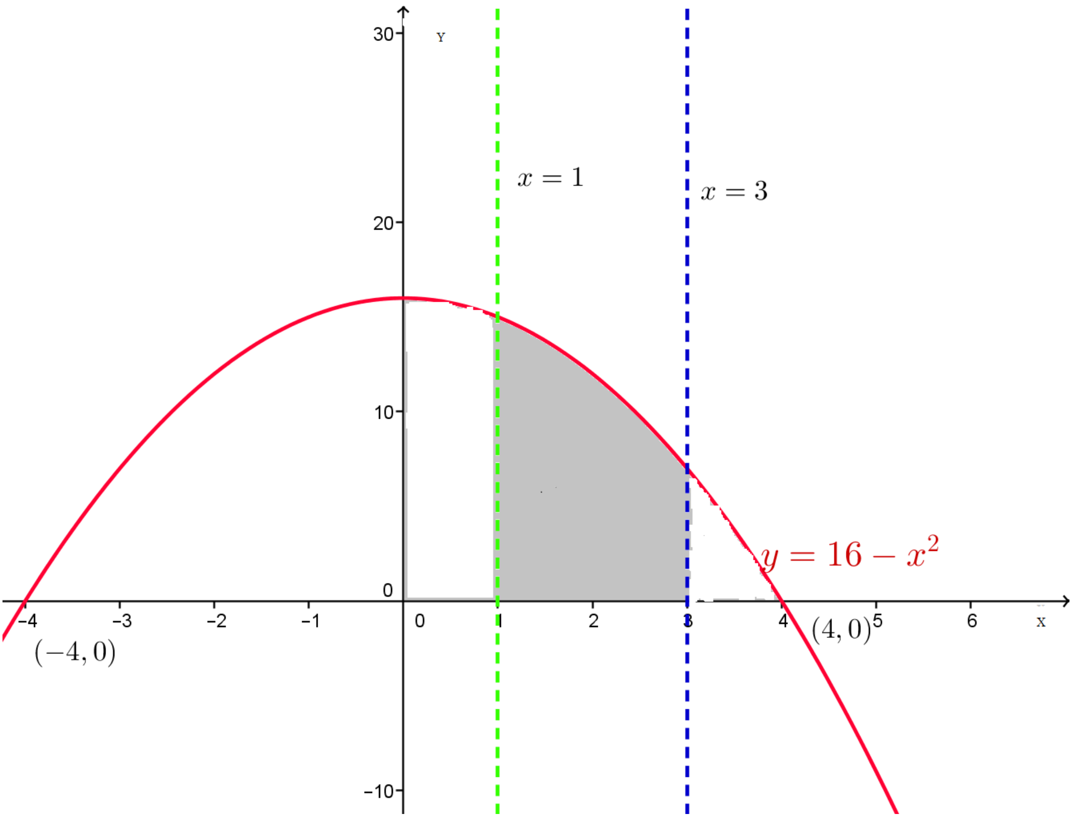
c) Learning activities:

- Form groups of learners and invite each group to collaboratively attempt activity 3.10 in the learner's book;
- Monitor how the students are performing the task and provide support where necessary.
- Invite representatives of groups to present learners' findings.
- Harmonize students' work and to correct those which are false, complete those which are incomplete, and confirm those which are correct.
- Help learners to connect what they learnt to real life situations by highlighting the importance of increasing the level of production in the financial management.
- Let learners work through example 3.10;
- Invite learners to discuss the second task and work through example 3.11 and

finally the example 3.12 before asking them to work individually the application activities 3.9 to assess the competences.

Solutions to activity 3.10

1) a)



From the above figure, the definite integral which represents the measure of the area bounded by the curve $y = 16 - x^2$, *the* $x-$*axis and* $x = 1,\ x = 3$ is $\int_1^3 \left(16 - x^2\right)dx$

b) *Hence* $A = \int_1^3 \left(16 - x^2\right)dx = \left[16x - \frac{x^3}{3}\right]_1^3$

$= \left[16\times 3 - \frac{3^3}{3}\right] - \left[16\times 1 - \frac{1^3}{3}\right] = (48-9) - \left(\frac{48-1}{3}\right)$

$= 39 - \frac{47}{3} = \frac{117-47}{3} = \frac{70}{3}$ *square unit*

2) Let $C(q)$ denote the total cost of producing q units. Then the marginal cost is the derivative $\frac{dC}{dq} = 3(q-4)^2$.

The cost of production raised from $6\ units\ to\ 10\ units$ is given by the definite integral

$$C(q)=\int_{6}^{10}3(q-4)^2\,dq=\left[(q-4)^3\right]_{6}^{10}=(10-4)^3-(6-4)^3=216-8=208\ \$$$

3) The spring is compressed starting its natural length $0\,m$ and finish at $0.25\ m$ from the natural length, so the lower limit of the integral is 0 and the upper limit is 0.25

Thus, the work done $=\int_{0}^{0.25}16x\,dx=\left[8x^2\right]_{0}^{0.25}=8(0.25)^2-0=0.5\ Joules$

Solution for application activity 3.10

1. Allow learners to find the intersection points of the graphs for functions involved.

Solving the equation $f(x)=g(x)$ yields $x^2+x-3=-x^2-2x+2$

$$2x^2+3x-5=0$$

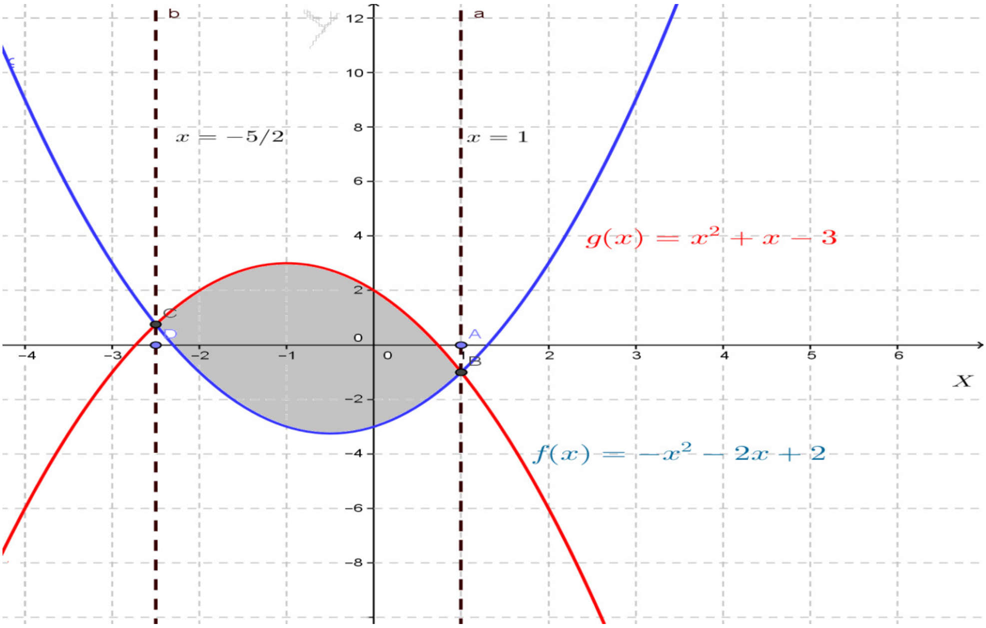
$$\Delta=b^2-4ac=3^2-4[2\times(-5)]$$

$$=9+40=49$$

$$x_1=\frac{-b+\sqrt{\Delta}}{2a}=\frac{-3+7}{4}=1$$

$$x_2=\frac{-b-\sqrt{\Delta}}{2a}=\frac{-3-7}{4}=\frac{-10}{4}=\frac{-5}{2}$$

Graphically

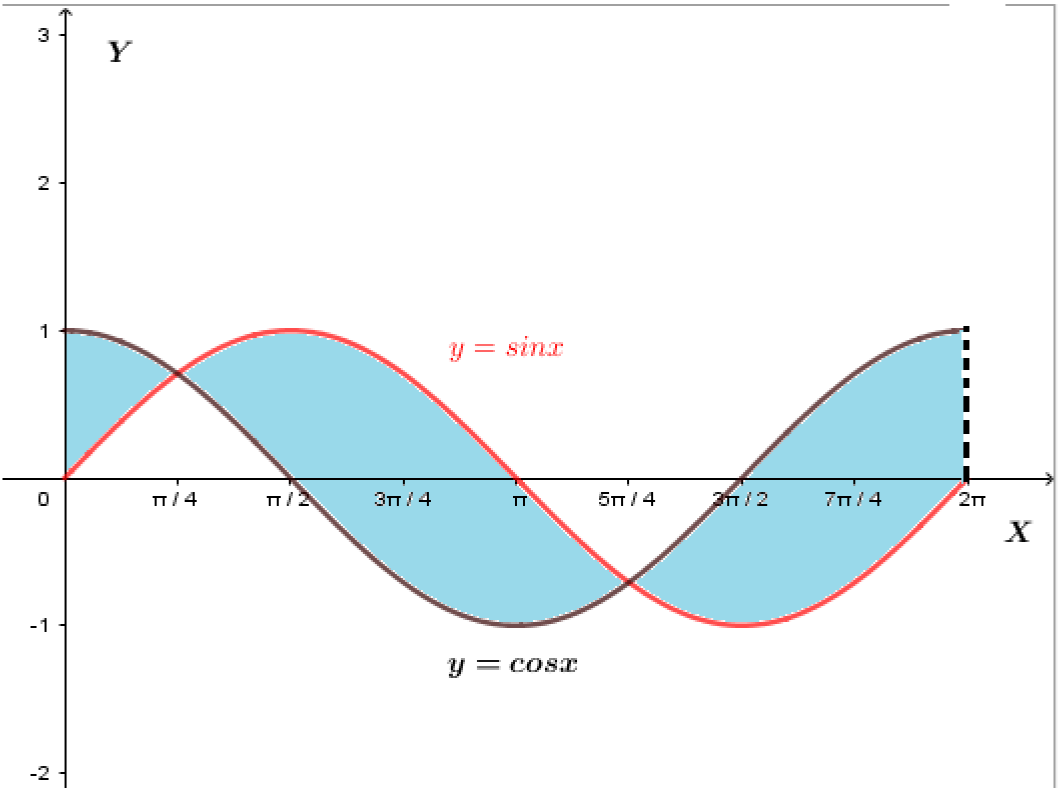


$$A = \int_{\frac{-5}{2}}^{1} \left[f(x) - g(x) \right] dx = \int_{\frac{-5}{2}}^{1} \left[\left(-x^2 - 2x + 2 \right) - \left(x^2 + x - 3 \right) \right] dx$$

$$= \int_{\frac{-5}{2}}^{1} \left(-2x^2 - 3x + 5 \right) dx = \left[\frac{-2}{3} x^3 - \frac{3}{2} x^2 + 5x \right]_{\frac{-5}{2}}^{1}$$

$$= \left(\frac{-2}{3} - \frac{3}{2} + 5 \right) - \left(\frac{250}{24} - \frac{75}{8} - \frac{25}{2} \right) = \frac{343}{24} UA$$

2) The figure bellow shows the graph of $y = \sin x$ *and* $y = \cos x$ in the same $XY-$ coordinates system



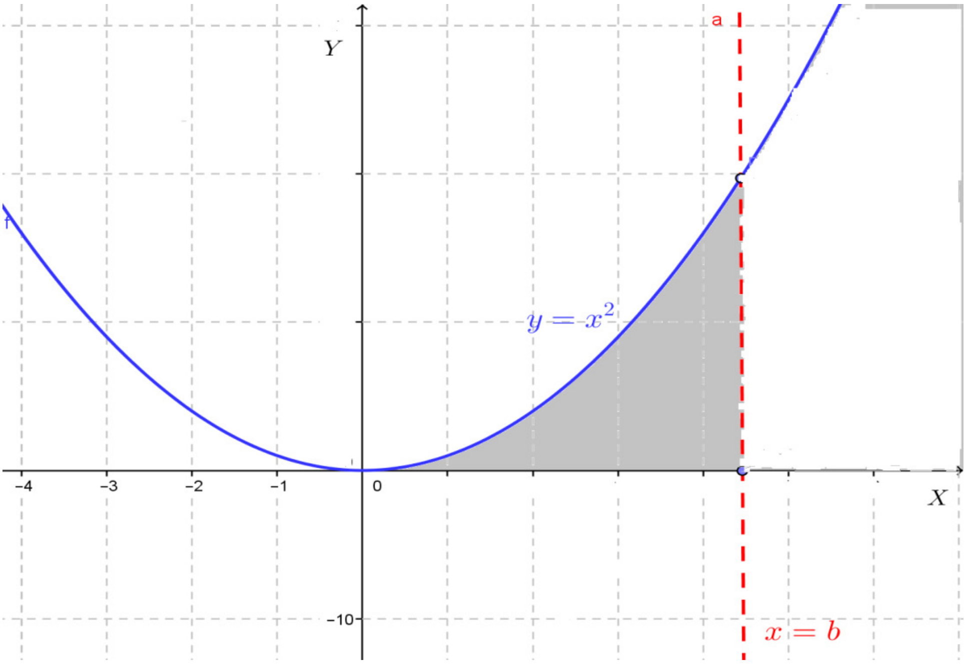
The graph of sine and cosine cross at $x = \frac{\pi}{4}$ and $x = \frac{5\pi}{4}$.

The required area is $A = \int_0^{\frac{\pi}{4}} (\cos x - \sin x)dx + \int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} (\sin x - \cos x)dx + \int_{\frac{5\pi}{4}}^{2\pi} (\cos x - \sin x)dx$

$$= [\sin x + \cos x]_0^{\frac{\pi}{4}} - [cox + \sin x]_{\frac{\pi}{4}}^{\frac{5\pi}{4}} + [\sin x + \cos x]_{\frac{5\pi}{4}}^{2\pi}$$

$$= \left(\sqrt{2} - 1\right) + \left(\sqrt{2} + \sqrt{2}\right) + \left(1 + \sqrt{2}\right) = 4\sqrt{2} \text{ square units}$$

3) The parabola



$$A = \int_0^b (x^2 - 0)\,dx = \left[\frac{x^3}{3}\right]_0^b = \frac{b^3}{3} \text{ square units}$$

4) Given $\frac{dP}{dx} = 40 - 3\sqrt{x}$, the change in profit is

$$P = \int_{100}^{121} \left(40 - 3\sqrt{x}\right)dx = \left[40x - 3\frac{x^{3/2}}{3/2}\right]_{100}^{121}$$

$$= \left(40\times121 - 2\times121^{3/2}\right) - \left(40\times100 - 2\times100^{3/2}\right)$$

$$= (4840 - 4000) - 2(1331 - 1000) = 840 - 662 = 178$$

3.6. Unit summary

1) The differential of a function

The rate of change $\frac{\Delta y}{\Delta x} = \frac{f(x+h) - f(x)}{h}$ means that $\Delta y = f'(x)\Delta x$.

When Δx becomes very small, the change in y can be approximated by the differential of y , that is, $\Delta y \approx dy$ and $\Delta x \approx dx$.

Therefore, the differential of a function $f(x)$ is the approximated increment of that function when the variation in x becomes very small. It is given by $dy = f'(x)dx$ .f ' (x) {\displaystyle f'(x)}.

2) Anti-derivatives

Let $y = f(x)$ be a continuous function of variable x .An anti-derivative of $f(x)$ is any function $F(x)$ such that $F'(x) = f(x)$. For any arbitrary constant C, $F(x)+C$ is also an anti-derivative of $f(x)$ because $\left(F(x)+c\right)' = F'(x)$

3) Indefinite integral

Let $y = f(x)$ be a continuous function of variable x . The **indefinite integral** of $f(x)$ is the set of all its anti-derivatives. If $F(x)$ is any anti-derivative of function $f(x)$, then the indefinite integral of $f(x)$ is denoted and defined as follows:

$\int f(x)dx = F(x)+C$ where C is an arbitrary constant called the **constant of integration**.

Thus, $\int f(x)dx = F(x)+C$ if and only if $F'(x) = f(x)$.

- **Properties of indefinite integral**

Let $y = f(x)$ and $y = g(x)$ be continuous functions and k a constant. Integration obeys the following properties:

1) $\int kf(x)dx = k\int f(x)dx$: the integral of the product of a constant by a function is equal to the product of the constant by the integral of the function.

2) $\int [f(x)+\text{g(x)}]dx = \int f(x)dx + \int g(x)dx$: the integral of a sum of two functions is equal to the sum of the integrals of the terms.

- **Basic integration formulae**

1) If k is constant, $\int kdx = kx + C$

2) $\int u^n du = \dfrac{1}{n+1}u^{n+1} + C$, where $n \neq -1$, n is a rational number

3) If $b \neq -1$, and u a differentiable function, $\int u^b du = \dfrac{u^{b+1}}{b+1} + C$

4) $\int x^{-1}dx = \int \dfrac{1}{x}dx = \ln|x| + c$ for x nonzero

5) $\int e^x dx = e^x + c$, the integral of exponential function of base e

6) $If\ a > 0\ and\ a \neq 1, \int a^x dx = \dfrac{a^x}{\ln a} + c$

7) $\int \dfrac{1}{x-1}dx = \ln|x-1| + C$

8) $If\ a \neq 0, \int \dfrac{dx}{ax+b} = \dfrac{1}{a}\ln|ax+b| + C$

9) $If\ \ a \neq 0, and\ \ n \neq -1, \int (ax+b)^n dx = \frac{(ax+b)^{n+1}}{a(n+1)} + C$

- **Integration involving trigonometric functions**

10) $\int \cos x dx = \sin x + C$

11) $\int \sin x dx = -\cos x + C$

12) $\int \frac{dx}{1+x^2} = Arc\tan x + C$

13) $\int \frac{dx}{\sqrt{1-x^2}} = Arc\sin x + C$

14) $\int \frac{dx}{\cos^2 x} = \tan x + C$

15) $\int \frac{dx}{\sin^2 x} = -\cot x + C$

16) $If\ \ a \neq 0, \int \sin(ax+b)dx = -\frac{1}{a}\cos(\mathrm{ax+b}) + C$

17) $\int \sec^2 x dx = \tan x + C$

18) $\int \mathrm{cosec}^2 \mathrm{x} dx = -\cot x + C$

19) $\int \sec^2(ax+b)dx = \frac{1}{a}\tan(\mathrm{ax+b}) + C$

20) $\int \mathrm{cosec}^2(ax+b)dx = -\frac{1}{a}\cot(\mathrm{ax+b}) + C$

21) $\int \cos(\mathrm{ax+b})dx = \frac{1}{a}\sin(ax+b) + C$

4)Techniques of integration of indefinite integrals

- **Integration by substitution**

It is the method in which the original variables are expressed as functions of other variables.

Generally, if we cannot integrate $\int h(x)dx$ directly, it is possible to find a new variable u and function $f(u)$ for which $\int h(x)dx = \int f(u(x))dx = \int f(u)du$

- **Integration by parts**

If u and v are two functions of x, the product rule for differentiation can be used to integrate the product udv or vdu in the following way. Since $d(uv) = udv + vdu$ it comes that $\int d(uv) = \int udv + \int vdu$. This leads to: $uv = \int udv + \int vdu$. Thus

$\int udv = u.v - \int v\text{du}$.

When using integration by parts, keep in mind that you are separating the integrand into two parts. One of these parts, corresponding to u will be differentiated and the other, corresponding to dv, will be integrated. Since you can differentiate easily both parts, you should choose a dv for which you know an anti-derivative to make easier the integration.

5) Definite integrals

Let f be a continuous function defined on a close interval $[a,b]$ and F be an anti-derivative of f. Thus, if $F(x)$ is an anti-derivative of $f(x)$, then $\int_a^b f(x)dx = [F(x)+c]_a^b$

$= [(F(b)+c)-(F(a)+c)] = [F(b)+c-F(a)-c] = F(b)-F(a)$

- **Fundamental theorem of integral calculus:**

Let $F(x)$ and $f(x)$ be functions defined on an interval $[a,b]$. If $f(x)$ is continuous and $F'(x) = f(x)$, then $\int_a^b f(x)dx = [F(x)]_a^b = F(b)-F(a)$.

- **Properties of definite integral**

If $f(x)$ *and* $g(x)$ are continuous functions on a closed interval $[a,b]$ then:

1. $\int_a^b 0dx \quad 0$

2. $\int_a^b f(x)dx = -\int_b^a f(x)dx$ (Permutation of bounds)

3. $\int_a^b [\alpha f(x) \pm \beta g(x)]dx = \alpha\int_a^b f(x)dx \pm \beta\int_a^b g(x)dx,\ \alpha \text{ and } \beta \in \mathbb{R}$ (Linearity)

4. $\int_a^a f(x)dx = 0$ (Bounds are equal)

5. $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx \text{ with } a < c < b$ (Chasles relation)

6. $\forall x \in [a,b], f(x) \le g(x) \Rightarrow \int_a^b f(x)dx \le \int_a^b g(x)dx$ it follows that

$f(x) \ge 0 \Rightarrow \int_a^b f(x)dx \ge 0 \text{ and } \left|\int_a^b f(x)dx\right| \le \int_a^b |f(x)|dx$ (Positivity)

7. $\int_{-a}^{a} f(x)dx = \begin{cases} 2\int_{0}^{a} f(x)dx, \text{ if } f(x), \text{ is even function} \\ 0 \text{ if } f(x) \text{ is an odd function} \end{cases}$

6. Techniques of integration of definite integrals

- **Integration by substitution**

The method in which we change the variable to some other variable is called **"Integration by substitution".**

When definite integral is to be found by substitution then change the lower and upper limits of integration. If substitution is $\varphi(x)$ and lower limit of integration is a and upper limit is b then new lower and upper limits will be $\varphi(a)$ and $\varphi(b)$ respectively.

- **Integration by parts**

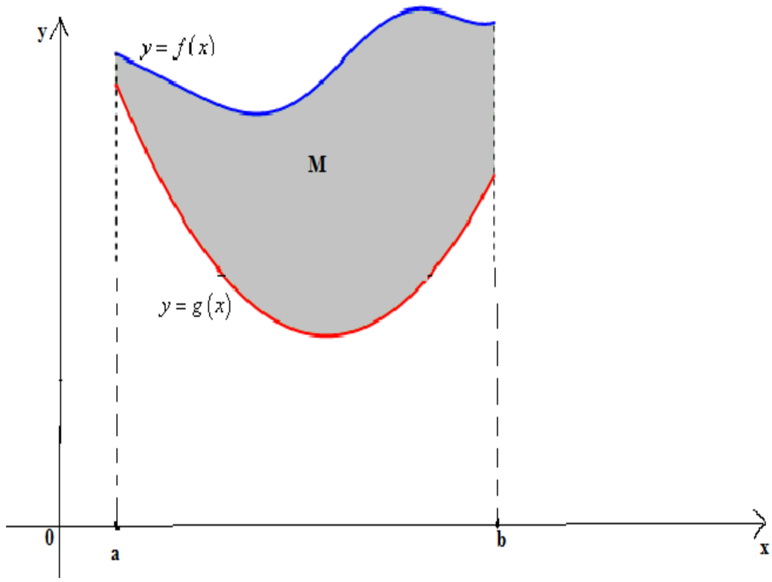
To compute the definite integral of the form $\int_{a}^{b} f(x)g(x)dx$ using integration by parts, simply set $u = f(x) \text{ and } dv = g(x)dx$.

Then $du = f'(x)dx$ and $v = G(x)$, antiderivative of $g(x)$ so that the integration by parts becomes: $\int_{a}^{b} udv = [uv]_{a}^{b} - \int_{a}^{b} vdu$

7. Application of definite integrals

- **Calculation of area between two curves**

Suppose that a plane region M is bounded by the graphs of two continuous functions $y = f(x) \text{ and } y = g(x)$ and the vertical straight lines $x = a \text{ and } x = b$ as shown in figure below



From the above figure, we see that $f(x) \geq g(x)$ for $a \leq x \leq b$. Thus, the area enclosed between those two functions, the vertical lines $x = a$, $x = b$ and the horizontal line $y = 0$ is calculated as follows: $A = \int_a^b [f(x) - g(x)]dx$. It means that $A = \int_a^b \left[\begin{pmatrix} upper \\ function \end{pmatrix} - \begin{pmatrix} lower \\ function \end{pmatrix} \right] dx$

- **Determination of the work done in Physics**

The **work** W done by a force F moving through x axis is given by $W = \int F(x)dx$.

Suppose that a force in the direction of the $x - axis$ moves an object from $x = a$ to $x = b$ on that axis and that force varies continuously with the position x of the object, that is $F = F(x)$ is a continuous function. The element of work done by the force in moving the object through a very short distance from x to $x + dx$ is $dW = F(x)dx$, so the total work done by the force is $W = \int_{x=a}^{x=b} dW = \int_a^b F(x)dx$

- **Determination of cost function in Economics**

The **cost** C (respectively the revenue R, utility U and profit P) is related to the marginal cost M (respectively marginal revenue, marginal utility, and marginal profit) by the formula $C(x) = \int M(x)dx$, where x is the number of units produced. The marginal cost is the additional cost to produce one extra unit.

3.7. Additional information for the teacher

A teacher should have a broad knowledge of the topic. The following may be useful:

Improper integrals

A definite integral with infinity for either an upper or lower limit of integration or a function that is discontinuous on the integration interval is called an improper integral. $\int_a^{\infty} f(x)dx$, $\int_{-\infty}^{b} f(x)dx$ and $\int_1^5 \frac{dx}{x-3}$ are improper integrals. For the two first integrals, ∞ is not a number and cannot be substituted for x in $\mathrm{D}(x)$. For the third, the integrand function is discontinuous on [1,3]. However, they can be defined as limits of other integrals, as shown below:

$$\int_a^{\infty} f(x)dx = \lim_{x\to\infty}\int_a^{b} f(x)dx, \quad \int_1^5 \frac{dx}{x-3} = \lim_{p\to 3^-}\int_1^{p}\frac{dx}{x-3} + \lim_{p\to 3^+}\int_p^{5}\frac{dx}{x-3}$$

If the limit exists, the improper integral is said to converge, and their integral has a definite value. If the limit does not exist, the improper integral diverges and is meaningless.

