SUBSIDIARY MATHEMATICS

FOR LFK, HLP & HGL

SENIOR 5

STUDENT'S BOOK

Experimental version

Kigali, 2023

FOREWORD

Dear Student,

Rwanda Basic Education Board (REB) is honoured to present the Subsidiary Mathematics book for Senior five (S5) students in the following combinations: Literature in English-French-Kinyarwanda and Kiswahili (LFK), History-Literature in English and Psychology (HLP), History-Geography and Literature in English (HGL). This book will serve as a guide to competence-based teaching and learning to ensure consistency and coherence in the learning of the Mathematics.

The Rwandan educational philosophy is to ensure that you achieve full potential at every level of education which will prepare you to be well integrated in society and exploit employment opportunities.

The government of Rwanda emphasizes the importance of aligning teaching and learning materials with the syllabus to facilitate your learning process. Many factors influence what you learn, how well you learn and the competences you acquire. Those factors include the relevance of the specific content, the quality of teachers' pedagogical approaches, the assessment strategies and the instructional materials available. In this book, we paid special attention to the activities that facilitate the learning process in which you can develop your ideas and make new discoveries during concrete activities carried out individually or with peers.

In competence-based curriculum, learning is considered as a process of active building and developing knowledge and meanings by the learner where concepts are mainly introduced by an activity, situation or scenario that helps the learner to construct knowledge, develop skills and acquire positive attitudes and values.

For efficiency use of this textbook, your role is to:

- Work on given activities which lead to the development of skills;
- Share relevant information with other learners through presentations, discussions, group work and other active learning techniques such as role play, case studies, investigation and research in the library, on internet or outside;
- Participate and take responsibility for your own learning;

- Draw conclusions based on the findings from the learning activities.

To facilitate you in doing activities, the content of this book is self-explanatory so that you can easily use it yourself, acquire and assess your competences. The book is made of units as presented in the syllabus. Each unit has the following structure: the key unit competence is given and it is followed by the introductory activity before the development of mathematical concepts that are connected to real world problems or to other sciences.

The development of each concept has the following points:

- It starts by a learning activity: it is a hand on well set activity to be done by students in order to generate the concept to be learnt;
- Main elements of the content to be emphasized;
- Worked examples; and
- Application activities: those are activities to be done by the user to consolidate competences or to assess the achievement of objectives.

Even though the book has some worked examples, you will succeed on the application activities depending on your ways of reading, questioning, thinking and grappling ideas of calculus not by searching for similar-looking worked out examples.

Furthermore, to succeed in Mathematics, you are asked to keep trying; sometimes you will find concepts that need to be worked at before you completely understand. The only way to really grasp such a concept is to think about it and work-related problems found in other reference books.

I wish to sincerely express my appreciation to the people who contributed towards the development of this book, particularly, REB staff, UR-CE Lecturers and Secondary School teachers for their technical support. A word of gratitude goes also to Head Teachers who availed their staff for various activities. Any comment or contribution would be welcome to the improvement of this text book for the next edition.

Dr. MBARUSHIMANA Nelson

Director General, REB

ACKNOWLEDGEMENT

I wish to express my appreciation to the people who played a major role in the development of this Subsidiary Mathematics book for Senior five (S5) students in the following combinations: Literature in English-French-Kinyarwanda and Kiswahili (LFK), History-Literature in English and Psychology (HLP), History-Geography and Literature in English (HGL). It would not have been successful without active participation of different education stakeholders.

I owe gratitude to different universities and schools in Rwanda that allowed their staff to work with REB in the in-house textbooks production initiative.

I wish to extend my sincere gratitude to lecturers and teachers whose efforts during writing exercise of this book were very much valuable.

Finally, my word of gratitude goes to the Rwanda Basic Education Board staffs who were involved in the whole process of in-house textbook elaboration.

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Head of CTLR Department

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UNIT 1: INTRODUCTION TO LOGIC

Key Unit competence: Use Mathematical logic as a tool of reasoning and decision making in daily life

1.0 Introductory activity 1

1) Discuss the meaning of propositional statement
2) In the following sentences, which are propositions and which are not propositions?
   a) Don't eat the daisies!
   b) My class is senior five.
   c) Do you enjoy reading novels?
   d) The jokes are great.
   e) Mozart composed classical music.
   f) The camera is not a Kodak.
   g) This statement is false.
   h) Use the quadratic formula on that one.

1.1 Simple statement and compound statement

Activity 1.1

From the following expressions, give your answer by using true or false

1) Every integer larger than 1 is positive.
2) Kampala is in Rwanda.
3) How old are you?
4) Every liquid is water.
5) Write down the names of the president of Uganda.
6) $1-x^2=0$.
7) Rwanda is an African country or Rwanda is a member of Commonwealth.

Content summary

A sentence which is either true or false but not both simultaneously is named **statement or proposition**. In the context of logic, a proposition or a statement is the sentence in the grammatical sense conveying a situation which is neither imperative, interrogative nor exclamatory.
A **proposition** is a declarative sentence that can be meaningfully classified as either true or false.

In activity 1.1, the expressions 1, 2, 4 and 7 are statements while 3, 5 and 6 are not. The 2^{nd} and 4^{th} statements are false while 1^{st} and 7^{th} statements are true.

- The expression "How old are you?" is not a proposition since you cannot reply by true nor false (grammatically this sentence is interrogative).
- The equality "$1-x^2=0$" is not a proposition because for some values of x the equality is true, whereas for others it is false.
- The expression "Write down the names of the president of Uganda" is not a proposition as the answer will be given by neither true nor false. It is a command.

A statement that cannot be broken into two or more sentences is called **simple statement**. Combining two or more simple statements we form a **compound statement.**

A **compound proposition** is a proposition formed by connecting two or more propositions or by negating a single proposition. The words and phrases (or symbols) used to form compound propositions are called **connectives.**

Examples

In activity 1.1, the 1^{st}, 2^{nd} and 4^{th} expressions are simple statements while the 7^{th} is a compound statement. "Rwanda is an African country or Rwanda is a member of Commonwealth" is a compound statement.

In this unit, statements will be denoted by small letters such as ***p***, ***q, r, ...***

The truth value is the attribute of a proposition as to whether the proposition is true or false. For example, the truth value for "7 is odd" is true, which can be denoted as T.
The truth value of "1 + 1 = 3" is false, which can be denoted as F.

The **logical statements** are required to have a definite **truth-value**, or to be either true or false, but never both, and to always have the same truth value.

The two truth values of a proposition are **true** and **false** and are denoted by the symbols **T** and **F** respectively.

Occasionally, **true** and **false** are also denoted by the symbols **1** and **0** respectively.

Application Activity 1.1

1) Find out which of the following sentences are statements and which are not. Justify your answer.

    a) Uganda is a member of East African Community.
    b) The sun is shining.
    c) Come to class!
    d) The sum of two prime numbers is even.
    e) It is not true that China is in Europe.
    f) May God bless you!

2) Write down the truth value (T or F) of the following statements

    a) Paris is in Italy.
    b) 13 is a prime number.
    c) Kigeri IV Rwabugiri was the King of the Kingdom of Rwanda
    d) Lesotho is a state of South Africa.
    e) Tanzania is in East of Rwanda and it is in SADC (Southern African Development Community).

1.2 Truth values and truth tables

Activity 1 .2

Are these sentences proposition? If yes, give their truth values

1) Uganda is a member of East African Community.
2) The sun shines
3) Paris is in England.
4) Come to class!
5) The sum of two prime numbers is even.
6) It is not true that Uganda is in Europe.

Content summary

The way we will define compound statements is to list all the possible combinations of the truth-values (abbreviated **T** or **1** and **F** or **0**) of the simple statements (that are being combined into a compound statement) in a table, called a **truth table**. The name of each statement is at the top of a column of the table.

Let us find the number of combinations of truth values we can consider when we have a compound statement with n distinct simple statements.

Example 1

One proposition ***p*** has two truth values ($2^1 = 2$ possible combinations), the truth table is

p
T
F

Or

p
1
0

Example 2

Suppose we are given two mathematical statements, named **p** and **q**, new mathematical statements that incorporate **p** and **q** are called **compound statements**. Their truth values will be determined solely by the truth-values of **p** and of **q**.

For two propositions p and q, we have $2^2 = 4$ possible combinations of truth values, the truth table is:

p	q
T	T
T	F
F	T
F	F

or

p	q
1	1
1	0
0	1
0	0

Any proposition can be represented by a truth table. It shows truth values for all combinations of its constituent variables

Example 3

Let us take the proposition r involving 2 true statements p and q where

p : I am at home

q : It is raining.

One combination of statements is "*I am at home and it is raining*" (T).

Others are the following: "*I am at home and it is not raining*" (F), "*I am not at home and it is raining*" (F) and "*I am not at home and it is not raining*" (F). All these results are summarized in the following table:

All possible combinations of truth values of 2 propositions p and q		Truth values of compound proposition r **r**
p	q	r
True	True	True
True	False	False
False	True	False
False	False	False

A proposition can involve any number of variables; each row corresponds to a possible combination of variables. With n variables the truth table has: $n+1$ column (one for each of the n variables and one for the compound expression); it has also 2^n rows plus a header.

Therefore, if the compound statement contains n distinct simple statements, we will consider 2^n *possible combinations of truth values in order to obtain the truth table*. The following lesson will develop how to make a compound statement using different connectives.

Application Activity 1.2

Write down the truth table for any:

1) Three propositions *p*, *q* and *r*

2) Four propositions *p*, *q*, *r* and s

1.3 Logical connectives

1.3.1 Negation "not"

Activity 1.3.1

1) Given statements *p*: "I am strong" and *q*: "I can jump"; make a compound statement formed by *p* and *q* in different ways. What are the different connecting words that can be used?

2) Let *p*, *q*, ... be the given propositions, put these propositions in negative form

   a) Jack is running.

   b) Ronald does not smile.

   c) She isn't a football player.

   d) -3 is a natural number.

   e) Mathematics is needed in languages education combinations.

Content summary

The negation of a proposition is **what is asserted when that proposition is denied.** The negation of a statement is the opposite of the given mathematical statement. The negation of a statement ***p*** is made by introducing the word "not" denoted by prefixing the statement ***p*** and it has opposite truth value of the statement ***p***. The negation of a statement P is denoted by $\neg p$ or $\overline{p}$ or $\sim p$

From this definition, it follows that the negation of a true statement is false while the negation of false statement is true; simply if p is true, then $\neg p$ is false and if p is false, then $\neg p$ is true.

Example 1

a) Let p : "Kamana is a student", then $\neg p$: "Kamana is not a student".

b) p : The earth is round, $\neg p$: The earth is not round

Example 2

Let p be a proposition. Construct the truth table of $\neg p$

Solution

p	$\neg p$
T	F
F	T

Note: The following propositions all have the same meaning:

p: All people are intelligent.

q: Every person is intelligent.

r: Each person is intelligent.

s: Any person is intelligent.

(a) The negation of the proposition *p: "*All students are intelligent" can be stated in any of the following ways:

$\neg \boldsymbol{p}$: Some students are not intelligent.

$\neg \boldsymbol{p}$: There exists a student who is not intelligent.

$\neg \boldsymbol{p}$: At least one student is not intelligent

Note that "No student is intelligent" is *not* a negation of *p*.

(b) The negation of the proposition *q: "*No student is intelligent" may be stated as:

$\neg \boldsymbol{q}$: At least one student is intelligent.

Note that "All students are intelligent" is *not* a negation of *q*.

Application activity 1.3.1

1) Write the negation of each of the following statements.
   a) Today is raining.
   b) The sky is blue
   c) My native country is Rwanda.
   d) Bony is smart and healthy.
2) Complete the following truth table

p	q	r	$\neg p$	$\neg q$	$\neg r$
T		T		F	
	T	F	F		
T	F	T			
T	F				T
	T	T	T		
F	T	F			
F				T	F
F	F	F			

1.3.2 Conjunction "and"

Activity 1.3.2

Given two propositions:

p : I am at school, q : it is raining. Discuss the truth value of the compound propositions:

a) "I am at school and it is raining"

b) "I am not at school and it is raining",

c) "I am not at school and it is not raining".

Content summary

If two simple statements p and q are connected by the word **"and"**, then the resulting compound statement ***p*** **and** ***q*** is called a conjunction of ***p*** and ***q*** and is written in symbolic form $p \wedge q$. It has the truth value **true** whenever both ***p*** and ***q*** have the truth value **true**; otherwise it has the truth value **false**. In language, the conjunction *and* joins two similar ideas together.

Example 1

Let ***p*** be "It is raining today" and ***q*** be "There are fifteen chairs in this class room". Assuming that p and q are true statements, construct simple sentences which describe each of those statements and construct the truth table of $p \wedge q$.

Solution

From the given two simple statements, one of the resulting compound statements is

(i) "It is raining today and there are fifteen chairs in this class room" which is True.

(ii) "It is not raining and there are not fifteen chairs in this classroom: Which is False.

You can give the remaining two sentences and the truth table of related compound statements as follows:

p	q	$p \wedge q$
T	T	T
T	F	F

F	T	F
F	F	F

Example 2: Consider the two statements:

p: Kigali is the capital of Rwanda. **True**

q: Rwamagana is the largest District in Rwanda. **False**

The compound proposition (p and q) is “Kigali is the capital of Rwanda and Rwamagana is the largest District.

Since the statement *q* is false, the compound statement is false even though *p* is true.

Example 3

Let p and q be propositions. Construct the truth table for

a) $\neg p \wedge q$

b) $(\neg p \wedge q) \wedge \neg p$

Solution

p	q	$\neg p$	$\neg p \wedge q$	$(\neg p \wedge q) \wedge \neg p$
T	T	F	F	F
T	F	F	F	F
F	T	T	T	T
F	F	T	F	F

Application activity 1.3.2

1) If **p** stands for the statement " It is cold" and **q** stands for the statement "It is raining", then what does $\neg q \wedge \neg q$ stands for? Construct its truth table.

2) Let *p* and *q* be two propositions. Construct the truth table of

   a) $p \wedge q$

   b) $\neg p \wedge q$

   c) $p \wedge \neg q$

   d) $\neg(p \wedge q)$

   e) $\neg q \wedge (\neg p \wedge q)$

3) Determine the truth value of each of the following statements

   a) Paris is in France **and** it is a Capital city

   b) $4+4=9$ **and** $5+8=11$

   c) Paris is in England **and** $3+4=7$

   d) Kigali is the Capital city of Burundi **and** $1+1=2$

   e) Alphabets are the basic of any languages **and** the digits is not the basic in counting

   f) m^2 is "the unit of area" **and** *kg* is "one of the units of weight.

1.3.3 Disjunction “or”

Activity 1.3.3

1) There were boys and girls in the classroom.

   a) If they ask you to choose two boys and two girls. How many students will you choose?

   b) If they ask you to select two girls or two boys how many students will you select?

2) Given the true proposition *p*: I am at home, *q*: it is raining. Give the truth value of:

   a) “I am at home or it is raining”.

   b) “I am not at home or it is not raining”.

   c) “I am at home or it is not raining”.

Content summary

If two simple statements p and q are connected by the word **“or”**, then the resulting compound statement *p* or *q* is called a **disjunction** of *p* and *q* and it is written in symbolic form by $p \vee q$.

It has the truth value **false** only when *p* and *q* have truth value **false**, otherwise it has **true** as the truth value.

Example 1

Let *p* be “Paris is in France” and *q* be “London is in England”. Construct simple verbal sentences which describe different related compound statements each of the following statements and construct the truth table of $p \vee q$.

Solution

From the given two simple statements, the resulting compound statement is

(i) “Paris is in France or London is in England” (True)

(ii) “Paris is in France or London is not in England” (Tue)

(iii) "Paris is not in France or London is in England" (True)

(iv) "Paris is not in France or London is not in England" (False).

The Truth table of related compound statement is as follows:

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

Example 2

Construct the truth tables for $\neg(p \vee q)$ and $\neg p \wedge \neg q$ then compare their results.

Solution

First, we make a table that displays all the possible combinations of truth-values for p and q, then we add two columns for the truth values of $p \vee q$ and $\neg(p \vee q)$ into its top cell and then start completion of your truth table with appropriate truth values.

p	q	$p \vee q$	$\neg(p \vee q)$
T	T	T	F
T	F	T	F
F	T	T	F
F	F	F	T

Note: Ambiguity of "or" in English language. In natural language **"or" has two meanings:**

The English word "or" can be used in two different ways—as an inclusive ("and/or") or exclusive ("either/or") disjunction. The correct meaning is usually inferred from the context in which the word is used. However, when it is important to be precise (as it often is in mathematics, business, science, etc.), we must carefully distinguish between the two meanings of "or."

i. **Inclusive or:** where $p \vee q$ is true if either p or q or both are true. This is the inclusive we explored here above.

Example

"Numbers or measurements may be taken as prerequisites for geometry". It means that take either one but you may also take both.

We define $p \vee q$ to be true if *at least one* of the propositions p, q is true. That is, $p \vee q$ is true if both p and q are true, if p is true and q is false, or if p is false and q is true. It is false only if both p and q are false.

ii. **Exclusive or:** denoted as $\oplus$: where $p \oplus q$ is true if either p or q but not both are true. It can also be noted as $p \veebar q$

Example

"You will be paid money or a computer". It m**eans that** do not expect to get both.

We define $p \veebar q$ to be true if exactly one of the propositions p, q is true. That is, $p \veebar q$ is true if p is true and q is false, or if p is false and q is true. It is false if both p and q are false or if both p and q are true.

Application activity 1.3.3

1) Translate each of the following compound statements into symbolic form
   a) Bwenge reads News Paper or Mathematics book.
   b) Rwema is a student or not a book seller.
2) Suppose that p is a false statement, and q is a true statement.
   a) What is the truth-value of the compound statement $\neg p \vee q$?
   b) What is the truth-value of the compound statement $p \vee \neg q$?
3) Let p and q be two propositions. Construct the truth table of
   a) $p \vee q$;
   b) $p \vee \neg q$;
   c) $\neg(p \vee q) \wedge (\neg p \vee \neg q)$
   d) $(\neg q \vee p) \wedge (\neg p \vee q)$

1.3.4 Conditional statement "if … then"

Activity 1.3.4

1) Given the following conditional statement: "If Amanda's health is good, then she will go to the party". Discuss and identify the hypothesis (premise) and conclusion (consequent) of the conditional statement.

2) Complete these sentences referring to the conditional statement given in question one.

   a) If Amanda's health is not good, ...

   b) Amanda will go to the party, ...

   c) Amanda will not go to the party, ...

Content summary

Consider the following statement: If you eat too much meat, then you will get fat. This is a conditional statement because there is a condition to get fat.

A conditional statement is a logic statement used when the statement is in the form "if ...then ...." It can be written as $"If\ p\, then\, q"$ or $p \to q$, (or $p \Rightarrow q$) read as $p\, implies\, q$.

In this case the proposition p is called **antecedent, hypothesis or premise** while the proposition q is the **conclusion** or **the consequent**. In language, p can be called the first clause and q is the second clause.

The conditional $p \Rightarrow q$ can also be read:

- If p, then q.
- q follows from p.
- q if p.
- p only if q.
- p is sufficient for q.

- q is necessary for p.

Let us consider the logical conditional as an obligation or contract, for example:

"If you get 100% on the final exam, then you will earn an A". For this compound statement we have the following simple statements:

p: If you get 100% on the final exam, q: you will earn an A. Symbolically, it is written as: $p \Rightarrow q$.

	q : earn an A	$\neg q$: don't get A
p : get 100%	$p \Rightarrow q$ is **true**	$p \Rightarrow q$ is **false**
$\neg p$: don't get 100%	$p \Rightarrow q$ is **true**	$p \Rightarrow q$ is **true**

This shows that $F \Rightarrow T$ does not violate the obligation, the only time the obligation is broken is when $T \Rightarrow F$. Therefore, the statement $p \Rightarrow q$ has the truth value True in all cases except when p is true while q is false. The related truth table is as follows:

p	q	$p \Rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

Example 1

Rephrase the sentence "*If it is Sunday, you go to church*", then construct the related truth values.

Solution

Here are some various ways of rephrasing the sentence:

- "You go to church if it is Sunday."
- "It is Sunday only if you go to church."
- "It can't be Sunday unless you go to church."

Let denote the given statements as: ***p***: It is Sunday, ***q***: you go to church.

The statement "*If it is Sunday, you go to church*" is symbolized as follow: $p \Rightarrow q$.

p	q	$p \Rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

Example 2

Construct the truth value of the following statement:

"If either John takes Calculus or Betty takes Sociology then Peter will take English."

Solution

Let denote the statements as:

p: John takes Calculus, *q*: Betty takes Sociology and *r*: Peter takes English.

The given statement can be symbolized as follow: $(p \vee q) \Rightarrow r$ and the truth values are in the following truth table

p	q	r	$p \vee q$	$(p \vee q) \Rightarrow r$
T	T	T	T	T
T	T	F	T	F
T	F	T	T	T
T	F	F	T	F
F	T	T	T	T
F	T	F	T	F
F	F	T	F	T
F	F	F	F	T

Note: As in the previous section, we express a compound proposition symbolically by replacing each component statement and connective by an appropriate symbol. For example, denoting "I study" and "I will pass" by *a* and *b*, respectively, the proposition "If I study, then I will pass" is written as $a \Rightarrow b$. This proposition can also be read as "A sufficient condition for passing is to study."

To understand "implication" better, look at an implication as a conditional promise.

If the promise is broken, the implication is false; otherwise, it is true. For this reason the only circumstance under which the implication $p \Rightarrow q$ is false is when p is true and q is false.

The word *then* in an implication merely serves to separate the conclusion from the hypothesis—it can be, and often is, omitted.

Application activity 1.3.4

1) Using the statements p: Mico is fat and p: Mico is happy.

Assuming that "not fat" is thin, write the following statements in symbolic form

a) If Mico is fat, then she is happy.

b) Mico is unhappy implies that Mico is thin.

2) Write the following statements in symbolic form and their truth values

a) If n is prime, then n is odd or n is 2.

b) If x is nonnegative, then x is positive or x is 0.

c) If Manzi is Keza's father, then Stephen is her uncle and Micheline is her aunt.

1.3.5 Converse and contra-positive of a conditional statement

Activity 1.3.5

1) Let p be "you are a guitar player" and let q be "you are a musician". Write each statement in symbols. Then decide whether it is true or false.
   a) If you are guitar player, then you are musician.
   b) If you are musician, then you are a guitar player.
   c) If you are not a musician, then you are not a guitar player.
2) Select the statement that is the converse of the following statement: "If you want to be on my team, then you like getting bossed around."
   a) If you don't like getting bossed around, then you don't want to be on my team.
   b) If you don't want to be on my team, then you don't like getting bossed around.
   c) If you like getting bossed around, then you want to be on my team.
   d) a), b), & c) are all correct.
   e) Stop complaining and get to work.

Content summary

a) Converse

The implication obtained by interchanging the antecedent and the consequent of an implication is called the **converse.** Thus, the converse of $p \Rightarrow q$ is $q \Rightarrow p$.

The implication *if q, then p* is called the **converse** of the implication *if p, then q*. That is $q \Rightarrow p$ is the converse of $p \Rightarrow q$.

p	q	$p \Rightarrow q$	$q \Rightarrow p$
T	T	T	T
T	F	F	T
F	T	T	F
F	F	T	T

Example 1

If you are a student, then you should study. The converse is "if you study, then you are a student". This is not always true.

Example 2: Consider the statements

p: You are a thief. *q*: You go to jail.

The implication $p \Rightarrow q$ states that "If you are a thief, you go to jail".

The converse of this implication, namely $q \Rightarrow p$, states that "If you go to jail, you are a thief".

To say that thieves go to jail is not the same as saying that everyone who goes to jail is a thief. This example illustrates that the truth of an implication does not imply the truth of its converse. Many common fallacies in thinking arise from confusing an implication with its converse. So, $p \Rightarrow q$ and $q \Rightarrow p$ are not equivalent; they do not give the same truth values.

b) Contra-positive

The contrapositive of a conditional statement is obtained by first writing the converse, then negate both antecedent (hypothesis) and the consequent (conclusion).

Thus, given the conditional $p \Rightarrow q$, the implication $\neg q \Rightarrow \neg p$ is called the **contra-positive** of $p \Rightarrow q$.

Example 1: Consider the statements *p*: You are a thief. *q*: You go to jail.
The implication $p \Rightarrow q$ states "If you are a thief, then you go to jail." The contra-positive, namely $\neg q \Rightarrow \neg p$ is "If you do not go to jail, then you are not a thief."

p	$\neg p$	q	$\neg q$	$p \Rightarrow q$	$q \Rightarrow p$	$\neg q \Rightarrow \neg p$
T	F	T	F	T	T	T
T	F	F	T	F	T	F
F	T	T	F	T	F	T
F	T	F	T	T	T	T

Example 2: If it is a PC, then it is a computer. The contrapositive is "if it is not a computer, then it is not a PC". This is true. Basing on the results of the two examples, we note that:

A conditional statement is logically equivalent to its contrapositive. A conditional statement can be replaced with its contrapositive and keeps its truth value.

Application activity 1.3.5

1) Consider the conditional proposition: "If a number is a multiple of 10, then the number is a multiple of 5".

(a) What is the truth value of that proposition?

(b) Deduce the converse of the proposition and the related truth value.

(d) Deduce the contrapositive of the proposition and the related truth value.

2) Write the converse and contrapositive of the false conditional statement below and determine whether each of the statements found is true or false.

"If x is an even number, then the last digit of x is 2.

1.3.6 Inverse of a conditional statement

Activity 1.3.6

1) Consider the conditional proposition: "If this year is a leap year, then February has 29 days ".
    a) What is the truth value of that proposition?
    b) Deduce the inverse of the proposition and the related truth value.
2) Write the inverse of the false conditional statement below and determine whether the statement found is true or false. "If a number is even, then its last digit is 4".

Content summary

Given the conditional proposition $p \Rightarrow q$, the implication $\neg p \Rightarrow \neg q$ is called its **inverse**. It is obtained by **negating the antecedent and negating the consequent**.

Example 1:

If it is a PC, then it is a computer, the inverse is "If it is not a PC, then it is not a computer". This is false.

Example 2: Consider the statements: p: You are a thief. q: You go to jail.

The implication $p \Rightarrow q$ states "If you are a thief, then you go to jail."The inverse of $p \Rightarrow q$ namely $\neg p \Rightarrow \neg q$, is "If you are not a thief, then you do not go to jail."

p	$\neg p$	q	$\neg q$	$p \Rightarrow q$	$\neg p \Rightarrow \neg q$
T	F	T	F	T	T
T	F	F	T	F	T
F	T	T	F	T	F
F	T	F	T	T	T

We have shown that an implication and its contra-positive are logically equivalent. Also, the converse and inverse of an implication are logically equivalent, but neither of these is logically equivalent to the original implication.

Application activity 1.3.5

1) Consider the conditional proposition: "If a number is a multiple of 10, then the number is a multiple of 5".

   (a) What is the truth value of that proposition?

   (b) Write its inverse and the related truth value.

2) Write the inverse of the conditional statement below and determine whether each of the statements found is true or false. “If x is an even number, then x is divisible by 2.

1.3.7. Bi-conditional statement “if and only if”

Activity 1.3.71bb

1. Let *p* be the statement: “Anne Maria is intelligent” and *q* be the statement: “Anne Maria is hard working”
   a. Express $p \Rightarrow q$ and its truth value.
   b. Express $q \Rightarrow p$ and its truth value
   c. Give the truth value of $(p \Rightarrow q) \wedge (q \Rightarrow p)$.
2. Let *r* be the statement: “$7 = 7$” and *s* be the statement: “$5 = 5$”
   a. Give the truth value of $r \Rightarrow s$
   b. Give the truth value of $s \Rightarrow r$
   c. Give the truth value of $(r \Rightarrow s) \wedge (s \Rightarrow r)$.

Content summary

Let us consider the proposition *p*: Two lines are perpendicular, *q*: Two lines form a right angle.

We have: $p \Rightarrow q$: If two lines are perpendicular, then the two lines form a right angle **(True)**

The converse is $q \Rightarrow p$ means: if two lines form a right angle, then the two lines are perpendicular. **(True).**

We see that a statement and its converse are both true statements; that is $p \Rightarrow q$ and $q \Rightarrow p$ are both true.

Generally $p \Rightarrow q$ is not the same as $q \Rightarrow p$. It may happen, however, that both $p \Rightarrow q$ and $q \Rightarrow p$ are true. The statement $p \Leftrightarrow q$ defined by $(p \Rightarrow q) \wedge (q \Rightarrow p)$ is compound statement with the **bi-conditional connective.**

For this reason, the double headed arrow $p \Leftrightarrow q$ is called the **bi-conditional**.

Example: Let *p* and *q* denote the statements

p: Cauchy is happy.

q: Cauchy is studying.
Then "A *necessary condition* for Cauchy to be happy is that Cauchy is studying" means "If Cauchy is happy, he is studying."
Moreover, "A *sufficient condition* for Cauchy to be happy is that Cauchy is studying" means "If Cauchy is studying, he is happy."

If both implications $p \Rightarrow q, q \Rightarrow p$ are true statements, then the bi-conditional is true.
In other words, Cauchy is happy if and only if he is studying.

We can restate the definition of the bi-conditional connective by using its truth table,

p	q	$p \Rightarrow q$	$q \Rightarrow p$	$(p \Rightarrow q) \wedge (q \Rightarrow p)$	$p \Leftrightarrow q$
T	T	T	T	T	T
T	F	F	T	F	F
F	T	T	F	F	F
F	F	T	T	T	T

The **bi-conditional** $p \Leftrightarrow q$, which we read "***p* if and only if *q***" or "***p* is equivalent to *q***" is **true** if both *p* and *q* have the same truth values and **false** if *p* and *q* have opposite truth values.

p	q	$p \Leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

Example 1

Let denote the statements as:

p: The number is divisible by 3

q: The sum of the digits forming the number is divisible by 3.

The compound statement "*The number is divisible by 3 if and only if the sum of the digits forming the number is divisible by three*";

This means that if the sum of the digits forming the number is divisible by 3, then the number is divisible by 3 and if the number is divisible by 3, then the sum of the digits forming the number is divisible by 3.

Symbolically $p \Leftrightarrow q$ means $(p \Rightarrow q) \wedge (q \Rightarrow p)$.

Given the statements: *p*: Two lines are perpendicular and *q*: Two lines form a right angle. Formulate 4 different compound statements and their truth values and deduce the truth table of $p \Leftrightarrow q$.

Solution

One of these compound statements is: "If two lines are perpendicular, then the two lines form a right angle". Make others and you will see the following table.

p	q	$p \Leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

Equivalent statements

Let us compare the truth values of $\neg(p \vee q)$ and $\neg p \wedge \neg q$.

a) Considering the truth values of $\neg p \wedge \neg q$, we can determine the case in which this statement $\neg p \wedge \neg q$ is true.

p	q	$\neg p$	$\neg q$	$\neg p \wedge \neg q$
T	T	F	F	F
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

The first two columns of the truth table give all possible combinations of the truth values of p and q , and the second two columns of the truth table are merely negations of the first two columns. The statement $\neg p \wedge \neg q$ is the conjunction of two statements $\neg p$ and $\neg q$. Therefore, the only case in which $\neg p \wedge \neg q$ is true is when both A and B are false.

b) The truth values of $\neg(p \vee q)$ can be summarized in the truth table:

Like the previous question, the truth table consists of four rows. The third column of the truth table gives the truth values for the disjunction $p \vee q$. The fourth column gives the truth values for the negation of disjunction. Therefore, $\neg(p \vee q)$ is true only when both p and q are false.

p	q	$p \vee q$	$\neg(p \vee q)$
T	T	T	F
T	F	T	F

F	T	T	F
F	F	F	T

Comparing the truth values of $\neg(p \vee q)$ and $\neg p \wedge \neg q$ in the last columns of the two tables, we find that their truth values are identical.

Whenever two statements have the same truth values, the statements are said to be logically equivalent. The symbol of equivalent statements is $\equiv$.

Thus we can write: $\neg(p \vee q) \equiv \neg p \wedge \neg q$.

In spoken language, the equivalence of two propositions means they are similar, when you express one you have the same idea with which expressed the other.

Example 2

Use a truth table to show that the conditional $p \Rightarrow q$ and its contrapositive $\neg q \Rightarrow \neg p$ are logically equivalent.

Solution

Let us construct the truth table containing both $p \Rightarrow q$ and its contrapositive $\neg q \Rightarrow \neg p$.

p	q	$p \Rightarrow q$	$\neg q$	$\neg p$	$\neg q \Rightarrow \neg p$
T	T	T	F	F	T
T	F	F	T	F	F
F	T	T	F	T	T
F	F	T	T	T	T

The order of $\neg q$ and $\neg p$ when organizing the columns of the table must be respected.

It is clear that $p \Rightarrow q$ and $\neg q \Rightarrow \neg p$ have the same truth values in the third and the sixth columns.

Note: Two propositions differ if you can find at least one row of the truth table where values differ.

Application activity 1.3.7

1) Suppose that r is a false statement, and s is a true statement.

a) What is the truth-value of the compound statement $(\neg r) \Leftrightarrow s$?

b) What is the truth-value of the compound statement $r \Leftrightarrow (\neg s)$?

c) What is the truth-value of the compound statement $r \Leftrightarrow s$?

d) What is the truth-value of the compound statement $\neg(r \Leftrightarrow (\neg s))$?

2) Construct the truth table for

a) $p \leftrightarrow q$ and $(p \to q) \wedge (q \to p)$

b) $p \leftrightarrow q$ and $(\neg p \vee q) \wedge (\neg q \vee p)$

c) $\neg(p \leftrightarrow q)$ and $(p \vee q) \wedge \neg(p \wedge q)$

d) $\neg(p \leftrightarrow q)$ and $(p \wedge \neg q) \vee (\neg p \wedge q)$

3) Show that neither the converse nor the inverse of an implication are equivalent to the implication.

4) Formulate a conditional statement made by two compound statements p and q. Use that statement to express different forms of $\neg p \vee q$ and $p \Rightarrow q$ and show that $\neg p \vee q$ and $p \Rightarrow q$ are equivalent. Relate to the daily life situation.

1.4 Tautology and contradiction

Activity 1.4

1) Consider the following propositional statements:

   a) It is raining.

   b) Either it is raining, or it is not.

   c) It is both raining and not raining.

   Determine their true value(s) and give comment from your results.

2) Let p and q be two propositions. Construct the truth table of

   a) $p \vee \neg p$ b) $p \wedge (\neg p)$ c) $\neg p \wedge (p \wedge q)$ d) $\neg (p \wedge q) \vee (p \vee q)$

   Do you find a column in which the truth value is always true or always false?

Content summary

Most propositional statements are true in some situations, and false in others. But some propositional statements are true in all situations, and others are false in all situations**.**

- A compound statement that is always **true** regardless of the truth values of the individual statements substituted for its statement variables is called a **tautology**.
- A **contradiction** is a compound statement that is false in all situations; that is, it is false for all possible values of its variables.

Example 1

The statement **"I will either get paid or not get paid"** is a tautology since it is always true.

We can use *p* to represent the statement "**I will get paid**" and not *p* (written ¬*p*) to represent "**I will not get paid**."

p : I will get paid

$\neg p$: I will not get paid

So, $p \vee (\neg p)$: I will either get paid or not get paid

A truth table for the statement would look like:

p	$\neg p$	$p \vee (\neg p)$
T	F	T
F	T	T

Looking at the final column in the truth table, one can see that all the truth values are T (for true). Whenever all of the truth values in the final column are true, the statement is a tautology.

So, our statement 'I will either get paid or not get paid' is always a true statement, a tautology.

Example 2

1) The statement "**I don't believe in reincarnation, but I did in my past life**", is always false.

2) The statement $p \wedge (\neg p)$ is always false, because p and $\neg p$ cannot both be true.

P	$\neg p$	$p \wedge \neg p$
T	F	F
F	T	F

Application activity 1.4

1) From the following compound statements, indicate which is tautology or contradiction

a) $p \wedge \neg(p \wedge q)$ b) $\neg q \wedge (q \wedge r)$ c) $(p \wedge q) \wedge \neg(p \vee q)$

d) $(p \vee r) \vee \neg r$ e) $p \wedge [(\neg q) \vee (\neg r)] \Rightarrow [p \Rightarrow (\neg q)]$

2) Show that the statement $(p \vee q) \wedge [(\neg p) \wedge (\neg q)]$ is a contradiction.

3) Show that the statement $[(p \vee q) \Rightarrow r] \vee [\neg(p \vee q) \Rightarrow r]$ is a tautology.

4) In your own words, formulate

   a) 2 propositional statements expressing the tautology and show where these tautologies can be avoiding in every day practice of argumentation.

   b) 2 propositional statements expressing the contradiction and show where these contradictions can be avoided in every day practice of argumentation.

1.5 Quantifiers and their negation

1.5.1 Universal quantifier "for all"

Activity 1.5.1

1) Let $A = \{1, 2, 3, 4, 5\}$. Determine the truth value of each of the following statements:
   a) For all $x \in A, x+3 < 9$
   b) For all $x \in A, x+3 \leq 7$
2) Determine the truth value of the following statement where $U = \{1, 2, 3\}$ is the universal set: There exist $x \in U$ such that $x+3 \leq 5$
3) What is the truth value of *p*, *q* if *p*: "All men eat banana", *q*: All men are

Content summary

Let *p(x)* be a propositional function defined on a set *A*. Consider the expression $(\forall x \in A)\ p(x)$ or $\forall x\ p(x)$ which is read by **"For every x in A, $p(x)$ is a true statement"** or, simply, **"For all x, $p(x)$."** The symbol universal quantifier "For all" symbolized by $\forall$ is read by **"for all"** or **"for every"** is called the ***universal quantifier***. The statement $(\forall x \in A)\ p(x)$ means that the truth set of $p(x)$ is the entire set A.

Example 1

All men are mortal. This is true if for every man, dying is applicable.

The expression $p(x)$ by itself is an open sentence or condition and therefore has no truth value. However, $(\forall x \in p(x)$, that is $p(x)$ proceeded by the quantifier $\forall$, does have a truth value which follows from the equivalence.

Specifically: Q_1: if $\{x / x \in A, p(x)\} = A$ **then** $\forall x\ p(x)$ **is True; otherwise** $\forall x\ p(x)$ **is False.**

Example 2

(a) The proposition $(\forall n \in \mathbb{N})(n+4 > 3)$ is true since $(n / n+4 > 3\} = \{1, 2, 3, ..\} = \mathbb{N}$

(b) The proposition $(\forall n \in \mathbb{N})(n+2 > 8)$ is false since $(n / n+2 > 8\} = \{7, 8, ...\} = \mathbb{N}$

Application activity 1.5.1

1) Given the statement p: "for every real number x, $x+y>10$." discuss the values of y which make p true.
2) Give the truth value for the following statements:
   a) Some rectangles are squares
   b) All squares are rectangles
   c) Every language student-teacher must take a logic mathematics course.
   d) Given $\forall x \in \mathbb{N}, x+4>4$.

1.5.2 Existence quantifier "there exists"

Activity1.5.2

1) Let $A=\{1,2,3,4,5\}$. Determine the truth value of each of the following statements:

a) There exist $x \in A,\ x+3=10$

b) *For some* $x \in A, x+3<5$

2. Let $A=\{1,2,3,...,8,9,10\}$. Consider each of the following sentences; determine the set of y for which the following statement is true. Such set is called a truth set.

a) $(\forall x \in A)(\exists y \in A)(x+y<14)$

b) $(\exists y \in A)(x+y<14)$.

Content summary

Let $p(x)$ be a propositional function defined on a set A. Consider the expression: there exists some $x \in A$ that satisfy $p(x)$.

This can be written as $(\exists x \in A)\, p(x)$ or $\exists x, p(x)$ or "There exists an x in A such that $p(x)$ is a true statement" or, simply, "For some x, $p(x)$."

The quantifier "there exist" or "for some" or "for at least one" symbolized by *∃ is an existential quantifier*. The statement $(\exists x)\, p(x)$ is read as "there exists an x such that $p(x)$ is true".

Specifically:

Q_2 : if $\{x / p(x)\} \neq 0$, then $\exists x\, p(x)$ is True; otherwise $\exists x\, p(x)$ is False,.

Example

(a) The proposition $(\exists n \in \mathbb{N})(n+4<7)$ is true since $\{n / n+4<7\} = \{1,2,\} \neq \varnothing$

(b) The proposition $(\exists n \in \mathbb{N})(n+6<4)$ is false since $\{n / n+6<4\} = \phi$.

c) Some men do not have wives, is true since some men such as Priests remain single.

d) There exist natural number between 3 and 6. This is statement is true because $\{4,5\}$ exists.

e) Some student teachers teach at university. This is false because the set of such students is empty.

Application Activit1.5.2

1. Translate the following into symbolic form:
   a) Somebody cried out for help and called the police.
   b) Nobody can ignore her.
2. Consider the predicates: $r(x): x-7 = 2$ *and* $s(x): x > 9$. If the universe of discourse is the real numbers, give the truth value of propositions: $(\exists x)\, s(x) \wedge \neg(\forall x)\, r(x)$
3. Formulate different statements with existential quantifiers and give their truth values.

1.5.3 Negation of quantifiers

Activity 1.5.3

Negate the following statements

1) All grapefruits have red colour.
2) Some celebrities are beautiful.
3) No one weighs more than five hundred Kilograms.
4) Some people are more than 2m tall.
5) All snakes are poisonous.
6) Some mammals can stay under water for two days without surfacing for air.
7) All birds can fly.

Consider the statement: "All dogs have tails". Its negation is "Not all dogs have tails" which means that: "There is at least one dog that does not have a tail."