Example

Evaluate the improper integrals:

$$\int_1^{\infty} 3x^{-2}dx$$

$$\int_1^{\infty} 3x^{-2}dx = \lim_{x\to\infty}\int_1^{b} 3x^{2}dx \text{ if limit exits}$$

$$\lim_{x\to\infty}\int_1^{b} 3x^{\ 2}dx = \lim_{x\to\infty}\left[\frac{-3}{x}\right]_1^b = \lim_{x\to\infty}\left[\frac{-3}{x}\right]_1^b = \lim_{b\to\infty}\left[\frac{-3}{b} - \frac{(-3)}{1}\right] = \lim_{x\to\infty}\left[\frac{-3}{b}+3\right] = 3$$

$$\text{Therefore } \int_1^{\infty} 3x^{-2}dx = 3$$

Probability density function

The probability P that an event will occur can be measured by the corresponding area under the probability density function. A probability density or frequency function is a continuous function $f(x)$ such that:

1. $f(x) \geq 0$ for all possible values of x
2. $\int_{all} f(x)dx = 1$, $P(a < x < b) = \int_a^b f(x)dx$

3.8. End unit assessment

This part provides the answers of end unit assessment activities designed in integrative approach to assess the key unit competence with cross reference to the textbook.

The following are standards (ST) which have been based on in setting end unit assessment questions:

ST5: Competently use integration as the inverse of differentiation; presenting explanations and conclusions; using language for precision.

ST6: Competently use integration in solving problems, selecting the appropriate mathematical operations, measurements and calculations.

Solutions

QUESTION ONE

a) $\int (9x^7 + \frac{1}{x-1} + \frac{2}{\cos^2 x} - \frac{1}{2}e^x)dx = \frac{9x^8}{8} + \ln|x-1| + 2\tan x - \frac{1}{2}e^x + c$

b) $\int \frac{x}{\sqrt{x+3}}dx =$

Set $\sqrt{x+3} = t$ or equivalently $x+3 = t^2$. This implies $x = t^2 - 3$, then $dx = 2tdt$.

$\int \frac{x}{\sqrt{x+3}}dx = \int \frac{(t^2-3)\times 2tdt}{t} = 2\int (t^2-3)dt = 2\frac{t^3}{3} - 6t + c$. Since $t = \sqrt{x+3}$,

we have $\int \frac{x}{\sqrt{x+3}}dx = 2\frac{\left(\sqrt{x+3}\right)^3}{3} - 6\sqrt{x+3} + c$

c) $\int \frac{1}{4}\sin 3x dx = -\frac{1}{12}\cos 3x + c$

QUESTION TWO

a) i) Total Cost Function $=\int \frac{x}{\sqrt{x^2+1600}}dx$

Set $\sqrt{x^2+1600}=t$ or equivalently $x^2+1600=t^2$. Thus $xdx=tdt$.

Total Cost Function $=\int \frac{x}{\sqrt{x^2+1600}}dx=\int \frac{tdt}{t}=\int dt=t+c$

Replacing t by $\sqrt{x^2+1600}$, gives the total cost function

$$C(x)=\int \frac{x}{\sqrt{x^2+1600}}dx=\sqrt{x^2+1600}+c$$

Given that the fixed Cost is $500FRW$ we get:

$500=\sqrt{0^2+1600}+c$

$500=400+c$

$c=100$

Total Cost Function $C(x)=\sqrt{x^2+1600}+100$

ii) An Average Cost $AC=\frac{C(x)}{x}=\frac{\sqrt{x^2+1600}+100}{x}=\sqrt{1+\frac{1600}{x^2}}+\frac{100}{x}$

b) $f(x)=4-\sqrt{x}$

i)The $y-$intercept is the point with abscissa $x=0$. We have $y=4-\sqrt{0}=4\Rightarrow A(0,4)$

The $x-$intercept, $y=0$. We have $0=4-\sqrt{x}\Rightarrow -\sqrt{x}=-4\Rightarrow x=16\Rightarrow B(16,0)$

ii) The graph

i) The shaded area in terms of a definite integral is expressed by

$$A=\int_0^{16} f(x)dx \Rightarrow A=\int_0^{16}\left(4-\sqrt{x}\right)dx$$

ii) $$A=\int_0^{16}\left(4-\sqrt{x}\right)dx=\left[4x-\frac{x^{3/2}}{\frac{3}{2}}\right]_0^{16}=\left[(4\times 16)-\frac{16^{3/2}}{\frac{3}{2}}\right]$$

$$=64-\frac{2}{3}\sqrt{4096}=64-\frac{2}{3}\times 64=64\left(1-\frac{2}{3}\right)=64\left(\frac{3-2}{3}\right)=\frac{64}{3}.$$

The area of the shaded region is $\frac{64}{3}$ *Square unit.*

QUESTION THREE

This is an open question, it can be given as a home work but the findings are to be presented and discussed in classroom.

3.9. Remedial, Consolidation and Extended activities

The teacher 's guide suggests additional questions and answers to assess the key unit competence.

a) Remedial activities:

Suggestion of questions and answers for remedial activities for slow learners.

Evaluate the following integrals:

1) $\int (4x^5+x+1)dx$

2) $\int_1^4 \left(e^x - x^{\frac{1}{2}}\right)dx$

3) $\int (3x^2-1)xdx$

4) $\int_1^2 (e^{3x}+3x^2)dx$

Answers for remedial activities

Question one:

$\int (4x^5+x+1)dx$

$\int (4x^5+x+1)dx = \int 4x^5dx + \int xdx + \int dx$

$= \frac{4x^6}{6} + \frac{x^2}{2} + x + C$

Question two:

$\int_1^4 \left(e^x - x^{\frac{1}{2}}\right)dx =$

$$\int_1^4 \left(e^x - x^{\frac{1}{2}}\right)dx = \int_1^4 e^x dx - \int_1^4 x^{\frac{1}{2}}dx = \left[e^x\right]_1^4 - \left[\frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1}\right]_1^4 = \left(e^4 - e^1\right) - \left(\frac{4^{\frac{3}{2}}}{\frac{3}{2}} - \frac{1^{\frac{3}{2}}}{\frac{3}{2}}\right)$$

$= \left(e^4 - e\right) - \left(\frac{16}{3} - \frac{2}{3}\right)$

$= e^4 - e - \frac{14}{3}$

Question three

$\int (3x^2-1)xdx$

Set $u = 3x^2 - 1$, $du = 6xdx$, $\frac{1}{6}du = xdx$

$\int (3x^2-1)dx = \frac{1}{6}\int udu = \frac{1}{6}\frac{u^2}{2} + C = \frac{1}{12}u^2 + C$. By substituting u by $3x^2-1$ we get

$\int (3x^2-1)xdx = \frac{1}{12}(3x^2-1)^2 + C$

Question four:

$$\int_1^2 (e^{3x}+3x^2)dx = \left[\frac{e^{3x}}{3}+x^3\right]_1^2 = \frac{e^6-e^3}{3}+7$$

b) Consolidation activities:

Suggestion of questions and answers for deep development of competences.

1) Evaluate $\int_0^3 x\sqrt{10-x^2}\,dx$

2) Determine the area of the region P bounded by $y=5+x^2$ and $y=8+2x$

3) Compute $\int 3x\sin x dx =$

4) A variable force $(F\ \ Newtons)$ modelled by the equation $F=8x-6$ is applied over a certain distance (x). What is the work $(W\ in\ Joules)$ done in moving the object for a displacement of $4m$ to $8m$

5) By the use of the method of your choice perform the integral $\int \frac{1-\tan x}{1+\tan x}dx$

Answers for consolidation activities

1) $\int_0^3 x\sqrt{10-x^2}\,dx$

$$10-x^2=t,\ \text{then} -2xdx=dt,\ or\ xdx=-\frac{1}{2}dt$$

$$when\ x=0,\ t=10,\ when\ x=3,\ t=10-9=1$$

$$\int_0^3 x\sqrt{10-x^2}\,dx=\int_{10}^1 -\sqrt{t}\frac{dt}{2}=\frac{1}{2}\int_1^{10}\sqrt{t}dt$$

$$=\frac{1}{2}\int_1^{10} t^{\frac{1}{2}}dt=\frac{1}{2}\left[\frac{t^{\frac{1}{2}+1}}{\frac{1}{2}+1}\right]_1^{10}=\frac{1}{2}\left[\frac{t^{\frac{3}{2}}}{\frac{3}{2}}\right]_1^{10}=\frac{1}{2}\times\frac{2}{3}\left[10^{\frac{3}{2}}-1^{\frac{3}{2}}\right]=$$

$$\Rightarrow \frac{1}{3}\left(\sqrt{10^3}-1\right)$$

$$\Rightarrow \frac{1}{3}\sqrt{1000}-1$$

$$\therefore \int_0^3 x\sqrt{10-x^2}\,dx=\frac{1}{3}\sqrt{1000}-1$$

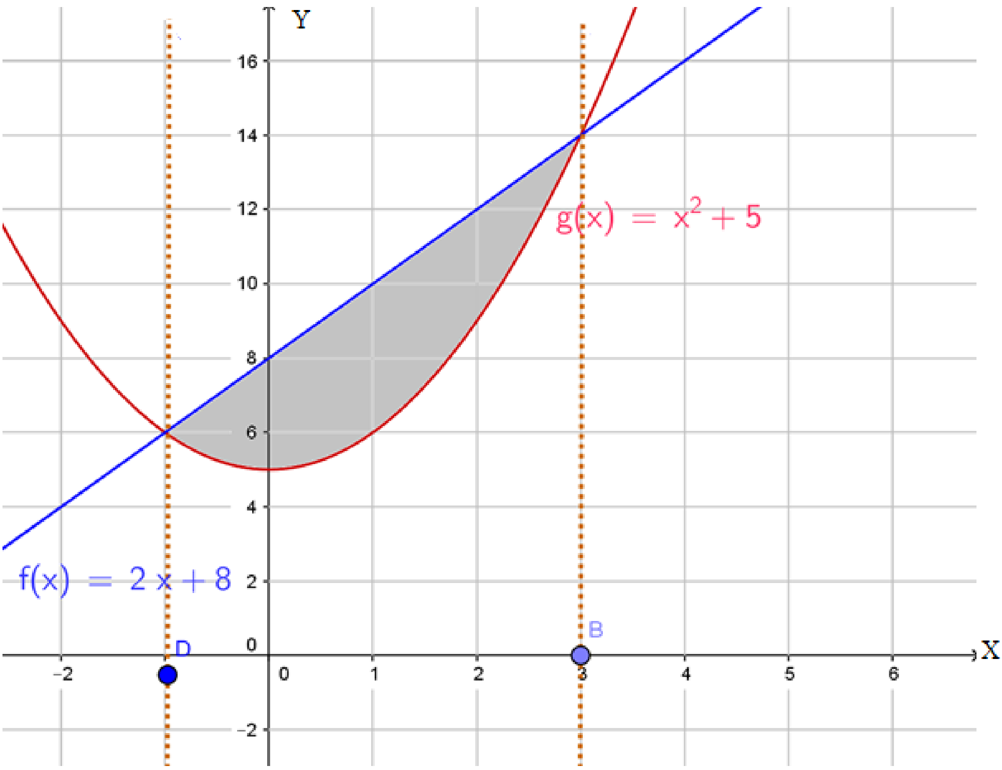
2)To find the intersection points, we have to equate the two equations: $5+x^2=8+2x$,

which gives $x^2-2x-3=0 \Leftrightarrow (x+1)(x-3)=0$.

So, the two curves will intersect when $x=-1$ *and* $x=3$,

The graph of g is a parabola while the graph of f is a line.

Graph of $g(x)=5+x^2$ *and* $f(x)=8+2x$



$$Area\ \ A=\int_D^B\left[\begin{pmatrix}upper\\function\end{pmatrix}-\begin{pmatrix}lower\\function\end{pmatrix}\right]dx$$

$$A=\int_{-1}^{3}\left[(2x+8)-(x^2+5)\right]dx=\int_{-1}^{3}(-x^2+2x+3)dx$$

$$=\left[-\frac{x^3}{3}+x^2 3x\right]_{-1}^{3}=\frac{50}{3}.$$

Therefore, A=$\frac{50}{3}$square unit

3)$\int 3x\sin x dx = 3\int x\sin x dx$

$\int 3x\sin x dx = 3I$ where $I = \int x\sin x dx$

Let's use integration by parts method formula $\int u dv = uv - \int v du$

Set $\begin{cases} u = x \\ dv = \int \sin x dx \end{cases}$ then $\begin{cases} du = dx \\ v = -\cos x \end{cases}$

We have $\int u dv = -x\cos x - \int(-\cos x)dx = -x\cos x + \sin x + C$

$\text{Finally, } I = \int x\sin x dx = -x\cos x + \sin x + C$

$\therefore \int 3x\sin x dx = 3\int x\sin x dx = 3I = 3(-x\cos x + \sin x) + C$

4) $W = \int_2^4 (8x-6)dx = \left[8\frac{x^2}{2} - 6x\right]_2^4 = \left[4x^2 - 6x\right]_2^4 = \left[4(16)-6(4)\right] - \left[4(4)-6(2)\right]$

$= \left[4(16)-6(4)\right] - \left[4(4)-6(2)\right]$

$= (64-24) - (16-12) = 40 - 4 = 36$

The work done is 36 joules.

5) $\int \frac{1-\tan x}{1+\tan x}dx = \int \frac{1-\frac{\sin x}{\cos x}}{1+\frac{\sin x}{\cos x}}dx$

$= \int \frac{\cos x - \sin x}{\cos x + \sin x}dx$

Assume that $\cos x + \sin x = u$, $(-\sin x + \cos x)dx = du$ or $(\cos x - \sin x)dx = du$

Then $I = \int \frac{du}{u} = \ln|u| + C$

$I = \int \frac{du}{u} = \ln|\cos x + \sin x| + C$

Therefore $\int \frac{1-\tan x}{1+\tan x}dx = \ln|\cos x + \sin x| + C$

Extended activities: Suggestion of questions and answers for gifted and talented learners.

1) Suppose that the consumers' demand function for a certain commodity is $D(q) = 4(25 - q^2)$ dollars per unit.

a) Find the total amount of the money consumers are willing to spend to get 3 units of commodity.

b) Sketch the demand curve showing the willingness to spend for 3 units as an area.

2) Let be given $f(t)=\frac{4}{81}t^3$ for $0\le t\le 3$

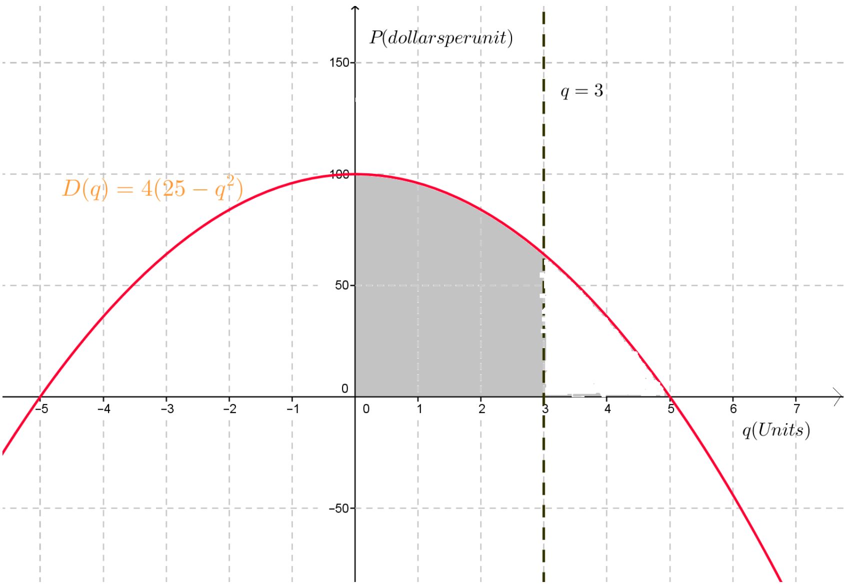
Prove that the function $f(t)$ represents a probability density function of a ceratin random variable t (in minutes) that represents a waiting time in traffic jamp. What is the probability of waiting in a traffic jam between 1 to 3 minutes.

Solution for extended activities 3.9.3

1) a) The total amount of money is given by: $T=\int_0^3 D(q)\,dq=\int 4(25-q^2)dq$

Therefore, $T=4(25q-\frac{1}{3}q^3)\Big|_0^3$ Thus, $T=264\,\$$

The demand curve showing the willingness to spend for 3 units as an area.



2) The given function is a probability density function if $\int_0^3 \frac{4}{81}t^3dt=1.$

Computing the given integral we find: $\int_0^3 \frac{4}{81}t^3dt=\frac{4}{81}\left[\frac{t^4}{4}\right]_0^3=\frac{1}{81}(81-0)=1.$

Thus, the function f(t) is a probability density.

The probability P of waiting in a traffic jam between 1 to 3 minutes is calculated as follows:

$$P=\int_1^3 \frac{4}{81}t^3dt\ \therefore P=\left[\frac{1}{81}t^4\right]_1^3=\frac{1}{81}(3)^4-\frac{1}{81}(1)^4$$

Then, $P=\frac{1}{81}(81)-\frac{1}{81}(1)=0.9$.

Therefore, the probability of waiting in a traffic jam between 1 to 3 minutes is 0.9.

UNIT 4: ORDINARY DIFFERENTIAL EQUATIONS

4

4.1. Key unit competence

Use ordinary differential equations of first and second order to model and solve related problems in Physics, Economics, Chemistry, Biology, etc.

4.2 Prerequisite knowledge and skills

Learners will perform well in this unit if they have a good background on: Exponential functions (Unit 2), Differentials and integration (Unit 3), exponential and polar form of a complex number (Unit 1), quadratic equations, simultaneous equations (Senior 4: Unit 3).

4.3 Cross-cutting issues to be addressed

- **Inclusive education:** promote the participation of all learners while teaching.
- **Peace and value Education:** During group activities, the teacher will encourage learners to help each other and to respect opinions of colleagues.
- **Gender:** During group activities try to form heterogeneous groups (with boys and girls) or when learners start to present their findings encourage both (boys and girls) to present.
- **Environment and Sustainability:** During the lesson on population growth, guide learners to discuss the effects of the high rate of population growth on the environment and sustainability.
- **Standardization culture:** During the lesson on application of differential equations in chemistry (**the quantity of a drug in the body)** guide learners to discuss advantages for respecting Doctor's instructions when taking drugs.

4.4 Guidance on the introductory activity

- Invite learners to form groups and let them to work independently on introductory activity to understand the concept of differential equation.
- Walk around to provide various pieces of advice where necessary.
- After a given time, invite learners to present their findings and through their works help them to have an idea on the differential equation.
- Harmonize their works and emphasize that they had a differential equation representing a situation of the population for a country and that it can be solved to obtain the formula for estimating the population of that country at any time t.

- Invite learners to discuss positive measures that should be taken to address the problem of exponential growth of the population.
- Ask learners to discuss the importance of studying how to solve differential equations.

Solution for the introductory activity:

The quantity $y(t)$ satisfies the exponential growth model: $\frac{dy}{y} = kdt$.

To find $y(t)$, let us integrate both sides of the equation $\frac{dy}{y} = kdt$;

$\int \frac{dy}{y} = \int kdt \Leftrightarrow \ln y = kt + c$ where c is an arbitrary constant.

We have the function $y = e^c e^{kt}$.Taking the constant $C = e^c$, we find $y(t) = Ce^{kt}$.

This is an exponential function that is increasing when the constant k is positive.

Assuming an exponential growth model of the population y and constant growth rate k, at initial time ($t = 0$), the population is $y(0) = Ce^{k0} = Ce^0 = C_0$.

If the population of a country is C_0 at time $t = 0$, this population with the growth rate k will be $y(t) = C_0 e^{kt}$ after the time t.

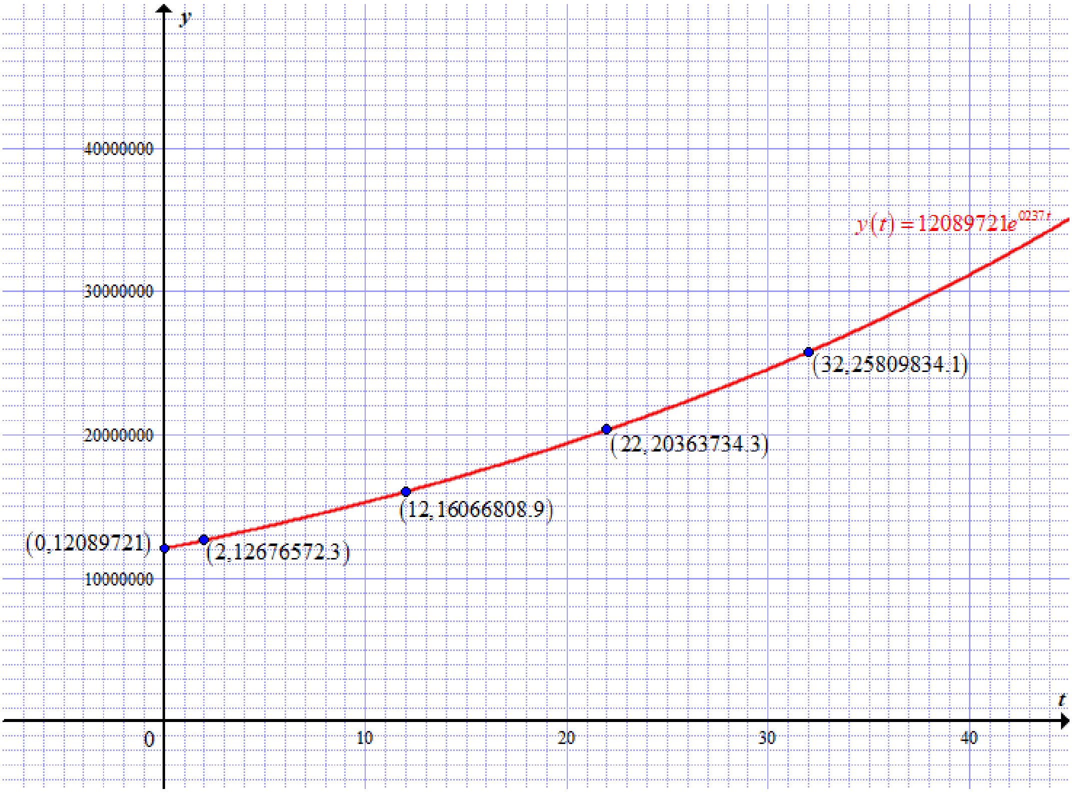
Therefore, given that the size of the Rwandan population is now (in the year 2018) estimated to $C_0 = 12,089,721$ with a growth rate of about $k = 2,37\% = 0.0237$ comparatively to the year 2017, the equation of Rwandan population becomes $y(t) = 12089721e^{0.0237t}$.

1) The national population at the beginning of the year 2020, 2030, 2040 and 2050:

From now in 2018 taken as initial time, in 2020 the time $t = 2$, in 2030 the time $t = 12$, in 2040 the time $t = 22$, in 2050 the time $t = 32$.

- Hence, the national population at the beginning of the year 2020 will be $y(2) = 12089721e^{(0.0237)2} = 12089721(0.04854134) = 12,676,572.3$ people.
- The national population at the beginning of the year 2030 will be $y(12) = 12089721e^{(0.0237)12} = 12089721(1.32896441) = 16,066,808.9$ people.
- The national population at the beginning of the year 2040 will be $y(22) = 12089721e^{(0.0237)22} = 20,363734.3$ people.
- The national population at the beginning of the year 2050 will be $y(32) = 12089721e^{(0.0237)32} = 25,809,834.1$ people.

2) Graph representing the increasing population $y(t) = 12089721e^{0.0237t}$



3) Our observation is that the national population is increasing with time but the surface where to live remains constant and we are not sure that the economy of the country is going to increase exponentially.

4) Pieces of advice: Police makers should adopt the family planning policy and sensitize the population as well as integrate the family planning programs into school curricula at **all levels of education.**

The teacher should encourage learners to provide more ideas.

4.5. List of lessons

UNIT TITLE: ORDINARY DIFFERENTIAL EQUATIONS (27 periods)			
Introductory activity: 1 period: 40 minutes			
No	Lesson title	Learning objectives (from the syllabus including knowledge, skills and attitudes):	Number of periods
1	Definitions and classification of differential equations	Extend the concepts of differentiation and integration to ordinary differential equations. State the order and the degree of an ordinary differential Equation.	1
2	Differential equations of first order with separable variables	Determine whether an ordinary differential equation of first order is with separable variables.	3
3	Linear differential equations of first order	differential equation of first order by “variation of constant” and by “integrating factor”	3
4.	Application of differential equations	Use differential equations to solve word problems related to the population growth	1
5	Application of differential equations	Use differential equations to solve word problems involving crime investigation	1
6	Application of differential equations	Use differential equations to solve word problems involving the quantity of a drug in the body	1
7	Application of differential equations	Use differential equations to solve problems in economics and finance	1
8	Application of differential equations	Use differential equations to solve problems of electricity (Series Circuits).	1

9.	Introduction to 2^{nd} order linear homogeneous differential equations	Express the auxiliary quadratic equation of a homogeneous linear differential equation of second order with constant coefficients	1
10.	Linear independence and superposition principle	• Appreciate the use of differential equations in solving problems occurring from daily life • Show patience, commitment and dedication when solving a differential equation or modeling a problem using differential equations • Solve a linear ordinary differential equation of second order	2
11.	Characteristic equation of a 2^{nd} order differential equations		1
12	Solving linear differential equations whose characteristic equation has two distinct real roots		2
13.	Solving linear differential equations whose characteristic equation has a double real root		2
14.	Solving linear differential equations whose characteristic equation has two complex roots		3
15	Applications of differential equations of second order	Use differential equations to model and solve problems in Physics (simple harmonic motion…), Economics	2
16	End unit assessment		1

Notice: *For application of mathematics content to other subjects, the teacher will consider the prerequisite of learners in this domain then act accordingly; the time spent and importance given to application activities depends on the learners' level of knowledge and interest.*

Lesson 1: Definitions and classification of ordinary differential equations

a) Prerequisites/Revision/Introduction:

Learners will perform better in this lesson if they refer to techniques of derivatives (S4).

b)Teaching resources: Textbooks, and Internet to facilitate research.

c) Learning activities:

This lesson will help learners to understand the concept of differential equation and their classifications through derivatives and the highest derivative degree.