Let us express these ideas symbolically:

Let $p(x)$ mean "dog x has a tail". Then all dogs have tails is represented by $\forall x, p(x)$ and dog x doesn't have a tail is $\neg p(x)$. So not all dogs have tails is $\neg[\forall x, p(x)]$. Abbreviating there is at least one dog that doesn't have a tail or there exists a tailed dog, we can say that there exists a tailless dog is $\exists x, p(x)$.

Since the last two statements have, in words, the same meaning; we can declare the predicate formulae representing them to be logically equivalent: $\neg[\forall x, p(x)] \equiv \exists x[\neg p(x)]$.

By generalizing the argument that we have used to produce two results, it can be shown that the logical equivalence holds true no matter what predicate is represented by $p(x)$. In fact, this is one of the generalized of the De Morgan's laws. For predicate logic, the other is similar:

$\neg[\exists x, p(x)] \equiv \forall x[\neg p(x)]$ that can be interpreted as "there are no dogs with tails" has the same meaning as "every dog is tailless."

Example 1

In statements that involve the words "all, every, some, none or no", forming the negation is not as easy. The following table shows examples

Statement	**Negation**
All men are mortal	Some men are not mortal
Some of the numbers are not positive	All of the numbers are positive
No birds are fish	Some birds are fish
Some women can jump	No women can jump
None of the small children worked	Some of the small children worked

This shows that*:*

"Negating a universally quantified statement changes it into an existentially quantified statement and vice-versa with the part of the statement after the quantifier becoming negated".

Example 2

Let $p(x)$mean "country x has a president." Interpret the following statements. Establish the equivalence of their negation.

a) $\forall x, p(x)$ b) $\exists x, p(x)$

Solution

a) Since $p(x)$mean "country x has a president, thus $\forall x, p(x)$ means "all countries have president" The negation is $\neg[\forall x, p(x)]$that is interpreted as "it is not true that all counties have presidents."

This sentence can be express as “there exist a country without a president” and symbolically it is written as $\exists x\left[\neg p(x)\right]$.

Thus, $\neg\left[\forall x, p(x)\right] \equiv \exists x\left[\neg p(x)\right]$

b) $\exists x, p(x)$ means “there is a country with a president.”

Negating the given statements, we get

$\neg\left[\exists x, p(x)\right]$ which means “it is not true that there is a country with a president.”

This sentence is equivalent to” all countries have not a president” symbolically $\forall x\left[\neg p(x)\right]$.

Hence $\neg\left[\exists x, p(x)\right] \equiv \forall x\left[\neg p(x)\right]$.

These **laws can be summarized** as follow:

“Negating a universally quantified formula changes it into an existentially quantified formula and vice-versa with the part of the formula after the quantifier becoming negated.”

The basic forms for negative statements that involve “all, every, some, none or no” can be summarized as follows:

Statement	**Negation**
All/every	Some .... not.
Some ... not	All/ every
None/no	Some
Some	None/No

Application Activity 1.5.3

Negate each of the following statements and write the answer in symbolic form:

1) Some students are mathematics majors.
2) Every real number is positive, negative or zero.
3) Every good boy does fine.
4) There is a broken desk in our classroom.
5) Lockers must be turned in by the last day of class.
6) Quickness makes waste.

1.6 Applications of logic in real life

1.6.1 Hypothetical syllogism

Activities 1.6.1

It is known that you must have the correct form and **true premises** to reason deductively toward a **true conclusion**.

The following are examples of arguments that need true conclusions. Try to conclude:

a) If you live in Nyarugenge, then you live in Kigali.

If you live in Kigali, then you live in Rwanda.

Therefore, if you live in Nyarugenge, ...

b) If you study the whole student book, then you will pass the exam.

If you pass the exam, you will be eligible to sponsorship.

Therefore, if you study the whole student book, ...

c) If God created the universe, then the universe will be perfect. If the universe is perfect, then there will be no evil. So, if God created the universe, ...

Content summary

A hypothetical syllogism is a valid argument form in logic. It is a special type of syllogism in which one or both of the premises are conditional (Hypothetical), usually beginning with the word "**if**".

The form of hypothetical syllogism is: "If p, then q. If q, then r. Therefore, if p, then r." It may also be written as:

$$p \Rightarrow q$$
$$q \Rightarrow r$$
$$\therefore p \Rightarrow r$$

p, q and r may represent any proposition.

The hypothetical syllogism can be written in three different ways:

If a, then b If b, then c Therefore, if a, then c	a implies b b implies c Therefore, a implies c	$a \Rightarrow b$ $b \Rightarrow c$ Therefore, $a \Rightarrow c$

In the hypothetical syllogism, the argument is correct even when one or both of the premises are false. The truth or falsehood of the premises does not affect the logic of the argument. Logic deals with the relationship between premises and conclusion, not the truth of the premises.

To say that a deductive argument is correct means that the premises are related to the conclusion in such a way that, if the premises are true, the conclusion must be true. A conclusion cannot be false if the logical form is correct and the premises are true. The following are examples of the hypothetical syllogism argument form:

Example 1

If it rains, we will not have a picnic.

If we don't have a picnic, we won't need a picnic basket.

Therefore, if it rains, we won't need a picnic basket.

Example 2

If you live in Rwamagana, then you live in Eastern province.

If you live in Eastern province, you live in Rwanda.

Therefore, if you live in Rwamagana, then you live in Rwanda.

Example 3

Is the following argument a hypothetical syllogism? Why or why not?

"If you have a party, you should invite your friends. If you are graduating from college, you should invite your friends. Therefore, if you are having a party, you are graduating from college."

Solution

The argument is not a hypothetical syllogism. The premises do not link properly. The conclusion of the first premise should be the hypothesis of the second premise, and no logical rearrangement can accomplish the proper linking of the statements.

Example 4

Even though the conclusion of this argument is true, explain why the following argument is a poor one.

If you are over 18 years old, then you can read. If you can read, you can vote. Therefore, if you are over 18 years old, then you can vote.

Solution

The argument has the form of hypothetical syllogism, so it is a correct argument. However, it is a poor argument, since neither of the premises is true; the argument does not actually prove its conclusion.

Application Activity 1.6.1

1) Deduce the conclusion of the following arguments
   a) "If the people elect their own government, then the political system is democratic." "If the political system is democratic, then the economy is prosperous."
   b) "If it snows, then I will study discrete math." "If I study discrete math, I will get an A"
   c) "If I do not wake up, then I cannot go to work." "If I cannot go to work, then I will not get paid."
2) Determine whether or not the following arguments are correct. For those which are not correct,
   a) Explain what is wrong with the argument;
   b) Change the minor premise and make a correct argument.
      (i) All good chess players wear glasses.
      Sylvia is a good chess player.
      Therefore, Sylvia wears glasses.
      (ii) ii) If ABCD is a square, it has four sides.
      If it has four sides, then it is a quadrilateral.
      Therefore, if ABCD is a square, it is a quadrilateral.

1.6.2 Affirming the Antecedent

Activities 1.6.2

Rephrase the following arguments, by completing the gaps

1) "If you were standing out in the rain, then you would be wet now.

   You must have been standing out in the rain; so, you _______

2) "If you have a driver's license, then you must have taken the driver's test. You do have a driver's license. So, you …

Content summary

Affirming the antecedent is a **valid argument** from which proceeds by affirming the truth of the first part (the "if" part, commonly called the antecedent) of a conditional, and concluding that the second part (the "then" part, commonly called the consequent) is true. This form of argument is called by the Latin phrase, "**modus ponens**".

It is written as:

If P, then Q.

P. Therefore, Q

Example 1

If Martha loves sugar, she will enjoy her cake. Martha loves sugar, thus she also loves her cake.

Example 2

If I study for 6 hours, I will pass the exam.

I studied for 6 hours.

Therefore, I will pass the exam.

Example 3

This classical argument is another example of affirming the antecedent.

All men are mortal.

Makuza is a man.

Therefore, Makuza is mortal.

This can be written so that the correct form is apparent.

If one is a man, then one is mortal.

Makuza is a man.

Therefore, Makuza is mortal.

Notice:

If an argument has the correct form, it is a logical argument. However, if it is to be a convincing argument with a true conclusion, its premises must also be true. You can affirm the antecedent to reason deductively if the argument has the correct form and true premises.

Example 4

Is the following argument a good one? Explain.

If you want to run a marathon, then you should train for the race.

Mukamurenzi wants to run a marathon.

Therefore, Mukamurenzi should train for the race.

Solution:

The argument has the correct form for affirming the antecedent. If we take its first premise as true because of commonly accepted notions about the physical stamina needed to run a marathon, the argument is a good one.

Even though the argument has the correct form of an argument using the technique of affirming the antecedent, the major premise is not true. Thus, it is a correct argument but it does not arrive at a true conclusion. You need both the correct form and true premises to ensure true conclusions.

Application Activity 1.5.2

1) From two propositional statements, draw the corresponding table for affirming the antecedent.

2) Give the conclusion, by affirming the antecedent

   a) If anyone is born of God, then he loves his brothers. Adolph is born of God.

   b) If the national debt is paid off with inflated money, then borrowers will benefit and lenders will be harmed. The national debt is being paid off with inflated money.

1.6.3 Denying the Consequent

Activities 1.6.3

Rephrase the following arguments, by completing the gaps

1) "If you have a driver's license, then you must have taken the driver's test.

   You have not taken the driver's test. So, you ________"

2) "If you were standing out in the rain, then you would be wet now.

   You must not be wet now. So, you ________."

Content summary

Denying the consequent is the form of **argument** that states that, given a first thing, a second thing is true, it then denies that the second thing is true. This form of argument is called by the Latin phrase, "**modus tollens**".

If ***A*** and ***B*** represent statements, an argument that denies the consequent has the following form:

Major premise: $A \Rightarrow B$

Minor premise: $\sim B$

Conclusion: $\therefore \sim A$

Example 1

If Keza is at school, then she has notebooks.

Keza does not have notebooks.

Therefore, Keza is not at the school.

Example 2

If you pay the bill on time, then you are not charged a penalty.

You are charged a penalty.

Therefore, you did not pay the bill on time.

Example 3

If I own a book, then I have read it. I have not read this book.

Example 4

Is the following argument a good one? Explain.

If a number is not positive, then the number is negative.

Five is not negative.

Therefore, five is positive.

Solution

The argument has the form of an argument using denying the consequent, so it is a correct argument. The conclusion is correct but the premise is not true. It should be "a number is not positive or zero".

Application activity 1.6.3

1) Give the conclusion, by denying the consequence
   a) If Bwenge is competent, she will get a promotion.
   b) If I have cash, I will loan your money. I am not loaning your money; So….
   c) If a world government is established, wars will cease. Wars are not ceasing. Therefore, ...
2) Determine whether or not the following arguments are correct. For those which are not correct,
   a) Explain what is wrong with the argument;
   b) Change the minor premise and make a correct argument.

      i. When it is midnight, I am asleep.
      I was asleep.
      Therefore, it was midnight.

      ii. All Rhode Island Red hens lay brown eggs.
      My hen, Motopa, is a Rhode Island Red.
      Therefore, Motopa lays brown eggs

      iii. If a triangle is equilateral, then it has three equal sides.
   ABC does not have three equal sides.
   Therefore, ABC is not equilateral.

1.6.4 Valid argument

Activity 1.6.4

From the two Premises:

(1) All college students are intelligent.

(2) All first-year students are college students

What should be the conclusion about first year students?

From this activity, the answer or the conclusion should be: “All first-year students are intelligent”.

In an argument or a proof we are not concerned with the *truth* of the conclusion but rather with whether the conclusion does or does not follow from the premises. If the conclusion follows from the premises, we say that our reasoning is *valid;* if it does not, we say that our reasoning is *invalid*.

Now the last statement (“All first-year students are intelligent”) certainly is not regarded generally as true, but the reasoning leading to it is valid.

If both of the premises are true, the conclusion is also true.

An **argument** or **proof** consists of a set of propositions $p_1, p_2, ..., p_n$ called the **premises** or **hypotheses,** and a proposition q, called the **conclusion.** An argument is **valid** if and only if the conclusion is true whenever the premises are all true.

An argument that is *not* valid is called a **fallacy** or an **invalid** argument.

An argument is *valid* if the truth of the premises logically guarantees the truth of the conclusion. The following argument is valid, because it is impossible for the premises to be true and the conclusion nevertheless to be false:

Elizabeth owns either a Honda or a Saturn.
Elizabeth does not own a Honda.
Therefore, Elizabeth owns a Saturn.

In a **direct proof** we go through a chain of propositions, beginning with the hypotheses and leading to the desired conclusion.

Example: Suppose each of the following statements is true:

Either John obeyed the law or John was punished, but not both

and

John was not punished.

Prove that John obeyed the law.

Solution: Let p and q denote the propositions

p: John obeyed the law.

q: John was punished.

The premises can be symbolized as $(\boldsymbol{p} \veebar \boldsymbol{q})$ and $\neg \boldsymbol{q}$

Since $\neg \boldsymbol{q}$ is true, then, by the Law of Contradiction, q is false.

Since $(\boldsymbol{p} \veebar \boldsymbol{q})$ is true, either p is true or q is true.

Since q is false, p must be true. So, John obeyed the law.

Valid Argument forms

"There are a great many valid argument forms, but as we saw it here above, we consider only four basic ones. They are basic in the sense that they occur in everyday use, and that all other valid argument forms can be derived from these four forms:

a) Affirming the Antecedent

If p then q.
p.
Therefore, q.

b) Denying the Consequent

If p then q.
Not-q.
Therefore, not-p.

c) Chain Argument

If p then q.
If q then r.
Therefore, if p then r.

d) Disjunctive Syllogism

Either p or q.
Not-p.
Therefore, q.

Whenever we find an argument whose form is identical to one of these valid argument forms, we know that it must be a valid argument."

Application activity 1.6.4

Determine whether the arguments are valid and justify your reasoning.

Hypotheses: When students study, they receive good grades.

These students do not study.

Conclusion: These students do not receive good grades.

Solution (in the TG):

The argument is invalid.

If the hypothesis $\boldsymbol{p \Rightarrow q}$ and $\neg \boldsymbol{p}$ are both true, then p is false.

In an implication, if p is false, q can be either true or false, so the conclusion $\neg \boldsymbol{q}$ could be either true or false.

1.7 End of unit assessment

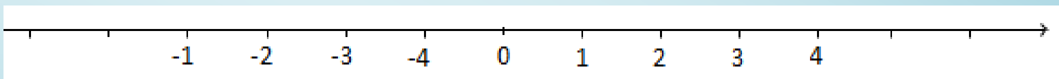
1) Which of the following sentences are propositions?
   a) Pretoria is the capital of South Africa
   b) Is this concept important?
   c) Wow, what a day!
2) Find the negation of the proposition “Today is Monday”
3) Find the conjunction of the propositions p and q, where p is the proposition “Today is Sunday” and q is the proposition “The moon is made of cheese”.
4) Find the disjunction of the propositions p and q, where p is the proposition “Today is Sunday” and q: “The moon is made of cheese”.
5) How many rows, not counting the top one, are needed to construct the truth table for a compound statement made from n statements?
6) Show that $\left[p\wedge\left(p\to q\right)\right]\to q$ is a tautology.
7) Find the truth value of the bi-conditional “The moon is made of cheese if and only if 1=2”.
8) Construct the truth table for the proposition $\left(\neg p\to q\right)\wedge r$

UNIT 2: POINTS, LINES AND GEOMETRIC SHAPES IN 2D

Key Unit competence: Represent geometric shapes in 2D and calculate their area.

2.0. Introductory activity 2

Rukundo uses a number line to locate the points with x-coordinates -4, -2, 3, and 4. Her classmate Isimbi notices that -4 is closer to zero than -2 as shown in the figure below.



Isimbi told him that he is mistaken. Rukundo replied that his is not sure about his diagram because he did not have time to repeat his course yesterday evening.

a) Use what you know about a vertical number line to determine if Rukundo made a mistake or not. Support your explanation with a number line diagram.

b) What is number line?

c) What is Cartesian plane?

2.1. Cartesian coordinates of a point

Activity 2.1

Consider the points $A(1,2)$ and $B(5,4)$.

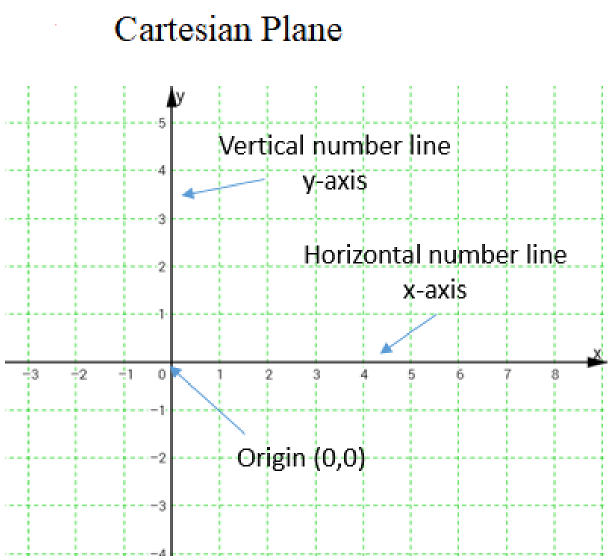
1) Represents these points in xy -plane

2) Draw a line segment from A to B

Content summary

In two dimensions, a point is represented by Cartesian coordinates (also called rectangular coordinates). Cartesian coordinates are a pair of numbers that specify signed distances from the coordinate axes. They are specified in terms of the *x* coordinates and the *y* coordinates. The origin is the intersection of the two axes.

The position of a point on the Cartesian plane is represented by a pair of numbers. The pair is called an ordered pair or coordinates (x, y). The first number, x, called the x-coordinate and the second number, y, is called the y-coordinate.



The origin is indicated by the ordered pair or coordinates (0, 0).

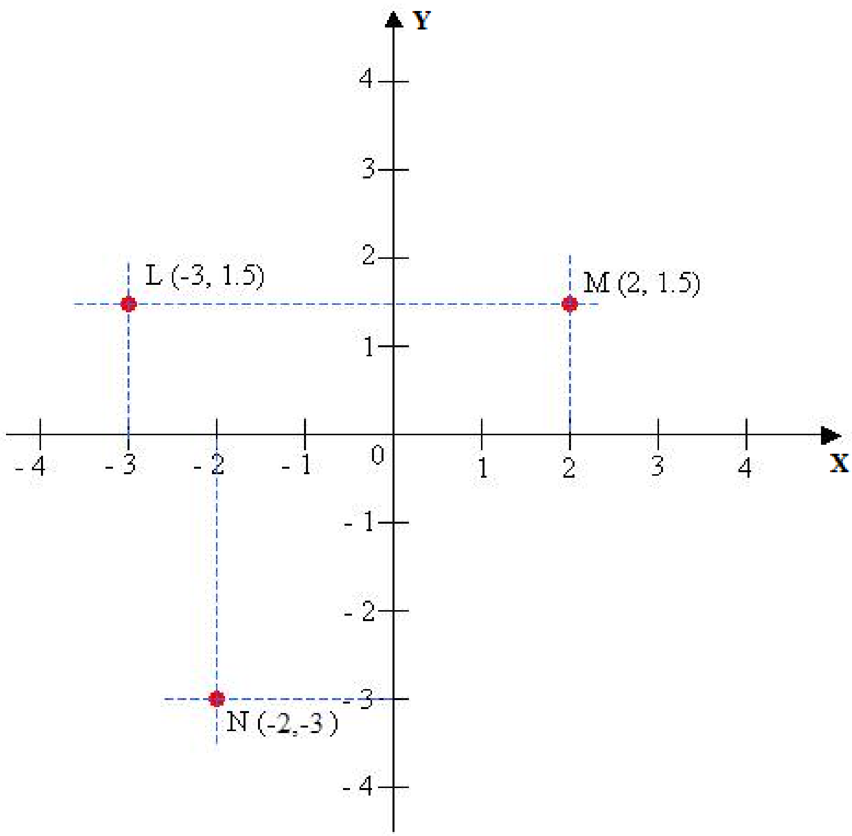
To get to the point (x, y) in the Cartesian plane, we start from the origin. If x is positive, then we move x units right from the origin otherwise if x is negative then we move x units left from the origin. Then, if y is positive, we move y units up otherwise if y is negative, we move y units down

Example:

Represent the points $M(2,1.5)$, $L(-3,1.5)$ and $N(-2,-3)$ in the Cartesian plane.

Solution

- Point M has coordinates $M(2,1.5)$. To get to point M, we move 2 units to the right of x-axis from the origin (positive side) and 1.5 units up on y-axis from the origin (positive side).
- Point L is represented by the coordinates $L(-3,1.5)$. To get to point L, we move 3 units to the left of x-axis from the origin (negative side) and 1.5 units up on y-axis from the origin (positive side).
- Point N has coordinates $N(-2,-3)$. To get to point N, we move 2 units to the left of the origin on x-axis (negative) and 3 units down on y-axis from the origin (negative side).



Points on Coordinate Plane

Application activity 2.1

Represent the points $A(-3,2)$, $B(4,2)$, $C(-3,-2)$, $D(3,-1)$ in $xy-plane$.

2.2. Distances between two points

Activity 2.2

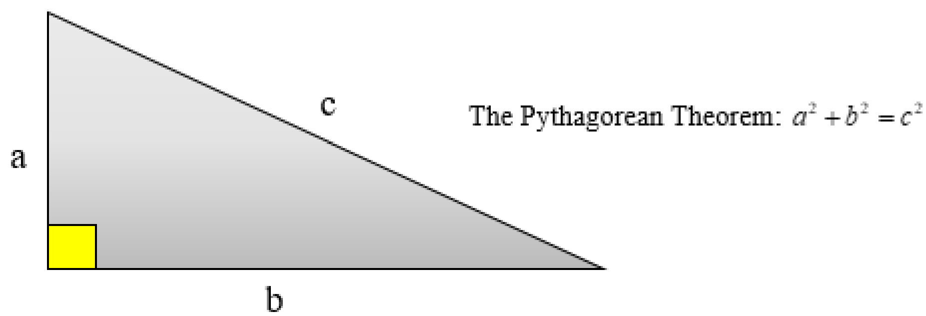
Kalisa and Mugisha live in the same Sector but at two different hills. Kalisa's house is located at $A(1,2)$ from the Sector and Mugisha's house is at $A(4,6)$.

a) If the office of the sector is considered as the origin, present this situation on Cartesian plan.

b) Use the ruler to calculate the distance between Kalisa's house and Mugisha's house.

Content summary

Recall from the Pythagorean theorem that, in a right triangle, the hypotenuse ***c*** and sides ***a*** and ***b*** are related by $a^2+b^2=c^2$. Conversely, if $a^2+b^2=c^2$, the triangle is a right (see figure below).



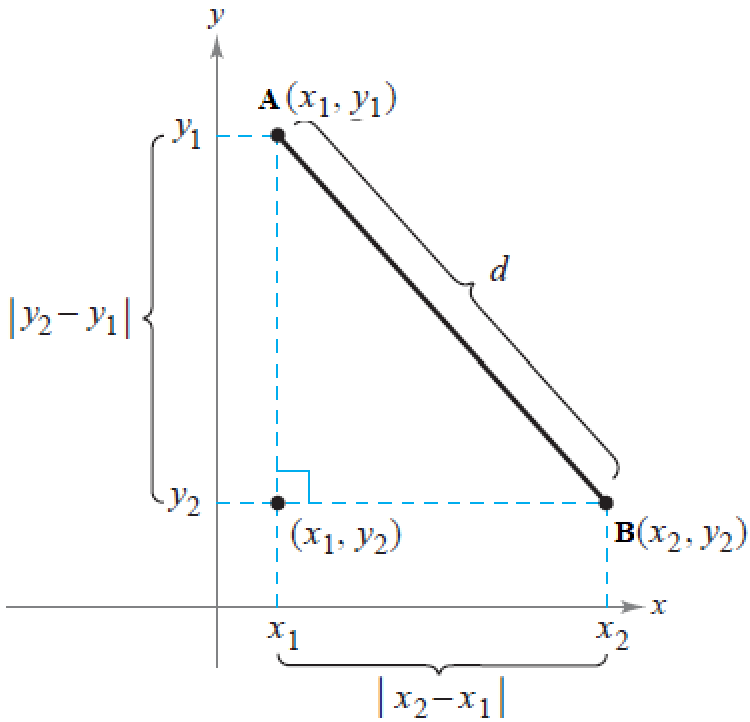
If A and B are two points on xy- coordinates, we can form a vector $\overrightarrow{AB}$ and the distance between these two points denoted $d(A,B)$ is given by $\left\|\overrightarrow{AB}\right\|$.

Suppose you want to determine the distance "d" between the two points (x_1,y_1) and (x_2,y_2) in the plane.

If the points lie on a horizontal line, then $y_1=y_2$ and the distance between the points is $d=|x_2-x_1|$.

If the points lie on a vertical line, then $x_1 = x_2$ and the distance between the point is $d = |y_2 - y_1|$.

If the two points do not lie on a horizontal or on a vertical line, they can be used to form a right triangle, as shown in the figure below.



The length of the vertical side of the triangle is $|y_2 - y_1|$, and the length of the horizontal side is $|x_2 - x_1|$. By Pythagorean theorem, it follows that:

$$d^2(A,B) = (x_2 - x_1)^2 + (y_2 - y_1)^2 \quad \text{and} \quad d(A,B) = \|\overline{AB}\| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

Thus, if $A(x_1, y_1)$ and $B(x_2, y_2)$ are points of the Cartesian plane, then

The distance "d" between the points $A(x_1, y_1)$ and $B(x_2, y_2)$ in the Cartesian plane, is given by: $d(A,B) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

Example 1

Consider the points $A(1,4), B(-2,-3)$ in **Cartesian** plane. Find the distance between the point A and B.

Solution

The distance is

$$d(A,B) = \|\overrightarrow{AB}\| = \sqrt{(-2-1)^2 + (-3-4)^2}$$
$$= \sqrt{9+49}$$
$$= \sqrt{58} \quad units$$

Example 2

Consider the points $C(k,-2)$ and $D(0,1)$ in Cartesian plane. Find the number k if the distance between the point A and B is 5 units.

Solution

$$d(C,D) = \sqrt{k^2+9}$$

$$\sqrt{k^2+9} = 5$$
$$\Leftrightarrow k^2 + 9 = 25$$
$$\Leftrightarrow k^2 = 16 \Rightarrow k = \pm 4$$

Thus the value of k is -4 or 4

Application activity 2.2

1. Calculate the distance between the points given below:

   a) $S(-2,-5)$ and $Q(7,-2)$ b) $A(2,7)$ and $B(-3,5)$ c) $A(x,y)$ and $B(x+4, y-1)$

2. The length of $CD = 5$. Find the missing coordinate if:

   a) $C(6, -2)$ and $D(x, 2)$. b) $C(4, y)$ and $D(1, -1)$.

2.3. Midpoint of a line segment

Activity 2.3

Three friends: Pascal, Steve, and Benjamin live on the same side of a street from Nyabugogo to Ruyenzi. Steve's house is halfway between Pascal's and Benjamin's houses. If the locations of Pascal can be given by the coordinates (2,5), Benjamin's house at (4,7);

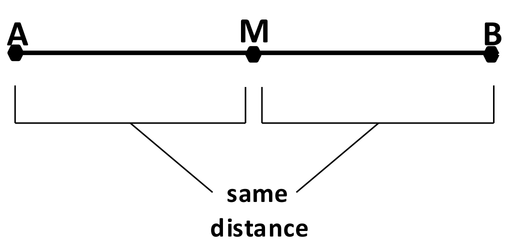
a) Draw a Cartesian plan and locate Pascal's and Benjamin's houses
b) Show the line segment joining the locations of Pascal's and Benjamin's houses;
c) Given that Steve's house is in the half way, locate his house, estimate the coordinate of that location and explain how to find it.

Content summary

A line segment from point *A* to point *B*, denoted $[AB]$ is the set of all points on the part of the line that joins *A* and *B*, including *A* and *B*. The midpoint of this segment is point *M* such that the distance $[AM]=[MB]$ and it is given by $M=\frac{1}{2}(A+B)$.

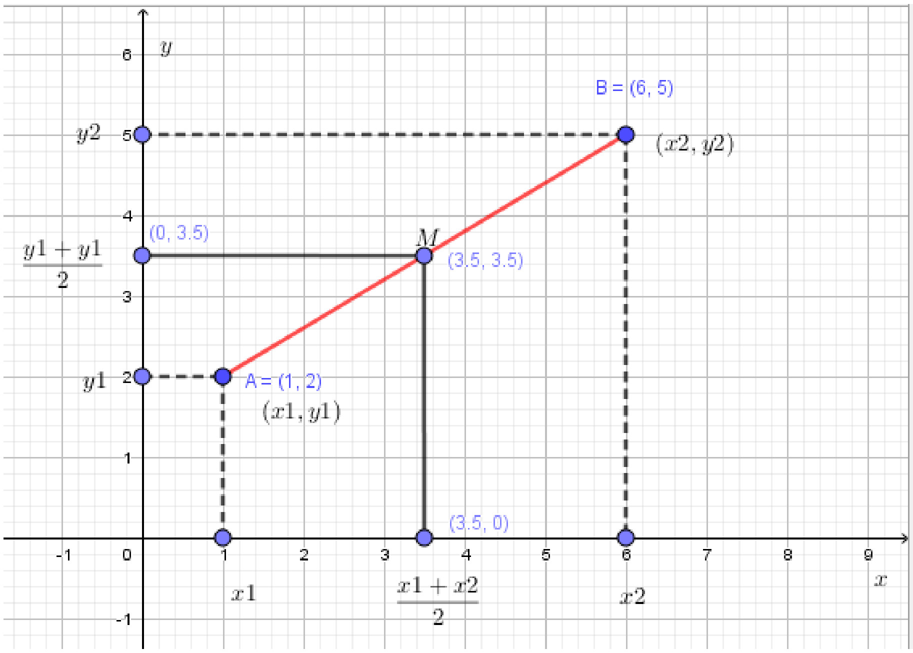
The midpoint of a segment is a point on that line segment which maintains the same distance from both of the endpoints of that line segment.

For example, consider segment to the left.



It has endpoints named A and B. The midpoint of the segment is labelled M. It is at the same distance from each of the endpoints.

Suppose that the coordinates of the line segment $\overline{AB}$, are $A(x_1, y_1)$ and $B(x_2, y_2)$.



The coordinates of the midpoint, are obtained using the midpoint formula, such that:

The midpoint formula:

Coordinates of midpoint = $\left(\frac{x_1+x_2}{2},\frac{y_1+y_2}{2}\right)$

Example 1

Find the midpoint of the segment joining points $A(3,0)$ *and* $B(1,8)$.

Solution

The midpoint is $M=\frac{1}{2}(A+B)=\frac{1}{2}(4,8)=(2,4)$.

Experimental version

Example 2

If $(-3,5)$ is the midpoint of $(2,6)$ and (a,b), find the value of a and b.

Solution

$$(-3,5)=\frac{1}{2}(2+a,6+b)$$
$$\Leftrightarrow(-3,5)=\left(\frac{2+a}{2},\frac{6+b}{2}\right) \Rightarrow \begin{cases}\frac{2+a}{2}=-3\\ \frac{6+b}{2}=5\end{cases} \Leftrightarrow \begin{cases}2+a=-6\\ 6+b=10\end{cases} \Rightarrow \begin{cases}a=-8\\ b=4\end{cases}$$

Application activity 2.3

Michael and Sarah live in different cities and one day they decided to meet up for lunch. Because they both wanted to travel as little as possible they decided to meet at a point halfway between their homes. If their positions are given by $(3100,500)$ and $(5120,125)$.

Which of the following coordinates represents the place where they should meet?

a) $(4110,312.5)$ b) $(4110,375)$ c) $(2020,375)$ d) $(8220,625)$

2.4. Vector in 2D and dot product

2.4.1. Vectors in 2D

Activity 2.4.1

In $xy-plane$:

1) Represent the points $A(1,2)$ and $B(-3,1)$
2) Draw arrow from point A to point B
3) Refer to the content of S2 and represent the Vector $\vec{AB}$

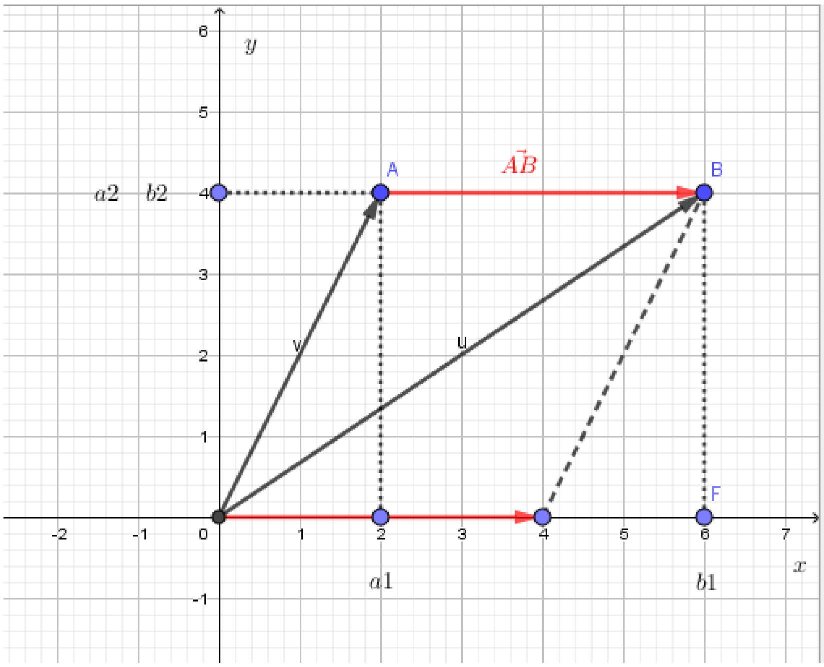
Content summary

Definitions and operations on vectors

A vector is a directed line segment; it is a quantity that has both magnitude and direction. A vector is represented by an arrow. The length of the arrow represents the **magnitude** of the vector, and the arrowhead indicates the **direction** of the vector. That is to say, a vector has a given length and a given direction.

The vector joining the point *A* and the point *B* is denoted by $\overrightarrow{AB}$ and to find its position, we subtract the coordinates of point *A* from the coordinates of point *B*.

For example the vector $\overrightarrow{AB}$ defined by two points $A(a_1,a_2)$ and $B(b_1,b_2)$



On the figure, it is clear that the vector

$$\vec{AB} = \vec{OB} - \vec{OA} = (b_1, b_2) - (a_1, a_2) = (b_1 - a_1, b_2 - a_2) = \begin{pmatrix} b_1 - a_1 \\ b_2 - a_2 \end{pmatrix}$$

Therefore, $\overrightarrow{AB} = (b_1 - a_1, b_2 - a_2)$, which can also be written in the form of a column vector $\vec{AB} = \begin{pmatrix} b_1 - a_1 \\ b_2 - a_2 \end{pmatrix}$.

The point A is called the **initial point** or tail of $\overrightarrow{AB}$ and B is called the **terminal point** or **tip**.

The **zero vector** is $(0,0)$ is denoted by $\vec{0}$.

Example 1

The vector defined by the point $A(1,2)$ and $B(4,3)$ is $\overrightarrow{AB} = (3,1)$

Example 2

The vector $\overrightarrow{CD} = (-4,2)$ is defined by the point $C(-3,4)$ and D.

Find the coordinates of the point D.

Solution

Let the point D be $D(x, y)$, then $\overrightarrow{CD} = (x+3, y-4) = (-4,2)$

$x+3=-4 \Rightarrow x=-7$
$y-4=2 \Rightarrow y=6$

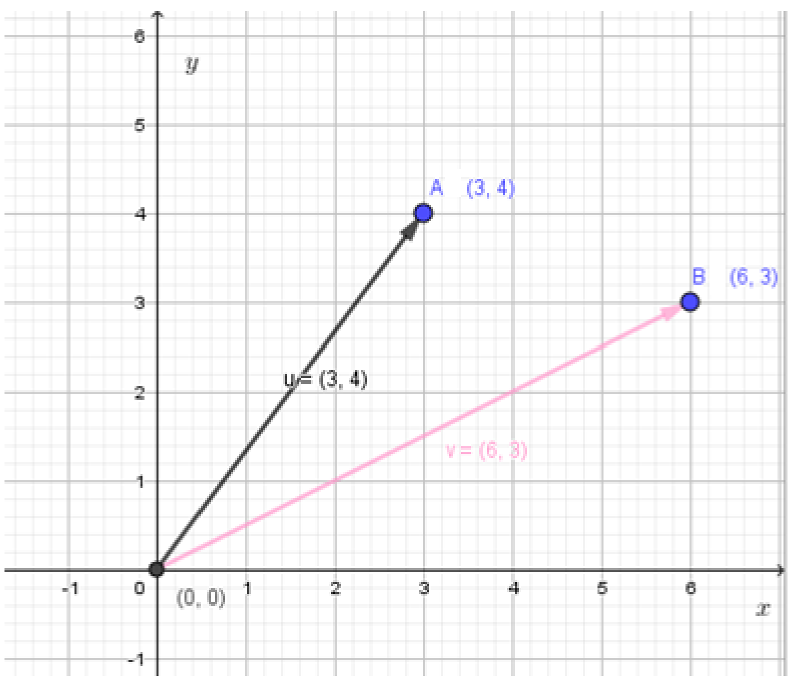
Thus, the point D has the coordinates (-7, 6).

Position Vector

In a Cartesian plane, an **algebraic vector** $\vec{v}$ is represented as $\vec{v}=(a,\mathrm{b})$ where a and b are real numbers (scalars) called the **components** of the vector $\vec{v}$.

We use a rectangular coordinate system to represent algebraic vectors in the plane.

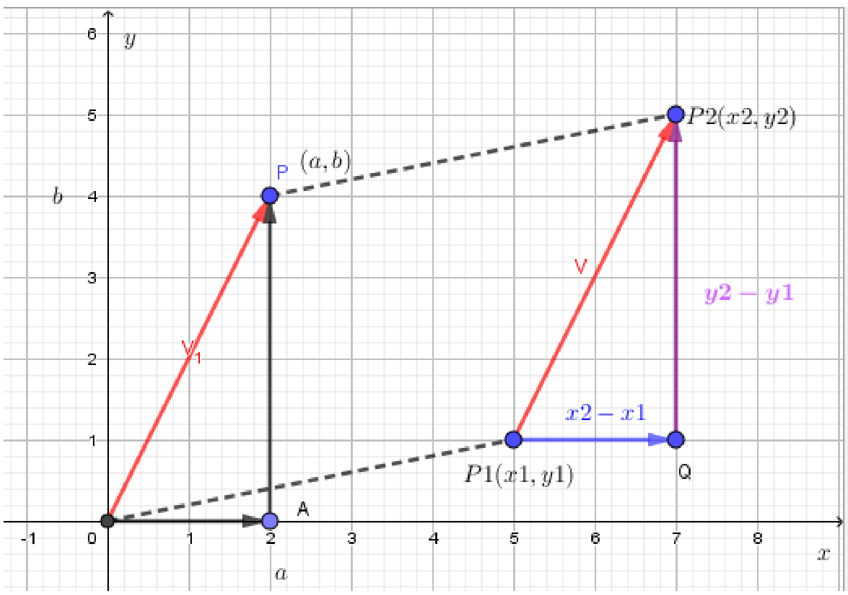
The vector $\vec{v}$ is called a **position vector** if $\vec{v}=(a,\mathrm{b})$ is an algebraic vector whose initial point is at the origin. See the figure below.



Notice that the terminal point of the position vector $\vec{v}=(a,\mathrm{b})$ is $\mathrm{A}(a,\mathrm{b})$. On the figure we have $\vec{u}=(3,4)$ and $\vec{v}=(6,3)$.

The next result states that any vector whose initial point is not at the origin is equal to a unique position vector. Suppose that $\vec{v}$ is a vector with **initial point** $P_1(x_1, y_1)$ not necessarily the origin, and **terminal point** $P_2(x_2, y_2)$.

If $\vec{v} = \overrightarrow{P_1P_2}$, then $\vec{v}$ is equal to the vector $\vec{v} = \overrightarrow{P_1P_2} = (x_2 - x_1, y_2 - y_1)$. See the figure below.



Triangle OPA and triangle P_1P_2Q are congruent. This is because the line segments have the same magnitude. So, $d(0, \mathrm{P}) = d(\mathrm{P}_1, P_2)$; and they have the same direction, so the angle $\angle POA = \angle P_2P_1Q$.

Since the triangles are right triangles, we have angle–side–angle congruence.

It follows that corresponding sides are equal. As a result, $x_2 - x_1 = a$ and $y_2 - y_1 = b$, and so $\vec{v}$ may be written as $\vec{v} = \overrightarrow{P_1P_2} = (a, b) = (x_2 - x_1, y_2 - y_1)$.

Because of this result, we can replace any algebraic vector $\vec{v} = \overrightarrow{P_1P_2} = (a, b) = (x_2 - x_1, y_2 - y_1)$ by a unique **position vector** $\vec{V_1}$ with **initial point** the origin $O(0, 0)$ and **terminal point** $P(a, b)$ and vice versa. On the figure above we have: $\vec{V_1} = (2, 4)$;

$P_1(5,1);\ P_2(7,5)$ *and* $\vec{V} = \overrightarrow{P_1P_2} = (7-5, 5-1) = (2,4)$.