- Form groups and ask learners to work out the activity 4.1.
- Let students work independently for somewhile.
- Facilitate learners to derivate and to guess the power of the highest derivative.
- Ask randomly some groups to present their findings to the whole class;
- Guide learners to interact about the findings, to conclude on the new concept and to write a short summary.
- Let learners work out example 4.1 under your guidance and work individually application activity 4.1 to assess their competences.

Solution for activity 4.1

$y = 4kx$. Using the differentiation, we have $\frac{dy}{dx} = 4k \Rightarrow k = \frac{dy}{4dx}$

The given equation becomes $y = \frac{dy}{dx}x$ *or* $y = y'x$. Order of the derivative is 1.

2) $y = kx + bx^2$

Differentiate to get $\frac{dy}{dx} = k + 2bx$. Solving for k yields to $k = y' - 2bx$ .Differentiating again: $\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2} = 2b$ .Then $b = \frac{d^2y}{2dx^2}$ *or* $b = \frac{y''}{2}$

Replace k and b by their values in $y = kx + bx^2$ to get:

$$y = \left[\frac{dy}{dx} - 2\left(\frac{d^2y}{2dx^2}x\right)\right]x + \frac{1}{2}\frac{d^2y}{dx^2}x^2 \text{ or } y = y'x - \frac{1}{2}y''x^2$$

This is a differential equation of 2nd order.

3) $y = k\cos 2x - b\sin 2x$

Differentiate y with respect to x to get: $y' = -2k\sin 2x - 2b\cos 2x$

Differentiating again, we get:

$y'' = -4k\cos 2x + 4b\sin 2x = -4(k\cos 2x - b\sin 2x) = -4y''$.

Then, $y'' = -4y$ or $y'' + 4y = 0$. This is the differential equation of 2nd order.

Solutions for application activities 4.1

a) $\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^4-4x+y=1 \Rightarrow$ This is differential equation of the 2^{nd} order and degree one

b) $\left(\frac{dy}{dx}\right)^3-2x=\cos y-2\sin x \Rightarrow$ This is a differential equation of the 1^{st} order and degree 3.

c) $(y'')^3+(y')-2y=x \Rightarrow$ This is a differential equation of the 2^{nd} order and degree 3.

d) $y\frac{d^2y}{dx^2}=-\cos x \Rightarrow$ This is a differential equation of the 2^{nd} order and degree one.

e) $x^2\left(\frac{d^2y}{dx^2}\right)^4+y\left(\frac{dy}{dx}\right)+y^4=0 \Rightarrow$ This is a differential equation of the 2^{nd} order and degree 4.

NB: Ask learners to explain in words their answers based on the definition of order and degree of differential equations.

Lesson 2: Differential equations of first order with separable variables

a) Prerequisites/Revision:

Students will learn better this lesson if they have a good understanding on concepts of integration learnt in unit 3.

b) Teaching resources:

Textbooks, Scientific calculators, graph papers.

c) Learning activities:

- Form groups and ask learners to attempt activity 4.2 in the learner's book.
- Ask them to discuss, participate actively and solve the activity 4.2.
- Let students work independently for some while,
- Thereafter, as they are working, walk around to ensure that they are performing the task effectively and cooperatively.
- Provide assistance where necessary by facilitating learners to separate variables and integrate both sides of the obtained equality after separation.
- Call upon groups to present their findings.
- Ask learners to follow attentively their classmates' presentations, interact with them about their findings and write a short summary.
- Harmonize their findings by leading students to solve differential equations with separable variables.

- Let learners go through the examples 4.2 and 4.3 under your guidance and work individually application activities 4.2 to reinforce their skills in solving 1st order differential equations with separable variables.

Solution for activity 4.2

1) $4y' - 2x = 0$

Solve for y' to get y : $4\frac{dy}{dx} = 2x \Rightarrow 4dy = 2xdx \Rightarrow dy = \frac{x}{2}dx$

Integrate both sides to deduce the value of the dependent variable y :

$\int dy = \frac{1}{2}\int xdx \Rightarrow y = \frac{x^2}{4} + c$ knowing that $y' = \frac{dy}{dx}$ we have.

$4\frac{dy}{dx} - 2x = 0$ *and* $4y' - 2x = 0$ are the same.

Replace the value of y in the given equation to get:

$4\left(\frac{x^2}{4} + c\right)' - 2x = 0 \Leftrightarrow 4\left(\frac{2x}{4}\right) - 2x = 0$ and it is clear that the equality remains correct.

2) a) $\sin xdx - \sin ydy = 0 \Rightarrow \sin ydy = \sin xdx$

By integrating both sides we get: $\int \sin ydy = \int \sin xdx \Leftrightarrow -\cos y = -\cos x + c$

Given that $-\cos y = -\cos x + c \Leftrightarrow \cos y = \cos x - c$, we get $y = \cos^{-1}(\cos x - c)$

b) $x\frac{dy}{dx} = 1$,

Separate variables: $dy = \frac{1}{x}dx \Leftrightarrow dy = \frac{dx}{x}$

Integrating both sides: $\int dy = \int \frac{dx}{x}$ thatgives $y = \ln|x| + c$

c) To solve $f(y)\frac{dy}{dx} = g(x)$ we separate variables to both sides of the equation and then integrate both sides to deduce the value of the dependent variable y .

Solution for application activity 4.2

Solution:

1)The general solution for:

a) $\frac{dy}{dx} = x\cos x$; separate variables to get: $dy = x\cos xdx$

Integrating both sides we get $y = \int x\cos xdx$; then integrate by parts right hand side

$y = x\sin x - \int \sin xdx = x\sin x + \cos x + c$ which is the required solution.

b) $x\frac{dy}{dx} = 2 - 4x^3$.

Rearranging $x\frac{dy}{dx} = 2 - 4x^3$ gives $\frac{dy}{dx} = \frac{2 - 4x^3}{x} = \frac{2}{x} - 4x^2$.

Integrating both sides gives: $y=\int\left(\frac{2}{x}-4x^2\right)dx=2\ln x-\frac{4}{3}x^3+c$

Thus, the required solution is $y=2\ln x-\frac{4}{3}x^3+c$

2) $(x+1)\frac{dy}{dx}=x\left(y^2+1\right)$.

Separate the variables to get: $(x+1)dy=x\left(y^2+1\right)dx$ or $\frac{}{y\ \ +1}dy=\frac{x}{x+1}dx$

Integrating both sides $\int\frac{}{y\ \ +1}dy=\int\frac{x}{x+1}dx$ or $\int\frac{1}{1-y^2}dy=\int\left(1-\frac{1}{x+1}\right)dx$

or $\int\frac{1}{1-y^2}dy=\int dx+\int\frac{1}{x+1}dx$ or $\tan^{-1}y=x-\ln(x+1)+c$

Putting the value of $y(0)=0$ in the general solution to get
$\tan^{-1}0=0-\ln(0+1)+c\Rightarrow c=0$

3) (a) Let $y=y(x)$ be the required function and $P(x,y)$ any point on the curve. The line OP has a slope $\frac{y}{x}$. The tangent to the curve at $P(x,y)$ that is perpendicular to the line OP has therefore the slope $\frac{-x}{y}$ since $\frac{y}{x}\left(\frac{-x}{y}\right)=-1$. We know that the slope of the tangent line to the curve $y=y(x)$ at $P(x,y)$ is defined by $\frac{dy}{dx}$. Therefore, $\frac{dy}{dx}=\frac{-x}{y}$ is the required.

The initial value problem is $\frac{dy}{dx}=\frac{-x}{y};\ y(0)=1$

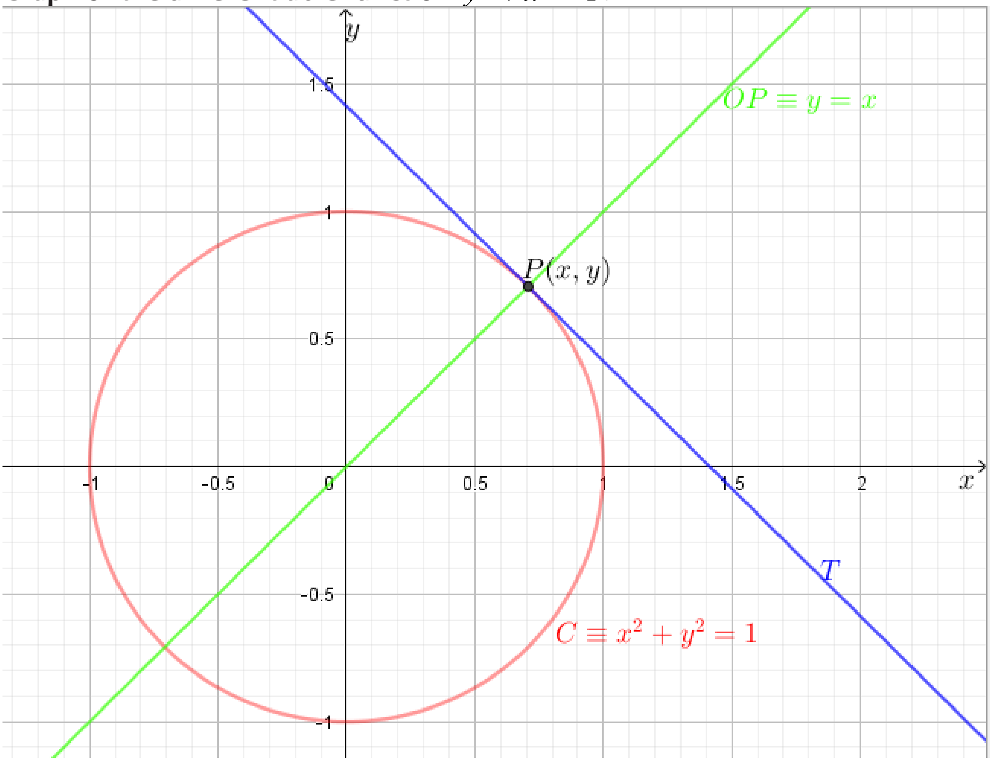
(b) The differential $\frac{dy}{dx}=\frac{-x}{y}$ is equivalent to $ydy=-xdx$ that is separable.

Let's solve the problem $ydy=-xdx\ \Rightarrow\ \int ydy=-\int xdx$

Simple integration gives $\frac{y^2}{2}+\frac{x^2}{2}=c$.

Replace the point $(0,1)$ to get $c=\frac{1}{2}$. So, $y^2+x^2=1$ is the final solution. The curve representing this solution is the circle centred at $(0,0)$ and radius 1.

Graph of the differentiable function $y^2+x^2=1$:



This confirms the theorem of geometry that states that a tangent to a circle forms a right angle with the circle's radius, at the point of contact of the tangent.

4) Rearranging the given equation, $\frac{y^2-1}{3y}dy = dt \Leftrightarrow \left(\frac{y}{3}-\frac{1}{3y}\right)dy = dt$

Direct integration yields $\frac{y^2}{6}-\frac{\ln y}{3}=t+c$ or $t=\frac{y^2}{6}-\frac{\ln y}{3}-c$ that is general equation.

Given that $y=1$ when $t=2\frac{1}{6}$, we have

$$2\frac{1}{6}=\frac{(1)^2}{6}-\frac{\ln 1}{3}-c \Leftrightarrow 2\frac{1}{6}=\frac{1}{6}-c \Rightarrow c=\frac{1-13}{6}=-2$$

Hence particular solution is $t=\frac{y^2}{6}-\frac{\ln y}{3}+2$.

5) Separating the variables gives $\frac{dy}{y}=\frac{x}{1+x^2}dx$. Integrating both sides gives $\ln y=\frac{1}{2}\ln\left(1+x^2\right)+c$. Given that $y=1$ when $x=0$, $\ln 1=\frac{1}{2}\ln 1+c \Rightarrow c=0$

The particular solution is $\ln y=\frac{1}{2}\ln\left(1+x^2\right)$ or $y=\sqrt{1+x^2}$

6) a) $\frac{dR}{d\theta}=\alpha \Leftrightarrow d\theta=\frac{dR}{\alpha R}$

$\int d\theta=\int\frac{dR}{\alpha R} \Rightarrow \theta=\frac{1}{\alpha}\ln R+c$

The general solution is $\theta=\frac{1}{\alpha}\ln R+c$. $R=R_o$ when $\theta=0^oC$, thus $0=\frac{1}{\alpha}\ln R_o+c \Rightarrow c=-\frac{1}{\alpha}\ln R_o$. Hence the particular solution is $\theta=\frac{1}{\alpha}\ln R-\frac{1}{\alpha}\ln R_o \Leftrightarrow \theta=\frac{1}{\alpha}\ln\frac{R}{R_o}$. Finally, $R=R_o e^{\alpha\theta}$

(b) Substituting $\alpha=38\times10^{-4}$, $\theta=50$ and 24.0Ω into $R=R_o e^{\alpha\theta}$ gives the resistance at 50^oC, $R=24.0e^{38\times10^{-4}\times50}=29.0\Omega$, $\alpha=38\times10^{-4}$, $\theta=50$ and 24.0Ω into $R=R_o e^{\alpha\theta}$ gives the resistance at $50^{o}C$, $R = 24.0e^{38\times10^{-4}\times50} = 29.0\Omega$

Lesson 3: Linear differential equations of first order

a) Prerequisites/Revision/Introduction:

Learners will learn better this lesson if they have a good understanding on technique of integration learnt in Senior 6, unit3.

b) Teaching resources:

Textbooks, Scientific calculators.

c) Learning activities:

- Organize learners into groups and let them attempt activity 4.3.
  - Ask them to discuss and solve the provided activity.
  - Let them work independently for some while,
  - Thereafter, move around to ensure all learners in groups participate actively.
  - Invite randomly some groups to present their findings to the whole class and after presentation facilitate learners to have a lesson summary through harmonization of their findings
  - Individually, invite learners to read through example 4.4 and then work out the application activities 4.3 to enhance their knowledge and skills about linear differential equations of the first order.

Solutions for activity 4.3

Given the differential equation $\frac{dy}{dx}+2xy = x \text{ or } \quad y'+2xy = x$ (1)

1) $\frac{dy}{dx}+2xy = x$ (1), $I(x) = e^{\int 2xdx} = e^{x^2}$. For convenience, we set the integration constant to 0.

2) Multiplying both sides in the differential equation (1) by $I(x) = e^{x^2}$ to get:
$e^{x^2}(y'+2xy) = xe^{x^2}$ Or $\frac{d}{dx}\left(e^{x^2}y(x)\right) = xe^{x^2}$

1) Integrating both sides and divide by integrating factor $I(x)$ to get $y(x)e^{x^2} = \frac{1}{2}e^{x^2}+c.$

4) Solve for $y(x)$ to get $y(x) = \frac{1}{2}+ce^{-x^2}$.

5) Replacing $y(x) = \frac{1}{2}+ce^{-x^2}$ and $y' = -2cxe^{-x^2}$ in (1) we find that $y(x)$ is a solution of (1)

$$y'+2xy = -2cxe^{-x^2}+2x(\frac{1}{2}+ce^{-x^2}) = -2cxe^{-x^2}+x+2cxe^{-x^2} = x$$

Solution for application activity 4.3

Solution:

1. a) $y'+\frac{y}{x}=1$ this is the form of $\frac{dy}{dx}+py=q$; $p=\frac{1}{x},q=1$

$y=uv$, $u=\int qe^{\int pdx}dx$; $v=e^{-\int pdx}$ thus $u=\int e^{\int\frac{1}{x}dx}dx=\int e^{\ln x}dx=\int xdx=\frac{x}{2}+c$

$\Rightarrow u=\frac{x^2}{2}+c$; $v=e^{-\int\frac{1}{x}dx}=e^{-\ln x}=\frac{1}{e^{\ln x}}=\frac{1}{x}$

Finaly $y(x)=uv=(\frac{x^2}{2}+c)\frac{1}{x}=\frac{x}{2}+\frac{c}{x}$, where c is a constant

b) $y'+xy=x$, this is the form of $\frac{dy}{dx}+py=q$, $p=x,q=x$

$y=uv$,

$u=\int qe^{\int pdx}dx=\int xe^{\int xdx}dx=\int xe^{\frac{x^2}{2}}dx$, let $t=\frac{x^2}{2}\Rightarrow dt=xdx$, we get $\int e^t dt=e^t+c$.

Then $u=e^{\frac{x^2}{2}}+c$ and $v=e^{-\int pdx}=e^{-\int xdx}=e^{-\frac{x^2}{2}}$.

Therefore, $y(x)=uv=\left(e^{\frac{x^2}{2}}+c\right)e^{-\frac{x^2}{2}}=1+ce^{-\frac{x^2}{2}}$, where c is a constant

c) $y'+\frac{\quad}{x}=x$ this is the form of $\frac{dy}{dx}+py=q$,where $p=\frac{1}{x},q=x$

$y=uv$;

$u=\int qe^{\int pdx}dx\Leftrightarrow u=\int xe^{\int\frac{1}{x}dx}dx=\int xe^{\ln x}dx=\int x^2dx=\frac{x^3}{3}+c\Leftrightarrow u=\frac{x^3}{3}+c$

$v=e^{-\int pdx}=e^{-\int\frac{1}{x}dx}=e^{-\ln x}=e^{\ln x^{-1}}=\frac{1}{x}$.

Then, $y=uv=(\frac{x^3}{3}+c)\frac{1}{x}=\frac{x^2}{3}+\frac{c}{x}$, where c is a constant

d) $y'+2y=e^x$, This is the form of $\frac{dy}{dx}+py=q$, where $p=2,q=e^x$

$y=uv$, $u=\int qe^{\int pdx}dx$; $u=\int e^x e^{\int 2dx}dx=\int e^x e^{2x}dx=\int e^{3x}dx=\frac{e^{3x}}{3}+c$

$v=e^{-\int pdx}=e^{-\int 2dx}=e^{-2x}=\frac{1}{e^{2x}}$

$y=uv=(\frac{e^{3x}}{3}+c)\frac{1}{e^{2x}}=\frac{e^{3x}}{3e^{2x}}+\frac{c}{e^{2x}}=\frac{e^x}{3}+\frac{c}{e^{2x}}$, where c is a constant

e) $y'-2x\,y=e^{x^2}$, $p(x)=-2x$, the integrating factor is $I(x)=e^{-\int 2xdx}=e^{-x^2}$. Multiplying each side of the differential equation by integrating factor $I(x)=e^{-x^2}$ you get: $e^{-x^2}(y'-2xy)=1$ which can be written as $\frac{d}{dx}\left(e^{-x^2}y(x)\right)=1$. Integrating both sides you get: $e^{-x^2}y(x)=x+c$

Solve for y to get: $y(x)=(x+c)e^{x^2}$, where is an arbitrary constant.

f) $y'+\frac{3}{x}y=\frac{\sin x}{x^3}$, $I(x)=e^{\int\frac{3}{x}dx}=e^{3\ln x}=e^{\ln x^3}=x^3$

Multiply by $I(x)=x^3$ to get: $\frac{d}{dx}\left(x^3y(x)\right)=\sin x$

$x^3 y(x) = -\cos x + c$

$\Rightarrow y(x) = \frac{c}{x^3} - \frac{\cos x}{x^3}$

2) $CR\frac{dV}{dt} + V = E \Leftrightarrow CR\frac{dV}{dt} = E - V \Leftrightarrow \frac{dV}{E-V} = \frac{1}{CR}dt$

$\Leftrightarrow \frac{dV}{E-V} = \frac{1}{CR}dt \Leftrightarrow -\ln(E-V) = \frac{t}{CR} + k$

$\Leftrightarrow \ln\frac{1}{E} = k, t = 0; V = 0 \Leftrightarrow -\ln(E-V) = \frac{t}{CR} + \ln\frac{1}{E}$

$\ln\frac{1}{E-V} - \ln\frac{1}{E} = \frac{t}{CR}$; $\ln\frac{E}{E-V} = \frac{t}{CR} \Rightarrow \frac{E-V}{E} = e^{-\frac{t}{CR}}$

$E - V = Ee^{-\frac{t}{CR}} \Rightarrow V = E - Ee^{-\frac{t}{CR}}$; $V = E - Ee^{-\frac{t}{CR}}$ or $V = E\left(1 - e^{-\frac{t}{CR}}\right)$

When $E = 25V$, $C = 20\times10^{-6}F$, $R = 200\times10^{3}\Omega$ and $t = 3.s$; we get $V \simeq 19.422 Volts$.

Lesson4: Differential equations and the population growth

a) Prerequisites/Revision/Introduction:

Learners will perform well in this lesson if they refer to: integration techniques learnt in the unit 3, differential equations of 1st order with separable variables and linear differential equations of 1st order seen in previous lessons (unit 4, lessons 1, 2 and 3).

b) Teaching resources:

Textbooks, Scientific calculators, graph drawing software such as GeoGebra, …

c) Learning activities:

- Ask learners to work in small groups and attempt activity 4.4, in the learner's book.
- Let them work independently for some while,
- Monitor the work of different groups to ensure that all learners understand better how to model and solve the differential equation $\frac{dP}{dt} = KP$, (P is Population, t is time and K is a positive constant).

- Choose randomly a group to present the findings to the whole class, while the audience is following attentively,
- Invite learners to exchange their views about the presentation and advise on how to face the exponential growth of the population.
- Harmonize their findings and initiate learners to summarize the key points of the presentation.
- Ask learners to read and revise the example 4.5 in pairs and work individually

application activity 4.4, in the learner's book to enhance skills they have acquired from the application of differential equations of 1st order.

Solutions for activity 4.4

1) Differential equation expressing this model is $\frac{dP}{dt} = KP$

2) Separating variables and integrating both sides we get:

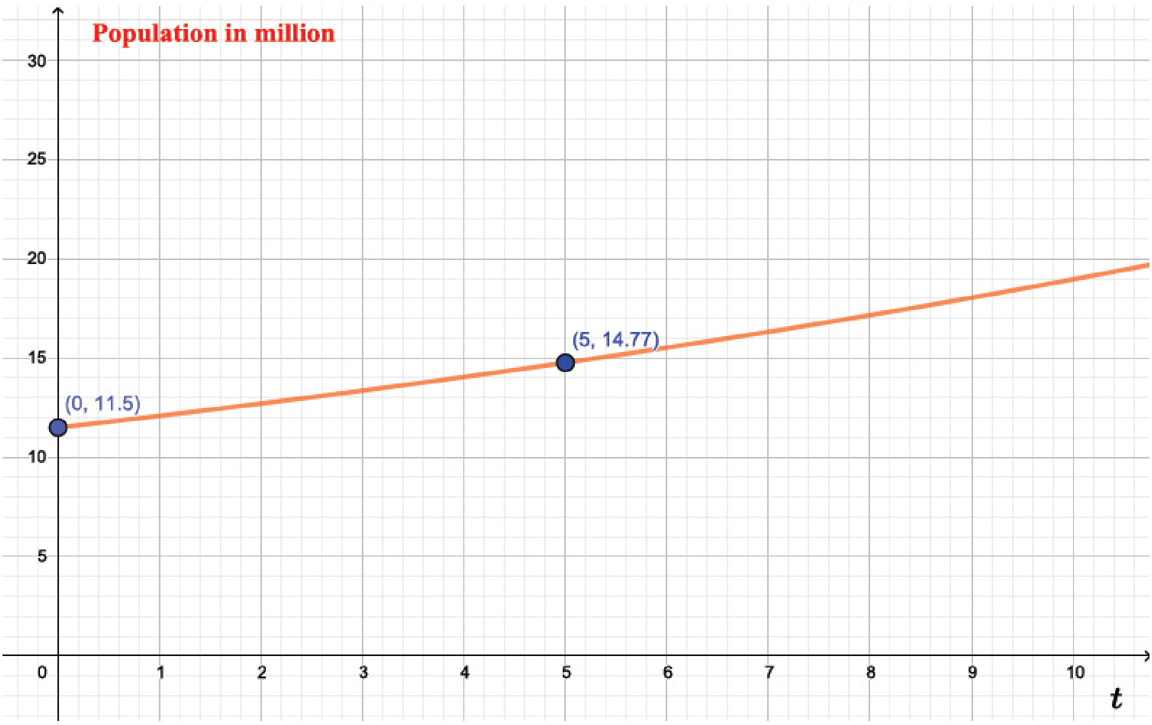
$\int \frac{dP}{P} = k\int dt \Rightarrow \ln P = Kt + c \Rightarrow P = ce^{Kt}$

If the initial population at time $t = 0$ is P_0, and $k = 0.05$ then $P_0 = ce^{(0.05)(0)} = c$

Therefore, $P_0 = c$ and we have $P = P_0 e^{0.05t}$

3) If the population $P_0 = 11,500,000$; respects the same variation $P = P_0 e^{0.05t}$ then $P = 11,500,000e^{0.05t}$.

After 5 years, this population will be $P(5) = 11,500,000e^{(0.05)5}$ that is 14,766,292 people.



Graph showing the population growth

The population is exponentially increasing and the policy makers of that town should think about family planning, environment protection, etc.

Solution for application activities 4.4

Solutions

1) Let P be the number of population; $\frac{dP}{dt} \sim P \Rightarrow \frac{dP}{dt} = \lambda P$

Separating variables you get $\frac{dP}{P} = \lambda dt$, then integrate both sides

$\int \frac{dP}{P} = \lambda \int dt \Rightarrow \ln P = \lambda t + c \Rightarrow P = e^{\lambda t + c} \Leftrightarrow P = e^{c} e^{\lambda t} \Leftrightarrow P = \alpha e^{\lambda t}$.

For $t = 0, P_0 = \alpha e^{0} \Leftrightarrow P_0 = \alpha$; Hence, $P = P_0 e^{\lambda t}$ *because* $\alpha = P_0$

$P = 2P_0$ *in* $t = 20$ *years with* $P = P_0 e^{\lambda t}$,

Thus, $2P_0 = P_0 e^{20\lambda} \Leftrightarrow 2 = e^{20\lambda} \Leftrightarrow 20\lambda = \ln 2 \Leftrightarrow \lambda = \frac{\ln 2}{20}$.

If P_0 is the initial population, then $t = ?$

If we have $P = 3P_0$ *then* $t = ?$ *with* $P = P_0 e^{\frac{\ln 2}{20} t}$

$3P_0 = P_0 e^{\frac{\ln 2}{20} t} \Rightarrow 3 = e^{\frac{\ln 2}{20} t} \Leftrightarrow \ln 3 = \frac{\ln 2}{20} t \Rightarrow t = \frac{\ln 3}{\ln 2} \times 20 = 31.699$ years

The population will triple after approximately 32 years

$(t = 31 years, 8 months, 11 days, 17h, 31 \min, 12 \sec)$.

2) Let P be the population of bacteria

$\frac{dP}{dt} \sim P \Rightarrow \frac{dP}{dt} = \alpha P$ *where* α *is the proportionality coefficient*

$$\frac{dP}{dt} = \alpha P \Leftrightarrow \frac{dP}{P} = \alpha dt \Rightarrow \int \frac{dP}{P} = \int \alpha dt$$
$$\Leftrightarrow \ln P = \alpha t + c$$
$$\Leftrightarrow P = e^{\alpha t + c} \Leftrightarrow P = e^{\alpha t} \times e^{c} \Leftrightarrow P = \delta e^{\alpha t} \,(*) \text{ where } \delta = e^{c}.$$

The number of bacteria is increasing from 1000 to 3000 in $10\,hours$.

For $t = 0, P = 1000$, for $t = 10, P = 3000$.

$P = 1000$ *correspond to* $t = 0$, *where* $p = \delta e^{\alpha t} \Leftrightarrow 1000 = \delta e^{0} \Leftrightarrow \delta = 1000$,

Hence $(*)$ *becomes* $P = 1000 e^{\alpha t}$.

$P = 3000$ *correspond to* $t = 10 \Leftrightarrow 3000 = 1000 e^{10\alpha} \Leftrightarrow e^{10\alpha} = 3 \Leftrightarrow 10\alpha = \ln 3 \Leftrightarrow \alpha = \frac{\ln 3}{10}$.

If $\delta = 1000$ *and* $\alpha = \frac{\ln 3}{10}$ *then* $p = 1000 e^{\frac{\ln 3}{10} t}$

b) If $t = 5 \Rightarrow P = ?$

$$P(5) = 1000 e^{\frac{\ln 3}{10} \times 5} = 1000 e^{\frac{\ln 3}{2}} = 1000 \times e^{\ln \sqrt{3}} = 1000 \times \sqrt{3}.$$

Hence, after 5 hours the number of bacteria is about 1732.

Lesson 5: Differential equations and Crime investigation

a) Prerequisites/Revision/Introduction:

Learners will perform well in this lesson if they refer to: integration techniques learnt in unit3, differential equations of 1st order with separable variables and linear differential equations of 1st order seen in previous lessons (unit 4, Lessons 1, 2 and 3).

b) Teaching resources:

Textbooks, Scientific calculators, graph drawing software such as GeoGebra, …

c) Learning activities:

- Instruct learners to form groups and work on activity 4.5,
- Let them work independently for somewhile.
- Follow up the working steps of different groups to give support where necessary and motivate learners to mention any difficulty met when modelling and solving the differential equation related to the given situation "crime investigation".
- Facilitate each group to present their findings.
- Harmonize the learners 'works and help them realize that the time of death of a murdered person can be determined with the help of differential equation $\frac{dT}{dt}+KT=KT_e$, with T: the temperature of the object and Te : the temperature of the environment surrounding the object.
- Individually, let learners go through the example 4.6 and work out application activities 4.5 to enhance their skills in the application of differential equations of 1st order.

Solution for activity 4.5

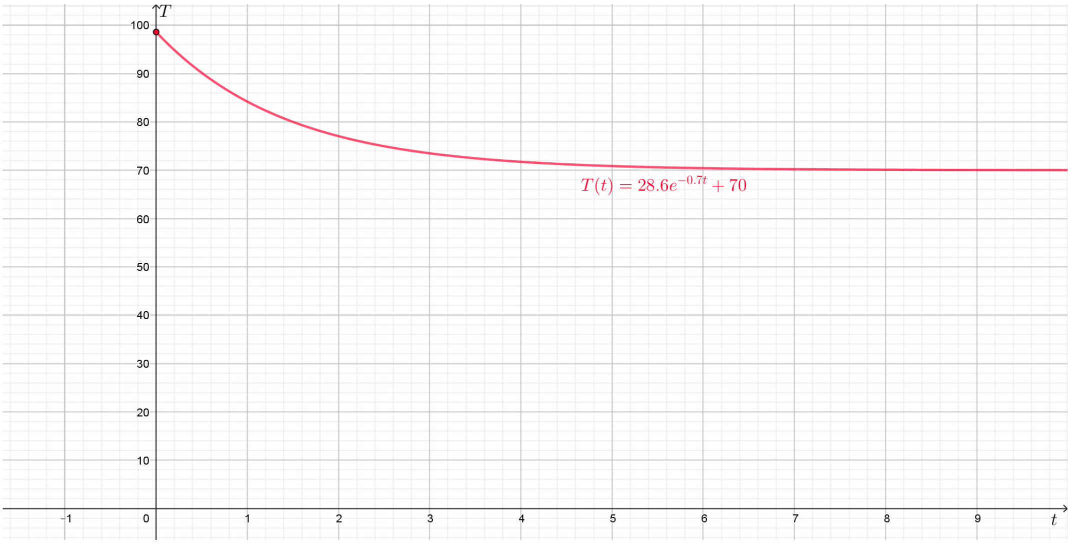
1) The differential equation expressing the model is $\frac{dT}{dt}=-K\left(T-T_e\right)$

2) $\frac{dT}{dt}=-K\left(T-T_e\right)\Rightarrow\frac{dT}{dt}+KT=KT_e$. This is a first order linear differential equation, its solution $T\left(t\right)=T_e+Be^{Kt}$ where B is a constant. Since $K=\ln\left(\frac{1}{2}\right)$; $T\left(t\right)=T_e+Be^{\ln\left(\frac{1}{2}\right)t}$

3) At $t=0, T=98,6^0F$, $T_e=98,6^0F$; our equation $T\left(t\right)=T_e+Be^{Kt}$ becomes $98,6=70+Be^0\Rightarrow B=28.6$

At time t, when $K=\ln\left(\frac{1}{2}\right)$; $T\left(t\right)=70+28.6e^{-0.7t}$

The graph of $T\left(t\right)=70+28.6e^{-0.7t}$



As the time elapses the temperature decreases and converges to 70 °F which is the temperature of the room.

Solution for application activity 4.5

Solution

According to Newton's law of cooling, the body will radiate heat energy into the room at a rate proportional to the difference in temperature between the body and the room.

If $T(t)$ is the body temperature at time t, then for some constant of proportionality K,

a) $\frac{dT}{dt} = K[T(t) - T_a]$

$\frac{dT}{dt} = K\left[T(t) - 70\right]$

b) Separating variables and integrating both sides,

$\int \frac{1}{T-70} dT = \int k dt \Rightarrow \ln|T-70| = Kt + C \quad \Rightarrow \quad |T-70| = e^{Kt+C} = Ae^{Kt}$, where $A = e^c$.

Then, $T - 70 = \pm Ae^{Kt} = Be^{Kt}$

Then, $T(t) = 70 + Be^{Kt}$

Constants K and B can be determined provided the following information is available:

Time of arrival of the police personnel, the temperature of the body just after his arrival and the temperature of the body after certain interval of time.

The officer arrived at 10.40 p.m. while the body temperature was 94.4 degrees. This means that if the officer considers 10:40 p.m. as $t = 0$ then $T(0) = 94.4 = 70 + B$ and so $B = 24.4$ giving $T(t) = 70 + 24.4e^{Kt}$.

Taking $K = 28.10^{-4}$, the officer has now temperature function $T(t) = 70 + 24.4e^{0.0028t}$

In order to find when the last time t the body was 98.6 (presumably the time of death), one has to solve for time the equation $T(t) = 98.6 = 70 + 24.4e^{0.0028t}$

We find approximately $t = -57.07$.

The death occurred approximately 57.07 minutes before the first measurement at 10.40 p.m, that is at 9.43 p.m. approximately.

Lesson 6: Differential equations and the quantity of a drug in the body

a) Prerequisites/Revision/Introduction:

Learners will perform well in this lesson if they have good understanding of integration techniques learnt in unit 3, differential equations of 1st order with separable variables and linear differential equations of 1st order seen in previous lessons (unit 4, Lessons 1, 2 and 3).

b) Teaching resources:

Textbooks, Scientific calculators, GeoGebra for graph drawing, …

c) Learning activities:

- Instruct learners to form groups and work on activity 4.6,
- Let them work independently for somewhile,
- Follow up the working steps of different groups to give support where necessary and motivate learners to mention any difficulty met when modelling and solving the differential equation related to the given situation "the quantity of a drug in the body".
- Facilitate each group to present their findings.
- Harmonize the learners' works and help them to find the quantity of drug $Q(t)$ left in the body at the time t .
- Invite learners to take decision after discussing their points of view on what happens when the patient does not respect the dose of medicine as prescribed by the Doctor.
- Individually, let learners go through the example 4.7 and work out application activities 4.6 to enhance their skills in the application of differential equations of 1st order.

Solution for activity 4.6

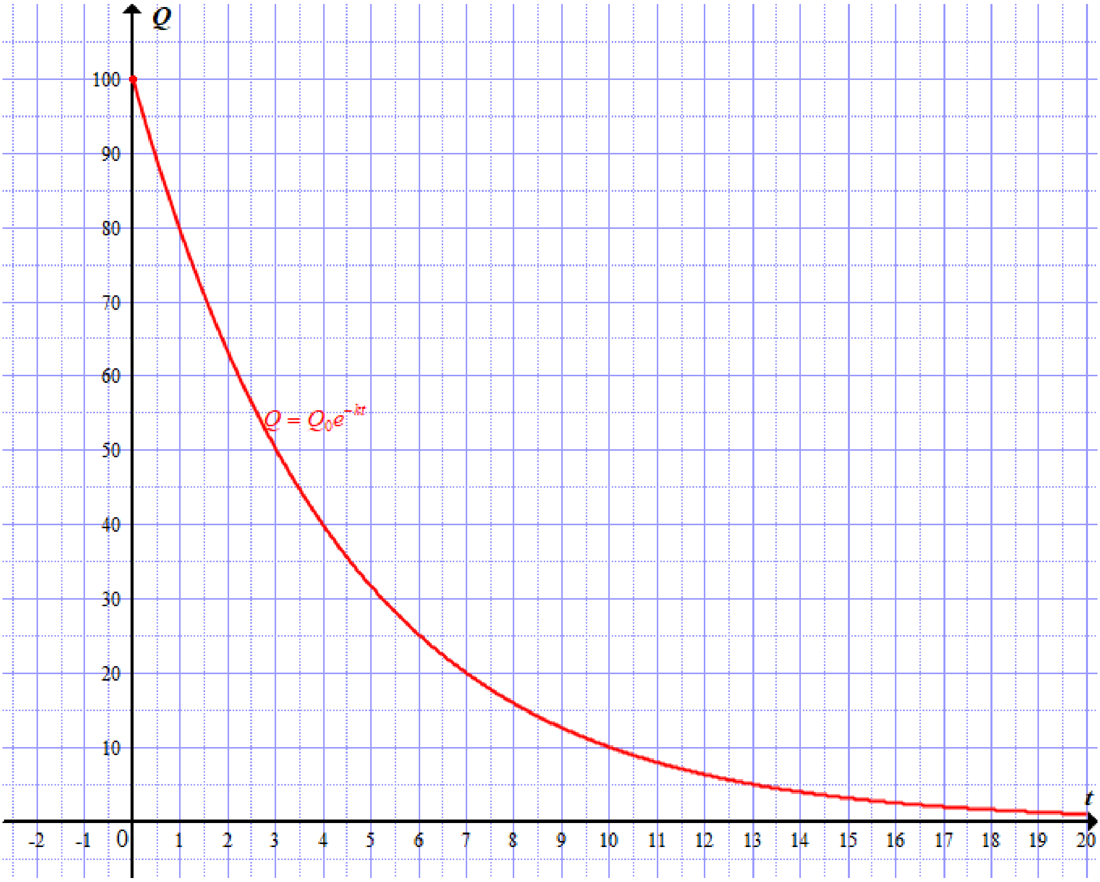
1) The equation for modelling the situation is $\frac{dQ}{dt} = -kQ$

2) $\frac{dQ}{Q} = -kdt \Rightarrow \int \frac{dQ}{Q} = -\int Kdt \Rightarrow \ln Q = -Kt$

3) The solution of this equation is $Q = Q_0 e^{-kt}$

2) When t = 0, the drug provided was $Q_0 = 100mg$ then $Q = 100e^{-kt}$

The graph of the equation $Q = 100e^{-kt}$



The graph shows that the drug in the human body decreases to 0 when the time of taking medicine increases.

When a patient does not respect the doctor's prescriptions he/she could suffer from the effects of medicine.

Solution for application ativity 4.6

Solution

(a) Since the half-life is 15 hours, we know that the quantity remaining $Q = \frac{1}{2}Q_0$ when t = 15. We substitute into the solution to the differential equation, $Q = Q_0 e^{-kt}$, and solve for k: $Q = Q_0 e^{-kt}$ thus $0.5Q_0 = Q_0 e^{-k(15)}$ or after dividing by Q_0 we get $0.5 = e^{-15k}$. By taking the natural logarithm of both sides: $\ln 0.5 = -15k$. Thus $k = \frac{-\ln 0.5}{15} = 0.0462$

(b) To find the time when 10% of the original dose remains in the body, we substitute $0.10Q_0$ in Q and solve for the time t. That is: $0.10Q_0 = Q_0 e^{-0.0462t}$

$0.10 = e^{-0.0462t}$, $\ln 0.10 = -0.0462t$. Then $t = \frac{\ln 0.10}{-0.0462} = 49.84$

There will be 10% of the drug still in the body at $t = 49.84$, or after about 50 hours.

Lesson 7: Differential equations in economics and finance

a) Prerequisites/Revision/Introduction:

Learners will perform well in this lesson if they have good understanding of integration techniques learnt in Senior 6 (unit3), differential equations of 1st order with separable variables and linear differential equations of 1st order seen in previous lessons (Senior 6: unit 4, Lessons 1, 2 and 3).

b) Teaching resources:

Textbooks, Scientific calculators, GeoGebra for graph drawing, etc.

c) Learning activities:

- Ask learners to work in small groups and attempt activity 4.7, in the learner's book.
- Let them work independently for somewhile,
- Monitor the work of different groups to ensure that all learners understand better how to model and solve the differential equation.
- Choose randomly some groups to present the findings to the whole class, while the audience is following attentively for interaction,
- Invite learners to exchange their views about the presentation of their classmates
- Harmonize their findings and initiate learners to summarize the key points of the presentation.
- Ask learners to read and revise the example 4.8 in pairs and work individually application activity 4.7, in the learner's book to enhance their skills about the application of differential equations of 1st order.

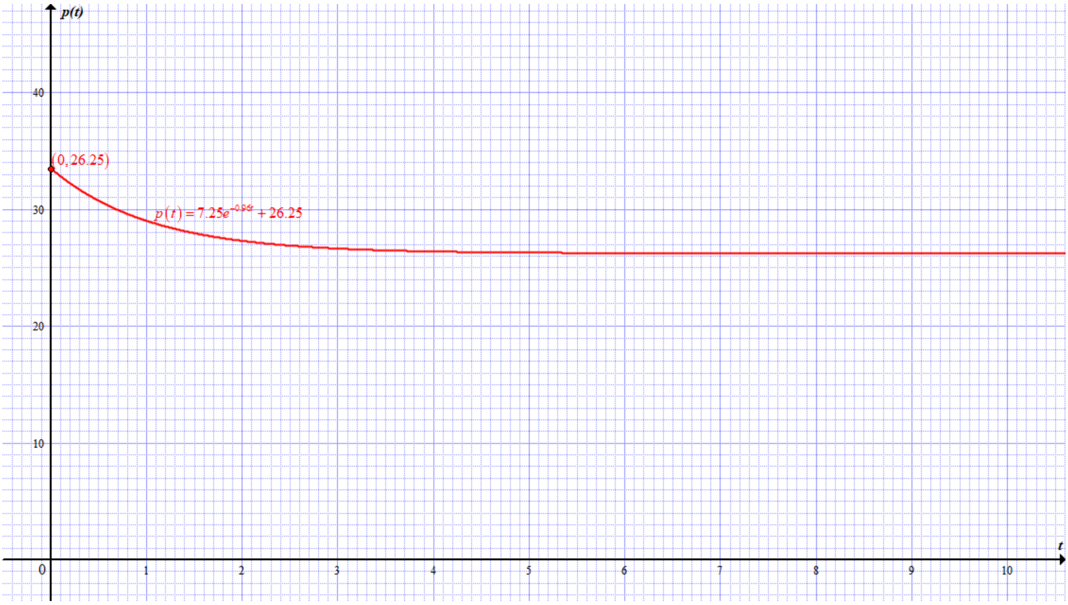
Solutions for activity 4.7

a) In general, we have $\frac{dP}{dt} = k(Q_d - Q_s)$, if $k = 0.08$ then our equation becomes

$$\frac{dP}{dt} = 0.08(Q_d - Q_s)$$

b) We have $\frac{dP}{dt} = k(Q_d - Q_s) \Rightarrow \frac{dP}{dt} = 0.08\left(280 - 4P - \left(-35 + 8P\right)\right)$ $\frac{dP}{dt} = 0.96P + 26.25$
which is a linear first-order differential equation.
Considering the initial condition $P(0) = 19$, we find $A = 7.25$. Therefore,
$P(t) = 7.25e^{-0.96t} + 26.25$

Graph of $P(t)=7.25e^{-0.96t}+26.25$.



c) At $t=1, P(1)=7.25e^{-0.96}+26.25=33.5$ and $\lim_{t\to\infty} p(t)=\lim_{t\to\infty}\left(7.25e^{-0.96t}+26.25\right)=26.25$.

If you compare those two situations, you find that the price is decreasing and tends to 26.25 as t gets larger.

Solution for application activity 4.7

Solutions/answers

1) $\frac{dP}{dt}=r(Q_d-Q_s)$

$\Rightarrow \frac{dP}{dt}=0.04\left(50-0.2P-(-10+0.3P)\right)$

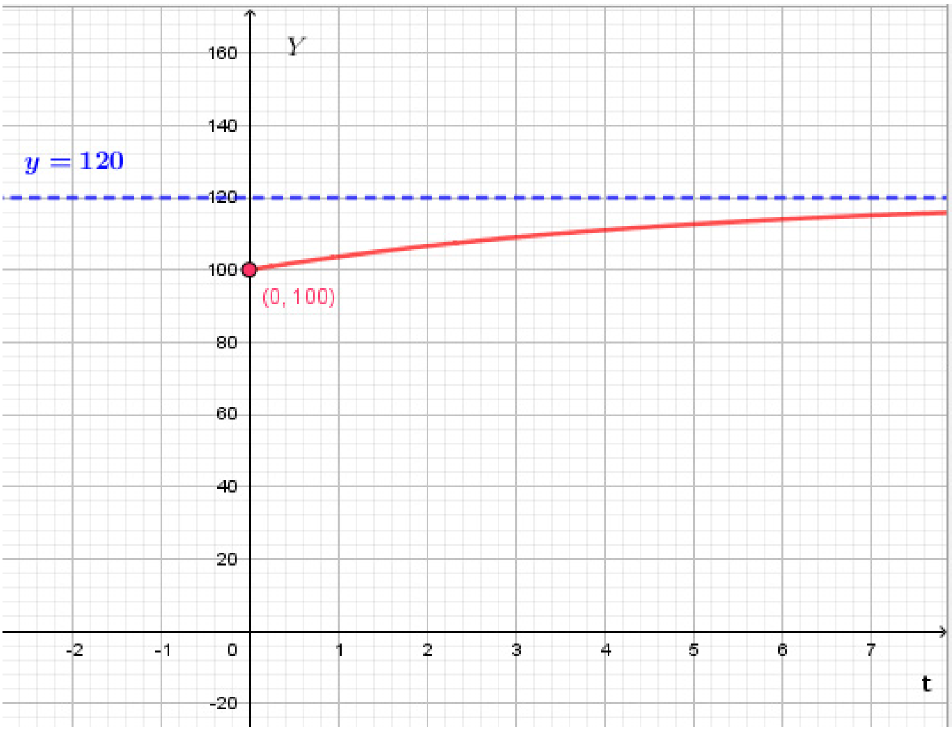
$\frac{dP}{dt}=-0.2P+24$ which is a linear first-order differential equation.

$P(t)=Ae^{-0.2t}+120$.

Considering the initial condition, $P(0)=100$, we find $A=-20$.

Therefore, $P(t)=-20e^{-0.2t}+120$.

The graph of $P(t) = -20e^{-0.2t} + 120$



We can say that this market is stable as the coefficient of t in the exponential function is the negative number -0.02

However, the graph shows that the convergence of $P(t)$ on its equilibrium value of 120 is relatively slow.

This time, price gradually approaches its equilibrium value *from one direction only*. A similar time path will occur in other similar market models with continuous price adjustment, although if the initial value is above the equilibrium then price will, obviously, approach this equilibrium from above rather than from below.

Lesson 8: Differential equations in electricity (Series Circuits)

a) Prerequisites/Revision/Introduction:

Learners will perform well in this lesson if they have good understanding of integration techniques learnt in unit 3, differential equations of 1^{st} order with separable variables and linear differential equations of 1^{st} order seen in previous lessons (unit 4, Lessons 1, 2 and 3).

b) Teaching resources:

Textbooks, Scientific calculators, GeoGebra for graph drawing, …

c) Learning activities:

- Ask learners to work in small groups and attempt activity 4.8, in the learner's book.
- Let them work independently for somewhile,
- Monitor the work of different groups to ensure that all learners understand better how to model and solve the differential equation
- Choose randomly a group to present the findings to the whole class, while the audience is following attentively,
- Invite learners to exchange their views about the presentation of their class-mates
- Harmonize their findings and initiate learners to summarize the key points of the presentation.
- Ask learners to read and revise the example 4.9 in pairs and work individually application activity 4.8, in the learner's book to enhance their skills about the application of differential equations of 1st order.

Solutions for activity 4.8

a) The situation is modelled by $L\frac{di}{dt}+Ri=E(t)$ or $\frac{di}{dt}+\frac{R}{L}i=\frac{1}{L}E(t)$

b) Since L, R and E are constant, the equation is linear of first-order in i.

The integrating factor is $I(t)=e^{\int\frac{Rdt}{L}}=e^{\frac{R}{L}t}$

The solution is $i(t)=\frac{\int\frac{E}{L}e^{\frac{R}{L}t}dt+K}{e^{\frac{R}{L}t}}=\frac{E}{R}+Ke^{-\frac{R}{L}t}$

Since the current was zero before switching on, it means $i(0)=0$, therefore $K=-\frac{E}{R}$.

Solutions for application activity 4.8

Solution:

a) The situation is modelled by $L\frac{di}{dt}+Ri=E(t)$ or $\frac{di}{dt}+\frac{R}{L}i=\frac{1}{L}E(t)$

b) Since L, R and E are constant, is linear equation of first-order in i with integrating

factor $I(t)=e^{\int\frac{Rdt}{L}}=e^{\frac{R}{L}t}$.The solution is $i(t)=\frac{\int\frac{E}{L}e^{\frac{R}{L}t}dt+K}{e^{\frac{R}{L}t}}=\frac{E}{R}+Ke^{-\frac{R}{L}t}$, therefore

$i=\frac{E}{R}+Ke^{-\frac{R}{L}t}$. Since the current was zero before switching on, it means $i(0)=0$, there fore $K=-\frac{E}{R}$. Putting this value of K in our solution, we find $i=\frac{E}{R}\left(1-e^{-\frac{R}{L}t}\right)$.

Taking $E=60$, $R=12$, we obtain $i=\frac{60}{12}\left(1-e^{-\frac{12}{4}t}\right)=5\left(1-e^{-3t}\right)$.

c) When t gets larger, we have $\lim_{t\to\infty} i(t)=\frac{E}{R}$, this looks like the current of the circuit is composed of the resistor R and the generator.

Lesson 9: Introduction to second order linear homogeneous differential equations

a) Prerequisites/Revision/Introduction:

Learners will get a better understanding of the content of this lesson if they have mastered how to find successive derivatives of a given function (Senior 4, unit 6).

b) Teaching resources:

Learner's book and other reference books to facilitate research

c) Learning activities:

- Form groups and ask learners to attempt activity 4.9, in the learner's book.
- Let them work independently for somewhile,
- Monitor learners while they calculate the derivative for a given function.
- Ask one group to present his/her findings to the whole class.
- Guide learners to follow attentively and interact about the findings and to write a short summary.
- Let learners work out example 4.10 under your guidance and work individually application activity 4.9 for assessment.