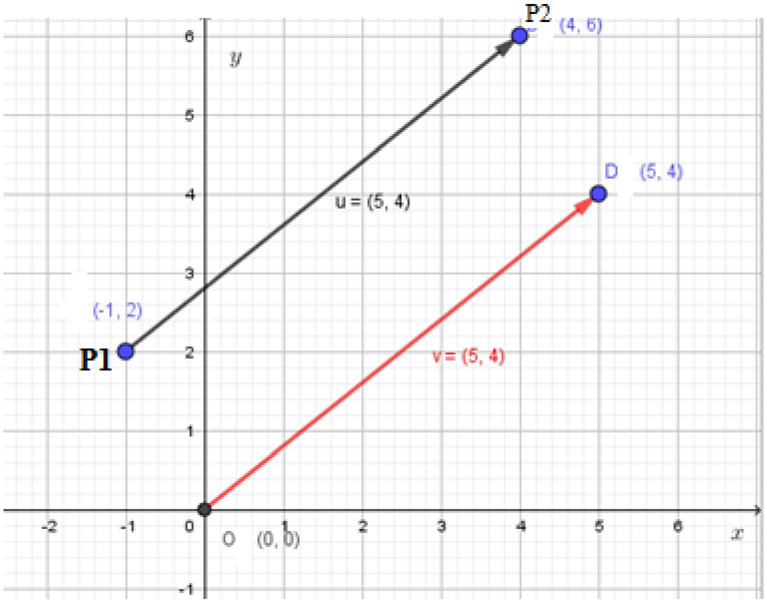
This shows that the vector $\vec{V}_1 = \vec{V} = (2,4)$ which means that $\vec{V}_1 = (2,4)$ is the position vector of the vector $\vec{V}$.

Example 3

Find the position vector of the vector $\vec{v} = \overrightarrow{P_1P_2}$ if $P_1(-1,2)$ and $P_2 = (4,6)$.

Solution

$\vec{v} = \overrightarrow{P_1P_2} = P_2 - P_1 = (4-(-1), 6-2) = (5,4)$.



Components of a vector

We now present an alternative representation of a vector in the plane that is common in the physical sciences. Let $\vec{i}$ denote the unit vector whose direction is along the positive *x*-axis; let $\vec{j}$ denote the unit vector whose direction is along the positive *y*-axis.

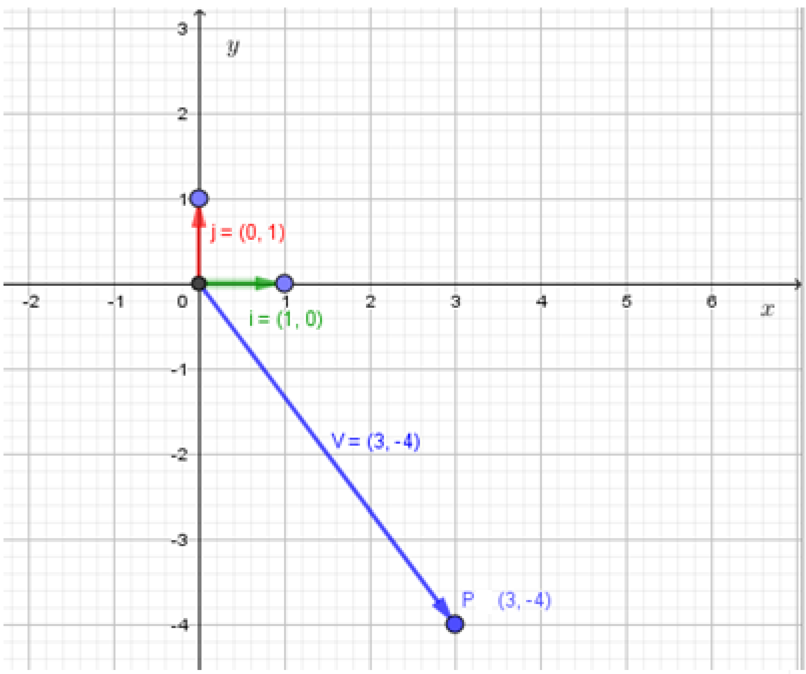
Then $\vec{i} = (1,0)$ and $\vec{j} = (0,1)$.

Any position vector $\vec{v} = (a,b)$ can be written using the unit vectors $\vec{i}$ and $\vec{j}$ as follows:

$$\vec{v}=(a,b)=a(1,0)+b(0,1)=a\vec{i}+b\vec{j}$$

We call a and b the **horizontal** and **vertical components** of the position vector $\vec{v}=(a,b)$, respectively.

For example, if $\vec{v}=(3,-4)=3\vec{i}-4\vec{j}$, then 3 is the **horizontal component** and -4 is the **vertical component**.



Note: It is now easy to show that if we have two points $A(a_1,a_2)$ and $B(b_1,b_2)$, the vector

$$\vec{AB}=\vec{OB}-\vec{OA}=(b_1,b_2)-(a_1,a_2)=(b_1-a_1,b_2-a_2)$$

Since,

$$\vec{OA}=a_1\vec{i}+a_2\vec{j},\ \vec{OB}=b_1\vec{i}+b_2\vec{j}$$

$$\vec{AB}=\vec{OB}-\vec{OA}=\left(b_1\vec{i}+b_2\vec{j}\right)-\left(a_1\vec{i}+a_2\vec{j}\right)$$

$$=b_1\vec{i}+b_2\vec{j}-a_1\vec{i}-a_2\vec{j}$$

$$=b_1\vec{i}-a_1\vec{i}+b_2\vec{j}-a_2\vec{j}$$

$$=(b_1-a_1)\vec{i}+(b_2-a_2)\vec{j}$$

$\overrightarrow{AB}=(b_1-a_1)\vec{i}+(b_2-a_2)\vec{j}=(b_1-a_1,b_2-a_2)$

Therefore, $\overrightarrow{AB}=(b_1-a_1,b_2-a_2)$ or $\overrightarrow{AB}=\begin{pmatrix} b_1-a_1 \\ b_2-a_2 \end{pmatrix}$

Application activity 2.4.1

1) In xy-plane, present the following vectors:

   a. $\vec{u}=(5,6)$, $\vec{v}=(-1,4)$

   b. $\vec{u}=(3,5)$, $\vec{v}=(-1,6)$

2) Plot a vector with initial point P (1,1) and terminal point Q (8,5) in the Cartesian plane and illustrate its position vector.

2.4.2 Dot product of vectors in 2 D and properties

Activity 2.4.2

1) In xy-plane

   a) Represent the points $A(1,2)$ and $B(-3,1)$

   b) Find the distance between point A to point B

2) Given that the product of two unit vectors is such that $\vec{i}.\vec{i}=1$, $\vec{i}.\vec{j}=0$ and $\vec{j}.\vec{j}=1$, apply the distributive property of multiplication under addition to evaluate the following products:

   a) a) $(1,2)(-3,1)=\left(1\vec{i}+2\vec{j}\right)\left(-3\vec{i}+1\vec{j}\right)=$

   b) $(-4,2).(1,2)=$

   c) From your results, deduce how to calculate $(a_1,a_2).(b_1,b_2)$.

Content summary

Dot product and properties

The dot product or scalar product (or sometimes inner product) is an algebraic operation that takes two coordinate vectors and returns a single number.

Algebraically, it is the sum of the products of the corresponding coordinates of the two vectors.

That is, the **scalar product** of vectors $\vec{u}=(a_1,a_2)$ and $\vec{v}=(b_1,b_2)$ of plane is defined by

$\vec{u}\cdot\vec{v}=a_1b_1+a_2b_2$

Example 1

$(2,4).(10,4)=2.10+4.4=20+16=36$ or

$$(2,4).(10,4)=\left(2\,\vec{i}+4\,\vec{j}\right)\left(10\,\vec{i}+4\,\vec{j}\right)$$

$$=2.10\,\vec{i}\,\vec{i}+2.4\,\vec{i}\,\vec{j}+4.10\,\vec{i}\,\vec{j}+4.4\,\vec{j}\,\vec{j}$$

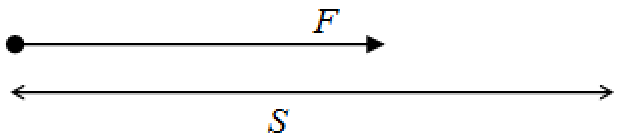
$$=20+0+0+16=36$$

They give the same result.

We can illustrate this scalar product in terms of work done by a force on the body:

Suppose that a person is holding a heavy weight at rest. This person may say and feel he is doing hard work but in fact none is being done on the weight in the scientific sense. Work is done when a force moves its point of application along the direction of its line of action.

Let the force $\vec{F}$ moves its point of application along the displacement $\vec{\boldsymbol{S}}$; If the constant force $\vec{F}$ and the displacement $\vec{\boldsymbol{S}}$ are in the same direction, we define the work W done by the force on the body by $W=\vec{\boldsymbol{F}}\cdot\vec{\boldsymbol{S}}$



Properties of scalar product

a. If $\vec{u}=\vec{0}$ or $\vec{v}=\vec{0}$, then $\vec{u}\cdot\vec{v}=\vec{0}$

b. If $\vec{u}\parallel\vec{v}$ and $\vec{u},\vec{v}$ have same direction, then $\vec{u}\cdot\vec{v}>0$

c. If $\vec{u}\parallel\vec{v}$ and $\vec{u},\vec{v}$ have opposite direction, then $\vec{u}\cdot\vec{v}<0$

d. If $\vec{u}\perp\vec{v}$, then $\vec{u}\cdot\vec{v}=0$

e. $\vec{u}\cdot\vec{v}=\vec{v}\cdot\vec{u}$

f. $\vec{u}\cdot\left(a\vec{v}+b\vec{w}\right)=a\vec{v}\cdot\vec{u}+b\vec{w}\cdot\vec{u}$, $\left(a\vec{u}+b\vec{v}\right).\vec{w}=a\vec{u}.\vec{w}+b\vec{v}.\vec{w}$

g. $\vec{u}\cdot\vec{u}>0$, $\vec{u}\neq\vec{0}$

We define the square of $\vec{u}$ to be $\vec{u}\cdot\vec{u}=\left(\vec{u}\right)^2$

Example 2

The scalar product of the vector $\vec{u}=(2,4)$ and vector $\vec{v}=(-5,0)$ is $\vec{u}\cdot\vec{v}=2(-5)+0=-10$

The square of the vector $\vec{u}=(10,4)$ is $\left(\vec{u}\right)^2=10(10)+4(4)=100+16=116$

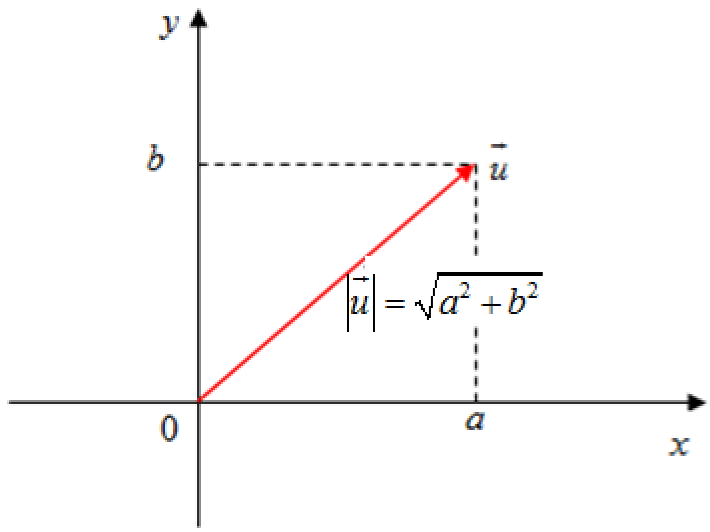
Notice

Two vectors are perpendicular if their scalar product is zero.

Magnitude or Modulus or norm of a vector

The magnitude of the vector $\vec{u}$ noted by $\left\|\vec{u}\right\|$ is defined as its length and is given by the square root of the sum of the squares of its components.

The magnitude of a vector $\vec{u}$ is also noted by $\left|\vec{u}\right|$.



That is $\left\|\vec{u}\right\| = \sqrt{\left(\vec{u}\right)^2}$ or $\left\|\vec{u}\right\|^2 = \left(\vec{u}\right)^2$.

Thus, if $\vec{u} = (a, b)$

$\vec{u} \cdot \vec{u} = \left(\vec{u}\right)^2$

$$\vec{u} = (a, b),\ \vec{u}.\vec{u} = (a, b)(a, b) = a^2 + b^2$$

$$\left(\vec{u}\right)^2 = a^2 + b^2$$

$$\|u\|^2 = a^2 + b^2$$

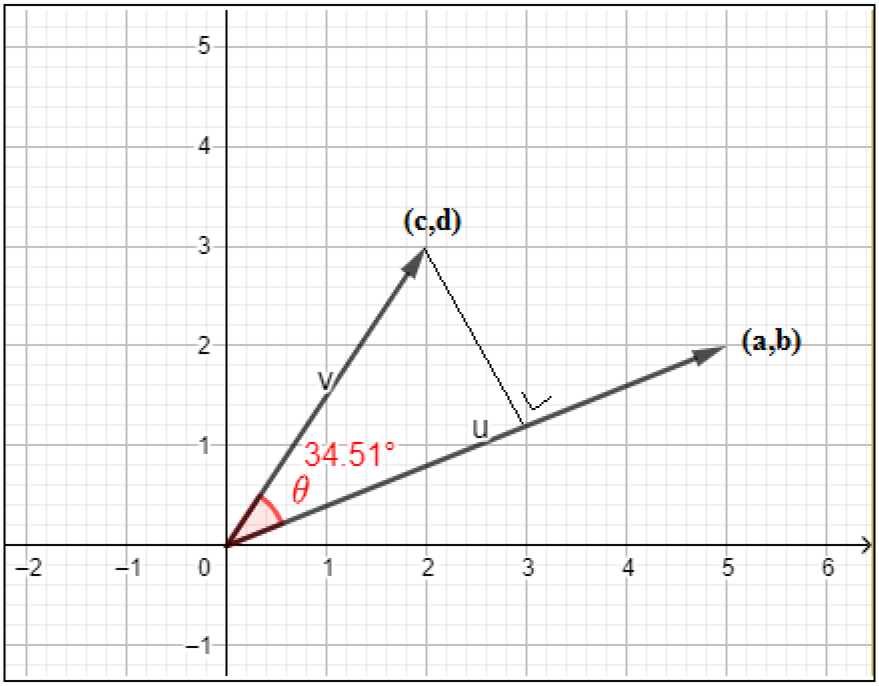
$$\|u\| = \sqrt{a^2 + b^2}$$

Consequences

a) If $\vec{u} = \vec{0}$ then $\left\|\vec{u}\right\| = 0$

b) If k is a real number, $\left\|k\vec{\hat{u}}\right\| = |k|\left\|\vec{\hat{u}}\right\|$.

c) Geometrically, the scalar product of two vectors $\vec{u} = (a,b)$ and $\vec{v} = (c,d)$ of a plane is given by $\vec{u} \cdot \vec{v} = \|\vec{u}\| \|\vec{v}\| \cos\theta$. Where θ is the angle between vectors $\vec{u}$ and $\vec{v}$.



From this relation we have $\cos\theta = \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\| \|\vec{v}\|}$ or $\cos\theta = \frac{a \cdot c + b \cdot d}{\sqrt{a^2 + b^2}\sqrt{c^2 + d^2}}$.

Example 3

Find the norm of the vector $\vec{v} = (3,4)$.

Solution

The norm is $\|\vec{v}\| = \sqrt{9+16} = 5$

Example 4

Find the norm of the vector $\vec{u} = (-1,4)$

Solution

The norm is $\|\vec{u}\| = \sqrt{1+16} = \sqrt{17}$

Application activity 2.4.2

1. Find the norm(magnitude) of the following vectors:
   a) $\vec{u} = (-3, 4)$
   b) $\vec{v} = (3,1)$
2. Consider the following points $A(3,4)$, $B(-2,3)$ and the vectors $\vec{u} = (4,5)$, $\vec{v} = (-3,1)$ in Cartesian plane. Find
   a) vector $\overrightarrow{AB}$
   b) vector $\vec{w} = 2\overrightarrow{AB} - 3\vec{u} + \vec{v}$
   c) the norm of vector $\overrightarrow{AB}$ and norm of vector $\vec{w}$
   d) the scalar product of vector $\vec{u}$ and vector $\vec{v}$
   e) the scalar product of vector $\vec{v}$ and vector $\vec{w}$

2.5. Equation of a straight line

2.5.1. Equation of a straight line passing through a point and parallel to a direction vector

Activity 2.5.1

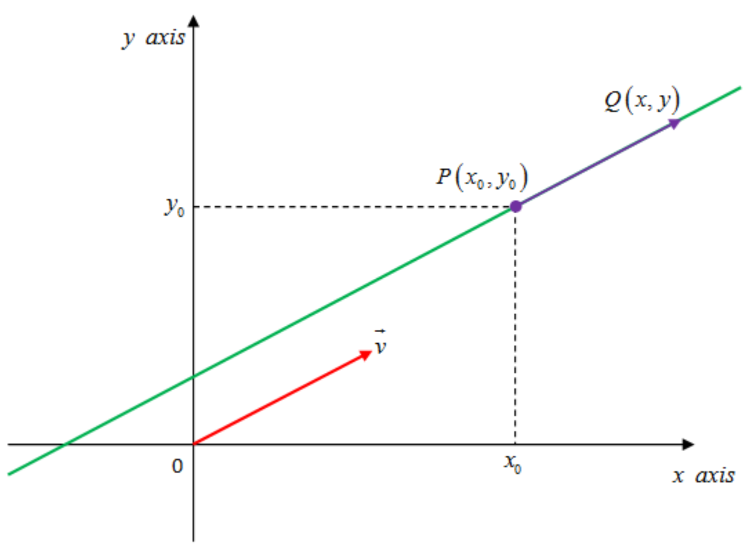
Given the vector $\vec{v} = (2, -3)$.

a) Determine the form of a vector $\vec{w} = (x, y)$ parallel to $\vec{v}$ and passing to the point $P(1, 6)$.

b) Give 2 examples of such vectors $\vec{w}$.

c) If D is a line with the direction vector $\vec{w}$, what should be the equation of D?

To determine the equation of the line passing through the point $P(x_0, y_0)$ and parallel to the direction vector, $\vec{v} = (a, b)$, we will use our knowledge that parallel vectors are scalar multiples. The vector $\overrightarrow{0P}$ is called the **position vector** of *P*.



Thus, the vector through $P(x_0, y_0)$ and any other point $Q(x, y)$ on the line is the product of a scalar and the direction vector $\vec{v} = (a, b)$.

$\overrightarrow{PQ} = r\vec{v}$ with r a parameter and $\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP}$

Hence, the **vector equation** of the line that is parallel to the vector $\vec{v} = (a, b)$ and which passes through the point *P* with position vector $\overrightarrow{0P} = (x_0, y_0)$ is given as

$\overrightarrow{OQ} = \overrightarrow{OP} + r\vec{v}$ where $Q(x, y)$ is any point of the line.

$(x, y) = (x_0, y_0) + r(a, b)$ or $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} + r\begin{pmatrix} a \\ b \end{pmatrix}$

The **parametric equations** of the line that is parallel to the vector $\vec{v} = (a, b)$ and which passes through the point *P* with position vector $\overrightarrow{0P} = (x_0, y_0)$ are given by:

$$\begin{cases} x = x_0 + ra \\ y = y_0 + rb \end{cases}$$

The **symmetric equation** or **Cartesian equation is** found after eliminating the parameter ***r***.

$$\frac{x - x_0}{a} = \frac{y - y_0}{b}$$

This can be expanded to the standard form $bx - ay = bx_0 - ay_0$.

Example

Find the vector equation of the straight line that is parallel to the vector $\vec{u} = (2, -1)$ and which passes through the point with position vector $\vec{v} = (3, 2)$.

Solution

The vector equation is $(x, y) = (3, 2) + r(2, -1)$, *r* is a parameter

The parametric equations:

$$\begin{cases} x = 3 + 2r \\ y = 2 - r \end{cases}$$

The Cartesian equation:

$$\frac{x-3}{2} = \frac{y-2}{-1} \quad \text{or} \quad 2y - 4 = -x + 3 \quad \text{or} \quad x + 2y = 7$$

In general

The **vector equation** of the line can be rewritten as

$(x, y) = (x_0, y_0) + r(a, b)$ or $x\vec{i} + y\vec{j} = x_0\vec{i} + y_0\vec{j} + r\left(a\vec{i} + b\vec{j}\right)$ where $\left\{\vec{i} = (1,0), \vec{j} = (0,1)\right\}$ form the standard basis of IR^2.

Application activity 2.5.1

1) Find the vector, parametric and Cartesian equation of the line passing through the point with position vector $(2,-3)$ and parallel to the line $(x, y) = (3,5) + r(1,6)$
2) Find the Cartesian equations of the lines whose vector equations are given below. Give your answers in the form $y = mx + c$
   a) $(x, y) = (-1, 2) + r(-2, 3)$
   b) $x\vec{i} + y\vec{j} = 3\vec{i} + 2\vec{j} + r\left(3\vec{i} - \vec{j}\right)$

2.5.2 Equation of a straight line given 2 points

Activity 2.5.2

Consider the line passing through points $A(1,4)$ and $B(3,-2)$.

a) Determine the vector $\overrightarrow{AB}$

b) Determine the form of a vector $\vec{w}=(\mathrm{x},\mathrm{y})$ parallel to $\overrightarrow{AB}$ and passing to the point $\mathrm{B}(3,-2)$

c) Give 2 examples of vectors represented by $\vec{w}$.

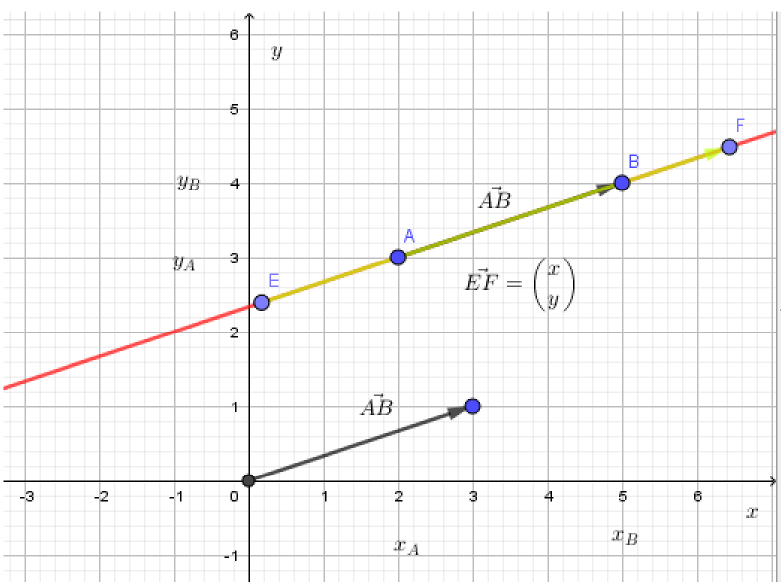
d) If D is a line with the direction vector $\vec{w}$, what should be the equation of D?

Content summary

Two points $A(x_A, y_A)$ *and* $B(x_B, y_B)$ form a vector $\overrightarrow{AB}=\begin{pmatrix} x_B-x_A \\ y_B-y_A \end{pmatrix}$.

A vector parallel to $\overrightarrow{AB}=\begin{pmatrix} x_B-x_A \\ y_B-y_A \end{pmatrix}$ and passing to the point $B(x_B, y_B)$ is of the form

$$\vec{w}=(x,y)=r\overrightarrow{AB}+A$$



Therefore,

The vector equation of the line is:

$(x, y) = \overrightarrow{0A} + r\overrightarrow{AB}$ or $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x_A \\ y_A \end{pmatrix} + r\begin{pmatrix} x_B - x_A \\ y_B - y_A \end{pmatrix}$, where $\overrightarrow{AB}$ is the direction vector.

The parametric equations of the line are

$$\begin{cases} x = x_A + r(x_B - x_A) \\ y = y_A + r(y_B - y_A) \end{cases}$$

The Cartesian equation is obtained in this way:

$$x = x_A + r(x_B - x_A), \quad r = \frac{x - x_A}{x_B - x_A}$$

$$y = y_A + r(y_B - y_A), \quad r = \frac{y - y_A}{y_B - y_A}$$

Equalizing the value of r we find the Cartesian equation:

$$\frac{x - x_A}{x_B - x_A} = \frac{y - y_A}{y_B - y_A}$$

This can be expanded to the standard form:

$$x(y_B - y_A) - y(x_B - x_A) = x_A(y_B - y_A) - y_A(x_B - x_A)$$

Example

Find the vector equation of the straight line passing through $A(3,2)$ and $B(4,1)$.

Solution

The direction vector is $\overrightarrow{AB} = (1,-1)$, r is a parameter

The vector equation is

$(x, y) = \overrightarrow{0A} + r\overrightarrow{AB}$ or $(x, y) = (3,2) + r(1,-1)$

The parametric equations: is

$$\begin{cases} x = 3 + r \\ y = 2 - r \end{cases}$$

The Cartesian equation, is

$$\frac{x-3}{1} = \frac{y-2}{-1} \text{ or } y-2 = -x+3 \text{ or } x+y=5$$

Application activity 2.5.2

1) Determine the equation of a straight line passing through $P(2,4)$ and $B(3,-2)$.
2) Find the vector equation of the straight line that passes through the point with position vector $2\vec{i}+3\vec{j}$ and which is perpendicular to the line $x\vec{i}+y\vec{j}=3\vec{i}+2\vec{j}+r\left(\vec{i}-2\vec{j}\right)$.

2.5.3 Equation of a straight line passing through given point and its gradient

Activity 2.5.3

Determine the slope and the y-intercept in the equations below. Hence draw the lines in the same graph.

a) $y=\frac{3}{2}x-2$

b) $y=-3x+\frac{5}{2}$

Consider a line having gradient m and passing through the point $P_1\left(x_1, y_1\right)$. Suppose that the point $P\left(x, y\right)$ is another point on the line. Then the gradient of the line is the rate at which the line rises (or falls) vertically for every unit across to the right. It is defined by the change in y to the change in x.

$$m=\frac{y-y_1}{x-x_1}$$

Thus, the equation of the line in point-slope form is defined by

$y - y_1 = m(x - x_1)$

From the equation above, if we take any other point $P_2(x_2, y_2)$ that lies on this line, then:

Vector equation

$\overrightarrow{OP} = \overrightarrow{OP_1} + r\overrightarrow{P_1P_2}$ or $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + r\begin{pmatrix} x_2 - x_1 \\ y_2 - y_1 \end{pmatrix}$,

$\overrightarrow{P_1P_2}$ is the direction vector.

Parametric equations

$$\begin{cases} x = x_1 + r(x_2 - x_1) \\ y = y_2 + r(y_2 - y_1) \end{cases}$$

Cartesian equation

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_2}$$

Example 1

Write the equation of the line that has slope -3 and passes through the point $(-1,7)$

Solution

$y - y_1 = m(x - x_1)$ use point-slope form

$y - 7 = -3(x - (-1))$ substitute $(-1,7)$ *for* (x_1, y_1) and -3 for m.

$y - 7 = -3(x + 1)$.

Note:

The general equation of the line is $Ax + By + C = 0$, where A, B, C are constants, and, $A \neq 0, B \neq 0$.

a) If $A = 0$ the line is horizontal.
b) If $B = 0$ the line is vertical.
c) If $C = 0$ the line passes through the origin.

On the other hand, the line has slope $m = \frac{-A}{B}$ and the y-intercept is $b = \frac{-c}{B}$

d) If the two lines are parallel, then their slopes/gradients are equal.

Therefore, $m_1 = m_2$.

Thus, the equations $Ax + By + C = 0$ and $Ax + By + D = 0$ are parallel.

e) If two lines are perpendicular, then the product of their slopes/gradients is equal to -1.

Therefore, $m_1 \times m_2 = -1$.

Thus, the equations $Ax + By + C = 0$ and $Bx - Ay + D = 0$ are perpendicular.

Example 2

Find the equation for the line that contains the point $(5,1)$ and is parallel to $y = \frac{3}{5}x + 3$.

Solution

Step1: Identify the slope of the given line

$$y = \frac{3}{5}x + 3 \Rightarrow m = \frac{3}{5}$$

Step2: Write the equation of the line through point $(5,1)$ and $m = \frac{3}{5}$

$$y - y_1 = m(x - x_1)$$

$$y - 1 = \frac{3}{5}(x - 5)$$

$$y = \frac{3}{5}x - 3 + 1$$

$$y = \frac{3}{5}x - 2$$

Example 3

Find the equation for the line that contains the point $(0,-2)$ and is perpendicular to $y = 5x + 3$?

Solution

Step1: Identify the slope of the given line and write its negative reciprocal.

$y = 5x + 3 \quad \Rightarrow m = 5$ and it has negative reciprocal $\frac{-1}{5}$

Step2: Write the equation of the line through point $(0,-2)$ and $m=\frac{-1}{5}$

$y-y_1=m(x-x_1)$

$y-(-2)=\frac{-1}{5}(x-0)$

$y=\frac{-1}{5}x-2$

Application activity 2.5.3

1) Write the equation in point-slope form for the line through the given point that has the given slope

a) $(3,-4); m=6$

b) $(-2,-7); m=\frac{-3}{2}$

c) $(-5,2); m=0$

d) $(4,2); m=\frac{-5}{3}$

e) $(4,0); m=1$

f) $(1,-8); m=\frac{-1}{5}$

2) Is $y-5=2(x-1)$ an equation of a line passing through $(4,11)$? Explain.

3) Write an equation of the line that contains the point $(-3,-5)$ and the same slope as $y+2=7(x+3)$

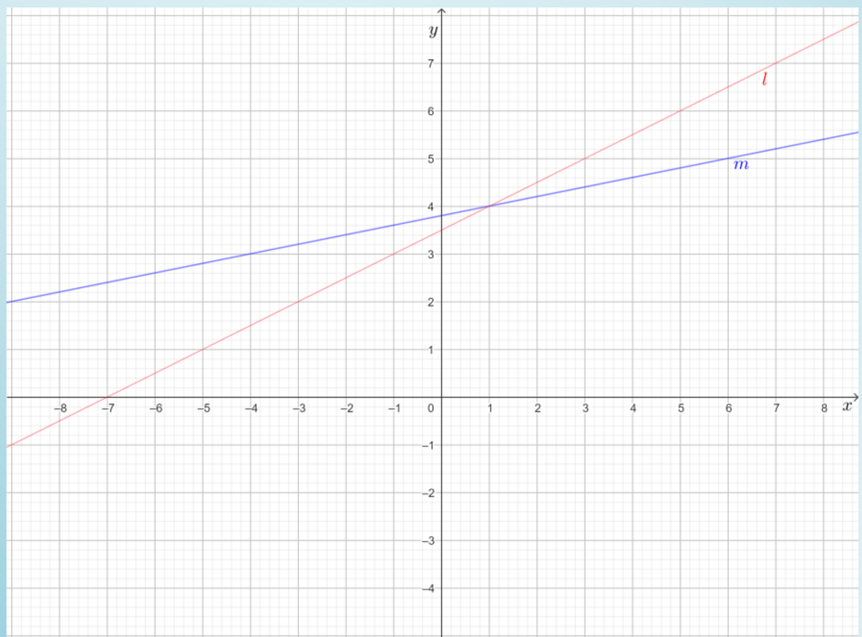
2.6. Problems on points and straight lines in 2D

2.6.1 Position of a point to the line

Activity 2.6.1

1) From the following points: $A(4,5)$, $B(6,2)$ and $C(8,7)$; select the one that is lying on the line with equation $y=\frac{1}{2}x+3$

2) The x and y values shown in the table represent points on one of the lines shown in the graph below. Which line is it?



x	-3	1	5
y	2	4	6

Experimental version

Content summary

A point represents position of an object only. It has zero size (that is, zero length, zero width, and zero height).

A line (straight line) can be thought of as a set of many points infinitely connected. A line has infinite length, zero width, and zero height.

To verify whether a point P with coordinates (x_1, y_1) lies on the line L with equation $ax+by+c=0$, verify if it satisfies the given equation by replacing x by x_1 and y by y_1 to get whether $ax_1+by_1+c=0$ (the line L passes through the point P). Otherwise, the point P is not on the line L (the line L does not pass through the point P).

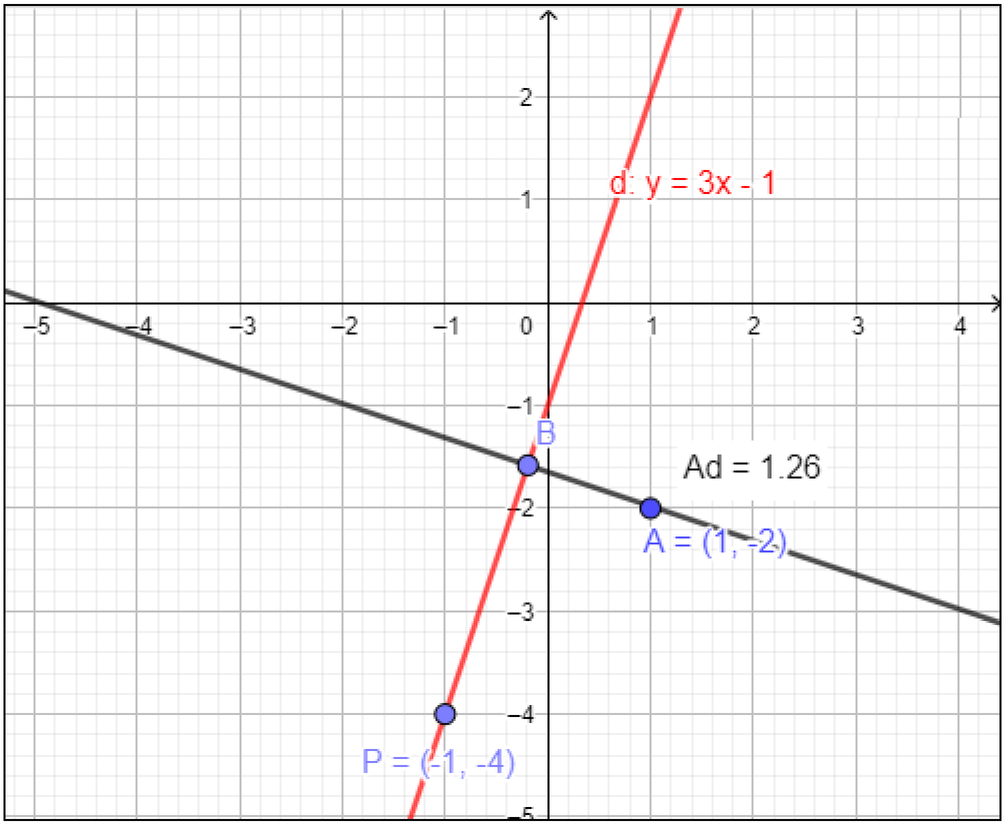
Example

Given a line **d** with equation $y=3x-1$. Show that the point P$(-1,-4)$ lies on that line and show that the point A(1,-2) is outside of that line.

Solution

-Replacing y by -4 and x by -1 in the equation $y=3x-1$ yields $-4=3(-1)-1 \Rightarrow -4=-4$

The left side is equal to the right side, thus the point $(-1,-4)$ lies on the line with equation $y=3x-1$.



- Replacing y by -2 and x by 1 in the equation $y = 3x - 1$ yields $-2 \neq (3\times1) - 1$ or $-2 \neq 2$.

This means that the point A is out of the line **d.**

Note: We can verify the position of a point and a line by calculating the distance from that line and the point.

The perpendicular distance from a point $A(x_1, y_1)$ to the line d of equation $ax + by + c = 0$ is given by $D(d, A) = \left|\frac{ax_1 + by_1 + c}{\sqrt{a^2 + b^2}}\right|$

If this distance is zero, the given point is on the line. If the distance is different from zero, the point is outside the line.

In our example, the distance between the line d and the point A is 1.2 units which is different from zero. So A is outside the line.

Application activity 2.6.1

1) From the following points, indicate those which do not lie on the line with equation $2x-3y+1=0$

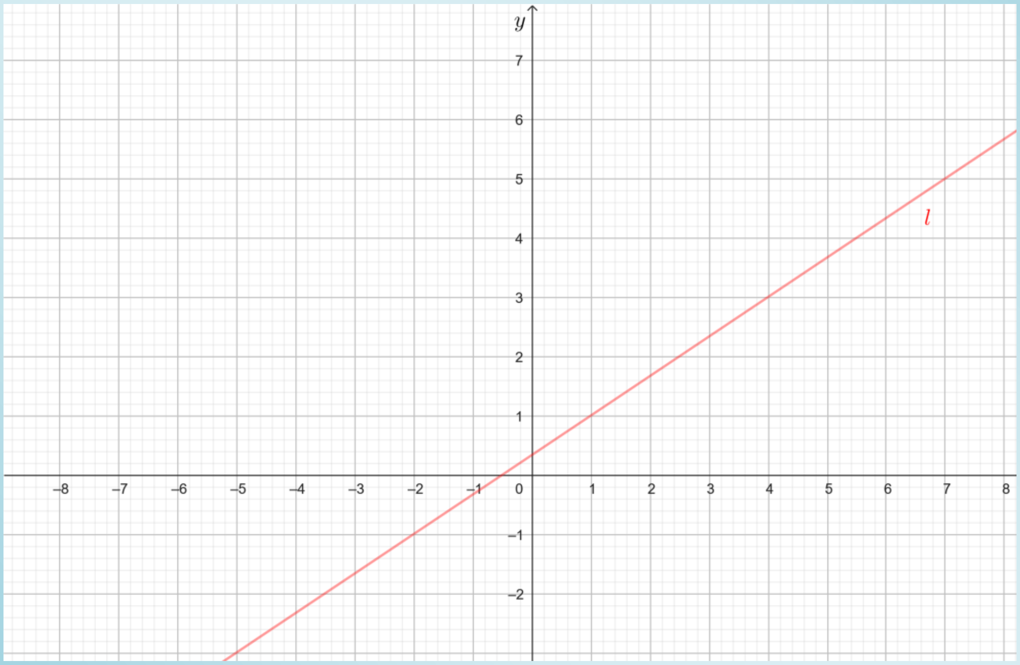
$A(-1,2)$, $B(1,0)$, $C(1,5)$, $D(-2,-1)$, $E(3,4)$, $F(4,3)$.

2) Select the point that is lying on the line with equation $y=\frac{3}{4}x+3$

$A(4,8)$ $B(6,1)$ $C(7,3)$

3) Given that the line with equation $y=3x+b$ passes through the point $(1,4)$, find the value of b

4) In the following graph, the ordered pairs from the table represent points on line l



Complete the missing values in the table.

x	-4	1	___	___
y	___	___	3	5

2.6.2 Position of two lines and condition of parallelism

Activities 2.6.2

1) In the same Cartesian plane, plot the straight lines containing the following points: $l_1:(0,4);\ (\frac{1}{2},2)$ and $l_2:(\frac{1}{2},2);\ (4,1)$. What is your opinion about the two lines in the Cartesian plane?
2) In the same Cartesian plane plot the straight lines containing the following points $l_1:(-3,-1);\ (0,3)$ and $l_2:(-2,-4);\ (1,0)$. Are the two lines intersecting or parallel?

Content summary

Intersection point of two lines

Let the equations of two intersecting straight lines be

$a_1x+b_1y+c_1=0$ ....(i) and $a_2x+b_2y+c_2=0$.......(ii)

Suppose the above equations of two intersecting lines meet at $P(x_1,y_1)$. Then, (x_1,y_1) will satisfy both the equations *(i)* and *(ii)*.

Therefore, $a_1x_1+b_1y_1+c_1=0$ and $a_2x_1+b_2y_1+c_2=0$

Solving the above two equations by using the method of cross-multiplication, we get,

$$\frac{x_1}{b_1c_2-b_2c_1}=\frac{y_1}{c_1a_2-c_2a_1}=\frac{1}{a_1b_2-a_2b_1}$$

Therefore, $x_1=\frac{b_1c_2-b_2c_1}{a_1b_2-a_2b_1}$ and $y_1=\frac{c_1a_2-c_2a_1}{a_1b_2-a_2b_1}$, $a_1b_2-a_2b_1\neq 0$

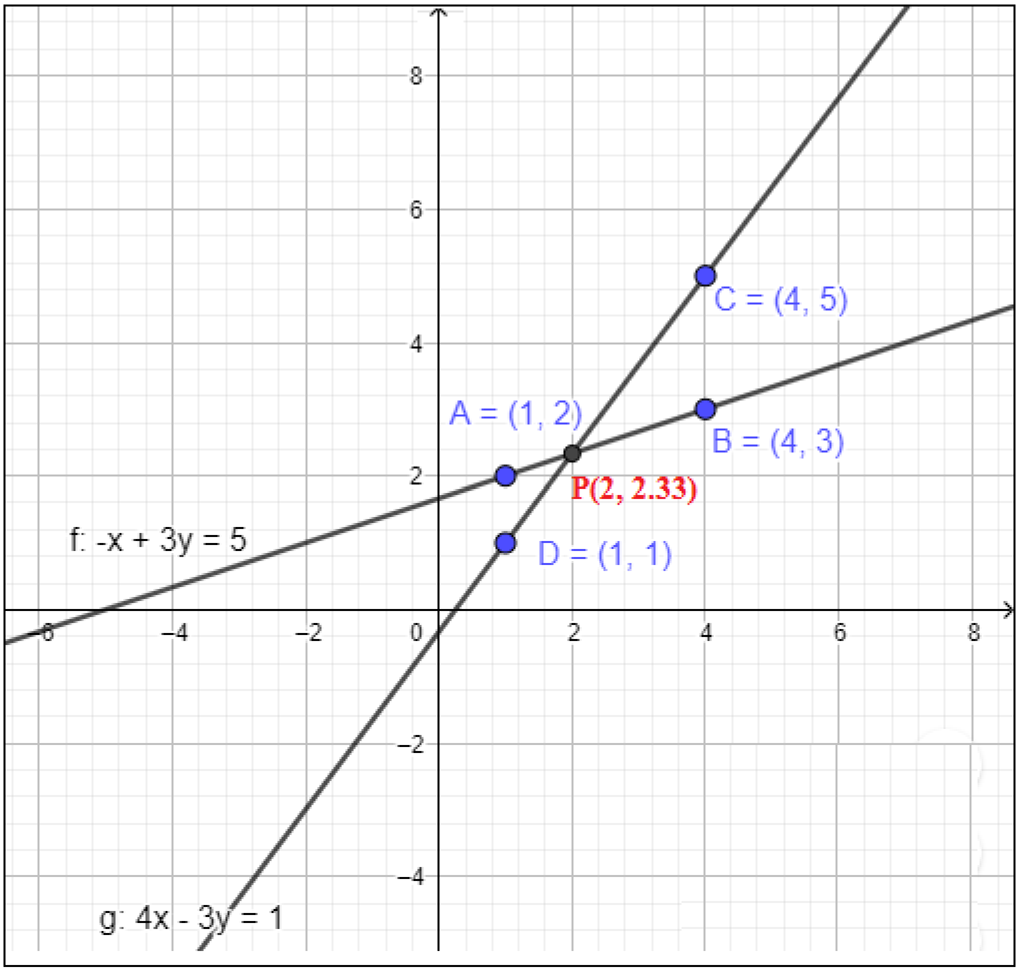
The two intersecting lines $a_1x+b_1y+c_1=0$ and $a_2x+b_2y+c_2=0$

meet at the point $P(x_1,y_1)$ if $(x_1,y_1)=\left(\frac{b_1c_1-b_2c_1}{a_1b_2-a_2b_1},\frac{c_1a_2-c_2a_1}{a_1b_2-a_2b_1}\right)$, $a_1b_2-a_2b_1\neq 0$

Notes:

To find the coordinates of the point of intersection of two non-parallel lines, we solve the given equations simultaneously and the values of x and y obtained determine the coordinates of the point of intersection.