Solutions for activity 4.9

1) Answers will be different

2) Given $y=A\cos\sqrt{\frac{1}{20}}t+B\sin\sqrt{\frac{1}{20}}t$, let us verify that $2\frac{d^2y}{dt^2}+0.1y=0$

$y=A\cos\sqrt{\frac{1}{20}}t+B\sin\sqrt{\frac{1}{20}}t$, $y'=-A\sqrt{\frac{1}{20}}\sin\sqrt{\frac{1}{20}}t+B\sqrt{\frac{1}{20}}\cos\sqrt{\frac{1}{20}}t$ and $y''=-\frac{1}{20}\left(A\cos\sqrt{\frac{1}{20}}t+B\sin\sqrt{\frac{1}{20}}t\right)$

$$2\frac{d^2y}{dt^2}+0.1y=-\frac{2}{20}\left(A\cos\sqrt{\frac{1}{20}}t+B\sin\sqrt{\frac{1}{20}}t\right)+0.1\left(A\cos\sqrt{\frac{1}{20}}t+B\sin\sqrt{\frac{1}{20}}t\right)$$

$$=-0.1\left(A\cos\sqrt{\frac{1}{20}}t+B\sin\sqrt{\frac{1}{20}}t\right)+0.1\left(A\cos\sqrt{\frac{1}{20}}t+B\sin\sqrt{\frac{1}{20}}t\right)=0$$

Therefore, $y=A\cos\sqrt{\frac{1}{20}}t+B\sin\sqrt{\frac{1}{20}}t$ is a general solution of $2\frac{d^2y}{dt^2}+0.1y=0$.

b) We compare the given equations basing on the coefficient of the derivative for the dependant variable y.

Equation	Order of the highest derivative	Coefficient of the second derivative	Coefficient of the first derivative	Coefficient of y	Function in the second side
$\frac{d^2y}{dx^2}+p(x)\frac{dy}{dx}+q(x)y=r(x)$	2	1	$p(x)$	$q(x)$	$r(x)$
$\frac{d^2y}{dt^2}+0.05y=0$	2	0.05	0	0.1	0

Note that the two functions are of the second order even though one has the independat variable x while the other has the independent variable t.

Solutions for application activity 4.9

Solution :

Equations (a) and (c) are linear homogeneous differential equations while (b) $x\frac{d^2y}{dx^2}+\cos x=0$ is not, because $r(x)=-\cos x$ according to the general form.

Lesson 10: Linear independence and superposition principle

a) Prerequisites/Revision/Introduction:

Through examples, let learners discuss about differentiation calculus especially exponential and trigonometric functions and how to calculate determinant of matrix of order 2.

b) Teaching resources:

- Learner's book and reference textbooks/ internet for developing learners' self-confidence through research activity.

c) Learning activities:

- Form groups and invite them to sit for activity 4.10 from learner's book.
- Let them work independently for some while,
- Walk around to all groups to guide them on the following:

1) For the function $y=e^{-x}$, facilitate learners in calculating its successive derivatives up to order 2 and then substitute them in $\frac{d^2y}{dx^2}+3\frac{dy}{dx}+2y$ to check if $\frac{d^2y}{dx^2}+3\frac{dy}{dx}+2y=0$ and then give conclusion.

2) Let learners multiply e^{-x} by any constant (let a and e^{-2x} by another constant (let b) and taking the sum of results. The corresponding results is $y=ae^{-x}+be^{-2x}$, $a,b\in\mathbb{R}$.

3) Guide learners in verifying if $y=ae^{-x}+be^{-2x}$, $a,b\in\mathbb{R}$ is a solution of $\frac{d^2y}{dx^2}+3\frac{dy}{dx}+2y=0$.

- Invite the group representatives to present their works and request their classmates to follow carefully.
- Harmonize their work emphasizing the key terms that e^{-x} and e^{-2x} are linearly

independent and the superposition principle leads to $y = ae^{-x} + be^{-2x},\ a,b \in \mathbb{R}$. Ask learners to go through the example 4.11 and work out application activities 4.10.

Solutions for activity 4.10

1) Given $y = e^{-x}$, we have $y' = -e^{-x}, y'' = e^{-x}$.

Putting these values into $\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y$, we get $e^{-x} - 3e^{-x} + 2e^{-x} = 0$

Thus, $y = e^{-x}$ is a solution of $\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = 0$. Similarly, for $y = e^{-2x}$, we have

$y' = -2e^{-2x}, y'' = 4e^{-2x}$ and $\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = \left(4e^{-2x}\right) - 6e^{-2x} + 2e^{-2x} = 0$.

Hence, $y = e^{-2x}$ is a solution of $\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = 0$

2) Multiplying e^{-x} by any constant (a, say) and e^{2x} by another constant (b) and taking the sum of results we get, $y = ae^{-x} + be^{-2x},\ a,b \in \mathbb{R}$.

3) From $y = ae^{-x} + be^{-2x}$, we have $y' = -ae^{-x} - 2be^{-2x}$ and $y'' = ae^{-x} + 4be^{-2x}$.

And then, $\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = \left(ae^{-x} + 4be^{-2x}\right) + 3\left(-ae^{-x} - 2be^{-2x}\right) + 2\left(ae^{-x} + be^{-2x}\right)$

$= (a - 3a + 2a)e^{-x} + (4b - 6b + 2b)e^{-2x} = 0$.

Therefore $y = ae^{-x} + be^{-2x}$, is also a solution of $\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = 0$.

If $y = e^{-x}$ and $y = e^{-2x}$ are solutions of $\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = 0$, thus their linear combination $y = ae^{-x} + be^{-2x}$ is also a solution of $\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = 0$.

Solutions for application activity 4.10

Solution:

1) Let $y_1 = 1 + \cos x \Rightarrow y' = -\sin x\ and\ y'' = -\cos x$ and $y_2 = 1 + \sin x \Rightarrow y' = \cos x\ and\ y'' = -\sin x$.

2) For $y_1 = 1 + \cos x$, $y'' + y = (1 + \cos x)'' + (1 + \cos x) = -\cos x + 1 + \cos x = 1$ as required.

3) For $y_2 = 1 + \sin x$, $y'' + y = (1 + \sin x)'' + (1 + \sin x) = -\sin x + 1 + \sin x = 1$ as required.

Therefore $y_1 = 1 + \cos x$ and $y_2 = 1 + \sin x$ are solutions of $y'' + y = 1$.

The sum of $y_1 = 1+\cos x$ and $y_2 = 1+\sin x$ is $y = 2+\sin x+\cos x$, and then

$$y''+y = (2+\sin x+\cos x)''+(2+\sin x+\cos x)$$
$$= (\cos x-\sin x)'+(2+\sin x+\cos x)$$
$$= -\sin x-\cos x+2+\sin x+\cos x = 2 \neq 1.$$

Therefore, $y = 1+\cos x$ and $y = 1+\sin x$ are solution of $y''+y=1$ but their sum is not a solution. In fact, equation $y''+y=1$ is not homogeneous linear differential equation; (the superposition principle holds on homogeneous linear differential equations)

1)

a) $\cos^2 x$ and $\sin^2 x$ are linear independent since $\frac{\sin^2 x}{\cos^2 x} = \tan^2 x \neq c$.

Or two functions are linear independent if their Wronskian is different from zero.

For our case $W(x) = \begin{vmatrix} \sin^2 x & \cos^2 x \\ 2\sin x\cos x & -2\cos x\sin x \end{vmatrix}$

$$= -2\cos x\sin x\sin^2 x - 2\cos x\sin x\cos^2 x$$
$$= -2\cos x\sin x(\sin^2 x+\cos^2 x) = -2\cos x\sin x \neq 0$$

b) Functions e^{-x} and e^{2x} are linear independent since $\frac{e^{-x}}{e^{2x}} = e^{-3x} \neq c$.

Or using Wronskian, $W(x) = \begin{vmatrix} e^{-x} & e^{2x} \\ -e^{-x} & 2e^{2x} \end{vmatrix} = 2e^{-x}\,\mathrm{e}^{2x}+e^{-x}\,\mathrm{e}^{2x} = 3\mathrm{e}^{x} \neq 0$

c) e^{ax} and $5e^{ax}$ are not linear independent since $\frac{e^{ax}}{5e^{ax}} = 5 = \text{constant}$.

Or using Wronskian, $W(x) = \begin{vmatrix} e^{\alpha x} & 5e^{\alpha x} \\ \alpha e^{\alpha x} & 5\alpha e^{\alpha x} \end{vmatrix} = 5\alpha e^{\alpha x} - 5\alpha e^{\alpha x} = 0$

d) $5\sin x\cos x$ and $4\sin 2x$ are not linear independent since

$\frac{\sin x\cos x}{4\sin 2x} = \frac{5\sin 2x}{4\sin 2x} = \frac{5}{4} = \text{constant}$ Alternatively using Wronskian,

$$W(x) = \begin{vmatrix} 5\sin x\cos x & 4\sin 2x \\ (5\sin x\cos x)' & (4\sin 2x)' \end{vmatrix} = \begin{vmatrix} \frac{5}{2}\sin 2x & 4\sin 2x \\ \left(\frac{5}{2}\sin 2x\right)' & 8\cos 2x \end{vmatrix} = \begin{vmatrix} \frac{5}{2}\sin 2x & 4\sin 2x \\ 5\cos 2x & 8\cos 2x \end{vmatrix} = 20\sin 2x\cos 2x - 20\cos 2x\sin 2x = 0$$

e) $\mathrm{e}^{ax}\cos 2x$ and $e^{ax}\sin 2x$ are linear independent since $\frac{\mathrm{e}^{ax}\cos 2x}{\mathrm{e}^{ax}\sin 2x} = \cot 2x \neq c$

Alternatively, using Wronskian, we have

$$W(x)=\begin{vmatrix} e^{\alpha x}\cos 2x & e^{\alpha x}\sin 2x \\ \left(e^{\alpha x}\cos 2x\right)' & \left(e^{\alpha x}\cos 2x\right)' \end{vmatrix}$$

$$=e^{\alpha x}\cos 2x\left(\alpha e^{\alpha x}\sin 2x-2e^{\alpha x}\cos 2x\right)-e^{\alpha x}\sin 2x\left(\alpha e^{\alpha x}\cos 2x-2e^{\alpha x}\sin 2x\right)$$

$$=e^{2\alpha x}\left(\alpha\sin 2x\cos 2x-\cos 2x\cos 2x-\alpha\cos 2x\sin 2x+2\sin 2x\sin 2x\right)$$

$$=e^{2\alpha x}\left(-\cos 2x\cos 2x+2\sin 2x\sin 2x\right)=-e^{2\alpha x}\cos 4x\neq 0$$

f) $\ln x$ and $\ln\sqrt{x}$ are not linear independent since $\frac{\ln x}{\ln\sqrt{x}}=\frac{\ln x}{\frac{1}{2}\ln x}=2=\text{constant}$.

Alternatively, from Wronskian, we have

$$W(x)=\begin{vmatrix} \ln x & \ln\sqrt{x} \\ (\ln x)' & \left(\ln\sqrt{2}\right)' \end{vmatrix}=\begin{vmatrix} \ln x & \frac{1}{2}\ln x \\ (\ln x)' & \left(\frac{1}{2}\ln x\right)' \end{vmatrix}=\begin{vmatrix} \ln x & \frac{1}{2}\ln x \\ \frac{1}{x} & \frac{1}{2x} \end{vmatrix}=\frac{\ln x}{2x}-\frac{\ln x}{2x}=0$$

g) e^{ax} and xe^{ax} are linear independent since $\frac{e^{ax}}{xe^{ax}}=\frac{1}{x}\neq\text{constant}$.

Using the Wronskian, $W(x)=\begin{vmatrix} e^{\alpha x} & xe^{\alpha x} \\ \left(e^{\alpha x}\right)' & \left(xe^{\alpha x}\right)' \end{vmatrix}=\begin{vmatrix} e^{\alpha x} & xe^{\alpha x} \\ \alpha e^{\alpha x} & (\alpha x+1)e^{\alpha x} \end{vmatrix}$

$$=(\alpha x+1)e^{2\alpha x}-\alpha xe^{2\alpha x}=e^{2\alpha x}\neq 0$$

h) $\frac{2\sin^2 x}{1-\cos^2 x}=\frac{2\sin^2 x}{\sin^2 x}=2=c^{te}$,

therefore $2\sin^2 x$ and $1-\cos^2 x$ are not linearly independent.

From Wronskian, $W(x)=\begin{vmatrix} 2\sin^2 x & 1-\cos^2 x \\ \left(2\sin^2 x\right)' & \left(1-\cos^2 x\right)' \end{vmatrix}=\begin{vmatrix} 2\sin^2 x & \sin^2 x \\ 4\sin x\cos x & 2\sin x\cos x \end{vmatrix}$

$$=4\sin x\cos x\sin^2 x-4\sin x\cos x\sin^2 x=0$$

Lesson 11: Characteristic equation of second order differential equations

a) Prerequisites/Revision/Introduction:

The students will perform well in this lesson if they have mastered the following concepts: derivative of exponential functions (unit 2), how to solve quadratic equations (Senior 4: unit 3) and linear differential equations of first order (Senior 6, unit 4, lesson 3).

b) Teaching resources:

Learner's book and other reference textbooks or internet (if possible) to facilitate research.

c) Learning activities:

- Organise learners into groups and let them attempt activity 4.11.
- Give clear instructions on the duration and how the group work is to be performed
- Check if every group member is participating and get opportunity of finding whether the learners remember how to solve linear differential equation of first order and give assistance where it is needed.
- After finding solution of $y^{'}-ky=0$, guide learners in differentiating the common solution and then substitute in $y''-3y'-4y=0$ to get a new equation in terms of k.
- Once the group discussion is over, ask a group, chosen randomly, to present his results while other learners are following and interacting. Inform learners that the new equation is called Characteristic equation or auxiliary equation of $y''-3y'-4y=0$.
- Choose randomly a group for presenting its work and request other learners to follow attentively.
- Ask learners to give their observations and determine characteristic equation or auxiliary equation of $ay''+by'+\text{c}\,y=0$.
- Let learners proceed to example 4.12 under your guidance and check their working against the solution proposed in the learner's book, and then request them to work individually application activity 4.11 to check the skills they have acquired.

Solutions for activity 4.11

1) $y^{'}-ky=0 \Leftrightarrow \frac{dy}{dx}=ky \Leftrightarrow \frac{dy}{y}=kdx \Leftrightarrow \int\frac{dy}{y}=\int kdx \Leftrightarrow \ln|y|=kx \Leftrightarrow y=e^{kx}$.

2) $y''-3y'-4y=0$

3) Given $y=e^{kx}$, it follows $y^{'}=ke^{kx}$ and $y^{''}=k^2e^{kx}$

Plug these values in $y''-3y'-4y=0$ to have $k^2e^{kx}-3ke^{kx}-4e^{kx}=0$.

This is equivalent to $\left(k^2-3k-4\right)e^{kx}=0$.
This relation is true if and only if $k^2-3k-4=0$ since e^{kx} cannot be zero.

Thus, the solution of $y'-ky=0$ is also a solution of $y''-3y'-4y=0$ if k is a root of $k^2-3k-4=0$. Therefore, the solution of the form e^{kx} is a solution of $y''-3y'-4y=0$.

Solutions for application activity 4.11

Solution:

1) For $y=\cos 2x$, $y''+4y=\left(\cos 2x\right)''+4\cos 2x=\left(-2\sin 2x\right)'+4\cos 2x$

$=-4\cos 2x+4\cos 2x=0$, hence $y=\cos 2x$ is a solution of $y''+4y=0$.

Similarly, for $y=2\sin x\cos x$,

$y''+4y=\left(2\sin x\cos x\right)''+8\sin x\cos x=\left(\sin 2x\right)''+4\sin 2x$

$=\left(2\cos 2x\right)'+4\sin 2x=\left(-4\sin 2x\right)+4\sin 2x=0,$

Thus, $y=\sin 2x$ is a solution of $y''+4y=0$.

Since $\cos 2x$ and $2\sin x\cos x$ are linearly independent, they form a basis and the general solution is $y=c_1\cos 2x+2c_2\sin x\cos x$.

The characteristic equation of $y''+4y=0$ is $\lambda^2+4=0$ $\lambda^2+4=0$: $\lambda=2i$ or $-2i$.

2) For $y=e^x$, $y''-2y'+y=\left(e^x\right)''-2\left(e^x\right)'+e^x=\left(e^x\right)'-2e^x+e^x=e^x-2e^x+e^x=0$,

thus $y=e^x$ is a solution of $y''-2y'+y=0$.

For $y=3e^x$, $y''-2y'+y=\left(3e^x\right)''-2\left(3e^x\right)'+3e^x$

$=3\left(e^x\right)'-2\left(3e^x\right)+3e^x=3e^x-6e^x+3e^x=0$;

Therefore, $xe^{-\frac{x}{2}}$ is a solution of $4y''+4y'+y=0$.

Since $e^{-\frac{x}{2}}$ and $xe^{-\frac{x}{2}}$ are both solutions of $4y''+4y'+y=0$, and linearly independent, they form a basis of solution of $4y''+4y'+y=0$, and corresponding general solution is $y=c_1e^{-\frac{x}{2}}+c_2xe^{-\frac{x}{2}}$.

Characteristic equation of $4y''+4y'+y=0$ is $4\lambda^2+4\lambda+1=0$

Roots $4\lambda^2+4\lambda+1=0$: $\Delta=16-16=0$ thus $\lambda=-\frac{4}{8}=-\frac{1}{2}$.

Lesson 12: Solving linear differential equations whose characteristic equation has two distinct real roots

a) Prerequisites/Revision/Introduction:

It will be helpful to the learners if they have mastered the following concepts: derivative of exponential functions (Senior 6: unit 2), how to solve quadratic equations (Senior 4: unit 3), linear independence, superposition principle and characteristic (auxiliary) equation of differential equations (Senior 6: unit 4).

b) Teaching resources:

Learner's book and other reference books or internet.

Scientific calculators or if possible, computers with mathematical software such as Microsoft Excel, Geogebra, graph… .

c) Learning activities:

- Organize learners into groups to attempt the activity 4.12.
- Let learners determine independently the roots of characteristic equation for the given DE.

Facilitate the slow learners for further explanation and provide assistance to any group in need.

- Guide learners in finding some solutions of the given DE and deduce the general solution.
- Once discussions are over, choose different groups to present their works when other learners are following attentively for interaction Ask learners to give constructive remarks and complements, in order to obtain a conclusion to be noted by all learners.
- Individually, let learners go through example1.13 and work out application activities 1.12, to emphasize their skills, knowledge and understanding in solving linear differential equation of second order.

Solutions for activity 4.12

Characteristic equation: $\lambda^2+7\lambda+6=0$.

Simple factorisation yields to $(\lambda+6)(\lambda+1)=0$. Thus $\lambda=-6$ or $\lambda=-1$. You can verify that $y=e^{-x}$ and $y=e^{-6x}$ are solutions of $y''+7y'+6y=0$ and they are linearly independent.

Therefore, the general solution is $y=c_1e^{-x}+c_2e^{-6x}$.

Solution for application activity 4.12

Solution:

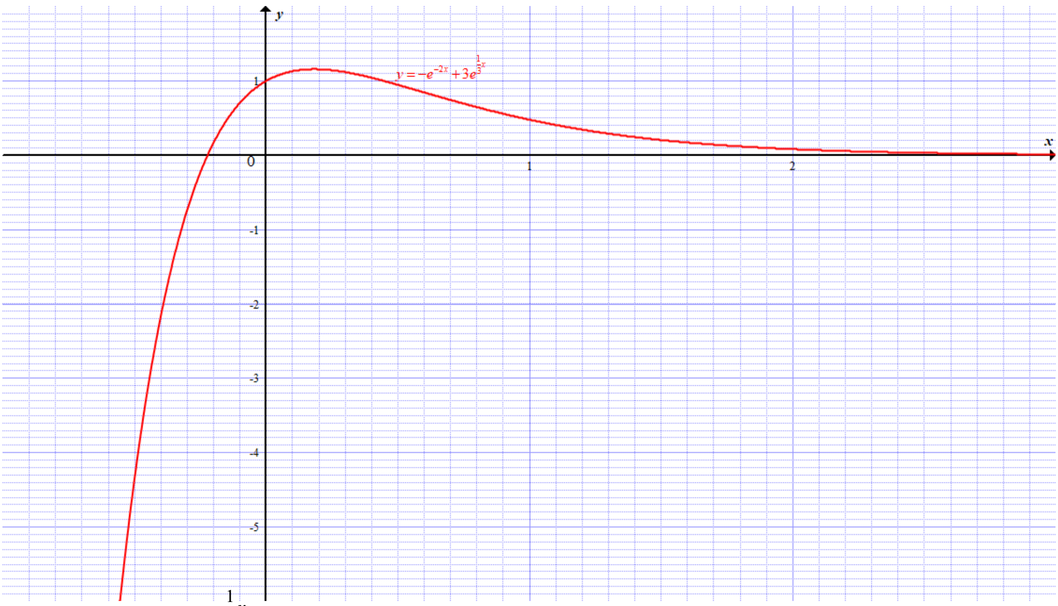
1) General solutions

a) $y = c_1 + c_2 e^{3x}$ b) $y = c_1 e^{2\sqrt{2}x} + c_2 e^{-2\sqrt{2}x}$ c) $y = c_1 e^{-6x} + c_2 e^{-x}$

d) $y = c_1 e^{x} + c_2 e^{-2x}$ e) $y = c_1 e^{2x} + c_2 e^{3x}$ f) $y = c_1 e^{-2x} + c_2 e^{\frac{x}{2}}$

2) Particular solutions

a) $y = 5e^{-2x} - 4e^{-3x}$



b) $y = -e^{-2x} + 3e^{\frac{1}{3}x}$

Lesson 13: Solving linear differential equations whose characteristic equation has a real double root

a) Prerequisites/Revision/Introduction:

The learners will perform well if they have a good package of the following concepts: derivative of exponential functions (Senior 6: unit 2), how to solve quadratic equations (Senior 4: unit 3), linear independence, superposition principle and characteristic (auxiliary) equation of differential equations (Senior 6: unit 4).

b) Teaching resources:

- Learner's book and other reference book or internet
- T-square, ruler, scientific calculator or where possible computer with mathematics software such as Microsoft excel, geogebra, Mathlab, graph….

c) Learning activities:

- Instruct learners to form groups and work on activity 4.13,
- Let them work independently for somewhile,
- Follow up the working steps of different groups to give support where necessary

a) Give assistance if learners mention any difficulties met when verifying whether

$f(x) = \frac{e^{mx} - e^{nx}}{m - n}$ is a solution of $y'' + py' + qy = 0$.

Setting $y=\frac{e^{mx}-e^{nx}}{m-n}$, let learners calculate y', y'' and then substitute these values in $y''+py'+qy$.

Stimulate learners to note that $m^2+pm+q=0=n^2+pn+q$ are characteristic equations of $y''+py'+qy=0$ and then deduce that

$f(x)=\frac{e^{mx}-e^{nx}}{m-n}$ is one of solution of $y''+py'+qy=0$.

b) Walk around to different groups and guide learners by providing further explanations where necessary and encouraging each member to participate actively in the discussion.

- Invite randomly a group to present its findings and request other learners to follow attentively to give their suggestions.
- Harmonize the learners works and help them realize that if n is a double root of auxiliary equation of $y''+py'+qy=0$, then $y=e^{nx}$ and $y=xe^{nx}$ are linearly independent which leads to general solution $y=c_1e^{nx}+c_2xe^{nx}$.

Individually, let learners go through the example 4.14 and work out application activities 4.13 to enhance their skills and knowledge in solving an ordinary linear differential equation of second order.

Solutions for activity 4.13

a) Let y be the function defined by $y(x)=\frac{e^{mx}-e^{nx}}{m-n}$, thus

$$y''+py'+qy=\left(\frac{e^{mx}-e^{nx}}{m-n}\right)''+p\left(\frac{e^{mx}-e^{nx}}{m-n}\right)'+q\frac{e^{mx}-e^{nx}}{m-n}$$

$$=\left(\frac{me^{mx}-ne^{nx}}{m-n}\right)'+p\frac{me^{mx}-ne^{nx}}{m-n}+q\frac{e^{mx}-e^{nx}}{m-n}$$

$$=\frac{m^2e^{mx}-n^2e^{nx}}{m-n}+p\frac{me^{mx}-ne^{nx}}{m-n}+q\frac{e^{mx}-e^{nx}}{m-n}$$

$$=\frac{m^2e^{mx}+pme^{mx}+qe^{mx}}{m-n}-\frac{n^2e^{nx}+pne^{nx}+qe^{nx}}{m-n}$$

$$=\frac{e^{mx}}{m-n}\left(m^2+pm+q\right)-\frac{e^{nx}}{m-n}\left(n^2+pn+q\right)=0$$

From characteristic equation of $y''+py'+qy=0$, $m^2+pm+q=0=n^2+pn+q$.

Thus, $y'' + py' + qy = \frac{e^{mx}}{m-n}\left(m^2 + pm + q\right) - \frac{e^{nx}}{m-n}\left(n^2 + pn + q\right) = 0 - 0 = 0$ and then

$y = f(x) = \frac{e^{mx} - e^{nx}}{m-n}$ is a solution of $y'' + py' + qy = 0$.

$\lim_{m \to n} f(x) = \lim_{m \to n} \frac{e^{mx} - e^{nx}}{m-n} = \frac{e^{nx} - e^{nx}}{m-n} = \frac{0}{0}$ which is indeterminate form.

c) Remove this indeterminate by Hospital rule:

$\lim_{m \to n} f(x) = \lim_{m \to n} \frac{\left(e^{mx} - e^{nx}\right)'}{(m-n)'} = \lim_{m \to n} \frac{xe^{mx} - 0}{1-0} = xe^{nx}$ as required.