For example, the line f: -x+3y+5 passing by the point A(1,2) and the point B(4,3) and the line g: 4x-3y =1 passing by the point C(4,5) and the point D(1,1) meet at the point P(2, 2.33):



If $a_1b_2 - a_2b_1 = 0$ then $a_1b_2 = a_2b_1$

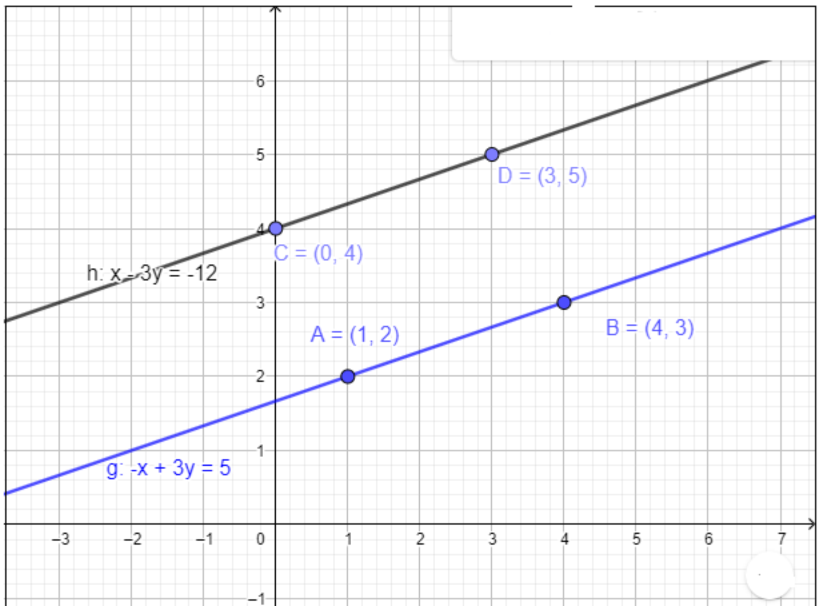
$$\Rightarrow \frac{a_1}{b_1} = \frac{a_2}{b_2}$$

$\Rightarrow -\frac{a_1}{b_1} = -\frac{a_2}{b_2}$ i.e., the slope of line (i) = the slope of line (ii)

Therefore, in this case the straight lines (i) and (ii) are **parallel**.

Two lines $a_1x+b_1y+c_1=0$ **and** $a_2x+b_2y+c_2=0$ **are parallel if they have the same slope:** $-\frac{a_1}{b_1}=-\frac{a_2}{b_2}$.

For example, the line ***h*** with equation $\boldsymbol{x}-3\boldsymbol{y}=-12$ and the line *g* with equation $\boldsymbol{x}-3\boldsymbol{y}=-5$ represented below are parallel. Their slope $\boldsymbol{m}=\frac{1}{3}$ is the same.



Example

Find the coordinates of the point of intersection of the lines $2x-y+3=0$ and $x+2y-4=0$

Solution

We know that the co-ordinates of the point of intersection of the lines $a_1x+b_1y+c_1=0$ and

$a_2x+b_2y+c_2=0$ are $\left(\frac{b_1c_1-b_2c_1}{a_1b_2-a_2b_1},\frac{c_1a_2-c_2a_1}{a_1b_2-a_2b_1}\right)$, $a_1b_2-a_2b_1\neq 0$

Given equations are

$2x-y+3=0$....(i)

$x+2y-4=0$.....(ii)

Here $a_1 = 2,\ b_1 = -1, c_1 = 3,\ a_2 = 1,\ b_2 = 2$ and $c_2 = -4$

$$\left(\frac{(-1).(-4)-(2)(3)}{(2).(2)-(1).(-1)}, \frac{(3)(1)-(-4)(2)}{(2)(2)-(1)(-1)}\right)$$

$$\Rightarrow \left(\frac{4-6}{4+1}, \frac{3+8}{4+1}\right)$$

$$\Rightarrow \left(\frac{-2}{5}, \frac{11}{5}\right)$$

Therefore, the co-ordinates of the point of intersection of the lines $2x - y + 3 = 0$ and $x + 2y - 4 = 0$ are $\left(\frac{-2}{5}, \frac{11}{5}\right)$

Alternatively, by solving simultaneous equations $2x - y + 3 = 0....(i)$ and $x + 2y - 4 = 0.....(ii)$ using different methods you get the same answer. $\left(\frac{-2}{5}, \frac{11}{5}\right)$.

Application activity 2.6.2

Find the position of the given pair of straight lines and their point of intersection if any.

a) $x + y + 9 = 0$ and $3x - 2y + 2 = 0$

b) $x - y + 3 = 0$ and $-x + y = 1$

c) $x = 3 - 2y$ and $2x - y = 3$

d) $3x + 6y = 9$ and $x + 2y = 3$

2.6.3 Angles between two lines and condition of perpendicularity

Activities 2.6.3

1) In the same Cartesian plane, draw the lines whose equations are $2x+3y=1$ and $-x+2y=2$; then measure the angles between these two lines.

2) Given two integers $m_1=1$ and $m_2=-2$ representing the slope of two lines intersecting at origin $(0,0)$, in the same Cartesian plane, draw these lines and measure the angles between them.

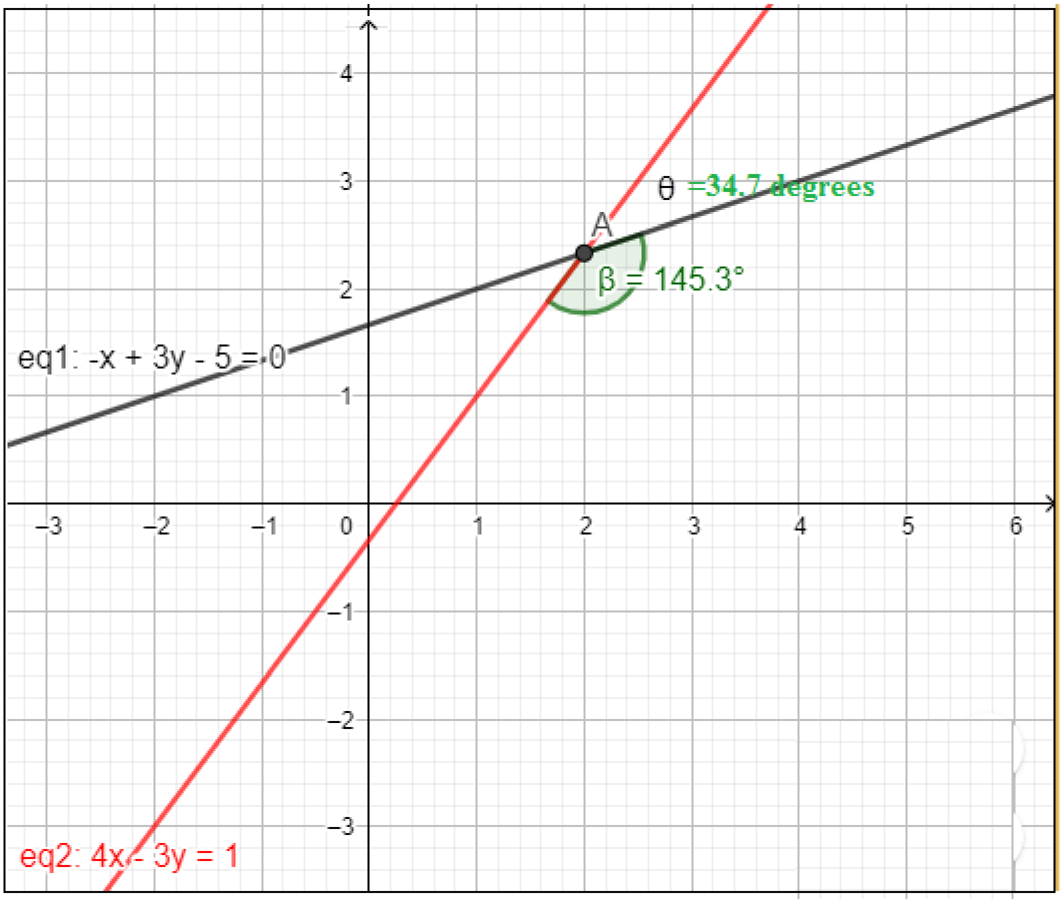
Content summary

The angle between two lines is the measure of the inclination between them. For two intersecting lines, there are two types of angles between them; the **acute angle** and the **obtuse angle**. Here we consider the acute angle between the lines to be the angle between two lines. Instead of drawing and measuring, the **angle between two lines** can be calculated **from the slopes of the lines or from the equation of the two lines**.

The simplest formula to find the angle between the two lines is from the slope of the two lines;

The angle between two lines with slopes m_1 and m_2 respectively can be computed by using the trigonometric tangent function. The **acute angle θ** between two lines is such that $\tan\theta=\left|\frac{m_1-m_2}{1+m_1m_2}\right|$.

For example, the line f with equation $-\boldsymbol{x}+3\boldsymbol{y}=5$ has the slop $\boldsymbol{m}_1=\frac{1}{3}$; The line g with equation $4\boldsymbol{x}-3\boldsymbol{y}=1$ has the slop $\boldsymbol{m}_2=\frac{4}{3}$. The angle **θ between them is** $\tan\theta=\left|\frac{m_1-m_2}{1+m_1m_2}\right|$



The second formula to find the angle between the two lines $l_1 \equiv a_1x + b_1y + c_1 = 0$, and $l_2 \equiv a_2x + b_2y + c_2 = 0$ is given by $\tan\theta = \left|\frac{a_2b_1 - a_1b_2}{a_1a_2 + b_1b_2}\right|$.

Also, the angle between the two lines can be computed using the following formula $\cos\theta = \left|\frac{a_1b_1 + a_2b_2}{\sqrt{a_1^2 + b_1^2} \times \sqrt{a_2^2 + b_2^2}}\right|$.

Example 1

Find the angle between two lines having slopes of $m_1 = 1$ and $m_2 = \frac{1}{2}$ respectively.

Solution

The gives slopes of the two lines are $m_1 = 1$ and $m_2 = \frac{1}{2}$.

The formula to find the angle between the two lines is $\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$,

or $\tan\theta = \left|\frac{1 - \frac{1}{2}}{1 + 1 \times \frac{1}{2}}\right| = \frac{\frac{1}{2}}{\frac{3}{2}} = \frac{1}{3}$

Therefore the angle between the lines is $\theta = \tan^{-1}\left(\frac{1}{3}\right) \simeq 18^0$

Example 2

Derive the condition for two lines with slopes m_1 and m_2 to be parallel and perpendicular using the angle between two lines formula.

Solution

The angle between two lines with slopes m_1 and m_2 is:

$$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|.$$

Note:

(i) When the lines are parallel, the angle between two parallel lines is 0° and $\tan 0^0 = 0$.

Substituting this in the above formula leads to $\frac{m_1 - m_2}{1 + m_1 m_2} = 0 \Rightarrow m_1 = m_2$.

(ii) When the lines are perpendicular, the angle between two perpendicular lines is 90°. We know that tan 90° is not defined.

The above formula is NOT defined when $1+m_1m_2=0 \Rightarrow m_1m_2=-1$. 1 + m_1 m_2 = 0

Therefore**, t**he condition for two lines to be **parallel** is $m_1=m_2$ and the condition for the two lines to be **perpendicular** is $m_1m_2=-1$.

Example 3

Find the angle between two straight lines having the equations $3x+4y-10=0$,
and $4x-5y+2=0$.

Solution

The given two equations of the lines are $3x+4y-10=0$, and $4x-5y+2=0$.

Here, we have $a_1=3, b_1=4, a_2=4$, and $b_2=-5$.

The angle between the two lines can be calculated using the formula $\tan\theta=\left|\frac{a_2b_1-a_1b_2}{a_1a_2+b_1b_2}\right|$.

Thus, $\tan\theta=\left|\frac{a_2b_1-a_1b_2}{a_1a_2+b_1b_2}\right|=\left|\frac{(4\times4)-3\times(-5)}{(3\times4)+4\times(-5)}\right|=\left|\frac{16+15}{12-20}\right|=\left|\frac{31}{-8}\right|=\frac{31}{8}$

Or $\theta=\tan^{-1}\left(\frac{31}{8}\right)\simeq 75.5^0$;

Therefore, the angle between the lines is about 75.5^0.

Alternative method:

The angle between the two lines can be calculated using the formula

$$\cos\theta = \left|\frac{a_1b_1 + a_2b_2}{\sqrt{a_1^2+b_1^2}\times\sqrt{a_2^2+b_2^2}}\right|.$$

Thus, $\cos\theta = \left|\frac{3\times 4 + 4\times(-5)}{\sqrt{9+16}\times\sqrt{16+25}}\right| = \left|\frac{-8}{5\sqrt{41}}\right| = \frac{8}{5\sqrt{41}}$

Or $\theta = \cos^{-1}\frac{8}{5\sqrt{41}} \simeq 75.5^0$;

Therefore, the angle between the lines is about 75.5^0.

Application activities 2.6.3

1) From the following pair of lines, determine which are parallel, perpendicular, or neither parallel nor perpendicular:

   a) $x - y + 4 = 0$ and $2x - 2y = -8$

   b) $-x + 2y + 1 = 0$ and $y = 1 - 2x$

   c) $x = 2 - y$ and $y = 1 - 2x - y$

   d) $x + 2y + 3 = 0$ and $2y = 5 - x$

2) Determine the angle between two lines, one of which is the $x - axis$ and the other line is $x - y + 4 = 0$.

3) Determine the angle between the lines whose slopes are $m_1 = -2$, and $m_3 = 3$ respectively.

4) Determine the angle between the following lines $2x - y + 4 = 0$, and $3 - x + 3y = 0$.

5) Determine the angle formed by the lines whose direction vectors are $\vec{u} = (3,4)$ and $\vec{v} = (-8,6)$.

2.7 Geometric shapes in 2D

2.7.1 Identification of the Geometric shapes in two dimensions

Activities 2.7.1

1) a) Discuss the difference between regular polygons and irregular polygons and give examples.

   b) Name the shapes with three sides and give at least 4 examples of materials in real life with three sides.

   c) Name the shapes with four sides and give at least 4 examples of materials in real life with four sides.

2) a) Plot the following points on the 2D coordinate plane, then use a ruler to connect the points in the order they are listed to form a polygon.

Polygon 1: $(8,6);(11,1);(8,6)$

Polygon 2: $(2,5);(6,5);(6,3);(2,3)$

Polygon 3: $(3,5);(7,2);(7,-5);(3,-2)$

Polygon 4: $(-2,5);(-2,-2),(3,-2),(5,5)$

Polygon 5: $(-2,-2);(2,-2);(2,2);(6,2),(6,-6);(-4,-6);(-4,-4);(-2,4)$

b) What is the name of each geometric figure formed?

c) Describe the similarities and differences of those figures.

Content summary

In geometry, a two-dimensional shape can be defined as a flat plane figure or a shape that has two dimensions: length and width. Length on $x-axis$ and width on $y-axis$ or vice versa.

Two-dimensional or 2-D shapes do not have any thickness and can be measured in only two faces. In the two –dimensional coordinate, geometric figures can be plotted:

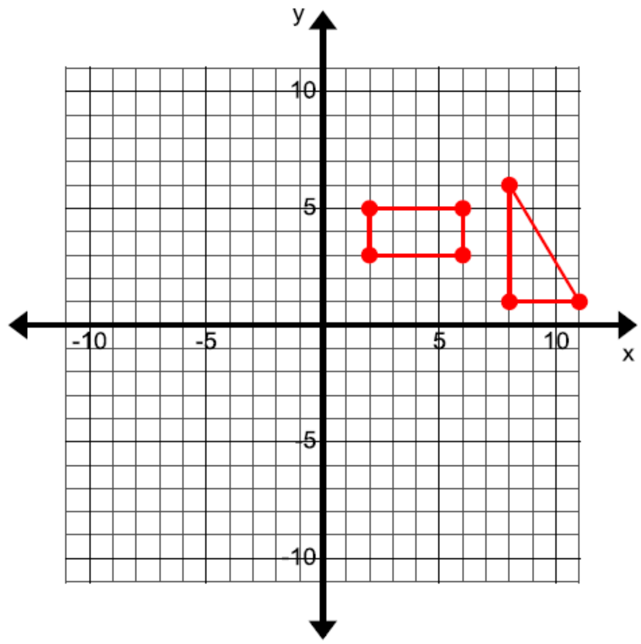
Example 1

The following points;

a) $(8,6);(11,1);(8,6)$

b) $(2,5);(6,5);(6,3);(2,3)$

Can make the following polygons in coordinates' graphs



Two dimensional shapes are for example: Triangle, square, rectangle, parallelogram, trapezium, rhombus, circle, pentagon, hexagon, etc.

It is also important to take note of the hierarchy, e.g. all rectangles are parallelograms but all parallelograms are not rectangles.

Example 2

Explain why square is rectangle but a rectangle is not a square?

Solution

Depending on their properties, a square is a quadrilateral with four right angles and equal sides. The two parallel sides are obviously equal, which allows a square to be a rectangle. Even though the two parallel sides of a rectangle are equal, the four sides are not equal; which means that a rectangle is not a square but a square is rectangle.

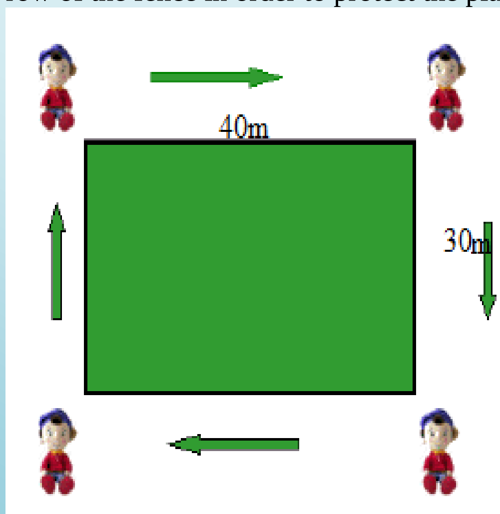
Application activities 2.7.1

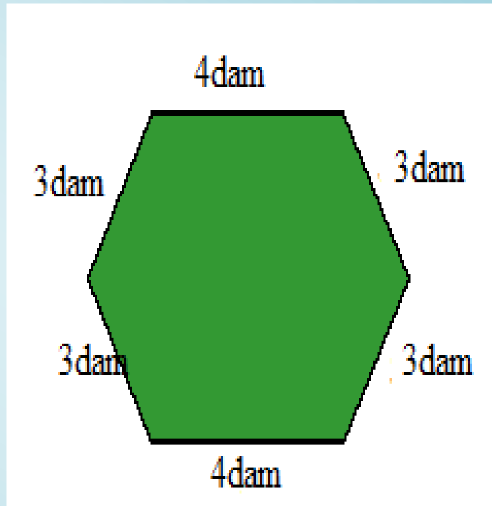
1) Given two points C (1,2) and P (5, -1).
   a) Determine the distance $r = \bar{CP}$
   b) Plot these points in the Cartesian plan, and draw a circle with centre C and radius $r = \bar{CP}$.
   c) Give other two points of the circle and their coordinates. Explain how you found those points.
2) Draw a polygon on the coordinate plane by plotting each set of vertices and connecting them in the order they are listed. Is it a regular or irregular polygon? Can you find its area?
   a) $(0,2),(0,-1),(5,-1),(5,-4),(8,-4),(8,5),(5,5),(5,2)$
   b) $(-5,7),(1,7),(1,2)(5,2),(5,-4),(-1,-4),(-1,1),-5,1)$

2.7.2 Perimeter and area of geometric shapes in 2D.

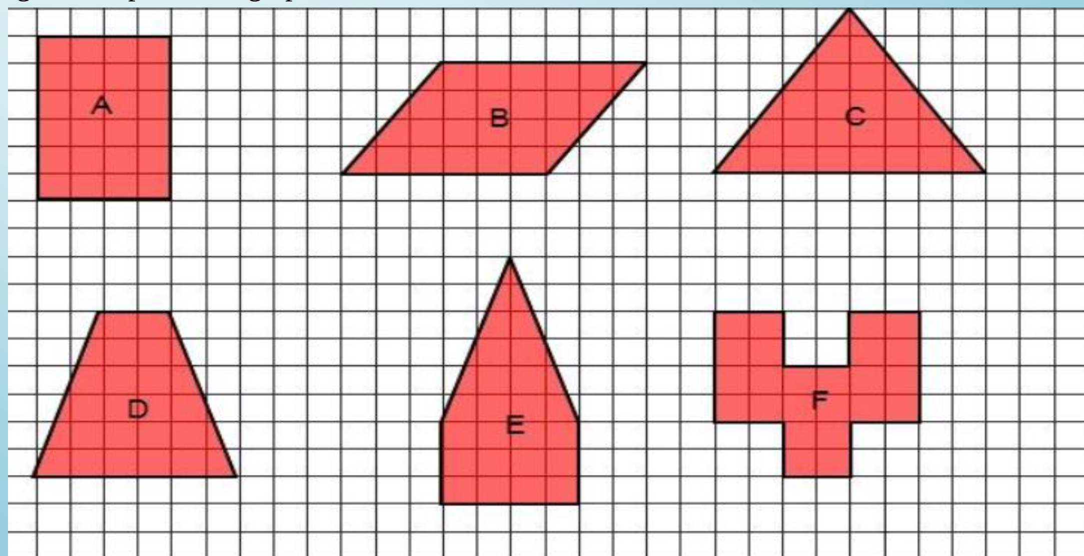
Activities 2.7.2

1) The following fields have two dimensional shapes; find the length of one wire that can be used on one row of the fence in order to protect the plants.



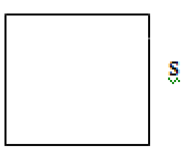
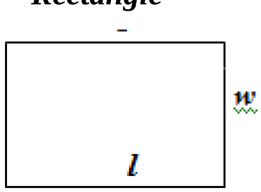


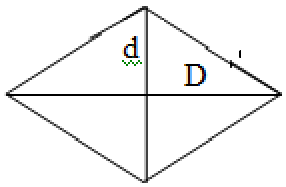
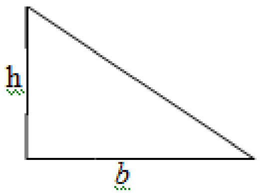
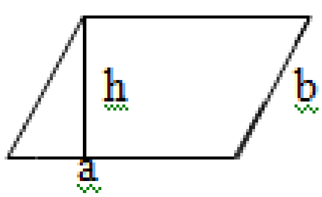
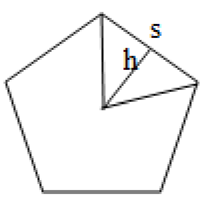
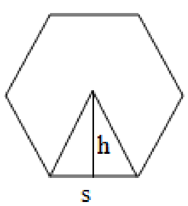
2) On the graph below, one square of the paper represents one square unit. Calculate the area of given shapes in the graph.

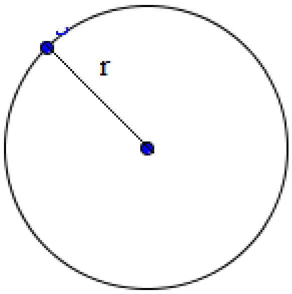


Content summary

As studied in previous levels (Primary and ordinary levels), the perimeter and area of two dimensional shapes can be summarised in the following table:

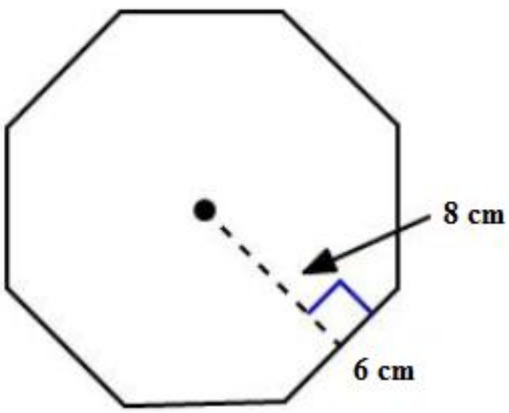
Shape	**Description**	**Perimeter formulas**	**Area Formulas**
Square 	A quadrilateral having all sides equal in length and forming right angles.	$4s$	s^2
Rectangle 	A 4-sided polygon with all right angles.	$2(w+l)$	$l\times w$
Rhombus	A 4-sided polygon with two equal opposites angles and 4 equal sides	$4s$	$\frac{D\times d}{2}$

Shape	**Description**	**Perimeter formulas**	**Area Formulas**
			
Triangle 	A 3-sided polygon (sum of internal angles = 180°)	Sum of its side	$\frac{b \times h}{2}$
Parallelogram 	4-sided polygon with two pairs of parallel sides.	$2(a+b)$	$b \times h$
Pentagon 	5-sided polygon (the graphic shows a regular hexagon with "regular" meaning each of the sides are equal in length)	$5s$	$5(\frac{s \times h}{2})$
Hexagon 	6-sided polygon	$6s$	$3(s \times h)$

Shape	Description	Perimeter formulas	Area Formulas
Circle 	Round Collection of infinite points Polygon with an infinite number of sides Any point on the circle is the same distance from the centre Made up of a closed curved line	$2\pi r$	πr^2

Example 1

Find the perimeter and the area of the regular octagon below. A regular octagon has all sides that are equal.



Solution

Given: $s = 6cm; \quad h = 8cm$

Asked: Perimeter =? Area = ?

a) Perimeter $= number\ of\ sides \times sides\ lenght = 6 \times 6in = 36in$

b) Area

$= Number\ of\ sides \times (area\ of\ triangle)$

$= 8(\frac{6cm \times 8cm}{2})$

$= 4 \times 48cm^2$

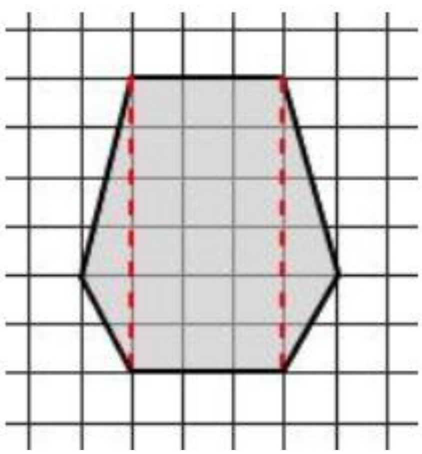
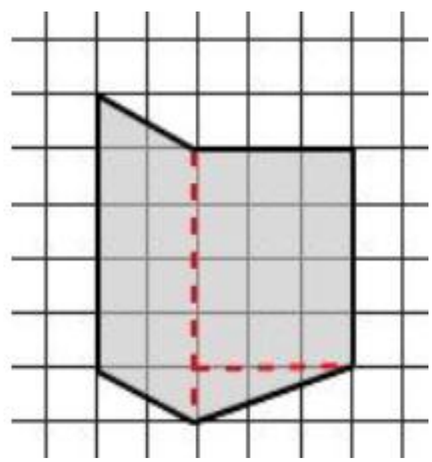
$= 192cm^2$

Irregular polygons

If the shape is irregular, it can be divided into regular or other simple figures to find its area. While the perimeter is found by adding the sum of sides for the original figure.

Example 2

Given that one square of the paper represents one square unit, find the area of shaded figures.

1) 	2) 

Solution

1) Area of the rectangle is $3(6) = 18$

Area of the two triangles is

$2 \times \frac{1}{2}(6 \times 1) = 2 \times 3 = 6$

Total area is

$18+6=24$ Units

2) Area of the rectangle is $3(4)=12$

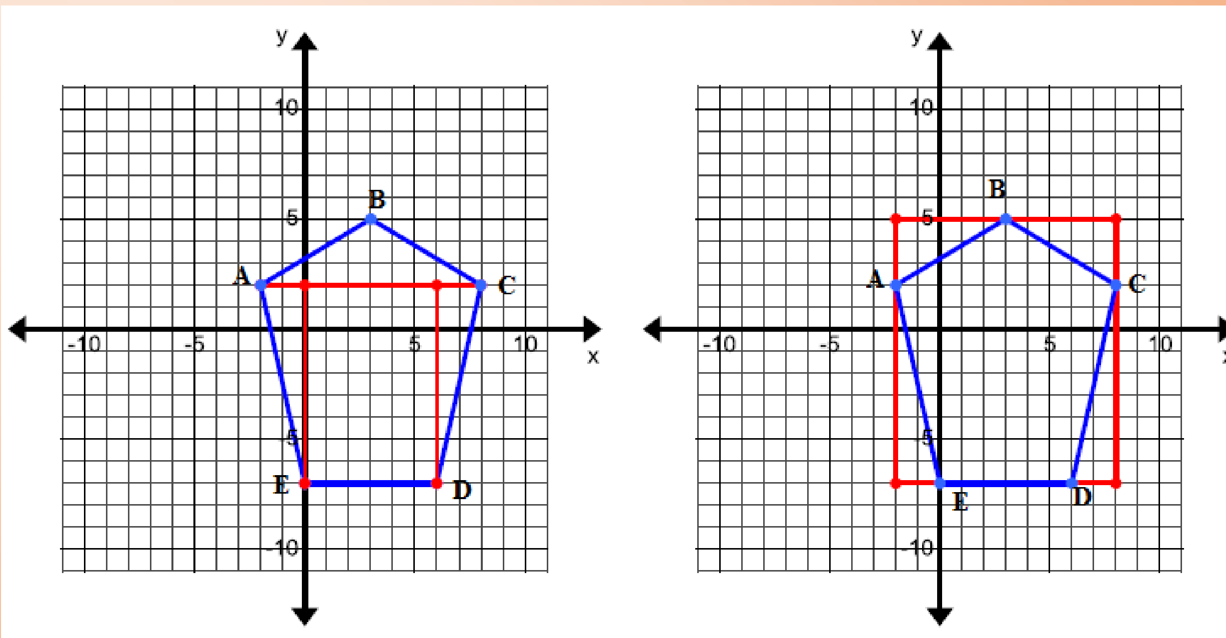
Area of the parallelogram is 5(2) =10

Area of the triangle is $\frac{1}{2}(1)(3)=1.5$

Total area is $12+10+1.5=23.5$ Units.

Application activity 2.7.2

Two different copies of a geometric figure were provided as follow:



a) Given that one square of the paper represents one square unit on each figure, find the area of the geometric figure ABCDE in blue.

b) Show and explain that different methods can be used when finding the area of an irregular polygon.

2.8 End unit assessment

1) A new park is being designed for your city. The plans for the design are being drawn on the coordinate plane below. The vertices given form a polygon that represents the location for 5 different features in the park.

   a) Plot each set of points in the order they are given to form a polygon. Label the feature that the polygon represents on the graph.

Playground:

$(-5,-2),(-8,-2),(-8,-5),(-11,-5),(-11,-8),(-8,-8),(-8,-11),(-5,-11)$

Splash Pad: $(10,4),(-4,8),(6,8)$

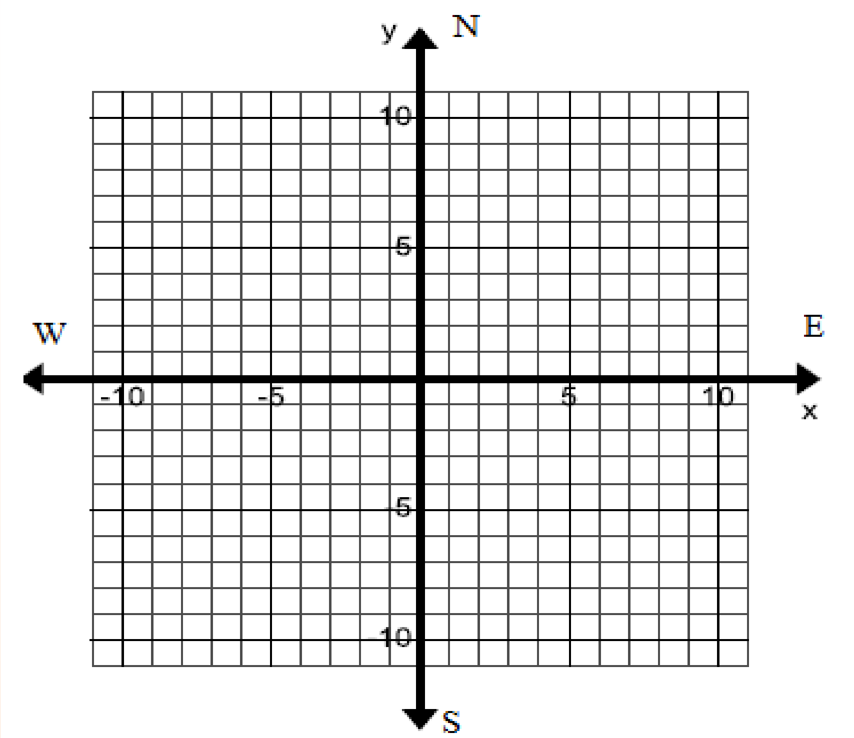
Restrooms: $(-4,5),(-4,8),(-7,10),(-10,7),(-7,5)$

Picnic Pavilion: $(8,-1),(6,-3),(4,-3),(2,-1),(2,1),(4,3),(6,3),(8,1)$.

If each square on the graph represents 1 square meter, answer the questions that follow.

   b) Cement needs to be poured to lay the foundation for the restroom and the splash pad. How many square meters of cement will the city need for the foundation of these two features?
   c) The playground is going to be covered with wood chips, how many square meters of wood chips will the city need for the playground?
   d) The citizens of the city have asked that the playground have a fence around its perimeter. How many meters of fencing will they need for the playground?
   e) The foundation of the pavilion picnic area is going to be covered with special pavers. How many square meters of pavers will they need for the pavilion?

2) A coordinate grid represents the map of a city. Each square on the grid represents one city block.



a) Helena's apartment is at the point $(5,7)$. She walks 4 blocks south, then 8 blocks west, then 4 blocks north, and then finally 8 blocks east back to her apartment. How many blocks did she walk total? Describe the shape of her path. Mark and label her apartment and highlight her walk.

b) Draw and describe in words at least two different ways you could walk exactly 20 blocks and end up back where you came from.

c) Carl lives at the point $(5,-5)$, Find the distance between Helena's house and Carl's house, and then mark and label Carl's house.

d) Establish the equation of the line passing through the points H(5,7) and C(5,-5). Given that one block has 10 m of length's side, calculate the

UNIT 3: FUNCTIONS AND GRAPHS

Key unit Competence: Apply graphical representation of functions in solving economics and financial models

3.0 Introductory activity 3

Suppose that average weekly household expenditure on food C depends on average net household weekly income y according to the relationship $C = 12 + 0.3y$.

a) Can you find a value of y for which C is not a real number?

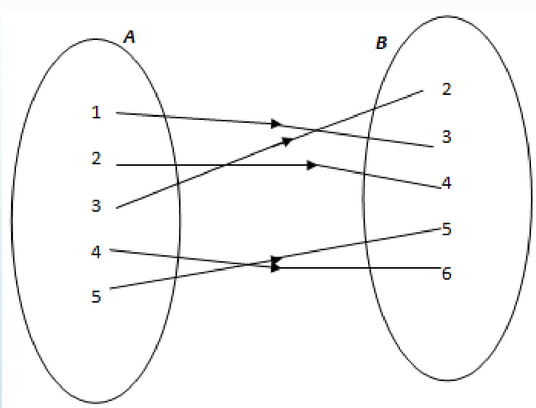
b) Complete a table of value of C from $y = 0$ to $y = 10$ and use it to draw the graph of $C = 12 + 0.3y$

c) If $y = 90$, what is the value of C?

3.1 Generalities on numerical functions

Activity 3.1

In the following Venn diagram, for each element of set A state which element of B is mapped to.



a) What is the set of elements of A which have images in B?

b) Determine the set of elements in B which have antecedent in A.

c) Is there any element of A which has more than one image?

Content summary

3.1.1 Function

A function is a rule that assigns to each element in a set *A* one and only one element in set *B*. We can even define a function as any relationship which takes one element of one set and assigns to it one and only one element of second set.

If x is an element in the domain of a function *f*, then the element that *f* associates with x is denoted by the symbol $f(x)$ (**read *f* of *x***) and is called the **image of *x* under *f*** or the **value of *f* at *x***.

The set of all possible values of $f(x)$ as x varies over the domain is called the **range** of *f* and it is denoted $R(f)$. The set of all values of A which have images in B is called **Domain of f** and denoted $Domf$.

We shall write $f(x)$ to represent the image of x under the function *f*. The letters commonly used for this purpose are *f*, *g* and *h*.

Example 1

Given that $f(x)=x^2$, find the values of $f(0), f(2), f(3), f(4) \, and \, f(5)$

Solution

$f(0)=0^2=0$ $\qquad$ $f(2)=2^2=4$

$f(3)=3^2=9$ $\qquad$ $f(4)=4^2=16$

$f(5)=5^2=25$

Note:

$f(x)=x^2$ can also be written as $f: x \to x^2$ which is read as

"*f* is a function which maps x onto x^2"

Example 2

Draw arrow diagrams for the functions. Use the domain $\{1,2,3\}$

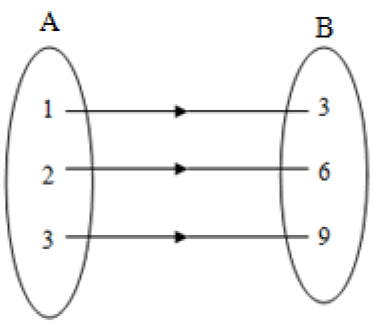
a. $f:x \to 3x$

b. $h:x \to x^2+1$

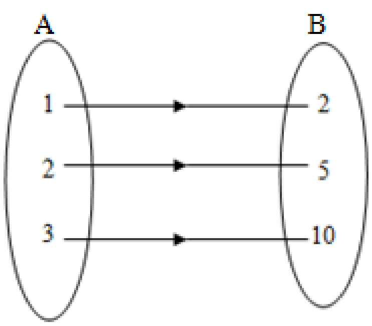
c. $g:x \to 2x+1$

Solution

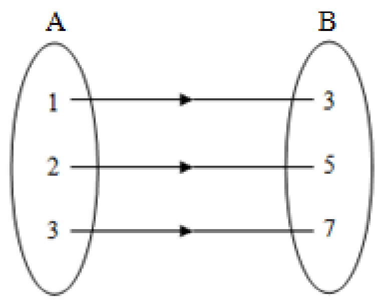
a. $f:x \to 3x$



b. $h:x \to x^2+1$



c. $g:x \to 2x+1$



Example 3

The functions f and g are given as $f(x) = x + 3 \;\; for \;\; x \geq 0$ and $g(x) = x^2 \;\; for \;\; -2 \leq x \leq 3$

State the range of each of these functions.

Solution

If $x \geq 0$, then $x + 3 \geq 3$. Thus, the range of *f* will be $f(x) \geq 3$

If $-2 \leq x \leq 3$, then $(-2)^2 \leq x^2 \leq (3)^2$. Thus, the range of *g* will be $4 \leq g(x) \leq 9$

3.1.2 Injective, surjective and bijective functions

Given sets A and B, a **function** defined from A to B is a correspondence, or a rule that associates to any element of A either one image in B, or no image in B.

A function such that every element of A has an image in B is called a **mapping**, thus, under a mapping any element of A has exactly one image in B (not less than one, and not more than one).

A mapping such that every element of B is image of either one element of A, or of no element of A, is called a **one- to- one mapping**, or an **injective mapping** or simply an **injection;** under a one-to-one mapping no two elements of A share the common image in B.

Mathematically, $(\forall x_1 \in A)(\forall x_2 \in A); \mathrm{f}(\mathrm{x}_2) = f(x_1) \Rightarrow x_2 = x_1$

A mapping such that every element of B is image of at least one element of A (image of one element of A, or image of more than one element of A), is called an **onto mapping**, or a **surjective mapping** or simply a **surjection**

Mathematically, $(\forall y \in B)(\exists x \in A); \mathrm{f(x)} = y$ or simply $B = f(A)$

A mapping satisfying properties of both one-to-one and onto is said to be **a bijective mapping**, or simply a bijection.

In particular, **linear function** $f : \mathbb{R} \to \mathbb{R}$

$x \mapsto f(x) = a\,\mathrm{x} + \mathrm{b}$, where $a \neq 0$, is bijective, there is no restrictions on the variables(independent or dependent).