Let $y = \lim_{m \to n} f(x) = xe^{nx}$, thus

$$y'' + py' + qy = \left(xe^{nx}\right)'' + p\left(xe^{nx}\right)' + qxe^{nx}$$

$$= \left(e^{nx} + nxe^{nx}\right)' + p\left(e^{nx} + nxe^{nx}\right) + qxe^{nx}$$

$$= ne^{nx} + ne^{nx} + n^2xe^{nx} + pe^{nx} + pnxe^{nx} + qxe^{nx}$$

$$= xe^{nx}\left(n^2 + pn + q\right) + e^{nx}(2n + p)$$

$= e^{nx}(2n + p)$ from auxiliary equation $n^2 + pn + q = 0$.

As n, is a double root of $n^2 + pn + q = 0$, then $n = -\frac{p}{2}$;

therefore $y'' + py' + qy = e^{nx}(2n + p) \Leftrightarrow y'' + py' + qy = 0$.

Hence, the function $y = xe^{mx}$ is a solution of $y'' + py' + qy = 0$ when $m \to n$ and $f(x) = \frac{e^{mx} - e^{nx}}{m-n}$ is a solution of the equation $y'' + py' + qy = 0$ (generally if m is a repeated root of the auxiliary equation $y = xe^{mx}$ is a solution of the given differential equation).

b) Let us check whether $y = \lim_{m \to n} f(x)$ and $y = e^{nx}$ are linearly independent;

Since $\frac{\lim_{m \to n} f(x)}{e^{nx}} = \frac{xe^{nx}}{e^{nx}} = x \neq c^{te}$, $y = \lim_{m \to n} f(x) = xe^{nx}$ and $y = e^{nx}$ are linearly independent that leads to general solution $y = c_1e^{nx} + c_2xe^{nx}$, n being a double root of auxiliary equation.

For $y'' - 2y' + y = 0$, the characteristic equation is $n^2 - 2n + 1 = 0$ whose double root $n = 1$ Therefore the general solution is $y = c_1e^x + c_2xe^x$.

Solution for application activity 4.13

Solution:

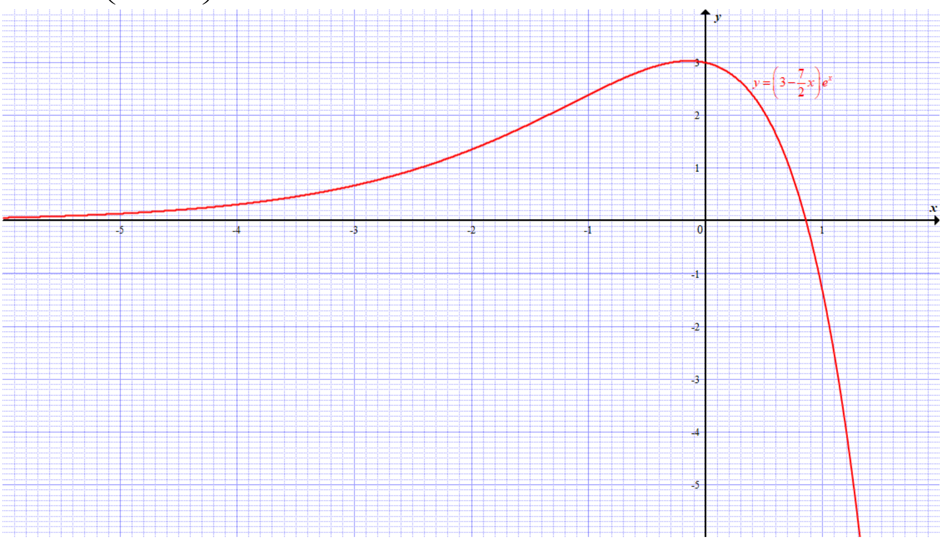
1) General solutions

a) $y=(c_1+c_2x)e^{-4x}$ b) $y=(c_1+c_2x)e^{\frac{3}{2}x}$ c) $y=(c_1+c_2x)e^{-\frac{x}{2}}$

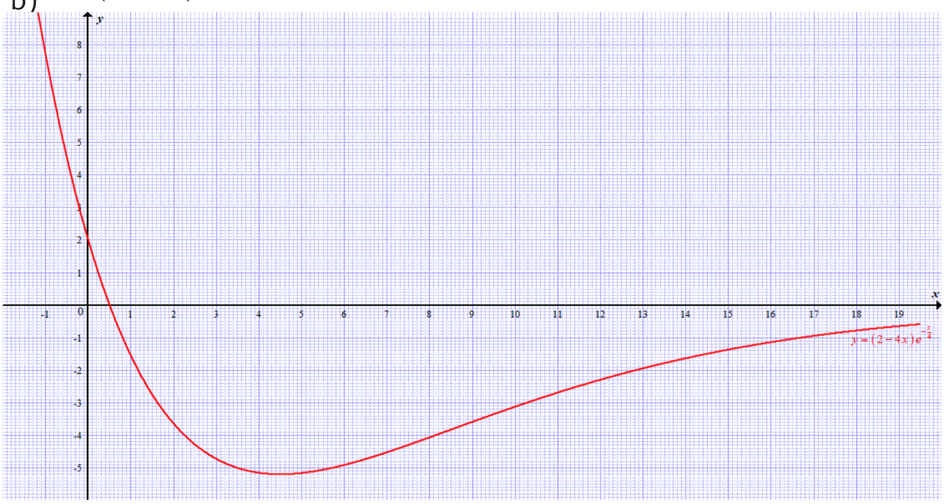
d) $y=(c_1+c_2x)e^{\frac{1}{6}x}$ e) $y=(c_1+c_2x)e^{\frac{\pi}{\ }x}$

2) Particular solutions

a) $y=\left(3-\frac{7}{2}x\right)e^{x}$



b) $y=(2-4x)e^{-\frac{1}{4}x}$



Lesson 14: Solving linear differential equations whose characteristic equation has complex roots

a. Prerequisites/Revision/Introduction:

Learners will get a better understanding of the content of this lesson if they have mastered how to find derivative of exponential functions (unit 2), solve quadratic equations whose roots are complex numbers (unit 1), linear independence, superposition principle and characteristic (auxiliary) equation of differential equations (unit 4).

b. Teaching resources:

Learner's book and other reference book or internet

T-square, ruler, scientific calculator or where possible computer with math draw software as

Microsoft excel, geogebra, Mathlab, graphcalc, etc.

c. Learning activities:

- From the activity 4.14, facilitate learners to work in small groups and find the solutions of the characteristic equation $y''-y'+5y=0$ and let them write down the two independent solutions y_1 and y_2.
- Let them work independently for some while,
- Facilitate learners to have a general idea on expressing the two independent solutions y_1 and y_2 in polar form.
- Invite some group members to present their findings.
- Harmonize the results by highlighting the real basis of solution of $y''-y'+5y=0$, then let learners deduce the general solution of $y''-y'+5y=0$.
- Guide learners to work through example 4.15, application activity 4.14: question 1) and 2) a) and work individually application activities 2.14: question 2) b) to assess the competences.

Solutions for activity 4.14

1) $y_1=e^{(1+2i)x}=e^x e^{2ix}$ and $y_2=e^{(1-2i)x}=e^x e^{-2ix}$

2) We know that $e^{\theta i}=\cos\theta+i\sin\theta$ and $e^{-\theta i}=\cos\theta-i\sin\theta$

From this we have $e^{2ix}=\cos 2x+i\sin 2x$ and $e^{-2ix}=\cos 2x-i\sin 2x$, then

$y_1=e^{(1+2i)x}=e^x e^{2ix}=e^x\left(\cos 2x+i\sin 2x\right)$ and

$y_2=e^{(1-2i)x}=e^x e^{-2ix}=e^x\left(\cos 2x-i\sin 2x\right)$.

3) Remember that **Euler formulae** are $\cos\theta\ \dfrac{e^{i\theta}\quad e^{\ i\theta}}{2}$ and $\sin\theta=\dfrac{e^{i\theta}-e^{-i\theta}}{2i}$.

By combining y_1 and y_2 we get

$y_1 + y_2 = e^x(\cos 2x + i\sin 2x) + e^x(\cos 2x - i\sin 2x) = 2e^x \cos 2x$

Or $e^x \cos 2x = \dfrac{y_1 + y_2}{2}$ is a real valued solution.

Also, $y_1 - y_2 = e^x(\cos 2x + i\sin 2x) - e^x(\cos 2x - i\sin 2x) = 2ie^x \sin 2x$

From $e^x \sin 2x = \dfrac{y_1 - y_2}{2i}$, omitting i, we get $e^x \sin 2x = \dfrac{y_1 - y_2}{2}$ which is another real valued solution.

Hence, real basis is formed by $y_1 = e^x \cos 2x$ and $y_2 = e^x \sin 2x$.

Therefore, the general solution is $y = c_1e^x \cos 2x + c_2e^x \sin 2x$ Or

$y = e\ (c_1 \cos 2x + c_2 \sin 2x)$.

Solution for application activities 4.14

Solution:

1) a) $y = c_1 \cos 5x + c_2 \sin 5x$

b) $y = e^{2x}(c_1 \cos x + c_2 \sin x)$

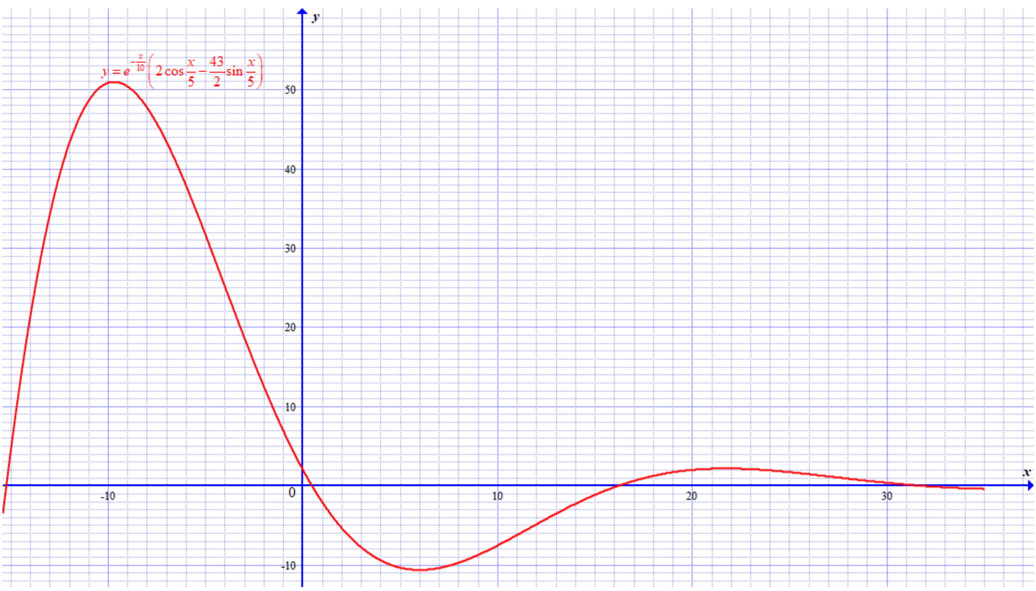
c) $y = e^{-2x}(c_1 \cos 3x + c_2 \sin 3x)$

d) $y = e^{-3x}\left(c_1 \cos\sqrt{2}x + c_2 \sin\sqrt{2}x\right)$

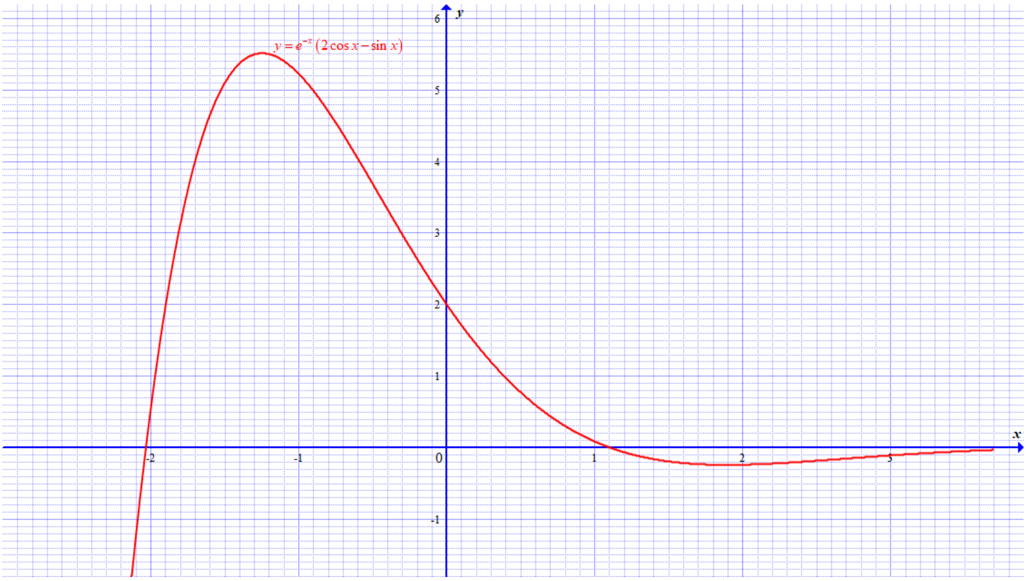
e) $y = e^x(c_1 \cos 3x + c_2 \sin 3x)$

f) $y = e^{-\frac{x}{10}}\left(c_1 \cos\dfrac{2}{5}x + c_2 \sin\dfrac{2}{5}x\right)$

2) a) $y = e^{-\frac{x}{10}}\left(2\cos\dfrac{x}{5} - \dfrac{43}{2}\sin\dfrac{x}{5}\right)$



b) $y = e^{-x}\left(2\cos x - \sin x\right)$



3) The summary of how to find solution of a second order linear homogenious differential equation:

Case	Roots	Basis	General solution
1	Distinct real roots: λ_1 and λ_2	$e^{\lambda_1 x}$ and $e^{\lambda_2 x}$	$y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$
2	Real double: λ	$e^{\lambda x}$ and $xe^{\lambda x}$	$y = \left(c_1 + c_2 x\right) e^{\lambda x}$
3	Complex conju-gate $\alpha \pm i\omega$	$e^{\alpha x}\cos \omega x$ and $e^{\alpha x}\sin \omega x$	$y = e^{\alpha x}\left(A\cos \omega x + B\sin \omega x\right)$

Lesson 15: Applications of second order linear homogeneous differential equation

a) Prerequisites/Revision/Introduction:

Learners will learn better the applications of second order linear homogeneous differential equation if they have a clear understanding of: Trigonometric functions (senior 5, unit 1 and 4), exponential functions (Senior 6, unit 2), differential equations (senior 6, unit 4: previous lessons), the concept of simple harmonic motion (Physics senior 2), Newton's Second Law of motion (Physics senior 5).

b) Teaching resources:

Learner's book and other reference book or internet

T-square, ruler, scientific calculator and where possible a computer with math draw software as Microsoft excel, Geogebra, Mathlab, graphcalc, etc.

c) Learning activities:

- Organize the learners into groups and let them attempt activity 4.15 from the learner's book.
- Let them work independently for some while,
- Guide them in modelling and solving the given problems
- Check how each member of the group is participating in the discussion.
- Chose randomly a group member to present her/his results while other learners are following attentively for interaction.
- Let learners work out example 4.16 under your guidance and work individually application activity 4.15 to check the skills they have acquired.

Solutions for activity 4.15

a) $m\frac{d^2x}{dt^2}=-kx \Leftrightarrow m\frac{d^2x}{dt^2}+kx=0$

Identifying the coefficients to $a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=0$ according the order of derivative, we get $a=m,\ b=0$ and $c=k$.

b) The simple harmonic motion has equation

$$m\frac{d^2x}{dt^2}+kx=0$$

Characteristic equation: $m\lambda^2+k=0 \Leftrightarrow \lambda=\pm i\sqrt{\frac{k}{m}}$

General solution is $x(t)=A\cos\sqrt{\frac{k}{m}}t+B\sin\sqrt{\frac{k}{m}}t \xrightarrow{\omega=\sqrt{\frac{k}{m}}} x(t)=A\cos\omega t+B\sin\omega t$.

c) For $m=2$, $k=128$ we have $2\frac{d^2x}{dt^2}+128x=0$ and then $x(t)=c_1\cos 8t+c_2\sin 8t$.

Given the initial conditions $\begin{cases}x(0)=0.2\\ x'(0)=0\end{cases}$, we find $c_1=\frac{1}{5}$ and $c_2=0$.

Therefore, $x(t)=\frac{1}{5}\cos 8t$.

Figure 4.6: Graph of $x(t)=\frac{1}{5}\cos 8t$

Solution for application activities 4.15

Solution:

1) The motion of that mass is modeled by the equation

$m\frac{d^2y}{dt^2}+c\frac{dy}{dt}+ky=0$, introducing the data given, $2\frac{d^2y}{dt^2}+40\frac{dy}{dt}+128y=0$

$\Leftrightarrow \frac{d^2y}{dt^2}+20\frac{dy}{dt}+64y=0$ whose characteristic equation is

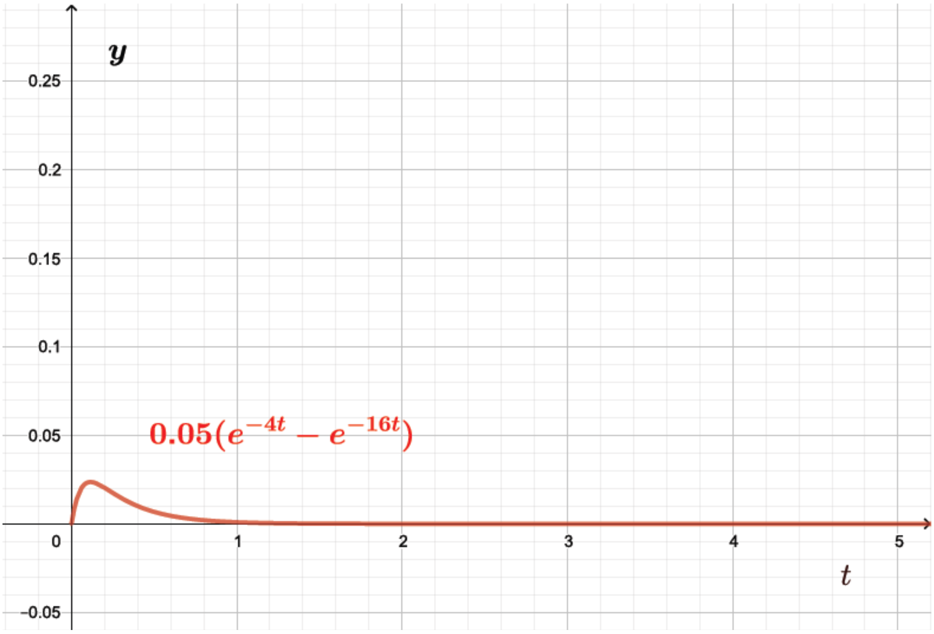
$\lambda^2+20\lambda+64=0 \Leftrightarrow (\lambda+4)(\lambda+16)=0$ and $\lambda_1=-4 \quad and \quad \lambda_2 -16$

As $\lambda_1=-4 \quad and \quad \lambda_2 -16$ are all real and different from each other, we have an over-damping spring and the general solution is $y(t)=C_1e^{-4t}+C_2e^{-16t}$

The initial conditions are $y(0)=0 \quad and \quad y^{'}(0)=0.6$

i.e $0=C_1+C_2$ and $0.6=-4C_1-16C_2$. These two equations give

$C_1=0.05 \quad and \quad C_2=-0.05$ Therefore, $y(t)=0.05\left(e^{-4t}-e^{-16t}\right)$.



Graph for $y(t) = 0.05\left(e^{-4t} - e^{-16t}\right)$

The spring is over-damped, it will oscillate only once.

2) a) If $R = 40$, $L = 1$, $C = 16.10^{-4}$ and $E(t) = 100\cos 10t$,

$L\dfrac{d^2Q}{dt^2} + R\dfrac{dQ}{dt} + \dfrac{1}{C}Q = E(t)$ will be written as $\dfrac{d^2Q}{dt^2} + 40\dfrac{dQ}{dt} + \dfrac{10^4}{16}Q = 100\cos 10t$ or

$$\frac{d^2Q}{dt^2} + 40\frac{dQ}{dt} + 625Q = 100\cos 10t$$

b) What type of equation obtained if you consider $E(t)$ for $t = \dfrac{\pi}{20}$?

For $t = \dfrac{\pi}{20}$, $E(t) = 100\cos 10t = 0$.

Therefore, $\dfrac{d^2Q}{dt^2} + 40\dfrac{dQ}{dt} + 625Q = 0$,
this is a second order homogeneous differential equation.
c) Determine the general solution for the equation obtained in (b).

$\dfrac{d^2Q}{dt^2} + 40\dfrac{dQ}{dt} + 625Q = 0$ and $\lambda^2 + 40\lambda + 625 = 0$, $\lambda_1 = -20 + 15i$ and
$\lambda_1 = -20 - 15i$

The general solution is $Q(t) = e^{-20t}\left(C_1\cos 15t + C_2\sin 15t\right)$.

4.6. Unit summary

4.6.1. Definition and classification of differential equations

An ordinary differential equation (ODE) for a dependent variable *y* (unknown) in terms of an independent variable *x* is any equation which involves first or higher order derivatives of *y* with respect to *x*, and possibly *x* and *y*.

The general differential equation of the 1st order is $F\left(x, y, \frac{dy}{dx}\right) = 0$ or $\frac{dy}{dx} = f(x, y)$

The order of a differential equation is the highest derivative present in the differential equation.

The degree of an ordinary differential equation is the algebraic degree of its highest ordered derivative after simplification.

4.6.2. First order of Differential equations with separable variables

A separable differential equation is an equation of the form $\frac{dy}{dx} = f(x)h(y)$. Before integrating both sides, such equation can be rewritten so that all terms involving y are on one side of the equation and all terms involving x are on the other. That is $\frac{dy}{h(y)} = f(x)dx$ and $\int \frac{dy}{h(y)} = \int f(x)dx + c$.

4.6.3. Linear differential equations of the first order

If p and q are functions in x or constants the general linear equation of first order can take the form $\frac{dy}{dx} + py = q$. To solve such equation, determine an integrating factor $I(x) = e^{\int pdx}$ taking the integrating constant $c = 0$ and then find $y(x) = \frac{\int I(x)q(x) + C}{I(x)}$.

Or let $y = uv$ where u and v are functions in x to be determined in the following ways:

$v = e^{-\int pdx}$ by taking the constant $c = 0$ and $u = \int q\, e^{\int pdx}\, dx$

The solution of the equation $\frac{dy}{dx} + py = q$ becomes $y = uv$ where $u = \int q\, e^{\int pdx}\, dx$ and $v = e^{-\ pdx}$.

4.6.4. Application of differential equations of first order

a) Differential equations are applied in the population growth:

If P is the population of a country, its variation is related to $\frac{dp}{dt}=KP$ where K is the timely growth rate.

b) Differential equations are applied in crime investigation:

The time of death of a murdered person can be determined by the police by measuring the variation of the temperature T with the help of the differential equation:
$\frac{dT}{dt}=K\left(T-T_e\right)$ where Te is the temperature of the environment surrounding the murdered person and K the constant of proportionality.

c) Differential equations are applied to determine the quantity of a drug in the body

The quantity of drug Q in the body of a patient in the time t is modelled by $\frac{dQ}{dt}=-kQ$ where k is a constant that depends on the specific type of drug.

d) Differential equations in economics and finance

If r represents the rate of adjustment of P in proportion to excess demand, we can write

$\frac{dP}{dt}=r\left(Q_d-Q_s\right)$. Where Q_d and Q_s are respectively the demand and supply functions.

e) Differential equations are applied in the Series Circuits

Given that the voltage drops across the resistor, inductor, and capacitor are respectively RI, $L\frac{dI}{dt}$ and $\frac{Q}{C}$ where Q is the charge.

Kirchhoff's second law saying that the voltage $V\left(t\right)$ in the circuit is the sum of the voltage drop across the components of the circuit is used to determine all variables of the circuit:

- For a series circuit containing only a resistor R and an inductor L, we have $Ri+L\frac{di}{dt}=E\left(t\right)$
- For a series circuit containing only a resistor R and capacitor with capacitance C, we have $R\frac{dQ}{dt}+\frac{Q}{C}=E\left(t\right)$ where Q is the electrical charge at the capacitor.

4.6. 5. Second order linear homogeneous differential equations

A second order linear homogeneous differential equation is of the form
$a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=0$ where *a*, *b*, *c* are constants (and $a\neq 0$).

4.6.5.1 Linearity of solutions of linear homogeneous differential equations:

If y_1 is solution of the homogeneous linear differential equation $a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=0$, the second solution y_2 is such that $y_1(x)$ and $y_2(x)$ are linearly independent where the determinant called the Wronskian of y_1 and y_2 denoted and defined by:

$W(y_1,y_2)=\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}=y_1y_2'-y_1'y_2$ is not zero.

4.6.5.2 Superposition principle:

If y_1 and y_2 are two solutions of the homogeneous linear differential equation $a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=0$, then any other linear combination $y=Ay_1+By_2$ of these two solutions is a solution of the equation.

4.6.5.3 Characteristic equation of a second order differential equation

The equation $a\lambda^2+b\lambda+c=0$ in λ is called the **auxiliary** or **characteristic** equation of the second order differential equation $a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=0$.

4.6.5.4 Solution of of a second order differential equation $a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=0$

- If the Characteristic equation $a\lambda^2+b\lambda+c=0$ has two distinct real roots λ_1 and λ_2, the corresponding general solution is $y=c_1e^{\lambda_1 x}+c_2e^{\lambda_2 x}$.
- If the Characteristic equation $a\lambda^2+b\lambda+c=0$ has a real double root/repeated root $\lambda_1=\lambda_2=\lambda$, we have $y_1=e^{\lambda x}$ and the second linearly independent solution will be $y_2=xe^{\lambda x}$.
- If the characteristic equation $a\lambda^2+b\lambda+c=0$ has complex roots, $\lambda_1=\alpha+i\beta$ and $\lambda_2=\alpha-i\beta$, the general solution $y=e^{\alpha x}\left(c_1e^{i\beta x}+c_2e^{-i\beta x}\right)$ is modified using Euler's formulae and it becomes $y=e^{\alpha x}\left(\mathrm{A}\cos\beta x+B\sin\beta x\right)$.

4.6.6. Applications of second order linear homogeneous differential equation

4.6.6.1 The oscillatory movement of a mass on a spring

- At the end of a simple spring with the spring constant k, the mass m makes a simple harmonic motion with the equation $m\frac{d^2x}{dt^2}+kx=0$;
- The mass m at the end of a simple spring with the spring constant k in the milieu of the damping constant C makes a motion modeled by the equation $m\frac{d^2x}{dt^2}+C\frac{dx}{dt}+kx=0$.

4.6.6.2 The Electric charge in an RLC series circuit

The voltage drops across the resistor, inductor, and capacitor are respectively

RI, $L\frac{dI}{dt}$ and $\frac{Q}{C}$.

The Kirchhoff's voltage law says that the sum of these voltage drops is equal to the supplied voltage $E(t)$ and expressed in the following equation:

$$L\frac{d^2Q}{dt^2}+R\frac{dQ}{dt}+\frac{1}{C}Q=E(t)\ .$$

Given that the current $i=\frac{dQ}{dt}$, the differential equation becomes

$$L\frac{d^2i}{dt^2}+R\frac{di}{dt}+\frac{1}{C}i=\frac{dE(t)}{dt}$$

This is a non-homogeneous second order differential equation because $E(t)\neq 0$.

4.7. Additional information for the teacher

4.7.1. Dynamics of Market Price in finance

Suppose that for a particular commodity, the demand and supply functions are as follows

$$Q_d=\alpha-\beta P,\quad (\alpha,\beta>0)$$

$$Q_s=-\gamma+\lambda P,\quad (\gamma,\lambda>0)$$

The equilibrium price is $P^*=\frac{\alpha+\gamma}{\beta+\lambda}$

If the initial price $P(0)=P^*$, the market will be in equilibrium already, and no dynamics analysis will be needed.

If not, the equilibrium price P^* is attainable only after a process of adjustment. Thus, the price and quantity variables are functions of time. Our question is that the time path P (t) tend to converge to P^* as $t\to\infty$.

We must find that time path P (t). In general, price changes are governed by the relative strength of the demand and supply forces in the market, that is,

$\frac{dP}{dt}=r(Q_d-Q_s)$, where $r>0$ represents a constant adjustment coefficient

Since P (0) and P^* are constants, and $r(\beta+\lambda)>0$, the time path P (t) will eventually converge to P^* , the intertemporal equilibrium (rather than the market clearing equilibrium), the equilibrium is said to be dynamically stable.

4.7.2. Height of the falling object above the ground

Consider a mass m falling under the influence of constant gravity, such as approximately found on the Earth's surface. Newton's law results in the equation $F = m.a$ where m is the mass of the object and F the sum of all forces acting on that mass. If we suppose that the resistance of the air is absent,
We have $m\frac{d^2y}{dt^2} = -mg$ where y is the height of the object above the ground, m is the mass of the object, and $g = 9.8m/\sec^2$ is the constant gravitational acceleration. As Galileo suggested,

the mass cancels from the equation, and $\frac{d^2y}{dt^2} = -g$
Here, the right-hand-side of the ode is a constant. The first integration, obtained by anti-differentiation, yields $\frac{d\ y}{dt} = A - gt$ with A the first constant of integration; and the second integration yields $y = B + At - \frac{1}{2}gt^2$ with B the second constant of integration.

The two constants of integration A and B can then be determined from the initial conditions.

If we know that the initial height of the mass is H_0 , and the initial velocity is V_0, then the initial conditions are $y(0) = H_0$ and $y'(0) = V_0$

Substitution of these initial conditions into the equations for dx/dt and x allows us to solve for A and B. The unique solution that satisfies both the ODE and the initial conditions is given by

$$y = H_0 + V_0 t - \frac{1}{2}gt^2$$

This is the formula that gives the height at which arrives a falling object left with the initial velocity V_0 downwards from the height H_0 after the time t as it was learnt in physics Senior 2.

4.7.3. Integrating factor $I(x)$ of a linear first order differential equation

We are given to solve a linear first order differential equation $\frac{dy}{dx} + py = q$ (1) where p and q are functions in x or constants.

Such equation is solved by the use of the integrating factor $I(x)$ which is a function we assume it exists.

Let us multiply both sides of the equation

(1) by $I(x)$. We get $I(x)\frac{dy}{dx} + I(x)p(x)y = I(x)q(x)$ (2). The function $I(x)$ plays a role such that $I(x)p(x) = I'(x)$ (3) Therefore, substituting (3) into (2) we get $I(x)\frac{dy}{dx} + I'(x)y = I(x)q(x)$ (4) The objective is to write the first side of (4) as a derivative for the product of $I(x)$ by $y(x)$.

That is $I(x)\frac{dy}{dx}+I'(x)y=(I(x)y(x))'=I(x)q(x)$

This implies that $(x)y(x))'dx=\int I(x)q(x)dx \Leftrightarrow I(x)y(x)+c=\int I(x)q(x)dx$
It is necessary to include the constant c because if it is left out, you will get the wrong answer every time.
The final step is then some algebra to solve for the solution $y(x)$,
$I(x)y(x)=\int I(x)q(x)dx-c \Leftrightarrow y(x)=\frac{\int I(x)q(x)dx-c}{I(x)}$
Given that the constant of integration C is an arbitrary constant, to make our life easier we will consider a positive constant C and this will not affect the final answer for the solution.
Therefore, $y(x)=\frac{\int I(x)q(x)dx+C}{I(x)}$
This is a general solution of the equation (1) but we need to determine the function $I(x)$, Let us start with the equality (3): $I(x)p(x)=I'(x)$.Separating $I(x)$ from $p(x)$, we get $\frac{I'(x)}{I(x)}=p(x)$.The first side is the derivative of $\ln(I(x))$ because $(\ln(I(x)))'=\frac{I'(x)}{I(x)}$
Our equality $\frac{I'(x)}{I(x)}=p(x)$ implies that $(\ln I(x))'=p(t)$.Integrating both sides gives $\ln I(x)=\int p(t)dt+k$ and $I(x)=Ke^{\int p(t)dt}$ where K is a constant.
The solution $y(x)=\frac{\int I(x)q(x)dx+C}{I(x)}=\frac{\int e^{\int pdx}q(x)dx+\frac{C}{K}}{e^{\int pdx}}$
$y(x)=\frac{\int e^{\int pdx}q(x)dx+C_1}{e^{\int pdx}}$ with C is an arbitrary constant.
This is the formula that was previously given in the unit summary.

4.8. End unit assessment

This part provides the answers of end unit assessment activities designed in integrative approach to assess the key unit competence with cross reference to the textbook.

The following are standards (ST) which have been based on in setting end unit assessment questions:

ST7: Accurately find the best solutions to problems related to ordinary differential equations

Solutions

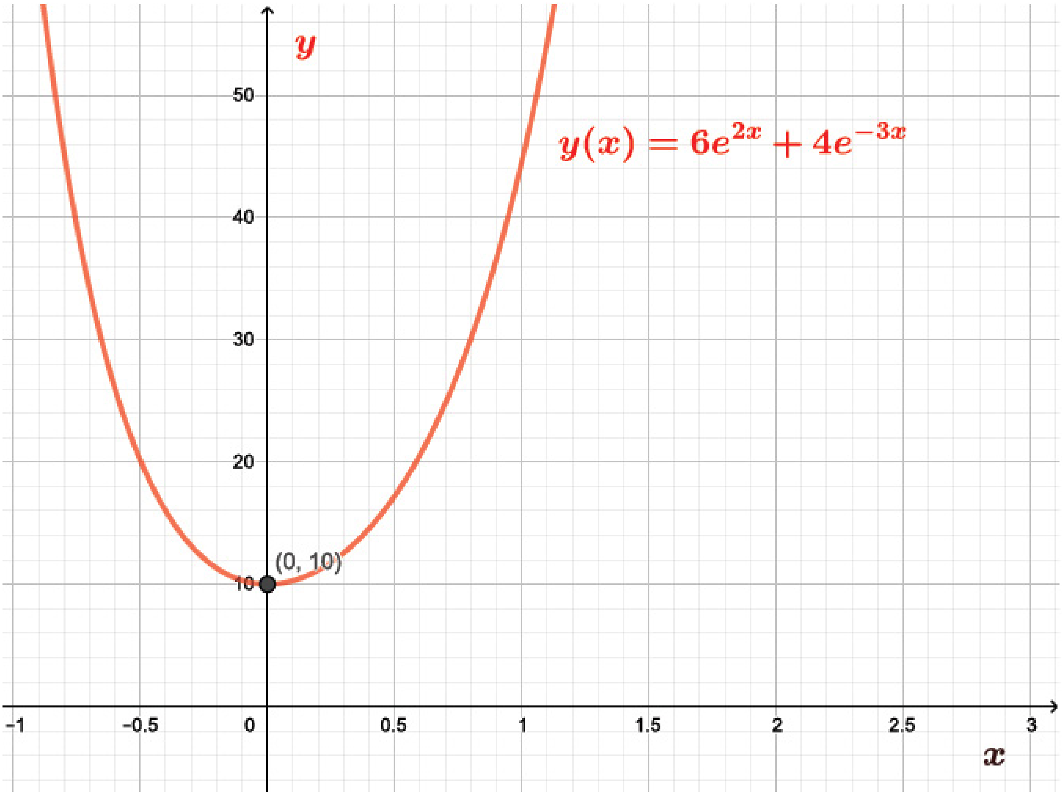
1) a) $\frac{d^2y}{dx^2}+\frac{dy}{dx}-6y=0,\ y(0)=10, y'(0)=0$ .The Differential equation is of order 2 and degree one.

b) Its characteristic equation is $\lambda^2+\lambda-6=0$ then $\lambda_{1,2}=\frac{-1\pm\sqrt{1^2+4.6}}{2}=\frac{-1\pm 5}{2}$, then $\lambda_1=2$ and $\lambda_2=-3$.The general solution $y(x)=c_1e^{2x}+c_2e^{-3x}$

c) Applying the initial conditions $y(0)=10, y'(0)=0$, we get $c_1=6$ and $c_2=4$

Therefore, the particular solution is $y(x)=6e^{2x}+4e^{-3x}$.

a) **Graphical presentation**



This function is decreasing on $]-\infty,0[$ but increasing on $]0,+\infty[$ and its minimum is the point $(0,10)$.

2) We have $\frac{dp}{dt}=0.2(Q_d-Q_s)\Rightarrow\frac{dp}{dt}=0.2\left(35-5P-(-23+6P)\right)$

a) $\frac{dP}{dt}=0.2(58-11P)\Leftrightarrow\frac{dP}{dt}+2.2P=11.6$

Which is a first order linear differential equation $p=2.2;\quad q=11.6$ and the initial condition is $P(0)=100$.

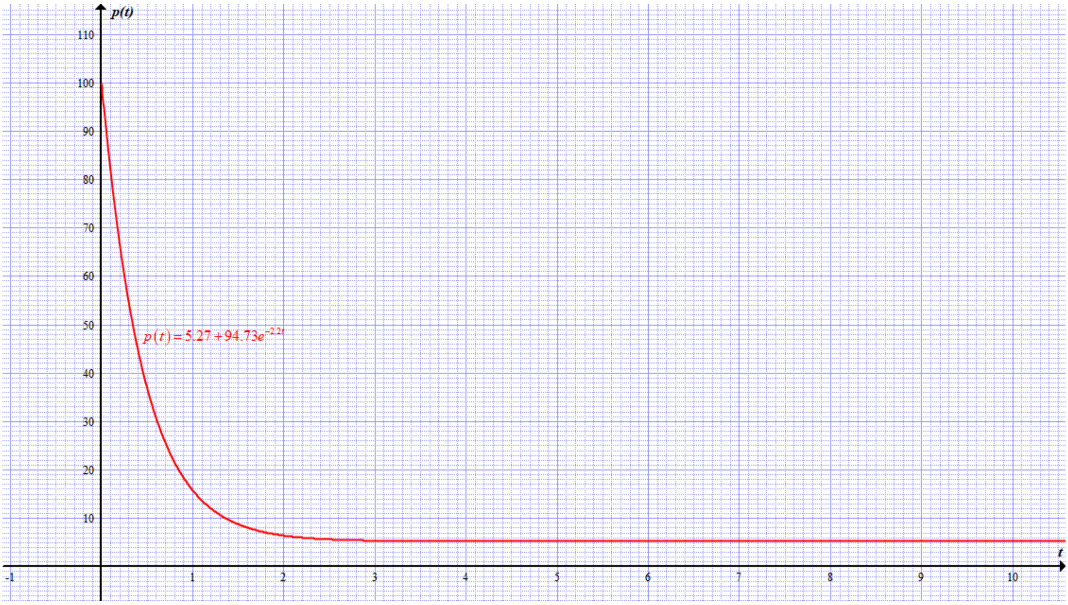
Let us determine an integrating factor $I(t)=e^{\int pdx}=e^{\int 2.2dt}=e^{2.2t}$

$P(t)=\frac{\int I(t)q(t)+C}{I(t)}=\frac{\int e^{2.2t}11.6+C}{e^{2.2t}}=\frac{11.6}{2.2}+Ce^{-2.2t}=5.27+Ce^{-2.2t}$.

Applying the initial condition is $P(0)=100\Leftrightarrow 5.27+C=100\Leftrightarrow C=94.73$,

$P(t)=5.27+94.73e^{-2.2t}$.

b) **The graph of** $P(t) = 5.27 + 94.73e^{-2.2t}$



This market is stable to the price of 5.27 when t becomes larger.

3) We are given a mass $m = 1kg$, on a spring of $k = 100$, $C = 25$ and initial conditions are $y(0) = -0.1$, $y'(0) = 0$.

a) The spring is modelled by the equation $m\frac{d^2y}{dt^2} + C\frac{dy}{dt} + ky = 0$

b) Let us solve $m\frac{d^2y}{dt^2} + C\frac{dy}{dt} + ky = 0$

The characteristic equation is $m\lambda^2 + C\lambda + k = 0$ where $\lambda_1 = \frac{-C + \sqrt{C^2 - 4mk}}{2m}$ and $\lambda_2 = \frac{-C - \sqrt{C^2 - 4mk}}{2m}$ depending on the value of $C^2 - 4mk$.

Or in our case $C^2 - 4mk = (25)^2 - 4.1.100 = 225 > 0$

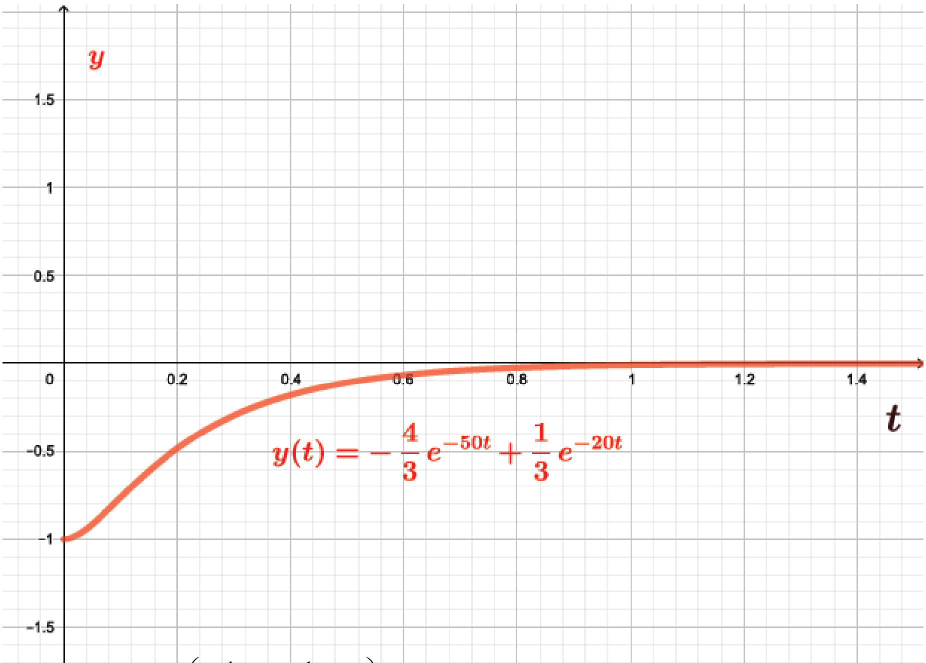
Then, $\lambda_1 = \frac{-25 + 15}{2} = -5$ and $\lambda_2 = \frac{-25 - 15}{2} = -20$

Therefore, $y(t) = c_1e^{\lambda_1 t} + c_2e^{\lambda_2 t} = c_1e^{-5t} + c_2e^{-20t}$

Applying initial conditions, $y(0) = -0.1$ and $y'(0) = 0$.

$$\begin{cases} C_1 + C_2 = -1 \\ -5C_1 - 20C_2 = 0 \end{cases} \Leftrightarrow \begin{cases} C_2 = \frac{1}{3} \\ C_1 = -\frac{4}{3} \end{cases}$$

$$y(t) = -\frac{4}{3}e^{-5t} + \frac{1}{3}e^{-20t}$$



c) $\lim_{t\to\infty} y(t) = \lim_{t\to\infty}\left(-\frac{4}{3}e^{-5t} + \frac{1}{3}e^{-20t}\right) = 0$. Both terms approach zero as $t \to +\infty$, physically speaking, after a sufficiently long time the mass will be at rest at the static equilibrium position $(y = 0)$.

4) a) $\frac{dT}{dt} = -K(T - T_e)$ Bottom of Form then $T(t) = ce^{-Kt} + T_e$

b) Initial Conditions: T (0) = 94.6°F and $T_e = 70$ imply that $c = 24.6$ and $T(t) = 24.6e^{-Kt} + 70$

Solving for T (1) = 93.4°F, we have $93.4 = 24.6e^{-Kt} + 70$ .Therefore, $K = 0.05$ and $T(t) = 24.6e^{-0.05t} + 70$

c) Assuming the body was 98.6°F when the murder occurred, we solve $98.6 = 24.6e^{-0.05t} + 70$

Solving for t gives the time $t = -3$ of death which means 3 hours before the initial temperature was taken.

4.9. Remedial, Consolidation and Extended activities

The teacher's guide suggests additional questions and answers to assess the key unit
competence.

A. **Remedial activities:** activities and answers for slow learners:

1) Solve the following differential equations

a) $4\frac{d^2y}{dx^2}-25y=0$
$\frac{d^2y}{dx^2}+2\pi y'+\pi^2y=0$

b) $\frac{d^2y}{dx^2}+25y=0; y(0)=4.6,\ y'(0)=-1.2$

c) $y''-y=0,\ y(0)=2,\ y'(0)=-2$

d) $y''-y=0,\ y(0)=2,\ y'(0)=-2$

2) Find an ordinary differential equation $a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=0$ whose solution is formed by:

(a) $e^{\sqrt{5}x} and\ xe^{\sqrt{5}x}$ (b) $\cos 2\pi x\ and \sin 2\pi x$ (c) $e^x \quad and \quad e^{-4x}$

(d) $e^{(-2+i)x}\ and\ e^{(-2+i)x}$ (e) $e^{3x}\cos 2x\ and\ e^{3x}\sin 2x$

Solution

1) a) For $4\frac{d^2y}{dx^2}-25y=0$, characteristic equation is $4\lambda^2-25=0$;
Roots are $\lambda=\frac{5}{2}$ and $\lambda=-\frac{5}{2}$; the corresponding general equation is $y=c_1e^{\frac{5}{2}x}+c_2e^{-\frac{5}{2}x}$
.

b) For $\frac{d^2y}{dx^2}+2\pi y'+\pi^2y=0;\ y(0)=0,\ y'(0)=13.137$, characteristic equation is $\lambda^2+2\pi\lambda+\pi^2=0$ where we have a repeated root $\lambda=-\pi$; the corresponding general solution is $y=c_1e^{-\pi x}+c_2xe^{-\pi x}$.

c) For $\frac{d^2y}{dx^2}+25y=0; y(0)=4.6,\ y'(0)=-1.2$, characteristic equation is $\lambda^2+25=0$;
Roots are $\lambda=5i$ and $\lambda=-5i$; the corresponding general solution is
$y=c_1\cos 5x+c_2\sin 5x$.

$y(0)=4.6 \Leftrightarrow c_1\cos 0+c_2\sin 0=4.6 \Leftrightarrow c_1=4.6$,
$y'=-5c_1\sin 5x+5c_2\cos 5x$ and

$y'(0)=-1.2 \Leftrightarrow -5c_1\sin 5x+5c_2\cos 5x=-1.2 \Leftrightarrow c_2=-0.24$.

Therefore, the particular solution is $y=4.6\cos 5x-0.24\sin 5x$

d) For $y''-y=0,\ y(0)=2,\ y'(0)=-2$, characteristic equation is $\lambda^2-1=0$ and the roots are $\lambda=1$ and $\lambda=-1$. Thus, the corresponding general solution is $y=c_1e^x+c_2e^{-x}$.

$y(0)=2 \Leftrightarrow c_1+c_2=2$; $y'=c_1e^x-c_2e^{-x}$ and $y'(0)=-2 \Leftrightarrow c_1-c_2=-2$.

Solving simultaneously $c_1+c_2=2$ and $c_1-c_2=-2$ yields $c_1=0,\ c_2=1$ and then, the particular solution is $y=2e^{-x}$.

e) For $y''+0.54\,y'+(0.0729+\pi)y=0,\ y(0)=0,\ y'(0)=1$, the characteristic equation is $\lambda^2+0.54\lambda+(0.0729+\pi)=0$ and the roots are $\lambda=-0.27+\sqrt{\pi}i$ and $\lambda=-0.27-\sqrt{\pi}i$. Then, the corresponding general solution is $y=e^{-0.27x}\left(c_1\cos\sqrt{\pi}x+c_2\sin\sqrt{\pi}x\right)$.

$y(0)=0 \Leftrightarrow c_1=0$ thus $y'=-0.27e^{-0.27x}c_2\sin\sqrt{\pi}x+\sqrt{\pi}e^{-0.27x}c_2\cos\sqrt{\pi}x$ and $y'(0)=1 \Leftrightarrow \sqrt{\pi}c_2=1 \Leftrightarrow c_2=\frac{1}{\sqrt{\pi}}$. Therefore, the particular solution is $y=\frac{1}{\sqrt{\pi}}e^{-0.27x}\sin\sqrt{\pi}x$.

2) The standard form of a quadratic equation in λ is $\lambda^2-s\lambda+p=0$ where "p" and "p" are the sum and product of its roots respectively.

(a) From the basis $e^{\sqrt{5}x},\ xe^{\sqrt{5}x}$, we note that $\sqrt{5}$ is a repeated root of characteristic equation $\lambda^2-s\lambda+p=0$ where $s=2\sqrt{5}$ and $p=\sqrt{5}=5$.

Hence, the corresponding differential equation is $\frac{d^2y}{dx^2}-2\sqrt{5}\frac{dy}{dx}+5y=0$.

(b) From the basis $\cos 2\pi x, \sin 2\pi x$, we note that $2\pi i$ and $-2\pi i$ are two conjugate roots of characteristic equation $\lambda^2-s\lambda+p=0$ where $s=0$ and $p=4\pi^2$.

Hence, the corresponding differential equation is $\frac{d^2y}{dx^2}+4\pi^2y=0$.

(c) From the basis $e^x,\ e^{-4x}$, we note that 1 and -4 are two distinct roots of characteristic equation $\lambda^2-s\lambda+p=0$ where $s=3$ and $p=-4$.

Hence, the corresponding differential equation is $\frac{d^2y}{dx^2}+3\frac{dy}{dx}-4y=0$.

(d) From the basis $e^{(-2+i)x}, e^{(-2-i)x}$, we note that $-2+i$ and $-2-i$ are two conjugate roots of characteristic equation $\lambda^2-s\lambda+p=0$ where $s=-4$ and $p=5$.

Hence, the corresponding differential equation is $\frac{d^2y}{dx^2}+4\frac{dy}{dx}+5y=0$.

(e) From the basis $e^{3x}\cos 2x, e^{3x}\sin 2x$, we note that $3+2i$ and $3-2i$ are two conjugate roots of characteristic equation $\lambda^2-s\lambda+p=0$ where $s=6$ and $p=13$.

Hence, the corresponding differential equation is $\frac{d^2y}{dx^2}-6\frac{dy}{dx}+13y=0$.

Consolidation activities: questions suggested and their answers for deep development of competences.

I) A population of bacteria in a culture grows according to the differential equation

$\frac{dN}{dt}=kN$ where k = 0.5, and N(t) is the number of bacteria present after t hours.

If at present time we approximately 5000 bacteria, estimate their number after 10 hours.

II) 1) Knowing that the time rate of change of a population $P(t)$ with constant birth and death rates is, in many simple cases, proportional to the size of the population, write an equation representing the expressed relation.

2) Suppose that $P(t)=ce^{kt}$ is the population of a colony of bacteria at time t and the population at time $t=0$ was 1000 while it doubled after one hour.

a) Respond to the following questions:

i. Find the value of c and k using the equation found in a)

ii. Re-write the equation found in a) using the values of and k

iii. Discuss the expression obtained in ii)

iv. Use the expression found in ii) to predict the future population of the bacteria colony after one and half hours and after six hours.

b) Draw the graphs of the obtained function

c) Discuss what you understand by solution of the equation in ii)

d) Discuss what you understand by initial conditions of the solution of the equation in ii)

e) Discuss the difference between general and particular solutions

III) For each of the following equations, determine the particular solution for initial value problems and give its graphical representation

a) $\frac{d^2y}{dx^2}-\frac{dy}{dx}-6y=0; \quad y(0)=1,\ y'(0)=0$

b) $2\frac{d^2y}{dx^2}+\frac{dy}{dx}-10y=0; \quad y(0)=0,\ y'(0)=1$

c) $\frac{d^2y}{dx^2}-4\frac{dy}{dx}+4y=0; \quad y(0)=0,\ y'(1)=1$

d) $\frac{d^2y}{dx^2}+4\frac{dy}{dx}+13y=0; \quad y(0)=0,\ y'\left(\frac{\pi}{2}\right)=1$

Solution

(I) $\frac{dN(t)}{dt}=kN(t)$, where k = 0.5.