Quadratic function

The function $f : \mathbb{R} \to \mathbb{R}$

$x \mapsto f(x) = a\,\mathrm{x}^2 + bx + c$, where $a \neq 0$, is not bijective, since some real numbers share images, or some real numbers are not images under function *f.*

But the following restrictions are bijective:

1) $f : [\frac{-b}{2a}, +\infty[\to [-\frac{\Delta}{4a}, +\infty[$

$x \mapsto f(x) = a\,\mathrm{x}^2 + bx + c$, $a > 0$;

2) $f :]-\infty, -\frac{b}{2a}] \to [-\frac{\Delta}{4a}, +\infty[$

$x \mapsto f(x) = a\,\mathrm{x}^2 + bx + c$, $a > 0$;

3) $f : [\frac{-b}{2a}, +\infty[\to]-\infty, -\frac{\Delta}{4a}]$

$x \mapsto f(x) = a\,x^2 + bx + c\ , a < 0$ and

4) $f :]-\infty, -\frac{b}{2a}] \to]-\infty, \frac{\Delta}{4a}]$

$x \mapsto f(x) = a\,x^2 + bx + c\ ,\ a < 0.$

The homographic function

$f : \mathbb{R} \to \mathbb{R}$

$x \mapsto f(x) = \frac{ax+b}{cx+d}$ is not bijective, since it is not a mapping ($x = -\frac{d}{c}$ has no image under function *f*, or $y = \frac{a}{c}$ is not image under function *f*).

But the restriction

$f : \mathbb{R} - \{-\frac{d}{c}\} \to \mathbb{R} - \{\frac{a}{c}\}$

$x \mapsto f(x) = \frac{ax+b}{cx+d}$ is bijective

Example 4

Consider the set of pigeons and the set of pigeonholes on the diagram below to answer the questions:

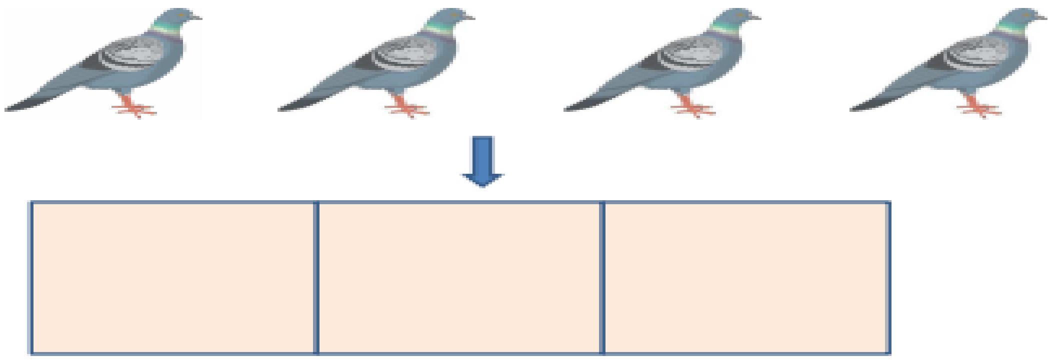
Determine whether it can be established or not between the two sets:

a. A mapping,

b. A one-to-one mapping,

c. An onto mapping,

d. A bijective mapping:



Solution

Let the pigeons be numbered *a, b, c,* d and the pigeonholes be numbered *1,2,3.*

a. It is possible to establish a mapping between the two sets. For example, $\{(a,1);(b,2);(c,3);(d,3)\}$.This function is a mapping since each pigeon is accommodated in exactly one pigeonhole, though pigeons *c* and *d* are in the same pigeonhole.

b. It is not possible to establish a one-to-one mapping, since sharing images is not allowed. A function from one finite set to a smaller finite set cannot be one-to-one: there must be at least two elements that have the same image

c. The example given in part (a) illustrates a mapping that is onto: no pigeonhole is empty.

d. It is impossible to define a bijection, since it is already impossible to establish a one-to-one mapping

Example 5

Determine whether function $f:\mathbb{R}\to\mathbb{R} \quad x\mapsto f(x)=3x-5$ is (or is not)

a. One-to-one

b. Onto

c. Bijective.

Solution

a. Let x_1 and x_2 be real numbers such that $f(x_1) = f(x_2)$.Then $3x_1 - 5 = 3x_2 - 5$

This is equivalent, successively to $3x_1 = 3x_2$ (by adding 5 on both sides);

$x_1 = x_2$ (Dividing both sides by 3)

Since the equality $f(x_1) = f(x_2)$ implies $x_1 = x_2$, the function is one-to-one.

b. Suppose *y* a real number. Let us look for real number *x*, if possible, such that $f(x) = y$.

Then $3x - 5 = y$.It follows that $x = \frac{y+5}{3}$; such *x* exists for any value of *y*; $f(\frac{y+5}{3}) = 3\left(\frac{y+5}{3}\right) - 5 = y$

Therefore, function *f is onto.*

c. Since, from points (a) and (b), *f* is one-to-one and onto, function *f* is bijective.

Example 6

Show that function *f* defined by $f: \mathbb{R} \to \mathbb{R} \quad x \mapsto f(x) = \mathrm{x}^2 + 2x - 3$ is neither one-to-one, nor onto

Solution

$f(-2) = (-2)^2 + 2(-2) - 3 = -3$ and $f(0) = (0)^2 + 2(0) - 3 = -3$

Since $f(-2) = f(0)$ and $-2 \neq 0$, the function is not one-to-one.

On the other side, there is no *x* such that $f(x) = -5$;

in fact, $f(x) = -5 \Leftrightarrow x^2 + 2x - 3 = -5$

$$\Leftrightarrow x^2 + 2x + 2 = 0$$

No such x since $\Delta = 2^2 - 4(1)(2) = -4 < 0$

Therefore, the function is not onto

Example 7

Consider the function f defined by $f: \mathbb{R} \to \mathbb{R}$

$$x \mapsto f(x) = -x^2 + 4x$$

Determine the greatest subsets A and B of $\mathbb{R}$ such that function

$f: A \to B$

$x \mapsto f(x) = -x^2 + 4x$ is bijective

Solution

The maximum value of the function occurs for $x = 2$ and $f(2) = 4$ .Therefore,

$A = [2, +\infty[$ and $B =]-\infty, 4]$,or $A =]-\infty, 2]$ and $B =]-\infty, 4]$

Example 8

Function $f: \mathbb{R} - \{a\} \to \mathbb{R} - \{b\}$

$$x \mapsto f(x) = \frac{2x-5}{3-x} \text{ is bijective}$$

a. Find the values of a and b

b. Show that f is one-to-one

c. Find the real number whose image is 2

Solution

a. f is bijective if $a=3$ and $b=-2$

b. Let $x_1 \neq 3$ and $x_2 \neq 3$ be such that $f(x_1)=f(x_2)$, that is $\frac{2x_1-5}{3-x_1}=\frac{2x_2-5}{3-x_2}$

Then $6x_1-2x_1x_2-15+5x_2=6x_2-2x_1x_2-15+5x_1$,

which is equivalent to $6(x_1-x_2)-5(x_1-x_2)=0 \Leftrightarrow x_1-x_2=0 \Leftrightarrow x_1=x_2$

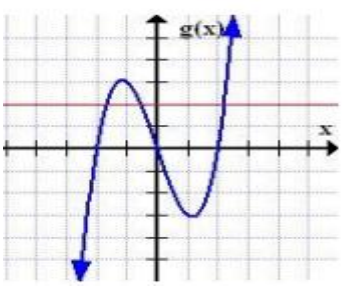
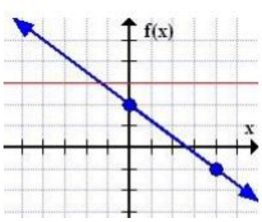
Therefore, f is one- to- one

c. Let x be the number. Then $f(x)=2 \Leftrightarrow \frac{2x-5}{3-x}=2$

Solving this equation, we get $x=2$

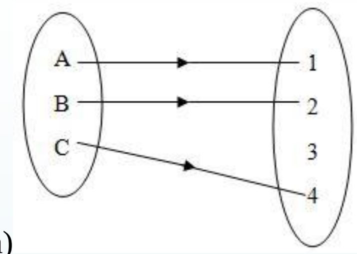
Horizontal Line Test

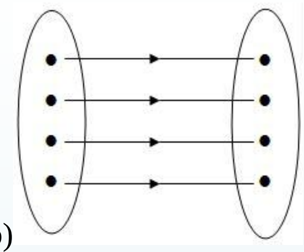
Horizontal Line Test states that a function is a one to one(injective) function if there is no horizontal line that intersects the graph of the function at more than one point.

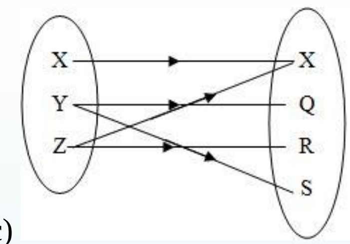
Graph representation	**Interpretation**	**Conclusion**
	You can see that for this graph, there are horizontal lines that intersect the graph more than once.	Not injective
	You can see that for this graph any horizontal line intersects the graph only once.	Injective function

Application activity 3.1

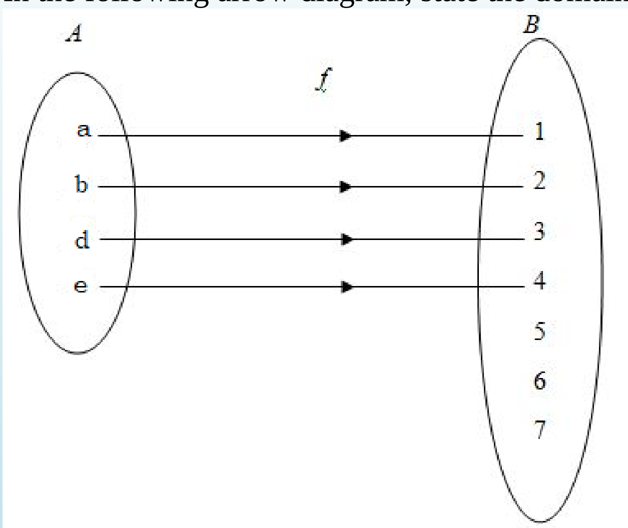
1) State which of the following relations shows a function

(a) 

(b) 

c) 

2) In the following arrow diagram, state the domain, co-domain and range



3) If $f(x) = 2x + 4$, find

a. $f(2)$

b. $f(-2)$

c. $f(d)$

4) Let W and Z be sets;

$W \times Z$ the Cartesian product of W and Z ;

C be the set of all correspondences (relations) from W to Z ;

F be the set of all functions from W to Z ;

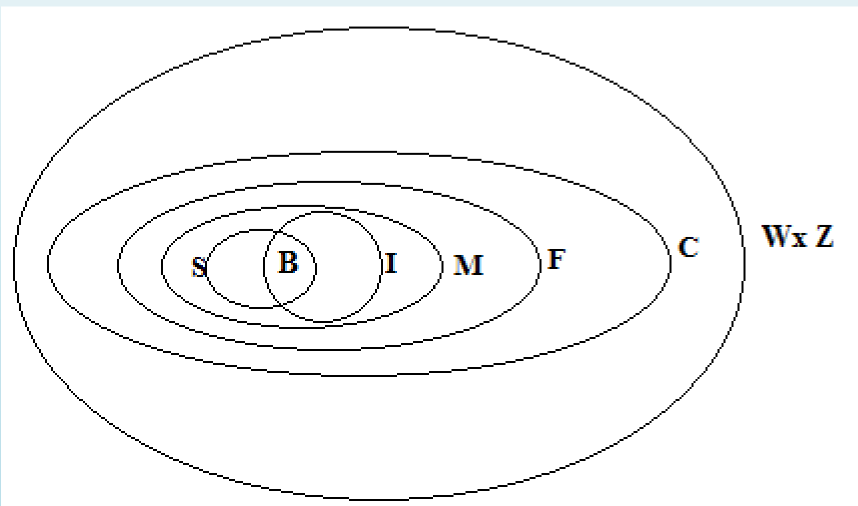
M be the set of all mappings from W to Z ;

I be the set of all one to one mappings from W to Z ;

S be the set of all onto mappings from W to Z ;

B be the set of all bijective mappings from W to Z ;

Then we have the following sequence of inclusion of sets



Using examples, explain in your own words the relationship amongst these sets and decide on the following inclusion: $S \subset M \subset F \subset C \subset (W \times Z)$.

3.2 Types of numerical functions

Activity 3.2

Differentiate rational from irrational numbers. Guess which of the following functions are polynomials, rational or irrational functions:

1. $f(x)=(x+1)^2$ 2. $h(x)=\frac{x^3+2x+1}{x-4}$ 3. $f(x)=\sqrt{x^2+x-2}$

Content summary

a) Constant function

A function that assigns the same value to every member of its domain is called **a constant function**. This is $f(x)=c$ where c is a given real number.

The function f given by $f(x)=3$ is constant.

Remark

The constant function that assigns the value c to each real number is sometimes called **the constant function *c***.

b) Identity: The identity function is of the form $f(x)=x$

c) Monomial

A function of the form cx^n, where c is constant and n a nonnegative integer is called a **monomial in *x***.

Examples

1) $f(x)=0.3x$ is a monomial in x.

2) $2x^3;\ \pi x^7;\ 4x^0;\ -6x\ and\ x^{17}$ are also monomials

The functions $4x^{\frac{1}{2}}$ and x^{-3} are not monomials because the powers of x are not nonnegative integers.

d) Polynomial

A function that is expressible as the sum of finitely many monomials in *x* is called

polynomial in *x*.

Examples

1) x^3+4x+7 and $17-\frac{2}{3}x$ are polynomials.

2) $(x-2)^3$ is a polynomial in x because it is expressible as a sum of monomials.

In general, f is a polynomial in x if it is expressible in the form $f(x)=a_0+a_1x+a_2x^2+......+a_nx^n$ where n is a nonnegative integer and $a_0, a_1, ..., a_n$ are real constants.

A polynomial is called

- **Linear** if it has the form $a_0+a_1x,\ a_1 \neq 0$, with degree 1;
- **Quadratic** if it has the form $a_0+a_1x+a_2x^2,\ a_2 \neq 0$, with degree 2;
- **Cubic** if it has the form $a_0+a_1x+a_2x^2+a_3x^3,\ a_3 \neq 0$ with degree 3;
- **n^th^ degree polynomial** if it has the form $a_0+a_1x^1+a_2x^2+a_3x^n+...+a_nx^n; a_n \neq 0$**,** with degree n.

e) Rational function

A function that is expressible as ratio of two polynomials is called **rational function**. It has the form $f(x)=\frac{a_0+a_1x+a_2x^2+......+a_nx^n}{b_0+b_1x+b_2x^2+......+b_mx^m}$.

Example

$f(x)=\frac{x^2+4}{x-1}$ *and* $g(x)=\frac{1}{3x-5}$ are rational functions

f) Irrational function

A function that is expressed as root extractions is called irrational function. It has the form $\sqrt[n]{f(x)}$, where $f(x)$ is a polynomial or rational function and n is positive integer greater or equal to 2.

Example

$f(x)=\frac{\sqrt{x^2+4}}{\sqrt[3]{x-1}}$, $g(x)=\sqrt{\frac{x}{x-5}}$ are irrational functions

Application activity 3.2

What is the type of the following function?

1) $f(x)=x^3+2x^2-2$ 2) $g(x)=-2$

3) $h(x)=\frac{x^3+2x^2-2}{x-5}$ 4) $f(x)=\frac{x^2-2}{x^2-8x+15}$

Provide other types of functions and explain your reasons with examples

3.3 Domain of definition for a numerical function

3.3.1 Domain of definition for polynomial function

Activity 3.3.1

For which value(s) of x, the following functions are not defined

1) $f(x)=x^3+2x+1$ 2) $f(x)=\frac{1}{x}$ 3) $g(x)=\frac{x+2}{x-1}$

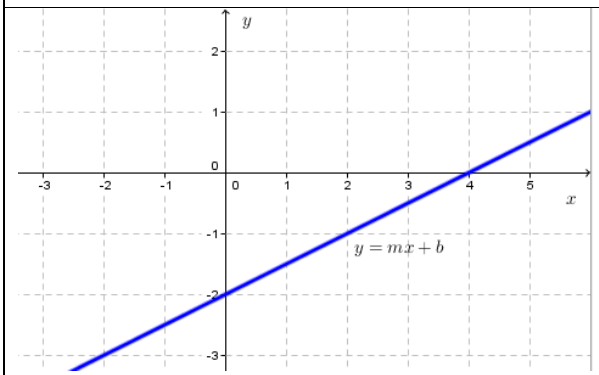
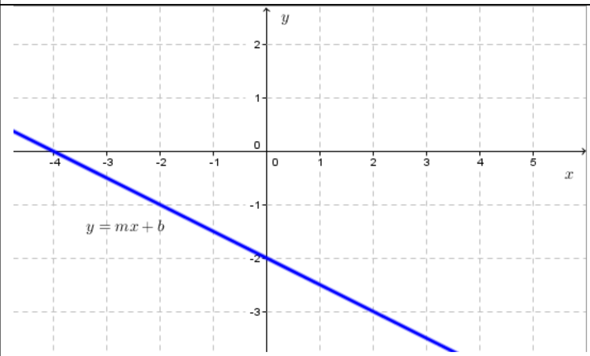
Content summary

The domain of any **linear function** f is the set of all real numbers, that is

$Domf=\Re$ or Dom $f=]-\infty,+\infty[$.

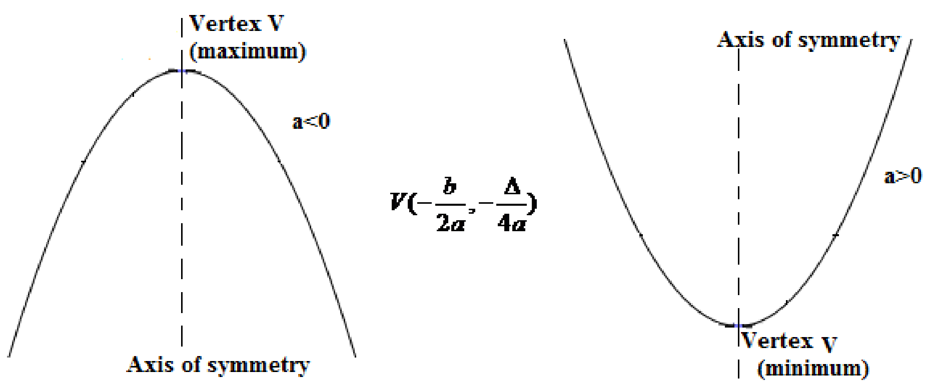
Similarly, the **range** (the set of all values $f(x)$ for all $x \in Domf$) of a linear function f, denoted $\operatorname{Im} f$ is the set of all real numbers, that is $\operatorname{Im} f =]-\infty, +\infty[$.

Depending on the sign of m *in the equation* $y = mx + b$, the trend of the graph is as follows:

$m > 0$	$m < 0$
	

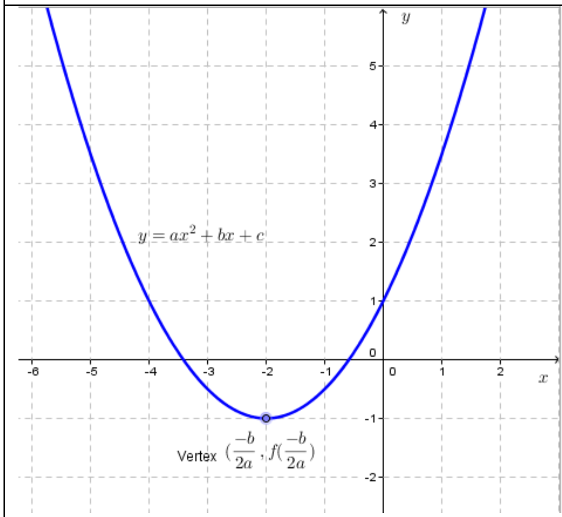
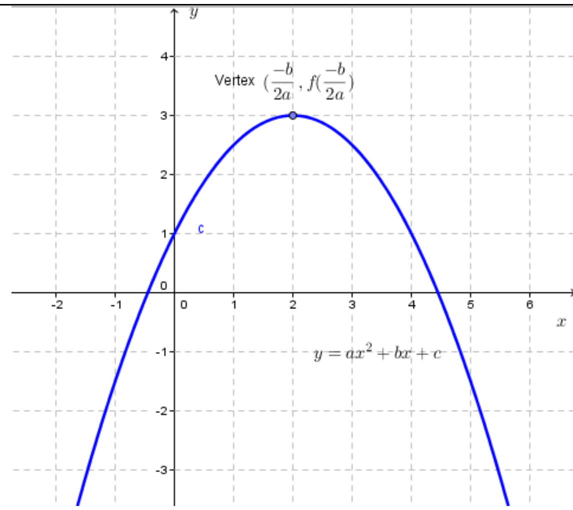
From the graphs, one can observe that each value of x has its corresponding y value.

For quadratic functions $y = ax^2 + bx + c$, the main features are summarized on the graph below:

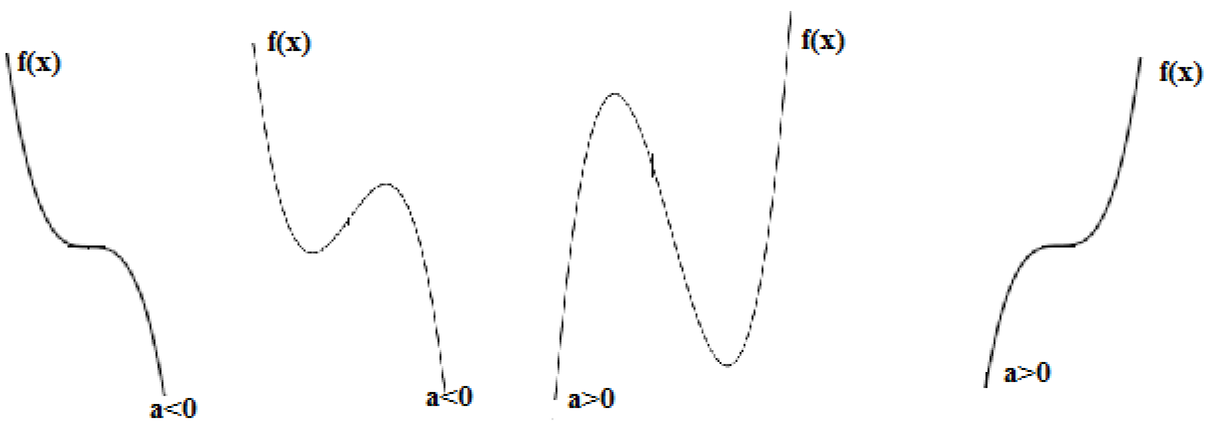


If $a > 0$, then the range of function $y = ax^2 + bx + c$ is $[-\frac{\Delta}{4a}, +\infty[$

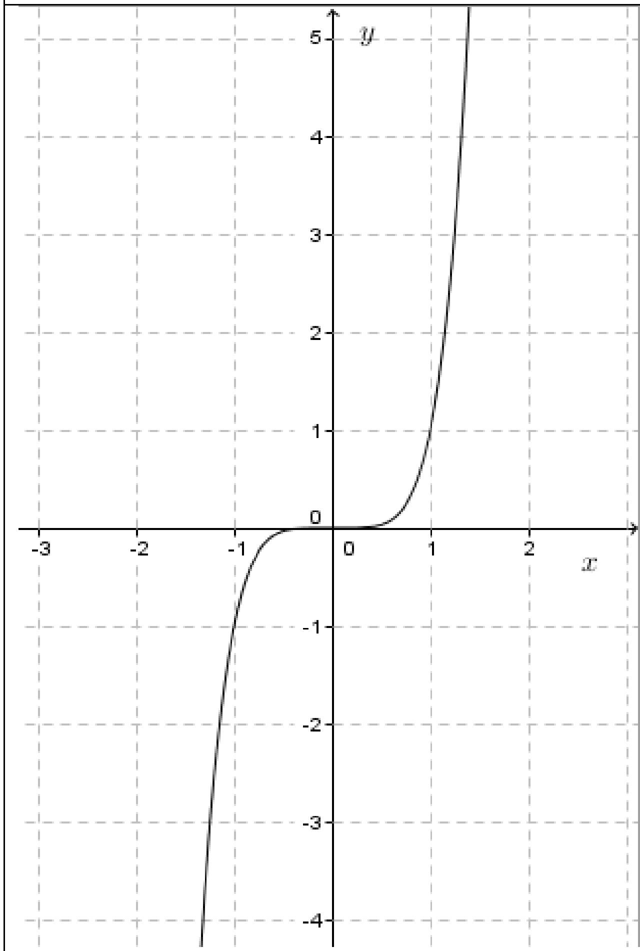
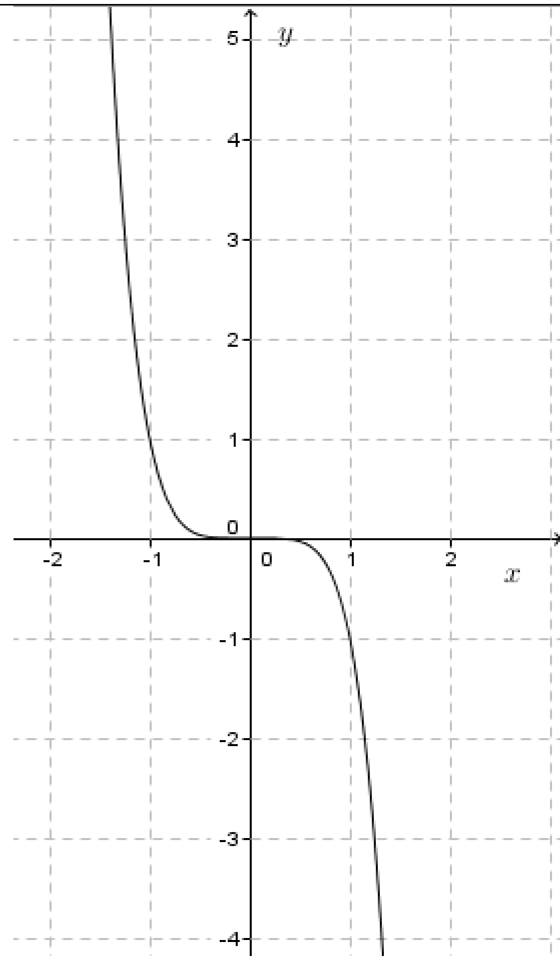
If $a < 0$, then the range of function $y = ax^2 + bx + c$ is $]-\infty, -\frac{\Delta}{4a}]$

$a > 0$, minimum	$a < 0$, maximum
	

For cubic functions $f(x) = ax^3 + bx^2 + cx + d; a \neq 0$, the trends of the graphs are as shown below:



In each case, the domain is $]-\infty, +\infty[$ and the range is $]-\infty, +\infty[$.

$a > 0$	$a < 0$
	

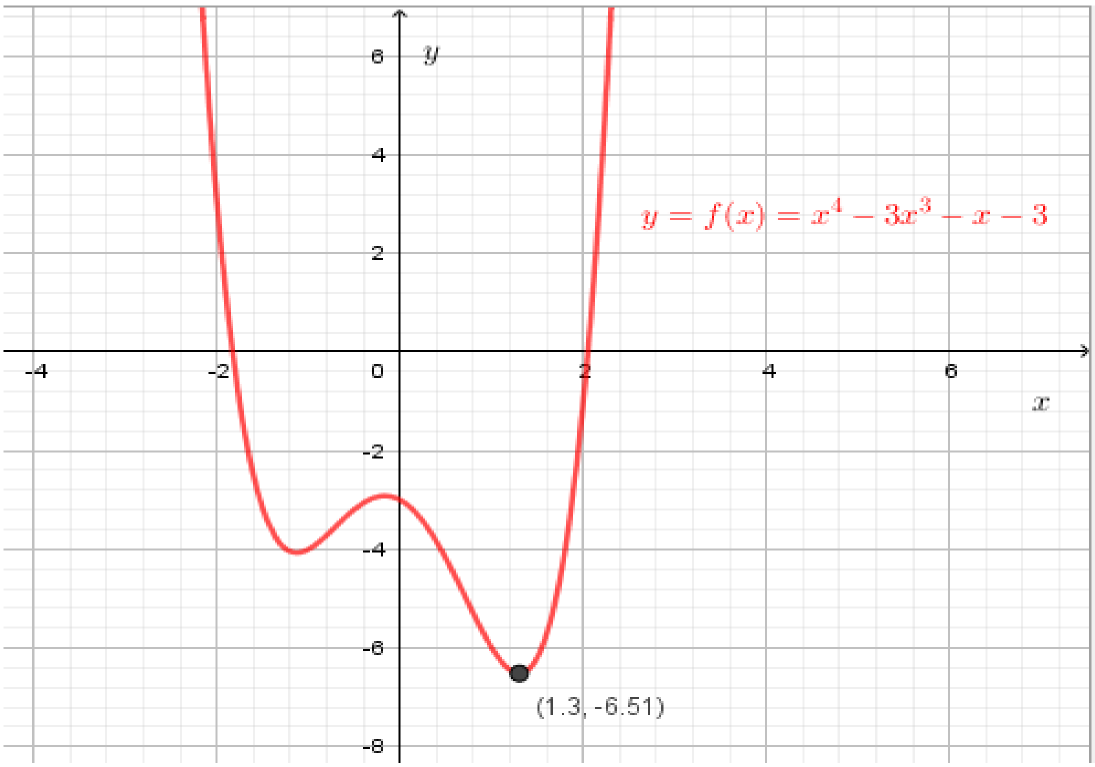
$D =]-\infty, +\infty[$ and $\mathbb{R} =]-\infty, +\infty[$.

Even though the range for polynomials of odd degrees is the set of all real numbers, it is not the case for polynomials of even degree greater or equal to 4.

The determination of the range is not easy unless the function is given by its graph; in this case, find by inspection, on the *y*-axis, the set of all points such that the horizontal lines through those points cut the graph.

Example 1

1) Determine the domain and range of $f(x) = x^4 - 3x^3 - x - 3$ shown on the graph below:



Solution

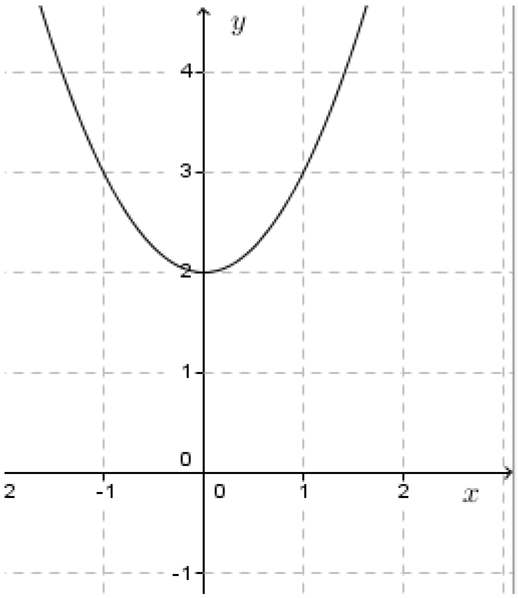
$domf =]-\infty, +\infty[$ and $\text{Im } f = [-6.51, +\infty[$

Example 2

Find the range for the function $f(x) = x^2 + 2$

Solution

Let $f : \mathbb{R} \to \mathbb{R}$ be the function given by $f(x) = x^2 + 2$. Then the range is $[2, +\infty[$ as shown on the graph below:



Application activity 3.3.1

Determine the domain of definition and range of the following functions

a) $f(x) = -x^3 + 2x - 1$

b) $f(x) = x^4 - 4$

c) $f(x) = 2x^5 - 3x^2 + 2x - 1$

d) $f(x) = x^2 + 3$

3.3.2 Domain of definition for a rational function

Activity 3.3.2

Find the value(s) of x for which the following functions are not defined

1) $f(x) = x^3 + 2x^2 - 2$

2) $g(x) = -2$

3) $h(x) = \dfrac{x^3 + 2x^2 - 2}{x - 5}$

4) $f(x) = \dfrac{x^2 - 2}{x^2 - 8x + 15}$

Content summary

Given that $f\left(x\right)=\dfrac{g\left(x\right)}{h\left(x\right)}$ where $g\left(x\right)$ and $h\left(x\right)$ are polynomials, then the domain of definition is the set of real numbers excluding all values where the denominator is zero. That is $Domf=\left\{x\in\mathbb{R}:h(x)\neq 0\right\}$

Example 1

Find the domain of each of the following functions:

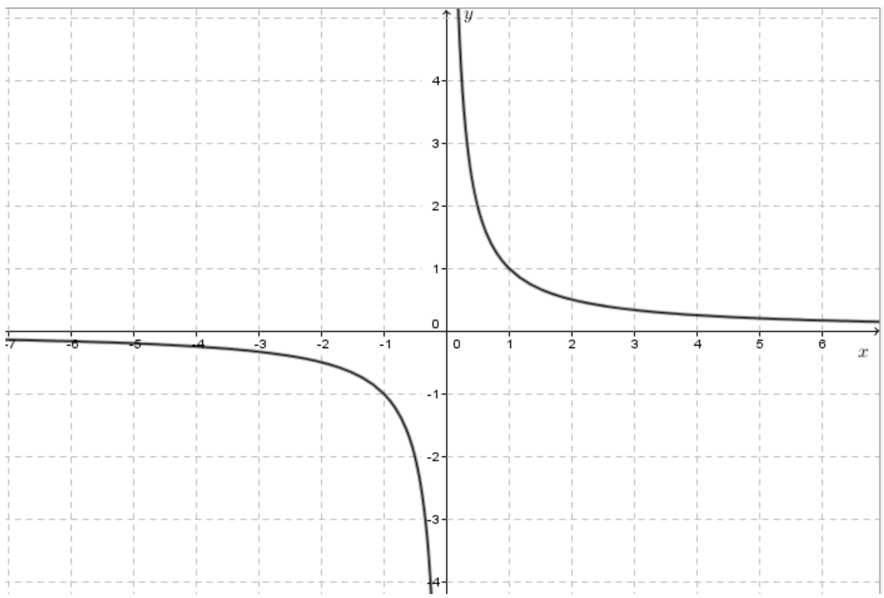
a) $f(x)=\dfrac{1}{x}$

b) $f(x)=\dfrac{x}{(x-1)(x+3)}$

c) $f(x)=\dfrac{x+1}{3x+6}$

Solution

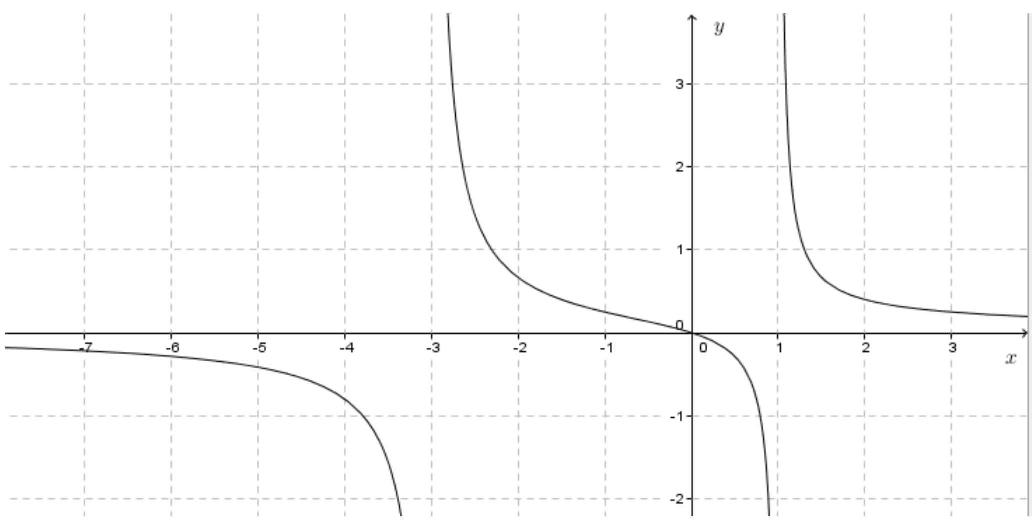
a) The denominator should be different from zero ($x\neq 0$), domain of definition is $\left\{x\in\mathbb{R}:x\neq 0\right\}$ or $\mathbb{R}^{*} or\, \mathbb{R}^{+}$. The domain can be written as an interval as follows: $\left]-\infty,0\right[\cup\left]0,+\infty\right[$. Observing the graph of the function $f(x)=\dfrac{1}{x}$, one can easily realize that the function has no value only if $x=0$.



b) $f(x)=\dfrac{x}{(x-1)(x+3)}$. The denominator should be different from zero, $(x-1)(x+3)\neq 0$.

Domain of definition is $\{x\in\mathbb{R}: x\neq 1 \text{ and } x\neq -3\}$ or simply $]-\infty,-3[\cup]-3,1[\cup]1,+\infty[$.

Observing the graph of the function $f(x)=\dfrac{x}{(x-1)(x+3)}$, one can early realize that the function has no value only if $x=1$ *and* $x=-3$.



c) Condition: $3x+6\neq 0$

$3x+6=0 \Rightarrow x=-2$

Then, $Domf=\mathbb{R}\setminus\{-2\}$ or $Domf=]-\infty-2[\cup]-2,+\infty[$

Example 2

Find the range for the function $f(x)=\dfrac{1}{x-2}$

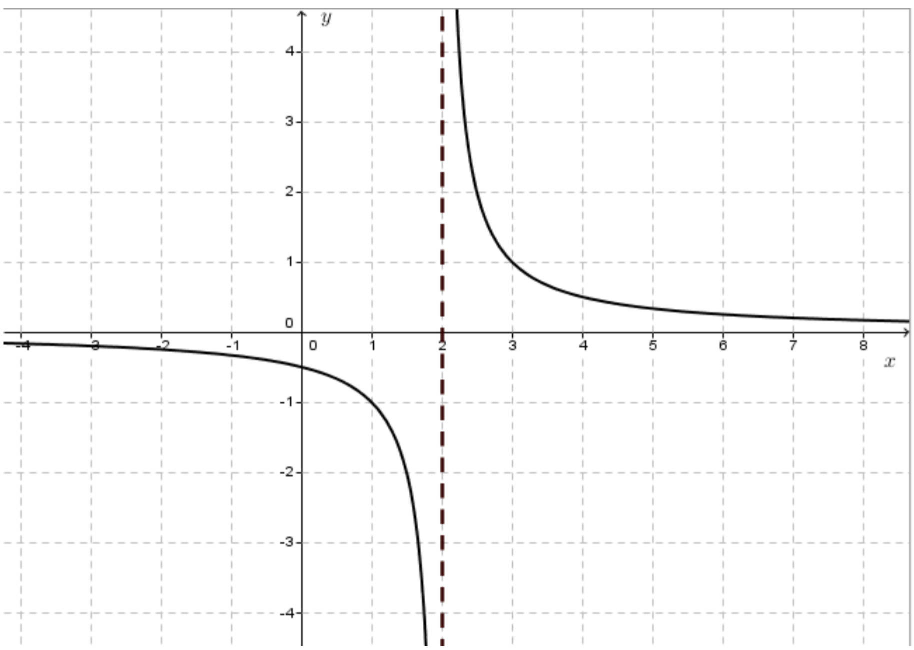
Solution

Put $y=f(x)=\dfrac{1}{x-2}$

Solve for x, $y=\dfrac{1}{x-2}\Leftrightarrow x=\dfrac{1}{y}+2$. Note that x can be solved if and only if $y\neq 0$.

The range of $f(x)$ is $\{y\in\mathbb{R}: y\neq 0\}=\mathbb{R}\setminus\{0\}$.

Alternatively, one can see on the graph that the range of $f(x)$ is $\mathbb{R}\setminus\{0\}$.



Example 3

Find the range for the function $f(x)=\dfrac{2x+1}{x^2+2}$

Solution

Put $y=f(x)=\dfrac{2x+1}{x^2+1}$

Experimental version

Solve for x, $y=\frac{2x+1}{x^2+1} \Leftrightarrow yx^2+y=2x+1$.

$yx^2+y=2x+1 \Leftrightarrow yx^2-2x+(y-1)=0$

$$x=\frac{2\pm\sqrt{4-4y(y-1)}}{2y} \text{ if } y\neq 0, \quad x=-\frac{1}{2} \text{ if } y=0$$

$$=\frac{1\pm\sqrt{1-y^2+y}}{y} \text{ if } y\neq 0$$

Comparing the two, we see that x exists in the set of real numbers if and only if $1-y^2+y\geq 0$, that is $y^2-y-1\leq 0$

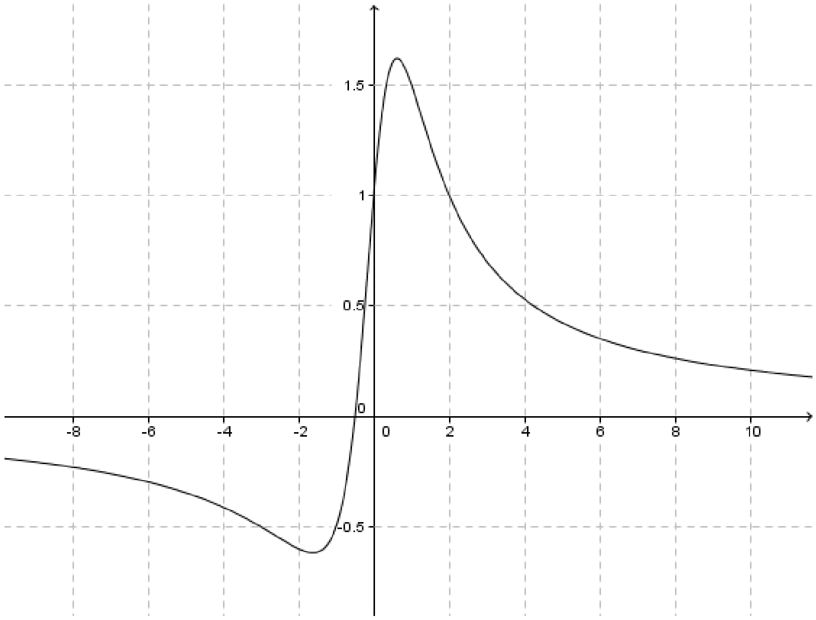
The range of $f(x)$ is $\{y\in\mathbb{R}: y^2-y-1\leq 0\}$. Solving the inequality $y^2-y-1\leq 0$, we get $y=\frac{1\pm\sqrt{5}}{2}$. Then studying the sign of the quadratic expression, $y^2-y-1=\left(y-\frac{1-\sqrt{5}}{2}\right)\left(y-\frac{1+\sqrt{5}}{2}\right)$.