This is a separable differential equation equivalent to $\frac{dN}{N}=kdt$, or k = 0.5

$\frac{dN}{N}=kdt \Leftrightarrow \frac{dN}{N}=0.5dt \Leftrightarrow \ln N=0.5t+c \Leftrightarrow N(t)=Ke^{0.5t}$ where K is a constant.

At the initial time $t=0$, there are approximately $N(0)=5000$ bacteria. Replacing this value in our equation we get $5000=K$.

Therefore, $N(t)=5000e^{0.5t}$.

Let us estimate their number after 10 hours; i.e $t=10$, $N(10)=5000e^{(0.5)10}=742065$

After 10 hours the number of bacteria is 742065.

(II) 1) $\frac{\Delta P}{\Delta t}\sim P$. As the time becomes smaller, $\frac{dP}{dt}=\lambda P$

$\Rightarrow \frac{dP}{P}=\lambda dt$ Integrating both sides we get

$\int \frac{dp}{P}=\int \lambda dt \Rightarrow \ln P=\lambda t+c$

$\Rightarrow P=e^{\lambda t+c} \Rightarrow P=e^{c}\times e^{\lambda t}$

$\Rightarrow P=\alpha e^{\lambda t}$

Where λ is a constant of proportionality and α the arbitrary constant.

2) $P(t)=ce^{kt}, t=0, P(t)=1000$

$\begin{cases}1000=ce^{0\times k}\ (i)\\ 2000=ce^{1\times k}\ (ii)\end{cases}$; $\quad From(i)\, c=1000$, $\quad From(ii)\, ce^{k}=2000$

$e^{k}=\frac{2000}{1000} \Leftrightarrow e^{k}=2 \Rightarrow k\ln e=\ln 2 \Leftrightarrow k=\frac{\ln 2}{\ln e} \Leftrightarrow k=\ln 2=0.693\simeq 0.7$

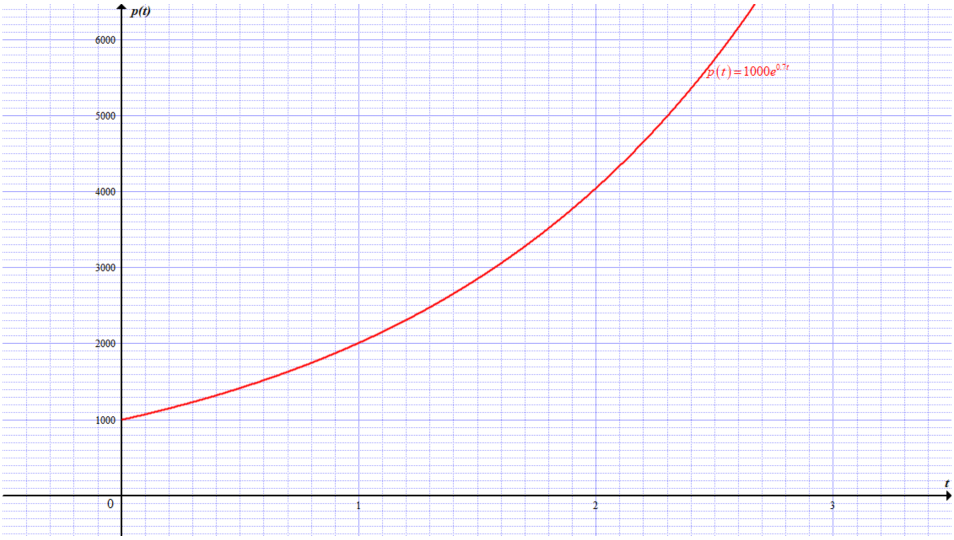
i) $c=1000\ and\ k=0.7$; ii) $P(t)=1000\times e^{0.7t}$

iii) The equation of population growth is an exponential equation.

iv) $P(t)=1000e^{0.7t}$. After one and half hours $P(t)=1000e^{0.7\times 1\frac{1}{2}}$

$P\left(1\frac{1}{2}\right)=1000e^{0.7\times 1\frac{1}{2}}=2858$ .After 6 hours: $P(6)=1000e^{0.7\times 6}=66686$

b) Draw the graphs of the obtained function $P(t)=1000\times e^{0.7t}$



c) Discuss what you understand by solution of the equation in ii)

The solution of the equation obtained in ii) is called the particular solution.

d) Initial conditions of the solution of the equation in ii): (See the definition)

e) The difference between general and particular solutions: (See the definition)

(III)

$\frac{d^2y}{dx^2}-\frac{dy}{dx}-6y=0; \quad y(0)=1,\ y'(0)=0$:

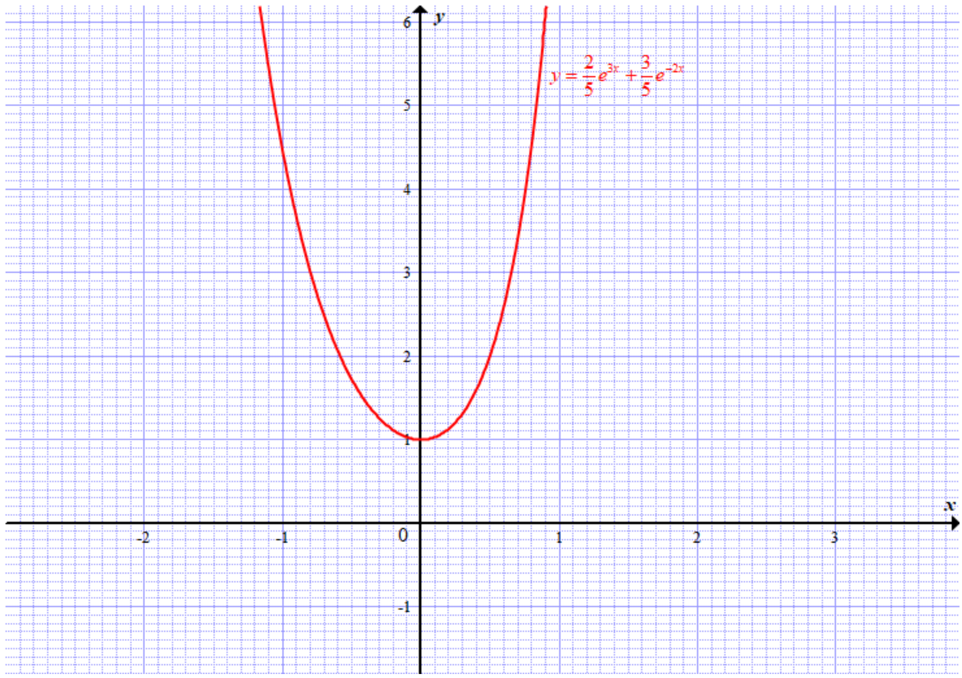
Characteristic equation is $\lambda^2-\lambda-6=0$;

Roots are $\lambda=3$ and $\lambda=-2$; the corresponding general solution is $y=c_1e^{3x}+c_2e^{-2x}$.

$y(0)=1 \Leftrightarrow c_1+c_2=1$

$y'=3c_1e^{3x}-2c_2e^{-2x}$ and $y'(0)=0 \Leftrightarrow 3c_1-2c_2=0$.

Solving simultaneously $c_1+c_2=1$ and $3c_1-2c_2=0$ yields $c_1=\frac{2}{5}$, $c_2=\frac{3}{5}$ and then, the particular solution is $y=\frac{2}{5}e^{3x}+\frac{3}{5}e^{-2x}$.



a) $2\frac{d^2y}{dx^2}+\frac{dy}{dx}-10y=0;\qquad y(0)=0,\ y'(0)=1$

Characteristic equation is $2\lambda^2+\lambda-10=0$;

Roots are $\lambda=2$ and $\lambda=-\frac{5}{2}$; the corresponding general solution is

$y=c_1e^{2x}+c_2e^{-\frac{5}{2}x}$.

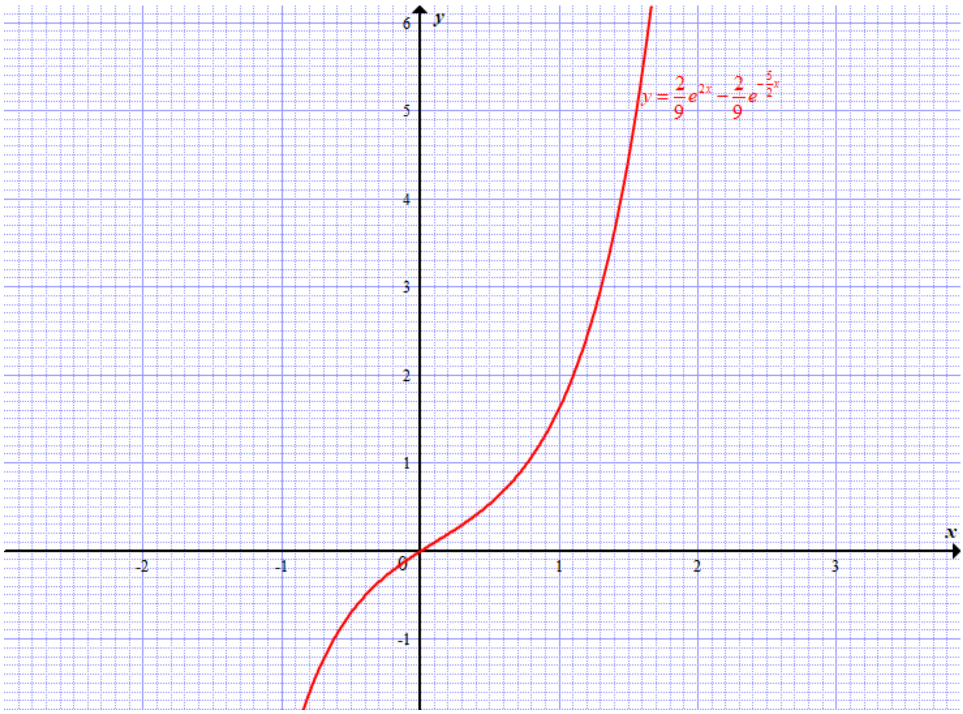
$y(0)=0 \Leftrightarrow c_1+c_2=0$

$y'=2c_1e^{2x}-\frac{5}{2}c_2e^{-\frac{5}{2}x}$ and $y'(0)=1 \Leftrightarrow 2c_1-\frac{5}{2}c_2=1$

Solving simultaneously $c_1+c_2=0$ and $2c_1-\frac{5}{2}c_2=1$ yields

$c_1=\frac{2}{9},\ c_2=-\frac{2}{9}$

Therefore, the particular solution is $y=\frac{2}{9}e^{2x}-\frac{2}{9}e^{-\frac{5}{2}x}$



b) $\frac{d^2y}{dx^2}-4\frac{dy}{dx}+4y=0; \quad y(0)=0,\ y'(1)=1$

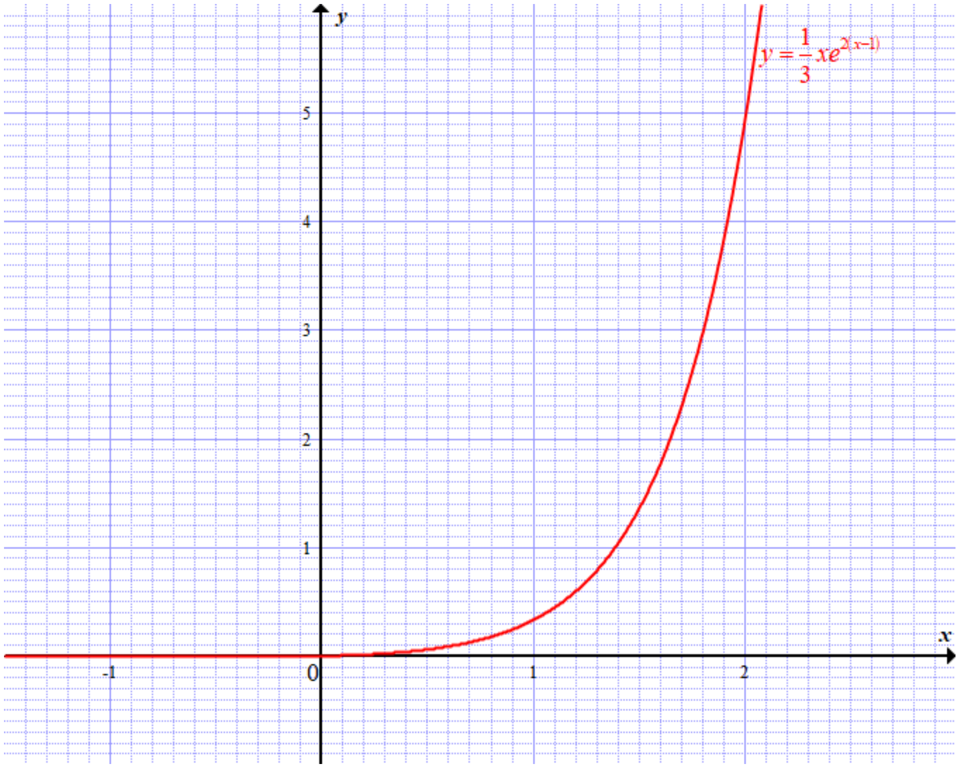
Characteristic equation is $\lambda^2-4\lambda+4=0$;

There is a repeated root $\lambda=2$; the corresponding general solution is $y=(c_1+c_2x)e^{2x}$.

$y(0)=0 \Leftrightarrow c_1=0$; $\quad y'=(2c_1+c_2+2xc_2)e^{2x}$ and

$y'(1)=1 \Leftrightarrow (2c_1+c_2+2c_2)e^2=1 \Leftrightarrow 3c_2e^2=1 \Leftrightarrow c_2=\frac{1}{3}e^2$;

The particular solution is $y=\frac{1}{3}xe^{2(x-1)}$.



c) $\frac{d^2y}{dx^2}+4\frac{dy}{dx}+13y=0; \quad y(0)=0,\ y'\left(\frac{\pi}{2}\right)=1$

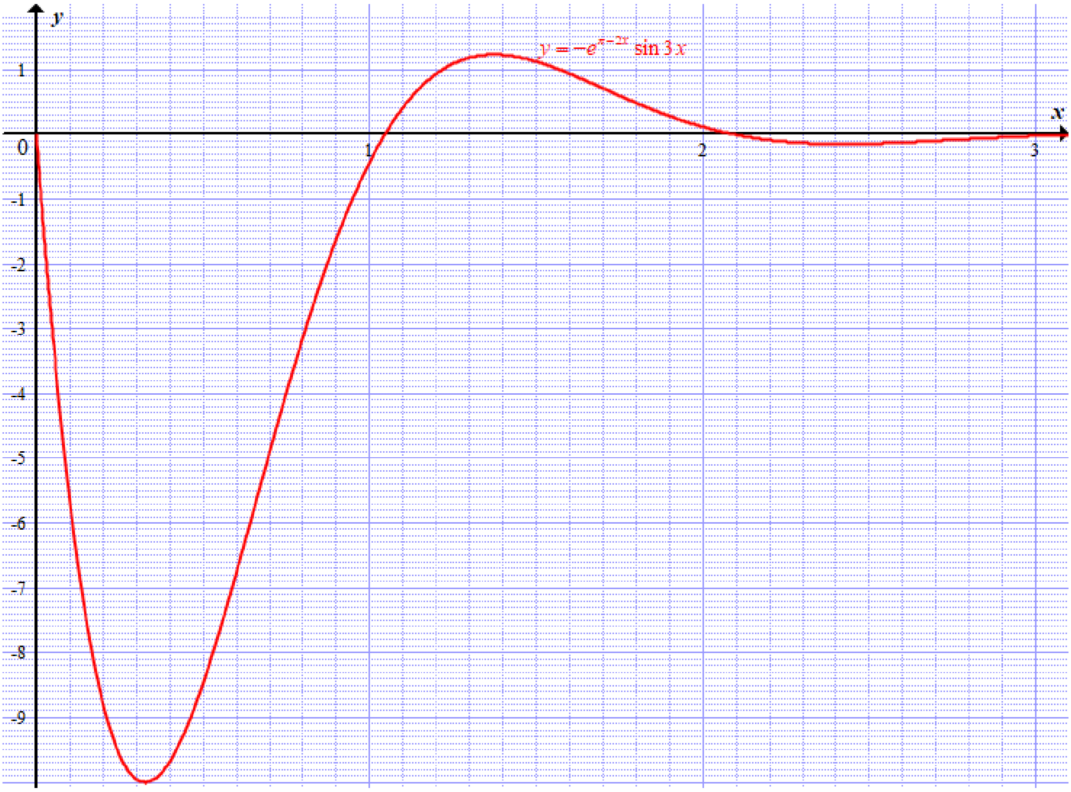
Characteristic equation is $\lambda^2+4\lambda+13=0$; Roots are $\lambda=-2+3i$ and $\lambda=-2-3i$; the corresponding general solution is

$y=e^{-2x}(c_1\cos 3x+c_2\sin 3x)$., $y(0)=0\Leftrightarrow c_1=0$

$y'=-2e^{-2x}(c_1\cos 3x+c_2\sin 3x)+3e^{-2x}(-c_1\sin 3x+c_2\cos 3x)$ and

$y'\left(\frac{\pi}{2}\right)=1\Leftrightarrow 2e^{-\pi}c_2=1\Leftrightarrow c_2=\frac{1}{2}e^{\pi}$.

Therefore, the particular solution i $y=\frac{1}{2}e^{\pi-2x}\sin 3x$.



d) Extended activities: Suggestion of Questions and Answers for gifted and talented learners.

1) Solve the following IVP: $\cos x\dfrac{dy}{dx}+y\sin x=-1+2\cos^3 x\sin x$, $y\left(\dfrac{\pi}{4}\right)=3\sqrt{2},\quad 0\le x<\dfrac{\pi}{2}$

2) A local company has 50 employees and one Director General. One day, a rumor began to spread among the employees that the DG was promoted to be a CEO of another company while he was in the mission to America. It is reasonable to assume that the rate of the spread of the rumour is proportional to the number of possible encounters between employees $y(t)$ who have heard the rumour and those $50-y(t)$ who have not after t days.

a) Given that $\dfrac{dy}{dt}=ky(50-y)$ where k is a constant of proportionality and $y(50-y)$ the number of possible meetings = (the number who've heard the rumour) times (the number who haven't);

Solve the equation considering that at the beginning 5 people participated in the same first meeting and had heard the rumour.

b) If 10 people had heard the rumour after one day, deduce $y(t)$ and plot its graph.

c) Use the calculation and the graph to estimate when 40 people will have heard the rumour.

d) What will happen as the time becomes larger?

Solution

1)

$$\cos x\frac{dy}{dx}+y\sin x=-1+2\cos^3 x\sin x,\ y\left(\frac{\pi}{4}\right)=3\sqrt{2}$$

$$\Leftrightarrow \frac{dy}{dx}+y\frac{\sin x}{\cos x}=\frac{-1}{\cos x}+2\cos^2 x\sin x$$

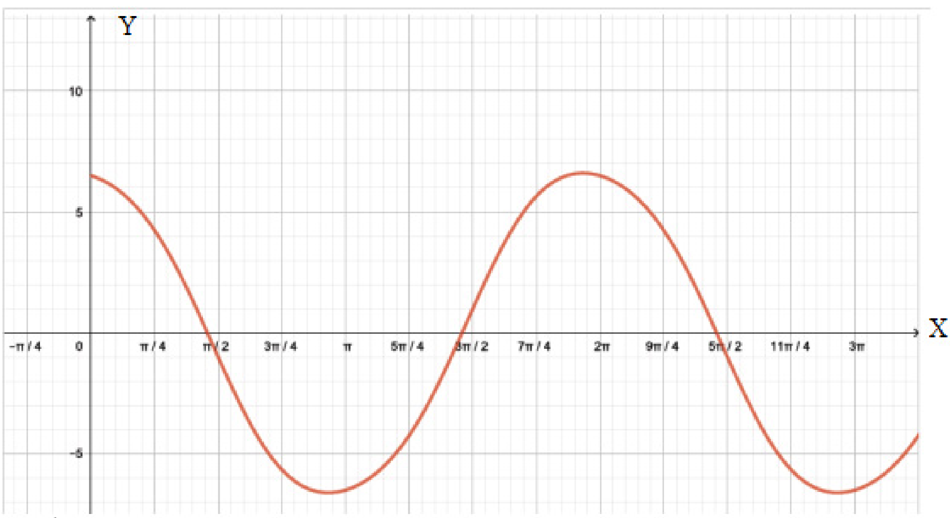
$$\Leftrightarrow \frac{dy}{dx}+y\tan x=2\cos^2 x\sin x-\sec x$$

Integrating factor $I(x)=e^{\int \tan x}=e^{\ln(\sec x)}=\sec x$. Therefore,

$y\sec x=\int\left(\sin 2x-\sec^2 x\right)dx=-\frac{1}{2}\cos 2x-\tan x+c$. This implies that $y=-\frac{1}{2}\cos x\cos 2x-\sin x+c\cos x$. Applying initial condition $y\left(\frac{\pi}{4}\right)=3\sqrt{2}$, we get $c=7$ and

$$y(x)=-\frac{1}{2}\cos x\cos 2x-\sin x+7\cos x$$

Graph of $y(x)=-\frac{1}{2}\cos x\cos 2x-\sin x+7\cos x$



2) $\frac{dy}{dt}=ky(50-y)$

It is a first order separable differential equation

$\frac{dy}{y(50-y)}=kdt \Leftrightarrow \frac{1}{50}\left(\frac{1}{y}+\frac{1}{50-y}\right)dy=kdt$. Integrating both sides, we have $\Leftrightarrow \frac{1}{50}\left(\int\frac{dy}{y}+\int\frac{dy}{50-y}\right)=k\int dt \Leftrightarrow \frac{1}{50}\left(\ln y-\ln(50-y)\right)=kt+c$

$$\Leftrightarrow \frac{1}{50}\ln\left(\frac{y}{50-y}\right)=kt+c \Leftrightarrow y(t)=50\left(\frac{Ce^{50kt}}{1+Ce^{50kt}}\right)$$

At the beginning 5 people participated in the same meeting and had heard the rumour, this means that $y(0)=5=50\left(\frac{C}{1+C}\right) \Leftrightarrow 5=50\left(\frac{C}{1+C}\right) \Leftrightarrow c=\frac{1}{9}$

Therefore, $y(t)=\frac{50e^{50kt}}{9+e^{50kt}}$

b) If 10 people had heard the rumour after one day, $y(1)=10=\frac{50e^{50k}}{9+e^{50k}}$

$$\Rightarrow k=\frac{\ln\left(\frac{9}{4}\right)}{50}=0.016$$

Therefore, $y(t)=\frac{50e^{0.81t}}{9+e^{0.81t}}$

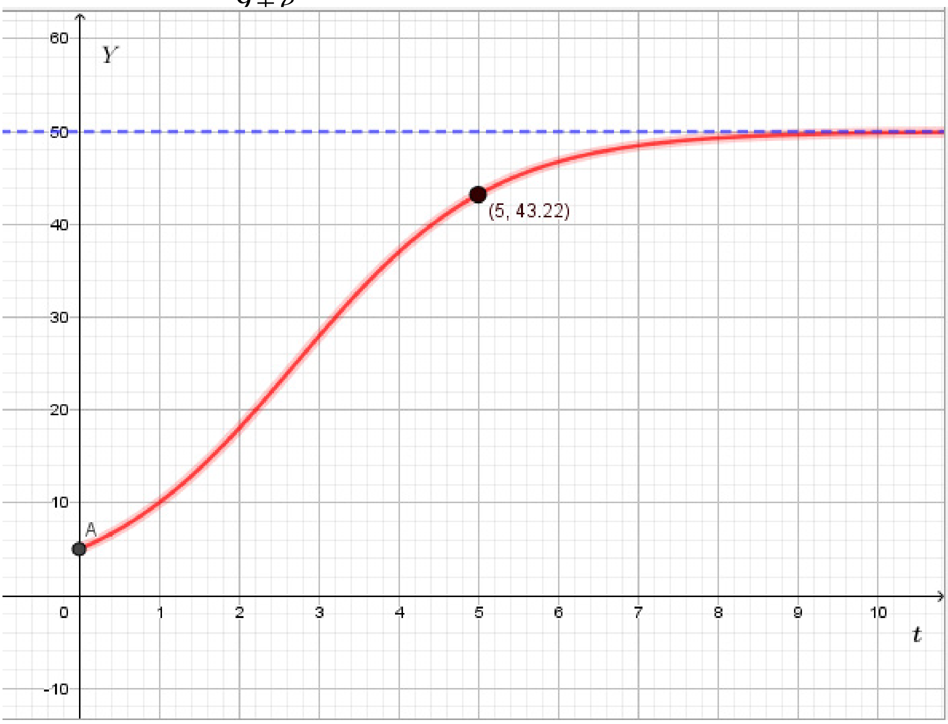


Figure: graph of $y(t)=\frac{50e^{0.81t}}{9+e^{0.81t}}$

c) When 40 people will have heard the rumour $y(t)=40=\frac{50e^{0.81t}}{9+e^{0.81t}} \Rightarrow t=?$

$40=\frac{50e^{0.81t}}{9+e^{0.81t}} \Leftrightarrow 36=e^{0.81t} \Leftrightarrow 36=e^{0.81t} \Leftrightarrow t=\frac{\ln(36)}{0.81}=4.424$. This means that after 5 days, 40 employees had heard the rumour.

d) As time becomes larger, we have $\lim_{t\to\infty} y(t)=\lim_{t\to\infty}\frac{50e^{0.81t}}{9+e^{0.81t}}=50$, this means that all 50 employees will hear the rumour.

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