	$y<\frac{1-\sqrt{5}}{2}$	$y=\frac{1-\sqrt{5}}{2}$	$\frac{1-\sqrt{5}}{2}<y<\frac{1+\sqrt{5}}{2}$	$y=\frac{1+\sqrt{5}}{2}$	$y>\frac{1+\sqrt{5}}{2}$
$y-\frac{1-\sqrt{5}}{2}$	$------$	0	$++++$	$+++++$	$+++$
$y-\frac{1+\sqrt{5}}{2}$	$-----$	$---$	$------$	0	$+++$
y^2-y-1	$++$	0	$---$	0	$++++$

From the table, we see that the range of the function $f(x)=\frac{2x+1}{x^2+2}$ is:

$$R(f)=\left\{y\in\mathbb{R}:\frac{1-\sqrt{5}}{2}\leq y\leq\frac{1+\sqrt{5}}{2}\right\}=\left[\frac{1-\sqrt{5}}{2},\frac{1+\sqrt{5}}{2}\right]$$

One can see on the graph that the range of $f(x)$ is $\left[\frac{1-\sqrt{5}}{2},\frac{1+\sqrt{5}}{2}\right] \simeq [-0.618034, 1.61803]$.



Application activity 3.3.2

Determine the domain of definition for the following functions

a) $f(x)=\frac{x}{1-x}$ b) $f(x)=\frac{x+1}{x^2-2}$ c) $f(x)=\frac{x^2+1}{x^2-1}$

d) $g(x)=1+\frac{3}{x^2-3x+2}$ e) $f(x)=3-\frac{x}{x\left(x^2+4x-5\right)}$

3.3.3 Domain of definition for a numerical function

Activity 3.3.3

For each of the following functions, give a range of values of the variable x for which the function is not defined

1) $f(x)=\sqrt{2x+1}$ 2) $f(x)=\sqrt[3]{x^2+x-2}$ 3) $g(x)=\sqrt{\frac{x-2}{x+1}}$

Content summary

Given that $f(x)=\sqrt[n]{g(x)}$ where $g(x)$ is a polynomial, there are two cases

a) If n is odd number, then the domain is the set of real numbers. That is $domf=\mathbb{R}$

b) If n is even number, then the domain is the set of all values of x such $g(x)$ is positive or zero. That is $domf=\{x\in\mathbb{R}: g(x)\geq 0\}$

Example 1

Given the function $g(x)=\sqrt{x^2-1}$, determine the domain.

Solution

Condition of existence: $x^2\text{-}1\geq 0$ this implies that we need to determine the interval where $x^2\text{-}1$ is positive. The corresponding equation is $x^2\text{-}1=0$, solving for the variable, we obtain:

$$x^2\text{-}1=0 \Leftrightarrow (x-1)(x+1)=0$$
$$x\text{-}1=0 \Rightarrow x=1$$
$$x+1=0 \Rightarrow x=\text{-}1$$

x	$-\infty$ -1 1 $+\infty$
$x+1$	- - - 0 + + + + + + +
$x-1$	- - - - - - - 0 + + + + + + + +
x^2-1	+ + + 0 - - - - 0 + + + + + + +

Therefore $Domf=]-\infty,-1]\cup[1,+\infty[$

Example 2

Find the domain of the function $h(x)=\dfrac{\sqrt{1-x^2}}{x}$

Solution

Conditions of existence $1-x^2 \geq 0$ and $x \neq 0$

- For the corresponding equation is $1-x^2=0$ and solving for the variable, we get:
  $(1-x)(1+x)=0 \Rightarrow \begin{cases} 1-x=0 \text{ or } x=1 \\ 1+x=0 \text{ or } x=-1 \end{cases}$

- For $x \neq 0$ all real numbers are accepted except from zero. Combining the two conditions we get:

x	$-\infty$ -1 0 1 $+\infty$
$1+x$	- - - - 0 + + + + + + + + + + +
$1-x$	+ + + + + + + + + + 0 - - - - -
$1-x^2$	- - - 0 + + + + + + 0 - - - - -
x	- - - - - - - 0 + + + + + + + + +
$f(x)=\dfrac{\sqrt{1-x^2}}{x}$	undefined 0 - - - || + + + + + 0 undefined

$-1 \leq x \leq 1$ and $x \neq 0$. Therefore, $domf = [-1,0[\cup]0,1]$

Example 3

Find domain of definition of $f(x)=\sqrt[3]{x+1}$

Solution

Since the index in radical sign is odd number, then $Domf = \mathbb{R}$

Example 4

Find the domain of definition of $g(x)$ if $g(x)=\sqrt[4]{x^2+1}$

Solution

Condition: $x^2+1\geq 0$

Clearly x^2+1 is always positive. Thus $Domg=\mathbb{R}$

Example 5

Find domain of $f(x)=\dfrac{x}{\sqrt{x^3-4x^2+x+6}}$

Solution

Conditions: $x^3-4x^2+x+6\geq 0$ and $x^3-4x^2+x+6\neq 0$.

The two conditions are combined in one: $x^3-4x^2+x+6\geq 0$ and $x^3-4x^2+x+6\neq 0$

$x^3-4x^2+x+6=(x+1)(x-2)(x-3)$

x	$-\infty$		-1		2		3		$+\infty$
$x+1$		-	0	+	+	+	+	+	
$x-2$		-	-	-	0	+	+	+	
$x-3$		-	-	-	-	-	0	+	
x^3-4x^2+x+6		-	0	+	0	-	0	+	

Then, $Domf=]-1,2[\cup]3,+\infty[$

Example 6

Find the range for the function $f(x) = \sqrt{1+5x}$

Solution

$1+5x \geq 0$ (Restrictions on x);

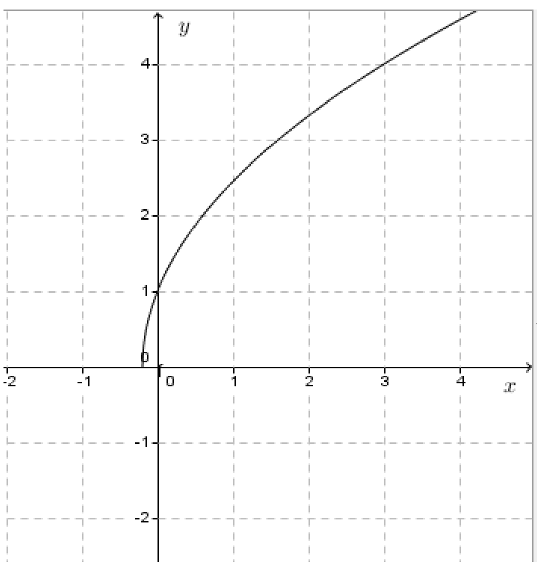
$\sqrt{1+5x} \geq 0$ (Taking the square roots);

But $f(x) = \sqrt{1+5x}$;

Therefore, $f(x) \geq 0$

The range of $f(x)$ is $\operatorname{Im} f = [0,+\infty[$.

The graph below illustrates the range:



Application activity 3.3.3

Find the domain of definition for each of the following functions

1. $f(x)=\sqrt{4x-8}$
2. $g(x)=\sqrt{x^2+5x-6}$
3. $h(x)=\dfrac{x^3+2x^2-2}{\sqrt[3]{x+4}}$
4. $f(x)=\dfrac{x-2}{\sqrt[4]{x^2-25}}$
5. $f(x)=\sqrt{\dfrac{(x-1)^2}{x+4}}$

3.4 Parity of a function (odd or even)

Activity 3.4

For each of the following functions, find $f(-x)$ and $-f(x)$. Compare $f(-x)$ and $-f(x)$ using $=$ or $\neq$

1) $f(x)=x^2+3$ 2) $f(x)=\sqrt[3]{x^3+x}$ 3) $f(x)=\dfrac{x^2-3}{x^2+1}$

Content summary

Even function

Let $f(x)$ be a numerical function whose domain is $Domf$.

$f(x)$ is said to be an **even function** if and only if:

i) $(\forall x\in Domf),-x\in Domf$

ii) $f(-x) = f(x)$, that is, any two opposite values of the independent variable have the same image under the function.

The graph of an even function is symmetrical about the *y*-axis.

Example 1

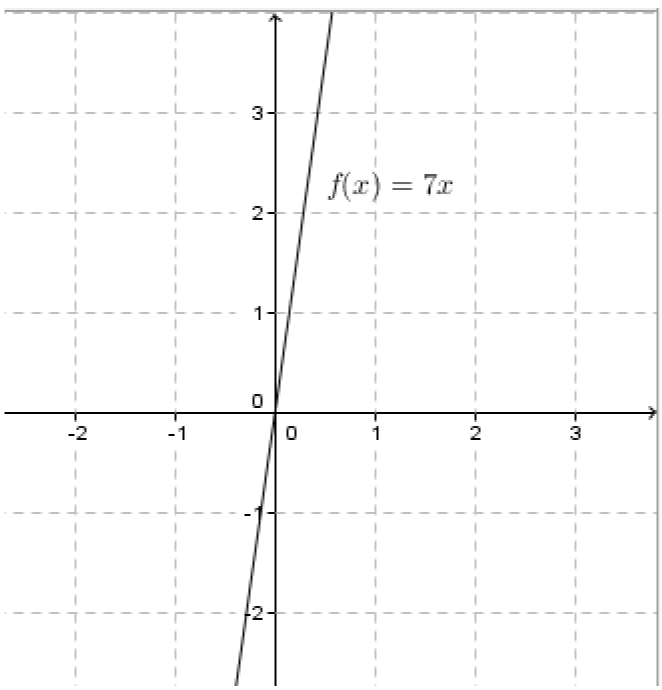
Determine whether the function $f(\mathrm{x}) = 7\mathrm{x}$ is even or not.

Solution

The domain of function *f* is the set of all real numbers. For any real number x, the opposite –x is also a real number and $f(-x) = -7x \neq 7x = f(x)$.

Since $f(-x) \neq f(x)$, function *f* is not even.

Graphically,



The graph is not symmetrical about the *y*-axis

Example 2

Determine whether the function $f(x) = 3x^2 - 4$ is even or not.

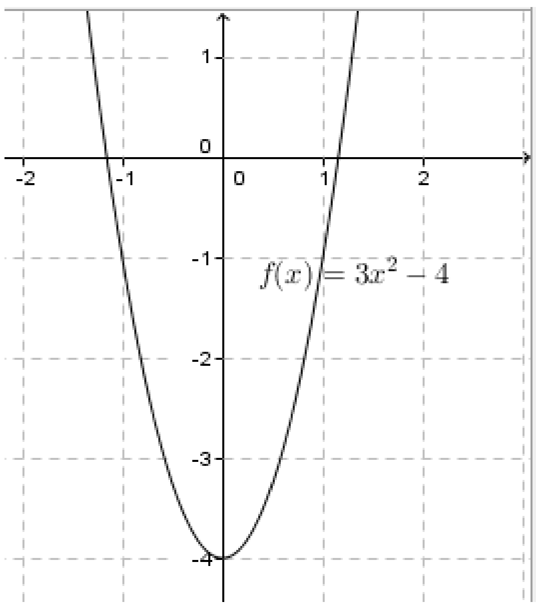
Solution

$f(-x) = 3(-x)^2 - 4 = 3x^2 - 4 = f(x)$

Since $f(-x) = f(x)$, function *f* is even.

Remember that $(-x)^n = \begin{cases} x^n, if \text{ n is even} \\ -x^n, if \text{ n is odd} \end{cases}$

The graph of the function is symmetrical about the y-axis as shown on the diagram below:



Example 3

Determine whether the function $g(x) = x^6 - x^4 + x^2 + 9$ is even or not.

Solution

$$g(-x) = (-x)^6 - (-x)^4 + (-x)^2 + 9$$
$$= x^6 - x^4 + x^2 + 9$$
$$= g(x)$$

Therefore, the function is even.

Example 4

Determine whether the function $f(x)=\dfrac{3x+1}{x^2-25}$ is even or not.

Solution

$f(-x)=\dfrac{3(-x)+1}{(-x)^2-25} \Leftrightarrow f(-x)=\dfrac{-3x+1}{x^2-25}$. Therefore the function is not even since $f(-x)\neq f(x)$

Example 5

Given functions $f(x)=\sqrt{x}$ and $g\left(x\right)=\sqrt{x-4}$, find the $f(x).g\ (x)$ and determine if the result is an even function or not.

Solution

Provided $x\geq 0$ and $x-4\geq 0$,

$f(x).g\left(x\right)=\sqrt{x}\ .\ \sqrt{x-4} \Leftrightarrow f(x).g(x)=\sqrt{x^2-4x}$

$f.g(-x)=\sqrt{(-x)^2-4(-x)} \Leftrightarrow f.g(-x)=\sqrt{x^2+4x}$. Therefore the function is not even since $f(-x)\neq f(x)$

Notice that the conclusion could have been drawn from the fact that $x\geq 4$ does not imply $-x\geq 4$, thus function f is not even

Odd function

Let $f(x)$ be a numerical function whose domain is $Domf$.

$f(x)$ is said to be an **odd function** if and only if:

i. $(\forall x\in Domf),-x\in Domf$

ii. $f(-x)=-f(x)$, that is, any two opposite values of the independent variable have opposite images under the function.

The graph of an even function is symmetrical about the *origin*.

Note: Some functions are neither even nor odd

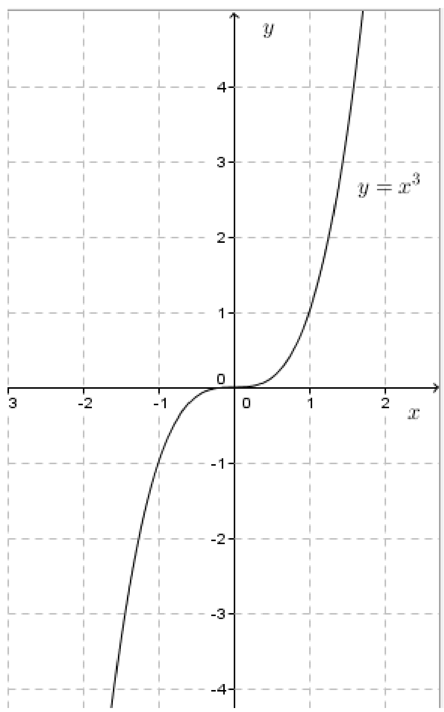
Example 6

Determine whether the function $f\left(x\right)=x^3$ is odd or not

Solution

$f(-x)=(-x)^3 \Leftrightarrow f(-x)=-x^3\ and -f(x)=-x^3$. Therefore, $f(-x)=-f(x)$ and the function $f(x)=x^3$ is odd.

Graphically, the point $(0,0)$ is the centre of symmetry for the graph of the function $f(x)=x^3$.



Example 7

Determine whether the function $f(x)=x^2+2x+1$ is odd ,even or neither

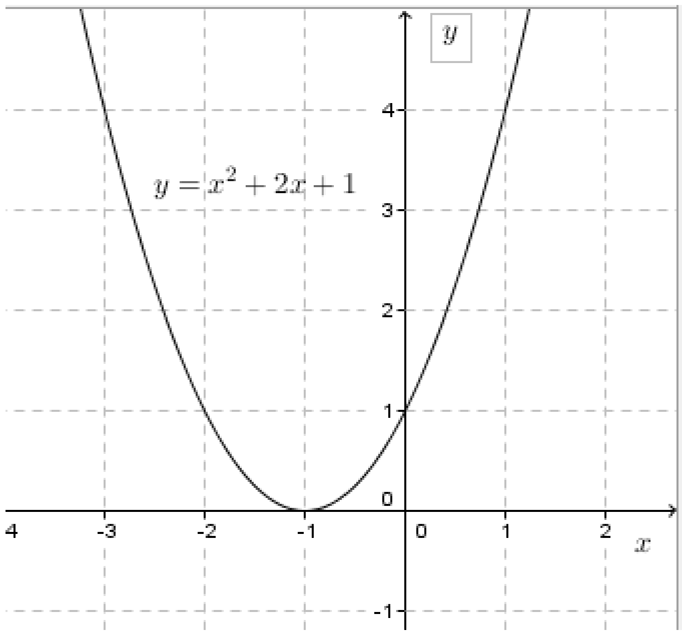
Solution

$f(-x)=(-x)^2+2(-x)+1 \Leftrightarrow f(-x)=x^2-2x+1 and - f(x)=-x^2-2x-1$.

$f(-x) \neq -f(x)$ and $f(-x) \neq f(x)$.

Therefore, the function $f(x) = x^2 + 2x + 1$ is not odd neither even.

Graphically, point $(0,0)$ is not the centre of symmetry for the graph of the function $f(x)$,and the line $x = 0$ is not the axis of symmetry for the graph of function $f(x) = x^2 + 2x + 1$.



Example 8

Determine if the function $f(x) = \frac{x^3}{x^2 - 1}$ is even , odd ,or neither and deduce the symmetry of its graph.

Solution

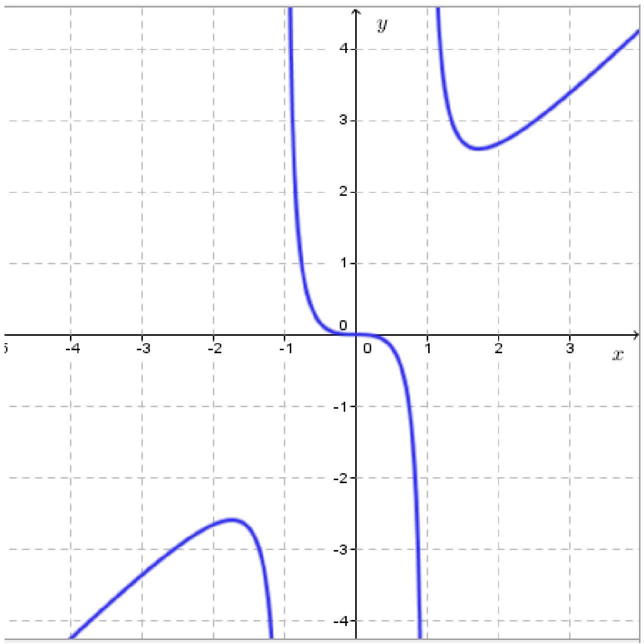
$$f(-x) = \frac{(-x)^3}{(-x)^2 - 1} = \frac{-x^3}{x^2 - 1}$$

$$\neq f(x)$$

Therefore, the function is not even

But, $f(-x) = -f(x)$;it follows that f is an odd function.

The graph of $f(x)=\dfrac{x^3}{x^2-1}$ is shown below. It can be seen that point $(0,0)$ is the centre of symmetry for the graph



Application activity 3.4

Study the parity of the following functions

1) $f(x)=2x^2+2x-3$

2) $f(x)=\dfrac{3x^3+2x^2+8}{x-5}$

3) $g(x)=x^3-x$

4) $h(x)=\dfrac{x^2+4}{x^2-4}$

5) $g(x)=x\left(x^2+x\right)$

3.5 Operations on functions

3.5.1 Addition, subtraction, multiplication and division of functions

Activity 3.5.1

Given the functions $f(x)=\frac{x+1}{2x-3}$ and $g(x)=x+1$, find

1) $f(x)+g(x)$

2) $f(x)-g(x)$

3) $f(x)\cdot g(x)$

4) $\frac{f(x)}{g(x)}$

Content summary

Just as numbers can be added, subtract, multiplied and divided to produce other numbers, there is a useful way of adding, subtracting, multiplying and dividing functions to produce other functions. These operations are defined as follows:

Given functions f and g, **sum** $f+g$**, difference** $f-g$**, product** $f\cdot g$ and **quotient** $\frac{f}{g}$ **,** are defined by

- $(f+g)(x)=f(x)+g(x)$
- $(f-g)(x)=f(x)-g(x)$
- $(f\cdot g)(x)=f(x)\cdot g(x)$
- $\left(\frac{f}{g}\right)(x)=\frac{f(x)}{g(x)}$

For the functions, $f+g$, $f-g$ and $f\cdot g$, the domain is defined to be the intersection of the domains of *f* and g and for $\frac{f}{g}$, as we have seen it, the domain is this intersection with **the points where** $g(x)=0$ **excluded.**

Example 1

Let f and g be the functions $f(x)=3x^4-5x^3+x-4$ and $g(x)=4x^3-3x^2+4x+3$. Find $(f+g)(x)$ and $(f-g)(x)$

Solution

$$
\begin{array}{rl}
+ & f(x)=3x^4-5x^3 \qquad +x \quad -4 \\
 & g(x)= \qquad 4x^3-3x^2+4x+3 \\
\hline
 & (f+g)(x)=3x^4-x^3-3x^2+5x-1
\end{array}
$$

$$
\begin{array}{rl}
- & f(x)=3x^4-5x^3 \qquad +x \quad -4 \\
 & g(x)= \qquad 4x^3-3x^2+4x+3 \\
\hline
 & (f-g)(x)=3x^4-9x^3+3x^2-3x-7
\end{array}
$$

Example 2

If $f(x)=\frac{9}{x+2}$ and $g(x)=x^3$. Find

a) $h(x)=f(x)+g(x)$

b) $t(x)=f(x)\times g(x)$

c) $k(x)=\frac{f(x)}{g(x)}$

Solution

$h(x)=f(x)+g(x)$	$t(x)=f(x)\times g(x)$ $=\frac{9}{x+2}\cdot x^3$ $=\frac{9x^3}{x+2}$	$k(x)=\frac{f(x)}{g(x)}$

$=\frac{9}{x+2}+x^3$ $=\frac{9+x^3(x+2)}{x+2}$ $=\frac{x^4+2x^3+9}{x+2}$		$=\frac{\frac{9}{x+2}}{x^3}$ $=\frac{9}{x+2}(\frac{1}{x^3})$ $=\frac{9}{x^2+2x^3}$

Application activity 3.5.1

1) Given the functions $f(x)=2x^3+5x-1$ and $g(x)=3x-4$. Find $(f+g)(x)$

2) Given the functions $f(x)=3x^3-5x^2+7x-4$ and $g(x)=2x^2-x+3$. Find $(f\cdot g)(x)$

3.5.2 Composite functions

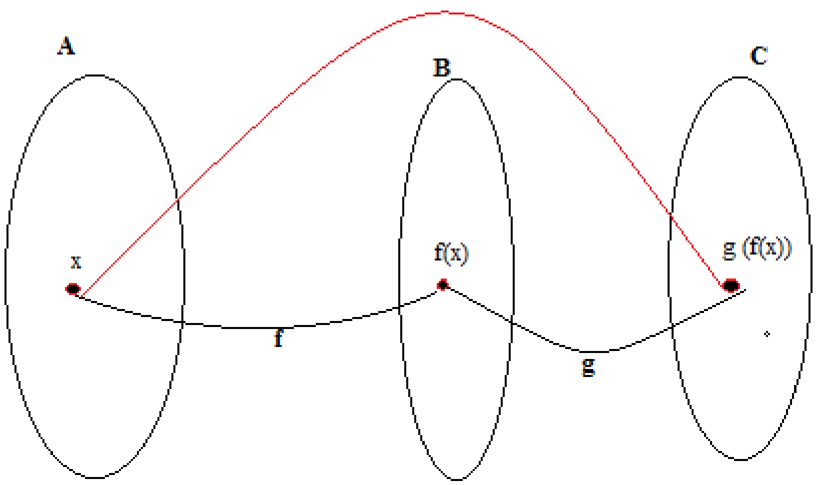
Activity 3.5.2

Consider two functions $f(x)=3x+2$ and $g(x)=x^2-1$:

1) Find the expressions $f(y)$ and $f(g(x))$
2) Find $g(f(x))$
3) Compare $f(g(x))$ and $g(f(x))$

Content summary

Consider the functions $f(x)=2x-1$ and $g(x)=x^2$.



$g_\circ f(x) = g(f(x))$

This combined or composite function is written $(g\circ f)(x)$ or $g[f(x)]$, simply gf. The function f is performed first and so is written nearer to the variable x. The set $\{1,3,5,7\}$ is the domain for the composite function and $\{1,25,81,169\}$ is the range.

Note that $(f\circ g)(x)\neq(g\circ f)(x)$

Example 1

If $f(x)=2x$ and $g(x)=3x+1$, express $g\circ f$ as a single function $h(x)$.

Solution

$f(x)=2x$ so $(g\circ f)(x)=g(2x)=3(2x)+1=6x+1$

$\therefore\ h(x)=6x+1$

Example 2

Let $f(x)=x-1$ and $g(x)=\sqrt{x}$, find $(f\circ g)(x)$ and $(g\circ f)(x)$

- $g(x)=\sqrt{x}$, *so* $fg(x)=f(\sqrt{x})=\sqrt{x}-1$ $\therefore\ (f\circ g)(x)=\sqrt{x}-1$
- $f(x)=x-1$, *so* $gf(x)=g(x-1)=\sqrt{x-1}$ $\therefore\ (g\circ f)(x)=\sqrt{x-1}$

Application activity 3.5.2

1) Let $f(x)=x^2$ *and* $g(x)=2x+1$. *find* $(fg)(x)$ *and* $(gf)(x)$
2) Given that

   $f(x)=x+3, g(x)=2x,$ *find* :

   a) $fg(x)$

   b) $gf(x)$
3) The functions

   $f(x)=2x-1$ and $g(x)=x^2+2$

   Find :

   a) $fg(x)$ b) $gf(x)$ c) $gf(3)$
4) Find $(fog)(x)$ and $(gof)(x)$

   a) if $f(x)=x^3-3x^2+1$ and $g(x)=2$

   b) if $f(x)=2x^2+x-3$ and $g(x)=6x$

3.5.3 The inverse of a function

Activity 3.5.3

Find the value of x in function of y if

1) $y = x + 1$
2) $y = 3x - 2$
3) $y = \frac{-x+3}{2x-1}$

Content summary

Consider a function f which maps each element x of the domain X onto its image y in the range Y that is $f : x \rightarrow y$ where $x \in X, y \in Y$. If this map can be reversed,

i.e. $f^{-1} : y \rightarrow x$ and the resulting relationship is a function, it is called the **inverse of the original function**, and is denoted by f^{-1}.

Only one-to-one functions can have an inverse function. To find the inverse of one-to-one functions, we change the subject of a formula.

Definition: Let $f(x)$ and $g(x)$ be two functions such that $f[g(x)] = x$ for each x in the domain of g and $g[f(x)] = x$ for each x in the domain of f.

Under these conditions, the function g is the inverse of f. The function g is denoted by $f^{-1}(x)$, which is read as "f - inverse". So $f\left(f^{-1}(x)\right) = x$ and $f^{-1}\left(f(x)\right) = x$

Note:

- If $f(x)$ is inverse of $g(x)$, then $g(x)$ is inverse of $f(x)$.
- The domain of f must be equal to the range of f^{-1}, and the range of f must be equal to the domain of f^{-1}.
- The notation $f^{-1}(x) \neq \frac{1}{f(x)}$

Steps to find inverse functions

Let $f : x \to \mathbb{R}$ be an injective function where $x \subseteq \mathbb{R}$. To find the inverse function of f means to find the domain of f^{-1} as well as a formula for $f^{-1}(y)$. If the formula for $f(x)$ is not very complicated, dom (f^{-1}) and $f^{-1}(y)$ can be found by solving the equation $y = f(x)$ for x.

(*Step 1*) Put $y = f(x)$.

(*Step 2*) Solve x in terms of y. The result will be in the form x = an expression in y.

(*Step 3*) From the expression in y obtained in Step 2, the range of f can be determined. This is the domain of f^{-1}. The required formula is $f^{-1}(y)$ = the expression in y obtained in Step 2.

Example 1

Find the inverse function of $f(x) = 2x + 3$.

Solution

Let us make x the subject of $y = 2x + 3$ as follows:

$y = 2x + 3$ (Solve for the variable x);

Then, $y - 3 = 2x$, thus, $x = \frac{y-3}{2}$

$\therefore f^{-1}(x) = \frac{x-3}{2}$

Example 2

Find the inverse of the function $f(x) = 3x - 1$

Solution

If $f(x) = 3x - 1$, we require $f^{-1}(y) = x$. If $y = 3x - 1$ then $x = \frac{y+1}{3}$

So, given *y*, we can return to *x* using the expression $\frac{y+1}{3}$. Thus, $f^{-1}(x)=\frac{x+1}{3}$

Application activity 3.5.3

Find the inverse of the following functions

a) $f(x)=5x+2$ b) $g(x)=-7x-2$ c) $h(x)=\frac{-2x+1}{x-2}$

3.6 Graphical representation and interpretation of linear and quadratic functions

3.6.1 Linear function

Activity 3.6.1

1) Copy and complete the tables below.

a)

x	-3	-2	-1	0	1	2	3
$y=2x-1$							

b)

x	-3	-2	-1	0	1	2	3
$y=x^2-1$							

2) Use the coordinates from each table to plot the graphs on separate Cartesian planes.

3) What is your conclusion about the shapes of the graphs?

Content summary

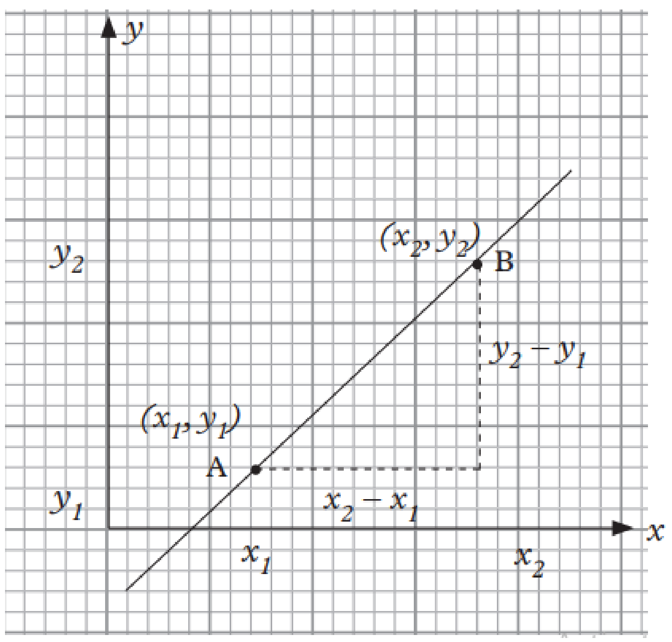
Definition: Any function of the form $f(x) = mx + b$, where m is not equal to 0 is called a linear function. The ***domain*** of this function is the set of all real numbers. The ***range*** of f is the set of all real numbers. The graph of f is a line with slope m and y intercept b.

Examples of linear functions

a) $a)\ y = x+1,\quad b)\ y = 2x-3, c)\ y = -3x+4, ...$

Graphs of linear functions

The ordered pair (x, y) represents coordinates of any point on the Cartesian plane.
Consider a line passing through the points $A(x_1, y_1)$ and $B(x_2, y_2)$.



From A to B, the change in the x-coordinate (horizontal change) is $x_2 - x_1$ and the change in the y-coordinate (vertical change) is $y_2 - y_1$.

By definition, gradient / slope is equal to $\frac{change\ in\ y-coordinates}{change\ in\ x-coordinates} = \frac{y_2 - y_1}{x_2 - x_1}$

In the Cartesian plane, the gradient of a line is the measure of its slope or inclination to the x-axis. It is defined as the ratio of the change in y-coordinate (vertical) to the change in the x-coordinate (horizontal).

When drawing a graph of a linear function, it is sufficient to plot only two points and these points may be chosen as the x and y intercepts of the graph. In practice, however, it is wise to plot three points. If the three points lie on the same line, the working is probably correct, if not you have a chance to check whether there could be an error in your calculation.

If we assign x any value, we can easily calculate the corresponding value of y.
Determine the x intercept, set $f(x)$ is equal to zero and solve for x and then determine the y intercept, set x equals zero to find $f(0)$.

Consider the equation y = 2x + 3.

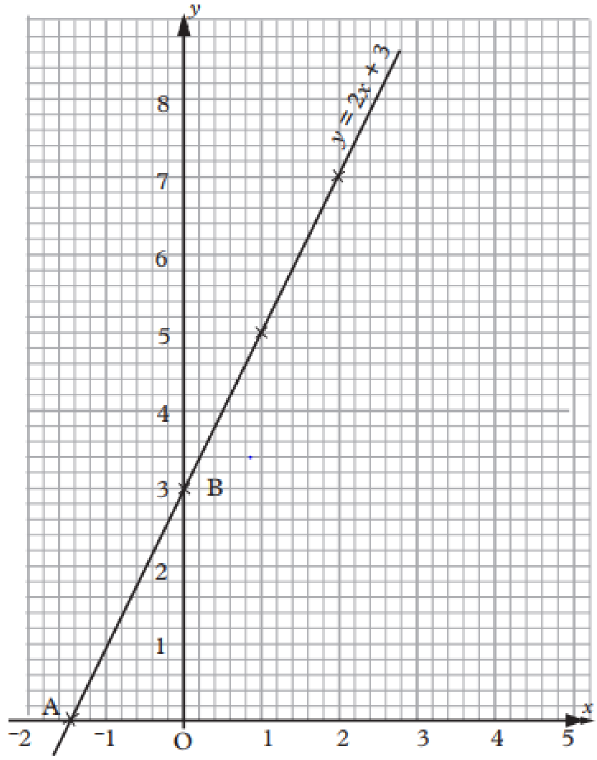
- When x = 0, y = 2 × 0 + 3 = 3
- When x = 1, y = 2 × 1 + 3 = 5
- When x = 2, y = 2 × 2 + 3 = 7 and so on.

For convenience and ease while reading, the calculations are usually tabulated as shown below in the table of values for y = 2x + 3.

x	0	1	2	3	4
$2x$	0	2	4	6	8
+3	3	3	3	3	3
$y=2x+3$	3	5	7	9	11

From the table the coordinates (x, y) are (0, 3), (1, 5), (2, 7), (3, 9), (4, 11) ....

When drawing the graph, the dependent variable is marked on the vertical axis generally known as the y – axis. The independent variable is marked on the horizontal axis also known as the x – axis.



3.6.2 Quadratic function

A polynomial equation in which the highest power of the variable is 2 is called a quadratic function. The expression $y = ax^2 + bx + c$, where a, b and c are constants and $a \neq 0$, is called a quadratic function of x or a function of the second degree (highest power of x is two).

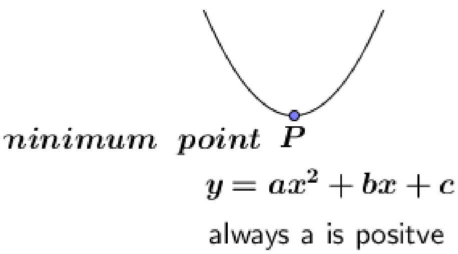
Table of values are used to determine the coordinates that are used to draw the graph of a quadratic function. To get the table of values, we need to have the domain (values of an independent variable) and then the domain is replaced in a given quadratic function to find range (values of dependent variables). The values obtained are useful for plotting the graph of a quadratic function. All quadratic function graphs are parabolic in nature.

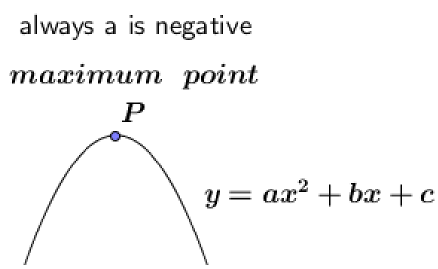
Any quadratic function has a graph which is symmetrical about a line which is parallel to the y-axis i.e. a line $x = h$ where h is constant value. This line is called **axis of symmetry**.

For any quadratic function $f(x) = ax^2 + bx + c$ whose axis of symmetry is the line $x = h$, the vertex is the point $(h, f(h))$.

The vertex of a quadratic function is the point where the function crosses its axis of symmetry.

If the coefficient of the x^2 term is positive, the vertex will be the lowest point on the graph, the point at the bottom of the U-shape. If the coefficient of the term x^2 is negative, the vertex will be the highest point on the graph, the point at the top of the ∩-shape. The shapes are as below.





Since the quadratic function written as $f(x) = ax^2 + bx + c$, then we can get the y-coordinate of the vertex by substituting the x-coordinate which is $x = -\frac{b}{2a}$ .So the vertex becomes $\left(-\frac{b}{2a}, f(-\frac{b}{2a})\right)$

The axis of symmetry of a quadratic function is the x-coordinate of the quadratic function and it is calculated from $x = -\frac{b}{2a}$.

The intercepts with axes are the points where a quadratic function cuts the axes.

There are two intercepts i.e. x-intercept and y-intercept.

x-intercept for any quadratic function is calculated by letting y = 0 and y- intercept is calculated by letting x = 0

Graph of a quadratic function

The graph of a quadratic function can be sketched without table of values as long as the following are known.

- The vertex
- The *x*-intercepts

- The *y*-intercept

Example

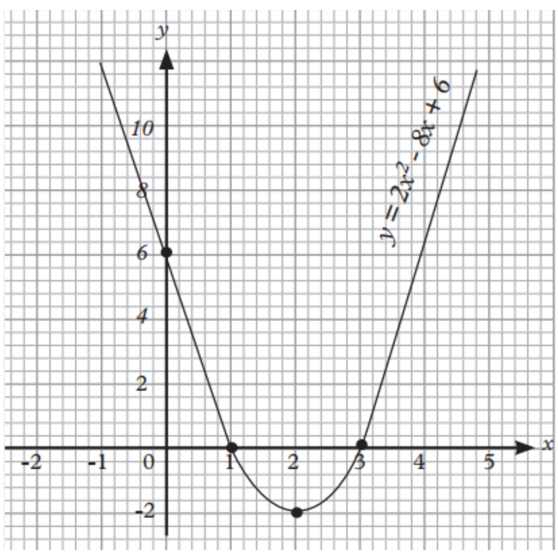
Find the vertex and axis of symmetry of the parabolic curve $y = 2x^2 - 8x + 6$

Solution

- The coefficients are a = 2, b = –8 and c = 6
- The x-coordinate of the vertex is $h = -\frac{b}{2a} = \frac{-(-8)}{2(2)} = \frac{8}{4} = 2$
- The y-coordinate of the vertex is obtained by substituting the x-coordinate of the vertex to the quadratic function. We get $y = 2(2)^2 - 8(2) + 6 = -2$
- The vertex is (2, –2) and the axis of symmetry is $x = 2$.
- When x = 0, $y = 2(0)^2 - 8(0) + 6 = 6$.
- The y-intercept is (0, 6)

When y = 0, $0 = 2x^2 - 8x + 6$, we therefore solve the quadratic equation for the values of x and we find the x-intercepts are (1,0) or (3,0)

The graph is as below.



Application activity 3.6

1) Using the table of values, sketch the graph of the following functions:

   a) $y = -3x + 2$

   b) $y = x^2 - 3x + 2$

2) Without tables of values, state the vertex, intercept with axis, axis of symmetry, and then, sketch the graph of $-3x^2 + 6x + 1 = y$

3.7 Applications of functions in economics and finance

Activity 3.7

1) Considering that C is the dependent variable, measured in the vertical axis, and Y is the independent variable, measured on the horizontal axis, Draw the graph of the function $C = 200 + 0.6Y$. Where C is consumer spending and Y is income. Note that the income cannot be negative.
   Determine the point (Y, C) at which the line cuts the vertical axis.
   What graph does tell about consumer spending and income?
2) Earl's Biking Company manufactures makes and sells bikes. Each bike costs \$40, and the company's fixed costs are \$5000. In addition, Earl knows that the price of each bike comes from the price function $P(x) = 300 - 2x$ Find:

   a) The company's cost function, $C(x)$.

   b) The company's revenue function, $R(x)$.

   c) The company's profit function, $P(x)$

Content summary

A function is a relationship between two or more variables such that a unique value of one variable is determined by the values taken by the other variables in the function.

Example 1

Suppose that average weekly household expenditure on food (C) depends on average net household weekly income (Y) according to the relationship $C = 12 + 0.3Y$. For any given value of Y, one can evaluate what C will be. For example, if $Y = 90$ then $C = 39$.

3.7.1 Cost function

The **cost function**, $C(q)$, gives the total cost of producing a quantity q of some good.

Cost is the total cost of producing output. The cost function consists of two different types of cost:

a. Variable costs, which depend on how many units are produced.
b. Fixed costs, which are incurred even if nothing is produced,

Variable cost varies with output (the number of units produced). The total variable cost can be expressed as the product of variable cost per unit and number of units produced. If more items are produced cost is more.

Fixed costs normally do not vary with output. In general, these costs must be incurred whether the items are produced or not.

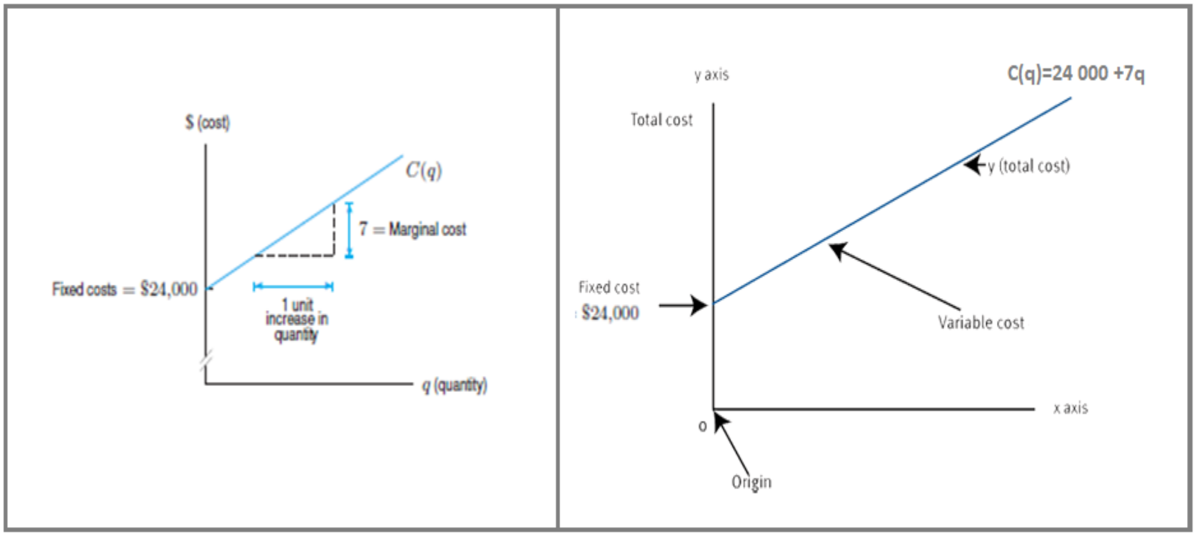
Cost Function $C(q) = F + Vq$, it is called **a linear cost function** where,

- $C =$ Total cost
- $F =$ Fixed cost
- $V =$ Variable cost Per unit
- $q =$ Number of units produced and sold

Example 2

Let's consider a company that makes radios. The factory and machinery needed to begin production with fixed costs which are incurred/earned even if no radios are made. The costs of labor and raw materials are variable costs since these quantities depend on how many radios are made. The fixed costs for this company are $\$24,000$ and the variable costs are $\$7$ per radio. Then, Total costs for the company = Fixed costs + Variable costs = 24,000 + 7$\times$(Number of radios), so, if q is the number of radios produced, $C(q) = 24,000 + 7q$.

This is the equation of a line with slope 7 and vertical intercept 24,000.



If $C(q)$ is a linear cost function,

- **Fixed costs** are represented by the vertical intercept.
- **Marginal cost** is represented by the slope.

Example 3

If a firm faces the total cost function $TC = 6 + x^2$, where x is output, what is TC when x is

(a) 14? (b) 1? (c) 0?

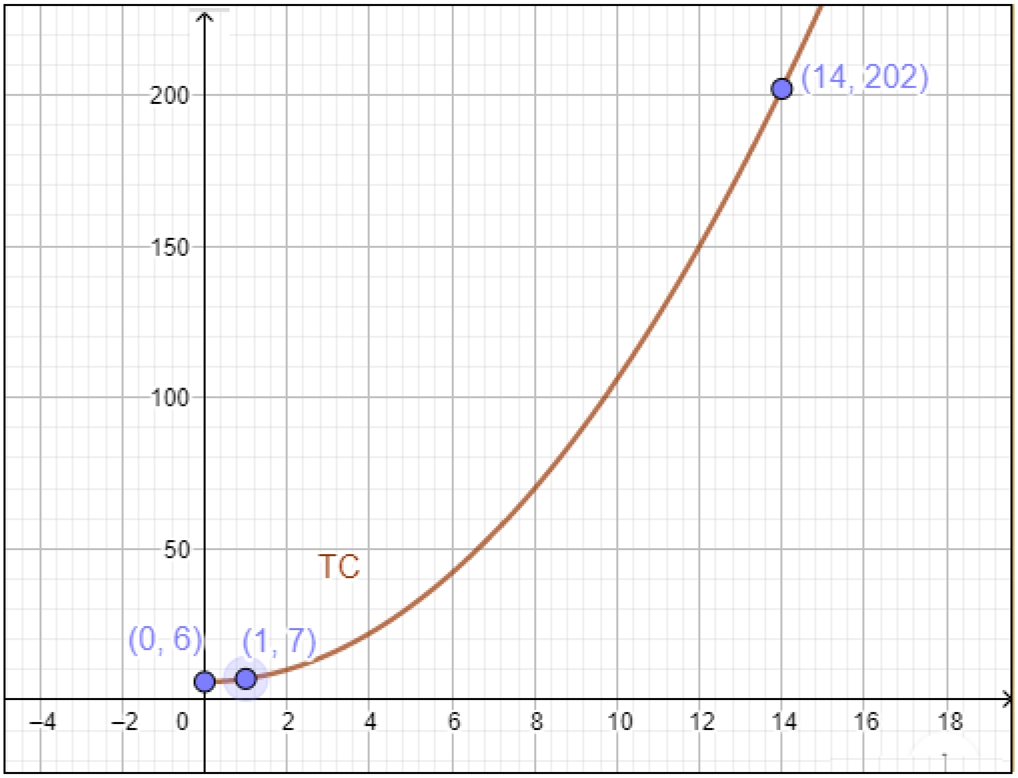
(d)What restrictions on the domain of this function would it be reasonable to make?

Solution

(a) When x =14, the value of TC is 202

(b) When x =1, the value of TC is 7

(c) When x =0, the value of TC is 6,



Graph of the function $TC(x) = 6 + x^2$

(d) The restrictions on the domain to make the cost function $TC = 6 + x^2$ being reasonable is only considering the values of $x \geq 0$.

3.7.2 Revenue function

Revenue is the total payment received from selling a good or performing a service. The **Revenue function**, $R(q)$, gives the total revenue received by a firm from selling a quantity, q, of some good. If the good sells for a price of p per unit, and the quantity sold is q, then

$\text{Revenue} = \text{Price} \times \text{quantity}$ or $R = pq$.

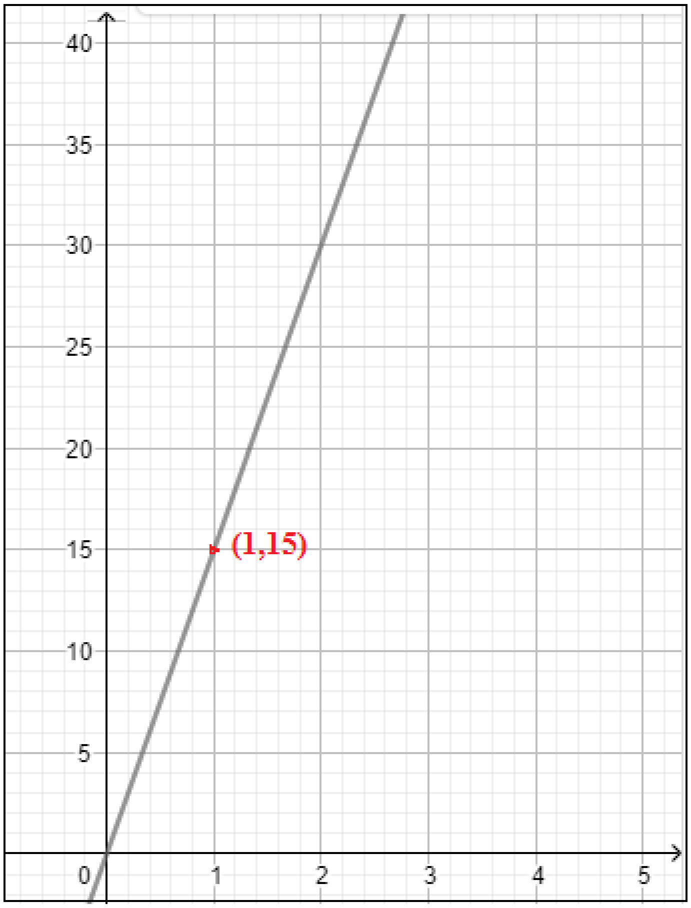
If the price does not depend on the quantity sold, so p is a constant, the graph of revenue as a function of q is a line through the origin, with slope equal to the price p.

Example 4

If radios sell for $15 each, sketch the manufacturer's revenue function. Show the price of a radio on the graph.

Solution

Since $R(q) = pq = 15q$, the revenue graph is a line through the origin with a slope of 15. See the figure. The price is the slope of the line.



Example 5

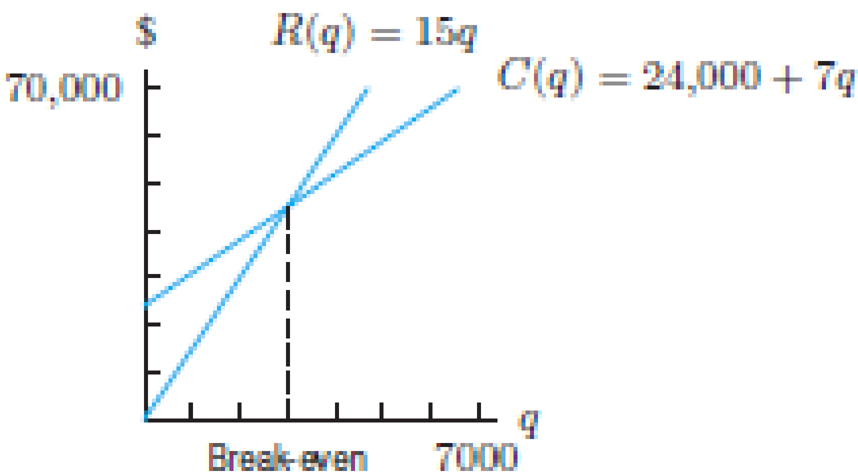
Graph the cost function $C(q) = 24{,}000 + 7q$ and the revenue function $R(q) = 15q$ on the same axes. For what values of q does the company make money?

Solution

The company makes money whenever revenues are greater than costs, so we find the values of q for which the graph of R(q) lies above the graph of C(q). We find the point at which the graphs of R(q) and C(q) cross: **Revenue = Cost**

$$15q = 24\ 000 + 7q$$
$$8q = 24\ 000$$
$$q = 3\ 000$$

The company makes a profit if it produces and sells more than 3 000 radios. The company loses money if it produces and sells fewer than 3 000 radios.



3.7.3 Profit function

The Profit Function $P(x)$ is the difference between the revenue function $R(x)$ and the total cost function $C(x)$. Thus $P(x)=R(x)-C(x)$

We have: **Profit = Revenue – Cost.**

The **break-even point** for a company is the point where the profit is zero and revenue equals cost.

Example 6

Let's consider a company that makes radios. The factory and machinery needed to begin production with fixed costs which are incurred/earned even if no radios are made. The costs of labor and raw materials are variable costs since these quantities depend on how many radios are made. The fixed costs for this company are $\$24,000$ and the variable costs are $\$7$ per radio while each radio is sold for $\$15$.

Find a formula for the profit function of the radio manufacturer. Graph it, marking the break-even point

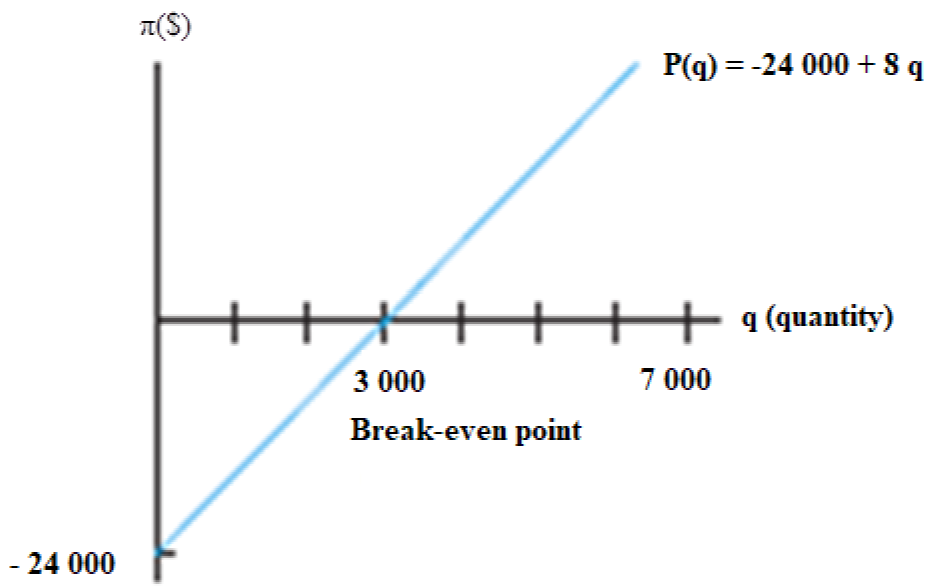
Solution

Since $R(q) = 15q$ and $C(q) = 24,000 + 7q$, we have

$P(q) = R(q) - C(q)$

$$= 15q - (24\,000 + 7q) = -2\,400 + 8q$$

Notice that the negative of the fixed costs is the vertical intercept and the break-even point is the horizontal intercept. See the figure;



3.7.4 Demand function

The demand function is in the form $P = a - bQ$, where a and b are parameters, P is the price and Q is the quantity demanded.

Example 7

Consider the function $P = 60 - 0.2Q$ where P is price and Q is quantity demanded. Assume that P and Q cannot take negative values, determine the slop of this function and sketch its graph.

Solution:

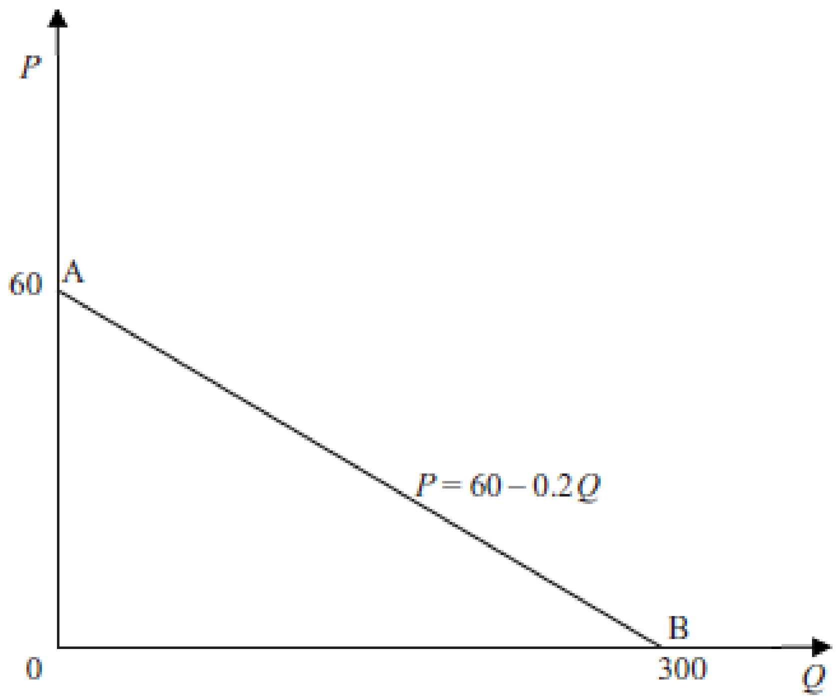
When $Q = 0$ then $P = 60$

When $P = 0$ then $0 = 60 - 0.2Q$

$0.2Q = 60$

$Q = \frac{60}{0.2} = 300$

Using these points: $(0, 60)$ and $(300, 0)$, we can find the graph as follows:



The slope of a function which slopes down from left to right is found by applying the formula

$slope = (-1)\dfrac{height}{base}$ to the relevant right-angled triangle. Thus, using the triangle 0BA, the slope of our function is

$$(-1)\frac{60}{300} = -0.2$$

This is the same as the coefficient of Q in the function $P = 60 - 0.2Q$.

Remember that in Finance and economics the usual convention is to measure the price P on the vertical axis of a graph. If you are given a function in the format $Q = f(P)$, then you would need to derive the inverse function to read off the slope.

Example 8

What is the slope of the demand function $Q = 830 - 2.5P$ when price P is measured on the vertical axis of a graph?

Solution

If $Q = 830 - 2.5P$; then $2.5P = 830 - Q$ or $P = 332 - 0.4Q$.

Therefore, the slope is the coefficient of Q, which is −0.4.

3.7.5 Point elasticity of demand

Elasticity can be calculated at a specific point on a linear demand schedule. This is called '*point elasticity of demand*' and is defined as

$e = (-1)\left(\frac{P}{Q}\right)\left(\frac{1}{slope}\right)$ where P and Q are the price and quantity at the point in question.

The slope refers to the slope of the demand schedule at this point although, of course, for a linear demand schedule the slope will be the same at all points.

Example 9

Calculate the point elasticity of demand for the demand schedule $P = 60 - 0.2Q$ where price is (i) zero, (ii) \$20, (iii) \$40, (iv) \$60.

Solution

This is the demand schedule referred to earlier and illustrated above as demand function. Its slope must be −0.2 at all points as it is a linear function and this is the coefficient of Q.

To find the values of Q corresponding to the given prices we need to derive the inverse function. Given that $P = 60 - 0.2Q$ then $0.2Q = 60 - P$ or $Q = 300 - 5P$.

i) When P is zero, at point B, then $Q = 300 - 5(0) = 300$. The point elasticity will therefore be

$$e = (-1)\frac{P}{Q}\left(\frac{1}{slope}\right) = -1\frac{0}{30}\left(\frac{1}{-0.2}\right) = 0$$

ii) When $P = 20$, then $Q = 300 - 5(20) = 200$.

$$e = (-1)\frac{P}{Q}\left(\frac{1}{slope}\right) = -1\frac{20}{200}\left(\frac{1}{-0.2}\right) = 0.5$$

iii) When $P = 40$, then $Q = 300 - 5(40) = 100$

$$e=(-1)\frac{P}{Q}\left(\frac{1}{slope}\right)=-1\frac{40}{100}\left(\frac{1}{-0.2}\right)=2$$

iv) When P = 60, then Q = 300 − 5(60) = 0.

If Q = 0, then $\frac{P}{Q}\to\infty$.

Therefore, $e=(-1)\frac{P}{Q}\left(\frac{1}{slope}\right)=-1\frac{60}{0}\left(\frac{1}{-0.2}\right)\to\infty$

3.7.6 Supply function

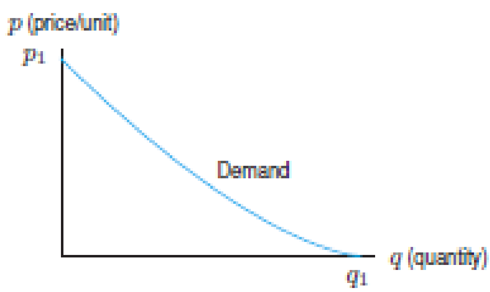
A. Price as function of quantity supplied

The quantity, q , of an item that is manufactured and sold depends on its price, p . As the price increases, manufacturers are usually willing to supply more of the product, whereas the quantity demanded by consumers falls.

The supply curve, for a given item, relates the quantity, q, of the item that manufacturers are willing to make per unit time to the price, p, for which the item can be sold.

The demand curve relates the quantity, q, of an item demanded by consumers per unit time to the price, p , of the item.

Economists often think of the quantities supplied and demanded Q as functions of price P. However, for historical reasons, the economists put price (the independent variable) on the vertical axis and quantity (the dependent variable) on the horizontal axis. (The reason for this state of affairs is that economists originally took price to be the dependent variable and put it on the vertical axis



B. Consumption as function of income

It is assumed that consumption C depends on income Y and that this relationship takes the form of the linear function $C = a + bY$.

Example 10

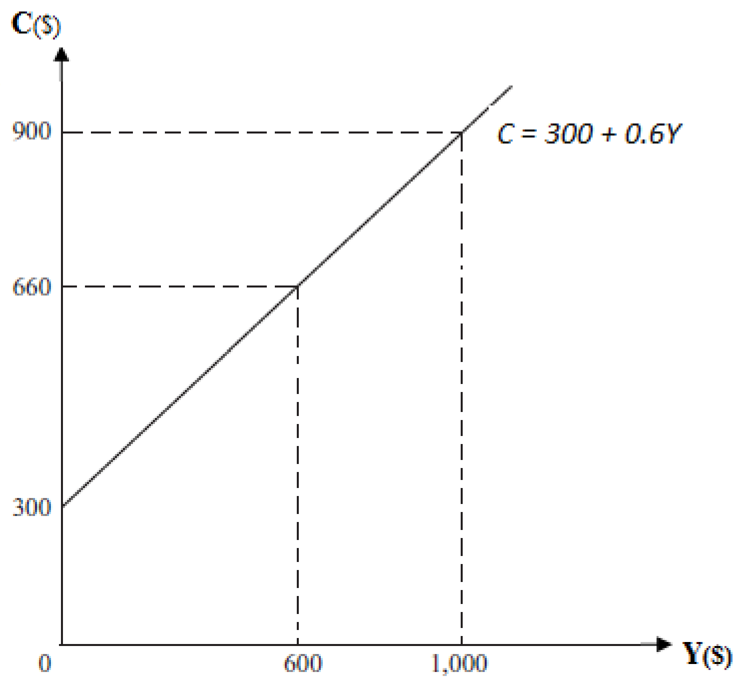
When *the income* is \$600, the consumption observed is \$660. When *the income* is \$1 000, the consumption observed is \$900. Determine the "consumption function of income".

Solution

As the consumption C depends on income Y and the linear function $C = a + bY$ gives their relationship, we expect b to be positive, i.e. consumption increases with income, and the function will slope upwards. As this is a linear function, then equal changes in Y will cause the same changes in C.

Therefore,

- A decrease in Y of \$400 from \$,000 to \$600 causes C to fall by \$240 from \$900 to \$660.
- If Y is decreased by a further \$600 (i.e. to zero), then the corresponding fall in C will be 1.5 times the fall caused by an income decrease of \$400, since \$600 = 1.5 × \$400.
- The fall in C is 1.5 × \$240 = \$360. This means that the value of C when Y is zero is \$660 − \$360 = \$300. Thus a = 300.
- A rise in Y of \$400 causes C to rise by \$240.
- A rise in Y of \$1 will cause C to rise by $\$\frac{240}{400} = \$0.6$. Thus, b = 0.6.
- The consumption function of income is specified as $C = 300 + 0.6Y$.



- The graph shows that when the income increases, the consumption increases also.

C. Graphical representation of demand and supply functions

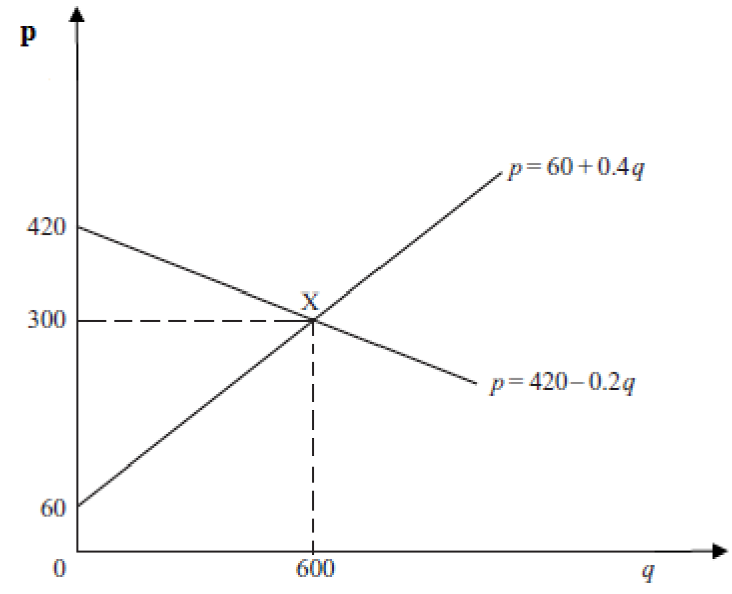
When only two or single variables and functions are involved, graphical solutions can be related to linear functions such as supply and demand analysis.

Example 11

Assume that in a competitive market the demand schedule is given by $p = 420 - 0.2q$ *a*nd the supply schedule is given by $p = 60 + 0.4q$, *solve* graphically for p and q and determine the point at which the market is in equilibrium. If this market is in equilibrium, the equilibrium price and quantity will be where the demand and supply schedules intersect. This requires to graphically solve the demand and supply functions at the same time. Its solution will correspond to a point which is on both the demand schedule and the supply schedule. Therefore, the equilibrium values of p and q will be such that both functions (1) and (2) hold.

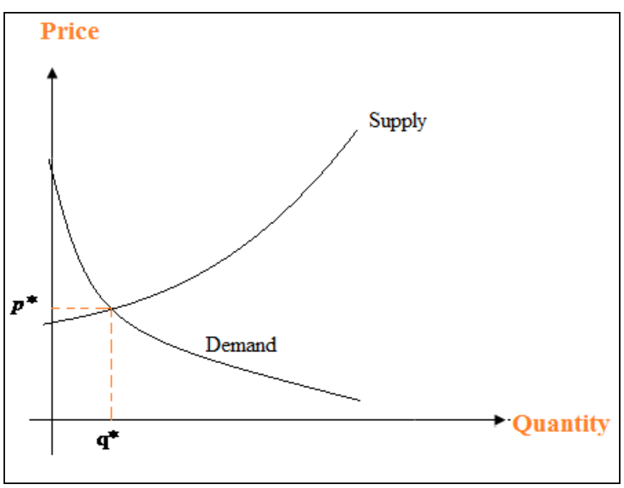
Solution

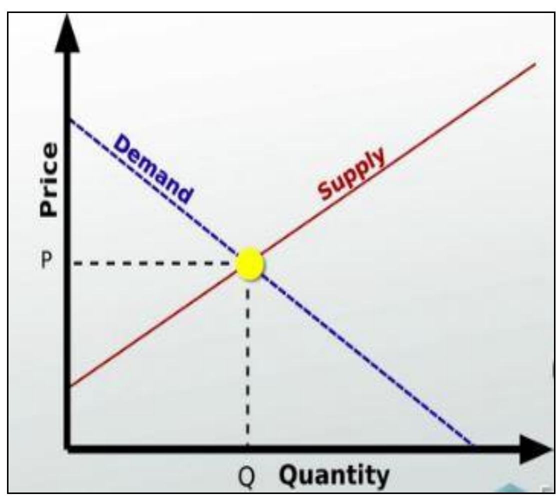
The two functional relationships $p = 420 - 0.2q$ *a*nd $p = 60 + 0.4q$ are plotted in the figure below and both hold at the intersection point $X(600, 300)$.



D. Equilibrium Price and Quantity

If we plot the supply and demand curves on the same axes, the graphs cross at the *equilibrium point*. The values p^* and q* at this point are called the *equilibrium price* and *equilibrium quantity*, respectively. It is assumed that the market naturally settles to this equilibrium point.





Example

Find the equilibrium price and quantity if Quantity supplied = 3p − 50 and

Quantity demanded = 100 − 2p.

Solution

To find the equilibrium price and quantity, we find the point at which

Supply = Demand

3p − 50 = 100 − 2p

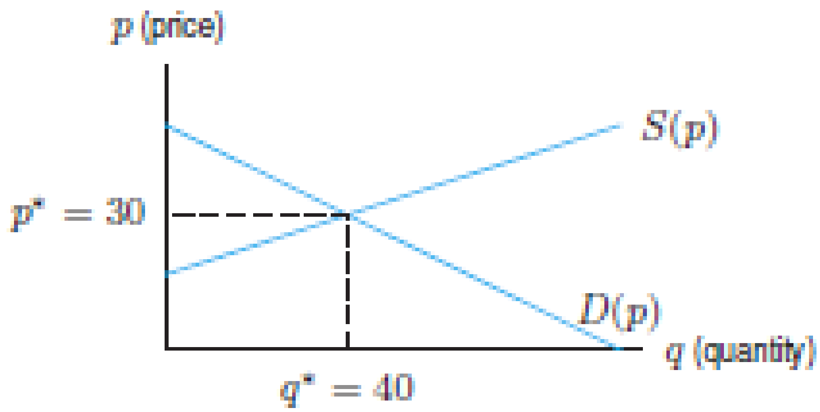
5p = 150

p = 30.

The equilibrium price is $30. To find the equilibrium quantity, we use either the demand curve or the supply curve. At a price of $30, the quantity produced is 100 − 2 (30) = 40 items.

The equilibrium quantity is 40 items.

In the figure, the demand and supply curves intersect at $p^* = 30$ and $q^* = 40$.



Application activity 3.7

1) Assume that consumption C depends on income Y according to the function $C = a + bY$, where a and b are parameters. If C is \$60 when Y is \$40 and C is \$90 when Y is \$80, If a= 30, sketch the graph of $C(Y)$ and interpret it.

2) Suppose that $q = f(p)$ is the demand curve for a product, where p is the selling price in dollars and q is the quantity sold at that price.
   a) What does the statement $f(12) = 60$ tell you about demand for this product?
   b) Do you expect this function to be increasing or decreasing? Why?

3) A demand curve is given by $75p + 50q = 300$, where p is the price of the product, in dollars, and q is the quantity demanded at that price. Find $p*$ and q* intercepts and interpret them in terms of consumer demand.

4) If a retail store has fixed cost of $15\ 000\ Frw$ and variable cost per unit is $17\ 500\ Frw$ and sells its product at $50000\ Frw$ per unit.
   a) Find the cost function $C(x)$
   b) What would the revenue function be?
   c) What would the profit function be?

3.8 End unit assessment

1) The total cost C for units produced by a company is given by $C(q) = 50000 + 8q$ where q is the number of units produced.
   a) What does the number 50000 represent?
   b) What does the number 8 represent?
   c) Plot the graph of C and indicate the cost when $q = 5$.
   d) Determine the real domain and the range of $C(q)$.
   e) Is $C(q)$ an odd function?
2) Bosco was working for his boss Kamana and they agreed to start a job where the monthly salary $f(t)$ was depending on the time t representing the t^{th} month Bosco spends on service. The salary $f(t)$ was the sum of a monthly bonus of 50,000Fr and product of 10,000Frw by the inverse of the time t.
   a) Give the function $f(t)$ which models the monthly salary of Bosco;
   b) Determine the domain of $f(t)$ and explain what it means
   c) Suppose that Bosco can continue to work indefinitely, determine the Maximum Salary and the Minimum Salary can Bosco get and deduce the Range of $f(t)$.
   d) Bosco has a monthly bonus, is this bonus motivating? Explain your answer.
   e) If you were Bosco, how many months can you work for Kamana?

UNIT 4: SEQUENCES

4.0 Introductory activity 4

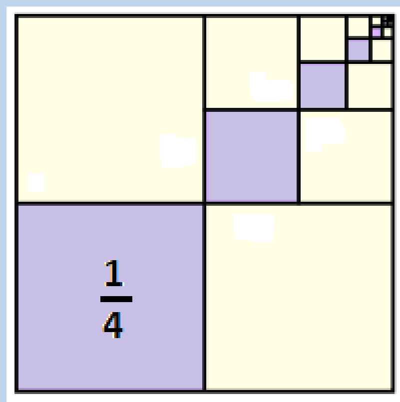
Suppose that your investments are growing in such a way that each new return per year is 2 times as large as the previous year. If there are 126 000 000 Frw in your investments, use a piece of paper (measured in million) and write down the number of investments that will be there in second, third, fourth, …n^{th} years. How can you classify this kind of investments financially and economically?

4.1 Definition of a sequence

Activity 4.1

Fold once a square paper. Write down a fraction corresponding to one part according to the original piece of paper.

Fold it twice, what is the fraction corresponding to one part according to the original piece of paper ?



- What is the fraction corresponding to one part according to the original piece of paper, if you fold it ten times?
- What is the fraction corresponding to one part, referring to the original piece of paper, if you fold it n times?
- Write a list of the fractions obtained starting from the first until the n^{th} fraction.

Content summary

Let us consider the following list of numbers: $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, ... \frac{1}{n}, ...$

The terms of this list are compared to the images of the function $f(x) = \frac{1}{x}$. The list never ends, as the ellipsis indicates. The numbers in this ordered list are called the **terms** of the sequence. In dealing with sequences, we usually use subscripted letters, such as u_1 to represent the first term, u_2 for the second term, u_3 for the third term, and so on such as in the sequence $f(n) = u_n = \frac{1}{n}$.

However, in the sequence such as $\{u_n\}: u_n = \sqrt{n-3}$, the first term is u_3 as the previous are not possible, in the sequence $\{u_n\}: u_n = 2n-5$, the first term is u_0.

Definition

A sequence is a function whose domain is the set of natural numbers. The terms of a sequence are the range elements of the function. It is denoted by $u_1, u_2, u_3, ..., u_{n-1}, u_n$ and shortly $\{u_n\}$. We can also write $\{u_1, u_2, u_3, ..., u_{n-1}, u_n\}$.

The dots are used to suggest that the sequence continues indefinitely, following the obvious pattern. The numbers $u_1, u_2, u_3, ..., u_{n-1}, u_n$ in a sequence are called **terms of the sequence**. The natural number n is called **term number** and value u_n is called a **general term** of a sequence and the term u_1 is the **initial term** or **the first term**.

As a sequence continues indefinitely, it can be denoted as $\{u_n\}_{n=1}^{\infty}$.

The number of terms of a sequence (possibly infinite) is called the **length of the sequence**.

Notice:

- Sometimes, the term number, *n*, starts from 0. In this case terms of a sequence are $u_0, u_1, u_2, ..., u_{n-1}, u_n, ...$ and this sequence is denoted by $\{u_n\}_{n=0}^{+\infty}$. In this case the initial term is u_0.
- A sequence can be finite, like the sequence $2, 4, 8, 16, ..., 256$.

The empty sequence $\{ \}$ is included in most notions of sequences, but may be excluded depending on the context. Usually a numerical sequence is given by some formula $u_n = f(n)$, permitting to find any term of the sequence by its number *n*; this formula is called a **general term formula**.

A second way of defining a sequence is to assign a value to the first (or the first few) term(s) and specify the n^{th} term by a formula or equation that involves one or more of the terms preceding it. Sequences defined in this way are said to be defined **recursively,** and the rule or formula is called a **recursive formula.**

Example 1

The sequence defined by $u_1 = 1, u_n = n.u_{n-1}$ is recursive.

Infinite and finite sequences

Consider the sequence of odd numbers less than 11: This is 1, 3,5,7,9. This is a finite sequence as the list is limited and countable. However, the sequence made by all odd numbers is:

$1, 3, 5, 7, 9, ...2n+1, ...$This suggests the definition that an i**nfinite sequence** is a sequence whose terms are infinite and its domain is the set of positive integers.

Note that it is not always possible to give the numerical sequence by a general term formula; sometimes a sequence is given by description of its terms.

Example 2

Numerical sequences:

$1, 2, 3, 4, 5, ...$ a sequence of natural numbers;

2,4,6,8,10, … a sequence of even numbers;

$1.4, 1.41, 1.414, 1.4142, \ldots$ a numerical sequence of approximate, defined more precisely values of $\sqrt{2}$.

For the last sequence it is impossible to give a general term formula, nevertheless this sequence is described completely.

Example 3

List the first five term of the sequence $\left\{2^n\right\}_{n=1}^{+\infty}$

Solution

Here, we substitute $n = 1, 2, 3, 4, 5, \ldots$ into the formula 2^n. This gives $2^1, 2^2, 2^3, 2^4, 2^5, \ldots$

Or, equivalently, $2, 4, 8, 16, 32, \ldots$

Example 4

Express the following sequences in general notation

a) $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots$

b) $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots$

Solution

a) $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots$

Beginning by comparing terms and term numbers:

Term number	1	2	3	4	$\ldots$
Term	$\frac{1}{2}$	$\frac{2}{3}$	$\frac{3}{4}$	$\frac{4}{5}$	$\ldots$

In each term, the numerator is the same as the term number, and the denominator is one greater the term number. Thus, the n^{th} term is $\frac{n}{n+1}$ and the sequence may be written as $\left\{\frac{n}{n+1}\right\}_{n=1}^{+\infty}$.

b) $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots$

Beginning by comparing terms and term numbers:

Term number	1	2	3	4	$\cdots$
Term	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{16}$	$\cdots$

Or

Term number	1	2	3	4	$\cdots$
Term	$\frac{1}{2^1}$	$\frac{1}{2^2}$	$\frac{1}{2^3}$	$\frac{1}{2^4}$	$\cdots$

In each term, the denominator is equal to 2 powers the term number. We observe that the n^{th} term is $\frac{1}{2^n}$ and the sequence may be written as $\left\{\frac{1}{2^n}\right\}_{n=1}^{+\infty}$.

Example 5

A sequence is defined by

$$\{u_n\}:\begin{cases} u_0 = 1 \\ u_{n+1} = 3u_n + 2 \end{cases}$$

Determine u_1, u_2 *and* u_3.

Solution

Since $u_0 = 1$ and $u_{n+1} = 3u_n + 2$, replace n by $0,1,2$ to obtain u_1, u_2, u_3 respectively.

$$n = 0, \quad u_{0+1} = u_1 = 3u_0 + 2$$
$$= 3\times 1 + 2$$
$$= 5$$

$$n = 1, \quad u_{1+1} = u_2 = 3u_1 + 2$$
$$= 3\times 5 + 2$$
$$= 17$$

$$n = 2, \quad u_{2+1} = u_3 = 3u_2 + 2$$
$$= 3\times 17 + 2$$
$$= 53$$

Thus,

$$\begin{cases} u_1 = 5 \\ u_2 = 17 \\ u_3 = 53 \end{cases}$$

Application activity 4.1

1) A sequence is given by $\{u_n\}: \begin{cases} u_0 = 1 \\ u_n = \dfrac{2n^2}{n^2+1} \end{cases}$

   Determine u_1, u_2 and u_3

2) List the first five terms of the sequence $\left\{\sqrt{n+1} - \sqrt{n}\right\}_{n=1}^{+\infty}$

3) Express the following sequence in general notation

1, 3, 5, 7, 9, 11, …

4.2 Arithmetic sequence and its general term

Activity 4.4

In each of the following sequence, each term can be found by adding a constant number to the previous. Guess that constant number.

a) Sequence $\{u_n\}$: 5,8,11,14,17,...

b) Sequence $\{v_n\}$: 26,31,36, 41, 46,...

c) Sequence $\{w_n\}$: 20,18,16,14,12, ...

Content summary

Let u_1 be an initial term of a sequence. If we add d successively to the initial term to find other terms, the difference between successive terms of a sequence is always the same number and the sequence is called **arithmetic.**

This sequence has the following term u_1, $u_2 = u_1 + d$, $u_3 = u_2 + d = u_1 + 2d$, $u_4 = u_3 + d = u_1 + 3d$, $\ldots u_n = u_{n-1} + d = u_1 + (n-1)d, \ldots$

An **arithmetic sequence** may be defined recursively as $u_n = u_1 + (n-1)d$ where u_1 and d are real numbers. The number u_1 is the first term, and the number d is called the **common difference.**

Examples

The following sequences are arithmetic sequences:

Sequence $\{u_n\}$: $5, 8, 11, 14, 17, \ldots$

Sequence $\{v_n\}$: $26, 31, 36, 41, 46, \ldots$

Sequence $\{w_n\}$: $20, 18, 16, 14, 12, \ldots$

Common difference

The fixed numbers that bind each sequence together are called the **common differences**. Sometimes mathematicians use the letter ***d*** when referring to these types of sequences.

d can be calculated by subtracting any two consecutive terms in an arithmetic sequence. That is $d = u_{n+1} - u_n$ or $d = u_n - u_{n-1}$.

Note: If three consecutive terms are in arithmetic sequence, the double of the middle term is equal to the sum of extreme terms. That is for an arithmetic sequence u_{n-1}, u_n, u_{n+1}, we have $2u_n = u_{n-1} + u_{n+1}$.

Proof:

If u_{n-1}, u_n, u_{n+1} form an arithmetic sequence, then

$u_{n+1} = u_n + d$ and $u_{n-1} = u_n - d$

Adding two equations, you get $u_{n+1} + u_{n-1} = 2u_n$.

General term of an arithmetic sequence

The n^{th} term, u_n of an arithmetic sequence $\{u_n\}$ with common difference *d* and initial term u_1 is given by $u_n = u_1 + (n-1)d$, which is **the general term of an arithmetic sequence.** If the initial term is u_0 then the general term of an arithmetic sequence becomes $u_n = u_0 + nd$

Generally, if u_p is any p^{th} term of a sequence then the n^{th} term is given by $u_n = u_p + (n-p)d$

Example 1

$4, 6, 8$ are three consecutive terms of an arithmetic sequence because $2 \times 6 = 4 + 8 \Leftrightarrow 12 = 12$

Example 2

If the first and tenth terms of an arithmetic sequence are 3 and 30, respectively, find the fiftieth term of the sequence.

Solution

$u_1 = 3$ and $u_{10} = 30$

But $u_n = u_1 + (n-1)d$,

$u_{10} = u_1 + (10-1)d$

Then $30 = 3 + (10-1)d \Leftrightarrow 30 = 3 + 9d \Rightarrow d = 3$

Now, $u_{50} = u_1 + (50-1)d = 3 + 49 \times 3 = 150$

Thus,

The fiftieth term of the sequence is 150.

Example 3

If the 3rd term and the 8th term of an arithmetic sequence are 5 and 15 respectively, find the common difference.

Solution

$u_3 = 5,\ u_8 = 15$

Using the general formula: $u_n = u_p + (n-p)d$

$u_3 = u_8 + (3-8)d$

$5 = 15 - 5d$

$\Leftrightarrow 5d = 15 - 5$

$\Rightarrow 5d = 10 \Rightarrow d = 2$

The common difference is 2.

Example 4

Consider the sequence $5,\ 8,\ 11,\ 14,\ 17,\ ...,\ 47$. Find the number of terms in this sequence

Solution

We see that $u_1 = 5$, $u_n = 47$ and $d = 3$.

But we know that $u_n = u_1 + (n-1)d$.

This gives

$47 = 5 + (n-1)3$

$\Leftrightarrow 42 = 3n - 3 \Rightarrow n = 15$

This means that there are 15 terms in the sequence.

Example 5

Consider the sequence which went from 20 to -26 with -2 as common difference. Find the number of terms.

Solution

We have

$$-26 = 20 + (n-1)(-2)$$
$$\Leftrightarrow -46 = -2n + 2 \Rightarrow n = 24$$

This means that there are 24 terms in the sequence.

Example 6

Show that the following sequence is arithmetic. Find the first term and the common difference.

$\{s_n\} = \{3n+5\}$

Solution

The n^{th} term and the $(n-1)^{\text{th}}$ term of the sequence $\{s_n\}$ are

$s_n = 3n+5 \quad and \quad s_{n-1} = 3(n-1)+5 = 3n+2$

The first term is $s_1 = 8$.

Their difference *d* is $s_n - s_{n-1} = (3n+5)-(3n+2) = 3$.

Since the difference of any two successive terms is the constant 3, the sequence $\{s_n\}$ is arithmetic and the common difference is 3.

Application activity 4.2

1) If $\frac{1}{a+b}, \frac{1}{a+c}, \frac{1}{b+c}$ are 3 consecutive terms of an arithmetic progression, show that it will be the same for a^2, b^2, c^2.

2) Form an arithmetic progression of three numbers such that the sum of its terms is 30 and the sum of the squares of its terms is 332.

3) Calculate x so that the squares of $1+x,\ q+x,\ and\ q^2+x$ will be three consecutive terms of an arithmetic progression where q is any given number.

4.3. Arithmetic Means of an arithmetic sequence

Activity 4.3

Suppose that you need to form an arithmetic sequence of 7 terms such that the first term is 2 and the seventh term is 20. Write down that sequence given that those terms are $2, A, B, C, D, E, 20$.

Content summary

If three or more than three numbers form an arithmetic sequence, then all terms lying between the first and the last numbers are called **arithmetic means**. If B is arithmetic mean between A and C, then $B = \frac{A+C}{2}$.

Let us see how to insert k terms between two terms u_1 and u_n to form an arithmetic sequence:

u_1	…	…	…	…	u_n

The terms to be inserted are called **arithmetic means** between two terms u_1 and u_n .

This requires to form an arithmetic sequence of $n = k + 2$ terms whose the first term is u_1 and the last term is u_n.

While there are several methods, we will use our n^{th} term formula $u_n = u_1 + (n-1)d$.

As u_1 and u_n are known, we need to find the common difference d taking $n = k + 2$ where k is the number of terms to be inserted and 2 stands for the first and last terms.

Example 1

Insert three arithmetic means between 7 and 23.

Solution

Here $k = 3$ and then $n = k + 2 = 5, u_1 = 7$ and $u_n = u_5 = 23$.

Then

$u_5 = u_1 + (5-1)d$

$\Leftrightarrow 23 = 7 + 4d \Rightarrow d = 4$

Now, insert the terms using $d = 4$, the sequence is $7, 11, 15, 19, 23$.

Example 2

Insert five arithmetic means between 2 and 20.

Solution

Here $k = 5$ and then $n = k + 2 = 7$, $u_1 = 2$ and $u_n = u_7 = 20$.

Then

$$u_7 = u_1 + (7-1)d$$
$$\Leftrightarrow 20 = 2 + 6d \Rightarrow d = 3$$

Now, insert the terms using $d = 3$, the sequence is $2, 5, 8, 11, 14, 17, 20$.

Application activity 4.3

1) Insert 4 arithmetic means between -3 and 7
2) Insert 9 arithmetic means between 2 and 32
3) Between 3 and 54, n terms have been inserted in such a way that the ratio of 8th mean and $(n-2)^{th}$ mean is $\frac{3}{5}$. Find the value of n.
4) There are n arithmetic means between 3 and 54 terms such that 8^{th} mean is equal to $(n-1)^{th}$ mean as 5 to 9. Find the value of n.
5) Find the value of n so that $\frac{a^{n+1} + b^{n+1}}{a^n + b^n}$ will be the arithmetic mean between a and b.

4.4 Sum of n first terms of an arithmetic sequence

Activity 4.4

Consider a finite arithmetic sequence $2, 5, 8, 11, 14, \ldots.$

a) What is its first term, the common difference d and the general term?

b) Determine the sum s_6 (in function of 6 and the first term 2) of the first 6 terms for this sequence taking that for each $u_n = u_1 + (n-1)d$ where for example $u_3 = u_1 + 2d$.

c) Try to generalize your results to determine the sum s_n for the first n terms of the arithmetic sequence $\{u_n\}$.

Content summary

The sum of first *n first* terms of a finite arithmetic sequence denoted by S_n, with initial term u_1 is given by

$$s_n = u_1 + u_2 + u_3 + \ldots + u_n$$
$$= u_1 + (u_1 + d) + (u_1 + 2d) + \ldots + (u_1 + (n-1)d)$$
$$= (\underbrace{u_1 + u_1 + \ldots u_1}_{n\ terms}) + (d + 2d + \ldots + (n-1)d)$$
$$= nu_1 + d[1 + 2 + 3 + \ldots + (n-1)] = nu_1 + d[\frac{n(n-1)}{2}]$$

$$s_n = nu_1 + \frac{n}{2}(n-1)d = \frac{n}{2}[2u_1 + (n-1)d]$$

$$= \frac{n}{2}[u_1 + u_1 + (n-1)d] = \frac{n}{2}(u_1 + u_n) \text{ or } \quad s_n = \frac{n}{2}(u_1 + u_n)$$

If the initial term is u_0, the formula becomes $S_n = \frac{n+1}{2}(u_0 + u_n)$

Example 1

Calculate the sum of first 100 terms of the sequence $2, 4, 6, 8, \ldots$

Solution

We see that the common difference is 2 and the initial term is $u_1 = 2$. We need to find $u_n = u_{100}$.

$$u_{100} = 2 + (100 - 1)2$$
$$= 2 + 198$$
$$= 200$$

Now,

$$S_{100} = \frac{100}{2}(u_1 + u_{100})$$
$$= 50(2 + 200)$$
$$= 10100$$

Example 2

Find the sum of first k even integers ($k \neq 0$).

Solution

$u_1 = 2$ and $d = 2$

$$u_n = u_k$$
$$u_k = 2 + (k - 1)2$$
$$u_k = 2k$$

$$S_n = S_k$$
$$S_k = \frac{k}{2}(2 + 2k)$$
$$S_k = k(k + 1)$$

Example 3

Find the sum: 60 + 64 + 68 + 72 + … + 120

Solution

This is the sum s_n of an arithmetic sequence u_n whose first term is $u_1 = 60$ and whose common difference is $d = 4$. The *n*th term is u_n.

We have $u_n = u_1 + (n - 1)d$ and

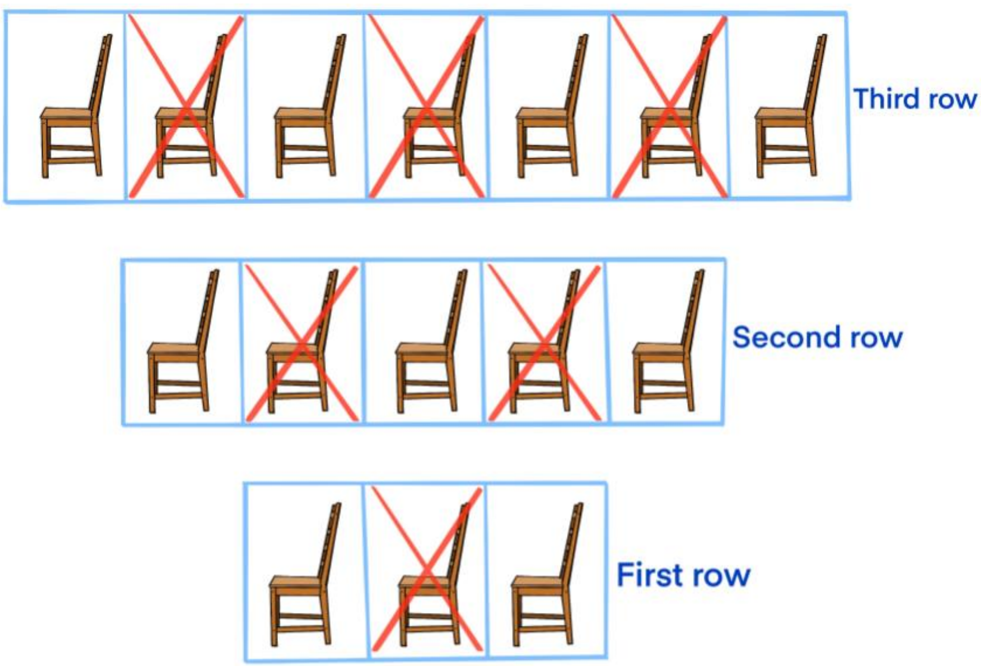
$$120 = 60 + (n - 1)4$$
$$60 = 4(n - 1)$$
$$n = 16$$

Now, the sum is

$$u_{16} = 60 + 64 + 68 + ... + 120 = \frac{16}{2}(60 + 120) = 1440$$

Example 4:

One of the measures for preventing the spread of Covid-19 is physical distancing. Consider a conference hall where people in the meeting are arranged in a way that chairs in each row are increasing by 2 from the previous line and between two people there must be a chair for social distancing as illustrated in the following figure.



Solution:

Order of line (n)	Number of authorized seats (U_n)	Total number of authorized seats from the first line (S_n)	S_n in terms of U_1 and d
1	2	$S_1 = U_1 = 2$	$S_1 = U_1$

2	3	$S_2 = S_1 + U_2$	$S_2 = S_1 + U_2 = U_1 + U_1 + d = 2U_1 + d$
3	4	$S_3 = S_2 + U_3$	$S_3 = S_2 + U_3 = 2U_1 + d + U_1 + 2d = 3U_1 + 3d$
4	5	$S_4 = S_3 + U_4$	$S_4 = S_3 + U_4 = 3U_1 + 3d + U_1 + 3d = 4U_1 + 6d$
5	6	$S_5 = S_4 + U_5$	$S_5 = S_4 + U_5 = 4U_1 + 6d + U_1 + 4d = 5U_1 + 10d$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
n	$U_1 + 2(n-$	$S_n = S_{n-1} + U$	$s_n = u_1 + u_2 + u_3 + \ldots + u_n$ $= u_1 + (u_1 + d) + (u_1 + 2d) + \ldots + (u_1 + (n-1)d)$ $= (\underbrace{u_1 + u_1 + \ldots u_1}_{n\ terms}) + (d + 2d + \ldots + (n-1)d)$ $= nu_1 + d[1 + 2 + 3 + \ldots + (n-1)] = nu_1 + d[\frac{n(n-1)}{2}]$ $= nu_1 + \frac{n}{2}(n-1)d = \frac{n}{2}[2u_1 + (n-1)d]$ $= \frac{n}{2}[u_1 + u_1 + (n-1)d]$ $= \frac{n}{2}(u_1 + u_n)$

Example 5:

Odd numbers are numbers that are not multiples of 2. Assuming we take the natural numbers, the odd numbers among them would be 1, 3, 5, 7, etc.

Find the sum of n first odd numbers.

Solution:

There is a need to know the total number of authorized chairs if there are n rows in the room.

Position of odd number (n)	1	2	3	4	5	6	7	8	9
Odd number as generated from the figure	1	3	5	7	9	11	13	15	17
Sum of odd number	1^2								
	$1+3=2^2$								
	$1+3+5=3^2$								
	$1+3+5+7=4^2$								
	$1+3+5+7+9=5^2$								
	$1+3+5+7+9+11=6^2$								
	$1+3+5+7+9+11+13=7^2$								
	$1+3+5+7+9+11+13+15=8^2$								
	$1+3+5+7+9+11+13+15+17=9^2$								
The sum of n first odd	$1+3+5+7+\cdots+(2n-1)=n^2$								

number	

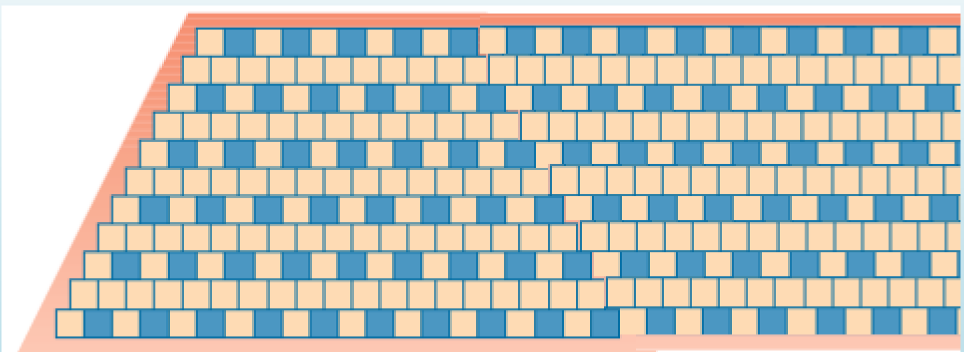
Application activity 4. 4

1) Consider the arithmetic sequence 8, 12, 16, 20, ... Find the expression for S_n

2) Sum the first twenty terms of the sequence 5, 9, 13, ...

3) The sum of the terms in the sequence 1, 8, 15, … is 396. How many terms does the sequence contain?

4) **Practical activity:** A ceramic tile floor is designed in the shape of a trapezium 10m wide at the base and 5m wide at the top as illustrated on the figure bellow:



The tiles, 10cm by 10cm, are to be placed so that each successive row contains one less tile than the preceding row. How many tiles will be required?

4. 5 Geometric sequence and its general term

Activity 4.5

Take a piece of paper with a square shape.

1) Cut it into two equal parts.
2) Write down a fraction corresponding to one part according to the original piece of paper.
3) Take one part obtained in step 2) and repeat step 1) and then step 2)
4) Continue until you remain with a small piece of paper that you are not able to cut into two equal parts and write down the sequence of fractions obtained.
5) Observe the sequence of numbers you obtained and give the relationship between any two consecutive numbers.

Content summary

Sequences of numbers that follow a pattern of multiplying a fixed number ***r*** from one term u_1 to the next are called **geometric sequences**.

The following sequences are geometric sequences:

Sequence $\{u_n\}$: $5, 10, 20, 40, 80, \ldots$

Sequence $\{v_n\}$: $2, 1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots$

Sequence $\{w_n\}$: $1, -2, 4, -8, 16, \ldots$

Common ratio

We can examine these sequences to greater depth, we must know that the fixed numbers that bind each sequence together are called the **common ratios**, denoted by the letter ***r***. This means if u_1 is the first term, $u_2 = ru_1$; $u_3 = r^2u_1; u_4 = r^3u_1; \ldots u_n = r^{n-1}u_1; \ldots$

The n^{th} term or the general term of a geometric sequence becomes $u_n = r^{n-1}u_1$.

If the first term is u_0, then the general term of an geometric sequence becomes $u_n = r^n u_0$ $u_n = r^n u_0$.

The common ration r can be calculated by dividing any two consecutive terms in a geometric sequence. That is

$r = \frac{u_{n+1}}{u_n}$ or $r = \frac{u_n}{u_{n-1}}$ or $u_n = ru_{n-1}$.

Generally, if u_p is the p^{th} term of the sequence, then the n^{th} term is given by $u_n = u_p r^{n-p}$.

Example 1

$6, 12, 24$ are consecutive terms of a geometric sequence because $(12)^2 = 6 \times 24 \Leftrightarrow 144 = 144$

Find b such that $8, b, 18$ will be in geometric sequence.

Solution

$b^2 = 8 \times 18 = 144$

$b = \pm\sqrt{144} = \pm 12$

Thus, $8, 12, 18$ or $8, -12, 18$ are in geometric sequence.

Example 2

The product of three consecutive numbers in geometric progression is 27. The sum of the first two and nine times the third is -79. Find the numbers.

Solution

Let the three terms be $\frac{x}{a}, x, ax$.

The product of the numbers is 27. So $\frac{x}{a} xax = 27 \Rightarrow x^3 = 27 \Rightarrow x = 3$

The sum of the first two and nine times the third is -79:

$\frac{x}{a} + x + 9ax = -79 \Rightarrow \frac{3}{a} + 3 + 27a = -79$

$27a^2 + 82a + 3 = 0 \Rightarrow a = -3 \text{ or } a = -\frac{1}{27}$

The numbers are: -1, 3, -9 or -81, 3, $-\frac{1}{9}$.

Example 3

If the first and tenth terms of a geometric sequence are 1 and 4, respectively, find the nineteenth term.

Solution

$u_1 = 1$ and $u_{10} = 4$

But $u_n = u_1 r^{n-1}$, then $4 = 1r^9 \Leftrightarrow r = \sqrt[9]{4}$ *or* $r = 4^{\frac{1}{9}}$

Now,

$$u_{19} = u_1 r^{19-1}$$
$$= 1\left(4^{\frac{1}{9}}\right)^{18}$$
$$= 16$$

Thus, the nineteenth term of the sequence is 16.

Example 4

If the 2nd term and the 9th term of a geometric sequence are 2 and $-\frac{1}{64}$ respectively, find the common ratio.

Solution

$u_2 = 2,\ u_9 = -\frac{1}{64}$

Using the general formula: $u_n = u_p\, r^{n-p}$

$$u_2 = u_9\, r^{2-9}$$

$$2 = -\frac{1}{64} r^{-7}$$

$$\Leftrightarrow 128 = -\frac{1}{r^7}$$

$$\Leftrightarrow r^7 = -\frac{1}{128}$$

$$\Leftrightarrow r = \sqrt[7]{-\frac{1}{128}} \Rightarrow r = -\frac{1}{2}$$

The common ratio is $r = -\frac{1}{2}$.

Example 5

Find the number of terms in sequence $2, 4, 8, 16, ..., 256$.

Solution

This sequence is geometric with common ratio 2, $u_1 = 2$ and $u_n = 256$

But $u_n = u_1 r^{n-1}$, then $256 = 2 \times 2^{n-1} \Leftrightarrow 256 = 2^n$ or $2^8 = 2^n \Rightarrow n = 8$.

Thus, the number of terms in the given sequence is 8.

Application activity 4.5

1) If the second and fifth terms of a geometric sequence are 6 and -48, respectively, find the sixteenth term.
2) If the third term and the 8^{th} term of a geometric sequence are $\frac{1}{2}$ and $\frac{1}{128}$ respectively, find the common ratio.
3) The 4^{th} term of a geometric sequence is square of its 2^{nd} term, and the first term is -3. Determine its 7^{th} term.
4) Find the fourth term from the end of geometric sequence $8, 4, 2, \cdots, \frac{1}{128}$
5) The fifth term of a geometric sequence is $\frac{81}{32}$ and the ratio of 3^{rd} and 4^{th} is $\frac{2}{3}$, write the geometric sequence and its 8^{th} term.
6) If p^{th} terms of two sequences $5, 10, 20, \cdots$ and $1280, 640, 320, \cdots$,are equal, find the value of p.

4.6. Geometric Means of geometric sequence

Activity 4.6

Suppose that you need to form a geometric sequence of 6 terms such that the first term is 1 and the sixth term is 243. Given that these terms are $1, A, B, C, D, 243$. Write down that sequence.

Content summary

To insert k terms called **geometric means** between two terms u_1 and u_n is to form a geometric sequence of $n = k + 2$ terms whose the first term is u_1 and the last term is u_n.

While there are several methods, we will use our n^th^ term formula $u_n = u_1 r^{n-1}$.

As u_1 and u_n are known, we need to find the common ratio r taking $n = k + 2$ where k is the number of terms to be inserted.

Example 1

Insert three geometric means between 3 and 48.

Solution

Here $k = 3$, then $n = 5, u_1 = 3$ and $u_n = u_5 = 48$

$u_5 = u_1 r^{n-1} \Leftrightarrow 48 = 3r^4 \Rightarrow r = 2$

Inserting three terms using common ratio $r = 2$ gives $3, 6, 12, 24, 48$

Example 2

Insert 6 geometric means between 1 and $-\frac{1}{128}$.

Solution

Here $k = 6$, then $n = 8, u_1 = 1$ and $u_n = u_8 = -\frac{1}{128}$

$u_8 = u_1 r^{n-1}$

$\Leftrightarrow -\frac{1}{128} = 1r^7$

$\Leftrightarrow r^7 = -\frac{1}{128}$

$\Leftrightarrow r^7 = -\frac{1}{(2)^7}$

$\Leftrightarrow r = \left[-\frac{1}{(2)^7}\right]^{\frac{1}{7}} = -\frac{1}{2}$

Inserting 6 terms using common ratio $r = -\frac{1}{2}$ gives $1, -\frac{1}{2}, \frac{1}{4}, -\frac{1}{8}, \frac{1}{16}, -\frac{1}{32}, \frac{1}{64}, -\frac{1}{128}$.

Application activity 4.6

1) Insert 5 geometric means between $\frac{1}{4}$ and $\frac{1}{256}$.
2) Insert 5 geometric means between 2 and $\frac{2}{729}$
3) Find the geometric mean between
   a) 2 and 98
   b) $\frac{3}{2}$ and $\frac{27}{2}$
4) Suppose that $4, 36, 324$ are in geometric progression. Insert two more numbers in this sequence so that it again forms a geometric sequence.
5) The arithmetic mean of two numbers is 34 and their geometric mean is 16. Find the numbers.

4.7 Sum of n first terms of a geometric sequence

Activity 4.7

During a competition of student teachers at the district level, 5 first winners were paid an amount of money in the way that the first got 100,000Frw, the second earned the half of this money, the third got the half of the second's money, and so on until the fifth who got the half of the fourth's money.

a) Discuss and calculate the money earned by each student from the second to the fifth.

b) Determine the total amount of money for all the 5 student teachers.

c) Compare the money for the first and the fifth student and discuss the importance of winning at the best place.

d) Try to generalize the situation and guess the money for the student who passed at the n^{th} place if more students were paid. In this case, evaluate the total amount of money for n students.

Content summary

The sum S_n of the first *n* terms of a geometric sequence $\{u_n\} = \{u_1 r^{n-1}\}$ is

$S_n = u_1 + ru_1 + ... + r^{n-1}u_1$ (1)

Multiply each side by *r* to obtain $rS_n = ru_1 + r^2u_1 + ... + r^n u_1$ (2)

Subtracting equation (2) from equation (1) we obtain

$$S_n - rS_n = u_1 - u_1 r^n$$
$$(1-r)S_n = u_1(1-r^n)$$

Since $r \neq 1$, we can solve for S_n and find $S_n = u_1 \frac{(1-r^n)}{(1-r)}$

If the initial term u_1 and the common ratio r are given, the sum $S_n = \frac{u_1(1-r^n)}{1-r}$ with $r \neq 1$

If the initial term is u_0, then the formula is $S_n = \frac{u_0(1-r^{n+1})}{1-r}$ with $r \neq 1$

If $r = 1$, $s_n = nu_1$

Also, the product of first n terms of a geometric sequence with initial term u_1 and common ratio r is given by $P_n = (u_1)^n r^{\frac{n(n-1)}{2}}$

If the initial term is u_0 then $P_n = (u_0)^{n+1} r^{\frac{n}{2}(n+1)}$

Example 1

Find the sum of the first 20 terms of a geometric sequence if the first term is 1 and common ratio is 2.

Solution

Here $u_1 = 1, r = 2, n = 20$

Then, $s_{20} = \frac{1(1-2^{20})}{1-2} = \frac{1-2^{20}}{-1} = 1048575$

Example 2

Consider the sequence $\{u_n\}$ defined by $u_0 = 0$ and $u_{n+1} = u_n + \frac{1}{2^n}$.

Consider another sequence $\{v_n\}$ defined by $v_n = u_{n+1} - u_n$.

a) Show that $\{v_n\}$ is a geometric sequence and find its first term and common ratio.

b) Express $\{v_n\}$ in term of n.

Solution

a) $u_0 = 0,\ \ v_0 = u_1 - u_0 = 1$

$u_1 = u_0 + \frac{1}{2^0} = 1,\ u_2 = u_1 + \frac{1}{2^1} = \frac{3}{2};$

$v_1 = u_2 - u_1 = \frac{1}{2}$, $v_2 = u_3 - u_2 = \frac{1}{4}$

$\{v_n\}$ is a geometric sequence if $v_1^2 = v_0 \cdot v_2$.

$v_1^2 = \frac{1}{4}$ and $v_0 \cdot v_2 = \frac{1}{4}$.

Thus, $\{v_n\}$ is a geometric sequence.

First term is $v_0 = 1$

Common ratio is $r = \frac{v_1}{v_0} = \frac{1}{2}$

b) General term

$$\begin{aligned} v_n &= v_0 r^n \\ &= \frac{1}{2^n} \end{aligned}$$

Thus, $\{v_n\}$ is defined by $v_n = \frac{1}{2^n}$

Example 3

Find the product of the first 20 terms of a geometric sequence if the first term is 1 and common ratio is 2.

Solution

$$P_n = (u_1)^n r^{\frac{n(n-1)}{2}}$$

Here $u_1 = 1, r = 2, n = 20$,

Then,

$$P_{20} = (1)^{20} 2^{\frac{20(19)}{2}} = 2^{190}.$$

Application activity 4.7

1) Find the sum of the first 8 terms of the geometric sequence 32, -16, 8, …
2) Find the sum of the geometric sequence with the first term 0.99 and the common ration is equal to the first term.
3) Find the first term and the common ratio of the geometric sequence for which

   $$S_n = \frac{5^n - 4^n}{4^{n-1}}$$

4) Find the product of the first 10 terms of the sequence in question 1.

5) Aloys wants to begin saving money for school. He decides to deposit $500 at the beginning of each quarter (January1, April 1, July1, and October1) in a saving account which pays an annual percentage of 6% compound interest quarterly. The interest for each quarter is posted on the last day of the quarter. Determine Aloys's account balance at the end of one year.

4.8 Application of sequences in real life

Activity 4.8

Carry out a research in the library or on internet and find out at least 3 problems or scenarios of the real life where sequences and series are applied.

Content summary

There are many applications of sequences. Sequences are useful in our daily lives as well as in higher mathematics. For example; the monthly payments made to pay off an automobile or home loan with interest portion, the list of maximum daily temperatures in one area for a month are sequences. Sequences are used in calculating interest, population growth, half-life and decay in radioactivity …

In economics and Finance, sequences and series can be used for example in solving problems related to:

a) Final sum, the initial sum, the time period and the interest rate for an investment

The amount A after t year due to a principal P invested at an annual interest rate r compounded n times per year is

$$A = P.\left(1+\frac{r}{n}\right)^{n.t}.$$

(i) If the compounding takes place Annually, n=1 and $A = P.\left(1+\frac{r}{1}\right)^{t}$

(ii) If the compounding takes place Monthly, n=12 and $A = P.\left(1+\frac{r}{12}\right)^{12t}$

(iii) If the compounding takes place Daily, n=365 and $A = P.\left(1+\frac{r}{365}\right)^{365t}$

In each case, **the Interest due** is $A-P$.

Note that From the compound interest formula $A = P\left(1+\frac{r}{n}\right)^{nt}$ If we let $k = \frac{n}{r}$,

then $n = kr$ and $nt = krt$, and we may write the formula as $A = P\left(1+\frac{1}{k}\right)^{krt} = P\left[\left(1+\frac{1}{k}\right)^{k}\right]^{rt}$.

For continuously compound interest, we may let n (the number of interest period per year) increase without bound towards infinity ($n \to \infty$), equivalently, by $k \to \infty$.

Using the definition of e, we see that $P\left[\left(1+\frac{1}{k}\right)^{k}\right]^{rt} \to P[e]^{rt}$ as $k \to \infty$. This result gives us the following formula: $A = Pe^{rt}$ where **P=Principal or initial value at t = 0** ; r is the Interest rate expressed as a decimal; r is the number of years P is invested; A is the amount after t years.

The amount A after t year due to a principal P invested at an annual interest rate r compounded continuously is

$A = P.\mathrm{e}^{rt}$.

b) Annual Equivalent Rate for part year investments and the nominal annual rate of return

For loan repayments the annual equivalent rate is usually referred to as the annual percentage rate (APR). If you take out a bank loan you will usually be quoted an APR even though you will be asked to make monthly repayments.

The **corresponding AER for any given monthly rate of interest** i_m can be found using the formula

$AER = (i + i_m)^{12} - 1$.

The APR on loans is the same thing as the annual equivalent rate and so the same formula applies.

The relationship between the daily interest rate i_d on a deposit account and the AER can be formulated as $AER = (i + i_d)^{365} - 1$.

c) The Present value of investment

The present value P of A money to be received after t years, assuming a per annum interest rate r compounded n times per year, is $P = A.\left(1+\frac{r}{n}\right)^{-n.t}$.

If the interest is compounded continuously, $P = A.e^{-n.t}$

d) Monthly repayments and the Annuity.

Suppose P is the deposit made at the end of each payment period for an annuity paying an interest rate of i per payment period.

The amount A of the annuity after n deposits is

$$A = P.\left[\frac{(1+i)^n - 1}{i}\right]$$

Example 1

A child building a tower with blocks uses 15 for the bottom row. Each row has 2 fewer blocks than the previous row. Suppose that there are 8 rows in the tower.

a) How many blocks are used for the top row?

b) What is the total number of blocks in the tower?

Solution

a) The number of blocks in each row forms an arithmetic sequence with $u_1 = 15$ and $d = -2$

$n = 8,\ u_8 = u_1 + (8-1)(-2)$. There is just one block in the top row.

b) Here we must find the sum of the terms of the arithmetic sequence formed with $u_1 = 15, n = 8, u_8 = 1$

$$S_8 = \frac{8}{2}(15+1) = 64$$

There are 64 blocks in the tower.

Example 2

An insect population is growing in such a way that each new generation is 1.5 times as large as the previous generation. Suppose there are 100 insects in the first generation.

a) How many will there be in the fifth generation?

b) What will be the total number of insects in the five generations?

Solution

a) The population can be written as a geometric sequence with $u_1 = 100$ as the first-generation population and common ratio $r = 1.5$. Then the fifth-generation population will be $u_5 = 100(1.5)^{5-1} = 506.25$. In the fifth-generation, the population will number about 506 insects.

b) The sum of the first five terms using the formula for the sum of the first n terms of a geometric sequence.

$$S_5 = \frac{100\left(1-(1.5)^5\right)}{1-1.5} = 1318.75$$

The total population for the five generations will be about 1319 insects.

Example 3

Find the accumulated value of $15,000 at 5% per year for 18 years using simple interest.

Solution

$P = 15000,\ r = 0.05, t = 18$

$$I = 15000(0.05)(18)$$
$$= 13500$$

A total of $13,500 in interest will be earned.

Hence, the accumulated value in the account will be 13,500 + 15,000 = $28,500.

Example 4

Suppose 20,000Frw is deposit in a bank account that pays interest at a rate of 8% per year compound Continuously. Determine the balance in the account in 5 years.

Solution: Applying the formula for Continuously compound interest with $P = 20,000; r = 0.08;\ and\ t = 5$, we have $A = Pe^{rt} = 20,000e^{0.08(5)} = 20,000\,\mathrm{e}^{0.4} = 29,836.49Frw$.

Example 5

Find the amount of an annuity after 5 deposits if a deposit of $100 is made each year, at 4% compounded annually. How much interest is earned?

Solution

The deposit is $P = \$100$. The number of deposits is $n = 5$ and the interest per payment period is $i = 0.04$. Using the formula, the amount *A* after 5 deposits is

$$A = P.\left[\frac{(1+i)^n - 1}{i}\right] = 100\left[\frac{\left((1+0.04)^5 - 1\right)}{0.04}\right] = 541.63$$

The interest earned is the amount after 5 deposits less the 5 annual payments of $100 each:

Interest earned $= A - 500 = 543.63 - 500 = 41.63$. It is $41.63.

Example 6

Mary decides to put aside $100 every month in a credit union that pays 5% compounded monthly. After making 8 deposits, how much money does Mary have?

Solution

This is an annuity with $P = \$100, n = 8$ deposits, and interest $i = \frac{0.05}{12}$ per payment period. Using the formula, the amount A after 8 deposits is

$$A = P.\left[\frac{(1+i)^n - 1}{i}\right] = 100\left[\frac{\left(\left(1+\frac{0.05}{12}\right)^5 - 1\right)}{\frac{0.05}{12}}\right] = 811.76$$

Mary has $811.76 after making 8 deposits.

Example 7

To save for her daughter's college education, Martha decides to put $50 aside every month in a bank guaranteed-interest account paying 4% interest compounded monthly.

She begins this savings program when her daughter is 3 years old. How much will she have saved by the time she makes the 180th deposit? How old is her daughter at this time?

Solution

This is an annuity with $P = \$50$, $\text{n} = 180$ deposits, and $i = \frac{0.04}{12}$. The amount A saved is

$$A = P.\left[\frac{(1+i)^n - 1}{i}\right] = 50\left[\frac{\left(\left(1+\frac{0.04}{12}\right)^{180} - 1\right)}{\frac{0.04}{12}}\right] = 12{,}304.52$$

Since there are 12 deposits per year, when the 180th deposit is made $\frac{180}{12} = 15$ have passed and Martha's daughter is 18 years old.

Application activity 4.8

1) If Linda deposits $1300 in a bank at 7% interest compounded annually, how much will be in the bank 17 years later?

2) The population of a city in 1970 was 153,800. Assuming that the population increases continuously at a rate of 5% per year, predict the population of the city in the year 2000.

3) To save for retirement, Manasseh, at age 35, decides to place 2000Frw into an Individual Retirement Account (IRA) each year for the next 30 years. What will the value of the IRA be when Manasseh makes his 30th deposit? Assume that the rate of return of the IRA is 4% per annum compounded annually.

4) A private school leader received permission to issue 4,000,000Frw in bonds to build a new high school. The leader is required to make payments every 6 months into a sinking fund paying 4% compounded semi-annually. At the end of 12 years the bond obligation will be retired. What should each payment be?

Practical activity or project:

Explore a mortgage problem and decide the type of loan you can take from a bank

When a person gets a loan (mortgage) from the bank, the mortgage amount M, the number of months required to repay the total amount of the loan n, the monthly amount of the payment P, and the interest rate r, it is proved that all the 4 components are related by the following formula:

$P = \frac{M \times i \times (1+i)^n}{(1+i)^n - 1}$ or $P = M\left[\frac{i}{1-(1+i)^{-n}}\right]$ is the monthly amount to pay.

Where: **M** is the mortgage amount of the loan, **i** is the monthly interest rate per payment period and **n** is the number of months required to repay the total amount of the loan. This is a concept-based practical activity of monthly payment of a mortgage, the total amount to pay, the related interest to be taken by the bank and decide the type of loan you can take from a bank.

Objective: To calculate the monthly payment of a mortgage, the total amount to pay, the related interest to be taken by the bank and decide the type of loan you can take from a bank.

List of required materials: Paper, Pencil or pen, scientific calculator, computer (excel sheet for loan amortization).

Illustration of the activity:

Formula to be used: $P = M\left[\frac{i}{1-(1+i)^{-n}}\right]$

Excel sheet to be used: Note that the dollar sign ($) in the excel sheet below does not stand for currency. It is by default; ignore it when you want to use Rwanda Francs (Frw) as our currency.

Loan Amortization Schedule

Enter values	
Loan amount	$ 9,045,000.00
Annual interest rate	18.50 %
Loan period in years	5
Number of payments per year	12
Start date of loan	11/15/2021
Optional extra payments	$ -

Loan summary	
Scheduled payment	$ 232,150.86
Scheduled number of payments	60
Actual number of payments	60
Total early payments	$ -
Total interest	$ 4,884,051.72

Lender name:

Pmt. No.	Payment Date	Beginning Balance	Scheduled Payment	Extra Payment	Total Payment	Principal	Interest	Ending Balance	Cumulative Interest
1	12/15/2021	$ 9,045,000.00	$ 232,150.86	$ -	$ 232,150.86	$ 92,707.11	$ 139,443.75	$ 8,952,292.89	$ 139,443.75
2	1/15/2022	$ 8,952,292.89	$ 232,150.86	$ -	$ 232,150.86	$ 94,136.35	$ 138,014.52	$ 8,858,156.54	$ 277,458.27
3	2/15/2022	$ 8,858,156.54	$ 232,150.86	$ -	$ 232,150.86	$ 95,587.62	$ 136,563.25	$ 8,762,568.93	$ 414,021.51
4	3/15/2022	$ 8,762,568.93	$ 232,150.86	$ -	$ 232,150.86	$ 97,061.26	$ 135,089.60	$ 8,665,507.67	$ 549,111.12
5	4/15/2022	$ 8,665,507.67	$ 232,150.86	$ -	$ 232,150.86	$ 98,557.62	$ 133,593.24	$ 8,566,950.05	$ 682,704.36
6	5/15/2022	$ 8,566,950.05	$ 232,150.86	$ -	$ 232,150.86	$ 100,077.05	$ 132,073.81	$ 8,466,873.00	$ 814,778.17
7	6/15/2022	$ 8,466,873.00	$ 232,150.86	$ -	$ 232,150.86	$ 101,619.90	$ 130,530.96	$ 8,365,253.10	$ 945,309.13
8	7/15/2022	$ 8,365,253.10	$ 232,150.86	$ -	$ 232,150.86	$ 103,186.54	$ 128,964.32	$ 8,262,066.55	$ 1,074,273.45
9	8/15/2022	$ 8,262,066.55	$ 232,150.86	$ -	$ 232,150.86	$ 104,777.34	$ 127,373.53	$ 8,157,289.22	$ 1,201,646.98
10	9/15/2022	$ 8,157,289.22	$ 232,150.86	$ -	$ 232,150.86	$ 106,392.65	$ 125,758.21	$ 8,050,896.56	$ 1,327,405.18

Procedures

Step 1: Read the problem for the old man who wants to take a loan of 9,045,000Frw but he is not sure if he will pay more interest to the bank when the number of periods (instalments) increases.

What monthly payment is necessary to pay off a loan of 9,045,000Frw at 18.5% per annum? in 2 years, in 3 years? And in 5 years.

What total amount is paid out for each loan?

What interest will the bank earn for each loan?

What advice can you give your colleague about the number of years and the interest to be given to the bank?

Step 2: Calculate the monthly payments using the formula and complete in the table of data recording.

Step 3: Use the excel sheet and find the monthly payments and the total interest and complete them in the table of data recording

Step 4: Evaluate the total amount paid and the total interest for each case.

Step 5: Consider the monthly payments and the interest to pay and then take the decision on whether you can choose a long-term loan or a short-term loan.

Data recording

	n=24	*n=36*	*n=60*
M			
i			
$1+i$			
$(1+i)^{-n}$			
$1-(1+i)^{-n}$			
$\frac{1}{1-(1+i)^{-n}}$			
P			
$P\times n$			
$(P\times n)-M$			

Does the calculation give the same results of P and total interest as the output of the excel sheet of the bank?

What advice can you give your colleague about the number of years and the interest to be given to the bank?

Expected answers

n	12	24	36	60
Loan	M	9045000	9045000	**9,045,000.00**
	i	0.015416667	0.015416667	0.015416667
	$1+i$	1.015416667	1.015416667	1.015416667
	$(1+i)^{-n}$	0.692687101	0.576508415	0.399339943
	$1-(1+i)^{-n}$	0.307312899	0.423491585	0.600660057
	$\frac{1}{1-(1+i)^{-n}}$	0.050166025	0.036403714	0.025666209
Monthly payment	P	453751.6994	329,271.60	**232,150.86**
Total	$P\times n$	10890040.79	11,853,777.45	**13,929,051.72**
Total of interest	$(P\times n)-M$	1,845,040.79	2,808,777.45	**4,884,051.72**

The calculation gives the same results as those shown by the excel sheet of the bank.

Interpretation of results and conclusion

- In a long-term loan, the monthly payment is low, but the total interest is high. This means that when your salary is low, you are obliged to take a long-term loan. For a loan of 9,045,000Frw in 12 months, the borrower is asked to pay 453,731.7Frw per month but in 60 months, the borrower will need to pay 232,150.9Frw per month.
- In a short-term loan, the monthly payment is high, but the total interest is low.

- The borrower will take a decision considering how much money S/he is able to pay per month and the total interest S/he do not want to give the bank.

Note:

When r is the interest rate, n the number of instalments per year, t the number of year, above formula can be transformed into

$P = \frac{M \times \frac{r}{n}}{1-\left(1+\frac{r}{n}\right)^{-nt}}$. This is the formula seen in ordinary level

Taking $i = \frac{r}{n}$ we come back to the formula $P = M\left[\frac{i}{1-(1+i)^{-n}}\right]$ where nt stands for the total number of instalments (number of all months).

4.9 End unit assessment

1) Find first four terms of the sequence

a) $\left\{\frac{1-n}{n^2}\right\}$ b) $\left\{\frac{(-1)^{n+1}}{2n-1}\right\}$ c) $\left\{2+(-1)^n\right\}$

2) Find the formula for the n^{th} term of the sequence

a) 1, -1, 1, -1, 1, …
b) 0, 3, 8, 15, 24, …
c) 1, 5, 9, 13, 17, …

3) Which of the following sequences converge, and which diverge? Find the limit of each convergent sequence.

a) $\left\{\sqrt{\frac{2n}{n+1}}\right\}$ b) $\left\{\frac{n}{2^n}\right\}$ c) $\left\{8^{\frac{1}{n}}\right\}$

4) A mathematical child negotiates a new pocket money deal with her unsuspecting father in which she receives 1 pound on the first day of the month, 2 pounds on the second day, 4 pounds on the third day, 8 pounds on the fourth day, 16 pounds on the fifth day, … until the end of the month. How much would the child receive during the course of a month of 30 days? (Give your answer to the nearest million pounds).

5) A culture of bacteria doubles every 2 hours. If there are 500 bacteria at the beginning, how many bacteria will there be after 24 hours?

6) You complain that the hot tub in your hotel suite is not hot enough. The hotel tells you that they will increase the temperature by 10% each hour. If the current temperature of the hot tub is 75º F, what will be the temperature of the hot tub after 3 hours, to the *nearest tenth* of a degree?

7) The sum of the interior angles of a triangle is 180º, of a quadrilateral is 360º and of a pentagon is 540º. Assuming this pattern continues, find the sum of the interior angles of a dodecagon (12 sides).

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